Mathematisches Forschungsinstitut Oberwolfach

Report No. 52/2022

DOI: 10.4171/OWR/2022/52

Mini-Workshop: Topological and Differential Expansions of o-minimal Structures

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27 November – 3 December 2022

Abstract. The workshop brought together researchers with expertise in areas of mathematics where model theory has had interesting applications. The areas of expertise spanned from expansions of o-minimal structures preserving tame geometric properties to expansions of specified fields by classical operators that preserve neo-stability properties. There were presentations and discussions on recent developments in definable groups and decompositions in relatively tame setups, the interplay of different notions of dimension and closure operators, and applications of the model theory of differential fields to diophantine geometry.

Mathematics Subject Classification (2020): 03C45, 03C60, 03C64, 03C98, 12H05, 11D61.

Introduction by the Organizers

The workshop "Topological and differential expansions of o-minimal structures" was organised by:
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We had 16 participants, all of them experts in model theory and its applications to other areas of mathematics. Within the participants, we had 3 experts in complex exponentiation, 3 experts in the model theory of differential fields, 3 experts in expansions of topological fields by operators, 4 experts in tame expansions of o-minimal structures, and 3 experts in geometric fields and neostability theory. Participants presented their recent research work during the scheduled morning and afternoon talks. The time after lunch and dinner permitted lively discussions where collaborations were initiated and/or further developed. There were 14 talks, including three survey talks, one on complex exponentiation and Zilber's quasi-minimality conjecture, one on strongly minimal sets in differentially closed fields with an eye for applications to diophantine geometry, and one on a classification project on expansions of an o-minimal dense linear order that do not define all bounded projective sets, up to various notions of tameness.

The overarching theme of the workshop was *model-theoretic tame geometry*, a new and exciting branch of mathematical logic with far reaching applications both in mathematics and in other areas. It has become one of the most active fields of research in model theory, and an almost indispensable tool for the study of many problems in algebra, real and complex geometry, number theory and analysis.

One of the (more specialised) focus was that of structures with NIP (not the independence property). This subject has had enormous growth in recent years. On the one hand, these structures include the two most established and extreme cases of model theoretic study, stable and o-minimal structures, and, on the other hand, they have found surprising applications to areas such as combinatorics and machine learning. Some of the presentations discussed how NIP structures can be obtained and how, in certain cases, NIP can be transferred to tame expansions. In other talks, descriptions of definable groups in NIP contexts were presented; this was quite interesting as such a sharp description is unavailable in general NIP theories (in contrast to the stable and o-minimal settings).

A second focus was the model-theoretic study of (topological) fields with a generic derivation. In particular, the model theory of closed ordered differential fields (CODF) was a recurrent topic, or rather prototype, in several discussions. Results vastly generalising what is known for CODF were a common theme in interconnected talks, some on differential expansions of topological fields of characteristic 0 with an open theory, respectively smooth theory and some differentially large fields (considering also the characteristic p case). On the other hand, there were several expositions of fields equipped with additional structure (of a topological flavour, say a valuation in a Hardy field) where there is a strong interaction between the derivation and the given structure.

The third focus was on fields with exponentiation, and in particular, on Zilber's program on a model theoretic analysis of $(\mathbb{C}, \exp)$ the complex exponential field (still to be fully understood), in connection with the quasi-minimality conjecture for $(\mathbb{C}, \exp)$. Zilber identified a possible axiomatization of $(\mathbb{C}, \exp)$ where a crucial role is played by Schanuel's conjecture in transcendental number theory. Modulo Schanuel's conjecture the strong exponential-algebraic closure axiom is the only axiom still unknown to be true in $(\mathbb{C}, \exp)$. This requires that certain algebraic complex varieties intersect the graph of exponentiation in generic points. Modulo Schanuel's conjecture many special cases of this axiom have been established in the last years by several authors. Recent results on the exponential closure for complex algebraic varieties corresponding to systems of exponential sum equations were reported. The proofs of these results use diophantine geometry, and in the latest results also tropical geometry arguments have been used.

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Abstracts

Tameness beyond o-minimality

Philipp Hieronymi

At the heart of model theory lies the observation there are certain objects that have to be considered tame, and others that have to be considered wild. Many foundational results in logic concerned the existence of objects considered wild. Gödel's proof of the undecidability of the theory of $(\mathbb{N},+,\cdot)$ established that this structure is wild from a logical viewpoint. Such results are negative in spirit pointing to the limitations of mathematical reasoning. However, model theorists also found a vast number of mathematical structures that exhibit no such wildness, and for often very different reasons are tame. General frameworks for such tame structures have been developed and their properties have been studied extensively. This program of identifying and analyzing tame classes of structures whose model theory can be understood, became to be known as *the geography of tame mathematics*, and in its various forms has dominated model theory throughout the last sixty years.

Here we discuss this program in the context of expansions of the real field. Although it is similar in spirit to Shelah's classification theory, the aim here is to classify structures over a fixed universe in terms of geometric properties of their definable sets rather than theories in terms of their combinatorial properties or numbers of models. While classification theory is motivated by Morley's theorem, our tame program here originated in the tremendous success of o-minimality.

Let $\overline{\mathbb{R}}$ denote the real field $(\mathbb{R},+,\cdot)$, and consider a collection $\mathcal{X}$ of subsets of various $\mathbb{R}^n$. We are studying the expansion of $\overline{\mathbb{R}}$ by predicates for each $X \in \mathcal{X}$, that is $(\overline{\mathbb{R}},(X)_{X\in\mathcal{X}})$. We say two expansions $\mathcal{R}_1$ and $\mathcal{R}_2$ are **interdefinable** if they define the same sets with parameters.

Naïve Goal. Classify expansions of $\overline{\mathbb{R}}$ up to interdefinability.

As Miller put it, the Naïve Goal is "*too vague at best and intractable at worst.*" So instead of classifying expansions up to interdefinability, we aim to sort such expansions in classes according to the geometric tameness of their definable sets. We focus on geometric tameness of the definable sets rather than model-theoretic tameness of the structure and its theory. Thus is a rather a definability-theoretic enterprise than a model-theoretic one.

Fix an expansion $\mathcal{R}$ of $\overline{\mathbb{R}}$. When we say a set is *definable*, we mean definable in $\mathcal{R}$ possibly with parameters. The core tameness notion that gave rise to this systematic study of tameness in expansions of $\overline{\mathbb{R}}$ is o-minimality. We say $\mathcal{R}$ is **o-minimal** if every definable subset of $\mathbb{R}$ either has interior or is finite. O-minimality was isolated by van den Dries in order to prove important results from semi-algebraic geometry in this generality, and developed by Pillay and Steinhorn as a tameness notion in the setting of dense linear orders. Among the many results that transfer from the semi-algebraic geometry are the monotonicity theorem for definable functions and the cell decomposition theorem. It is not our goal to discuss o-minimal structures, rather we focus on surveying the larger framework of geometric tameness on $\mathbb{R}$. For that, we note that o-minimality implies model-theoretic tameness; something that fails for all other tameness notions considered here. By the cell decomposition theorem, if a structure is o-minimal, then every elementarily equivalent structure is also o-minimal. Furthermore, an o-minimal structure is NIP, even dp-minimal, and distal. In model-theoretic universe, o-minimality plays a similar role in the ordered setting as strong minimality in the stable setting.

The tameness conditions we consider are of the following form: every definable subset of $\mathbb{R}$ has interior or is *small*. In the case of o-minimality *small* means finite. A subset of $\mathbb{R}$ is finite if and only it is closed, bounded and discrete. Thus the most obvious first step towards weaker notions of smallness, is to drop one of the three conditions, say boundedness. We say $\mathcal{R}$ is **weakly d-minimal** if every definable subset of $\mathbb{R}$ either has interior or is finite union of discrete sets. We say $\mathcal{R}$ is **d-minimal** if for every $m \in \mathbb{N}$ and every definable subset $A \subseteq \mathbb{R}^{n+1}$ there is $N \in \mathbb{N}$ such that for all $x \in \mathbb{R}^m$ either A_x has interior or is the union of N discrete sets. In the case of weak d-minimality, being *small* means being the finite union of discrete sets. It is an easy exercise to see that if *small* is defined as discrete, then $\mathcal{R}$ is actually o-minimal. It is an open question whether weak d-minimality implies d-minimalilty[1]. As pointed out in [10], it follows from van den Dries [2] that $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$ is d-minimal, and this is the typical example of a d-minimal structure. In contrast to o-minimality, d-minimality only implies geometric tameness, but not model-theoretic tameness. Let $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})^{\#}$ be the expansion of $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})$ by every subset of every cartesian power of $2^{\mathbb{Z}}$. By Friedman and Miller [7], the expansion $(\overline{\mathbb{R}}, 2^{\mathbb{Z}})^{\#}$ is d-minimal, yet defines an isomorphic copy of $(\mathbb{Z}, +, \cdot)$. Thus its theory is undecidable, and it fails Shelah-style combinatorial tameness conditions like NIP and NTP2.

