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Complex Geometry and Dynamical Systems

Organized by
Tien-Cuong Dinh, Singapore
George Marinescu, Köln
Valentino Tosatti, Montréal
Elizabeth Wulcan, Gothenburg

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Abstract. The workshop focused on recent developments in holomorphic dynamics, several complex variables and complex geometry. The topics of the talks included dynamics of holomorphic and rational maps, the theory of currents, the Bergman kernel, together with applications in geometry, dynamics, foliations and mathematical physics.

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Introduction by the Organizers

The workshop on Complex Geometry and Dynamical Systems was organized by Tien-Cuong Dinh (Singapore), George Marinescu (Köln), Valentino Tosatti (New York) and Elizabeth Wulcan (Gothenburg).

Complex geometry is a highly attractive branch of modern mathematics that witnesses active and successful research. Due to its interactions with various other fields (differential, algebraic, and arithmetic geometry, but also mathematical physics), it has become an area with many facets. The connection to Dynamical Systems is particularly fruitful. These subjects are advancing on many fronts due to several recent developments in pluripotential theory, Kähler geometry, and intersection theory for currents. The meeting focused on recent progress on pluripotential theory (including the non-Kähler setting), Monge-Ampère equations, foliation theory, Levi-flat hypersurfaces, complex dynamics and theory of currents, Bergman kernels, statistical properties of holomorphic maps and zeros of holomorphic sections.

The workshop had 47 on-site participants, and 12 online participants. Altogether we had 24 talks (45 minutes each), covering a broad range of topics. There were intensive interactions and lively discussions among the participants after the talks and during the breaks.

The opening talk was given by *Duong H. Phong*, who described his recent work proving uniform estimates for a very large class of geometric PDEs on complex manifolds, including the complex Monge-Ampère equation, using direct PDE methods as opposed to classical pluripotential techniques. In particular, he explained a new proof of a celebrated L^∞ estimate of Kołodziej, and its extensions to many other geometric PDEs. Complex Monge-Ampère equations also featured prominently in the talk by *Vincent Guedj*, in the context of non-Kähler Hermitian manifolds. In this realm a major challenge is that one loses a priori control on the Monge-Ampère volumes (due to the non-closedness of the Hermitian form) and there is a priori the possibility that these volumes might approach 0 or ∞. The main result presented rules out these scenarios if the Hermitian form is pluriclosed, and interesting examples are furnished by 3-dimensional nilmanifolds. Another talk on complex geometry was given by *Daniel Greb*, who presented recent results about the cycle space of the period domain of complex $K3$ surfaces. Unlike the $K3$ period domain, this cycle space has many nice properties, and over a large portion of it there is a marked family of $K3$ surfaces, which can be interpreted in terms of deformations of complex hyperkähler metrics. *Min Ru* has focused his talk on the recent interaction between geometry (namely the K-stability), the Nevanlinna theory and Diophantine approximation.

Foliation theory was originally introduced by Poincaré in order to study equations from physics that one cannot solve explicitly. The goal is to study directly the properties of the solutions which are represented by leaves of foliations in the phase space. Holomorphic foliations are a classical area where holomorphic dynamics and several complex variables interact fruitfully, and several talks were devoted to this topic. *Judith Brinkschulte* discussed codimension 1 foliations on compact complex manifolds (of dimension 3 or higher) with ample normal bundle, and showed that every leaf (which must necessarily be noncompact) accumulates on the singularities of the foliation, as was conjectured by Brunella. The talk by *Lucas Kaufmann* took a fresh look at a classical topic, the Baum-Bott residue classes for holomorphic foliations with singularities, which localize characteristic classes of the foliation around the singularities. These residue classes can be computed using Grothendieck residues when the leaves are complex curves and the singularities are isolated, but are more mysterious in general. The speaker explained a new construction of residue currents which naturally represent the Baum-Bott classes, which in particular allows for an explicit description of these currents in the rank 1 case when the singularities are not isolated. Foliations also featured in the talk by *Takayuki Koike* who studied compact Kähler manifolds which admit a semipositive definite smooth real $(1,1)$-form whose cohomology class contains at least another such form. In this case he constructs a codimension 1 holomorphic foliation away from a (possibly empty) closed set, and in particular deduces the existence of uncountably many compact real-analytic Levi-flat hypersurfaces in the manifold. He also mentioned applications to the study of embeddings of hypersurfaces in complex manifolds with numerically trivial normal bundle, which was directly related to the main topic of the talk by *Laurent Stolovitch*. In this talk, the speaker considered a complex torus embedded in a larger complex manifold, with numerically trivial normal bundle. Generalizing classical work of Arnol'd and Il'yashenko-Pyartli, they show that if the normal bundle satisfies a non-resonant Diophantine condition (which is almost-surely the case) then the torus has a holomorphic tubular neighborhood. Also related to the theory of foliations, *Jun-Muk Hwang* presented his study with Qifeng Li on $(2, 3, 5)$-distributions in the holomorphic setting and relate them to nondegenerate lines on holomorphic contact manifolds of dimension 5. This continues a celebrated work by E. Cartan and can be extended to higher dimension.

Another classical topics at the intersection of complex analysis and dynamics is the theory of complex currents. While smooth forms can be pulled back via holomorphic maps, there is no general pullback map for currents, and constructing one under suitable hypotheses is often needed for applications in dynamics. *Håkan Samuelsson Kalm* explained how to define the pullback of a large class of currents under an arbitrary holomorphic map between complex manifolds, so that the pullback enjoys many desirable properties. The talk by *Duc-Viet Vu* discussed the positive intersection product of closed positive $(1, 1)$-currents in big cohomology classes on compact Kähler manifolds, as introduced by Boucksom-Eyssidieux-Guedj-Zeriahi. It is known that some mass can be lost when taking this product, and a very natural question is to characterize those currents for which there is no loss of mass. For the top self-product of one such current there is a well-developed theory, however there were no such results known for general positive products of possibly different currents. The speaker presented a result that shows that in this case at least one of the currents has as small as possible Lelong number along any given analytic subvariety.

A topic appearing in several talks was the asymptotic expansion as $p \to \infty$ of the Bergman kernel of tensor powers L^p of a positive line bundle L. This asymptotic expansion plays a crucial role in the work of Tian, Donaldson and many others, where the existence of Kähler metrics with constant scalar curvature is shown to be closely related to the Chow-Mumford stability. The talk by *Viet-Anh Nguyen* studied the space of holomorphic sections of L^p, which vanish to order comparable to p along finitely many given analytic sets. The speaker discussed estimates of the partial Bergman kernel, convergence of the Fubini-Study currents and their potentials, and the equilibrium distribution of currents of integration on zero divisors of random holomorphic sections as $p \to \infty$. *Xiaonan Ma* presented an asymptotic version of Bismut's local family index theorem for the Bergman kernel by using superconnections, and showed that the curvature operator of the direct images associated with a fiberwise positive line bundle is a generalized Toeplitz operator. One consequence is a new proof of Berndtsson's result about the positivity of direct image bundles. This description is applied in the talk by *Nikhil Savale* to obtain a generalization of earlier results of Marinescu-Savale for semipositive line bundles to families of Bochner Laplacians. *Chin-Yu Hsiao* presented the asymptotic expansion near the boundary of the Bergman kernel of the $\overline{\partial}$-Neumann Laplacian of a complex manifold admitting a holomorphic $\mathbb{R}$-action. This has applications to compactification of pseudoconcave manifolds. The talk of *Siarhei Finski* dealt with semiclassical asymptotics of Ohsawa-Takegoshi extension operator associated with high tensor powers of a positive line bundle. This result is needed in Demailly's approach to the invariance of plurigenera for Kähler families. *Xiaojun Huang* explained the solution of a long standing conjecture by Cheng stating that if the Bergman metric of a smoothly bounded strongly pseudoconvex domain is Kähler-Einstein, then the domain is biholomorphic to the ball. Moreover, he showed that a Stein space with compact strongly pseudoconvex boundary must have spherical boundary and when the boundary is algebraic, then this Stein space has to be a finite ball quotient. *Sung-Yeon Kim* discussed proper holomorphic maps between bounded symmetric domains which extend to the Shilov boundary and gave a general form from and derived criteria for their rationality.

Concerning the fast developing theory of complex dynamics in higher dimension, *Fabrizio Bianchi* explained a unified approach to obtain the statistical properties of holomorphic maps on projective spaces. Soft techniques have been introduced to handle dynamical objects such as Green currents and equilibrium measure which are known to be rigid. *Jasmin Raissy* gave a talk about her study of holomorphic endomorphisms which are tangent to the identity at a fixed point. The goal is to obtain an example for which the immediate basin of attraction of the fixed point has an infinite number of distinct invariant connected components, where the orbits converge to the fixed point without being tangent to any direction (spiralling domain). *Keiji Oguiso* presented his recent results about some applications of dynamics in algebraic geometry. Namely, special automorphisms of algebraic varieties are used to study the finiteness and non-finiteness of real structures of these varieties. *Serge Cantat* talked about a recent development on random dynamics of automorphisms of a complex projective surface which can be seen as an analogue of some homogeneous dynamical systems. To study meromorphic dynamical systems on a compact Kähler surface, the hypothesis on the algebraic stability is often assumed in several methods. *Jeffrey Diller* showed that one can sometimes get around the stability hypothesis. Finally, inspired by complex dynamics, *Charles Favre* presented an extension of Gromov's upper bound of topological entropy for holomorphic maps to the case of dynamical systems defined over a non-Archimedean metrized field. Non-Archimedean dynamics is also a fast growing research direction.

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Workshop: Complex Geometry and Dynamical Systems

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Abstracts

Estimates for the complex Monge-Ampère equation and other fully non-linear equations

Duong H. Phong
(joint work with Bin Guo, Freid Tong)

In 1998, Kolodziej [14] had obtained sharp L^∞ estimates for the complex Monge-Ampère equation, considerably refining the estimates originally established by Yau [15] in his solution of the Calabi conjecture. Kolodziej's methods relied on pluripotential and capacity theory, which is specific to Monge-Ampère equations. It had been an open question since whether his sharp estimates can be obtained instead by PDE methods, with possible generalizations to other fully non-linear equations. This question was answered positively by the authors [9] in 2021. In this talk, we describe the answer, and we also survey many of the consequences of the new methods in [9].

Let (X, ω_X) be a compact n-dimensional Kähler manifold. If ω is another Kähler metric, and φ is a smooth function, we let $\omega_\varphi = \omega + i\partial\bar\partial\varphi$, and define h_φ to be the relative endomorphism $h_\varphi = (\omega_X)^{-1}\omega_\varphi$, or in components, $(h_\varphi)^j{}_k = \omega_X^{j\bar m}(\omega_\varphi)_{\bar m k}$. Let $\lambda[h_\varphi]$ be the unordered set of eigenvalues of h_φ. We shall consider non-linear operators $f : \Gamma \to \mathbf{R}_+$ defined on a cone Γ satisfying the following properties:

(1) $\Gamma \subset \mathbf{R}^n$ is a symmetric cone containing the first octant $\Gamma_n = \{\lambda \in \mathbf{R}^n; \lambda_j > 0, 1 \leq j \leq n\}$ and contained in $\lambda_1 + \cdots + \lambda_n > 0$;

(2) $f(\lambda)$ is symmetric in λ and homogeneous of degree 1;

(3) $\frac{\partial f}{\partial \lambda_j} > 0$ for each j and $\lambda \in \Gamma$;

(4) There is a $\gamma > 0$ such that

$$\prod_{j=1}^n \frac{\partial f}{\partial \lambda_j} > \gamma,\ \ \lambda \in \Gamma.$$

The Monge-Ampère equation corresponds to $f(\lambda) = (\prod_{j=1}^n \lambda_j)^{\frac{1}{n}}$. It has been shown by Harvey and Lawson [13] that the class of functions $f(\lambda)$ satisfying all the above conditions is quite large, and includes all invariant Garding-Dirichlet operators. They also noted that the condition (4) was introduced independently in [1] for the purpose of $W^{2,p}$ interior regularity. For any such $f(\lambda)$, we consider the second order PDE given by

$$f(\lambda[h_\varphi]) = c_\omega e^{F_\omega}, \qquad \sup_X \varphi = 0, \qquad h_\varphi \in \Gamma \tag{1}$$

where F_φ and c_φ are defined by the normalization $\int_X e^{nF_\omega}\omega_X^n = \int_X \omega_X^n$. We also set $V_\omega = \int_X \omega^n$. We have then the following theorem:

Theorem 1 [9]. Assume that $\omega \leq \kappa\,\omega_X$ for some constant $\kappa > 0$. Fix any $p > n$. Then any C^2 solution φ of the equation (1) must satisfy

$$\sup_X |\varphi| \leq C$$

where C is a constant depending only on $\omega_X, n, p, \gamma, \kappa$ and upper bounds for the following three quantities

$$\frac{c_\omega^n}{V_\omega}, \quad E(\omega) = \frac{c_\omega^n}{V_\omega}\int_X (-\varphi)e^{nF_\omega}\omega_X^n, \quad Ent_p(\omega) = \int_X e^{nF_\omega}|F_\omega|^p\omega_X^n. \tag{2}$$

Perhaps as important as the theorem itself is its proof. It is based on comparing the solution φ of (1) to the solution ψ of an auxiliary Monge-Ampère equation

$$(\omega + i\partial\bar\partial\psi_k)^n = \frac{\tau_k(-\varphi - s)}{A_{s,k}} f(\lambda[h_\varphi])\omega_X^n \tag{3}$$

with normalization $\sup_X \psi_k = 0$, and $\tau_k(x)$ is a sequence of strictly positive functions decreasing monotonically to the function $xH(x)$ where $H(x)$ is the Heaviside function. The coefficient $A_{s,k}$ is defined by $A_{s,k} = \int_X \tau_k(-\varphi - s)f(\lambda[h_\varphi])\omega_X^n$, so that the compatibility condition for the equation (3) is satisfied, and it admits a unique solution by Yau's theorem. Applying the maximum principle, we find the following relation between φ and ψ_k

$$-\varphi - s \leq \epsilon(-\psi + \Lambda)^{\frac{n}{n+1}} \tag{4}$$

if the following choices are made

$$\epsilon = \left(\frac{n+1}{n^2}\right)^{\frac{n}{n+1}} A_{s,k}^{\frac{1}{n+1}} \gamma^{-\frac{1}{n+1}}, \quad \Lambda = \frac{1}{(n+1)n^{n-1}}\frac{A_{s,k}}{\gamma}.$$

A comparison of φ with ψ_k seems naively ineffective, since ψ_k itself depends on φ. But it turns out that to get to the required L^∞ estimate for φ from (4), we need to know very little about ψ_k besides the exponential estimate resulting from the simple fact that ψ is ω-plurisubharmonic.

In [9], it was shown that, in the case of a fixed background metric ω, the energy $E(\omega)$ can be estimated by $\frac{c_\omega^n}{V_\omega}$ and the Nash entropy $Ent_p(\omega)$. The proof of this step was similar to the one of X.X. Chen and J.R. Cheng [4] for constant scalar curvature metrics, and required the Alexandrov-Bakelman-Pucci maximum principle [2], which worked only for fixed ω. Very recently, a more effective proof using a similar auxiliary Monge-Ampère equation as in the proof of Theorem 1 was found that works uniformly in ω, giving:

Theorem 2 [6]. Assume as before that $\omega \leq \kappa\,\omega_X$ for a fixed constant κ. Then for any $p > 0$, any C^2 solution φ of the equation (1) must satisfy the inequality

$$\int_X (-\varphi)^{pq} e^{nF_\omega}\omega_X^n \leq C$$

where the exponent q is defined to be $q = \frac{n}{n-p}$ if $p < n$, and can be any strictly positive exponent if $p \geq n$. The constant C is computable, and depends only on $\omega_X, n, p, q, \gamma, \kappa$ and upper bounds for $\frac{c_\omega^n}{V_\omega}$ and $Ent_p(\omega)$.

The method of proof of Theorem 1 has now been shown to extend to a large number of situations. We would like to mention in particular:

- Stability estimates for the Monge-Ampère and Hessian equations [10];
- L^∞ estimates for these equations in the case of nef classes [11];

- Estimates for the modulus of continuity of solutions to Monge-Ampère equations in non-Hölder cases [12];
- Estimates for parabolic equations [5];
- Lower bounds for the Green's function [8];
- L^∞ estimates for the Monge-Ampère and other fully non-linear equations on Hermitian manifolds [6]. Here the auxiliary Monge-Ampère equation is a Dirichlet problem on a ball, whose solvability is guaranteed by the classic theorem of Caffarelli-Kohn-Nirenberg-Spruck [3];
- L^∞ estimates for form equations [6]. In both this application as well as the preceding one, our method gives a much simplified proof of the known cases, together with extensions to a much wider class of equations;
- A general theory of estimates for diameters in Kähler geometry [7].

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Semiclassical Ohsawa-Takegoshi extension theorem and applications

Siarhei Finski

The results presented in the talk are based on the recent papers of the speaker [7], [8], [9], [10]. The main goal of those papers is to give an asymptotic version of Ohsawa-Takegoshi extension theorem when the power of the twisting positive line bundle tends to infinity (i.e. in the so-called semiclassical limit) and derive several consequences of such asymptotics.

More precisely, we fix two complex manifolds X, Y of dimensions n and m respectively. For the sake of simplicity, we assume that X and Y are compact, although our results work in a more general setting of manifolds and embeddings of bounded geometry. We fix also a complex embedding $\iota : Y \to X$, a positive line bundle (L, h^L) over X and an arbitrary Hermitian vector bundle (F, h^F) over X. In particular, we assume that for the curvature R^L of the Chern connection on (L, h^L), the closed real $(1,1)$-differential form $\omega := \frac{\sqrt{-1}}{2\pi} R^L$ is positive. We denote by g^{TX} the Riemannian metric on X so that its Kähler form coincides with ω. We denote by g^{TY} the induced metric on Y. We denote by dv_X, dv_Y the associated Riemannian volume forms on X and Y.

For smooth sections f, f' of $L^p \otimes F$, $p \in \mathbb{N}$, over X, we define the L^2-scalar product using the pointwise scalar product $\langle \cdot, \cdot \rangle_h$, induced by h^L and h^F as

$$\langle f, f' \rangle_{L^2(X)} := \int_X \langle f(x), f'(x) \rangle_h dv_X(x).$$

Similarly, using dv_Y, we introduce the L^2-scalar product for sections over Y.

A standard argument based on short exact sequences and Serre vanishing theorem implies that there is $p_0 \in \mathbb{N}$, such that for any $p \geq p_0$, $g \in H^0(Y, \iota^*(L^p \otimes F))$, there is $f \in H^0(X, L^p \otimes F)$, extending g, i.e. satisfying $f|_Y = g$.

We define the *optimal extension operator*

$$\mathrm{E}_p : H^0(Y, \iota^*(L^p \otimes F)) \to H^0(X, L^p \otimes F),$$

by putting $\mathrm{E}_p g = f$, where $f|_Y = g$, and f has the minimal L^2-norm among those $f' \in H^0(X, L^p \otimes F)$ satisfying $f'|_Y = g$. Clearly, the minimizing f exists and it is unique. Moreover, the operator E_p is linear since the minimality of the L^2-norm of f among different extensions is characterized by a linear condition, requiring f to be orthogonal to the space of holomorphic sections vanishing along Y. Our main goal is to find an explicit asymptotic expansion of the operator E_p, as $p \to \infty$.

We identify the normal bundle N of Y in X as orthogonal complement of TY in TX (with respect to g^{TX}) so that we have the orthogonal decomposition $TX|_Y \to TY \oplus N$. We denote by g^N the metric on N induced by g^{TX}.

For $y \in Y$, $Z_N \in N_y$, let $\mathbb{R} \ni t \mapsto \exp_y^X(tZ_N) \in X$ be the geodesic in X in direction Z_N. This map induces a diffeomorphism of $r_\perp$-neighborhood of the zero section in N with a tubular neighborhood U of Y in X for a certain $r_\perp > 0$. We use this identification, called *geodesic normal coordinates*, implicitly.

