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Representations and degenerations

Ilya Dumanski[1] • Valentina Kiritchenko[2]

In this snapshot, we explain two important mathematical concepts (representation and degeneration) in elementary terms. We will focus on the simplest meaningful examples, and motivate both concepts by study of symmetry.

1 Degeneration as a mathematical term

Mathematical terms sometimes sound as usual words but have a completely different meaning. As Humpty Dumpty explains to Alice in "Through the looking glass" (when she is puzzled by his words): "when I use a word, it means just what I choose it to mean – neither more nor less". Degeneration is one of such puzzling mathematical terms. According to the Oxford English dictionary, the word "degeneration" is synonymous with "decline" or "loss of function", which does not sound like a good thing at all. However, the mathematical meaning of "degeneration" is more positive. We first explain this meaning using art rather than mathematics.

Suppose we want to reconstruct a complicated real-life object, for example we would like to draw a realistic cat. We may first try to build a simplified version of the object focusing on its key features, and then gradually add details to this simplified version to come as close as possible to its real-life version.

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Speaking in artistic terms, we will first draw a sketch (just a few basic shapes), and then add details (see Figure 1). Professional artists are able to make simple sketches that capture the essence of an object (a famous example is the series of lithographs "The Bull" (Le Taureau) by Pablo Picasso, another example is the evolution of the Starbucks logo (Starbucks Siren)) and we may say that a sketch is a "degeneration" because it does not have all the qualities of a real-life object.

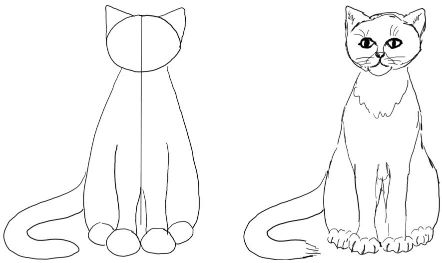

Figure 1: From sketch to final drawing.

In mathematics, degeneration is a way to produce a realistic yet simple sketch out of a complicated mathematical object, just like the artistic process we have described. We now illustrate the concept of degeneration with a mathematical example.

1.1 Degeneration of hyperbolas

A *hyperbola* is the set of all points (x, y) in the plane that satisfies the equation $xy = 1$. We will degenerate this hyperbola to a pair of lines. The main idea is to consider the hyperbola not on its own but as a member of the family $\{H_t \,|\, t \in \mathbb{R}\}$, where H_t is the set of all points (x, y) satisfying $xy = t$, which we call again hyperbola, see Figure 2.

All hyperbolas for $t \neq 0$ look similar, and are, in fact, a scaled version of our original hyperbola H_1. Because of this, H_t for $t \neq 0$ is called a *generic fiber* of the family. However, there is a black sheep or a special fiber in this family: the element H_0 described by the equation $xy = 0$, which is not a hyperbola but the union of the two lines $x = 0$ and $y = 0$. As t approaches 0, the generic hyperbola H_t approaches the special fiber H_0. Note that if we zoom out far enough from the origin, H_1 is almost indistinguishable from H_0. So the special fiber indeed yields a good sketch of a generic fiber.

The whole picture can be visualized by a surface P in 3-dimensional space given by the equation $z = xy$, see Figure 3. The intersections of this surface with the planes $z = t$ are exactly the hyperbolas H_t. So the family of hyperbolas

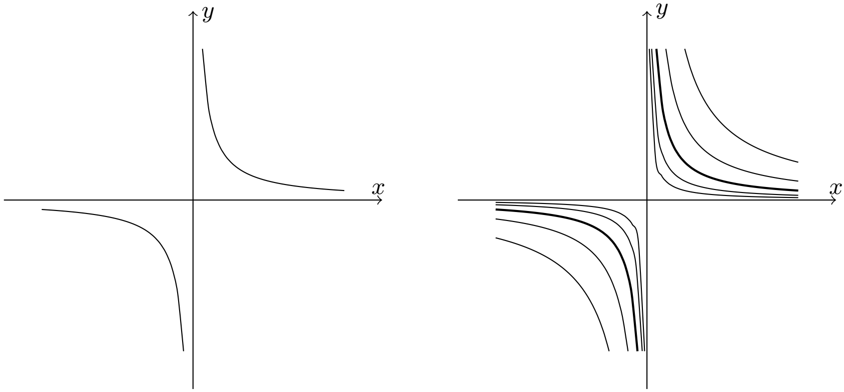

Figure 2: The hyperbola H_1 on the right and the family H_t on the left. The thick line represents H_1 as member of the family.