Of course, we can further increase the class of sets we want to consider *small*. We say $\mathcal{R}$ is **noiseless** if every definable subset of $\mathbb{R}$ either has interior or is nowhere dense. As a tameness condition, this noiselessness is first studied in [10], although the name *noiseless* was only suggested later by Miller[2]. It is not easy to see that there are expansions of $\overline{\mathbb{R}}$ that are noiseless, but not d-minimal. A good example of a set without interior that is not a finite union of discrete sets, is a **Cantor set**[3]. Friedman et al. [6] produce a Cantor set K such that every definable subset of $\mathbb{R}$ either has interior or is Hausdorff null. Hence $(\overline{\mathbb{R}}, K)$ is noiseless.

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[1] When Miller introduced d-minimality in [10], he expected this not to be case.

[2] The rationale behind the name is that the condition is equivalent to the statement that $\mathcal{R}$ does not define a set X such that $X \cap I$ is dense and co-dense in an interval $I \subseteq \mathbb{R}$. More inline with the aforementioned conditions, Fornasiero [3] used the term *i-minimal* (for interior-minimal) instead.

[3] A Cantor set is a compact, nonempty subset of $\mathbb{R}$ that has neither isolated nor interior points.

So what about wildness? Consider $(\overline{\mathbb{R}}, \mathbb{Z})$. Not only is its theory undecidable by Gödel, it defines every Borel subset, and hence every projective subset of $\mathbb{R}^n$. Thus geometrically wild phenomena like space-filling curves and nowhere differentiable continuous functions appear in this structure. The question whether every set definable in $(\overline{\mathbb{R}}, \mathbb{Z})$ is Lebesgue measurable, is independent of ZFC. Because of such complications, the nondefinability of $\mathbb{Z}$ is clearly necessary for an expansion of $\overline{\mathbb{R}}$ to be considered tame. Miller has championed a research program determining whether this nondefinability is also sufficient to enforce some form of well-defined tameness in such expansions. This is an ambitious program as it is far from obvious that any geometric pathologies can be ruled out. The most ambitions conjecture is that the **open core**[4] of such a structure is noiseless. Substantial progress has been made towards this conjecture in [3,4,8], in particular regarding the antifractal nature of closed sets in such structures. This can not be detailed here, but these results provide evidence for the viability of the following program:

Miller's tameness program. Can we classify expansions of $\overline{\mathbb{R}}$ that do not define $\mathbb{Z}$, up to some common notion of tameness?

Progress in this program will surely also give new insights about the precise nature of o-minimality. We focus here on expansions of $\overline{\mathbb{R}}$, but similar questions make perfect sense and produce similar answers for expansions of $(\mathbb{R}, <, +)$ (see [5, 9]). There is also a much older (predating o-minimality by multiple decades) similar program on studying expansions of $(\mathbb{N}, +)$, see Bès [1] for an excellent survey.

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[4] The open core of a structure is the reduct of all open definable sets.

An unbounded version of Zarankiewicz's problem

Pantelis E. Eleftheriou
(joint work with Aris Papadopoulos)

Basit-Chernikov-Tao-Starchenko-Tran [1] proved, among others, a linear version of Zarankiewicz's problem, Theorem 2 below. Let us first fix some notation/terminology. We let $\mathcal{M} = \langle M, \dots \rangle$ be a structure and $E \subseteq M^{d_1} \times \cdots \times M^{d_r}$ an r-ary relation. By a grid, we mean a set of the form $B = B_1 \times \cdots \times B_r$, where each $B_i \subseteq M^{d_i}$. If B is finite, we write $n_B = \max |B_i|$. So $|B| \leq n_B^r$. We are interested in classes of grids whose intersections with E have better asymptotic bounds than n_B^r.

Definition 1. Let $\mathcal{C}$ be a class of finite grids. We say that E has *linear Zarankiewicz bounds (Z-bounds) for* $\mathcal{C}$ if there is $\alpha \in \mathbb{R}^{>0}$, such that for every $B \in \mathcal{C}$,

$$|E \cap B| \leq \alpha n_B^{r-1}.$$

(The terminology is motivated by the fact that if E is the edge relation of a graph on B (so $r = 2$), these bounds are indeed linear.)

Let us call E *k-free*, where $k \in \mathbb{N}$, if E contains no complete k-grid B; that is, a grid B with all $|B_i| = k$. (If we saw E as the edge relation of a hypergraph on $M^{d_1} \times \cdots \times M^{d_r}$, then '$k$-free' is what is known in the literature as '$K_{k,\dots,k}$-free'.)

Theorem 2 ([1])**.** *Let $\mathcal{M} = \langle M, <, \dots \rangle$ be an o-minimal structure. Then the following are equivalent:*

(1) For every definable relation E, if E is k-free for some $k \in \mathbb{N}$, then E has linear Z-bounds.
(2) $\mathcal{M}$ does not define an infinite field (equivalently, $\mathcal{M}$ is 'linear'.)

In work in progress, we extend this theorem so that (2) weakens to having no definable field on the whole of M, as follows. Let us first give a definition.

Definition 3. Let $\mathcal{M} = \langle M, <, +, \dots, 0 \rangle$ be an o-minimal expansion of an ordered group, and $m \in M^{>0}$. We call a grid $B \subseteq M^{d_1} \times \cdots \times M^{d_r}$ *m-distant*, if for every i and $x, y \in B_i$, $|x - y| < m$. (Here, $|x - y|$ denotes the distance $\sum_{j=1}^{d_i} |x_j - y_j|$.)

Theorem 4 ([5])**.** *Let $\mathcal{M} = \langle M, <, +, \dots, 0 \rangle$ be an o-minimal structure. Then the following are equivalent:*

(1) For every definable relation E, there is $m \in M^{>0}$, such that if E is k-free for some $k \in \mathbb{N}$, then E has linear Z-bounds for the class of all m-distant finite grids.
(2) $\mathcal{M}$ does not define an unbounded field (equivalently, there is no definable field on M; equivalently, $\mathcal{M}$ is 'semibounded'.)

Examples of o-minimal semibounded structures include any o-minimal expansion of the real ordered group by bounded sets, such as the restriction of multiplication, analytic functions, or exponential, to bounded domains.

Remark 5. More can be said:

- E does not have to be k-free, but only free of m-distant complete k-grids.
- In case $\mathcal{M}$ is sufficiently saturated, the phrase 'there is m' can be placed before E; that is, the same m works for all E.

In the rest of this document we sketch the proof of Theorem 4(2)⇒(1), for $\mathcal{M}$ semibounded nonlinear (if linear, then any $m \in M^{>0}$ would yield the result, by Theorem 2).

This is done in three steps. Before describing them, let us note that Theorem 2 was proved in [1] by exploiting the fact that acl in $\mathcal{M}$ is weakly locally modular if and only if $\mathcal{M}$ is linear ([2]). In Step I, we extract from [1] an abstract version of Zarankiewicz's problem for a sufficiently saturated structure $\mathcal{M}$, a closure operator cl and a class of finite grids $\mathcal{C}$ satisfying certain conditions (Theorem 6 below). In Step II, we apply this abstract version to a semibounded o-minimal structure, the *short closure operator* scl from [4] and the class $\mathcal{C}$ of all *tall* finite grids. This proves Theorem 4 for a sufficiently saturated $\mathcal{M}$. In Step III, we deduce Theorem 4 for any $\mathcal{M}$, by moving to a saturated elementary extension, and applying a trick for replacing sufficiently distant grids by tall grids.

STEP I: We extract from [1] the following theorem:

Theorem 6. *Let $\mathcal{M}$ be a sufficiently saturated structure,* $\mathrm{cl} : \mathcal{P}(M) \to \mathcal{P}(M)$ *a pregeometric closure operator whose induced independence relation* $\downarrow$ *satisfies extension and non-degeneracy, and $\mathcal{C}$ a class of finite grids. Denote by* cl-dim(X) *the corresponding dimension of definable sets. Assume the following conditions:*

(C1) (Weak Local Modularity). *For all small sets $A, B \subseteq M$ (that is, smaller than saturation cardinality of $\mathcal{M}$), there is some small $C \subseteq M$ such that:*

$$C \underset{\varnothing}{\downarrow} AB \text{ and } A \underset{\mathrm{cl}(AC)\cap \mathrm{cl}\, BC}{\downarrow} B.$$

(C2) (cl-Definability). *Let $a, b \in M$. If $a \in \mathrm{cl}(Ab)$, then there is an A-definable set X that contains (a, b), such that for every $(a', b') \in X$, we have $a' \in \mathrm{cl}(Ab')$.*

(C3) ($\mathcal{C}$-Uniform Bounds) *Let $\{X_b\}_{b\in I}$ be a definable family of sets in M^r. Then there is $N \in \mathbb{N}$ such that for every $b \in I$ and every $Y \in \mathcal{C}$, if* $\mathrm{cl\text{-}dim}(X_b) = 0$ *and $Y \subseteq X_b$, then $|Y| \le N$.*

Then: for every definable relation E, if E is k-free for some $k \in \mathbb{N}$, then E has linear Z-bounds for $\mathcal{C}$.