We denote by $\pi : U \to Y$ the natural projection $(y, Z_N) \mapsto y$. Over U, we identify L, F to $\pi^*(L|_Y), \pi^*(F|_Y)$ by the parallel transport with respect to the

Chern connections along the geodesic $[0,1] \ni t \mapsto (y, tZ_N) \in X$, $|Z_N| < r_\perp$, where the norm $|Z_N|$, $Z_N \in N$, is with respect to g^N.

We define the *trivial extension operator* $\mathrm{E}_p^0 : H^0(Y, \iota^*(L^p \otimes F)) \to L^2(X, L^p \otimes F)$: for $g \in H^0(Y, \iota^*(L^p \otimes F))$, we let $(\mathrm{E}_p^0 g)(x) = 0$, for $x \notin U$, and in U, we let

$$(\mathrm{E}_p^0 g)(y, Z_N) = g(y) \exp\Big(- p\frac{\pi}{2}|Z_N|^2\Big)\rho\Big(\frac{|Z_N|}{r_\perp}\Big),$$

where the bump function $\rho : \mathbb{R}_+ \to [0,1]$ satisfies

$$\rho(x) = \begin{cases} 1, & \text{for } x < \frac{1}{4}, \\ 0, & \text{for } x > \frac{1}{2}. \end{cases}$$

The Gaussian integral calculation gives us the following asymptotics

$$\big\|\mathrm{E}_p^0\big\| \sim \frac{1}{p^{\frac{n-m}{2}}}, \tag{1}$$

as $p \to \infty$, where $\|\cdot\|$ denotes the operator norm. Now, the section $\mathrm{E}_p^0 g$ satisfies $(\mathrm{E}_p^0 g)|_Y = g$, but it is not holomorphic over X (unless g is null). Nevertheless, as our main result says, $\mathrm{E}_p^0 g$ approximates very well the holomorphic section $\mathrm{E}_p g$. More precisely, we have the following result.

Theorem. *There are $C > 0$, $p_1 \in \mathbb{N}^*$, such that for any $p \geq p_1$, we have*

$$\big\|\mathrm{E}_p - \mathrm{E}_p^0\big\| \leq \frac{C}{p^{\frac{n-m+1}{2}}}.$$

Remark.

a) By (1), the theorem tells that the principal asymptotic term of the optimal extension operator is given by the trivial extension operator.

b) Our result refines Randriambololona [14, Théorème 3.1.10], stating that for any $\epsilon > 0$, there is $p_1 \in \mathbb{N}^*$, such that $\big\|\mathrm{E}_p\big\| \leq \exp(\epsilon p)$ for $p \geq p_1$.

Theorem above appears in [7, Theorem 1.1] as almost direct consequence of more precise results about the asymptotics of the *Schwartz kernel* $\mathrm{E}_p(x, y) \in (L^p \otimes F)_x \otimes (L^p \otimes F)_y^*$, $x \in X$, $y \in Y$, of E_p with respect to dv_Y. More precisely, in [7, Theorem 1.5], we show that $\mathrm{E}_p(x, y)$ has exponential decay with respect to the distance between the parameters. In particular, to understand fully the asymptotics of $\mathrm{E}_p(x, y)$, it suffices to do so for x, y in a neighborhood of a fixed point $(y_0, y_0) \in Y \times Y$ in $X \times Y$. In [7, Theorem 1.6], we show that after a reparametrization, given by a homothety with factor $\sqrt{p}$ in the so-called Fermi coordinates around (y_0, y_0), the Schwartz kernel $\mathrm{E}_p(x, y)$ admits a complete asymptotic expansion in integer powers of $\sqrt{p}$, as $p \to \infty$. The first two terms of this expansion can be easily calculated explicitly, and the first term corresponds to the Schwartz kernel of the optimal extension operator of the Bargmann space. This term corresponds in geodesic normal coordinates precisely to the first asymptotic term of the trivial extension operator. Establishing the above theorem then becomes a routine exercise.

Theorem above has several applications. Among those are the proof of the asymptotically optimal L^∞-bound on the extension of holomorphic sections [7, Theorem 1.10], refining previous results of Zhang [15, Theorem 2.2] and Bost [4, Theorem A.1], the proof of asymptotic transitivity of the optimal holomorphic extension with respect to a tower of submanifolds [8, Theorem 1.1], the proof of the asymptotic isometry for the space of holomorphic jets along the submanifold [9, Theorem 1.3] and the algebraic characterization of L^2-metrics on the section ring of an ample line bundle [10, Theorems 1.5 and 1.6].

The general strategy for dealing with semi-classical limits in our papers is inspired by Bismut [1] and Bismut-Vasserot [3]. Our methods are based on spectral techniques inspired by Bismut-Lebeau [2] and the proofs of the extension theorem due to Ohsawa-Takegoshi [13] and Demailly [6]. Technically, our work relies on exponential estimate for the Bergman kernel due to Ma-Marinescu [12], off-diagonal asymptotic expansion of the Bergman kernel due to Dai-Liu-Ma [5], and an asymptotic characterization of Toeplitz operators due to Ma-Marinescu [11].

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Topological entropy of endomorphisms of projective varieties over a general metrized field

Charles Favre
(joint work with Tuyen Trung Truong, Junyi Xie)

This report contains a summary of the results discussed during my talk based on [FTX22]. For pedagogical reasons, I chose to restrict my attention to analytic maps. We refer the interested reader to op. cit. for the more general case of meromorphic maps.

The most basic dynamical invariant associated to a continuous self-map of a compact space $f\colon X \to X$ is arguably its topological entropy $h_{\mathrm{top}}(f) \in [0, +\infty]$ which measures in a rough way its dynamical complexity. Recall that it is defined as the exponential growth of the complexity of the sequence of open covers $\mathfrak{U}_n := \mathfrak{U} \vee \cdots \vee f^{-n}\mathfrak{U}$ where $\mathfrak{U}$ is a fixed open cover of X. Our aim is to compute (or at least estimate) the topological entropy of a regular self-map $f\colon X \to X$ of a projective variety defined over some field k.

A first elementary observation is that $h_{\mathrm{top}}(f) = 0$ when X is endowed with its Zariski topology. This reflects the fact that the Zariski topology is too crude so that the dynamics of f does not exhibit chaos.

The first interesting case to consider is when $k = \mathbb{C}$, and one considers the map induced by f on the complex analytification X^{an} of X (which is compact). A famous theorem by Yomdin [Yo87] and Gromov [Gr03] states that in this situation

$$h_{\mathrm{top}}(f) = \max_{0 \le i \le d} \{\log \lambda_i(f)\} \ . \tag{1}$$

Here $\lambda_i(f) \ge 1$ is the dynamical degree of f firsyt introduced by Russakovskii and Shiffman [RS97], and defined as the spectral radius of the action of f by pull-back on the Dolbeault cohomology $H^{i,i}(X)$.

Let us briefly explain how Gromov obtained the upper bound in (1). First one observes that it is sufficient to estimate the complexity $N(n, \epsilon)$ of the cover $\mathfrak{U}_n$ when $\mathfrak{U}$ is the open cover given by the set of all balls of radius $\le \epsilon$ for some fixed $\epsilon > 0$. For each n, consider the Zariski closure $\Gamma_n \subset X^n$ of the set of orbits of length n, $(x, \cdots, f^{n-1}(x))$. Observe that Γ_n is an algebraic variety of dimension $d := \dim(X)$ for all n. Fix an ample line bundle $L \to X$, and let L_n be the restriction to Γ_n of the line bundle $\sum_{i=0}^{n-1} \pi_i^* L$ where π_i is the projection $X^n \to X$ onto the i-th factor.

We may choose a smooth positive metrization on L whose curvature defines a Kähler form ω on X. By Wirtinger's theorem the volume of Γ_n for the Kähler form $\omega_n := \sum_{i=0}^{n-1} \pi_i^* \omega$ can be computed in cohomological terms, and we get $\mathrm{Vol}_{\omega_n}(\Gamma_n) = \deg_{L_n}(\Gamma_n) := c_1(L_n)^{\wedge d}$. By a clever calculation, Gromov proves that $\limsup_n \frac{1}{n} \log \deg_L(\Gamma_n) \le \max_{0 \le i \le d}\{\log \lambda_i(f)\}$. On the other hand, the minimality of complex subvarieties (or a theorem by Lelong) implies the volume $\mathrm{Vol}_{\omega_n}(\Gamma_n)$ to be no less than $N(n, \epsilon)$ up to a uniform constant which implies the upper bound of (1).

Now suppose $(k, |\cdot|)$ be a complete metrized non-Archimedean field. Defining the topological entropy of f raises some difficulties in this context, since the set $X(k)$ of k-points in X can be endowed with a metric topology coming from the norm on k, but this space is not locally compact if $d \geq 1$ except if k is a local field. To get around this problem, one considers the Berkovich analytification X^{an} of X, see [Ber90]. This space has good topological properties and is in particular compact. Our main theorem can be stated as follows.

Theorem. *Let $(k, |\cdot|)$ be any complete metrized non-Archimedean field, and let $f\colon X \to X$ be any algebraic self-map of a projective variety X defined over k. Then the topological entropy of the map induced by f on the Berkovich analytification of X is bounded from above by* $\max_{0\leq i\leq \dim(X)}\{\log \lambda_i(f)\}$.

Here the dynamical degrees are computed in an algebraic way as the limit $\lambda_j(f) := \lim_n (f^{n*}c_1(L)^j \cdot c_1(L)^{d-j})^{1/n}$. It is a theorem by Dinh and Sibony [DS05] in characteristic 0 and by Tuyen [Tr20] in positive characteristic that this limit exists. The definition is also consistent with the one given previously in the complex case. Note that the theorem was obtained by the first author and Rivera-Letelier in the case $X = \mathbb{P}^1_k$, see [FR10, Théorème C].

Our proof in higher dimension follows very closely Gromov's argument. Definitions of a Kähler form have been proposed in [BFJ15, §2.3] and [Yu18, §3] in the non-Archimedean case based on the notion of models, but the general theory of these objects is not developed yet. In particular, it is unclear what the analog of Wirtinger's theorem should be. A model of X over $k^\circ = \{|z| \leq 1\} \subset k$ is by convention a flat projective scheme $\mathfrak{X} \to \operatorname{Spec} k^\circ$ whose generic fiber is isomorphic to X. In our argument, the choice of a relatively ample line bundle $\mathfrak{L} \to \mathfrak{X}$ replaces the choice of a Kähler form in the complex setting.

The first observation is that any fixed model $\mathfrak{X}$ defines an open cover $\mathfrak{U}$ and that it is sufficient to estimate the growth of the complexity $N(n, \mathfrak{X})$ of $\mathfrak{U}_n$. The second observation is that $N(n, \mathfrak{X})$ is bounded from above by the number Q_n of irreducible components of the model $\mathfrak{G}^{\langle n\rangle}$ of Γ_n obtained by taking the closure of Γ_n inside $\mathfrak{X}^n$. Next we prove that

$$Q_n \leq c_1(\mathfrak{L}_n)^{\wedge d} \cdot \mathfrak{G}_s^{\langle n\rangle}$$

where $\mathfrak{L}_n = \sum_{i=0}^{n-1} \pi_i^* \mathfrak{L}$, and π_i is the projection $\mathfrak{X}^n \to \mathfrak{X}$ onto the i-th factor. Using the invariance of intersection numbers under flat morphisms, we get $Q_n \leq c_1(L_n)^{\wedge d}$. We then adapt Gromov's calculation using ideas from [D20] to relate $c_1(L_n)^{\wedge d}$ to the dynamical degrees, and our proof is complete.

A large part of our article is then devoted to the characterization of self-maps whose topological entropy is 0. In dimension 1, such maps are exactly the ones conjugated to maps having good reduction, see [FR10]. We prove that any map having good reduction has topological entropy is 0 in arbitrary dimension. I refer to [FTX22] for more informations.

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Spiralling domains in dimension 2

Jasmin Raissy

(joint work with Xavier Buff)

In this joint work in progress with Xavier Buff, we study the dynamics of holomorphic endomorphisms $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ fixing the origin and *tangent to the identity* at the origin $\mathbf{0}$, i.e., such that $D_{\mathbf{0}}F = \mathrm{Id}$. In particular, we are interested in the *basin of attraction* of the origin, i.e., the set of points whose orbit under iteration of F converges to the origin:

$$\mathcal{B} = \left\{ \mathbf{z} \in \mathbb{C}^2 \mid F^{\circ n}(\mathbf{z}) \underset{n\to+\infty}{\longrightarrow} \mathbf{0} \right\}.$$

This set is a priori neither open nor closed. It may have nonempty interior, as it happens for the map $F(z,w) = (z - z^2, w - w^2)$. But it may also have empty interior, and this is the case if F is a complex Hénon map. A *parabolic domain P at the origin* for F is a connected component of the interior of $\mathcal{B}$ which is invariant under F (i.e., $F(P) \subseteq P$).

In our main result, we show the existence of polynomial maps $F\colon \mathbb{C}^2 \to \mathbb{C}^2$ which are tangent to the identity at the origin and have infinitely many *spiralling* parabolic domains, i.e., parabolic domains with orbits converging to the origin without being tangent to any direction.

Spiralling parabolic domains are a purely higher dimensional phenomenon. In fact, in dimension one, the classical Leau-Fatou flower Theorem ensures that there are only finitely many parabolic domains and neither of them can be spiralling.

Theorem 1 (Leau-Fatou flower Theorem [4, 6])**.** *Let $f(z) = z + az^{k+1} + O(z^{k+2})$, with $k \geq 1$ and $a \neq 0$, be a holomorphic function fixing the origin and tangent to the identity at the origin. Then there are k distinct* attracting *directions $v_1, \ldots, v_k$, i.e., unit vectors $v_j \in \mathbb{S}^1$ so that $a \cdot v_j^k$ is real and negative, and k disjoint parabolic domains $P_1, \ldots, P_k$ such that in each P_j orbits converge to the origin tangentially to the attracting direction v_j.*

Applying the Leau-Fatou flower Theorem to the inverse of f, we obtain a description of the dynamics in a full neighbourhood of the origin.

Given $F: \mathbb{C}^2 \to \mathbb{C}^2$ fixing the origin and tangent to the identity at the origin, it is natural to take inspiration from the one-dimensional picture, and look at the first non-linear term V in the homogeneous expansion of F at the origin. One can search for *preferred* directions for the dynamics, but can also interpret V as a homogeneous vector field, and try to understand orbits of F by studying the real-time trajectories of the vector field V.

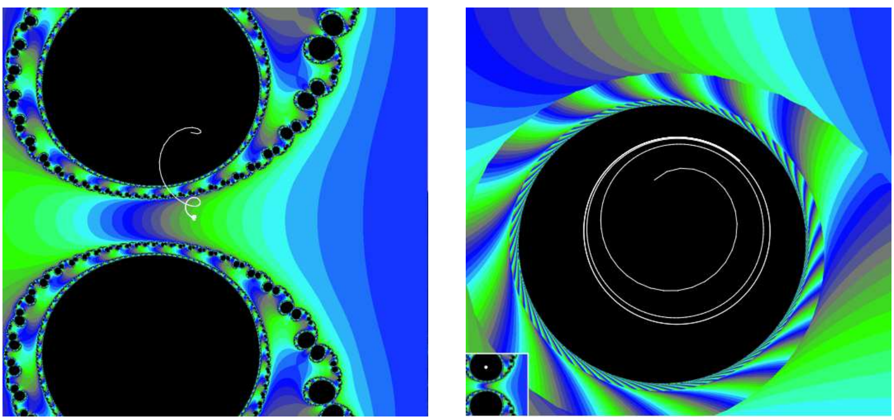


FIGURE 1. Left: slice $\{z = -w\}$ for $F(z,w) = (z - z^2, w + w^2 + 4z^2)$. In black, points of the slice whose orbit stays bounded; in white, projection on $\{z = -w\}$ of an orbit starting in the black region. Right: in white, radial projection of the directions of the points in the orbit on the left.

If an orbit of F converges to the origin tangentially to a complex direction $[t] \in \mathbb{P}^1(\mathbb{C})$, then the complex line passing through the origin and directed by $[t]$ is invariant under the homogeneous map V (see [5] for a proof). Such a direction is called *characteristic*, and Écalle [3] and Hakim [5] gave the first sufficient conditions ensuring the existence of parabolic domains where orbits converge to the origin tangentially to a characteristic direction.

Rivi constructed in [7] a first example of orbits converging to the origin without being tangent to any direction, and Rong provided in [8] a first example of spiralling domain. In these examples orbits converge spiralling *around* a characteristic direction (see figure 1).

In the previous examples, one could prove the existence of *finitely many* parabolic domains. The precise statement of our main result is the following (see figures 2 and 3).

Theorem 2 (Buff-R., in progress)**.** *For $a \in \mathbb{R}\setminus\{0\}$, the polynomial endomorphism $F_a\colon \mathbb{C}^2 \to \mathbb{C}^2$ defined by*

$$F_a\begin{pmatrix} z \\ w \end{pmatrix} = \begin{pmatrix} z \\ w \end{pmatrix} + \begin{pmatrix} w^2 \\ z^2 \end{pmatrix} + a\begin{pmatrix} z(z-w) \\ w(z-w) \end{pmatrix} \tag{1}$$

has infinitely many distinct spiralling parabolic domains.

To prove such result, we study the real-time trajectories of homogeneous vector fields, geodesics on affine surfaces (following ideas from [1]) and triangular billiards.

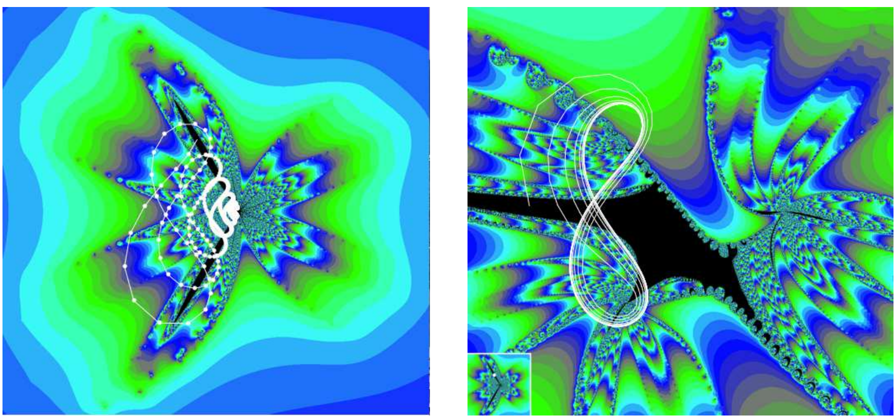
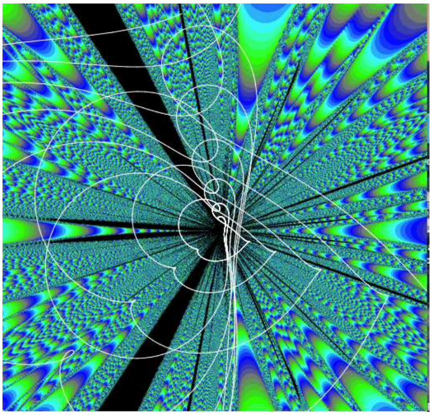


FIGURE 2. Left: slice $\{z = -w\}$ for F_a as in (1) with $a = 0.1$. In black, points of the slice whose orbit stays bounded; in white, projection on $\{z = -w\}$ of an orbit starting in a black region. Right: in white, radial projection of the directions of the points in the orbit on the left.



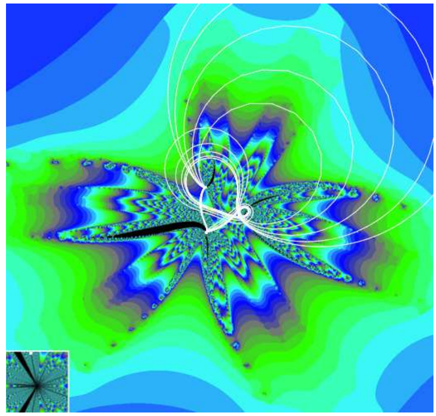


Figure 3. Left: zoom in the slice $\{z = -w\}$ for F_a as in (1) with $a = 0.1$. In black, points of the slice whose orbit stays bounded; in white, projection on $\{z = -w\}$ of an orbit starting in a black region different from that in figure 2. Right: in white, radial projection of the directions of the points in the orbit on the left.

