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Mini-Workshop: Quantization of Complex Symplectic Varieties

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Abstract. The mini-workshop featured two main series of lectures: *Functoriality in non-abelian Hodge theory* by Tony Pantev, and *Quantization of the Hitchin system and the analytic Langlands program* by Jörg Teschner. In addition, four senior mathematicians and physicists gave two talks each on their recent mysterious discoveries related to the theme of the workshop. Three junior mathematicians also gave a talk based on their fresh results. All talks by mathematicians and physicists were coordinated to form a common ground of understanding. The smallness of the size of workshop promoted deeper discussions and helped to create friendly and inclusive atmosphere.

Mathematics Subject Classification (2020): 14-XX.

Introduction by the Organizers

The MFO welcomed the maximum number (16) of in-person participants and three online participants with a wide range of generations and diverse background to the mini-workshop *Quantization of Complex Symplectic Varieties*, organized by John Alexander Cruz Morales (Bogotá), Olivia M. Dumitrescu (Chapel Hill), Motohico Mulase (Bonn/Davis), and Katrin Wendland (Dublin). Participants included one junior faculty member, three postdoctoral scholars (one each from Germany, U.K., and U.S.), four graduate students (two each from Germany and U.S.), and a Ukrainian undergraduate student studying in Germany at the time of the workshop.

The smallness of the mini-workshop allowed the participants to create a welcoming, inclusive, and friendly atmosphere. Everybody became a member of a tightly knitted happy group of highly charged researchers with ambitions. Senior participants shared their life's stories with junior participants, and junior participants felt comfortable to ask any mathematical questions throughout the workshop, including during the excursion and after dinner hours.

The initial motivation of the organizers was to learn something new to solve problems they were struggling with, such as the problem of identifying the relationship between *real quantizations* and *complex symplectic geometry* of a mirror symmetric pair $(X, \widehat{X})$ of hyperkähler manifolds. A particularly interesting case is when X is the *Dolbeault* moduli space $\mathcal{M}_{\mathrm{Dol}}^G$ consisting of G-Higgs bundles defined on a smooth projective curve C of genus $g(C) > 1$ with appropriate stability conditions. Here, G is a complex reductive group. This moduli space was first introduced by Hitchin. A hyperkähler rotation changes $\mathcal{M}_{\mathrm{Dol}}^G$ to the *de Rham* moduli space $\mathcal{M}_{\mathrm{dR}}^G$ consisting of holomorphic G-connections on C. Two hyperkähler manifolds $\mathcal{M}_{\mathrm{dR}}^G$ and $\mathcal{M}_{\mathrm{dR}}^{^LG}$, the latter defined with the Langlands dual group ^LG, are expected to be *mirror symmetric* in the sense of Strominger-Yau-Zaslow.

The organizers chose speakers according to their original motivation, and then let the speakers determine the dynamics of the research discussions. As participants' excitement being heated and discussions deepened, Tony Pantev decided to give six lectures, and Jörg Teschner four, exceeding the numbers the organizers had originally asked. The flexibility of the mini-workshop allowed these spontaneous evolution to happen. As a result, the participants became more familiar with the newly presented concepts to them, and were able to understand the theories far better.

Teschner's proposal [7] has origins in conformal field theory and a method known as *separation of variables* in physics. It has led mathematicians [3] and physicists [4] to come up with a theory to understand the vision of Langlands [5], leading to an *analytic form* of geometric Langlands correspondence. This correspondence predicts a relation between ^LG-holomorphic connections on a curve C and certain $\mathcal{D}$-modules on the moduli stack of G-bundles on C, denoted by $Bun(G)$. The series of four lectures by Teschner highlighted this correspondence in its analytic form for the case of $G = SL(2)$ with concrete constructions in terms of *opers* and separation of variables.

For the case of Higgs bundles on a curve, the canonical construction of a holomorphic connection from a given Higgs bundle due to Carlos Simpson is known as *non-abelian Hodge correspondence* (NAH). As a map between two moduli spaces, it is a homeomorphism, but not biholomorphic. One of the ideas to establish geometric Langlands correspondence is to utilize a high dimensional version of non-abelian Hodge correspondence for the construction of $\mathcal{D}$-modules on $Bun(G)$. To this end, one needs a general theory of NAH [6]. Pantev's series of powerful lectures, based on his expertise [2] and a forthcoming paper with Donagi and Simpson, were aimed at explaining the *functoriality* of general non-abelian Hodge correspondence, describing its compatibility with Grothendieck's *six operations.*

Because of the leisurely scheduled pace of these series of lectures, organizers were able to understand the right point of view to attack their original problems.

A mathematician may wish to view the geometric Langlands correspondence as a direct construction. It was a revelation to see the physicist's way of understanding the same effect through conformal field theory from Teschner's lectures. Similarly, the functoriality of NAH is a hard subject to grasp for a mathematician with different background, because the correspondence never leads to a holomorphic map of moduli spaces. The categorical equivalence and its functoriality as explained by Pantev brought deeper understanding to the participants and organizers why the new way of thinking is necessary when the natural map does not exhibit a would-be-nice property (such as holomorphicity).

Participants Murad Alim, David Baraglia (online), Ana Peón-Nieto (online) and Piotr Sułkowski gave two lectures each, presenting their breakthrough results. These results are also filled with mysteries, indicating a new chapter of research frontiers to begin. The two-hour allocation helped both speakers and the audience, to explain and to grasp, the concepts new to most everybody in the audience.

Talks by junior speakers, Jennifer Brown, John Alexander Cruz Morales, and Emre Sertöz, were equally energetic and filled with surprises. For example, the result of Sertöz is a discovery of new phenomena in the very spot where classical experts had once declared nothing interesting could happen.

After the mini-workshop, several new collaborations are spontaneously taking place. We hope to be able to report on these new developments near future.

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Mini-Workshop: Quantization of Complex Symplectic Varieties

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Abstracts

Non-perturbative quantum geometry, resurgence and BPS structures

Murad Alim
(joint work with L. Hollands, A. Saha, J. Teschner, I. Tulli)

1. Introduction

BPS invariants of certain physical theories correspond to Donaldson-Thomas (DT) invariants of an associated Calabi-Yau geometry. The notion of BPS structures refer to the data of the DT invariants together with their wall-crossing structure. On the same Calabi-Yau geometry another set of invariants are the Gromov-Witten (GW) invariants. These are organized in the GW potential, which is an asymptotic series in a formal parameter λ and can be obtained from topological string theory. An example is the Gromov-Witten potential for the resolved conifold geometry which is a CY threefold given by the total space of the rank two bundle:

$$\mathbf{X} := \mathcal{O}(-1) \oplus \mathcal{O}(-1) \to \mathbb{P}^1\,, \tag{1}$$

over the projective line and corresponds to the resolution of the conifold singularity in $\mathbb{C}^4$. The GW potential for this geometry was determined, in physics [GV98, GV99] as well as in mathematics [FP00], with the outcome

$$F^{\mathrm{top}}(\lambda,t) = \sum_{g=0}^{\infty} \lambda^{2g-2} F_g(t) = \frac{1}{\lambda^2}\mathrm{Li}_3(Q) + \sum_{g=1}^{\infty} \lambda^{2g-2}\, \frac{(-1)^{g-1}B_{2g}}{2g\,(2g-2)!}\,\mathrm{Li}_{3-2g}(Q)\,, \tag{2}$$

for the non-constant maps, where $Q=\exp(2\pi i t)$.