For $\mathcal{M}$ linear, cl = acl and $\mathcal{C}$ the class of all finite grids, the above theorem gives Theorem 2.

STEP II: Here we verify the conditions of Theorem 6 in the semibounded setting. Let us recall some notions from semibounded geometry ([4, 6]). Let $\mathcal{M} = \langle M, <, +, \dots, 0\rangle$ be an o-minimal semibounded structure which is not linear. Then there is an interval $R \subseteq M$, such that $\mathcal{M}$ defines a field with domain R. For $A \subseteq M$, we denote

$$\mathrm{scl}(A) = \mathrm{dcl}(AR).$$

Call an interval I *short* if I is in definable bijection with R. Call a definable set $X \subseteq M^n$ *short* if it is in definable bijection with a subset of R^n. It is proved in [4] that a definable set is short if and only if $\text{scl-dim}(X) = 0$. Call an element $a \in M$ short if $(0, |a|)$ is short, and tall otherwise.

We define here a grid B to be *tall* if it is m-distant for some tall m.

Theorem 7. *Conditions (C1) – (C3) hold for $\mathcal{M}$ semibounded nonlinear,* $\text{cl} = \text{scl}$ *and $\mathcal{C}$ the class of all tall finite grids. Hence, for m tall, Theorem 4 holds.*

Proof. (C1) We prove that $\text{dcl}(-R) = \text{dcl}_{lin}(-\text{dcl}(R))$, where dcl_{lin} is the definable closure operator in the reduct $\mathcal{M}_{lin} = \langle M, <, +, 0, \{\lambda\}_{\lambda \in \Lambda}\rangle$, where Λ is the set of all partial $\varnothing$-definable (in $\mathcal{M}$) endomorphisms of $\langle M, <, +, 0\rangle$. Since dcl_{lin} is weakly locally modular, so is $\text{dcl}(-R)$.

(C2) essentially follows from [4], whereas the proof of (C3) exploits the fact that definably connected short definable sets cannot contain two elements x, y whose difference $x - y$ is tall. □

STEP III: To obtain Theorem 4 for any $\mathcal{M}$, we need the following proposition. We take a saturated elementary extension $\mathcal{N}$ of $\mathcal{M}$ and consider the interpretation $E^{\mathcal{N}}$ of $E \subseteq M^n$ in $\mathcal{N}$. Denote again $E = E^{\mathcal{N}}$. By cone decomposition ([3, 4]), we may assume E is a *cone* (essentially $E = S + V$, where S is short and V is linear and unbounded.) Take any $m \in M^{>0}$ such that $S \subseteq (-m, m)^n$. (The precise choice of m is a bit more elaborate, and we omit the details.)

Proposition 8. *Let B be an m-distant finite grid. Then there is a tall finite grid B' such that $|E \cap B| = |E \cap B'|$ and $n_{B'} \leq n_B$.*

We can now finish as follows. Let $\alpha \in \mathbb{R}^{>0}$ witness the linear Z-bounds of E for the class of all tall finite grids in $\mathcal{N}$, provided by Theorem 4. It follows that for any m-distant finite grid B,

$$|E \cap B| = |E \cap B'| \leq \alpha n_{B'}^{r-1} \leq \alpha n_B^{r-1},$$

as needed.

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Definable groups in tame theories of differential topological fields

Françoise Point

1. Topological fields (of characteristic 0) and tame theories T.

Let $\mathcal{L}$ be a (possibly multi-sorted) language containing the ring language $\mathcal{L}_{rings} := \{+,-,\cdot,0,1\}$ together with a symbol for the inverse function $\{^{-1}\}$ ($\mathcal{L}_{fields}$). Let τ denote a definable topology (given by an $\mathcal{L}$-formula χ); τ is assumed to be Hausdorff, non discrete and cartesian products are endowed with the product topology. A *tame* $\mathcal{L}$-theory T of topological fields (K,τ) (of characteristic 0) is a complete, geometric $\mathcal{L}$-theory, namely acl satisfies the exchange. We denote by dim the dimension induced by acl on definable sets (automatically, one has the elimination of $\exists^\infty$ [13]).

In [11] and [5], we considered open theories of topological fields (of characteristic 0) by further requiring that on the field sort $\mathcal{L}$ is a relational extension of $\mathcal{L}_{fields}$ (plus possibly additional constants) and definable sets are finite unions of Zariski closed sets intersected with definable open sets. In that case, acl coincides with the field algebraic closure and dim coincides with the topological dimension. As examples of open theories T, we have the theory of real-closed fields, p-adically closed fields, algebraically closed valued fields (of characteristic 0), real-closed valued fields and using a multi-sorted language, Henselian valued fields of characteristic 0 [8].

In [6], we obtained a cell decomposition theorem, where a cell is either a definable open set, a finite set, or graphs of continuous correspondences which are closed under the projection under initial subsets of coordinates. (Correspondences are needed since one doesn't necessarily have finite Skolem functions).

Theorem 1. [6] *Let T be an open $\mathcal{L}$-theory of topological fields and let $\mathcal{K}$ be a model of T. Let X be an A-definable subset of K^n, then X is a finite disjoint union of A-definable cells.*

2. Existentially closed differential expansions

We consider the theory T_δ of differential expansions of models of T by a derivation δ, on which we impose no interactions with the topology; $T_\delta := T \cup \{\delta(x+y) = \delta(x)+\delta(y),\ \delta(xy) = x\delta(y)+\delta(x)y\}$. One axiomatizes the existentially closed models of T_δ, under an additional condition of largeness on T. For $T = RCF$, it was done by M. Singer [18], who described the theory of closed ordered differential fields ($CODF$). Then by M. Tressl [19] in a general setting using the notion of large fields [17].

Let T_δ^* be the theory T_δ together with the following scheme of axioms (DL):

for $\mathcal{K} \models T_\delta$, for every differential polynomial $P(x) \in K\{x\}$ in one variable, of order $m \geqslant 1$, for $y = (y_0,\dots,y_m)$, we have that for any neighbourhood W of 0 in

K^{m+1}, letting P^* is the algebraic polynomial associated with P,

$$(\forall y)\big((P^*(y) = 0 \wedge s^*_P(y) \neq 0) \rightarrow \exists x \tag{2.1}$$

$$\big(P(x) = 0 \wedge s_P(x) \neq 0 \wedge (\bar{\delta}^m(x) - y) \in W, \big)\big) \tag{2.2}$$

where s_P denotes the separant of P ($s_P := \partial_{\delta^m(x)} P(x)$) and $s^*_P := \partial_{y_n} P^*$. When consistent, the theory T^*_δ axiomatizes the theory of existentially closed models of T_δ [11]; [5]. Furthermore if T is r.e. and T^*_δ is complete, it is decidable.

We get transfer from T to T^*_δ of the properties: quantifier elimination, completeness, NIP, [11], [6] , distality [4], see also [2], NTP_2 [16]. Also note that if $\mathcal{K} \models T^*_\delta$, then the subfield C_K is dense and co-dense in K and $C_K \preceq_{\mathcal{L}} K$. So some of our transfer results can be applied to dense (elementary) pairs of models of T [9].

Next we improve a cell decomposition theorem proven for $CODF$ [3] and show a general transfer result for elimination of imaginaries. A useful notion is the one of of linked triple: we associate with an $\mathcal{L}_\delta$-definable set X, an $\mathcal{L}$-definable set where tuples consisting of successive derivatives (up to some order) of elements of X are dense. It enables us to first show that T^*_δ has $\mathcal{L}$-open core [6] (property introduced in [7]), then to define a δ-cell and show that every $\mathcal{L}_\delta$-definable subset in a model of T^*_δ is the disjoint union of finitely many δ-cells [6].

Theorem 2 ([6])**.** *Suppose that T admits elimination of imaginaries in an expansion $\mathcal{L}^{\mathcal{G}}$ of $\mathcal{L}$. Then the theory T^*_δ admits elimination of imaginaries in $\mathcal{L}^{\mathcal{G}}_\delta$.*

The proof uses the open core property of T^*_δ and the properties of the dimension on $\mathcal{L}$-definable subsets X in models of T, in particular that $\dim(Fr(X)) < \dim(X)$.

3. Finite-dimensional definable groups

Let $X \subset K^n$ be an $\mathcal{L}_\delta(k)$-definable set, k a differential subfield of K. One may define a (well-behaved) δ-dimension [12] and when $\dim_\delta(X) = 0$, one defines $\operatorname{ord}(X) := \max\{ord(a/k) : a \in X\}$, where $\operatorname{ord}(a/k) := trdeg_k k\{a\}$, where $k\{a\} = k(\nabla_\infty(a))$ and $\nabla_\infty(a)$ is the infinite tuple consisting of successive derivatives of the tuple a.