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Semi-classical Bergman kernel asymptotics on complex manifolds with boundary

Chin-Yu Hsiao

(joint work with Xiaoshan Li, George Marinescu)

Let M be a relatively compact connected open subset with smooth connected boundary X of a complex manifold M' of complex dimension n. Let $(L, h^L) \to M'$ be a holomorphic line bundle over M'. Let

$$H^0(\overline{M}, L) := \{u \in \mathcal{C}^\infty(\overline{M}, L);\, \overline{\partial} u = 0\},$$

where $\overline{\partial}$ is the Cauchy-Riemann operator acting on sections of L. Let $H^0_{(2)}(\overline{M}, L) \subset L^2(M, L)$ be the L^2 closure of $H^0(\overline{M}, L)$ and let

$$\Pi : L^2(M, L) \to H^0_{(2)}(\overline{M}, L)$$

be the orthogonal projection (Bergman projection). The study of boundary behavior of Π is a classical subject in several complex variables. When M is strongly pseudoconvex and L is trivial, Fefferman [Fef74] obtained an asymptotic expansion of Π on the diagonal of the boundary. Subsequently, Boutet de Monvel-Sjöstrand [BS76] described the singularities of the distribution kernel of Π by showing that it is a complex Fourier integral operator (see also [Hsiao], [HM19]). When M is not strongly pseudoconvex, for example, if M is strongly pseudoconcave, the space $H^0(\overline{M}, L)$ could be trivial. In general, the usual methods (L^2-estimates for $\overline{\partial}$) cannot be used to construct holomorphic section in $H^0(\overline{M}, L)$, if M is not pseudoconvex. To get non-trivial holomorphic sections of M and a rich and flexible theory, we can consider high tensor powers L^k and study asymptotic behavior of the space $H^0_{(2)}(\overline{M}, L^k)$ and the orthogonal projection

$$\Pi_k : L^2(X, L^k) \to H^0_{(2)}(\overline{M}, L^k).$$

The study of large k behaviour of Π_k is closely related to the problem of extending the Kodaira embedding theorem and deformation theory to complex manifolds with boundary. and is a fundamental question in the study of complex manifolds with boundary. The difficulty of the study of Π_k comes form the fact that we do not know if the associated $\overline{\partial}$-Neumann Laplacian has spectral gap. The spectral gap property plays an important role of the study of Bergman kernel asymptotic expansions on complex manifolds without boundary (see [MM07]). The boundary X is a compact CR manifold and L is a CR line bundle over X. In [HLM], [HHL17], it was shown that if X admits a transversal and CR $\mathbb{R}$-action and (L, h^L) is a $\mathbb{R}$-invariant positive CR line bundle, then the associated Kohn Laplacian has *partial spectral gap* and the associated Szegő kernel admits a full asymptotic expansion. Therefore, it is natural to study Π_k when M' admits a holomorphic $\mathbb{R}$-action. A big difference between CR case and complex manifolds with boundary case is that even M' admits a holomorphic $\mathbb{R}$-action, it is still very difficult to see if the associated $\overline{\partial}$-Neumann Laplacian has spectral gap or *partial spectral gap*. From the Szegő kernel asymptotic expansion obtained in [HLM], [HHL17] and by carefully study semi-classical Poisson operator and using some kind of reduction to the boundary technique, we establish an asymptotic expansion for the Bergman kernel of the $\overline{\partial}$-Neumann operator on M with respect to high powers of a positive line bundle L under under certain natural assumptions.

More precisely, in this work [HLM22], we assume that M' admits a holomorphic $\mathbb{R}$-action and the $\mathbb{R}$-action preserves the boundary of M and is CR-transversal to the boundary. Under these assumptions and some curvature conditions, we establish an asymptotic expansion for the Bergman kernel of the $\overline{\partial}$-Neumann operator on $\overline{M}$ with respect to high powers of L.

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Bergman-Einstein Metrics on a Normal Stein Space

Xiaojun Huang

This talk is taken from two recent papers, one jointly with M. Xiao at UCSD [2] and one with X. Li from Wuhan University [1].

For any bounded domain in $\mathbb{C}^n$, there exists a canonical Kähler metric-the Bergman metric. Cheng-Yau proved that there exists a complete Kähler-Einstein metric on a bounded pseudoconvex domain in $\mathbb{C}^n$ with a C^2-smooth boundary. A well-known open question by Cheng asked in 1979 asks when the Bergman metric on a bounded pseudoconvex domain with smooth boundary is Kähler-Einstein. It is well known that the Bergman metric on the unit ball is Kähler-Einstein. A long standing conjecture given by Cheng states that if the Bergman metric of a smoothly bounded strongly pseudoconvex domain is Kähler-Einstein, then the domain is biholomorphic to the ball. This conjecture was solved by Fu-Wong and Nemirovski-Shafikov in the case of complex dimension two. In the first paper mentioned above, Huang-Xiao gave a confirmative answer to this conjecture in higher dimension. In the same paper, Huang-Xiao generalized the classical Kerner theorm to a stein space with possible isolated singularities. A crucial ingredient of Huang-Xiao's proof of the Cheng's conjecture is a classical uniformization result of Lu which is generalized to stein space. In the same paper, they asked if the generalization of Cheng's conjecture on Stein spaces with compact strongly pseudoconvex boundaries is still true or not. In the second paper mentioned above, Huang-Li proved that such a Stein space must have spherical boundary. When the boundary is algebraic, then this Stein space has to be a finite ball quotient. The general conjecture remains open.

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On dynamical aspects of holomorphic foliations with ample normal bundle

Judith Brinkschulte
(joint work with Masanori Adachi)

Let X be a complex manifold of dimension $n \geq 2$. We discuss holomorphic foliations in the following sense:

Definition. We say that a collection of holomorphic 1-forms $\mathcal{F} = \{\omega_\mu\}$, where $\omega_\mu \in \Omega^1(U_\mu)$ and $\mathcal{U} = \{U_\mu\}$ is an open covering of X, define a *codimension one holomorphic foliation* on X if they satisfy the following conditions: for any μ and ν,

(1) There exists $g_{\mu\nu} \in \mathcal{O}^*(U_\mu \cap U_\nu)$ such that $\omega_\mu = g_{\mu\nu}\omega_\nu$ on $U_\mu \cap U_\nu$;
(2) The analytic set $\{p \in U_\mu \mid \omega_\mu(p) = 0\}$ has codimension ≥ 2;
(3) The integrability condition is fulfilled: $\omega_\mu \wedge d\omega_\mu = 0$ on U_μ.

Here Ω^1 denotes the sheaf of germs of holomorphic 1-forms on X. From the first condition, we see that $\{g_{\mu\nu}\}$ enjoys the cocycle condition and defines a holomorphic line bundle over X. We call it the *normal bundle* of $\mathcal{F}$ and denote it by $N_\mathcal{F}$.

From the first and second condition, the zero sets of the ω_μ's glue together and define an analytic set of codimension ≥ 2 on X. We call it the *singular set* of $\mathcal{F}$ and denote it by $\operatorname{Sing}(\mathcal{F})$. We also denote $X^\circ := X \setminus \operatorname{Sing}(\mathcal{F})$ for a given foliation $\mathcal{F}$.

On X°, the kernels of the ω_μ's define an integrable holomorphic subbundle of $T^{1,0}_{X^\circ}$ of corank one, called the *tangent bundle* of $\mathcal{F}$, denoted by $T_\mathcal{F}$. It follows that X° is decomposed into the union of all the maximal integral submanifolds of $T_\mathcal{F}$, the leaves of $\mathcal{F}$. Therefore, we may think of a foliation as a higher dimensional analogue of flows on X.

From this perspective, it would be natural to seek for a Poincaré–Bendixson type property for foliations.

When X is compact and $N_\mathcal{F}$ is ample, Baum–Bott theory tells us that $\operatorname{Sing}(\mathcal{F})$ cannot be empty. It is also not difficult to see that $\mathcal{F}$ does not have any compact leaf.

Our main result in [1] is that $\mathcal{F}$ admits no nontrivial minimal sets if $N_\mathcal{F}$ is ample.

Main Theorem. Let X be a compact complex manifold of dimension ≥ 3. Let $\mathcal{F}$ be a codimension one holomorphic foliation on X with ample normal bundle $N_\mathcal{F}$. Then every leaf of $\mathcal{F}$ accumulates to $\operatorname{Sing}(\mathcal{F})$.

This result was conjectured by Brunella in [3].

In the special case of $X = \mathbb{CP}^n,\ n \geq 3$, the result goes back to Lins Neto [6]; note that in this situation, the normal bundle $N_{\mathcal{F}}$ is automatically ample since $\mathbb{CP}^n$ has positive holomorphic bisectional curvature. In [4], Brunella gave an affirmative answer to his conjecture when X is a complex torus or, more generally, a compact homogeneous manifold. Also, under the assumption that $\mathrm{Pic}(X) = \mathbb{Z}$, Brunella and Perrone confirmed the conjecture in [5].

If $\dim X = 2$, the problem becomes more difficult, and even for the special case $X = \mathbb{CP}^2$, no answer is known.

On the other hand, the Main Theorem might be seen as a further generalization of nonexistence theorems for compact Levi-flat real hypersurfaces that have attracted a great interest in the field of complex analysis over the last decades. Several results concerning the nonexistence of smooth real hypersurfaces invariant by a holomorphic foliation or, more generally, a Levi-flat real hypersurface, related to positivity of the normal bundle can be found in [6], [8], [3], [7] and [2]. In this setting, however, the assumption $\dim X \geq 3$ is crucial: examples for $\dim X = 2$ can be found in [3].

The proof of the Main Theorem is by contradiction. If there exists a leaf $\mathcal{L}$ of $\mathcal{F}$ such that $M := \overline{\mathcal{L}}$ does not intersect $\mathrm{Sing}(\mathcal{F})$, then Brunella's convexity result proved in [4] implies that $X \setminus M$ is strongly pseudoconvex. Then we use analytic tools, L^2 Hodge theory on complete Kähler manifolds, which permits to localize $c_1(N_{\mathcal{F}})$, the first Chern class of the normal bundle, to the maximal compact analytic set $A \subset X \setminus M$. Once we can localize $c_1(N_{\mathcal{F}})$ to A, a contradiction easily follows from the $\partial\overline{\partial}$-lemma and the maximum principle for strictly plurisubharmonic functions.

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A spectral gap for the transfer operator on complex projective spaces

Fabrizio Bianchi
(joint work with Tien-Cuong Dinh)

We develop a new method, based on pluripotential theory, to study the transfer operator induced on $\mathbb{P}^k = \mathbb{P}^k(\mathbb{C})$ by a generic holomorphic endomorphism and a suitable continuous weight. This method not only allows us to prove the existence and uniqueness of the equilibrium state and conformal measure for very general weights (due to Denker-Przytycki-Urbański in dimension 1 and Urbański-Zdunik in higher dimensions, both in the case of Hölder continuous weights), but, most importantly, to establish the existence of a spectral gap for the transfer operator and its perturbations on various functional spaces, which is new even in dimension 1, and to deduce a complete statistical study of the equilibrium states.

Given a *weight* $\phi\colon \mathbb{P}^k \to \mathbb{R}$, the *pressure* $P(\phi)$ is defined as $\sup\big\{\operatorname{Ent}_f(\nu) + \langle \nu, \phi\rangle\big\}$, where the supremum is taken over all f-invariant probability measures ν and $\operatorname{Ent}_f(\nu)$ denotes the metric entropy of ν. An *equilibrium state* for ϕ is a measure μ_ϕ realizing a maximum in the above formula. The *transfer* (or *Perron-Frobenius*) operator $\mathcal{L}$ with weight ϕ is defined as

$$\mathcal{L}g(y) := \mathcal{L}_\phi g(y) := \sum_{x\in f^{-1}(y)} e^{\phi(x)} g(x), \tag{1}$$

where $g\colon \mathbb{P}^k \to \mathbb{R}$ is a continuous test function and the points x in the sum are counted with multiplicity. A *conformal measure* is an eigenvector for the dual operator $\mathcal{L}^*$ acting on positive measures.

When ϕ is Hölder continuous and $\Omega(\phi) := \max\phi - \min\phi < \log d$, the existence and uniqueness of equilibrium states and conformal measures in any dimension was established by Urbański-Zdunik [4], under a genericity assumption for f. When ϕ is constant, the operator $\mathcal{L}$ reduces to a constant times the push-forward operator f_* and the equilibrium state is the measure of maximal entropy. For an account of the known results in this case, see, e.g., [3]. Our first result is as follows.

Theorem 1 ([1])**.** *Let f be a generic endomorphism of $\mathbb{P}^k$ of algebraic degree $d \geq 2$. Let ϕ be a real-valued $\log^q$-continuous function on $\mathbb{P}^k$, for some $q > 2$, such that $\Omega(\phi) < \log d$. Then ϕ admits a unique equilibrium state μ_ϕ, which is K-mixing and mixing of all orders, and repelling periodic points of period n (suitably weighted) are equidistributed with respect to μ_ϕ as n goes to infinity. Moreover, there is a unique conformal measure m_ϕ associated to ϕ. We have $\mu_\phi = \rho m_\phi$ for some strictly positive continuous function ρ on $\mathbb{P}^k$ and the preimages of points by f^n (suitably weighted) are equidistributed with respect to m_ϕ as n goes to infinity.*

We say that a function is $\log^q$-continuous if its oscillation on a ball of radius r is bounded by a constant times $(\log^\star r)^{-q}$.

A reformulation of Theorem 1 is the following: given ϕ as in the statement, there exist a number $\lambda > 0$ and a continuous function $\rho = \rho_\phi\colon \mathbb{P}^k \to \mathbb{R}$ such that, for every continuous function $g\colon \mathbb{P}^k \to \mathbb{R}$, the following uniform convergence holds:

$$\lambda^{-n}\mathcal{L}^n g(y) \to c_g\rho \tag{2}$$

for some constant c_g depending on g. By duality, this is equivalent to the convergence, uniform on probability measures ν, $\lambda^{-n}(\mathcal{L}^*)^n\nu \to m_\phi$, where m_ϕ is a conformal measure associated to the weight ϕ. The equilibrium state μ_ϕ is then given by $\mu_\phi = \rho m_\phi$, and we have $c_g = \langle m_\phi, g\rangle$.

The main idea of our method is as follows. Let us just consider g and ϕ of class $\mathcal{C}^2$ (the general case is handled with suitable approximations). First we prove that the ratio $\max \mathcal{L}^n g/\min \mathcal{L}^n g$ stays bounded with n. This allows us to define the good scaling ratio λ and to get that the sequence $\lambda^{-n}\mathcal{L}^n g$ is uniformly bounded. Next, we prove that this sequence is equicontinuous. This, together with other technical arguments, implies the existence and uniqueness of the limit function ρ.

In order to establish the above controls, we study the sequence of $(1,1)$-currents given by $dd^c\mathcal{L}^n g$. First we prove that suitably normalized versions of these currents are uniformly bounded by a common positive closed $(1,1)$-current R. This is the core of our method which replaces all controls on the distortion of inverse branches of f^n in the geometric method of [4] by a unique, global, and flexible estimate. Namely, for every $n \in \mathbb{N}$ we can get an estimate of the form

$$\Big|dd^c\frac{\mathcal{L}^n g}{c_n}\Big| \lesssim \sum_{j=0}^{\infty}\Big(\frac{e^{\Omega(\phi)}}{d}\Big)^j \frac{(f_*)^j\omega_{\mathrm{FS}}}{d^{(k-1)j}} \quad \text{with} \quad c_n := \|g\|_{\mathcal{C}^2}\langle \omega_{\mathrm{FS}}^k, \mathcal{L}^n 1\rangle. \tag{3}$$

Here, ω_{FS} denotes the usual Fubini-Study form on $\mathbb{P}^k$ normalized so that ω_{FS}^k is a probability measure. We then establish some general criteria which allow one to bound the oscillation of $c_n^{-1}\mathcal{L}^n g$ in terms of the oscillation of the potentials of the current in the RHS of (3), which is controllable.

Combining all these ingredients, the existence and uniqueness of the equilibrium state and conformal measure, as well as the equidistribution of preimages and the equality $P(\phi) = \log\lambda$, follow from more standard arguments.

Without extra arguments, the convergence in (2) is not uniform in g. Our next and main goal is to establish an exponential speed of convergence in (2). This requires to build a suitable (semi-)norm for which the operator $\lambda^{-1}\mathcal{L}$ is a contraction. The following statement is then our main result.

Theorem 2 ([2])**.** *Let f, q, ϕ, m_ϕ be as in Theorem 1 and $\mathcal{L}, \lambda$ the above transfer operator and scaling factor. Fix $A > 0$ and $0 < \Omega < \log d$. For every constant $0 < \gamma \le 1$, there exist two explicit equivalent norms for functions on $\mathbb{P}^k$: $\|\cdot\|_{\diamond_1}$, depending on f, γ, q, and $\|\cdot\|_{\diamond_2}$, depending on f, ϕ, γ, q, such that*

$$\|\cdot\|_\infty + \|\cdot\|_{\log^q} \lesssim \|\cdot\|_{\diamond_1} \simeq \|\cdot\|_{\diamond_2} \lesssim \|\cdot\|_{\mathcal{C}^\gamma},$$

and a positive constant $\beta = \beta(f, \gamma, q, A, \Omega) < 1$ such that when $\|\phi\|_{\diamond_1} \le A$ and $\Omega(\phi) \le \Omega$ we have

$$\big\|\lambda^{-1}\mathcal{L}g\big\|_{\diamond_2} \le \beta\|g\|_{\diamond_2} \quad \textit{for every function } g\colon \mathbb{P}^k \to \mathbb{R} \textit{ with } \langle m_\phi, g\rangle = 0.$$

The construction of the norms $\|\cdot\|_{\diamond_1}$ and $\|\cdot\|_{\diamond_2}$ is quite involved. We use ideas from the theory of interpolation between Banach spaces combined with techniques from pluripotential theory and complex dynamics. Roughly speaking, an idea from interpolation theory allows us to reduce the problem to the case where $\gamma = 1$. The definition of the above norms in this case requires a control of the derivatives of g (in the distributional sense), and this is where we use techniques from pluripotential theory. This also explains why these norms are bounded by the $\mathcal{C}^1$ norm. Note that we should be able to bound the derivatives of $\mathcal{L}g$ in a similar way. A quick expansion of the derivatives of $\mathcal{L}g$ using (1) gives an idea of the difficulties that one faces. The existence of these norms is still surprising to us.

Let us highlight two among these difficulties. First, the objects from complex analysis and geometry are too rigid for perturbations with a non-constant weight: none of the operators f_*, d, and dd^c commutes with the operator $\mathcal{L}$. In particular, the dd^c-method developed by Dinh-Sibony (see for instance [3]) cannot be applied in this context, even for small perturbations of the weight $\phi = 0$. Moreover, we have critical points on the support of the measure, which cause a loss in the regularity of functions under the operators f_* and $\mathcal{L}$.

Our solution to these problems is to define a new invariant functional space and norm in this mixed real-complex setting, taking into account both the regularity of the function (to cope with the rigidity of the complex objects) and the action of f (to take into account the critical dynamics). The construction of this norm requires the definition of several intermediate semi-norms and the precise study of the action of the operator f_* with respect to them.

A spectral gap for the transfer operator and its perturbations is one of the most desirable properties in dynamics. We deduce an exponential speed for the equidistribution of the backward orbits of points, and a full list of statistical properties for the equilibrium states: K-mixing, mixing of all orders, exponential mixing, CLT, Berry-Esseen theorem, local CLT, ASIP, LIL, LDP, almost sure CLT. Many of these properties are new even in dimension one, some even in the case of $\phi = 0$ (i.e., for the measure of maximal entropy). Properties such as the local CLT seem to be unattainable without a spectral gap.

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Holomorphic sections of line bundles vanishing along subvarieties

Viet-Anh Nguyen

(joint work with Dan Coman, George Marinescu)

Let (X, L) be a polarized projective manifold of dimension n, let Σ be a complex hypersurface of X, and let τ be a positive real number. The study of holomorphic sections of L^p which vanish to order at least $p\tau$ along Σ received much attention in the past few years. The density function of this space, called partial Bergman kernel, appears in a natural way in several contexts, and especially in Kähler geometry and pluripotential theory, linked to the notion of extremal quasiplurisubharmonic (qpsh) functions with poles along Σ, see e.g. [Be1, RoS, PS, RWN1, RWN2, CM3, ZZ]. One of the motivations is the notion of slope of the hypersurface Σ in the sense of Ross-Thomas [RT06] and its relation to the existence of a constant scalar curvature Kähler metric in $c_1(L)$.