can be encoded by the surface P together with the function $f\colon P \to \mathbb{R}$ defined by $f(x,y,z) = z$. Then the preimage $f^{-1}(t)$ coincides with H_t. Mathematically speaking, we have a family of algebraic curves H_t over the line $\mathbb{R}$, and $t \neq 0$ is a generic point of $\mathbb{R}$ (that is, the fiber over this point is generic). In contrast, the point $t = 0$ belongs to the family but it is non-generic, since the fiber H_0 over this point is a special one. Also in 3-dimensional space, if we zoom out far enough from the origin, the hyperbola H_1 will be very similar to H_0. Hence we can regard H_0 as a degeneration of H_1.

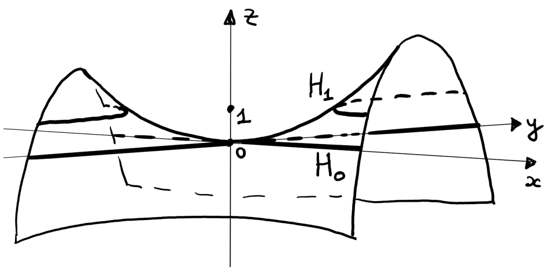

Figure 3: The surface P and the hyperbolas H_0 and H_1.

2 Representations of groups

In contrast to degeneration, representation as a mathematical term is much closer to the usual meaning of the word "representation". According to the Oxford English dictionary, representation is the description or portrayal of something or someone in a particular way. In order to convert this into a mathematical definition, we just have to say explicitly what we are going to represent and by which means. Below we explain the concept of *representation* of a *group*[3] by means of *matrices*.

An example of a group is the set of real numbers $\mathbb{R}$ considered together with the operation of addition $+$. Generally speaking, a group is a pair $(S, \star)$ where S is a set and $\star$ is an operation on S assigning an element $a \star b \in S$ to every pair of elements $a, b \in S$, and having properties which mimic the properties of the addition on $\mathbb{R}$. However, in contrast to $+$, the operation $\star$ does not need to be commutative, that is, we do not necessarily have $a \star b = b \star a$. In fact, non-commutative operations occur naturally, for example when S is a set of symmetries of a geometric object. Symmetries, groups, and group actions are central notions in mathematics. Below we focus on a concrete example of a symmetry group (for a more formal treatment of symmetry groups, we refer the reader to Snapshot 3/2018 *Computing with Symmetries* by C. M. Roney-Dougal and to Snapshot 5/2019 *Algebra, matrices, and computers* by A. S. Detinko, D. L. Flannery, and A. Hulpke).

Let S be the set of transformations of the plane that preserve a given equilateral triangle Δ. Such transformations are called *symmetries* of the triangle. There are six different symmetries, see Figure 4, and every symmetry can be realized by a sequence of rotations r and reflections f. Here r rotates Δ by $120°$ counterclockwise around the center of the triangle Δ, and f flips Δ about its vertical axis of symmetry. By e we denote the *identity* symmetry, which takes every point of Δ to itself. Traditionally, the notation rf means that we first apply f and then r.[4] The symmetry rf, also denoted $r \circ f$, is called the *composition* of r and f. Composition $\circ$ is a non-commutative operation on the set of symmetries S. In fact, with the help of Figure 4, the reader can check that $r \circ f \neq f \circ r$. The set $S = \{e, r, rr, f, rf, rrf\}$ together with the operation $\circ$ of composition is called the *symmetry group* of Δ. The elements of S are uniquely defined, they are the transformations in Figure 4, but there are different ways to write them mathematically. For example, the identity

[3] Apart from groups, there are other mathematical objects such as algebras (in particular, Lie algebras) whose representations play an important role in mathematics and physics.