Theorem 3 ([15])**.** *Let $\mathcal{K} \models T^*_\delta$ and assume that $\mathcal{K}$ is sufficiently saturated. Let $\mathfrak{G} := (\Gamma, f_\times, f_{-1}, e)$ be an $\mathcal{L}_\delta$-definable group in $\mathcal{K}$ (possibly with parameters) with $\Gamma \subset K^n$. Assume that $\dim_\delta(\mathfrak{G}) = 0$. Then a large subset of some linked triple for Γ is the domain of $\mathcal{L}$-definable group $\mathcal{G} := (G, F_\times, F_{-1})$, in which $\mathfrak{G}$ embeds with dense image and the group operation $F_\times$ (respectively F_{-1}) coincides with $f_\times$ (respectively f_{-1}) on differential tuples.*

Let $V \subset K^s$, $s > 0$, be a K-irreducible affine variety and denote by $\tau(V) \subset K^{2s}$ the prolongation (or torsor) of V. In particular we have for $a \in V$, then $\nabla(a) = (a, \delta(a)) \in \tau(V)$.

Let W be an $\mathcal{L}$-definable subset of $\mathcal{K}$ of the form $\bigcup_{i \in I} V_i \cap O_i$, where I is finite, O_i a definable open subset of K^s, $V_i \subset K^s$ a K-irreducible affine variety, equal to the Zariski closure of $V_i \cap O_i$. Then the prolongation of W, $\tau(W) := \bigcup_{i \in I} \tau(V_i) \cap (O_i \times K^s)$.

In models of T_δ, we will use the notion of a δ-compatible $\mathcal{L}(\varnothing)$-definable C^1-function as introduced by A. Fornasiero and E. Kaplan [10] for o-minimal expansions of the field of reals. We also need to extend this notion to functions whose domains are not open sets.

Definition 4 ([20])**.** Let f be a definable map from $D \subset K^m$ to K^n. Then f is a δ-compatible definable C^1-map if there are an open set $D \subset U \subset K^m$ and a δ-compatible C^1-definable map $F : U \to K^n$ such that $F \upharpoonright D = f$. Further for $x \in D$, we define $f^\delta(x)$ as $F^\delta(x)$. If the map F is C^2, the map $f^\delta : D \to K^m$ is again a definable C^1-map δ-compatible.

Denote by $(\text{IFTA})_{\mathcal{P}}$ the implicit function theorem restricted to polynomial functions. Recall that $\mathcal{K}$ is a $\aleph_1$-saturated (non-discrete) valued field, an implicit function theorem holds for power series $K[[x,y]]$ [1]; it relies on Weierstrass preparation theorem. In particular it shows that $\mathcal{K}$ satisfies $(\text{IFTA})_{\mathcal{P}}$.

Let $\mathcal{K}$ be a model of T_δ^* where $(\text{IFTA})_{\mathcal{P}}$ holds. Let f be an $\mathcal{L}$-definable function $f : K^n \rightrightarrows K^m$. Then f is a δ-compatible definable C^1-map.

Following [14], but replacing regular maps by δ-compatible definable C^1-maps, we extend definable maps on $\mathcal{L}_K$-definable sets to maps on their prolongations.

Let $\mathcal{G} := (G, \star, {}^{-1})$ be an $\mathcal{L}$-definable group and suppose there an $\mathcal{L}$-definable group section $s : G \to \tau(G)$, namely letting π the projection from $\tau(G)$ to G, we have $\pi(s(g)) = g$ for any $g \in G$. Then the pair $(\mathcal{G}, s)$ is called an $\mathcal{L}$-definable D-group.

Theorem 5. [15] *Let $\mathcal{K}$ be a model of T_δ^* where* $(\text{IFTA})_{\mathcal{P}}$ *holds. Let $\mathfrak{G}$ be an $\mathcal{L}_\delta$-definable group with* $\dim_\delta(\mathfrak{G}) = 0$. *Then there is an $\mathcal{L}$-definable D-group $(\mathcal{G}, s)$ and an $\mathcal{L}_\delta$-group embedding from $\mathfrak{G}$ to G whose image is $\{g \in G : s(g) = \nabla(g)\}$.*

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Generic derivations on topological fields

Antongiulio Fornasiero

Let K be a first-order structure expanding a field, in a language L. We assume that K is endowed with a definable topology (in the sense of [Pil87]), making it a topological field (more precisely, the topology should be a V-topology).

In this context, it makes sense to say that a definable set $X \subseteq K^n$ is a $\mathcal{C}^1$-manifold.

Definition 1. A *derivation* on K is a function $\delta : K \to K$ such that

$$\delta(x+y) = \delta x + \delta y$$
$$\delta(xy) = x\delta y + y\delta x.$$

We say that δ is *compatible* with the structure of K if the following happens:
For every $X \subseteq K^n$ such that X is a $\mathcal{C}^1$ manifold which is L-definable over the empty set, and for every $\bar{a} \in X$, $\delta\bar{a}$ is tangent to X in $\bar{a}$.

If T is the theory of K, we denote by T^δ the $L(\delta)$-theory expanding T by the axioms imposing that δ is a compatible derivation (and call any δ satisfying those axioms a T-derivation).

We will give some sufficient conditions on T for T^δ to have a model completion. Those conditions will include the case studied in [FK20] for compatible derivations on o-minimal structures; related questions were studied in [KP19] and [ST20] among other.

Definition 2. Let $X \subseteq K^n$. The *dimension* of X, denoted by $\dim(X)$, is the largest integer d such that, after some permutation of variables, $\Pi_d(X)$ has non-empty interior inside K^d, where $\Pi_d : K^n \to K^d$ is the projection onto the first d coordinates.

The main novelty is the following definition:

Definition 3. Let $f : K^n \rightharpoonup K^m$ be a partial function. We say that f is L∅d (locally definable without parameters) if:

- f is $L(K)$-definable;
- the domain of f is an open subset U of K^n;
- there exist an open set $V \subseteq K^m$ and a $\varnothing$-definable set X, such that

$$\Gamma(f) = X \cap (U \times V).$$

We call K a *smooth* structure (and its theory T a smooth theory) if it satisfies the following conditions:

(1) dim is a dimension function (in the sense of [vdD89]). We denote by cl the associated matroid, and by K^* the monster model.
(2) Let $a \in K^*$ and $\bar{b} \in K^{*n}$. If $a \in cl(\bar{b})$, then there exists a $\mathcal{C}^1$ L∅d partial function $f : K^{*n} \rightharpoonup K^*$ such that $a = f(\bar{b})$

Theorem 4. *If T is smooth and model complete, then T^δ has a model completion T^δ_g.*

An axiomatization of T^δ_g is given by the following axiom scheme:
For every $X \subseteq K^n \times K^n$ which is L-definable with parameters, if $\dim(\Pi_n(X)) = n$, then there exists $\bar{a} \in K^n$ such that $(\bar{a}, \delta\bar{a}) \in X$.

Theorem 5. *If K is smooth, then every set which is L-definable (with parameters) is a finite disjoint union of definable $\mathcal{C}^1$ manifolds.*

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Vector spaces with a dense-codense generic module

Alexander Berenstein

(joint work with Christian d'Elbée and Evgueni Vassiliev)

This talk brings together ideas of dense-codense expansions of geometric structures [1–3] with ideas about generic expansions by groups by D'Elbée [4].

The framework is the following. Suppose that $\mathbb{F}$ is a field of characteristic zero and that R is a subring of $\mathbb{F}$. Let $\mathcal{L}_0 = \{+, 0, \{\lambda\cdot\}_{\lambda\in\mathbb{F}}\}$ and let $\mathcal{L} \supset \mathcal{L}_0$ be an extension. Let T be an $\mathcal{L}$-theory expanding the theory of vector spaces over $\mathbb{F}$

which has quantifier elimination in $\mathcal{L}$ for which dcl = acl = $\text{span}_{\mathbb{F}}$ and such that it eliminates the quantifier $\exists^\infty$. We will denote $\hat{R} = Frac(R)$, the fraction field of R.

Let G be a unary predicate and for each formula $\phi(\vec{x})$ in the language $\mathcal{L}_{R-mod} = \{+, 0, (r\cdot)_{r\in R}\}$ of R-modules, let $P_\phi(\vec{x})$ be a new predicate. Let $\mathcal{L}_G$ be the expansion of $\mathcal{L}$ by G and P_ϕ for all formula ϕ in $\mathcal{L}_{R-mod}$. We will consider pairs (V, G) in the language $\mathcal{L}_G$ such that $V \models T$ and that also satisfy the following first order conditions:

(1) G is a proper R-submodule of the universe, and for all $\vec{a} \in V$, $P_\phi(\vec{a})$ holds if and only if $\vec{a} \in G$ and $G \models \phi(\vec{a})$ as an R-module.