In this paper we consider a compact normal complex space X of dimension n, a holomorphic line bundle L over X and the space $H^0_0(X, L^p)$ of holomorphic sections vanishing to order at least $p\tau_j$ along irreducible proper analytic subsets $\Sigma_j \subset X$, $j = 1, \dots, \ell$. We study algebraic and analytic objects associated to $H^0_0(X, L^p)$, especially the partial Bergman kernels, the Fubini-Study currents and their potentials.

We first give an analytic characterization for $H^0_0(X, L^p)$ to be big, which means by definition that $\dim H^0_0(X, L^p) \sim p^n$, $p \to \infty$. This criterion, stated in terms of singular Hermitian metrics with positive curvature current in the spirit of the Ji-Shiffman/Bonavero/Ta-kayama criterion for big line bundles, involves a desingularization of X where the Σ_j's become divisors.

Next we prove that under natural hypotheses the Fubini-Study currents associated to $H^0_0(X, L^p)$ and their potentials converge as $p \to \infty$. The limit of the sequence of Fubini-Study potentials is the push-forward φ_{eq} of a certain equilibrium envelope with logarithmic poles defined on a desingularization. The sequence of the Fubini-Study currents converge to the corresponding equilibrium current T_{eq}. These are analogues of Tian's theorem [T] which applies for smooth Hermitian metrics with positive curvature. In the context of singular Hermitian metrics they were introduced in [CM1, CM2]. The convergence of the Fubini-Study currents/potentials is based on the asymptotics of the logarithm of the partial Bergman kernel (see also [CM1, CM2, CMM, DMM] for results of this type concerning the full Bergman kernel).

Returning to the case of a polarized projective manifold (X, L), Shiffman-Zelditch [SZ] showed how Tian's theorem can be applied to obtain the distribution of the zeros of random holomorphic sections of $H^0(X, L^p)$. Dinh-Sibony [DS] used meromorphic transforms to obtain an estimate on the speed of convergence of zeros to the equilibrium distribution (see also [DMS] for the non-compact setting). Random polynomials or more generally holomorphic sections in high tensor powers of a holomorphic line bundle and the distribution of their zeros represent a classical subject in analysis (see e.g. [BP, ET, H, Ka]). The result of [SZ] was generalized for singular metrics whose curvature is a Kähler current in [CM1] and for sequences of line bundles over normal complex spaces in [CMM] (see also [CM2, DMM]). We show here that the equilibrium distribution of random zeros of sections from $H^0_0(X, L^p)$ is the equilibrium current T_{eq} and we give an estimate on the convergence speed.

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Plurisigned hermitian metrics

Vincent Guedj
(joint work with Daniele Angella, Hoang Chinh Lu)

The study of complex Monge-Ampère equations on compact hermitian (non-Kähler) manifolds has gained considerable interest in the last decade. Tosatti-Weinkove [TW10] and then Székelyhidi-Tosatti-Weinkove [STW17] have resolved the Gauduchon-Calabi-Yau conjecture, extending to the hermitian setting Yau's fundamental result [Yau78]. Associated degenerate complex Monge-Ampère equations have been systematically studied by Dinew, Kołodziej, and Nguyen (see e.g. [Din16, KN19]), as well as in [LPT21, GL21, GL21b].

By comparison with the setting of Kähler manifolds, a key new difficulty lies in the uniform control of Monge-Ampère volumes. Given X a compact complex manifold of complex dimension n equipped with a hermitian metric ω, it is of crucial importance to decide whether

$$v_+(\omega) := \sup \left\{ \int_X (\omega + dd^c\varphi)^n \ : \ \varphi \in \mathcal{C}^\infty(X) \text{ and } \omega + dd^c\varphi > 0 \right\}$$

is finite, and whether

$$v_-(\omega) := \inf \left\{ \int_X (\omega + dd^c\varphi)^n \ : \ \varphi \in \mathcal{C}^\infty(X) \text{ and } \omega + dd^c\varphi > 0 \right\}$$

is bounded away from zero. Here $d = \partial + \overline{\partial}$ and $d^c = \frac{1}{2i}(\partial - \overline{\partial})$.

It follows from Stokes theorem that $v_-(\omega) = v_+(\omega) = \int_X \omega^n$ when ω is closed or, more generally, when $dd^c\omega = 0$ and $dd^c\omega^2 = 0$. The latter conditions are however rather restrictive and it is an important open problem to decide whether $v_+(\omega)$ (resp. $v_-(\omega)$) is always finite (resp. positive). We refer the reader to [GL21, Theorem C] for an illustration of how the finiteness of $v_+(\omega)$ is related to a transcendental form of Demailly's holomorphic Morse inequalities, while [GL21b] strongly motivates the condition $v_-(\omega) > 0$.

It has been shown in [GL21, Theorem A] that the condition $v_+(\omega) < +\infty$ (resp. $v_-(\omega) > 0$) is independent of the choice of hermitian metric and is a bimeromorphic invariant. We further study these conditions in the joint work [AGL22], testing them on various classes of examples. We establish the finiteness of $v_+(X)$ (resp. positivity of $v_-(X)$) when X admits special plurisigned hermitian metrics.

Theorem A. *Let X be a compact complex manifold of dimension n.*

(1) *If there exists a pluripositive hermitian metric ω then $v_+(\omega) < +\infty$.*
(2) *If there exists a plurinegative hermitian metric ω then $v_-(\omega) > 0$.*

A hermitian metric ω is pluripositive if there exists $\varepsilon > 0$ such that $dd^c\omega \geq 0$ and $dd^c\omega^q \geq \varepsilon\omega \wedge dd^c\omega^{q-1}$, for $2 \leq q \leq n-2$. The plurinegativity condition is more involved; it can be reduced to $dd^c\omega \leq 0$ in dimension ≤ 3. In particular if $n = 3$ and ω is pluriclosed then $0 < v_-(\omega) \leq v_+(\omega) < +\infty$.

These conditions are always fulfilled when $n \leq 2$, so we initiate a systematic study of the 3-dimensional case. One can show that plurisigned hermitian metrics cannot coexist in dimension 3, i.e. the following conditions are mutually exclusive:

- X admits a hermitian metric ω such that $dd^c\omega \geq 0$ and $dd^c\omega \neq 0$;
- X admits a hermitian metric ω such that $dd^c\omega = 0$;
- X admits a hermitian metric ω such that $dd^c\omega \leq 0$ and $dd^c\omega \neq 0$;
- X does not admit any hermitian metric ω such that $dd^c\omega$ has a sign.

Each case does occur as we show by analyzing several classes of examples: (e.g. twistor spaces of K3 surfaces admit pluripositive hermitian metrics; Vaisman manifolds admit plurinegative hermitian metrics; non-Kähler manifolds from the class $\mathcal{C}$ of Fujiki do not admit any plurisigned hermitian metric).

We take a closer look at nilmanifolds $X = \Gamma\backslash G$ of (real) dimension 6, where G is a connected and simply connected nilpotent Lie group, and Γ is a discrete co-compact subgroup. There are 18 isomorphism classes of nilpotent Lie algebras in dimension 6 which admit a complex structure. Following [COUV16] we gather them in four large families (Np), (Ni), (Nii), (Niii) and show the following.

Theorem B. *Consider a six-dimensional nilmanifold $X = \Gamma\backslash G$ endowed with a left-invariant complex structure. There is always a plurisigned hermitian metric. More precisely if X is not a complex torus, then*

- *either X belongs to one of the classes (Np), (Nii), (Niii) and then any left-invariant hermitian metric is pluripositive but not pluriclosed.*
- *or X belongs to the class (Ni) and –depending on the complex structure– it admits a left-invariant hermitian metric wich is either pluriclosed, or pluripositive but not pluriclosed, or else plurinegative but not pluriclosed.*

This analysis largely generalizes the influential work of Fino-Parton-Salamon [FPS04] who characterized the existence of pluriclosed metric in this context.

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A pullback on a class of currents

Håkan Samuelsson Kalm

Let $f\colon X \to Y$ be a holomorphic mapping from a complex manifold X to a complex Hermitian manifold Y. We explain how the usual pullback f^* of smooth forms can be extended to a certain class of currents. Similar pullback problems turn up in, e.g., intersection theory (pullback of rational equivalence classes) and arithmetic geometry (pullback of Green currents). In complex geometry it has mainly been considered for closed positive (p,p)-currents; for instance by Dinh–Sibony in several papers, Meó, Truong and others, often with applications in complex dynamics in mind.

One cannot expect to have a reasonable pullback of all currents unless f is very special. If f is a submersion, then there is a well-known natural pullback of all currents, but since we allow arbitrary f we need to restrict to a subclass. We say that a current is a $\mathcal{PS}$-current if it locally is a finite sum of direct images of smooth compactly supported differential forms (of arbitrary bidegree) under an arbitrary holomorphic mapping. For instance, the Lelong integration current of any subvariety or cycle is a $\mathcal{PS}$-current. All $\mathcal{PS}$-currents have order 0, and the class of $\mathcal{PS}$-currents is closed under multiplication by smooth forms, under d, $\bar\partial$, and ∂, and under conjugation. Our class neither contains nor is contained in the class of closed positive currents, but there is a significant overlap. Our first result is

Theorem 1. *There is a linear bidegree preserving mapping $f^*\colon \mathcal{PS}(Y) \to \mathcal{PS}(X)$ extending the usual pullback of smooth forms such that f^* commutes with d, $\bar\partial$, and ∂; if φ is a smooth form in Y and $\mu \in \mathcal{PS}(Y)$, then $f^*(\varphi\wedge\mu) = f^*\varphi \wedge f^*\mu$; f^* is locally defined; if f is a submersion, then $f^*\mu$ coincides with the usual pullback of a current under a submersion.*

Some consequences of this result are: $\operatorname{supp} f^*\mu \subset f^{-1}\operatorname{supp}\mu$; if μ is smooth in an open $U \subset Y$, then $f^*\mu$ is the usual pullback in $f^{-1}U$; if $\widetilde{U} \subset X$ is open and $f|_{\widetilde{U}}$ is a submersion, then $f^*\mu$ is the usual pullback under a submersion in $\widetilde{U}$. We remark that $f^*\mu$ in general depends on the Hermitian structure of Y.

Our second result says that f^* is cohomologically sound. For proof-technical reasons (but probably not necessary ones) we assume that Y is *good*, by which we mean that there is a holomorphic vector bundle $F \to Y \times Y$ and a global holomorphic section of F vanishing precisely on the diagonal in $Y \times Y$ to first order. For instance, all projective manifolds are good, and any submanifold of a good manifold is again good.

Theorem 2. *Assume additionally that X is compact and that Y is compact and good. If $\mu \in \mathcal{PS}(Y)$ is closed, then*

$$[f^*\mu]_{dR} = f^*[\mu]_{dR},$$

where $[\cdot]_{dR}$ means de Rham cohomology class and $f^*[\mu]_{dR}$ is the usual pullback of a cohomology class. In particular, $[f^*\mu]_{dR}$ is independent of the choice of Hermitian structure on Y.

Our pullback is not functorial in general, and in fact one cannot expect it to be. Indeed, if we have

$$X_1 \xrightarrow{f_1} X_2 \xrightarrow{f_2} Y,$$

then $f_1^* f_2^* \mu$ depends in general on Hermitian structures on both X_2 and Y whereas $(f_2 \circ f_1)^*\mu$ only depends on the Hermitian structure on Y. However, since pullback of cohomology classes is functorial it follows from Theorem 2 that

$$[f_1^* f_2^* \mu]_{dR} = [(f_2 \circ f_1)^*\mu]_{dR}.$$

The basic idea to construct our pullback is standard; it is the following formal calculation. Let $\Gamma \subset X \times Y$ be the graph of f and let $\pi_1 \colon X \times Y \to X$ and $\pi_2 \colon X \times Y \to Y$ be the natural projections. If μ is a current in Y, then $\pi_2^*\mu$ is a well-defined current in $X \times Y$ since π_2 is a submersion. Suppose that the product

$$[\Gamma] \wedge \pi_2^*\mu \tag{1}$$

makes reasonable sense, where $[\Gamma]$ is the integration current along Γ. Then one can make the definition

$$f^*\mu := (\pi_1)_*([\Gamma] \wedge \pi_2^*\mu). \tag{2}$$

Notice that if μ is a smooth form, then (1) makes canonical sense and the right-hand side of (2) coincides with the usual pullback of μ.

The novelty in the construction of our f^* is our definition of the product (1) when $\mu \in \mathcal{PS}(Y)$. To indicate how this product is defined we assume for simplicity that Y is good. Then there is a holomorphic vector bundle $E \to X \times Y$ and a global holomorphic section Ψ of E vanishing to first order precisely on Γ. Let $N \to \Gamma$ be the normal bundle of Γ equipped with the Hermitian metric induced by the canonical isomorphism $N \simeq TY$. A key observation is the following formula, which follows from [2].

$$[\Gamma] = \hat{c}(N) \wedge M^\Psi,$$

where $\hat{c}(N)$ is the full Chern form of N and M^Ψ is a certain current introduced by Andersson in [1]. This current is a Monge–Ampère type product with support in Γ and it can be obtained as the weak limit of explicit smooth forms $M^{\Psi,\epsilon}$. It turns out that if $\mu \in \mathcal{PS}(Y)$, then the limit

$$M^\Psi \wedge \pi_2^*\mu := \lim_{\epsilon \to 0} M^{\Psi,\epsilon} \wedge \pi_2^*\mu$$

exists, has support in Γ, and is in $\mathcal{PS}(X \times Y)$. In general, even if μ has pure bidegree, $\hat{c}(N) \wedge M^\Psi \wedge \pi_2^*\mu$ has components of various different bidegrees. We let $(\hat{c}(N) \wedge M^{\Psi} \wedge \pi_2^*\mu)_{eb}$ be the components of the "expected bidegree", which is $(\dim Y, \dim Y) + \operatorname{bideg}\mu$. Our definition of (1) then is

$$[\Gamma] \wedge \pi_2^*\mu := (\hat{c}(N) \wedge M^{\Psi} \wedge \pi_2^*\mu)_{eb}.$$

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Holomorphic foliation associated with a semi-positive class of numerical dimension one

Takayuki Koike

Let X be a connected compact Kähler manifold and $\alpha \in H^{1,1}(X,\mathbb{R})(:= H^{1,1}(X,\mathbb{C}) \cap H^2(X,\mathbb{R}))$ be a class such that $\#\mathrm{SP}(\alpha) > 1$ and $\mathrm{nd}(\alpha) = 1$, where $\mathrm{SP}(\alpha)$ is the set of all C^∞'ly smooth d-closed semi-positive $(1,1)$-forms on X which represents the class α, and

$$\mathrm{nd}(\alpha) := \max\{k \in \{0,1,2,\dots,\dim X\} \mid \alpha^{\wedge k} \neq 0 \text{ in } H^{k,k}(X,\mathbb{C})\}.$$

For such a class α, we denote by K_α the closed subset of X defined by

$$K_\alpha := \bigcap_{\theta\in \mathrm{SP}(\alpha)} \bigcap_{\psi \in \mathrm{PSH}^\infty(X,\theta)} \{x \in X \mid (d\psi)_x = 0\},$$

where $\mathrm{PSH}^\infty(X,\theta)$ is the set of all the θ-plurisubharmonic functions of class C^∞ for a C^∞'ly smooth $(1,1)$-form θ on X: i.e. $\mathrm{PSH}^\infty(X,\theta) := \{\psi\colon X \to \mathbb{R} : C^\infty \mid \theta + \sqrt{-1}\partial\overline{\partial}\psi \geq 0\}$. Note that it follows from the $\partial\overline{\partial}$-lemma that $\mathrm{PSH}^\infty(X,\theta) \neq \mathbb{R}$ holds for $\theta \in \mathrm{SP}(\alpha)$ and $K_\alpha \neq X$, since $\#\mathrm{SP}(\alpha) > 1$. For such X and α, we show the following:

Theorem 1. *Let X and α be as above. Then there uniquely exists a non-singular holomorphic foliation $\mathcal{F}_\alpha$ on $X \setminus K_\alpha$ of complex codimension 1 such that $i_{\mathcal{L}}^*\theta \equiv 0$ for any $\theta \in \mathrm{SP}(\alpha)$ and any leaf $\mathcal{L}$ of $\mathcal{F}_\alpha$, where $i_{\mathcal{L}}\colon \mathcal{L} \to X$ is the holomorphic immersion.*

We investigate how large can $\mathcal{F}_\alpha$ be analytically extended by classifying the connected components of K_α from the view point of the existence of an $\mathcal{F}_\alpha$-*adaptive function* in the following sense on a suitable neighborhood: We say that a continuous function $h\colon \overline{W} \to [-\infty,+\infty]$ on the closure of a domain (connected open subset) W of X is $\mathcal{F}_\alpha$-adaptive if $h|_W$ is a $\mathbb{R}$-valued non-constant pluriharmonic function, $h|_{W\setminus K_\alpha}$ is $\mathcal{F}_\alpha$-leafwise constant, and the preimage $h^{-1}(\{\max_{\overline{W}} h, \min_{\overline{W}} h\})$ coincides with the boundary ∂W of W, where the topology of $[-\infty,+\infty]$ is the one such that $[-\infty,+\infty]$ is homeomorphic to the interval $[0,1] \subset \mathbb{R}$.

Definition. A connected component K' of K_α is said to be *an essential component* if there does **not** exist a connected open neighborhood W of K' in X such that $W \cap K_\alpha$ is a relatively compact subset of W and that there exists an $\mathcal{F}_\alpha$-adaptive function on $\overline{W}$. The union of all the essential components of K_α is denoted by K_α^{ess}.

Our second main result is the following:

Theorem 2. *Let X and α be as above. Then the holomorphic foliation $\mathcal{F}_\alpha$ on $X \setminus K_\alpha$ as in Theorem 1 can be extended to $X \setminus K_\alpha^{\mathrm{ess}}$ as a (maybe singular) holomorphic foliation. Moreover, one of the following holds:*

- **Case I:** *There exists a surjective holomorphic map $\Phi\colon X \to R$ to a compact Riemann surface R and a Kähler class α_R of R such that $\alpha = \Phi^*\alpha_R$. In this case, $\mathcal{F}_\alpha$ is the foliation defined by the fibration Φ, $K_\alpha^{\mathrm{ess}} = \emptyset$, K_α is included in the set of all the critical points of Φ, and the set of all the singular points $(\Phi^{-1}(p))_{\mathrm{sing}}$ of the (set-theoretical) fiber $\Phi^{-1}(p)$ is included in K_α for any point $p \in R$.*
- **Case II:** *Not in the case I and $K_\alpha^{\mathrm{ess}} = \emptyset$. In this case, there exist an open covering $\{U_1, U_2\}$ of X consisting of two domains and $\mathcal{F}_\alpha$-adaptive functions $h_j\colon \overline{U_j} \to [-\infty, +\infty]$ for each $j = 1, 2$ such that, on each connected component W of $U_1 \cap U_2$, there exist constants $a_W, b_W \in \mathbb{R}$ such that $h_2 = a_W h_1 + b_W$ holds on W. The foliation $\mathcal{F}_\alpha$ is defined on X, its tangent bundle is perpendicular to ∂h_j on U_j, and $K_\alpha \cap U_j = \{x \in U_j \mid (dh_j)_x = 0\}$ holds for $j = 1, 2$.*
- **Case III:** *$K_\alpha^{\mathrm{ess}} \neq \emptyset$. In this case, $X \setminus K_\alpha^{\mathrm{ess}}$ is a domain of X and there exists an $\mathcal{F}_\alpha$-adaptive function $h_\alpha\colon \overline{X \setminus K_\alpha^{\mathrm{ess}}} \to [-\infty, +\infty]$. The tangent bundle of the foliation $\mathcal{F}_\alpha$ on $X \setminus K_\alpha^{\mathrm{ess}}$ is perpendicular to ∂h_α, and $K_\alpha \setminus K_\alpha^{\mathrm{ess}} = \{x \in X \setminus K_\alpha^{\mathrm{ess}} \mid (dh_\alpha)_x = 0\}$ holds.*

As a corollary, one has the following:

Theorem 3. *Let X be a connected compact Kähler manifold. Assume that there exists a $(1,1)$-class $\alpha \in H^{1,1}(X, \mathbb{R})$ with $\#\mathrm{SP}(\alpha) > 1$ and $\mathrm{nd}(\alpha) = 1$. Then X admits uncountably many compact Levi-flat hypersurfaces of class C^ω (i.e. real analytic).*

Let Y be a non-singular hypersurface of X such that the normal bundle $N_{Y/X} = [Y]|_Y$ is unitary flat (i.e. $N_{Y/X} \in H^1(Y, \mathrm{U}(1))$, where $\mathrm{U}(1) := \{t \in \mathbb{C} \mid |t| = 1\}$), where $[Y]$ is the holomorphic line bundle on X which corresponds to the divisor Y. Note that the first Chern class $c_1([Y])$ of $[Y]$ satisfies $\mathrm{nd}(c_1([Y])) = 1$ in this case. Our motivation comes from the study of the relation between *the semi-positivity of* $[Y]$ (i.e. non-emptiness of $\mathrm{SP}(c_1([Y]))$) and the complex analytic structure of a neighborhood of Y. In [K1, Conjecture 2.1], we conjectured that $[Y]$ is semi-positive if and only if the pair (Y, X) is of class (β') or (β'') in the classification of Ueda [U]. The following corollary, which follows from [K2, Theorem 1.4] and the argument in the proof of Theorem 3, gives an affirmative answer to [K1, Conjecture 2.1] when Y is non-singular.