[4] This notation looks misleading as we usually read from left to right. However, it agrees with the standard notation $F(G(x))$ for the composition of functions. For instance, if $F(x) = \sin x$, and $G(x) = x^2$, then $F(G(x)) = \sin(x^2)$ and not $\sin^2(x)$.

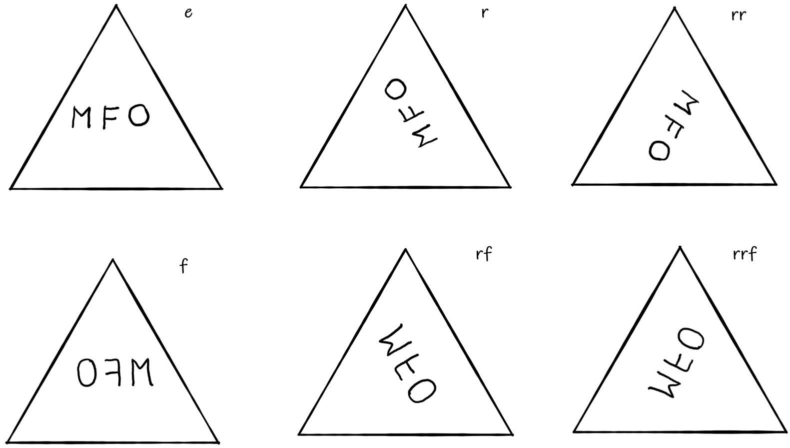

Figure 4: Symmetries of an equilateral triangle.

transformation e can be obtained by applying the reflection f twice ($e = ff$), and we may also write $S = \{ff, r, rr, f, rf, rrf\}$.

In general, the pair $(S, \star)$ might be quite hard to work with, even when $(S, \star)$ is a symmetry group of some mathematical object. In essence, representation theory is an attempt to linearize complicated symmetries and group actions, meaning that we replace them by linear transformations (such as reflections and rotations) of vector spaces. Since linear transformations are best captured by matrices, they are one of the main tools of representation theory.

We now briefly explain what a *representation by matrices* is. Suppose we assign a square matrix $R(a)$ to every element $a \in S$ so that the operation $\star$ corresponds to the matrix multiplication [5]

$$R(a \star b) = R(a)R(b).$$

This identity should hold for all pairs of elements a, $b \in S$ and, in particular, the matrices $R(a)$ and $R(b)$ should have the same number of columns and rows. If R is a representation of $(S, \star)$ by $n \times n$ matrices with real entries, we call it a *real representation of dimension* n. [6] Note that matrix multiplication is a straightforward operation that uses only addition and multiplication of

[5] The definition of matrices and matrix multiplication as well as their relation with linear transformations can be found in Snapshot 5/2019 by A. S. Detinko, D. L. Flannery, and A. Hulpke.

[6] Similarly, we can define complex representations by considering matrices whose entries are complex numbers.

real numbers (there are also online calculators that perform this task). Hence, working with a matrix representation R of $(S,\star)$ might be easier than working with $(S,\star)$ itself. Below we consider examples of matrix representations.

2.1 Representations of $\mathbb{R}$

For example, consider $(S,\star) = (\mathbb{R},+)$. By the definition we gave above, a one-dimensional real representation of $(\mathbb{R},+)$ is a function $R\colon \mathbb{R} \to \mathbb{R}$ such that $R(a+b) = R(a)R(b)$. For instance, the exponential function e^x will do the trick as $e^{a+b} = e^a e^b$. Another example is the *trivial representation* $R(a) = 1$ for all a.