(2) If $\lambda_1, \ldots, \lambda_n \in \mathbb{F}$ are $\hat{R}$-linearly independent, then for all $g_1, \ldots, g_n \in G$

$$\lambda_1 g_1 + \cdots + \lambda_n g_n = 0 \implies \bigwedge_i g_i = 0.$$

(3) (Density Property) for all $r \in R \setminus \{0\}$, rG is dense in the universe. This is a first order property that can be axiomatized through the scheme: for every $\mathcal{L}$-formula $\phi(x, \vec{y})$, add the sentence $\exists^\infty x \phi(x, \vec{y}) \to \exists x(\phi(x, \vec{y}) \wedge rG(x))$;

(4) (Extension/co-density property) for any $\mathcal{L}$-formulas $\phi(x, \vec{y})$ and $\psi(x, \vec{y}, \vec{z})$ and $n \geq 1$, the following sentence

$$(\exists^\infty x \phi(x, \vec{y}) \wedge \forall \vec{z} \exists^{\leq n} x \psi(x, \vec{y}, \vec{z})) \to \exists x(\phi(x, \vec{y}) \wedge \forall \vec{z}(G(\vec{z}) \to \neg\psi(x, \vec{y}, \vec{z}))).$$

We write T^G for the theory consisting of the models (V, G) satisfying the scheme $(1), (2), (3), (4)$ and let $(V, G) \models T^G$. We will say a tuple $\vec{a} \in V$ is *G-independent* if $dim(\vec{a}/G(V)) = dim(\vec{a}/G(V) \cap \vec{a})$, the idea being that all information that G provides about $\vec{a}$ is coded in $\vec{a} \cap G(V)$. With these ingredients we get then a good description of types:

Theorem 1. *Suppose (V, G), (W, G) are models of T^G, $\vec{a} \in V$ and $\vec{b} \in W$ are G-independent tuples, $\text{tp}_{\mathcal{L}}(\vec{a}, G(\vec{a})) = \text{tp}_{\mathcal{L}}(\vec{b}, G(\vec{b}))$ and $\text{tp}_{R-mod}(G(\vec{a})) = \text{tp}_{R-mod}(G(\vec{b}))$. Then $\text{tp}_{\mathcal{L}_G}(\vec{a}) = \text{tp}_{\mathcal{L}_G}(\vec{b})$.*

Since all module formulas can be reduced to boolean combinations of existential formulas, if T has quantifier elimination, we obtain as a corollary that the theory T^G is near model complete.

Using the result above to characterize the induced structure on G we can show our main preservation result:

Theorem 2. *Suppose $Th(V)$ has stable (respectively NIP, simple, NTP$_2$, NSOP$_1$) then so is $Th(V, G)$.*

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Expansions by r-regular sets of reals: the real additive group versus the real field

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(joint work with Jason Bell)

1. Introduction

A finite automaton is a machine with finitely many states, some subset of which are called accept states, with the remaining states being reject states. The machine takes finite-length strings over a finite alphabet Σ as inputs, and, beginning at a "start state" it moves from state to state based its transition function as it reads the string from left-to-right. The machine then either accepts or rejects a string depending upon whether or not it arrives in an accept state after it has finished reading the word.

Büchi automata differ from classical finite automata in that they take infinite-length strings over Σ as input. Though otherwise like finite automata, a Büchi automaton accepts a string $w \in \Sigma^\omega$ if a run of the automaton on input w enters some accepting state infinitely often. For both kinds of automata, we say that an automaton *recognizes* a set X (either a subset of Σ^* or, if it is a Büchi automaton, a subset of Σ^ω) if every element of X is accepted by the automaton, and every element of X^c, the complement, is not accepted.

Below, let $r \in \mathbb{N}$ be greater than one, and set $[r] := \{0, \dots, r-1\}$. We also let $[r]^\omega$ denote the set of all functions from the ordinal ω to the set $[r]$.

Definition 1. Say that $A \subseteq [0,1]$ is *r-regular* if there is a Büchi automaton $\mathcal{A}$ with alphabet $\{0, \dots, r-1\}$ that recognizes a set $L \subseteq \{0, \dots, r-1\}^\omega$ such that $(w_i)_{i<\omega} \in L$ if and only if there is $x \in A$ such that

$$x = \sum_{i=0}^{\infty} \frac{w_i}{r^{i+1}}.$$

Moreover, if this holds say that $\mathcal{A}$ *recognizes* A.

Definition 2. Say that $A \subseteq [0,1]$ is *r-sparse* if it is r-regular, the automaton $\mathcal{A}$ recognizes A, and the set of strings in $\{0, \dots, r-1\}^n$ with an infinite prolongation that is accepted by $\mathcal{A}$ grows at most polynomially in n.

Below, set $\mathcal{R}_A := (\mathbb{R}, <, +, 0, 1, A)$, and set $\mathcal{R}_{r,\ell} := (\mathbb{R}, <, +, 0, 1, r^{-\ell\mathbb{N}})$.

Theorem 3. *Let $r > 1$ be a natural number, and suppose $A \subseteq [0,1]$ is r-sparse. Then A is $\varnothing$-definable in $\mathcal{R}_{r,\ell}$, and the set $r^{-\mathbb{N}}$ is $\varnothing$-definable in $\mathcal{R}_A$.*

Conversely, we also show the following:

Theorem 4. *If $A \subseteq [0,1]$ is r-sparse, the structures $\mathcal{R}_A$ and $\mathcal{R}_{r,1}$ define the same sets.*

Recall that $X \subseteq \mathbb{R}^d$ is called a *Cantor set* if it is compact, has not isolated points, and no interior. From d-minimality, we can conclude that for an r-sparse set A the structure $\mathcal{R}_A$ does not define a Cantor set. Below, for $X \subseteq \mathbb{R}$ let $d_H(X)$ denote the Hausdorff dimension of X.

Theorem 5. *If A is a closed r-regular subset of $[0,1]$ such that $0 < d_H(A) < 1$, then there is a Cantor set definable in $\mathcal{R}_A$.*

The above theorem demonstrates the connection of fractal dimension, namely Hausdorff dimension, to definability of a more "pathological" set, in this case a Cantor set. The definition of a "Cantor set" here is a compact subset of $\mathbb{R}$ that has no isolated points and no interior. What makes a Cantor set "pathological" in this setting is that work of Hieronymi and Walsberg in [1] shows that the expansion of $(\mathbb{R}, <, +, 0)$ by a Cantor set is not model-theoretically tame. Connecting their work the theorem above, we obtain a theorem that solidifies the notion that for these structures, tameness in the model-theoretic sense and tameness in the sense of fractal geometry completely coincide.

Theorem 6. *For $A \subseteq [0,1]^d$ an r-regular set such that $d_H(\overline{\pi_i(A)}) < 1$ for all $i \in [d]$, the following are equivalent:*

(1) A is r-sparse;
(2) $d_H(\overline{\pi_i(A)}) = 0$ for all $i \in [d]$;
(3) $\mathcal{R}_A$ is d-minimal;
(4) $\mathcal{R}_A$ has NIP;
(5) $\mathcal{R}_A$ has NTP_2.

Note that in this final theorem, we have dropped the assumption that A is closed, illustrating how well the topological closures of these r-regular sets reflect or control the behavior of even the non-closed r-regular sets.

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Monotone T-convex T-differential fields

Elliot Kaplan

(joint work with Nigel Pynn-Coates)

Let T be a complete, model complete o-minimal theory extending the theory RCF of real closed ordered fields in an appropriate language $\mathcal{L} \supseteq \{0, 1, +, \cdot, -, <\}$. We assume that T is **polynomially bounded**: for every definable unary function f, there is some $n \in \mathbb{N}$ such that $|f(x)| < x^n$ for all sufficiently large x. All of the theorems below also hold for power bounded T.

Let $K \models T$. A T**-derivation** on K, introduced by Fornasiero and the first author [5], is a map $\partial \colon K \to K$ which satisfies the identity

$$\partial f(u) \;=\; \frac{\partial f}{\partial Y_1}(u)\partial u_1 + \cdots + \frac{\partial f}{\partial Y_n}(u)\partial u_n.$$

for all $u = (u_1, \dots, u_n) \in K^n$ and all $\mathcal{L}(\varnothing)$-definable functions f which are $\mathcal{C}^1$ at u. Let $K = (K, \partial)$ be a *T-differential field*, that is, as a model of T equipped with a T-derivation. We say that ∂ is **generic** if for all $\mathcal{L}(K)$-definable functions $f: U \to K$ with $U \subseteq K^n$ open, there is $a \in K$ with $(a, a', \dots, a^{(n-1)}) \in U$ and $a^{(n)} = f(a, \dots, a^{(n-1)})$, where $a^{(n)} = \partial^n(a)$. The theory of generic T-differential fields is the model completion of the theory of T-differential fields [5]. Note that any generic derivation is **linearly surjective**: for any $a_0, \dots, a_r \in K$, not all zero, there is $y \in K$ with $a_0 y + a_1 y' + \dots + a_r y^{(r)} = 1$.