2. Difference equations

A difference equation was derived from the asymptotic expansion of the Gromov-Witten potential of the resolved conifold in [Ali22] following methods of [IKT19]:

$$F^{\mathrm{top}}\left(\lambda, t+\check{\lambda}\right) + F^{\mathrm{top}}\left(\lambda, t-\check{\lambda}\right) - 2F^{\mathrm{top}}\left(\lambda,t\right) = -\mathrm{Li}_1(Q)\,, \quad \check{\lambda} = \frac{\lambda}{2\pi}\,. \tag{3}$$

Furthermore a solution in terms of the triple sine function of this difference equation was considered in [AS21]:

$$F^{\mathrm{top}}_{\mathrm{np}}(\lambda,t) := \left(\frac{\pi i}{6}\, B_{3,3}\left(t+\check{\lambda}\,|\,\check{\lambda},\check{\lambda},1\right)\right) + \log\left(\sin_3\left(t+\check{\lambda}\,|\,\check{\lambda},\check{\lambda},1\right)\right). \tag{4}$$

The non-perturbative content of this solution was analyzed in [Ali21] and in [ASTT21] it was identified as the Borel summation of the asymptotic series along a distinguished ray on the real axis in the Borel plane. A further difference equation for $F^{\mathrm{top}}_{\mathrm{np}}(\lambda,t)$ was found in [AHT22]:

$$F^{\mathrm{top}}_{\mathrm{np}}(\lambda,,t+1) - F^{\mathrm{top}}_{\mathrm{np}}(\lambda,t) = \frac{1}{2\pi i}\frac{\partial}{\partial\check{\lambda}}\left(\check{\lambda}\,\mathrm{Li}_2(e^{2\pi i t/\check{\lambda}})\right)\,. \tag{5}$$

3. Resurgence and DT invariants

In [ASTT21] the techniques of [GK20] were used to study the Borel resummation of the Gromov-Witten potential $F^{\text{top}}(\lambda, t)$ for the resolved conifold. Earlier results on the Borel resummation for the resolved conifold with different techniques and scope were obtained in [PS10, HO15]. The Borel transform has infinitely many singularities organized along rays coinciding with the rays $\pm\mathbb{R}_{<0}Z_\gamma$, where Z_γ denotes the central charge of a BPS state of charge γ. Different Borel resummations were defined along rays which avoid the singularities, and it was was found that they experience Stokes jumps across the rays $\pm\mathbb{R}_{<0}Z_\gamma$, with the BPS charge $\gamma \in \Gamma$ contributing to the jump by [ASTT21]:

$$\Delta_\gamma F^{\text{top}}_{\text{Borel}}(\lambda, t) = \frac{\Omega(\gamma)}{2\pi \mathrm{i}} \partial_{\check{\lambda}} \left(\check{\lambda} \mathrm{Li}_2(e^{Z_\gamma/\check{\lambda}}) \right) , \quad \check{\lambda} = \frac{\lambda}{2\pi} , \tag{6}$$

where $\Omega(\gamma)$ correspond to the Donaldson-Thomas (DT) invariant. The identification of the DT invariants was established by providing the link to a Riemann-Hilbert problem put forward by Bridgeland in [Bri19] and applied to the resolved conifold in [Bri20].

The Borel analysis furthermore allowed to connect to previous proposals for definitions of non-perturbative topological string theory and elucidate their overlaps of validity. The Borel summation along a distinguished ray for instance gave an expression previously proposed in [HMnMO14, HO15], while a limiting expression obtained from the latter through infinitely many jumps gave the Gopakumar-Vafa expression for the resummation of the free energies. In the work of Bridgeland it was suggested that a Tau-function, obtained as a solution of a Riemann-Hilbert problem defined from the wall-crossing structure of Donaldson-Thomas invariants, provides a non-perturbative completion of the Gromov-Witten potential.

One may note that the right-hand side of the difference equation (5) equals Stokes jump of the Borel resummation of $F^{\text{top}}(\lambda, t)$ obtained in [ASTT21]. Indeed, the difference equation can be given the interpretation as a relation between the Borel resummations in the different Stokes sectors, this was discussed in [AHT22].

4. Quantum curves and exact WKB

In [HK18, HRS21], it was realized that Borel resummation plays a central role in the geometric formulation of the effective twisted superpotential $\mathcal{W}^{\text{eff}}$ of a four-dimensional $\mathcal{N} = 2$ theory of class S in the $\frac{1}{2}\Omega$-background. The twisted superpotential in this setting is given by the Nekrasov-Shatashvili (NS) limit of the refined version of topological string theory. In [AHT22] the non-perturbative quantum geometry of the open and closed topological string on the resolved conifold and its mirror was studied in this refined setting. New finite difference equations were found which govern the open and closed moduli dependence of the refined topological string theory as well as its NS limit. Distinguished analytic solution for the refined difference equation were found which reproduce the expected non-perturbative content of the refined topological string. These solutions were compared to the Borel analysis of the free energy in the NS limit. In the open setting, the finite difference equation corresponds to a canonical quantization of the mirror curve. This difference equation was analyzed using Borel analysis and exact WKB techniques and the 5d BPS states in the corresponding exponential spectral networks were identified. The resurgence analysis in the open and closed setting was furthermore related. This gave a five-dimensional extension of the Nekrasov-Rosly-Shatashvili [NRS11] proposal, in which the NS free energy is computed as a generating function of q-difference opers in terms of a special set of spectral coordinates.

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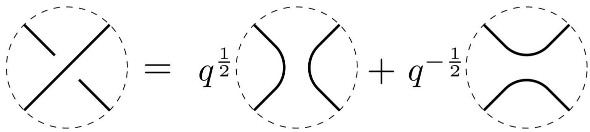
Skein Theory and Quantizing Character Varieties

Jennifer Brown
(joint work with David Jordan)

A skein algebra of a surface Σ is generated by isotopy classes of links in $\Sigma \times [0,1]$, subject to locally defined *skein relations*. The Kauffman bracket skein relation (shown below) is the most famous of such relations and produces the much-studied Kauffman bracket skein algebra [1].



The terms in these relations are typically described as small parts of knots, but we can think of them as morphisms in a category of tangles, **Tan**. Composition in **Tan** is given by gluing endpoints and tangles are thought of as maps between their endpoints. The above relation is between elements of the endomorpism algebra of two points in the category of tangles. A knot or link is an endomorphism of the empty set.

A framed oriented tangle is known as a ribbon graph, and the associated category **Rib** is the motivating example of a *ribbon category.* Finite dimensional representations of quantum groups $U_q\mathfrak{g}$ give another class of examples. We will denote such categories $\mathbf{Rep}_q G$, since their braiding morphisms

$$c_{V,W} : V \otimes W \to W \otimes V \tag{1}$$

are a q-deformation of the symmetric monoidal structure in the corresponding representation categories $\mathbf{Rep}\, G$.

There is a uniquely defined braided functor from the category of ribbon graphs colored by a ribbon category $\mathcal{A}$, denoted $\mathbf{Rib}_{\mathcal{A}}$, to $\mathcal{A}$. In the case $\mathcal{A} = \mathbf{Rep}_q G$, this functor is used to define quantum topological invariants [2]. The kernel of this map induces skein relations on the morphism spaces of $\mathbf{Rib}_{\mathcal{A}}$. Taking the quotient by this kernel leads to the definition of a skein category $\mathbf{SkCat}_{\mathcal{A}}(\Sigma)$. Considering again endomorphisms of the empty set, we get general skein algebras $\mathrm{SkAlg}_{\mathcal{A}}(\Sigma)$ of surfaces. This definition recovers the Kauffman bracket when $\mathcal{A} = \mathbf{Rep}_q \mathrm{SL}_2\mathbb{C}$.

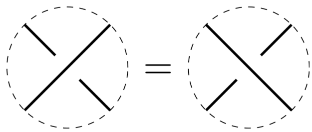
Skein theory gives a quantization (q-deformation) of character varieties, which are likewise built from the data of a reductive group G and a topological space X. We are interested in quantizing their coordinate rings, which are given as a ring of invariants

$$\mathcal{O}(\chi_G(X)) = \mathcal{O}\left(\mathrm{Hom}(\pi_1(X), G)\right)^G. \tag{2}$$

When $G = \mathrm{SL}_n\mathbb{C}$ and $X = \Sigma$ is a surface, these rings are generated by trace functions over elements of the fundamental group $\mathrm{tr}_\gamma : \rho \mapsto \mathrm{tr}(\rho(\gamma))$ [3, 4]. This inspires a homomorphism

$$\mathcal{O}\left(\chi_G(\Sigma)\right) \to \mathrm{SkAlg}_{\mathbf{Rep}\, G}(\Sigma) \tag{3}$$

sending tr_γ to the isotopy class of $\gamma \hookrightarrow \Sigma$ colored by the fundamental representation V of G. One hint that the above map is a homorphism is that both the skein relation induced by $\mathbf{Rep}\, G$ (shown below) and multiplication in the coordinate ring are commutative.



Changing the category of colors from $\mathbf{Rep}\, G$ to its q-deformation $\mathbf{Rep}_q G$ produces a non-commutative algebra $\mathrm{SkAlg}_{\mathbf{Rep}_q G}(\Sigma)$ as a quantization of the $\mathbf{Rep}\, G$ skein algebra. On a categorical level, this provides a quantization of quasicoherent sheaves on the character variety.

The author and her collaborators are putting this approach to quantization to use in defining a quantum version of the A-polynomial [5]. By recreating the original construction in an appropriate skein theoretic context, it should be possible to give a q-difference operator $\hat{A}_q$ whose $q \to 1$ limit recovers the original A-polynomial. One advantage of this approach is that it sets the quantum A-polynomial in the same context as the colored Jones polynomial, hopefully giving some insight into their close relation as suggested by the AJ conjecture [6].