We now construct a two-dimensional representation by setting

$$R(a) = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}.$$

This again satisfies the property $R(a+b) = R(a)R(b)$ as

$$\begin{pmatrix} 1 & a+b \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix}.$$

This representation is an example of a *reducible* but not *completely reducible* representation (we discuss this notion in more detail in the next section). Informally speaking, this representation shows us that classifying all representations [7] of $\mathbb{R}$ is not an easy task.[8]

2.2 Representations of the symmetric group S_3

Let $(S_3,\circ)$ be the group of symmetries of an equilateral triangle considered above (also called symmetric group on three elements). This group has two one-dimensional representations: the trivial representation and the *sign representation*. The latter assigns -1 to all elements that change the orientation of the triangle (that is, change the abbreviation "MFO" on Figure 4 to its mirror image), and assigns $+1$ to all elements that preserve orientation. There is also a natural two-dimensional representation, where we encode rotations and reflections by 2×2 matrices. Let us introduce Cartesian coordinates (x, y) so that the origin coincides with the center of the triangle, and the x- and y-axis

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[7] Here and later we consider only continuous representations, that is, entries of the matrix $R(a)$ are continuous functions of a.

[8] An advanced result from linear algebra, the Jordan normal form theorem, is required as a tool. In particular, the two-dimensional representation of $\mathbb{R}$ constructed above corresponds to a 2×2 matrix with the same non-zero entries on the diagonal, entry equal to 1 in the upper corner, and entry equal to zero in the lower corner (this form of matrix is known as Jordan block).

are the (usual) horizontal and vertical line meeting perpendicularly at the origin. In these coordinates, the rotation r sends the vectors $(1, 0)$ and $(0, 1)$ to the vectors $(-\frac{1}{2}, \frac{\sqrt{3}}{2})$ and $(-\frac{\sqrt{3}}{2}, -\frac{1}{2})$, respectively, see Figure 5.

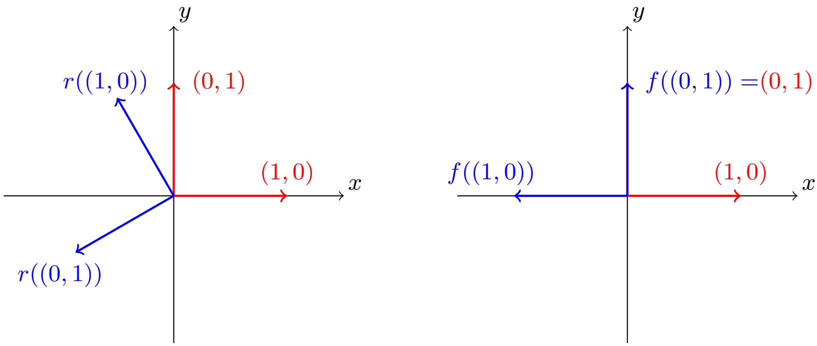

Figure 5: The rotation r on the left and the reflection f on the right.

Therefore, the matrix associated to r is $R(r) = \begin{pmatrix} -\frac{1}{2} & -\frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}$. The reflection f sends the vector $(1, 0)$ to $(-1, 0)$ and preserves the vector $(0, 1)$, see again Figure 5, so its associated matrix is $R(f) = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$. Following this process, the matrices associated to the remaining linear transformations e, rr, rf, rrf are

$$R(e) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad R(rr) = \begin{pmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix},$$

$$R(rf) = \begin{pmatrix} \frac{1}{2} & \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}, \quad R(rrf) = \begin{pmatrix} \frac{1}{2} & -\frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{pmatrix}.$$

Since the matrix representation of a linear transformation depends on the choice of coordinates, it is not unique. Indeed, if we changed the x- and y-axis, for example choosing them to be the axis obtained by rotating the original horizontal and vertical line by a certain angle, we would use vectors different from $(1, 0)$ and $(0, 1)$ to write the matrix associated to the six linear transformation above, and we would then obtain different matrices. However, the old and new matrices will be related: one can be obtained from the other by conjugation by a third matrix, which is the same for all pairs of old and new matrices. In this case, we say that the two matrix representations are similar. In representation theory, representations are usually classified up to such similarity. So far, we listed three representations of $(S_3, \circ)$. Since S_3 is a group with finitely many elements (called a *finite group*), it is not hard to classify all its representations.[9] Every representation of a finite group is *completely reducible*, that is, can be decomposed into simplest parts, which are called *irreducible* representations. It turns out that our list already contains all irreducible representations of $(S_3, \circ)$ (this follows from a purely combinatorial description which can be found in Section 5 of the Snapshot 5/2016 *Symmetry and characters of finite groups* by E. Giannelli and J. Taylor).