A **T-convex valuation ring**, defined by van den Dries and Lewenberg [2], is a convex subset $\mathcal{O} \subseteq K$ such that $f(\mathcal{O}) \subseteq \mathcal{O}$ for all continuous $\mathcal{L}(\varnothing)$-definable functions $f: K \to K$. Suppose that K is equipped with both a T-convex valuation ring $\mathcal{O}$ and a T-derivation ∂, and suppose in addition that K has **small derivation**: the unique maximal ideal $\smallo$ of $\mathcal{O}$ is closed under ∂. Then we call K a **T-convex T-differential field**. We may naturally view the residue field $\operatorname{res}(K) := \mathcal{O}/\smallo$ as a T-differential field; See [2, Remark 2.16] and [10, Section 3]. Any T-derivation is a derivation and any T-convex valuation ring is a convex valuation ring; the converse holds if $T = \mathrm{RCF}$.

Example 1. Suppose $\mathbb{R}$ admits an expansion to a model $\mathcal{R} \models T$. An **$\mathcal{R}$-Hardy field** is a Hardy field $\mathcal{H}$ which is closed under all $\mathcal{R}$-definable functions; see [3, Section 5]. The subring of germs of bounded functions is a T-convex valuation ring of $\mathcal{H}$, and the usual Hardy field derivation is a T-derivation. The maximal ideal (the germs of functions which tend to zero) is closed under this derivation, so $\mathcal{H}$ is a T-convex T-differential field.

Example 2. Let T_{an} be the elementary theory of $\mathbb{R}_{\mathrm{an}}$, the expansion of $\mathbb{R}$ by all functions which are real analytic on a neighborhood of $[-1,1]^n$, restricted to $[-1,1]^n$. This theory is model complete, o-minimal, and polynomially bounded [1, 6]. The following are examples of T_{an}-convex T_{an}-differential fields:

(1) The field of Puiseux series $\bigcup_{n>0} \mathbb{R}((t^{1/n}))$, ordered so that $0 < t < \mathbb{R}^{>}$. This field admits a canonical expansion to a model of T_{an} where each restricted analytic function is interpreted via Taylor expansion. The convex hull of $\mathbb{R}$ is a T_{an}-convex valuation ring. Put $x := 1/t$. Then the derivation "with respect to x", which maps $\sum_q r_q x^q$ to $\sum_q r_q q x^{q-1}$, is a T_{an}-derivation.
(2) The field $\mathbb{T}$ of logarithmic-exponential transseries also admits a canonical expansion to a model of T_{an} using Taylor expansion [4, Corollary 2.8]. The convex hull of $\mathbb{R}$ in $\mathbb{T}$ is T_{an}-convex and the usual derivation on $\mathbb{T}$ is a T_{an}-derivation.
(3) Let $\boldsymbol{k} = (\boldsymbol{k}, \partial_{\boldsymbol{k}})$ be a T_{an}-differential field and let Γ be a divisible ordered abelian group. Consider the Hahn field $\boldsymbol{k}[[t^\Gamma]]$, ordered so that $0 < t < \boldsymbol{k}^{>}$. We expand $\boldsymbol{k}[[t^\Gamma]]$ to a model of T_{an} using Taylor expansion (see [9, Proposition 2.13] for details), and we let $\mathcal{O}$ be the convex hull of $\boldsymbol{k}$, so $\mathcal{O}$ is T_{an}-convex. Let $c: \Gamma \to \boldsymbol{k}$ be an additive map. We use c to define a T_{an}-derivation ∂ on $\boldsymbol{k}[[t^\Gamma]]$ as follows:
$$\partial \sum_\gamma f_\gamma t^\gamma \; := \; \sum_\gamma (\partial_{\boldsymbol{k}} f_\gamma + f_\gamma c(\gamma)) t^\gamma.$$

This derivation is the unique T_{an}-derivation which extends $\partial_{\boldsymbol{k}}$, commutes with infinite sums, and satisfies the identity $\partial t^\gamma = c(\gamma)t^\gamma$ for each $\gamma \in \Gamma$. Let us denote this expansion of $\boldsymbol{k}[[t^\Gamma]]$ by $\boldsymbol{k}[[t^\Gamma]]_{\mathrm{an},c}$.

A T-convex T-differential field K is said to be **monotone** if $\partial a/a \in \mathcal{O}$ for each $a \in K$. Note that monotonicity implies small derivation. While some $\mathcal{R}$-Hardy fields are monotone, any maximal $\mathcal{R}$-hardy field is not. The field of Puiseux series and the Hahn series fields $\boldsymbol{k}[[t^\Gamma]]_{\mathrm{an},c}$ in Example 2 are monotone, but $\mathbb{T}$ is not.

Theorem 3. *The theory of monotone T-convex T-differential fields has a model completion. This model completion is complete, and it has quantifier elimination so long as T has quantifier elimination and a universal axiomatization. A monotone T-convex T-differential field K is a model of this model companion if:*

(1) K is nontrivially valued: $\mathcal{O} \neq K$.
(2) The induced derivation on the residue field $\mathrm{res}(K)$ *is* ***generic****.*
*(3) K is T^∂**-henselian**: the induced derivation on* $\mathrm{res}(K)$ *is linearly surjective and for every $\mathcal{L}(K)$-definable function $f\colon K^r \to K$, if $a \in K$ is an approximate zero of the function $y \mapsto y^{(r)} - f(y, \ldots, y^{(r-1)})$, and if this function is well-approximated by a linear differential operator on a neighborhood of a, then it has an actual zero in this neighborhood.*

Defining precisely what an "approximate zero" and a "well-approximated function" are in the definition of T^∂-henselianity is a bit beyond the scope of this extended abstract. It is worth mentioning that our definition is inspired by Rideau's definition of *σ-henselianity* for analytic difference valued fields [11].

Underpinning Theorem 3 is an Ax-Kochen/Ershov theorem for monotone T^∂-henselian T-convex T-differential fields. To formalize this AKE theorem, we recall that the **the field of exponents** of T is, roughly speaking, the subfield $\Lambda \subseteq K$ consisting of those $\lambda \in K$ for which the map $x \mapsto x^\lambda\colon K^> \to K^>$ is $\mathcal{L}(\varnothing)$-definable. For example, the field of exponents of T_{an} is $\mathbb{Q}$ [1]. We consider 3-sorted structures $\mathcal{K} = (K, \boldsymbol{k}, \Gamma; \pi, v, c)$ where

(1) K and $\boldsymbol{k}$ are T-differential fields and Γ is an ordered Λ-vector space.
(2) $v\colon K^\times \to \Gamma$ is a surjective valuation, and the corresponding valuation ring $\mathcal{O} \coloneqq \{a \in K : a = 0 \text{ or } v(a) \geq 0\}$ is T-convex.
(3) The derivation on K is monotone, and $\pi\colon \mathcal{O} \to \boldsymbol{k}$ induces a T-differential field isomorphism $\mathrm{res}(K) \to \boldsymbol{k}$.
(4) $c\colon \Gamma \to \boldsymbol{k}$ is Λ-linear, and for all $\gamma \in \Gamma$ there is $a \in K^>$ with $va = \gamma$ and $\pi(a^\dagger) = c(\gamma)$.

Any monotone T-convex T-differential field K admits an expansion of this form: take $\boldsymbol{k}$ and Γ to be the differential residue field and value group of K, take π and v to be the natural quotient maps, and for $\gamma \in \Gamma$, set $c(\gamma) \coloneqq \pi(s(\gamma)^\dagger)$ where $s\colon \Gamma \to K^>$ is a Λ-linear section of v.

Theorem 4. *Suppose $\mathcal{K}$ and $\mathcal{K}^*$ are T^∂-henselian. If $(\boldsymbol{k}, \Gamma; c) \equiv (\boldsymbol{k}^*, \Gamma^*; c^*)$, then $\mathcal{K} \equiv \mathcal{K}^*$. If $\mathcal{K} \subseteq \mathcal{K}^*$ and $(\boldsymbol{k}, \Gamma; c) \preccurlyeq (\boldsymbol{k}^*, \Gamma^*; c^*)$, then $\mathcal{K} \preccurlyeq \mathcal{K}^*$.*

This theorem is closely related to earlier work of Scanlon [12] and Hakobyan [8]. The proof relies on the following key ingredients.

(A) Lifting the differential residue field: If K is T^{∂}-henselian, then K contains a T-differential subfield $E \subseteq \mathcal{O}$ such that the restriction of the residue map to E is a T-differential field isomorphism onto $\operatorname{res}(K)$.

(B) Uniqueness of spherical completions: If K is monotone and $\operatorname{res}(K)$ is linearly surjective, then K has a *unique* spherically complete immediate T-convex T-differential field extension. The existence of a spherically complete immediate T-convex T-differential field extension, even without the assumption of monotonicity, was shown by the first author [10].