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A physical approach to Dubrovin conjecture

John Alexander Cruz Morales

(joint work with Mauricio Romo, Jin Chen)

Let X be a Fano manifold. Dubrovin's conjecture (see [3] for an initial formulation of the conjecture and [2] for a more precise statement) relates the Frobenius manifold coming from the big quantum cohomology of X, $QH^*_{\mathrm{big}}(X)$, with the bounded derived category of coherent sheaves of X, $D^b(X)$. More precisely, Dubrovin's conjecture establishes that $QH^*_{\mathrm{big}}(X)$ is (generically) semisimple if and only if $D^b(X)$ has a full exceptional collection $\mathcal{E}$. In addition, it is conjectured that the Stokes matrix of the quantum differential equation coincides with the Gram matrix of $\mathcal{E}$[1]. This conjecture has been completely verified for projective spaces and Grassmannians and partial results has been obtained in other cases, e.g., the Pezzo surfaces andsome Fano 3-folds.

Despite the work in the conjecture many things around it deserve more explanation. In that sense, the aim of our ongoing work is to give an approach to the conjecture from a physical point of view using Gauged Linear Sigma Models (GLSM) and the hemisphere partition functions introduced by Hori and Romo in [4]. We expect that this might shed light in order to understand deeply the content of the conjecture and produce a more conceptual proof than testing case by case as it has been done so far. Firstly, we will focus on the case of projective spaces.

A GLSM is a tuple $(G, \rho : G \to GL(V), W, t_I, R)$ where V is a finite dimensional vector space, G is a compact Lie group, ρ is a faithful unitary representation, $W : V \to \mathbb{C}$ is holomorphic, G-invariant polynomial, called superpotential, t_I is a set of complex parameters such that $\exp(t)$ is a group homomorphism from $\pi_1(G)$ to $\mathbb{C}^\times$ that is invariant under the adjoint action of G and $R : U(1) \to GL(V)$ is a symmetry such that $W(R(\lambda).\phi) = \lambda^2 W(\phi)$ where ϕ are the coordinates in V.

For the purposes of our work we consider $W = 0$. It can be defined a category of B-branes on a GLSM, i.e., the (G-equivariant graded) matrix factorizations of $W = 0$, denoted by $\mathcal{MF}_G(W = 0)$. Physics provides maps, $\pi_\zeta : \mathcal{MF}_G(W = 0) \to D^b(Y_\zeta)$, where $\zeta = \mathrm{Re}(t)$ and $Y_\zeta := \mu_\rho^{-1}(\zeta)/G$. Here μ_ρ is the moment map associated to the representation ρ. Y_ζ is called the Higss branch.

Let us consider the $\mathbb{CP}^n$ case. In this situation, $G = U(1)$, $\rho(g) = \bigoplus_{i=1}^{n+1} \mathbb{C}(1)$, $W = 0$, $R = e$. Since in this case we do not have a Calabi-Yau condition in the GLSM , we have that $Y_\zeta = \mathbb{CP}^n$ for $\zeta > 0$ and $Y_\zeta = \emptyset$ for $\zeta < 0$. So, the map defined above is $\pi_\zeta : \mathcal{MF}_G(W = 0) \to D^b(\mathbb{CP}^n)$ for $\zeta > 0$. What about $\zeta < 0$?. We need to study the Coulomb branch.

Inside $\mathcal{MF}_G(W = 0)$ we have a subcategory $\mathcal{W}_l$ defined by objects such that all the weights w belong to an internal I_l of length n. We have maps $\pi_\zeta : \mathcal{MF}_G(W = 0) \to \mathcal{C}$ for $\zeta < 0$ and maps $D^b(\mathbb{CP}^n) \to \mathcal{W}_l$ and $\mathcal{W}_l \to \mathcal{C}$. This construction predicts that the category $\mathcal{C}$ has a semi-orthogonal decomposition, which in our case is in fact a full exceptional collection and also establishes that there is an equivalence between $D^b(\mathbb{CP}^n)$ and $\mathcal{C}$, so, we have that $D^b(\mathbb{CP}^n)$ admits a full exceptional collection from the fact that we have an empty Higgs branch for the case $\zeta < 0$. We have to remark that there is no a mathematical description for $\mathcal{C}$ and this is constructed by purely physical considerations.

We also have that physics predicts a map $\mathcal{MF}_G(W = 0) \to \mathbb{C}$ which sends $\mathcal{E} \mapsto \mathcal{Z}(\mathcal{E})$. This $\mathcal{Z}(\mathcal{E})$ is the hemisphere partition function. An important fact is that $\mathcal{Z}(\mathcal{E})$ satisfies a differential equation. In the $\mathbb{CP}^n$ case the differential equations is $((z\frac{d}{dz})^{n+1} - (n+1)z^{n+1})\mathcal{Z}(\mathcal{E}) = 0$. This is the quantum differential

[1]There is a third part of the conjecture that relates the connection matrix of the quantum differential equation of X with some data coming from $\mathcal{E}$, but we are not interested in this part of the conjecture for the moment

equation for $\mathbb{CP}^n$. More generally, we will conjecture that $\mathcal{Z}(\mathcal{E})$ satisfies the quantum differential equation for a Fano manifold X that is the target for the GLSM. By using, some results in [4] we can compute the Stokes matrices for the quantum differential equation of $\mathbb{CP}^n$ by using its hemisphere partition function and the exceptional collection obtained before. In this way, we get the relation between the Stokes date of the quantum differential equation and the geometric-algebraic data coming from the exceptional collection.

We can enrich the model adding twisted masses, i.e., incorporating equivariant parameters given by the action of $(\mathbb{C}^\times)^{n+1}/\mathbb{C}^\times$ on $\mathbb{CP}^n$. In this case, we can define an equivariant hemisphere partition function $\mathcal{Z}^{eq}(\mathcal{E})$ which also satisfies a differential equation. In the $\mathbb{CP}^n$ situation the differential equation is $(\prod_{i=1}^{n+1}(z\frac{d}{dz} - m_i) - z)\mathcal{Z}^{eq}(\mathcal{E}) = 0$, where m_i are the equivariant parameters and they should satisfy $\sum_{i=1}^{n+1} m_i = 0$. Now,the problem is to study the 'equivariant' Stokes matrices. The answer is that the entries of the Stokes matrices must be replaced by various $SU(n+1)$-characters. More precisely $\chi_q^{n+1}(m) = \sum_{1\leq i_1 \leq \ldots \leq i_q \leq n+1} e^{2\pi i(m_{i_1}+\ldots+m_{i_q}})$. It is intersting to note that some specifications of the equivariant parameters produce the Stokes matrices for the $\mathbb{Z}_n$-symmetric models studied by Cecotti and Vafa in [1]. The precise relations between the $\mathbb{CP}^n$ model with twisted masses and the $\mathbb{Z}_n$-symmetric models have to be clarified and also whether it is possible or not to formulate a Dubrovin's type conjecture for more general situations suggested by working in the equivariant setting.

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Functoriality in non-abelian Hodge theory

Tony Pantev

(joint work with Ron Donagi and Carlos Simpson)

One approach to constructing categorified Hecke actions, mirror functors, or actions of string dualities on boundary conditions, is by computing integral transforms and Fourier-Mukai functors for variations of twistor structures. The implementation of this approach rests on certain strong functoriality properties of the non-abelian Hodge correspondence.

The key ingredient here is a rather surprising compatibility of non-abelian Hodge theory with Grothendieck's six operations. In an ongoing joint work with R. Donagi and C. Simpson initiated in [DPS16] we explored this compatibility in concrete geometric terms. In my series of lectures I explain our main result - an explicit formula which, in the tamely ramified case, computes L^2 cohomology and L^2 pushforwards of harmonic bundles and twistor D-modules in a purely algebraic manner. The formula generalizes previous work of Simpson in the unramified case and has a much wider scope of applicability.

The minicourse presents the natural operations one can perform on parabolic flat bundles or parabolic Higgs bundles: pullbacks [Moc06, IS07, IS08], tensor products [Moc06, IS07, IS08], and pushforwards [DPS16]. I also explain the relevant notions of stability or semistability and the way these natural operations are intertwined by the non-abelian Hodge theorems of Corlette-Simpson and Mochizuki.

Concretely, let $f : Y \to X$ be a morphism between smooth projective varieties. Given a semistable tame parabolic Higgs bundle $(F_\bullet, \varphi)$ we want to understand the derived pushforward of $(F_\bullet, \varphi)$ via f in the category of Higgs sheaves. By definition this pushforward is the complex of tame Higgs bundles which is the specialization at zero of the f-pushforward (in the sense of [Sab05, Moc07a]) of the twistor D-module corresponding to $(F_\bullet, \varphi)$. In [DPS16] we gave the following direct algebraic description of such pushforwards.