Corollary *Let X be a connected compact Kähler manifold and $Y \subset X$ be a non-singular connected hypersurface such that $N_{Y/X}$ is unitary flat. Then $[Y]$ is semi-positive if and only if there exists a neighborhood V of Y such that $[Y]|_V$ is unitary flat: i.e. there exists a non-singular holomorhic foliation on V which has Y as a leaf along which the holonomy is* $\mathrm{U}(1)$*-linear.*

Note that Ohsawa pointed out in [O, Remark 5.2] that Corollary for a surface X can be shown by combining [K2, Theorem 1.4] and Siu's solution [Si] of Grauert–Riemenschneider's conjecture (Kählerness assumption is not needed in his proof). Note also that this kinds of results can be regarded as a generalization of Brunella's theorem [B] for the blow-up of the projective plane at general nine points.

The foliation $\mathcal{F}_\alpha$ is constructed by considering the eigenvectors which belongs to the eigenvalue zero of each element of $\mathrm{SP}(\alpha)$, or equivalently, by considering *the Monge–Ampère foliation* for each element $\psi \in \mathrm{PSH}^\infty(X,\theta)$ for an element $\theta \in \mathrm{SP}(\alpha)$. In the proof, we show that $\sqrt{-1}\partial\overline{\partial}\psi = g_\psi \cdot \sqrt{-1}\partial\psi \wedge \overline{\partial}\psi$ holds for an $\mathcal{F}_\alpha$-leafwise constant function g_ψ on a suitable domain of X essentially by a linear-algebraic arguments. When g_ψ is a non-constant function on some level set of ψ, we show that the situation is in Case I. When g_ψ is constant on any level set of ψ, $g_\psi = \chi \circ \psi$ holds on a domain for some real function χ. By considering a solution G of a suitable ordinary differential equation concerning on χ, one can see that $h_0 := G \circ \psi$ is a pluriharmonic function. In this case, we show that the situation is either Case Ⅱ or Ⅲ by considering the analytic continuation of h_0.

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Dynamics of large groups of automorphisms of complex projective surfaces

Serge Cantat

(joint work with Romain Dujardin)

Consider a group Γ of automorphisms (i.e., holomorphic diffeomorphisms) of a complex projective surface X, and assume that Γ contains a non-abelian free group acting faithfully on the cohomology of X. Can we describe the asymptotic distribution of the Γ-orbits? This is related to a more technical question, namely the description of all $\nu-$stationary measures μ on X, where ν is a probability measure on $\mathrm{Aut}(X)$, the support of which generates Γ. In this talk, I explained results of the following types (under adequate hypotheses):

- stiffness results, showing that any stationary measure is in fact invariant;
- classification of invariant measures;
- finiteness results, for instance a finiteness result for the number of finite orbits.

The techniques rely on holomorphic dynamics, Hodge theory, ergodic and Pesin theory (in particular the work of Ledrappier and of Brown and Rodriguez-Hertz), and arithmetic dynamics (in particular, Yuan's arithmetic equidistribution theorem). Of course, we also use Furstenberg's theory of random products of matrices to study the dynamics of Γ in the cohomology of X.

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The cycle space of the K3 period domain and deformations of complex hyperkähler metrics

Daniel Greb

(joint work with Martin Schwald)

The Global Torelli Theorem implies that the "moduli space" of biholomorphism classes of complex (unpolarised) K3 surfaces can be described via Hodge theory as follows: Let

$$\Omega := \Big\{[x] \in \mathbb{P}(\Lambda_{\mathbb{C}}) \mid \langle x, x\rangle = 0, \langle x, \overline{x}\rangle > 0\Big\} \subset \Big\{[x] \in \mathbb{P}(\Lambda_{\mathbb{C}}) \mid \langle x, x\rangle = 0\Big\} =: Q,$$

where $(\Lambda, \langle\cdot,\cdot\rangle)$ is the even, unimodular lattice of signature $(3, 19)$. The set Ω is an open orbit of the real Lie group $G_0 = SO_0(3, 19)$ on the quadric Q. If $\Gamma = O^+(\Lambda)$ denotes the index two subgroup in the discrete subgroup $O(\Lambda)$ consisting of transformations whose real extension preserves the orientation on every positive three space inside $\Lambda_{\mathbb{R}}$, then the set-theoretic moduli space is Ω/Γ. In my talk, I recalled that in contrast to the situation e.g. of Riemann surfaces or polarised Abelian varieties at this point the following closely related problems appear:

(1) the Γ-action on Ω is not properly discontinuous, the quotient Ω/Γ is very non-Hausdorff;

(2) any two points in the period domain can be connected with chains of compact one-dimensional submanifolds constructed either as orbits of maximal

compact subgroups inside G_0 or as period maps of twistor families of hyperkähler metrics on the real four-manifold M underlying any K3 surface, in particular, Ω is maximally far from being Stein, as it does not carry non-constant holomorphic functions;

(3) the moduli space $\widetilde{\Omega} \to \Omega$ of marked K3 surfaces, the so-called Burns-Rapoport space, is a non-Hausdorff éspace étalé over Ω.

While twistor lines form a 57-dimensional real-analytic family, these submanifolds actually deform in a 57-dimensional complex analytic family, parametrised by one connected component $\mathcal{C}_1(\Omega)$ of the cycle space or Douady space of Ω. I explained that it is relatively easy to see that in contrast to Ω, the parameter space $\mathcal{C}_1(\Omega)$ is a bounded symmetric domain, in particular Stein, and that consequently, the induced Γ-action on it is properly discontinuous. It is hence a very natural and interesting problem to investigate the meaning of $\mathcal{C}_1(\Omega)$ and its quotient $\mathcal{C}_1(\Omega)/\Gamma$ in terms of moduli theory of certain geometric structures on K3-surfaces. This question was independently raised by Fels-Huckleberry-Wolf in [3] and Looijenga in [5].

In this direction, one first observes that $\mathcal{C}_1(\Omega)$ has further nice desirable properties: if

$$\mathcal{C}_1(\Omega) \xleftarrow{p} \mathcal{C} \xrightarrow{q} \Omega$$

is the universal family of cycles over $\mathcal{C}_1(\Omega)$, and if $\mathcal{F}$ is the sheaf (of sets) on Ω corresponding to $\widetilde{\Omega}$, then $\widehat{\mathcal{F}} := p_*(q^*(\mathcal{F}))$ is a Hausdorff sheaf on $\mathcal{C}_1(\Omega)$. Additionally, motivated by observations made by R. Kobayashi [4] and partially building on results obtained in [1] we show that this sheaf has a Γ-invariant section $\sigma_{\mathbb{R}}$ over the part $\mathbf{T}_{\mathbb{R}}$ of the cycle space parametrising honest twistor lines that extends to a (still invariant) section σ in $\widehat{\mathcal{F}}$ over a Γ-invariant neighbourhood of $\mathbf{T}_{\mathbb{R}}$ in $\mathcal{C}_1(\Omega)$; i.e., there exists an equivariant marked family of K3 surfaces over a significant part of $\mathcal{C}$. This family can be seen as a marked complex deformation of the twistor spaces associated with hyperkähler metrics on M, and in fact there is an interpretation in terms of deformations (and potentially also moduli theory) of complex hyperkähler metrics on M or rather its complexification $M^{\mathbb{C}}$ via Penrose's Nonlinear Graviton Construction, cf. [2].

One of the questions that remain to be investigated is whether the section σ and the family of marked K3 surfaces it induces can be analytically continued to the part of the cycle space $\mathcal{C}_1(\Omega)$ parametrising cycles not contained in any of the hyperplane sections of Ω defined by orthogonality to a (-2)-class in Λ.

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Superconnection and family Bergman kernels

Xiaonan Ma
(joint work with Weiping Zhang)

We establish an asymptotic version of Bismut's local family index theorem for the Bergman kernel. The key idea is to use the superconnection as in the local family index theorem. We only formulate the results in the fiberwise positive holomorphic line bundle case, while the main results hold also in the fiberwise symplectic case.

Let W, S be smooth compact complex manifolds with S being connected. Let $\pi : W \to S$ be a holomorphic submersion with compact fiber X and $\dim X = n$.

Let L, E be holomorphic vector bundles on W and the rank $\mathrm{rk}(L)$ of L is 1. Let h^L, h^E be Hermitian metrics on L, E. Let ∇^L, ∇^E be the Chern (i.e., holomorphic Hermitian) connections on (L, h^L), (E, h^E) with curvatures R^L, R^E. Set

$$\omega := \frac{\sqrt{-1}}{2\pi} R^L.$$

Then ω is a smooth real 2-form of complex type $(1,1)$ on W. We suppose that ω defines a fiberwise Kähler form along the fiber X, i.e.,

$$g^{T_{\mathbb{R}}X}(u,v) = \omega(u, J^{T_{\mathbb{R}}X} v)$$

defines a Riemannian metric on $T_{\mathbb{R}}X$. This simply means that (L, h^L) is a fiberwise positive line bundle on W.

For a differential form ϑ on S, we will denote by $\vartheta^{(i)}$ its component in $\Lambda^i(T^*_{\mathbb{R}}S)$.

By the Riemann-Roch-Grothendieck theorem and the Kodaira vanishing theorem, there exists $p_0 > 0$ such that for $p > p_0$, $H^0(X, L^p \otimes E)$ is a holomorphic vector bundle on S and we have in $H^\bullet(S, \mathbb{R})$

$$\mathrm{ch}(H^0(X, L^p \otimes E)) = \int_X \mathrm{Td}(T^{(1,0)}X)\,\mathrm{ch}(E)\,\mathrm{ch}(L^p).$$

By (), in $H^2(S, \mathbb{R})$, we have

$$c_1(H^0(X, L^p \otimes E)) = \left[\int_X \mathrm{Td}(T^{(1,0)}X)\,\mathrm{ch}(E)\,\mathrm{ch}(L^p)\right]^{(2)}$$
$$= \mathrm{rk}(E)\int_X \frac{c_1(L)^{n+1}}{(n+1)!}p^{n+1} + \int_X \Big(c_1(E) + \frac{\mathrm{rk}(E)}{2}c_1(T^{(1,0)}X)\Big)\frac{c_1(L)^n}{n!}p^n + \mathcal{O}(p^{n-1}).$$

Now, from the local index theory point of view [1], it is nature to ask whether one can refine () to an equality of differential forms via Chern-Weil representatives.

For $s \in S$, let $P_{p,s}$ be the orthogonal projection from $\mathcal{C}^\infty(X_s, L^p \otimes E)$ onto $H^0(X_s, L^p \otimes E)$. Note that P_p is the Bergman projection, and its smooth kernel is called Bergman kernel, and the asymptotic expansion was studied by a lot of people.

Let $\nabla^{H^0(X,L^p\otimes E)}$ be the Chern connection on $H^0(X, L^p \otimes E)$ with curvature $R^{H^0(X,L^p\otimes E)}$. The main result of our talk is as follows.

Theorem [2]**.** *The curvature operators*

$$\frac{1}{p}R^{H^0(X,L^p\otimes E)} \in \Omega^2(S, \mathrm{End}(H^0(X, L^p \otimes E)))$$

are Toeplitz operators for any $s \in S$, i.e., there exists $R_r \mathcal{C}^\infty(W, \pi^*(\Lambda^2(T^*_{\mathbb{R}}S)) \otimes \mathrm{End}(E))$, $(r \in \mathbb{N})$ such that for any $k \in \mathbb{N}$, when $p \to +\infty$, under the operator norm of the morphisms of vector bundles: $H^0(X, L^p \otimes E) \to \Lambda^2(T^*_{\mathbb{R}}S) \otimes H^0(X, L^p \otimes E)$ over S, we have

$$\frac{1}{p}R^{H^0(X,L^p\otimes E)} = \sum_{r=0}^{k} T_{R_r,p}\, p^{-r} + \mathcal{O}(p^{-k-1}), \ \textit{with}\ T_{R_r,p} = P_p R_r P_p,$$
$$R_0 = -\, 2\pi\sqrt{-1}\frac{(\omega^{n+1})^{(2)}}{(n+1)(\omega^n)^{(0)}}\,\mathrm{Id}_E\,.$$

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Real forms of a smooth complex projective surface

Keiji Oguiso
(joint work with Tien-Cuong Dinh, Cécile Gachet, Hsueh-Yung Lin, Long Wang, Xun Yu)

Let $G := \mathrm{Gal}\,(\mathbf{C}/\mathbf{R}) = \{\mathrm{id}_{\mathbf{C}}, c\}$ be the Galois group of the field extension $\mathbf{C}/\mathbf{R}$.

It is quite recent that negative answers are given to the following long standing natural questions:

Question. Let V be a smooth complex projective variety of dimension ≥ 2. Are real forms of V, i.e., systems of homogeneous equations with real coefficients defining V, finite up to isomorphisms over $\mathbf{R}$?

The first negative answer to this question is given by Lesieutre [Le18]. He constructs a smooth complex projective variety V of dimension 6 with Kodaira dimension $\kappa(V) = -\infty$ with infinitely many real forms. Expanding his idea, Dinh and Oguiso ([DO19]) construct a smooth complex projective variety V of any dimension ≥ 2 with $\kappa(V) \geq 0$ with infinitely many real forms and Dinh, Oguiso and Yu ([DOY21], [DOY22]) construct a smooth complex projective rational variety V of any dimension ≥ 2 with infinitely many real forms.

In the talk, being based on [BS64], we confirm a few basic criteria of the finiteness of real forms of a given smooth complex projective variety, in terms of the Galois cohomology set of the discrete part of the automorphism group, the cone conjecture and the topological entropy as follows:

Theorem 1. *Let V be a complex projective variety with at least one real form $V_{\mathbf{R}}$. Then the non-isomorphic real forms of V are at most countable. Moreover, the non-isomorphic real forms of V are at most finite provided that the cohomology set $H^1(G, \mathrm{Aut}\,(V)/\mathrm{Aut}^0\,(V))$ is finite. Here we naturally regard $\mathrm{Aut}\,(V)/\mathrm{Aut}^0\,(V)$ as a G-group via a real form $V_{\mathbf{R}}$ chosen.*

This is asserted by [DIK00, Appendix D] under the assumption that $\mathrm{Aut}^0\,(V)$ is linear algebraic group (and ignored the abelian variety factor in [DIK00, Appendix D]). We also compare the first statement with a recent remarkable result by [Bo21].

Theorem 2. *Let V be a complex projective variety and satisfies the cone conjecture in the sense that the natural action of*

$$\mathrm{Aut}\,(V)^* := \mathrm{Im}\,(\mathrm{Aut}\,(V) \to \mathrm{GL}(\mathrm{NS}\,(V)/\mathrm{torsion})$$

on the rational hull $\mathrm{Nef}^+\,(V)$ of the nef cone $\mathrm{Nef}\,(V) \cap \mathrm{NS}\,(V)_{\mathbf{Q}}$ has a rational polyhedral fundamental domain Σ, or more generally, $\mathrm{Nef}^+\,(V)$ contains a rational polyhedral cone Σ' satisfying

$$\mathrm{Aut}\,(V) \cdot \Sigma' \supset \mathrm{Amp}\,(V).$$

Then V has at most finitely many non-isomorphic real forms. In particular, this is the case where V is a minimal surface of Kodaira dimension zero ([Ka97]).

This is also a generalization and clarification of the assertion of [CF20] and suggests some unexpected relation with real form problems and cone conjecture (whose origin goes back to the mirror symmetry).

Theorem 3. *Let V be a complex projective variety of dimension ≥ 2. If $\mathrm{Aut}\,(V)^*$ is virtually solvable, then V admits at most finitely many non-isomorphic real forms. In particular, this is the case where V is smooth and all automorphisms of V is of zero-entropy ([DLOZ22]).*

This is a slight generalization and clarification of the assertion of [Ki20] and suggests again some unexpected relation with the real form problems and algebraic dynamics.

We then apply them to show that a smooth complex projective surface has at most finitely many mutually non-isomorphic real forms unless it is either rational or a non-minimal surface birational to either a K3 surface or an Enriques surface. We finally construct an Enriques surface whose blow-up at one point admits infinitely many mutually non-isomorphic real forms, which answers a question of Kondo to us and also, together with [DOY21], shows the three exceptional cases really occur.

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Rationality of proper holomorphic maps between bounded symmetric domains

Sung-Yeon Kim

The study of proper holomorphic maps between bounded symmetric domains dates back to the work of Poincaré, who discovered that any biholomorphic map between two connected open pieces of the the unit sphere in $\mathbb{C}^2$ is a restriction of (the extension to $\overline{\mathbb{B}}^2$ of) an automorphism of the 2-dimensional unit ball $\mathbb{B}^2$. Later, Alexander and Henkin-Tumanov generalized his result to higher dimensional unit balls and higher rank bounded symmetric domains respectively. For unit balls of different dimensions, proper holomorphic maps have been studied thoroughly by many mathematicians: Cima-Suffridge([CS90]), Faran([Fa86]), Forstnerič([Fo86, Fo89]), Huang-Ji([HJ01]), Webster([W78, W79]) and many others.

In [Fo89], Forstnerič proved that if a proper holomorphic map $f : \mathbb{B}^n \to \mathbb{B}^N$, $(N > n > 1)$ extends to a C^{N-n+1} map on $\overline{\mathbb{B}}^n$, then f is a rational map with degree at most $N^2(N-n+1)$ and has no pole on $\partial\mathbb{B}^n$. This theorem implies that the space of all rational proper mappings $f : \mathbb{B}^n \to \mathbb{B}^N$ is finite dimensional. The proof relies on the method of Segre varieties and Reflection Principle developed first by Webster in [W78]:

Let Ω be a domain with real analytic boundary M in $\mathbb{C}^n$, $n > 1$. For each point $z_0 \in M$ there is a neighborhood U of z_0 in $\mathbb{C}^n$ and a real valued C^ω function

$r(z,\overline{z})$ on U with nonvanishing gradient such that

$$M \cap U = \{z \in U : r(z,\overline{z}) = 0\}.$$

After shrinking U if necessary, we define for each $w \in U$ a complex hypersurface $Q_w \subset U$ by

$$Q_w = \{z \in U : r(z,\overline{w}) = 0\}.$$

These hypersurfaces were first introduced by Segre [S31] and were later used by many authors to study holomorphic maps preserving real analytic CR manifolds. Let $\Omega' \subset \mathbb{C}^N$ be another domain with real analytic boundary M' and let $f : U \to U'$ be a holomorphic mapping taking $M \cap U$ into $M' \cap U'$. Then $f(Q_w) \subset Q'_{f(w)}$, where $Q'_{w'}$ is the Segre variety associated to M', implying that

$$f(z) \in \bigcap_{w \in Q_z} Q'_{f(w)}.$$

Hence if the right hand side is a finite set, then the value $f(z)$ is determined by finite jet of f at $w \in Q_z \cap \Omega$. If Ω and Ω' are balls, then this is true if $f(\Omega)$ is not contained in a proper linear subspace of $\mathbb{C}^N$.