3 How degenerations meet representations

So far, we used matrices (and linear transformations) as a tool to study symmetry. We now go further and study symmetries hidden in matrices. An important example of a matrix group is the *general linear group* $\mathrm{GL}_n(\mathbb{R})$, that consists of all *invertible* $n \times n$ matrices with real entries. The operation on this group is the matrix multiplication (we will not use any symbol for this operation and just write AB for the product of matrices A and B). A matrix A is called invertible if there exists another matrix A^{-1}, called the *inverse* of A, such that $AA^{-1} = I$, where I is the *identity matrix* (namely, the matrix with diagonal elements equal to 1 and all other elements equal to 0). For example, the matrix below is invertible:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 2 \end{pmatrix}; \quad A^{-1} = \begin{pmatrix} -2 & 1 \\ 1 & 0 \end{pmatrix}; \quad AA^{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I.$$

Similarly, we can define $\mathrm{GL}_n(\mathbb{F})$ by considering matrices with entries in a set of numbers $\mathbb{F}$, such as $\mathbb{F} = \mathbb{Z}$ (integer numbers) or $\mathbb{F} = \mathbb{C}$ (complex numbers).

It turns out that the representation theory of the group $\mathrm{GL}_n(\mathbb{C})$ is simpler than the one of $\mathrm{GL}_n(\mathbb{R})$ (this sounds counterintuitive, but complex numbers are often used to simplify things). While $\mathrm{GL}_n(\mathbb{C})$ is an infinite group, its representation theory can be managed by almost the same methods as in the case of finite groups. Despite their non-real nature, complex matrices are not far from physical reality. For instance, a model of the hydrogen atom in quantum mechanics relies on representation theory of $\mathrm{GL}_2(\mathbb{C})$ (as explained in [11]).[10] More precisely, measurable physical characteristics of the atom can

[9] Classifying representations of a finite group is always a manageable task, see the Snapshot 5/2016 by E. Giannelli and J. Taylor for more details on why and how it is possible.

[10] Strictly speaking, it relies on representation theory of the groups $\mathrm{SO}_3(\mathbb{R})$ (rotations in 3-dimensional space) and its close relative Spin(3). There is a beautiful relation between $\mathrm{SO}_3(\mathbb{R})$ and $\mathrm{SU}_2(\mathbb{C})$ (special unitary 2×2 matrices with complex entries), which identifies Spin(3) with $\mathrm{SU}_2(\mathbb{C})$ and reduces the representation theory of $\mathrm{SO}_3(\mathbb{R})$ to that of $\mathrm{SU}_2(\mathbb{C})$.

be neatly encoded by the decomposition of a representation into its irreducible components.[11]

We now explain the representation theory of $\mathrm{GL}_n(\mathbb{C})$ for $n = 1$ and $n = 2$. We will consider complex representations, that is, we would like to represent matrices from $\mathrm{GL}_n(\mathbb{C})$ by matrices from $\mathrm{GL}_m(\mathbb{C})$. Note that we get an irreducible representation of dimension $m = n$ for free, since $\mathrm{GL}_n(\mathbb{C})$ is by definition represented by $n \times n$ matrices. This representation is called *standard* or *tautological*. It is interesting that irreducible representations with $m > n$ can be built from the tautological representation by methods of tensor algebra.[12] In particular, representation theory of $\mathrm{GL}_n(\mathbb{C})$ is a good playground to master computational skills with tensor, symmetric, and exterior products.