Assume Y and X are equipped with simple normal crossings divisors $\mathsf{Par}_Y \subset Y$ and $\mathsf{Par}_X \subset X$. Decompose Par_Y as a sum $\mathsf{Par}_Y = \mathsf{Par}_Y^{\mathsf{ver}} + \mathsf{Par}_Y^{\mathsf{hor}}$ of a vertical and horizontal part. That is $\mathsf{Par}_Y^{\mathsf{ver}}$ is the sum of all components of Par_Y which map to proper subvarieties in X, and $\mathsf{Par}_Y^{\mathsf{hor}}$ is the sum of all components that dominate X. Additionally we will assume that $\mathsf{Par}_Y^{\mathsf{ver}} = f^* \mathsf{Par}_X$, and that $\mathsf{Par}_Y^{\mathsf{hor}} = \sum_{k \in \mathsf{K}} \mathfrak{D}_k$ is a sum of disconnected smooth components $\mathfrak{D}_k$ each of which is also smooth over X.

Given a parabolic level t on X we will write $\mathsf{up}(t)$ for the parabolic level on Y which assigns 0 to each component of $\mathsf{Par}_Y^{\mathsf{hor}}$, while to each component of $\mathsf{Par}_Y^{\mathsf{ver}}$ it assigns the value of the level t on the image of that vertical component under f. Now for each component $\mathfrak{D}_k$ of the horizontal divisor we can consider the associated graded $\mathrm{gr}^{\mathfrak{D}_k} F_{\mathsf{up}(t)}$ of $F_{\mathsf{up}(t)}$ with respect to the parabolic filtration along $\mathfrak{D}_k$. By definition this a vector bundle on $\mathfrak{D}_k$ given by $\mathrm{gr}^{\mathfrak{D}_k} F_{\mathsf{up}(t)} = F_{\mathsf{up}(t)}/F_{\mathsf{up}(t)-\varepsilon\boldsymbol{\delta}_k}$, where $\boldsymbol{\delta}_k$ is the characteristic function of k (viewed as a parabolic level on Y), and $\varepsilon > 0$ is a small real number. Note that by construction $\mathrm{gr}^{\mathfrak{D}_k} F_{\mathsf{up}(t)}$ is a quotient of the vector bundle $F_{\mathsf{up}(t)|\mathfrak{D}_k}$ and that the residue $\mathsf{res}_{\mathfrak{D}_k} \varphi : F_{\mathsf{up}(t)|\mathfrak{D}_k} \to F_{\mathsf{up}(t)|\mathfrak{D}_k}$ descends to an endomorphism $\mathrm{gr} - \mathsf{res}_{\mathfrak{D}_k} \varphi \in \mathrm{End}\left(\mathrm{gr}^{\mathfrak{D}_k} F_{\mathsf{up}(t)}\right)$. The nilpotent part of this endomorphism induces a monodromy weight filtration $W_\bullet\left(\mathrm{gr}^{\mathfrak{D}_k} F_{\mathsf{up}(t)}\right)$ on the vector bundle $\mathrm{gr}^{\mathfrak{D}_k} F_{\mathsf{up}(t)}$. The vector bundle $\mathrm{gr}^{\mathfrak{D}_k} F_{\mathsf{up}(t)}$ on $\mathfrak{D}_k$ can be viewed as a torsion sheaf on Y supported on $\mathfrak{D}_k$ and so we get a torsion sheaf

$$\mathrm{gr}^{\mathsf{Par}_Y^{\mathsf{hor}}} F_{\mathsf{up}(t)} := \bigoplus_{k \in K} \mathrm{gr}^{\mathfrak{D}_k} F_{\mathsf{up}(t)}$$

Let $W_\ell(\mathsf{hor}, F_{\mathsf{up}(t)})$ be the pullback of $W_\ell \, \mathrm{gr}^{\mathsf{Par}_Y^{\mathsf{hor}}} F_{\mathsf{up}(t)} \subset \mathrm{gr}^{\mathsf{Par}_Y^{\mathsf{hor}}} F_{\mathsf{up}(t)}$. Then $W_\bullet(\mathsf{hor}, F_{\mathsf{up}(t)})$ is a filtration of $F_{\mathsf{up}(t)}$ by locally free subsheaves which are equal to $F_{\mathsf{up}(t)}$ away from $\mathsf{Par}_Y^{\mathsf{hor}}$.

Tensoring the global and relative residue maps with $W_0(\mathsf{hor}, F_{\mathsf{up}(t)})$ gives maps

$$
\begin{aligned}
&(1) \qquad W_0(\mathsf{hor}, F_{\mathsf{up}(t)}) \otimes \Omega^i_Y(\log \mathsf{Par}_Y) \longrightarrow W_0(\mathsf{hor}, F_{\mathsf{up}(t)})_{|\,\mathsf{Par}^{\mathsf{hor}}_Y} \otimes \Omega^{i-1}_{\mathsf{Par}^{\mathsf{hor}}_Y},\\
&W_0(\mathsf{hor}, F_{\mathsf{up}(t)}) \otimes \Omega^i_{Y/X}(\log \mathsf{Par}_Y) \longrightarrow W_0(\mathsf{hor}, F_{\mathsf{up}(t)})_{|\,\mathsf{Par}^{\mathsf{hor}}_Y} \otimes \Omega^{i-1}_{\mathsf{Par}^{\mathsf{hor}}_Y/X}.
\end{aligned}
$$

We define locally free sheaves

$$
\begin{aligned}
W_{-2,0}\left(\mathsf{hor}, F_{\mathsf{up}(t)} \otimes \Omega^i_Y(\log \mathsf{Par}_Y)\right) &\subset W_0(\mathsf{hor}, F_{\mathsf{up}(t)}) \otimes \Omega^i_Y(\log \mathsf{Par}_Y)\\
W_{-2,0}\left(\mathsf{hor}, F_{\mathsf{up}(t)} \otimes \Omega^i_{Y/X}(\log \mathsf{Par}_Y)\right) &\subset W_0(\mathsf{hor}, F_{\mathsf{up}(t)}) \otimes \Omega^i_{Y/X}(\log \mathsf{Par}_Y)
\end{aligned}
$$

as the preimages of

$$
\begin{aligned}
W_{-2}(\mathsf{hor}, F_{\mathsf{up}(t)})_{|\,\mathsf{Par}_Y} \otimes \Omega^{i-1}_{\mathsf{Par}_Y}(\log \mathsf{Par}_Y) &\subset W_0(\mathsf{hor}, F_{\mathsf{up}(t)})_{|\,\mathsf{Par}_Y} \otimes \Omega^{i-1}_{\mathsf{Par}_Y}(\log \mathsf{Par}_Y)\\
W_{-2}(\mathsf{hor}, F_{\mathsf{up}(t)})_{|\,\mathsf{Par}_Y} \otimes \Omega^{i-1}_{\mathsf{Par}_Y/X}(\log \mathsf{Par}_Y) &\subset W_0(\mathsf{hor}, F_{\mathsf{up}(t)})_{|\,\mathsf{Par}_Y} \otimes \Omega^{i-1}_{\mathsf{Par}_Y/X}(\log \mathsf{Par}_Y)
\end{aligned}
$$

under the maps (1). These subsheaves are preserved by φ and so we get weight modified global and relative Dolbeault complexes for $(F_{\mathsf{up}(t)}, \varphi)$, e.g.

$$
(2) \qquad \mathrm{DOL}^{\mathrm{par}}(Y, F_{\mathsf{up}(t)}) := \begin{array}{c} \left[\begin{array}{c}
W_0(\mathsf{hor}, F_{\mathsf{up}(t)})\\
\downarrow \wedge\varphi\\
W_{-2,0}\left(\mathsf{hor}, F_{\mathsf{up}(t)} \otimes \Omega^1_Y(\log \mathsf{Par}_Y)\right)\\
\downarrow \wedge\varphi\\
W_{-2,0}\left(\mathsf{hor}, F_{\mathsf{up}(t)} \otimes \Omega^2_Y(\log \mathsf{Par}_Y)\right)\\
\downarrow \wedge\varphi\\
\vdots\\
\downarrow \wedge\varphi\\
W_{-2,0}\left(\mathsf{hor}, F_{\mathsf{up}(t)} \otimes \Omega^{\dim_Y}_Y(\log \mathsf{Par}_Y)\right)
\end{array} \right] \end{array}
\begin{array}{c} 0 \\ \\ 1 \\ \\ 2 \\ \\ \vdots \\ \\ \dim_Y \end{array}
$$

and similarly defined relative complex $\mathrm{DOL}^{\mathrm{par}}(f, F_{\mathsf{up}(t)})$ We also define inductively subcomplexes $I^k(F_{\mathsf{up}(t)})$ in the global Dolbeault complex by setting:

$$
\begin{aligned}
I^0(F_{\mathsf{up}(t)}) &:= \mathrm{DOL}^{\mathrm{par}}(Y, F_{\mathsf{up}(t)}),\\
I^{k+1}(F_{\mathsf{up}(t)}) &:= \mathrm{im}\left[I^k(F_{\mathsf{up}(t)}) \otimes f^*\Omega^1_X(\log \mathsf{Par}_X) \to \mathrm{DOL}^{\mathrm{par}}(Y, F_{\mathsf{up}(t)})\right].
\end{aligned}
$$