The proofs of (A) and (B) use the *Jacobian property* in an essential way. This property, shown to hold for models of T by García Ramírez [7], tells us that $\mathcal{L}(K)$-definable functions are locally well-approximated by linear functions. We can use Theorem 4 to completely characterize the $T^{\partial}_{\mathrm{an}}$-henselian monotone T_{an}-convex T_{an}-differential fields up to elementary equivalence:

Corollary 5. *Let K be a monotone T_{an}-convex T_{an}-differential field with differential residue field $\boldsymbol{k}$ and value group Γ. If K is $T^{\partial}_{\mathrm{an}}$-henselian, then K is elementarily equivalent to some Hahn field model $\boldsymbol{k}[[t^{\Gamma}]]_{\mathrm{an},c}$.*

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Forking degree and the Borovik-Cherlin conjecture

James Freitag

1. The Borovik-Cherlin conjecture

Given a definable set X, how transitively can a definable group action on X be? The Borovik-Cherlin conjecture proposes an answer to this question in the setting of finite Morley rank for the notion of generic k-transitivity. A definable group action (G, X) is *generically k-transitive* if the induced diagonal action of G on X^k has an orbit $\mathcal{O}$ such that $X^k \setminus \mathcal{O}$ has Morley rank less than the Morley rank of X^k. The Borovik-Cherlin conjecture predicts that if G acts definably and generically k-transitively on X, then $k \leq RM(X) + 2$, and if equality holds, then the action is definably isomorphic to the action of PGL_n on $\mathbb{P}^{n-1}$.

Generic k-transitivity weakens the classical notion of k-transitivity by replacing the condition that there be an orbit $\mathcal{O}$ in X^k which is everything except the generalized diagonals to asking that there is a *large* orbit. Of course, many variations are possible. For instance, in the o-minimal context, replacing Morley rank by o-minimal dimension, the analog of the Borovik-Cherlin conjecture was established by [10].[1] One can formulate the problem the problem directly in the Kolchin topology (where large means Kolchin dense). In the finite rank case for the Kolchin topology, the analog of Borovik-Cherlin has been solved by work in progress of Freitag, Jimenez, and Moosa.

In a series of recent works, [6–8], the relationship between transitivity of *binding groups* and *forking* was exploited. More general versions of this work are expected to require new cases (e.g. outlined above) of Borovik-Cherlin type problems.

2. The forking degree

Let T be a superstable theory, and $p(x)$ be a complete type over A relative to the theory T with $RU(p) > \alpha$. Then we define the *α-forking degree of p* to be the least n such that there is a Morley sequence $a_1, \dots, a_n$ in the type of p such that p has a forking extension q over $A \cup \{a_1, \dots, a_n\}$ with $RU(q) \geq \alpha$. We denote the α-forking degree of p by $F_\alpha(p)$.

In a series of recent works, the special case when $\alpha = 1$ was considered, in which case the notion is called the *degree of nonminimality*. In [8], under mild assumptions on the theory, the degree of nonminimality was shown to be bounded by $RU(p) + 2$, assuming the truth of the Borovik-Cherlin conjecture. Additionally, the existence of *some bound* depending only on the rank of the type p, rather than the type itself was shown to follow from work of Borovik and Cherlin [1]. The Borovik-Cherlin conjecture was verified for algebraic groups of characteristic zero, and the case was shown to be sufficient to obtain the bound unconditionally for types in the theory of differentially closed field of charactistic zero and compact complex manifolds. In [7] ideas from [8] to show transcendence results for solutions of a differential equation and families of compact complex manifolds. In

[1] Interestingly, [10] predates the Borovik-Cherlin conjecture.

[6], the bounds for differentially closed fields and compact complex manifolds were improved to show the degree of nonminimality is at most two. When the type is defined over the constants, the degree of nonminimality was shown to be at most one in these theories.

The original central motivation for investigating the degree of nonminimality was to develop a general purpose method for establishing the minimality of differential equations. In that direction, [5] resolved a long-standing problem of Poizat to show that generic differential equations of sufficiently high degree are strongly minimal. In [9], a new proof of the strong minimality of various Painlevé equations was given. In [3], a new short proof of the strong minimality of the equations satisfied by automorphic functions of triangle groups is given. In each of these applications, the degree of nonminimality plays a crucial role.

3. Problems and questions

We hope the following questions and problems give significant interesting directions for future work around forking degree and variants of the Borovik-Cherlin conjecture.

(1) Can one control α-forking degree for arbitrary (perhaps infinite) ordinals using an appropriate version of the Borovik-Cherlin conjecture?

(2) Generalizations of generic k-transitivity have been studied in the context of algebraic group actions. For instance, [4] studied actions of PGL which have a dense orbit in products of (not necessarily the same) Grassmannians, while [11] studied when the action of an algebraic group has *finitely many orbits,* a stronger condition than generic transitivity. Are there natural model theoretic applications for these stronger or more general variants of the Borovik-Cherlin setup?

(3) Formulate and prove a version of the Borovik-Cherlin conjecture in the context of supersimple groups and apply the results via the theory $ACFA$ to functional transcendence results for solutions of difference equations.

(4) In [6–8], the functional transcendence applications always allowed for the reduction to the case in which any binding group associated with the bounds could be assumed to be *definably primitive.* Bounds for the sizes of bases (see e.g. [2]) are naturally associated with internality problems, where sometimes taking the definable quotients which allow for reduction to the primitive case is a more delicate matter. Can definable primitivity of the binding group be related to currently studied notions from geometric stability theory, differential algebraic geometry or some notion from the theory of compact complex manifolds?

(5) Formulate and prove versions of Borovik-Cherlin for Kolchin polynomials in various settings. Borovik and León-Sánchez conjecture (private communication) that if X is definable in $DCF_{0,m}$ and G acts definably with generically n-transitive action then n is at most $M+2$ with M the sum of the absolute value of the coefficients of the Kolchin polynomial of X.

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Some well behaved classes of differential fields

Omar León Sánchez

We let $\mathcal{L}_\delta$ denote the language of differential fields. Recall the following $\mathcal{L}_\delta$-axioms for the theory DCF_0 of differentially closed fields in characteristic zero (due to L. Blum).

Theorem 1. *Let (K,δ) be a differential field of characteristic zero. Then, the following are equivalent:*

(1) (K,δ) is differentially closed
(2) for every pair of nonzero differential polynomials $f,g \in K\{x\}$ with $ord(g) < ord(f)$ there exists $a \in K$ with $f(a)=0$ and $g(a)\neq 0$.

In this presentation we consider two modifications of these axioms: one yields the class of differentially large fields in characteristic zero, and the second moves in the direction of positive characteristic introducing the class of separably differentially closed fields.

1. Differentially large fields

In this section all fields are of chacteristic zero. Recall that a field K is large if every K-irreducible algebraic variety with a smooth K-point has a Zariski-dense set of K-points; equivalently, K is existentially closed (e.c.) in the Laurent series field $K((t))$.

For a differential polynomial $f \in K\{x\}$, we denote by f^* the (algebraic) polynomial in $ord(f)+1$ variables obtained by symply viewing f in the variables $x, \delta x, \delta^2 x, \ldots$. Also, recall that s_f denotes the separant of f; namely, the partial derivative of f with respect to $\delta^n x$ where $n = ord(f)$.

Consider the following first-order conditions:
(B_0) for every pair of nonzero differential polynomials $f, g \in K\{x\}$ with $ord(g) < ord(f)$, if the algebraic system

$$f^* = 0, \quad g^* \neq 0, \quad s_f^* \neq 0$$

has a solution, then there exists $a \in K$ with $f(a) = 0$ and $g(a) \neq 0$.

The following result describes the models of the theory B_0 (in the case the field involved is large).

Theorem 2 (Leon Sanchez and Tressl, 2019)**.** *Let K be a large field of characteristic zero. Then, the following are equivalent*

(1) $(K, \delta) \models B_0$
(2) for every differential field extension (L, δ), if K is e.c. in L (as fields) then (K, δ) is e.c. in (L, δ) (as differential fields).
(3) (K, δ) is e.c. in $(K((t)), \delta)$, where the derivation in $K((t))$ is the unique extension of δ that commutes with meaningful sums and $\delta(t) = 1$.
(4) for every K-irreducible D-variety (V, s), if V has a K-point, then $(V, s)^\#$ has a K-point.
(5) for every K-irreducible D-variety (V, s), if V has a smooth K-point, then $(V, s)^\#$ has Zariski dense many K-points.

A large field (of characteristic zero) equipped with a derivation satisfying any of the equivalent conditions of Theorem 2 is said to be *differentially large*. For further details on differentially large fields, in particular explicit constructions of them, the reader is advised to consult [1].

The theory B_0 does have some model-theoretic transfer properties... let us be more precise. Recall that the theories ACF_0, RFC, p-CF, $\mathrm{Psf}_0(\mathrm{c})$ are all examples of model-complete theories of large fields. Now let T be an arbitrary model-complete theory of large fields of characteristic zero (in the language of fields possibly extended by some constant symbols, like in the case $\mathrm{Psf}_0(\mathrm{c})$).