In the case of bounded symmetric domain Ω with rank greater or equal to two, the Segre varieties associated to the each boundary orbit may be 0-dimensional and therefore give no further information on the map f. On the other hand boundary of Ω has so called 'fine structure' introduced by Wolf([Wo72]). Using this structure, Mok and Tsai([MT92]) constructed a moduli map from the moduli space of characteristic symmetric subdomains to that of characteristic symmetric subdomains of a fixed rank in the target domain, and the moduli map was proven to admit a rational extension between moduli spaces of characteristic symmetric subspaces. This moduli map plays an important role in studying the properties of given proper holomorphic maps.

By modifying Mok-Tsai's method, we construct a map $f^\sharp$ between moduli space of invariantly totally geodesic subspaces with the property that

$$f(p) \in \bigcap_{\sigma \in \mathcal{Z}_p} X'_{f^\sharp(\sigma)},$$

where $\mathcal{Z}_p$ is the set of all invariantly totally geodesic subspaces X_σ of a fixed type that passes through p and $X'_{f^\sharp(\sigma)}$ is the minimal invariantly totally geodesic subspace that contains $f(X_\sigma)$. In the moduli space of invariantly totally geodesic subspaces, the sets of boundary components are homogeneous CR manifolds with mixed Levi signature and the restriction of moduli map to them are CR maps between them. Under certain conditions, it forces the moduli map to satisfy strong local differential-geometric constraints. Using this, we obtain the following theorems:

An irreducible bounded symmetric domains of type one is a domain defined by

$$\Omega_{p,q} := \{Z \in \mathbb{C}^{p\times q} : I_q - Z\overline{Z}^t > 0\}, \quad p \geq q \geq 1,$$

where $\mathbb{C}^{p\times q}$ is the set of p by q matrices.

Theorem. *Let $\Omega = \Omega_{p,q}$ and $\Omega' = \Omega_{p',q'}$ be irreducible bounded symmetric domains of type one with $q, q' \geq 2$ and let $f : \Omega \to \Omega'$ be a proper holomorphic map that extends $C^{p' \cdot q' - (p-q) \cdot q}$ up to the boundary. Suppose that f maps Shilov boundary to Shilov boundary. Then f is of the form $f = \imath \circ F$, where*

$$F = F_1 \times F_2 \colon \Omega \to \Omega'_1 \times \Omega'_2,$$

Ω'_1 and Ω'_2 are bounded symmetric domains, $F_1 \colon \Omega \to \Omega'_1$ is a proper rational map and $\imath : \Omega'_1 \times \Omega'_2 \hookrightarrow \Omega'$ is a holomorphic totally geodesic isometric embedding of a reducible bounded symmetric domain $\Omega'_1 \times \Omega'_2$ into Ω' with respect to canonical Kähler-Einstein metrics. Moreover, if Ω is of non-tube type, then f is a rational map.

Corollary *Let $f : \Omega_{p,q} \to \Omega_{p',q'}$ be a proper holomorphic map that extends $C^{p' \cdot q' - (p-q) \cdot q}$ up to the boundary. Suppose*

$$3 \leq q \leq q' \leq 2q - 1.$$

Then f is rational or of the form $f = \imath \circ F$, where

$$F = F_1 \times F_2 \colon \Omega \to \Omega'_1 \times \Omega'_2,$$

Ω'_1 and Ω'_2 are bounded symmetric domains, $F_1 \colon \Omega \to \Omega'_1$ is a standard embedding and $\imath : \Omega'_1 \times \Omega'_2 \hookrightarrow \Omega'$ is a holomorphic totally geodesic isometric embedding of a reducible bounded symmetric domain $\Omega'_1 \times \Omega'_2$ into Ω' with respect to canonical Kähler-Einstein metrics. If $q = 2$, then the same conclusion holds under an additional assumption that f maps Shilov boundary to Shilov boundary.

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Residue currents for holomorphic foliations

Lucas Kaufmann
(joint work with R. Lärkäng and Elizabeth Wulcan)

This aim of this talk was to present recent results on the use of the theory of residue currents to study singularities of holomorphic foliations.

Let M be a compact complex manifold of dimension n and $\mathcal{F}$ be a holomorphic foliation on M. It is given by a coherent subsheaf $T\mathcal{F}$ of TM, called the *tangent sheaf* of $\mathcal{F}$, which is involutive, i.e. closed under the Lie bracket, and such that the *normal sheaf* $N\mathcal{F} := TM/T\mathcal{F}$ is torsion free. The *singular set* of $\mathcal{F}$ is, by definition, the smallest subset $\operatorname{sing} \mathcal{F} \subset M$ outside of which $N\mathcal{F}$ is locally free. The generic rank of $T\mathcal{F}$, which will be denoted by k, is called the *rank of the foliation* $\mathcal{F}$.

A fundamental vanishing theorem of Baum-Bott asserts that, when $\mathcal{F}$ is non-singular, that is, when $\operatorname{sing} \mathcal{F}$ is empty, all the characteristic classes of $N\mathcal{F}$ of degree ℓ vanish when $n - k < \ell \leq n$, see [3]. This follows from the existence of a *basic connection* on the normal bundle $N\mathcal{F}$, which is a connection compatible with both the holomorphic and foliation structure. Such connections always exist in the non-singular case.

When $\mathcal{F}$ is singular, the above vanishing result implies the following important index theorem: for every connected component Z of $\operatorname{sing} \mathcal{F}$ and any homogeneous symmetric polynomial $\Phi \in \mathbb{C}[z_1, \dots, z_n]$ of degree ℓ with $n-k < \ell \leq n$, there exists a cohomology class $\operatorname{Res}^{\Phi}(\mathcal{F}; Z) \in H^{2\ell}(M, \mathbb{C})$ depending on the local behavior of $\mathcal{F}$ around Z such that

$$\sum_{Z \subset \operatorname{sing} \mathcal{F}} \operatorname{Res}^{\Phi}(\mathcal{F}; Z) = \Phi(N\mathcal{F}) \quad \text{in} \quad H^{2\ell}(M, \mathbb{C}),$$

where $\Phi(N\mathcal{F})$ is the corresponding characteristic class of $N\mathcal{F}$, see [3]. This should be seen as a localization formula for $\Phi(N\mathcal{F})$ around the singularities of $\mathcal{F}$. The classes $\operatorname{Res}^{\Phi}(\mathcal{F}; Z)$ are called *Baum-Bott residue classes.*

When $k = 1$ and the singular set consists of isolated points, the Baum-Bott classes are actually numbers, called Baum-Bott indices. In this case, Baum-Bott already showed that they can be computed using the so called Grothendieck residues. This makes the use of Baum-Bott Theory an effective tool in the study of one-dimensional foliations. However, when the singularities are not isolated or when $k \geq 2$, the situation is more delicate and there is no general effective way to compute Baum-Bott classes or find explicit representatives. So far, the available results are limited and rely on a reduction to the above case of rank one foliations with isolated singularities. See for instance [3, 5, 4]

In our work [6], we have shown that the above Baum-Bott classes can be naturally represented by currents of residue type supported by the corresponding singular component. The precise statement is as follows.

Theorem. Let M be an n-dimensional projective manifold and $\mathcal{F}$ be a holomorphic foliation of rank k on M. Then, for every connected component Z of sing $\mathcal{F}$ and any homogeneous symmetric polynomial $\Phi \in \mathbb{C}[z_1, \ldots, z_n]$ of degree ℓ with $n-k < \ell \leq n$, there exists a pseudomeromorphic current R_Z^{Φ} on M of degree 2ℓ supported by Z whose cohomology class coincides with the Baum-Bott residue class $\mathrm{Res}^{\Phi}(\mathcal{F}; Z) \in H^{2\ell}(M, \mathbb{C})$.

In the above statement, a pseudomeromorphic current is a pushforward of a current of the form $\tau = \omega \wedge \frac{1}{t_1^{m_1}} \cdots \frac{1}{t_k^{m_k}} \overline{\partial} \frac{1}{t_{k+1}^{m_{k+1}}} \wedge \cdots \wedge \overline{\partial} \frac{1}{t_r^{m_r}}$, under a sequence of holomorphic modifications, projections and open inclusions, where $t_1, \ldots, t_r$ are local holomorphic coordinates. This is a natural class of currents in multivariable residue theory. In particular, this class contains several known constructions of residue currents, see [1, 2] and the references therein.

The first step of the proof of the above theorem is to consider a resolution of the normal sheaf of the form

$$0 \to E_N \xrightarrow{\varphi_N} E_{N-1} \xrightarrow{\varphi_{N-1}} \ldots \xrightarrow{\varphi_2} E_1 \xrightarrow{\varphi_1} E_0 = TM \xrightarrow{\varphi_0} N\mathcal{F} \to 0$$

equipped with suitable connections over $M \setminus \mathrm{sing}\, \mathcal{F}$ *with controlled singularities* along $\mathrm{sing}\, \mathcal{F}$ as explained below. Here, each E_k is a holomorphic vector bundle, φ_k are holomorphic and the induced vector bundle complex is exact out of $\mathrm{sing}\, \mathcal{F}$.

Once we fix a hermitian metric on each vector bundle E_k, $k = 0, \ldots, N$, there are minimal inverses σ_k of the above morphisms φ_k, which are defined outside $\mathrm{sing}\, \mathcal{F}$. This allow us to start from basic connection D_{basic} on $N\mathcal{F}$ over $M \setminus \mathrm{sing}\, \mathcal{F}$ and produce connections on each E_k that are compatible with the maps of the complex. Via a cut-off procedure one can produce, for any $\epsilon > 0$, connections D_k^{ϵ} on each E_k over M such that, by Baum-Bott theory, polynomials of the form $r_Z^{\Phi}(\epsilon) := \Phi(\Theta(D_N^{\epsilon}), \ldots, \Theta(D_0^{\epsilon}))$ are 2ℓ-forms supported by an ϵ-neighborhood of Z representing the residue class $\mathrm{Res}^{\Phi}(\mathcal{F}; Z)$. Here $(\Theta(D_k^{\epsilon})$ stands for the curvature matrix of $\Theta(D_k^{\epsilon})$.

A key technical step is to prove that the limit as $\epsilon \to 0$ of the forms $r_Z^{\Phi}(\epsilon)$ described above exists and yield a well-defined residue current supported by Z and representing $\mathrm{Res}^{\Phi}(\mathcal{F}; Z)$. This is done by working in the category of *almost semi-meromorphic singularities* in the sense of [1, 2]. We show that there exists a basic connection D_{basic} on $N\mathcal{F}$ over $M \setminus \mathrm{sing}\, \mathcal{F}$ with almost semi-meromorphic singularities along Z and observe that the minimal inverses σ_k described above also have the same singularity type. This allow us to use resolutions of singularities à la Hironaka and show that $r_Z^{\Phi}(\epsilon)$ tends to a well defined current R_Z^{Φ} when $\epsilon \to 0$.

When $\mathcal{F}$ is of rank one and has isolated singularities, we recover the known formula involving Gorthendieck residues, but now expressed in terms of residue currents. Still in the rank one case, when the singular set is of higher dimension, one can express residue currents using currents of Bochner-Martinelli type.

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On neighborhoods of embedded complex tori

Laurent Stolovitch
(joint work with Xianghong Gong)

Our talk is dedicated to the following

Theorem [GSb22]**.** *Let C be an n-dimensional complex torus embedded in a complex manifold M of dimensional $n+d$. Assume that T_CM, the restriction of TM on C, splits as $TC \oplus N_C$. Suppose that the normal bundle of C in M admits transition functions that are Hermitian matrices and satisfy a non-resonant Diophantine condition. Then a neighborhood of C in M is biholomorphic to a neighborhood of the zero section in the normal bundle.*

A complex torus C can be identified with the quotient of $\mathbb{C}^n$ by a lattice Λ spanned by the standard unit vectors $e_1, \ldots, e_n$ in $\mathbb{C}^n$ and n additional vectors $e'_1, \ldots, e'_n$ in $\mathbb{C}^n$, where $\mathrm{Im} e'_1, \ldots, \mathrm{Im} e'_n$ are linearly independent vectors in $\mathbb{R}^n$. Let Λ' be the lattice in the cylinder $\mathcal{C} := \mathbb{R}^n/\mathbb{Z}^n + i\mathbb{R}^n$ spanned by $e'_1, \ldots, e'_n \mod \mathbb{Z}^n$. There are two coverings for the torus $C = \mathbb{C}^n/\Lambda = \mathcal{C}/\Lambda'$: the universal covering $\pi\colon \mathbb{C}^n \to C$ and the covering by cylinder, $\pi_{\mathcal{C}}\colon \mathcal{C} \to C$ that extends to a covering $\mathcal{M}$ over M. In section two we recall some facts about factors of automorphy for vector bundles on C via the covering by $\mathbb{C}^n$. In section three, we study the flat vector bundles on C. The pull back of the flat vector bundle N_C to the cylinder $\mathcal{C}$ is the normal bundle $N_{\mathcal{C}}$ of $\mathcal{C}$ in $\mathcal{M}$. We show that $N_{\mathcal{C}}$ is always the holomorphically trivial vector bundle $\mathcal{C} \times \mathbb{C}^d$. By "vertical coordinates", we mean "coordinates on $\mathbb{C}^d$", the normal component of the normal bundle N_C, while "horizontal coordinates" mean the tangential components of N_C.

Since $\mathcal{C}$ is a Stein manifold, a theorem of Siu says that a neighborhood of $\mathcal{C}$ in $\mathcal{M}$ is biholomorphic to a neighborhood of the zero section in its normal bundle, which is trivial as mentioned above. We show that the holomorphic classification of neighborhoods M of C with flat N_C is equivalent to the holomorphic classification of the family of the deck transformations of coverings $\mathcal{M}$ of M in a neighborhood of $\mathcal{C}$. These deck transformations are "higher-order" (in the vertical coordinates) perturbations $\tau_1, \ldots, \tau_n$ of $\hat{\tau}_1, \ldots, \hat{\tau}_n$, where the latter are the deck transformations of the covering of $\widetilde{N}_C$ over N_C. In order to find a biholomorphism between a neighborhood of C in M and a neighborhood of its zero section in N_C, it is sufficient to find a biholomorphism that conjugates $\{\tau_1, \ldots, \tau_n\}$ to $\{\hat{\tau}_1, \ldots, \hat{\tau}_n\}$.

There are two useful features. First, since the fundamental group of C is abelian, the deck transformations $\tau_1, \dots, \tau_n$ commute pairwise. Second, we can also introduce suitable coordinates on $\mathcal{C}$ so that the "horizontal" components of deck transformations have diagonal linear parts. In such a way the classification of neighborhoods of C is reduced to a more attainable classification of deck transformations. While the full theory for this classification is out the scope of this paper, we study the case when N_C admits Hermitian transition functions. Since a Hermitian transition matrix must be locally constant, we call such an N_C *Hermitian flat*. Finally, we have : $\hat{\tau}_j(h,v) = (T_j h, M_j v)$ with $M_j := \operatorname{diag}(\mu_{j,1}, \dots, \mu_{j,d})$ and $T_j := \operatorname{diag}(\lambda_{j,1}, \dots, \lambda_{j,n})$ with $\lambda_{j,k} := e^{2\pi i e'_{jk}}$, $j = 1, \dots n$. The normal bundle N_C, is said to be *non-resonant Diophantine* if for all $(Q,P) \in \mathbb{N}^d \times \mathbb{Z}^n$, $|Q| > 1$ and all $i = 1, \dots, n$, and $j = 1, \dots, d$,

$$\min\left(\max_{\ell\in\{1,\dots,n\}}\left|\lambda_\ell^P \mu_\ell^Q - \lambda_{\ell,i}\right|, \max_{\ell\in\{1,\dots,n\}}\left|\lambda_\ell^P \mu_\ell^Q - \mu_{\ell,j}\right|\right) \quad > \quad \frac{D}{(|P|+|Q|)^\tau}.$$

The proof of the theorem relies on a Newton rapid convergence scheme adapted to our situation based on an appropriate Diophantine condition among the lattice and the normal bundle. At step k of the iteration scheme, let δ_k be the error of the deck transformations $\{\tau_1^{(k)}, \dots, \tau_n^{(k)}\}$ defined on domain $D^{(k)}$ to $\hat{\tau}_1, \dots, \hat{\tau}_n$ in suitable norms. By an appropriate transformation $\Phi^{(k)}$, we conjugate to a new set of deck transformations $\{\tau_1^{(k+1)}, \dots, \tau_n^{(k+1)}\}$ of which the error to the linear ones is now δ_{k+1} on a slightly smaller domain $D^{(k+1)}$. Using our Diophantine conditions, related to the lattice Λ and the normal bundle, we show that the sequence $\Phi^{(k)} \circ \cdots \circ \Phi^{(1)}$ converges to a holomorphic transformation Φ on an open domain $D^{(\infty)}$ where we linearize $\{\tau_1, \dots, \tau_n\}$.

We now describe closely related previous results. Our work is motivated by work of Arnol'd and Ilyashnko-Pyartli, somehow related to Grauert's "Formal Prinzip" question, the study of which has a long history. Our main theorem was proved in [Arn76] when C is an elliptic curve ($n = 1$) and N_C has rank one ($d = 1$). Il'yashenko-Pyartli [IP79] extended Arnol'd's result to the case when the torus is the product of elliptic curves together with a normal bundle which is a direct sum of line bundles, while our result deals with general complex tori. Our "non-resonant Diophantine" assumption on N_C is also weaker than theirs. . Also, see some recent work [Hwa19, Koi21, LTT19]. We refer to [GSa22] for some references and a different approach to this range of questions.

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Bochner Laplacian and the Bergman kernel expansion for families

Nikhil Savale
(joint work with Xiaonan Ma & George Marinescu)

1. Introduction

Let $\left(Y, h^{TY}\right)$ be a compact, complex Hermitian manifold of dimension n equipped with a holomorphic, Hermitian vector bundle $\left(E, h^{E}\right)$. The Bergman projector is the orthogonal projector $\Pi_E : L^2\left(Y, E\right) \rightarrow H^0\left(Y; E\right)$ from from the space of square integrable holomorphic sections of E onto the square integrable holomorphic ones. The Bergman kernel $\Pi_E\left(y, y'\right)$ is its Schwartz kernel. More generally, a Toeplitz operator can be defined via $T_f := \Pi_E f \Pi_E$ as the quantization of an arbitrary smooth function f on the manifold.

Tian [10] first described the leading asymptotics of the Bergman kernel on-diagonal $\Pi_{L^k}\left(y, y\right)$, associated to tensor powers $L^k := L^{\otimes k}$ of a positive line bundle in the semiclassical limit as $k \rightarrow \infty$. This was extended to a full asymptotic expansion in [3, 11] as an application of the fundamental parametrix of Boutet de Monvel-Sjöstrand [2] for the Szegő kernel of a strongly pseudoconvex CR manifold. Later a new method for the asymptotic expansion was developed in the articles of Dai-Liu-Ma [4] and Ma-Marinescu [6]. It is based on the analytic localization technique of Bismut-Lebeau [1]. A key step in the latter method is the study of the asymptotics of the smallest positive eigenvalue of the associated Bochner and Kodaira Laplacians. A similar on-diagonal asymptotic expansion can be proved for the kernel of a Toeplitz operator.

The present work, and this talk, are motivated by the following natural question.

Question. Can one generalize the classical Bergman kernel asymptotics by

(1) relaxing the positivity assumption (eg. to semi-positive bundles)
(2) considering vector bundles of higher rank (eg. symmetric powers $\mathrm{Sym}^k E$, for rk $E > 1$)

The CR analog of the above problem is also outstanding. Namely to construct a parametrix for the Szegő kernel of a weakly pseudoconvex CR manifold and a CR manifold of high co-rank.