With this notation we get a short exact sequence of complexes

$$
\begin{array}{c}
0 \\
\downarrow \\
\mathrm{DOL}^{\mathrm{par}}(f, F_{\mathsf{up}(t)})[-1] \otimes f^*\Omega^1_X(\log \mathsf{Par}_X) \\
\downarrow \\
\mathrm{DOL}^{\mathrm{par}}(Y, F_{\mathsf{up}(t)})/I^2(F_{\mathsf{up}(t)}) \\
\downarrow \\
\mathrm{DOL}^{\mathrm{par}}(f, F_{\mathsf{up}(t)}) \\
\downarrow \\
0
\end{array} \tag{3}
$$

which we can view as a morphism $\mathfrak{d}(\varphi) : \mathrm{DOL}^{\mathrm{par}}(f, F_{\mathsf{up}(t)}) \to \mathrm{DOL}^{\mathrm{par}}(f, F_{\mathsf{up}(t)}) \otimes f^*\Omega^1_X(\log \mathsf{Par}_X)$ in the derived category $D^b_{\mathrm{coh}}(Y, \mathcal{O}_Y)$.

For every parabolic level t on X consider the sheaf theoretic pusforward of the pair $(\mathrm{DOL}^{\mathrm{par}}(f, F_{\mathsf{up}(t)}), \mathfrak{d}(\varphi))$. This gives a parabolic Higgs complex

$$f_*\mathfrak{d}(\varphi) : f_*\mathrm{DOL}^{\mathrm{par}}(f, F_{\mathsf{up}(\bullet)}) \longrightarrow f_*\mathrm{DOL}^{\mathrm{par}}(f, F_{\mathsf{up}(\bullet)}) \otimes \Omega^1_X(\log \mathsf{Par}_X) \tag{4}$$

on X which conjecturally coincides with Higgs pushforward $f_*(F_\bullet, \varphi)$ defined above via pushing forward the twistor $\mathcal{D}$-module corresponding to $(F_\bullet, \varphi)$. In [DPS16] we proved this conjecture in the case when the residue of ϕ is nilpotent.

I also discuss some applications to geometric representation theory and mirror symmetry.

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On wobbly and very stable bundles

Ana Peón-Nieto

1. Very stable and wobbly bundles in the nilpotent cone

Since their definition by Laumon [La], very stable bundles and their counterpart, wobbly bundles, have gained a more and more prominent role in the theory of vector bundles and its applications. To define them, let X be a Riemann surface of genus at least 2. Let K be its canonical bundle.

Definition 1.1. A vector bundle E is very stable if it admits no non zero nilpotent Higgs field $\varphi \in H^0(\mathrm{End}(E)\otimes K)$. A vector bundle which is not very stable is called wobbly [DP1].

Very stable bundles are stable [La]. Let $\mathrm{N}(n,d)$ denote the moduli space of vector bundles of rank n and degree d, and let $\mathrm{N}(n,d)^v \subset \mathrm{N}(n,d)^s$ be the very stable and stable loci respectively. Similarly, let $\mathrm{W}^s = \mathrm{N}(n,d)^s \setminus \mathrm{N}(n,d)^v \subset \mathrm{W} = \mathrm{N}(n,d) \setminus \mathrm{N}(n,d)^v$ be the (stable) wobbly locus in the moduli space.

A basic yet crucial aaplication of very stable bundles is the computation of multiplicities of the irreducible compoments of the nilpotent cone. Let $\mathrm{M}(n,d)$ be the moduli space of Higgs bundles, and let

$$h\,:\,\mathrm{M}(n,d) \longrightarrow \bigoplus_{i=1}^{n} H^0(X,K^i)$$

be the Hitchin map. The nilpotent cone $h^{-1}(0)$ has multiple irreducible components, most of them non reduced. One of these has underlying reduced scheme isomorphic to $\mathrm{N}(n,d)$. Very stable bundles were used by Beauville–Narasimhan–Ramanan [BNR] to compute the multiplicity of $\mathrm{N}(n,d)$ inside the nilpotent cone in the moduli space of Higgs bundles $\mathrm{M}(n,d)$. For the other components, the multiplicities remained unknown until recently.

Definition 1.2. [HH] A fixed point $\mathcal{E} \in \mathrm{M}(n,d)$ is very stable if

$$\mathcal{E}^+ := \{(F,\psi) \in \mathrm{M}(n,d)\,:\,\lim_{t\to 0} t(F,\psi) = \mathcal{E}\}$$

intersects the nilpotent cone at a unique point $\{\mathcal{E}\}$.

Now, $\mathcal{E}$ is very stable if and only if the Hitchin map h restricted to $\mathcal{E}^+$

$$h_{\mathcal{E}} : \mathcal{E}^+ \longrightarrow B$$

is proper [PPe1, Z, HH]. As a result, the multiplicity of the corresponding component containing $\mathcal{E}$ can be computed as the generic cardinality of $h_{\mathcal{E}}^{-1}(b)$ or the rank of $h_{\mathcal{E},*}\mathcal{O}_{\mathcal{E}^+}$ [HH]. Unfortunately, not all fixed point components contain very stable points. In fact, one may define an invariant of the fixed point components (the virtual equivariant multiplicity [HH]) which for very stable points is a polynomial and recovers the actual multiplicities. It however may fail to be a polynomial for components with no very stable points. In rank three for example, two thirds of the fixed components of type $(2,1)$ and $(1,2)$ fail to have polynomial virtual equivariant multiplicities [HH]. It was conjectured by Hausel–Hitchin that this corresponds exactly to components having no stable bundles.

Proposition 1.3 ([PPe2]). Let $n = 3$. Then, the fixed point components of type $(1, 2)$ are labelled by an invariant $g-1 \leq \delta \leq 4g-4$ [G]. They contain very stable points if and only if $3g-3 \leq \delta \leq 4g-4$.

2. Wobbly bundles and Drinfeld's conjecture

The first motivation to study (stable) wobbly bundles is already pointed at by Laumon. He announces pure codimensionality of wobbly bundles, attributing the result to Drinfeld. This conjecture has been proven constructively in rank two and three [PPa, PPe2], and in general by means of Fano geometry [P].

When studying the question for fixed points instead of vector bundles, one finds irregular behavior. For example:

Proposition 2.1. [PPe2] Let $n = 3$. Then:

(1) The wobbly locus has pure codimension one if and only if the fixed point is of type $(1, 1, 1)$ and the component is not maximal or $\delta = 3g-3$.
(2) The wobbly locus is empty of the fixed point is of type $(1, 1, 1)$ and the component is maximal.
(3) The wobbly locus has codimension one if and only if the fixed point is of type $(2, 1)$ or $(1, 2)$ and $\delta \geq 3g-3$.
(4) The wobbly locus is the whole component if and only if the fixed point is of type $(2, 1)$ or $(1, 2)$ and $\delta < 3g-3$.

We have already pointed out the importance of this result. Indeed, wobbliness of a component (that is, non existence of very stable bundles therein) gives an obstruction to computing the multiplicities through the known methods.

3. Very stable and wobbly bundles in geometric Langlands

One of the first applications of wobbly bundles came through the programme by Donagi–Pantev to prove geometric Langlands from abelianisation of Higgs bundles [DP1]. The idea is to translate geometric Langlands to the abelian case by interpreting Hecke eigensheaves as parabolic Higgs bundles over the moduli space of bundles with a suitable divisor. One may then study them via the corresponding spectral data. The appropriate divisor is the so called shaky locus, and it is defined in terms of unstable bundles as follows.

Definition 3.1. Let $(n, d) = 1$. Consider the rational map

$$\mathrm{M}(n, d) \overset{r}{\dashrightarrow} \mathrm{N}(n, d)$$

given by forgetting the Higgs field. A bundle $E \in \mathrm{N}(n, d)$ is shaky if it is in the image of the exceptional divisor obtained by resolving r by successive blowups along the unstable locus in $\mathrm{M}(n, d)$, consisting of Higgs bundles with underlying unstable bundle.