3.1 Representations of $\mathrm{GL}_1(\mathbb{C})$

The group $\mathrm{GL}_1(\mathbb{C})$ is just the group $\mathbb{C}^*$ of all non-zero complex numbers with the operation of multiplication. Note that $\mathbb{C}^*$ is commutative. Because of commutativity, all irreducible representations of $\mathbb{C}^*$ have dimension one.[13] A complete list of irreducible representations is infinite, but still quite simple: its elements χ_0, $\chi_{\pm 1}$, $\chi_{\pm 2}, \dots$ are labelled by integer numbers and any representation χ_n, for $n \in \mathbb{Z}$, is given by the formula $\chi_n(z) = z^n$. For instance, $\chi_{-1}(z) = z^{-1} = \frac{1}{z}$. The map χ_n is a representation since $\chi_n(zw) = (zw)^n = z^n w^n = \chi_n(z)\chi_n(w)$ for any pair of complex numbers z, $w \in \mathbb{C}^*$.

Any other representation of $\mathbb{C}^*$ is completely reducible (here the difference between $\mathbb{C}$ and $\mathbb{R}$ comes into play). We will explain two important ideas behind this fact, since these ideas also work in a more general setting. In fact, they are also used in representation theory of *reductive groups* such as $\mathrm{GL}_n(\mathbb{C})$ and $\mathrm{SO}_n(\mathbb{C})$ for larger n.

First, since $z \in \mathbb{C}$ is given by $z = x + \mathrm{i}y$ for $x, y \in \mathbb{R}$, it is possible to identify $\mathbb{C}$ with $\mathbb{R}^2$ by the correspondence $z \mapsto (x, y)$. With this correspondence, $\mathbb{S}^1 = \{z \in \mathbb{C} \mid |z| = 1\}$ is identified with the circle of radius one and center at the origin. Trigonometry tells us that any point p on this circle can be written as $(\cos(\theta), \sin(\theta))$, where θ is the angle formed by the x-axis and the line through the origin and p. If one considers any other point $(0, 0) \neq (x, y) \in \mathbb{R}^2$, it belongs to a circle of radius $r > 0$ and center at the origin. Similarly as before, this point can be written as $(r\cos(\theta), r\sin(\theta)) = r(\cos(\theta), \sin(\theta))$. Going back to $\mathbb{C}^*$,

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This relation is also used in computer 3D graphics to encode spatial rotations ($\mathrm{SO}_3(\mathbb{R})$) by means of unit quaternions ($\mathrm{SU}_2(\mathbb{C})$).

[11] For simplicity, we consider representations of finite dimension in this snapshot. However, infinite-dimensional representations occur naturally in applications of representation theory to physics.

[12] The reader may find more details about this construction in [7, Lectures 11, 12, and 15].

[13] This sentence is, in essence, Schur's Lemma, which can be found in [3].

we then have the correspondence $z \mapsto (x, y) \mapsto r(\cos(\theta), \sin(\theta))$. Therefore we could think of $\mathbb{C}^*$ as the product of $\{r \in \mathbb{R} \,|\, r > 0\}$ with $\mathbb{S}^1$. An elegant argument (called *Weyl's unitarian trick* [13]) implies that representations of $\mathbb{C}^*$ can be fully reconstructed from representations of $\mathbb{S}^1$. In short, $\mathbb{C}^*$ and $\mathbb{S}^1$ have the same representation theory.

Second, note that the set $\mathbb{S}^1$ is *compact* (that is, it is a closed and bounded subset of the plane) and so $\mathbb{S}^1$ is a compact group.[14] Representation theory of compact groups mirrors the representation theory of finite groups. In particular, continuous representations of $\mathbb{S}^1$ can be classified by essentially the same method as representations of finite commutative groups, up to some technical modifications in the argument. This means that the representation theory of $\mathbb{S}^1$ is manageable.