2. Main results

Both generalizations mentioned in our question are well addressed in the setting of families. Namely, consider a holomorphic submersion $\left(W, h^{TW}\right) \xrightarrow{\pi} \left(Y, h^{TY}\right)$ of compact complex Hermitian manifolds with compact fibers. Let $\left(L, h^{L}\right) \rightarrow W$ be a holomorphic, Hermitian line bundle. Under the assumption that the curvature R^{L} of the fiber bundle is fiber-wise positive, it follows from Kodaira vanishing and Riemann-Roch that the the direct image $\mathcal{E}_k := R^0 \pi_* L^k$, $k \gg 0$, is a well-defined holomorphic vector bundle on the base. Its fiber above $y \in Y$ is the fiber-wise cohomology of the line bundle $\mathcal{E}_{k,y} := H^0\left(W_y, \left.L^k\right|_{W_y}\right)$ and carries the Hermitian metric $h^{\mathcal{E}_k}$ obtained from the fiber-wise L^2 metric. We denote by $\nabla^{\mathcal{E}_k}$ the associated Chern connection and by $\Delta_{\mathcal{E}_k} := \left(\nabla^{\mathcal{E}_k}\right)^* \nabla^{\mathcal{E}_k}$ the Bochner Laplacian associated to it.

To state our first result, one introduces the horizontal order of vanishing of the curvature $r : W \rightarrow \mathbb{R}$ via

(1)
$$r(w) - 2 := \operatorname{ord}_w^H\left(R^L\right) = \min\left\{l \mid J^l_{T^H W}\left(\Lambda^2 T^* Y\right) \ni j^l_{T^H W} R^L \neq 0\right\}, \quad r_w \geq 2,$$

in terms of the horizontal jet bundles. One now has the following.

Theorem 1. *Let $\left(W, g^{TW}\right) \xrightarrow{\pi} \left(Y, h^{TY}\right)$ be a holomorphic submersion of compact complex Hermitian manifolds with compact fibers. Let $\left(L, h^{L}\right) \rightarrow W$ be a holomorphic Hermitian line bundle on the total space. Assume that its curvature R^L is fiber-wise positive with a fiber-wise constant horizontal order of vanishing* (1) *of maximal value r.*

The first eigenvalue $\lambda_0(k)$ of the Bochner Laplacian $\Delta_{\mathcal{E}_k}$ on the direct image bundle $\mathcal{E}_k := R^0 \pi_* L^k$ then satisfies

$$\lambda_0(k) \sim C k^{2/r}, \tag{2}$$

for some positive constant C. Moreover the first eigenfunction ψ_0^k concentrates on the locus $Y_r := \{y \in Y \mid r_y = r\}$ where the order of vanishing is maximized :

$$\left|\psi_0^k(y)\right| = O\left(k^{-\infty}\right);\ y \in Y \setminus Y_r. \tag{3}$$

The above result is closely related to the spectral theory of the sub-Riemannian Laplacian [9].

Our next result concerns the Bergman kernel of the direct image bundle $\mathcal{E}_k$. To state it, consider the horizontal metric complement $T^H W := \left(T^V W\right)^{\perp}$ of the bundle of vertical fibers on the total space and let $T^{H,(1,0)}W$ be its holomorphic part. We introduce the horizontal curvature tensor $\dot{R}^{L,H} \in \operatorname{End}\left(T^{H,(1,0)}W\right)$ via $R^{L,H}(.,.) := h^{TY}\left(., \dot{R}^{L,H}.\right)$. Denote by $\operatorname{Spec}\left(\dot{R}^{L,H}_w\right) = \{a_1(w), \ldots a_{n-1}(w)\}$ its eigenvalues for each $w \in W$. The curvature is said to be horizontally semipositive if $a_j(w) \geq 0$, for each $j = 1, \ldots, n$ and $w \in W$. It is said to have *comparable eigenvalues* if there exist positive constants $c_1, c_2 > 0$ such that $c_1 a_i(w) < a_j(w) < c_2 a_i(w)$ for each $i, j = 1, \ldots, n$ and $w \in W$. One now has the following.

Theorem 2. *Let* $\left(W, g^{TW}\right) \xrightarrow{\pi} \left(Y, h^{TY}\right)$ *be a holomorphic submersion of compact complex Hermitian manifolds with compact fibers. Let* $\left(L, h^{L}\right) \to W$ *be a holomorphic Hermitian line bundle on the total space. Assume that its curvature* R^L *is fiber-wise positive and horizontally semipositive with comparable eigenvalues. Furthermore suppose that the curvature has a fiber-wise constant and finite horizontal order of vanishing.*

Then the point-wise Bergman kernel of the direct image is a generalized Toeplitz operator

$$\begin{aligned}\Pi_k^Y (y,y) &= k^{2n/r_y} \left[\sum_{j=0}^{N} k^{-2j/r_y} T_{g_j}\right] + O_{L^2 \to L^2}\left(k^{-(2N+2-2n)/r_y}\right) \\ &\in \mathit{End}\ (\mathcal{E}_{k,y}) \coloneqq \mathit{End}\ H^0\left(W_y; L^k|_{W_y}\right),\end{aligned}$$

for each $y \in Y$, $N \in \mathbb{N}$ *and some* $g_j \in C^\infty (W_y)$, $j = 0, 1, \ldots$, *on each fiber of the direct image bundle.*

To see how the above theorem answers the question that we raised, we consider two specializations of the above.

(1) In the case when the fibers of the submersion π are points, the direct image bundles are tensor powers $\mathcal{E}_k = L^k$. In this case our theorem Theorem 2 gives the Bergman kernel expansion for line bundles whose curvature is semi-positive with comparable eigenvalues and a finite order of vanishing. This particularly includes the case of semi-positive line bundles on a Riemann surface [8], in which case the comparable eigenvalues condition is automatic. The CR analog of this was also recently analyzed by the author in [5].
(2) Direct image bundles particularly include *highest weight families*. Namely, consider a principal G bundle $P \to Y$ with holomorphic connection A, where G is a compact Lie group. Let $\mathfrak{t} \subset \mathfrak{g}$ be the Lie algebra of a maximal torus $T \subset G$ and $\nu \in \mathfrak{t}^*$ be a dominant integral weight. Each dominant weight corresponds to an irreducible representation, giving rise to the family of associated Hermitian, holomorphic vector bundles $\left(E^{k\nu}, \nabla^{k\nu}\right)$ corresponding to multiples of the dominant weight $k\nu \in \mathfrak{t}^*$, $k \in \mathbb{N}$. Particularly, when G is the unitary group and ν the dominant weight corresponding to its standard representation, the highest weight family $E^{k\nu} = \mathrm{Sym}^k E^\nu$ corresponds to symmetric powers of a higher rank bundle. By the Borel-Weil-Bott construction, highest weight families can be realized as direct image bundles with the fibers of the submersion corresponding to flag manifolds G/T.

Finally, we end with a remark on the proof. It is based on a result of Ma-Zhang [7] describing the curvature of the direct image $R_y^{\mathcal{E}_k} \in \mathrm{End}\ (\mathcal{E}_{k,y})$ as a fiber-wise generalized Toeplitz operator. The Chern connection and the Bochner/Kodaira Laplacians all have expressions in terms of the curvature in suitable geodesic coordinates and a parallel frame. The analytic localization technique is now applied to differential operators whose coefficients are valued in the algebra of Toeplitz operators of a given fiber.

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Contact lines and higher-dimensional generalizations of (2,3,5)-distributions

Jun-Muk Hwang

(joint work with Qifeng Li)

A holomorphic distribution $D \subset TM$ on a complex manifold M is *regular* if there is a sequence of vector bundles,

$$D = \partial^{(0)} D \subset \partial^{(1)} D \subset \partial^{(2)} D \subset \cdots \subset \partial^{(d)} D = \partial^{(d+1)} D$$

such that their associated sheaves satisfy

$$\partial^{(i+1)} D = [\partial^{(i)} D, D] + \partial^{(i)} D$$

for each $0 \leq i \leq d$. In this case, for each $x \in M$, the graded vector space

$$\mathrm{symb}_x(D) := \oplus_{i=1}^{d} (\partial^{(i)} D)_x / (\partial^{(i-1)} D)_x$$

has a natural structure of a nilpotent graded Lie algebra induced by Lie brackets of local sections, called the *symbol algebra* of D at x.

A distribution $H \subset TX$ of rank $2m+2$ on a $(2m+3)$-dimensional complex manifold X is a *contact distribution* if its symbol algebra at each point is isomorphic to the Heisenberg algebra. For a contact distribution $H \subset TX$, a smooth rational curve $C \subset X$ is a *contact line* if $TX|_C$ is semipositive and the degree of C with respect to the line bundle TX/H is 1. In this case, for each $x \in C$, there is an m-dimensional deformations of C in X fixing x whose tangent vectors define a germ of m-dimensional submanifold $\mathcal{C}_x \subset \mathbb{P}T_xX$. We say that C is a *nondegenerate contact line* if the Gauss map $\mathcal{C}_x \to \mathrm{Gr}(m+1; T_xX)$ is immersive at $[T_xC]$ for some $x \in C$.

Let $(c_{ijk}, 1 \leq i,j,k \leq m)$ be a symmetric cubic form on $\mathbb{C}^m$. We assume that it is nondegenerate in the sense that

$$\{(t^i) \in \mathbb{C}^m \mid \sum_i t^i c_{ijk} = 0 \text{ for all } j,k\} = 0.$$

A *Lie algebra associated to the cubic form* (c_{ijk}) is a graded Lie algebra $\mathfrak{s}_1+\mathfrak{s}_2+\mathfrak{s}_3$ with bases $p_1,\ldots,p_m,q_1,\ldots,q_m$ for $\mathfrak{s}_1$, $r^1,\ldots,r^m$ for $\mathfrak{s}_2$ and u,v for $\mathfrak{s}_3$ such that

$$[p_i,q_j] = \sum_k c_{ijk} r^k,\ [p_i,r^j] = \delta_i^j u,\ [q_i,r^j] = \delta_i^j v.$$

Our main result is the following.

Theorem. For each integer $m \geq 1$, there is a canonical one-to-one correspondence between germs of distributions $D \subset TM$ of rank $2m$ on a complex manifold M of dimension $3m+2$ with $\mathrm{symb}_x(D)$ isomorphic to a Lie algebra associated to a nondegenerate cubic form $(c_{ijk}(x))$ (possibly depending on $x \in M$) and germs of nondegenerate contact lines $C \subset (X,H)$ on a complex manifold X of dimension $2m+3$ equipped with a contact distribution H.

When $m=1$, this was proved in Theorem 1.2 of [5]. The proof of Theorem is an elaborate extension of that argument in [5]. One of the key ideas of the proof comes from the theory of abnormal extremals in geometric control theory (e.g. [7]).

When $m=1$, there is a unique (up to isomorphisms) nondegenerate cubic form on the 1-dimensional vector space $\mathbb{C}$ and any distribution $D \subset TM$ of rank 2 on a 5-dimensional complex manifold M satisfying $\mathrm{rank}(\partial^{(1)}D) = 3$ and $\partial^{(2)}D = TM$ has the symbol algebra at each point isomorphic to the Lie algebra associated to a nondegenerate cubic form. Such distributions are called (2,3,5)-distributions and have been extensively studied as a distinguished class of geometric structures on 5-dimensional manifolds. In particular, the pioneering work of É. Cartan ([1]) has been developed into a natural theory of connections on such geometric structures by N. Tanaka ([6]). Via this theory, our Theorem has the following consequence.

Corollary. A nondegenerate contact line $C \subset (X,H)$ on a 5-dimensional manifold X with a contact structure H satisfies the formal principle, namely, given any rational curve $\widetilde{C} \subset \widetilde{X}$ on a 5-dimensional complex manifold whose formal neighborhood is formally isomorphic to the formal neighborhood of C in X, the germs of $C \subset X$ and $\widetilde{C} \subset \widetilde{X}$ are biholomorphic.

In Corollary, the normal bundle of $C \subset X$ is semipositive, but not positive. When the normal bundle of a rational curve is positive, the rational curve satisfies the formal principle by the works of Commichau-Grauert ([2]) and Hirschowitz

([3]). However, when the normal bundle is semipositive, but not positive, it is difficult to check the formal principle explicitly (see [4]). To generalize Corollary to higher-dimensions, we need to develop the structure theory of distributions with symbols $(c_{ijk}(x))$.

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Equidistribution without stability for rational surface maps

Jeffrey Diller

(joint work with Roland Roeder)

In this talk, I discuss work with Roland Roeder concerning equidistribution of preimages of curves under a rational self-map $f : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ of the complex projective plane. We focus on cases where the map f cannot be made algebraically stable by birational conjugacy. A prototype for our main result is the following theorem [Sib2] of Sibony.

Theorem 1. *If $f : \mathbf{P}^k \dashrightarrow \mathbf{P}^k$ is a dominating and algebraically stable rational map with $\deg(f) > 1$, then there exists a positive closed $(1,1)$ current T^* on $\mathbf{P}^2$ such that for almost every hypersurface $H \subset \mathbf{P}^k$, one has weak convergence*

$$\frac{f^{-n}(H)}{\deg(f^n)} \to (\deg H) \cdot T^*,$$

where the preimages on the left are regarded as currents of integration.

A rational map is given in homogeneous coordinates, $f = [f_0, \dots, f_k]$, where the components f_j are homogeneous polynomials, all with the same degree, and with no non-constant factor common to all of them. The *degree* $\deg(f)$ is then the degree of the polynomials f_j. One calls f *algebraically stable* if these components do not develop common factors under iteration, so that $\deg(f^n) = (\deg f)^n$ for all $n \geq 0$. With or without algebraic stability, the *(first) dynamical degree* $\lambda(f) := (\deg(f^n))^{1/n} \leq \deg f$ always exists. If $d_{top}(f)$ denotes the number of preimages of a general point $p \in \mathbf{P}^k$, then one also has the lower bound $\lambda(f)^k \geq d_{top}(f)$.

Theorem 1 was established earlier in dimension $k = 1$ by Lyubich [Lju] and Frere-Lopes-Manẽ [FLMn]. In this case hypersurfaces are points, algebraic stability always holds, and the limiting current T^* is an f-invariant, ergodic probability measure supported on the Julia set of f. In higher dimensions $k \geq 2$, one expects $\operatorname{supp} T^*$ to continue to play a role similar that of the Julia set.

When $k = 2$, algebraic stability of f is equivalent to the geometric condition that the full forward orbit of every curve $C \subset \mathbf{P}^2$ is well-defined. That is, algebraic stability fails precisely when f contracts some $C \subset \mathbf{P}^2$ to a point $p = f(C)$ whose forward orbit meets one of the (at most) finitely many points at which f is ill-defined. It was shown in [DDG1] that Theorem 1 continues to hold for a rational map $f : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ satisfying $\lambda(f)^2 > d_{top}(f)$ provided only that there exists a rational surface X and a birational map $\varphi : X \dashrightarrow \mathbf{P}^2$ such that the lift $\varphi^{-1} \circ f \circ \varphi : X \dashrightarrow X$ is algebraically stable in the geometric sense just described.

Unfortunately there are examples of plane rational maps that cannot be made algebraically stable by birational conjugacy. These include some monomial maps [Fav2] and, more generally [DL1], many maps f which are *toric* in the sense that they have constant Jacobian $\delta(f)$ relative to the natural holomorphic two-form $\frac{dx_1 \wedge dx_2}{x_1 x_2}$ on the algebraic torus $(\mathbf{C}^*)^2$. Indeed, with the recent exception of some skew product examples discovered by Birkett, all the known examples of unstabilizable plane rational maps are toric. For toric rational maps we introduce a much weaker condition than algebraic stability that we call 'internal stability'. Roughly speaking, a toric map is internally stable if it becomes algebraically stable on the inverse limit $\widehat{(\mathbf{C}^*)^2}$ of all toric surfaces obtained from $\mathbf{P}^2$ by blowing up. Internally stable toric maps include most that cannot be stabilized by birational conjugacy and in particular the recent examples from [BDJ] of rational maps with transcendental first dynamical degree. Our main result may be stated a little imprecisely as follows.

Theorem 2. *Suppose that $f : \mathbf{P}^2 \dashrightarrow \mathbf{P}^2$ is an internally stable toric rational map obtained as a finite composition of monomial maps and birational toric maps. If f is not itself a monomial map, then there exists a positive closed $(1,1)$ current T^* on $\mathbf{P}^2$ satisfying the conclusion of Theorem 1.*

Interestingly enough, the exclusion of monomial maps is essential here since they have only a slightly weaker equidistribution property. On the other hand, we do not know of any toric rational maps that are *not* finite compositions of birational toric and monomial maps.

The proof of Theorem 2 involves several steps. Following [BFJ] we first define reasonable notions of both $(1,1)$ cohomology classes and positive closed $(1,1)$ currents on $\widehat{(\mathbf{C}^*)^2}$. We show in particular that every nef $(1,1)$ class is canonically represented by a positive closed current that is *internal*, in the sense that it has full mass on $(\mathbf{C}^*)^2$ and *homogeneous* in the sense that it has maximal symmetry related to the multiplicative action of $(\mathbf{C}^*)^2$ on itself. We then define an induced linear pullback f^* on classes and currents and invoke the main theorem of [BFJ] to extract a unique nef class $\alpha^* \in H^{1,1}(\widehat{(\mathbf{C}^*)^2})$ that is maximally expanded by pullback, satisfying $f^*\alpha^* = \lambda(f)\alpha^*$. Finally, we establish a very weak lower bound on the extent to which iterates of a map satisfying the hypotheses of Theorem 2 can shrink the volume of an open set in the torus $(\mathbf{C}^*)^2$. Combining all these ingredients we then prove L^1 convergence for potentials of the currents $\frac{f^{n*}(C)}{\deg(f^n)} - C$, which implies the theorem.

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Lelong numbers of currents of full mass intersection

Duc-Viet Vu

Let X be a compact Kähler manifold of dimension n. Let $1 \leq m \leq n$ be an integer. Let $\alpha_1, \dots, \alpha_m$ be pseudoeffective cohomology $(1,1)$-classes on X. Let T_j be a closed positive current in α_j for $1 \leq j \leq m$. Let $\langle T_1 \wedge \cdots \wedge T_m \rangle$ be the non-pluripolar product of $T_1, \dots, T_m$. It generalizes the classical product of $(1,1)$-currents of bounded potentials.

Let T'_j be closed positive $(1,1)$-currents in α_j for $1 \leq j \leq m$ such that T_j is more singular than T'_j, *i.e,* potentials of T_j is smaller than those of T'_j modulo an additive constant. Recall that

$$\{\langle T_1 \wedge \cdots \wedge T_m \rangle\} \leq \{\langle T'_1 \wedge \cdots \wedge T'_m \rangle\}, \tag{1}$$

where for a closed positive current S, we denote by $\{S\}$ the cohomology class of S. In general one does not have equality. This in particular says that the non-pluripolar products do not preserve masses. If $T_{j,\min}$ is a current of minimal singularity in α_j, then the cohomology class of $\langle T_{1,\min} \wedge \cdots \wedge T_{n,\min} \rangle$ is independent of the choice of $T_{j,\min}$. We denote the last class by $\langle \alpha_1 \wedge \cdots \wedge \alpha_m \rangle$. When α_j's are big, the class $\langle \alpha_1 \wedge \cdots \wedge \alpha_m \rangle$ coincides with the standard product of big classes introduced by Boucksom-Essydieux-Guedj-Zeriahi [1].

As a direct consequence of (1), we get

$$\{\langle T_1 \wedge \cdots \wedge T_m \rangle\} \leq \langle \alpha_1 \wedge \cdots \wedge \alpha_m \rangle.$$

When the equality in the above inequality occurs, the currents $T_1, \ldots, T_m$ are said to be *of full mass intersection.* The last notion lies at the heart of the theory of non-pluripolar products. The following question comes naturally:

Problem. Characterize currents of full mass intersection. More precisely if $T_1, \ldots, T_m$ are of full mass intersection, does T_j have the same singularity as the currents of minimal singularity in α_j.

Previously this problem was studied mainly in the case where $m = n$ and T_j's are the same for every j, as part of the theory of complex Monge-Ampère equations. The key tools in this case is the characterization of potentials of full Monge-Ampère masses in terms of plurisubharmonic envelopes by Darvas, Di Nezza, and Lu [2]. Our goal is to report a recent result in the mixed setting, *i.e,* when T_j's are not necessarily the same or m is not necessarily equal to n.