It is conjectured in [DP1] that the shaky and wobbly loci are the same.

Proposition 3.2 ([Pe])**.** Let $(n, d) = 1$. Then, the shaky locus and the wobbly locus coincide.

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Limit periods and Néron–Tate heights

Emre Can Sertöz

(joint work with Spencer Bloch and Robin de Jong)

1. Introduction

Consider a degeneration of curves $\pi\colon X \to \Delta$, where Δ is a complex disk. This means in particular that the curves $X_t := \pi^{-1}(t)$ are smooth for $t \neq 0$ and the central fiber $X_0 := \pi^{-1}(0)$ is possibly singular. More precisely, the total family X is smooth, the morphism π is flat, and π is smooth over $\Delta \setminus 0$.

Consider the family of pure Hodge structures $\mathrm{H}^1(X_t)$ induced by the smooth fibers. There are two key ingredients that define this structure: the underlying integral structure $\mathrm{H}^1(X_t, \mathbb{Z})$ give by Betti cohomology and the space of holomorphic forms $F^1\,\mathrm{H}^1(X_t) = \mathrm{H}^0(X_t, \Omega^1_{X_t}) \subset \mathrm{H}^1(X_t, \mathbb{C})$. The intersection pairing on integral cohomology is also important but we will suppress from notation. The famous Torelli theorem for curves states that the position of the g-dimensional subspace $F^1\,\mathrm{H}^1(X_t)$ in the $2g$-dimensional ambient space $\mathrm{H}^1(X_t, \mathbb{C})$ with respect to an integral basis (and its intersection pairing) is an invariant fine enough to recover the curve X_t.

There is also a *mixed* Hodge structure $\mathrm{H}^1(X_0)$ defined by the central fiber X_0. Suppose now for simplicity that X_0 is irreducible with a single node x. Let $C \to X_0$ be the normalization of the curve X_0 and $p, q \in C$ are the preimages of x. If C is not hyperelliptic then the mixed Hodge structure $\mathrm{H}^1(X_0)$ determines not only C, but also both p and q—thereby, recovering the central fiber X_0 [3].

2. The height of the limit mixed Hodge structure

It is well understood that the "limit" of the pure Hodge structures $\mathrm{H}^1(X_t)$ determines $\mathrm{H}^1(X_0)$. However, this limit carries precisely one extra dimensional information which seems to have escaped interpretation. This is partially because the limiting process obscures this extra quantity. In fact, in the beautiful book [2, p.35], this quantity is referred to as having "no significance" and we will see a significance emerges when X_0 is defined over a number.

The "limit" for these Hodge structures $\mathrm{H}^1(X_t)$ can not simply be the limiting position of $[F^1\,\mathrm{H}^1(X_t)] \in \mathrm{Gr}(g, \mathrm{H}^1(X_t) \otimes \mathbb{C})$. Any meaningful attempt to identify the moving ambient space $\mathrm{H}^1(X_t, \mathbb{C})$ with a fixed space $\mathrm{H}^1(X_{t_0}, \mathbb{C})$ introduces logarithmic singularities into the position of the g-dimensional space as $t \to 0$. However, there is a limiting *mixed* Hodge structure exists in the sense of Schmid [5]. Roughly, the limit is taken by reading the constant term in a logarithmic extension of the periods.

It is possible to choose a basis $\omega_1(t), \ldots, \omega_g(t)$ for $F^1\,\mathrm{H}^1(X_t)$ consistently around $t = 0$, using the vector bundle $\pi_*\omega_\pi$ associated to the relative dualizing sheaf of the family $\pi\colon X \to \Delta$. The problem is in choosing the integral basis $\gamma_1(t), \ldots, \gamma_{2g}(t) \in \mathrm{H}^1(X_t, \mathbb{Z})$ consistently around zero as there is a monodromy action on them. Nevertheless, we can canonically carry the homology basis to every fiber in a simply connected neighbourhood of t in $\Delta \setminus 0$. We can then express the coordinates of $\omega_i(t)$'s against $\gamma_j(t)$'s: $\omega_i(t) = \sum_j P_{ij}(t)\gamma_j(t)$. The indeterminacy in carrying the homology around 0 will show as logarithmic terms in the periods.

By choosing our bases carefully, we can arrange the period matrix $P(t) = (P_{ij}(t))$ to be of the form

$$\left(\begin{array}{c|c|c} b & P_C & 0 \\ \hline c + \log(t) & a & 2\pi\mathtt{i} \end{array}\right) + O(t), \tag{1}$$

where all entries except $\log(t)$ are constants and where we use the big-O notation. Here $a \in \mathbb{C}^{g-1}, b \in \mathbb{C}^{(g-1)\times 1}$, $c \in \mathbb{C}$, and $P_C \in \mathbb{C}^{(g-1)\times(g-1)}$. The limit mixed Hodge structure is determined by the period matrix

$$\left(\begin{array}{c|c|c} b & P_C & 0 \\ \hline c & a & 2\pi\mathtt{i} \end{array}\right). \tag{2}$$

Note that the matrix P_C is the period matrix of C and a, b are (two different representations of) the Abel–Jacobi image of $p - q$.

The extra information is stored in the corner entry c. However, it also depends on the parameter t used to take the limit. If t is replaced by λt then c changes to $c - \log \lambda$. To be more precise, we note that the limit mixed Hodge structure L_ξ depends on the choice of a cotangent vector $\xi = \mathrm{d}t|_0 \in \Omega^1_{\Delta,0}$. Moreover, the "corner entry" c is better expressed via Hain's [4] biextension height $\mathrm{ht}(L_\xi) \in \mathbb{R}$ which is independent of the coordinates for homology/cohomology. There is an explicit identity of the form

$$\mathrm{ht}(L_\xi) = \Re(c) + \text{an expression involving } a, b, P_C, \tag{3}$$

where $\Re(c)$ denotes the real part of c.

3. Interpreting the height

The question is, is there an interpretation for the height of L_ξ from the point of the central fiber? We prove that if we add a little bit of structure to the central fiber then there is indeed a meaning to $\mathrm{ht}(L_\xi)$. We need only assume that C and p, q are defined over a number field K. With this assumption, the "arithmetic complexity" of the point $p - q \in \mathrm{Jac}(C)(K)$ is expressed by the Néron–Tate height $\mathrm{ht}_{NT}(p - q) \in \mathbb{R}$ which plays a central role in number theory. We claim that $\mathrm{ht}(L_\xi)$ essentially computes the Néron–Tate height of $p - q$.

To make the statement precise, we first demonstrate that a choice for the cotangent ξ can essentially be fixed by our new arithmetic condition that (C, p, q) be defined over K. This is done by the Kodaira–Spencer map which gives an identification

$$\kappa\colon \Omega^1_{\Delta,0} \simeq \Omega^1_{C,p} \otimes \Omega^1_{C,q}. \tag{4}$$

The right hand side admits a natural 1-dimensional K-subvector space of differentials defined over K. But even better, by choosing a model of C defined over the ring of integers $\mathcal{O}_K$ of K, we can find a rank 1 $\mathcal{O}_K$-lattice inside.

Asking for a cotangent direction ξ such that $\kappa(\xi)$ is essentially a generator—more precisely, does not vanish over any prime of $\mathcal{O}_K$—fixes the value of $\mathrm{ht}(L_\xi)$, that is, other generators will give the same height. For simplicity, we can also ask that the Zariski closures of p, q are disjoint over the regular model $\mathcal{C}$ of C.

Theorem 3.1 (Bloch, De Jong, Sertöz 2022 [1])**.** *With the above choice $\mathcal{C}$ of regular model for C and the integral cotangent ξ, we have the following identity*

$$\mathrm{ht}_{NT}(p - q) = \mathrm{ht}(L_\xi) + \mathrm{ht}_{trop}(p - q) \tag{5}$$

where $\mathrm{ht}_{trop}(p - q)$ is a purely combinatorial (i.e. tropical) height assigned to $p - q$ on the dual graphs of the bad fibers of $\mathcal{C}$.

Recently, we also found a generalization of this statement where the central curve X_0 of the degeneration can have arbitrarily many nodes and components.

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Knots, quivers, and quantization of A-polynomials

Piotr Sułkowski

In this short note we summarize the relation between knots and quivers, which we refer to as the knots-quivers correspondence [1, 2]. This relation also makes contact with quantization of an interesting class of knot invariants called A-polynomials, and in particular imposes interesting constraints on the form quantum A-polynomials. For this reason, it is interesting to discuss the knots-quivers correspondence in the context of quantization of complex varieties, which is the main theme of this workshop.