3.2 Representations of $\mathrm{GL}_2(\mathbb{C})$

The group $\mathrm{GL}_2(\mathbb{C})$ is not commutative as the following example shows:

$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}; \quad B = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}; \quad AB = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \neq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} = BA. \quad (1)$$

It turns out that for every positive integer m, there exists an irreducible representation R_m of $\mathrm{GL}_2(\mathbb{C})$ of dimension m. The representations R_m are constructed using the map

$$\mathrm{GL}_2(\mathbb{C}) \times \mathbb{C} \to \mathbb{C}, \qquad (A, z) \mapsto \frac{az + b}{cz + d}, \quad (2)$$

identifying the matrix A with the linear fractional transformation $z \mapsto \frac{az+b}{cz+d}$. For example, the matrices A and B in (1) correspond to the transformations $z \mapsto \frac{1}{z}$ and $z \mapsto -z$, respectively. However, the transformation $z \mapsto \frac{1}{z}$ is not well-defined when $z = 0$. To solve this issue, we extend the map in (2) to $\mathrm{GL}_2(\mathbb{C}) \times \mathbb{CP}^1$, where $\mathbb{CP}^1$ [15] is called *Riemann sphere* and can be thought of as $\mathbb{C}$ with the additional point ∞. In this way, $z \mapsto \frac{1}{z}$ interchanges 0 and ∞.

The Riemann sphere $\mathbb{CP}^1$ plays an important role in complex analysis and algebraic geometry and it can also be used to reconstruct representations of $\mathrm{GL}_2(\mathbb{C})$ as follows. Let $(x_0 : x_1)$ be *homogeneous coordinates* on $\mathbb{CP}^1$. Roughly speaking, a point $z \in \mathbb{CP}^1$ has "homogeneous coordinate" $(x_0 : x_1)$ if $z = \frac{x_0}{x_1}$ (unless $x_1 = 0$). In this way, $(x_0 : x_1)$ and $(tx_0 : tx_1)$ represent the same

[14] This is the simplest interesting compact group, in particular, periodic functions such as $\sin x$ can be regarded as functions on $\mathbb{S}^1$.
[15] The notation $\mathbb{CP}^1$ means projective line over $\mathbb{C}$.

point for all $t \neq 0$. When the map (2) is extended to $\mathrm{GL}_2(\mathbb{C}) \times \mathbb{CP}^1$, it can be rewritten in homogeneous coordinates as

$$(x_0 : x_1) \mapsto \begin{pmatrix} ax_0 + bx_1 \\ cx_0 + dx_1 \end{pmatrix}.$$

In other words, if we regard $(x_0 : x_1)$ as coordinates of a vector, then the matrix $A \in \mathrm{GL}_2(\mathbb{C})$ acts on this vector by a linear transformation.

Consider all polynomials $f(x_0, x_1)$ of degree m in the variables x_0, x_1 such that all non-zero terms are of the form $a_i x_0^i x_1^{m-i}$ for $i = 0, \ldots, m$ for some coefficients a_i. We call these polynomials *homogeneous*. The composition of the above map with the polynomial f gives us the representations R_{m+1}, once we make a linear change of variables by the inverse matrix of A (a linear change of variables does not change the degree of a polynomial). We explain this for the case $m = 2$. We consider the quadratic polynomial $f(x_0, x_1) = px_0^2 + qx_0x_1 + rx_1^2$ which, by the inverse of A, is mapped to the polynomial

$$f(ax_0+bx_1, cx_0+dx_1) = p(ax_0+bx_1)^2 + q(ax_0+bx_1)(cx_0+dx_1) + r(cx_0+dx_1)^2 =$$

$$= (a^2p + acq + c^2r)x_0^2 + (2abp + adq + bcq + 2cdr)x_0x_1 + (b^2p + bdq + d^2r)x_1^2.$$

This transformation yields the representation R_3 given by [16]

$$R_3\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{-1}\right) = \begin{pmatrix} a^2 & ac & c^2 \\ 2ab & ad + bc & 2cd \\ b^2 & bd & d^2 \end{pmatrix}.$$

3.3 Degenerations for representations

The general idea behind the construction of the representation R_{m+1} in the previous example is to consider a geometric object X (such as $X = \mathbb{CP}^1$) together with a group G (such as $G = \mathrm{GL}_2(\mathbb{C})$) and a map $G \times X \to X$, and to extract representation theory of G from algebraic and geometric structures on X. We will refer to the map $G \times X \to X$ as the action of G on X. Informally speaking, X contains all the information about representations of G, and we just have to look at X at the right angle to read them.