Let S be a closed positive current on X and V be an irreducible analytic subset in X. Let α be a pseudoeffective $(1,1)$-class on X. Let $T_{\alpha,\min}$ be a current with minimal singularities in α. We denote by $\nu(\alpha, V)$ the generic Lelong number of $T_{\alpha,\min}$ along V. This number is independent of the choice of $T_{\alpha,\min}$. It is clear that for every current $S \in \alpha$, we have $\nu(S, V) \geq \nu(\alpha, V)$.

Theorem A. ([3]) Let $1 \leq m \leq n$ be an integer. Let $\alpha_1, \ldots, \alpha_m$ be big cohomology classes in X and let T_j be a closed positive $(1,1)$-currents in α_j for $1 \leq j \leq m$. Let V be a proper irreducible analytic subset of X of dimension $\geq n-m$. Assume that $T_1, \ldots, T_m$ are of full mass intersection. Then there exists an index $1 \leq j \leq m$ such that

$$\nu(T_j, V) = \nu(\alpha_j, V).$$

The key ingredient in the proof of Theorem A is a new notion of products of pseudoeffective classes. This new product of pseudoeffective classes is bounded from below by the positive product introduced in [1]. The feature is that this new product also captures some pluripolar part of "total intersection" of classes. This explains why we have a better control on masses.

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K-stability and Nevanlinna-Diophantine approximation

Min Ru

In this talk, we explore the still somewhat mysterious connection of the notion of K-stability with the Nevanlinna theory (Diophantine approximation).

$\diamond$ **Nevanlinna theory**. Nevanlinna theory is an extension of the "fundamental theorem of algebra". In 1929, Nevanlinna proved the following so-called the "Second Main Theorem": *Let f be meromorphic (non-constant) on $\mathbb{C}$ and $a_1, ..., a_q \in \mathbb{C} \cup \{\infty\}$ be distinct. Then, for any $\epsilon > 0$, $(q-2-\epsilon)T_f(r) \leq_{exc} \sum_{j=1}^q N_f(r, a_j)$, or equivalently $\sum_{j=1}^q m_f(r, a_j) \leq_{exc} (2+\epsilon)T_f(r)$, where $\leq_{exc}$ means that the inequality holds for $r \in [0, +\infty)$ outside a set E with finite measure. Here $m_f(r, a) = \frac{1}{2\pi}\int_0^{2\pi} \log^+ \frac{1}{|f(re^{i\theta})-a|}d\theta$.*

$\diamond$ **Diophantine approximation**. Roth's theorem states that every irrational algebraic number α has approximation exponent equal to 2. More precisely, Roth in 1955 proved the following result: *Let α be an algebraic number of degree≥ 2. For any given $\varepsilon > 0$, we have $\left|\alpha - \frac{p}{q}\right| > \frac{1}{q^{2+\epsilon}}$ for all, but finitely many, coprime integers p and q.* More generally "Roth's Theorem" holds over an arbitrary number field: *Let k be a number field and S be finite set of places on k. Let $a_1, \dots, a_q$ distinct points in $\mathbb{P}^1(k)$. Then, for any given $\varepsilon > 0$, $\sum_{j=1}^q \sum_{v \in S} \log^+ \frac{1}{\|x-a_j\|_v} \leq_{exc} (2+\epsilon)h(x)$, where $\leq_{exc}$ means that the inequality holds for all but finitely many x.* Denote by $m_S(x, a) := \sum_{v\in S} \log^+ \frac{1}{\|x-a\|_v}$. Then we have $\sum_{j=1}^q m_S(x, a_j) \leq_{exc} (2+\epsilon)h(x)$.

$\diamond$ **The result of Ru-Vojta**. In extending the above results, we introduce some notations: Let X be a complex projective variety and D be an effective Cartier divisor. Let s_D be the canonical divisor of $[D]$ (i.e. $[s_D = 0] = D$) and $\{h_\alpha\}$ be an Hermitian metric, i.e. $\|s\|^2 = |s_\alpha|^2 h_\alpha$. By Poincare-Lelong formula, $-dd^c \log \|s_D\|^2 = -D + c_1([D])$. Let $f\colon \mathbb{C} \to X$ be a holomorphic map. Applying $\int_1^t \frac{dt}{t}\int_{|z|<t}$, we get the so-called "First Main Theorem":

$$m_f(r, D) + N_f(r, D) = T_{f,D}(r) + O(1),$$

where $m_f(r, D) = \int_0^{2\pi} \lambda_D(f(re^{i\theta}))\frac{d\theta}{2\pi}$ with $\lambda_D(x) = -\log\|s_D(x)\|$ (*Weil-function*), and $T_{f,L}(r) := \int_1^r \frac{dt}{t}\int_{|z|<t} f^*c_1(L)$. Note that if $D_1 \geq D_2$, then $\lambda_{D_1} \geq \lambda_{D_2}$. In proving the Second Main Theorem, Ru-Vojta introduced the following *β-constant*: Let L be a line bundle on X and D be an effective Cartier divisor on X, we define

$$\beta(L, D) := \limsup_{m\to\infty} \frac{\sum_{t\geq 1} h^0(mL - tD)}{mh^0(mL)}.$$

Theorem ([1], analytic version). *Let X be a smooth complex projective variety and let $D_1, \dots, D_q$ be effective Cartier divisors in general position. Let L be a line sheaf on X with $h^0(L^N) \geq 1$ for N big enough. Let $f : \mathbb{C} \to X$ be a holomorphic* *map with Zariski image. Then, for every $\epsilon > 0$,*

$$\sum_{j=1}^{q} \beta_j(L, D_j) m_f(r, D_j) \leq_{exc} (1+\epsilon) T_{f,L}(r).$$

Theorem ([1], arithmetic version). *Let X be a projective variety over a number field k, and $D_1, \ldots, D_q$ be effective Cartier divisors intersecting properly on X. Let L be a line bundle on X with $h^0(L^N) \geq 1$ for N big enough. Let $S \subset M_k$ be a finite set of places. Then, for every $\epsilon > 0$, the inequality*

$$\sum_{i=1}^{q} \beta(L, D_j) m_S(x, D_j) \leq (1+\epsilon) h_L(x)$$

holds for all k-rational points outside a proper Zariski-closed subset of X.

$\diamond$ **The algebro-geometric K-stability criterion**. It turns out the β-constant introduced above appeared in the recent algebro-geometric stability criterion. Recall the following **K-E question**: *Does there always exists a Kähler from ω_{KE} on X such that Ric $(\omega_{KE}) = \lambda \omega_{KE}$?* Note, in the class level, $c_1(X) = \lambda[\omega]$ for $\lambda = 0, 1, -1$. The case $\lambda = 0$, it is true by the solution of Calabi conjecture, when $\lambda = -1$, it was proved by Aubin and Yau independently. The case when $\lambda = 1$, X is called *Fano*. In this case, problem becomes more subtle and there is no definite answer. It was recently established that the existence of K-E metric in the Fano case is equivalent to the K-stability condition, whose notion was introduced by Tian in 90's and it was later reformulated in a purely algebro-geometric form by Donaldson. In 2015, Fujita showed that if (Fano) X is K-(semi) stable, then $\beta(-K_X, D) < 1$ (resp. $\beta(-K_X, D) \leq 1$) for any nonzero effective divisor D on X. Later Fujita and C. Li (with a technical assumption, which were removed by Blum-Xu) independently proved that it is indeed an equivalence condition if one goes to the birational model, i.e. *the $\mathbb{Q}$-fano variety X is K- stable if and only if $\frac{A_X(E)}{\beta(-K_X, E)} > 1$ for any prime divisors E over X (i.e. E is a prime divisor on a birational model $\pi : \tilde{X} \to X$), where $A_X(E) := 1 + ord_E(K_{Y/X})$ and is called the log discrepancy.* We call $\delta(L) = \inf_E \frac{A_X(E)}{\beta(L,E)}$ the *stability threshold*. We summarize the following *valuative criterion of K-stability*:

1. X is uniformly K-stable (resp. semi-satble) if and only if $\delta(-K_X) > 1$ (resp. ≥ 1) (Fujita-Li).
2. X is K-stable if and only if $A_X(E) > \beta(-K_X, E)$ for any E (Blum-Xu).

$\diamond$ **The m-basis**. Blum-Jonsson (also Fujita-Odaka) used *m-basis type* to describe the stability threshold $\delta(L)$. For m sufficient large, we say D is a *m-basis type divisor* if $D = \frac{1}{mN_m}((s_1) + \cdots + (s_{N_m}))$ where $\{s_1, \ldots, s_{N_m}\}$ forms a basis of $H^0(X, mL)$. Let $S_m(D) := \sup_{\{s_1, \ldots, s_{N_m}\}} \frac{1}{N_m} \sum_{i=1}^{N_m} \mathrm{ord}_D(s_i)$, where *sup* runs all basis. Note that the sup is achieved by a basis of filtration $\mathcal{F}_m^t = H^0(X, mL - tD), t \geq 0$ of $H^0(X, mL)$. Define $\delta_m(L) := \inf_E \frac{A_X(E)}{S_m(E)}$, Blum-Jonsson proved that $\lim_{m \to} \delta_m(L) = \delta(L)$. It turns out, in Nevanlinna theory, we can estimate the Weil functions which are associated to the m-basis divisors. More precisely, we have the following theorem:

Theorem (m-base estimate). *Let X be a complex projective variety and let L be a line bundle on X with $\dim H^0(X, L) \geq 1$. Let $s_1, \ldots, s_q \in H^0(X, L)$. Let $f : \mathbb{C} \to X$ be a holomorphic map with Zariski-dense image. Then, for any $\epsilon > 0$,*

$$\int_0^{2\pi} \max_J \sum_{j\in J} \lambda_{s_j}(f(re^{i\theta}))\frac{d\theta}{2\pi} \leq_{exc} (\dim H^0(X, L) + \epsilon)T_{f,L}(r)$$

where the set J ranges over all subsets of $\{1, \ldots, q\}$ such that the sections $(s_j)_{j\in J}$ are linearly independent. Hence, Nevanlinna-Diophantine provided one more step which gives the estimate for m-base-type divisors. In the proof of Blum-Jonsson's result, they used the standard filtration $\mathcal{F}^t_m = H^0(X, mL-tD), t \geq 0$ of $H^0(X, mL)$. Ru-Vojta however used a more sophisticated filtration (multi-parameter filtration) in their proof. It is hoped that such filtration can be used in the study of algebro-geometric notion of k-stability.

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Reporter: Fabrizio Bianchi

Participants

Prof. Dr. Marco Abate
Dipartimento di Matematica
Università di Pisa
Largo Bruno Pontecorvo, 5
56127 Pisa
ITALY

Marc Abboud
I. R. M. A. R.
Université de Rennes I
Campus de Beaulieu
35042 Rennes Cedex
FRANCE

Prof. Dr. Mats Andersson
Department of Mathematics
Chalmers University of Technology
412 96 Göteborg
SWEDEN

Razvan Apredoaei
Institut de Mathématiques de
Jussieu-Paris Rive Gauche
Université Paris-Cité
8 Place Aurélie Nemours
P.O. Box Boite 7012
75013 Paris 75013
FRANCE

Dr. Hugues Auvray
Laboratoire de Mathématique d'Orsay,
Faculté des Sciences, Université
Paris-Saclay
307 rue Michel Magat
F91405 Orsay Cedex
FRANCE

Dr. Enchao Bi
School of Mathematics and Statistics
Qingdao University
308 Ningxia Road
Qingdao 266071
CHINA

Dr. Fabrizio Bianchi
CNRS, Univ. Lille
UMR 8524 - Laboratoire Paul Painlevé
59655 Villeneuve d'Ascq
FRANCE

Dr. Severine Biard
Université Polytechnique
Hauts-de-France
Le Mont Houy
59313 Valenciennes Cedex 9
FRANCE

Prof. Dr. Judith Brinkschulte
Mathematisches Institut
Universität Leipzig
Postfach 10 09 20
04009 Leipzig
GERMANY

Dr. Serge Cantat
I.R.M.A.R.
Université de Rennes I
Campus de Beaulieu
35042 Rennes Cedex
FRANCE

Dr. Junyan Cao
Laboratoire J.-A. Dieudonné, Université
Côte d'Azur
Parc Valrose
P.O. Box 06108
06108 Nice Cedex 2
FRANCE

Dr. Ionut Chiose
Institute of Mathematics"Simion
Stoilow"
of the Romanian Academy
P.O. Box 1-764
014 700 Bucharest
ROMANIA

Prof. Dr. Nguyen-Bac Dang
Université de Paris-Sud
Centre d'Orsay
Batiment 425
91400 Orsay Cedex
FRANCE

Dr. Ved Datar
Department of Mathematics
Indian Institute of Science
CV Raman Road
Bangalore, Bengaluru 560012
INDIA

Dr. Jeffrey Diller
Department of Mathematics
University of Notre Dame
Notre Dame, IN 46556-5683
UNITED STATES

Prof. Dr. Tien-Cuong Dinh
Department of Mathematics
National University of Singapore
10 Lower Kent Ridge Road
119076 Singapore 119076
SINGAPORE

Prof. Charles Favre
Centre de Mathématiques Laurent
Schwartz
École Polytechnique
91128 Palaiseau
FRANCE

Prof. Dr. Simion Filip
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago IL 60637-1514
UNITED STATES

Dr. Siarhei Finski
Centre de Mathématiques Laurent
Schwartz
École Polytechnique
91128 Palaiseau
FRANCE

Prof. Dr. John Erik Fornaess
Department of Mathematics
The Norwegian University of Science
and Technology
Sentralbygg 2
Alfred Getz vei 1
7491 Trondheim
NORWAY

Prof. Dr. Daniel Greb
Fakultät für Mathematik
Universität Duisburg-Essen
45117 Essen
GERMANY

Prof. Dr. Vincent Guedj
Institut de Mathématiques de Toulouse
Université Paul Sabatier
118, route de Narbonne
31062 Toulouse Cedex 9
FRANCE

Prof. Dr. Ursula Hamenstädt
Mathematisches Institut
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. Hendrik Herrmann
Room G.15.17
FB 4: Mathematik u.
Naturwissenschaften
Bergische Universität Wuppertal
Gaußstraße 20
42119 Wuppertal
GERMANY

Dr. Chin-Yu Hsiao
Institute of Mathematics
Academia Sinica
No.1 Roosevelt Rd. Sc.4
Taipei 10617
TAIWAN

Prof. Dr. Xiaojun Huang
Department of Mathematics
Rutgers University
Hill Center, Busch Campus
110 Frelinghuysen Road
Piscataway, NJ 08854-8019
UNITED STATES

Prof. Dr. Jun-Muk Hwang
IBS Center for Complex Geometry
55, Expo-ro, Yuseong-gu,
Daejeon 34126
KOREA, REPUBLIC OF

Seung uk Jang
E105
Department of Mathematics
The University of Chicago
5734 South University Avenue
Chicago IL 60637-1514
UNITED STATES

Prof. Dr. Cezar Joita
Institute of Mathematics"Simion Stoilow"
of the Romanian Academy
Calea Grivitei 21
P.O. Box 1-764
014 700 Bucharest
ROMANIA

Dr. Lucas Kaufmann
Institute for Basic Science
55 Expo-ro, Yuseong-gu
Daejeon 34126
KOREA, REPUBLIC OF

Dr. Sung Yeon Kim
Institute for Basic Science
55 Expo-ro, Yuseong-gu
Daejeon 34126
KOREA, REPUBLIC OF

Dr. Semyon Klevtsov
Institut de Mathématiques
Université de Strasbourg
7, rue René Descartes
67084 Strasbourg Cedex Cedex
FRANCE

Dr. Takayuki Koike
Department of Mathematics
Graduate School of Science
Osaka Metropolitan University
3-3-138 Sugimoto, Sumiyoshi-Ku
Osaka 558-8585
JAPAN

Dr. Bingxiao Liu
Mathematisches Institut
Universität zu Köln
Weyertal 86 - 90
50931 Köln
GERMANY

Dr. Hoang Chinh Lu
Laboratoire de Mathématiques
Université Paris Sud (Paris XI)
Batiment 307
91405 Orsay Cedex
FRANCE

Dr. Wen Lu
School of Mathematics & Statistics
Huazhong University of Science and Technology
Wuhan 430 074
CHINA

Prof. Dr. Xiaonan Ma
Université de Paris
Institut de Mathématiques de Jussieu -
Paris Rive Gauche
Bâtiment Sophie Germain
P.O. Box Case 7012
75205 Paris Cedex 13
FRANCE

Prof. Dr. George Marinescu
Mathematisches Institut
Universität zu Köln
Weyertal 86 - 90
50931 Köln
GERMANY

Dr. Shin-ichi Matsumura
Mathematical Institute
Tohoku University
Aramaki, Aoba-Ku 6-3
Sendai 980-8578
JAPAN

Luo Muhan
Department of Mathematics
National University of Singapore
10 Lower Kent Ridge Road
Singapore 119 076
SINGAPORE

Dr. Viet Anh Nguyen
Université Lille I
Laboratoire de mathématiques Paul
Painlevé
CNRS U.M.R. 8524
UFR de Mathématiques
59655 Villeneuve d'Ascq Cedex
FRANCE

Prof. Dr. Keiji Oguiso
Department of Mathematical Sciences
University of Tokyo
3-8-1 Komaba, Meguro-ku
Tokyo 153-8914
JAPAN

Aryaman Patel
Fakultät für Mathematik
Universität Duisburg-Essen
Thea-Leymann-Straße 9
45127 Essen
GERMANY

Prof. Dr. Mihai Paun
Mathematik
Universität Bayreuth
Postfach 101251
95412 Bayreuth
GERMANY

Prof. Dr. Duong H. Phong
Department of Mathematics
Columbia University
2990 Broadway
New York, NY 10027
UNITED STATES

Prof. Dr. Jasmin Raissy
Institut de Mathématiques de Bordeaux
Université de Bordeaux
Institut Universitaire de France
351, cours de la Libération
33405 Talence
FRANCE

Prof. Dr. Min Ru
Department of Mathematics
University of Houston
Houston, TX 77204-3008
UNITED STATES

Prof. Dr. Jean Ruppenthal
Fachgruppe Mathematik und Informatik
Bergische Universität Wuppertal
Gaußstraße 20
42119 Wuppertal
GERMANY

Prof. Dr. Håkan Samuelson Kalm
Department of Mathematical Sciences
Chalmers University of Technology &
University of Gothenburg
412 96 Göteborg
SWEDEN

Dr. Nikhil Savale
Mathematisches Institut
Universität zu Köln
Weyertal 86 - 90
50931 Köln
GERMANY

Dr. Lars Martin Sektnan
Department of Mathematics
University of Gothenburg
412 96 Göteborg
SWEDEN

Dr. Martin Sera
Kyoto University of Advanced Science
Faculty of Engineering
18 Gotanda-cho, Yamanouchi, Ukyo-ku
615-8577 Kyoto
JAPAN

Ood Shabtai
Department of Mathematics
School of Mathematical Sciences
Tel Aviv University
P.O.Box 39040
Ramat Aviv, Tel Aviv 69978
ISRAEL

Wei-Chuan Shen
Mathematisches Institut
Universität zu Köln
Weyertal 86 - 90
50931 Köln
GERMANY

Dr. Laurent Stolovitch
CNRS-Laboratoire J.-A. Dieudonne
Université Côte d'Azur
Parc Valrose
06108 Nice Cedex 2
FRANCE

Prof. Dr. Valentino Tosatti
Dept. of Mathematics and Statistics
McGill University
805, Sherbrooke Street West
Montréal QC H3A 0B9
CANADA

Prof. Dr. Duc Viet Vu
Mathematisches Institut
Universität zu Köln
Weyertal 86 - 90
50931 Köln
GERMANY

Prof. Dr. Elizabeth Wulcan
Department of Mathematical Sciences
Chalmers University of Technology and
University of Göteborg
412 96 Göteborg
SWEDEN

Dr. Weixia Zhu
Fakultät für Mathematik
Universität Wien
Oskar-Morgenstern-Platz 1
1190 Wien
AUSTRIA