Let us introduce first the objects of our interest and relevant notation, and then give the statement of the knots-quivers correspondence. Regarding knots, we are primarily interested in colored HOMFLY-PT polynomials and superpolynomials. In this note, by *color* we mean a symmetric representation S^r, and we denote corresponding objects by a label r. For a given knot, its colored superpolynomials are Poincaré characteristics of HOMFLY-PT homological spaces $\mathcal{H}_{ijk}^{S^r}$, which depend on variables a, q and t

$$P_r(a,q,t) = \sum_{i,j,k} a^i q^j t^k \dim \mathcal{H}_{ijk}^{S^r}. \tag{1}$$

In particular, we write the uncolored superpolynomials as

$$P_1(a,q,t) = \sum_{i,j,k} a^i q^j t^k \dim \mathcal{H}_{ijk}^{S^1} = \sum_{i=1}^{m} a^{a_i} q^{q_i} t^{t_i}, \tag{2}$$

where the summation runs over m generators of uncolored HOMFLY-PT homology and a_i, q_i, t_i are integer powers. Colored HOMFLY-PT polynomial is an Euler characteristic and can be obtained as $t = -1$ specialization of superpolynomial

$$P_r(a,q) = \sum_{i,j,k} a^i q^j (-1)^k \dim \mathcal{H}_{ijk}^{S^r} \equiv P_r(a,q,-1). \tag{3}$$

In what follows we also consider a generating series of colored HOMFLY-PT polynomials

$$P(x,a,q)=\sum_{r=0}^{\infty}\frac{x^r}{(q^2;q^2)_r}P_r(a,q), \tag{4}$$

where q-Pochhammer symbols is defined as $(x;q)_k=\prod_{i=0}^{k-1}(1-xq^i)$. A factorization of this generating series into quantum dilogarithms

$$P(x,a,q)=\prod_{r\geq 1;i,j;k\geq 0}\left(1-x^r a^i q^{j+2k+1}\right)^{N_{r,i,j}} \tag{5}$$

encodes conjecturally integer LMOV invariants (open BPS invariants) $N_{r,i,j}$.

In turn, we discuss quivers. A quiver is a graph that consists of nodes and arrows between them. Quivers that arise in the knots-quivers correspondence are symmetric, meaning that for each arrow connecting two different nodes there is an arrow in the opposite direction. An arrow that connects a node to itself is called a loop. We denote the number of arrows between nodes i and j by C_{ij} and assemble these numbers into a (symmetric) matrix C.

In quiver representation theory we are interested in the structure of moduli spaces of quiver representations. Consider a symmetric quiver with m nodes and assign to each node i a complex vector space of dimension d_i and a generating parameter x_i. The vector $\boldsymbol{d}=(d_1,\ldots,d_m)$ is referred to as the dimension vector, and we also denote $\boldsymbol{x}=(x_1,\ldots,x_m)$. To such a quiver we assign a motivic generating series

$$P_C(\boldsymbol{x},q)=\sum_{d_1,\ldots,d_m\geq 0}(-q)^{\sum_{i,j=1}^m C_{ij}d_i d_j}\frac{x_1^{d_1}\cdots x_m^{d_m}}{(q^2;q^2)_{d_1}\cdots(q^2;q^2)_{d_m}}. \tag{6}$$

A product decomposition of this generating series encodes motivic Donaldson-Thomas invariants $\Omega_{d_1,\ldots,d_m;j}$, which are interpreted as Betti numbers of moduli spaces of representations, and are non-negative integers

$$P_C(\boldsymbol{x},q)=\prod_{(d_1,\ldots,d_m)\neq 0}\prod_{j\in\mathbb{Z}}\prod_{k\geq 0}\Big(1-(x_1^{d_1}\cdots x_m^{d_m})q^{2k+j+1}\Big)^{(-1)^{j+1}\Omega_{d_1,\ldots,d_m;j}}. \tag{7}$$

The knots-quivers correspondence is the statement that for a given knot there exists a corresponding quiver, such that various invariants of the knot are expressed in terms of invariants of this quiver. In particular, the generating series of colored HOMFLY-PT polynomials $P(x,a,q)$ is related to the motivic generating series of the corresponding quiver (whose structure is captured by the matrix C)

$$P(x,a,q)=P_C(\boldsymbol{x},q) \tag{8}$$

upon the identification of the generating parameters

$$x_i=xa^{a_i}q^{q_i-t_i}(-1)^{t_i}, \tag{9}$$

where a_i,q_i and t_i are powers that arise in the uncolored superpolynomial of the knot in question (2). This statement has various non-trivial consequences. First, from the equality of product forms of both generating series $P(x, a, q)$ and $P_C(\boldsymbol{x}, q)$ it follows that LMOV invariants $N_{r,i,j}$ are expressed in terms of motivic Donaldson-Thomas invariants $\Omega_{d_1,\ldots,d_m;j}$, so that integrality of the latter invariants implies integrality of LMOV invariants, proving an important conjecture mentioned above (at least for those knots, for which the corresponding quiver is identified). Second, the relation (8) also means, that an infinite family of colored HOMFLY-PT invariants $P_r(a, q)$ is encoded in the finite amount of data, i.e. the quiver matrix C and parameters a_i, q_i, t_i. It also turns out that diagonal elements of the matrix C, i.e. the numbers of loops, agree with homological t-degrees introduced in (2): $C_{ii} = t_i$. Further consequences, subtleties and open problems related to the knots-quivers correspondence are discussed, among others, in papers mentioned below.

The knots-quivers correspondence was originally formulated [1, 2]. It was further developed and discussed from various perspectives in [3, 4, 5, 6, 7, 8]. In [9, 10] it was generalized to toric Calabi-Yau manifolds, and in [11, 12] its version for 3-manifolds that are knot complements was proposed. It is an interesting research direction, which still offers more interesting questions than answers.

Having stated the knots-quivers correspondence, let us finally comment on its relation to A-polynomials. Originally, an A-polynomial for a knot was defined as an $SL(2, \mathbb{C})$ character variety of the knot complement. The so-called AJ-conjecture states that there exists a quantum A-polynomial, which imposes recursion relations for colored Jones polynomials (which are $a = q^2$ specializations of colored HOMFLY-PT polynomials (3)), and whose symbol is the original A-polynomial. It turns out that colored HOMFLY-PT polynomials (3) also satisfy recursion relations, which can be encoded in difference operators that are generally called quantum A-polynomials, and whose classical limits can be interpreted as algebraic curves. Now, as the knots-quivers correspondence predicts that colored HOMFLY-PT polynomials have some particular structure captured by (8), it follows that this structure must also impose some non-trivial constraints on corresponding quantum A-polynomials, and thus also on corresponding classical A-polynomials. We believe that unraveling these constraints and their consequences is an important issue, and encourage everyone to join our efforts to tackle it.

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Lectures on the quantisation of the Hitchin system and the analytic Langlands program

Jörg Teschner

The first part of the lectures reviewed the approach to the geometric Langlands Correspondence pioneered by Beilinson and Drinfeld, and how this approach is related to the quantisation of the Hitchin system, following the lectures [Fr07].

The Hitchin moduli space $\mathcal{M}_{\rm Hit}(C)$ is the space of pairs $(\mathcal{E},\varphi)$, with $\mathcal{E}$ being a holomorphic vector bundle on a Riemann surface C, and $\varphi \in H^0(C,\mathrm{End}(\mathcal{E})\otimes K)$. $\mathcal{M}_{\rm Hit}(C)$ has a canonical Poisson structure from Serre-duality between the tangent space $H^1(C,\mathrm{End}(\mathcal{E}))$ to Bun_G at $\mathcal{E}$ and $H^0(C,\mathrm{End}(\mathcal{E})\otimes K)$. It has the structure of an algebraically integrable system. To simplify the presentation we shall restrict attention to the cases where $\mathcal{E}$ is a holomorphic $G = SL(2)$-bundle in the following. The structure of an integrable system can be exhibited using $\theta := \frac{1}{2}\mathrm{tr}(\varphi^2) \in H^0(C,K^2)$. The expansions $\theta(z) = \sum_{r=1}^{3g-3} H_r Q_r(z)$, with $\{Q_r; r = 1,\dots,3g-3\}$ being a basis for $H^0(C,K^2)$, define Hamiltonians H_r satisfying $\{H_r,H_s\} = 0$ and defining the integrable structure on $\mathcal{M}_{\rm Hit}(C)$.