However, X can be a complicated object to study. In this situation, it is useful to consider a degeneration X_0 of X carrying an action of a related group G_0. This is already the case for $\mathrm{GL}_3(\mathbb{C})$, for which the corresponding geometric

[16] It is a good exercise in linear algebra to show that if $\det A = 1$ then $R_3(A)$ preserves the quadratic form $q^2 - 4pr$ (the discriminant of $px_0^2 + qx_0x_1 + rx_1^2$). By a linear change of coordinates this form can be transformed into the canonical form (sum of squares). This can be used to represent $A \in \mathrm{SU}_2(\mathbb{C})$ by a matrix from $\mathrm{SO}_3(\mathbb{R})$, that is, by a spatial rotation.

object X (called *flag variety*) is considerably more complicated than the Riemann sphere $\mathbb{CP}^1$. We can then degenerate X to a simpler object X_0 (called *toric variety*), whose algebraic and geometric structures are easier to visualize. While the action of $\mathrm{GL}_3(\mathbb{C})$ does not extend to X_0, there is a commutative group $G_0 = (\mathbb{C}^*)^3$ (an *algebraic 3-torus*) that acts on X_0.[17] Looking at X_0 rather than at X we get more insight about the structure of the representations of G. In particular, by degenerating X to X_0 we may regard representations of G as representations of G_0. This is already meaningful in the case of $\mathrm{GL}_2(\mathbb{C})$. In this case, $X = X_0$ (it is hard to degenerate the Riemann sphere since it is already a very simple object) and $G_0 = \mathbb{C}^*$. The action of $t \in \mathbb{C}^*$ on $\mathbb{CP}^1$ takes a point $(x_0 : x_1)$ to $(tx_0 : x_1)$. The resulting representation of $\mathbb{C}^*$ on the homogeneous polynomials of degree m decomposes into $(m+1)$ representations of dimension 1, namely, χ_0, $\chi_1, \ldots, \chi_m$ (see subsection 3.1 for the definition of χ_i). Indeed, any homogeneous polynomial of degree m splits into a sum of simple terms $a_i x_0^i x_1^{m-i}$ for $i = 0, 1, \ldots, m$. The action of $t \in \mathbb{C}^*$ takes $a_i x_0^i x_1^{m-i}$ to $a_i (tx_0)^i x_1^{m-i} = t^i (a_i x_0^i x_1^{m-i})$, that is, multiplies it by t^i. Speaking in terms of linear algebra, we constructed a special basis in the vector space of polynomials by decomposing this space into one-dimensional subspaces invariant under the action of $\mathbb{C}^*$. A similar technique can be applied to representations of $\mathrm{GL}_n(\mathbb{C})$. In the last decades, many interesting results in representation theory and algebraic geometry were obtained using this technique. Some of these results are mentioned in a recent survey by E. Feigin [6].

4 Further reading

Textbook [1] contains excellent chapters on symmetry and representation theory. Paper [5] surveys recent results on degeneration techniques in representation theory, and paper [9] is a good first introduction to toric degenerations. Slides [4] provide a short and entertaining introduction to the same topic.

[17] For arbitrary n, the flag variety X has dimension $d = \frac{n(n-1)}{2}$, and X_0 is a toric variety with an action of an algebraic d-torus $(\mathbb{C}^*)^d$.

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Ilya Dumanski *is a Ph. D. student in pure mathematics at Massachusetts Institute of Technology.*

Valentina Kiritchenko *is an associate professor of pure mathematics at the National Research University Higher School of Economics.*

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Connections to other fields
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