Hitchin's Hamiltonians have been quantised in the work of Beilinson and Drinfeld on the geometric Langlands correspondence. This means the following: There exist global differential operators H_r, $r = 1,\dots,3g-3$ on the line bundle $K^{1/2}$ on the moduli space Bun_G of G-bundles such that the following holds:

- The differential operators H_r generate the commutative algebra $\mathfrak{D}$ of global differential operators acting on $K^{1/2}$, and
- the symbols of the differential operators H_r coincide with the Hamiltonians H_r defined above.

Beilinson and Drinfeld put the quantisation of the Hitchin in relation to the geometric Langlands correspondence, in the case of interest being a correspondence

$$(1)\qquad \boxed{{}^{\rm L}\mathfrak{g}\text{-opers}} \quad\longleftrightarrow\quad \boxed{\mathcal{D}-\text{modules on }\mathrm{Bun}_G,}$$

involving the following objects. ${}^{\rm L}\mathfrak{g}$ is the Langlands dual of the Lie algebra $\mathfrak{g}$ of G. In the case $\mathfrak{g} = \mathfrak{sl}_2$ one may represent ${}^{\rm L}\mathfrak{g}$-opers as pairs $\chi = (\mathcal{E},\nabla)$, where

$\mathcal{E} = \mathcal{E}_{\rm op}$ is the unique up to isomorphism extension $0 \to K^{\frac{1}{2}} \to \mathcal{E}_{\rm op} \to K^{-\frac{1}{2}} \to 0$, and the connection ∇ is locally gauge equivalent to the form $\nabla = dz\big(\partial_z + \left(\begin{smallmatrix} 0 & t \\ 1 & 0 \end{smallmatrix}\right)\big)$, with t being a projective connection on C. Beilinson-Drinfeld use methods from conformal field theory and a result of Feigin and Frenkel on the center of the universal enveloping algebra of the affine Lie algebra $\widehat{\mathfrak{g}}_k$ at the critical level $k = -h^\vee$ to construct a canonical isomorphism of algebras between the algebra $\mathrm{Fun}\,\mathrm{Op}(C)$ of functions on the space of ${}^{\mathrm{L}}\mathfrak{g}$-opers on C and the algebra $\mathfrak{D}$ of differential operators. Fixing an oper χ defines a homomorphism $\mathrm{Fun}\,\mathrm{Op}(C) \to \mathbb{C}$. Using the isomorphism $\mathrm{Fun}\,\mathrm{Op}(C) \simeq \mathfrak{D}$ one gets a homomorphism $\rho_\chi : \mathfrak{D} \to \mathbb{C}$. To each oper χ one may assign the $\mathcal{D}$-module Δ_t on Bun_G defined as

$$\Delta_t = \mathfrak{D}/\mathrm{ker}\rho_\chi \cdot \mathfrak{D}. \tag{2}$$

This is the $\mathcal{D}$-module associated to the oper χ by the correspondence (1). It corresponds to the following system of differential equations on Bun_G,

$$\mathsf{H}_r f = E_r f, \qquad E_i = \rho_\chi(\mathsf{H}_r), \qquad r = 1, \dots, 3g-3. \tag{3}$$

The results of Beilinson and Drinfeld yield important special cases of the geometric Langlands correspondence. By now there exist stronger and more general versions, especially due to Arinkin and Gaitsgory. However, the construction of Beilinson and Drinfeld still serves as a foundation for many approaches to the geometric Langlands correspondence and related developments.

The second part of the lectures discussed a variant of the geometric Langlands correspondence called analytic Langlands correspondence following Etingof, Frenkel and Kazhdan [EFK1], motivated by a suggestion of Langlands, and by the work [T18], which had proposed a variant of (1) schematically represented as

$$\boxed{\text{real } {}^{\mathrm{L}}\mathfrak{g}\text{-opers on } C} \leftrightarrow \boxed{\text{single-valued Hitchin eigenfunctions,}} \tag{4}$$

where real $\mathfrak{sl}_2$ opers are opers with holonomy in $\mathrm{PSL}(2,\mathbb{R})$. The objects on the right are required to satisfy the pairs of eigenvalue equations

$$\mathsf{H}\,\Psi = \rho_\chi(\mathsf{H})\Psi, \quad \forall\, \mathsf{H} \in \mathfrak{D}, \qquad \bar{\mathsf{K}}\,\Psi = \bar{\rho}_{\bar{\chi}}(\bar{\mathsf{K}})\Psi, \quad \forall\, \bar{\mathsf{K}} \in \bar{\mathfrak{D}}, \tag{5}$$

where $\chi \in \mathrm{Op}(C)$, and $\bar{\chi} \in \overline{\mathrm{Op}}(C)$, the complex conjugate of $\mathrm{Op}(C)$. One may look for smooth solutions to (5) on $\mathrm{Bun}_G^{\rm vs}$, the complement of a divisor of singularities of the Hitchin Hamiltonians in Bun_G, locally of the form

$$\Psi(\mathbf{x},\bar{\mathbf{x}}) = \sum_{r,s} C_{rs}\psi_r(\mathbf{x})\bar{\psi}_s(\bar{\mathbf{x}}), \qquad \begin{aligned} \mathsf{H}\,\psi_r(\mathbf{x}) &= \rho_\chi(\mathsf{H})\psi_r(\mathbf{x}), \quad \forall\, \mathsf{H} \in \mathfrak{D}, \\ \bar{\mathsf{K}}\,\bar{\psi}_s(\bar{\mathbf{x}}) &= \bar{\rho}_{\bar{\chi}}(\bar{\mathsf{K}})\bar{\psi}_s(\bar{\mathbf{x}}), \quad \forall\, \bar{\mathsf{K}} \in \bar{\mathfrak{D}}, \end{aligned}$$

which are furthermore *single-valued*.

In order to construct and classify the solutions to this problem, the paper [T18] proposed to use the Separation of Variables (SOV) method pioneered by Sklyanin. Application of this method yields invertible integral transformations of the form

$$\Psi(\mathbf{x},\bar{\mathbf{x}}) = \int d\mathbf{u}d\bar{\mathbf{u}}\; K(\mathbf{x},\bar{\mathbf{x}}\,|\mathbf{u},\bar{\mathbf{u}})\Phi(\mathbf{u},\bar{\mathbf{u}}), \tag{6}$$

$\mathbf{u} = (u_1, \ldots, u_d)$, $\bar{\mathbf{u}} = (\bar{u}_1, \ldots, \bar{u}_d)$, $d = \dim(\mathrm{Bun}_G)$, such that (5) is equivalent to

$$\Phi(\mathbf{u}, \bar{\mathbf{u}}) = \prod_{r=1}^{d} \phi(u_r, \bar{u}_r), \qquad \begin{aligned} &(\partial_u^2 - t(u))\phi(u, \bar{u}) = 0, \\ &(\bar{\partial}_{\bar{u}}^2 - \bar{t}(\bar{u}))\phi(u, \bar{u}) = 0. \end{aligned} \tag{7}$$

Single-valuedness of the kernel K implies that the functions $\Psi(\mathbf{x}, \bar{\mathbf{x}})$ are single-valued if and only if the functions $\phi(u, \bar{u})$ appearing in (7) have this property. It can be shown that single valued solutions $\phi(u, \bar{u})$ to the system of equations in (7) exist iff the connection $\nabla = dz\big(\partial_z + \left(\begin{smallmatrix} 0 & t \\ 1 & 0 \end{smallmatrix}\right)\big)$ has real holonomy, and therefore defines a real projective structure. The transformation (6) would thereby furnish an explicit realisation of the correspondence (4). Real projective structures on closed surfaces C have been classified in [Go].

A part of the lectures explained joint work in progress with Duong Dinh realising the SOV method for Hitchin systems associated to higher genus Riemann surfaces. This will allow us to realise the approach proposed in [T18] for this class of Riemann surfaces.[1]

The work of Etingof, Frenkel and Kazhdan deepens the correspondence (4) considerably by adding elements of functional analysis. A Hilbert space $\mathcal{H}$ of half-densities is defined, and the Hitchin-Hamiltonians are realised as unbounded operators acting on dense subspaces of $\mathcal{H}$. Etingof, Frenkel and Kazhdan formulate a set of conjectures, including

- The eigenspaces $\mathcal{H}_{\chi,\bar{\chi}}$ generated by single-valued solutions to the pair of eigenvalue equations with eigenvalues $(\chi, \bar{\chi})$ are contained in $\mathcal{H}_G$, at most one-dimensional, and non-vanishing only if $\bar{\chi}$ is complex conjugate to χ.
- The Hilbert spaces $\mathcal{H}$ admit an orthogonal decomposition into the spaces $\mathcal{H}_{\chi,\bar{\chi}}$ (completeness),

and prove them in some cases [EFK2]. A combination of these techniques with the SOV method may help to establish similar results in larger generality.

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