Theorem 3.

(1) $T \cup B_0$ is model-complete (in the language $\mathcal{L}_\delta$).
(2) If T has q.e. (in some expansion by definitions), then $T \cup B_0$ has q.e. (in the same expansion).
(3) If T is stable, then $T \cup B_0$ is DCF_0.
(4) If T has NIP, then $T \cup B_0$ has NIP. In particular, CODF has NIP, as we know.

Here are some interesting questions:

(1) Is there a transfer principle for NSOP or NTP? How about elimination of imaginaries?

(2) Suppose (L,δ) is an extension of (K,δ). Also, assume K and L are large, L/K is a finite extension, and (L,δ) is differentially large. Does it follow that (K,δ) is differentially large?

2. Separably differentially closed fields

In this section all fields are of characteristic $p > 0$. Consider now the following modification of Blum's axioms:
(B_1) for every pair of nonzero differential polynomials $f, g \in K\{x\}$ with $ord(g) < ord(f)$ and $s_f \neq 0$ there exists $a \in K$ with $f(a) = 0$ and $g(a) \neq 0$.

If to B_1 we add the first-order condition $C_K = K^p$ (where C_K denotes the δ-constants of (K,δ)), we recover the axioms of C. Wood of DCF_p differentially closed fields of characteristic $p > 0$. However, without the axiom $C_K = K^p$, the class of models of B_1 contains more than differentially closed fields. What the next result shows is that this new class is the differential analogue of separable closed fields.

Theorem 4 (Ino and Leon Sanchez)**.** *Let (K,δ) be a differential field of characteristic $p > 0$. Then, the following are equivalent:*

(1) $(K,\delta) \vDash B_1$
(2) for every extension (L,δ), if L/K is separable (as fields) then (K,δ) is e.c. in (L,δ) (as differential fields).
(3) (K,δ) is constrainedly closed (in the sense of Kolchin in positive characteristic).

A differential field satisfying any of the conditions of Theorem 4 is called *separably differentially closed*. By condition (1) they form an elementary class and the theory is denoted SDCF_p.

As in the algebraic case (where the theory SCF_p is not complete), the theory SDCF_p is not complete. What we need to specify is the what we call the *differential degree of imperfection*. Namely, for each differential field (K,δ) there is a unique $\epsilon \in \mathbb{N}_0 \cup \{\infty\}$ such that

$$[C_K : K^p] = p^\epsilon.$$

The number ϵ is called the differential degree of imperfection of (K,δ). After adding a sentence specifying the differential degree of imperfection, we denote the theory by $\mathrm{SDCF}_{p,\epsilon}$.

Theorem 5. *For each $\epsilon \in \mathbb{N}_0 \cup \{\infty\}$, the theory $SDCF_{p,\epsilon}$ is consistent and complete.*

We note that $\mathrm{SDCF}_{p,0}$ has the same models as DCF_p.

There is a natural language in which the theory $\mathrm{SDCF}_{p,\epsilon}$ has quantifier elimination – using the differential-lambda functions – . For a differential field (K,δ) a subset $A \subseteq K$ is said to be differentially p-independent if $A \subseteq C_K$ and the p-monomials of A are linearly independent over K^p. For each $n \in \mathbb{N}$, define the differential-lambda functions $\ell_{n,i} : K^n \times K \to K$ for $i = 0, \dots, p^n - 1$ as follows: for $(\bar{a}, b) \in K^n \times K$, $\ell_{n,i}(\bar{a}, b) = 0$ if one of the entries of $(\bar{a}, b)$ is not in C_K or if $\bar{a}$ is not differentially p-independent or if $(\bar{a}, b)$ is differentially p-independent; otherwise, they are uniquely defined by

$$b = (\ell_{n,0}(\bar{a}, b))^p\, m_0(\bar{a}) + \cdots + (\ell_{n,s}(\bar{a}, b))^p\, m_s(\bar{a})$$

where the $m_i(\bar{a})$'s are the p-monomials of $(a_1, \ldots, a_n)$ and $s = p^n - 1$.

After expanding the language by the differential-lambda functions and writing axioms expressing the above defining properties, we denote this theory by $\mathrm{SDCF}^{\ell}_{p,\epsilon}$.

Theorem 6. *The theory $SDCF^{\ell}_{p,\epsilon}$ admits quantifier elimination, is stable (but not superstable), and prime model extension exist and are unique (up to isomorphism).*

This is the starting point of an investigation of separably differentially closed fields, there are several direction of exploration. For instance,

(1) What other model theoretic properties does SDCF have? (taking the cue from SCF)?
(2) Can we describe types of $\mathrm{SDCF}^{\ell}_{p,\epsilon}$ in terms of separable ideals in some differential polynomial ring? Is there a form of Zilber's dichotomy in this context?

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Groups of automorphisms of saturated models are simple

Zoé Chatzidakis
(joint work with T. Blossier, C. Hardouin, A. Martin-Pizarro)

1. Introduction

D. Lascar proved in 1995 a very striking and surprising result: $\mathrm{Aut}(\mathbb{C}/\mathbb{Q}^{alg})$ is simple ([6]). It was actually the continuation of an earlier paper (1992, [5]), on automorphism groups of countable saturated strongly minimal structures, and where the result was announced assuming $\aleph_1 = 2^{\aleph_0}$. The proof given in the 1992 paper used very much topology (Polish group, Baire category), the proof in the 1995 paper was much more combinatorial.

Other similar results were proved by K. Tent and M. Ziegler on the isometry group of the Urysohn space (2013, [7]). (Simplicity of that group modulo the normal subgroup of bounded isometries). R. Konnerth (2002, [4]) extended Lascar's result to the automorphism group of any uncountable saturated differentially closed fields $\mathcal{U}$ of characteristic 0 which fix all elements of $\mathcal{U}$ which are differentially algebraic over $\mathbb{Q}$. (I.e., what we call below $\mathrm{cl}(\varnothing)$).

We wanted to extract from the proof of Lascar and of Konnerth what makes things works, and how this can be used to extend the existing results to other fields with operators. This is done by listing several fundamental properties, and proving a few lemmas. It also builds on existing work by four authors (in various combinations).

2. Fields with operators

Our theory T is a complete theory of fields (in the language $\{+,\cdot,0,1,^{-1}\}$, with a good notion of dimension or rank, and with some operators, i.e., in the language we have functions $K^n \to K$, which satisfy certain properties. Here are the structures we will discuss more in detail:

(1) The theory of algebraically closed fields of a given characteristic. No operator. ACF_p with $p = 0$ or p a prime.
(2) The theory of differentially closed fields of characteristic 0, one or several commuting derivations are the operators. DCF_0, $\mathrm{DCF}_{0,m}$.
(3) The theory of existentially closed difference fields of characteristic 0, with prescribed action of the automorphism σ on $\bar{\mathbb{Q}}$. The operators are σ and σ^{-1}. ACFA.
(4) Separably closed fields, together with the λ-functions as operators. I did discuss that example in more details, and why our results might or might not apply.

We use very much the existence (and uniqueness) of the generic type of the additive group, its precise description in the four examples. The generics of examples 1-3 are regular, but not those of example 4. There are good notions of basis in examples (1-3) and (4b).

We use these types to define a notion of closure, denoted cl. This notion, in contrast with algebraic closure, depends on the ambient model, and the closure of $\varnothing$ may be uncountable. In examples 1-3 it is easy to describe: $\mathrm{cl}(A)$ is the set of elements which satisfy some non-trivial algebraic/differential/difference equation with coefficients in the structure generated by A (-/difference/differential field). In example 4b, it is the maximal subfield of the ambient model which does not increase the p-basis. In example 4a, it is harder to describe, and to tell the truth, we do not have a nice description of it. The non-regularity of the generic is what poses problem.

The theories of examples 1,2 and 4 are stable, so that uncountably saturated models exist (with some restriction on the cardinalities in example 4). The theories of example 3 are however unstable, with IP and the maximal number of types over any set. However, essentially the only source of instability comes from the fixed field, and under suitable saturation hypotheses on the fixed field of the algebraically closed difference field K, the results of Shelah on existence and uniqueness of κ-prime models over K do extend ([3]).

3. The result

Theorem 1. *Let T be one of the theories (1-3), M a model of T, and $\kappa \geq \aleph_1$. Assume that M is κ-prime over $A := \mathrm{cl}_M(\varnothing)$. Then $\mathrm{Aut}(M/A)$ is simple.*

In particular we have:

Corollary 2. *Let T be one of the theories (1-3), M an uncountable model of T which is saturated. Then $\mathrm{Aut}(M/\mathrm{cl_M}(\varnothing))$ is simple.*

A notion playing an important role in the proof is that of *unbounded* automorphism. Lascar [6], then Blossier, Hardouin and Martin-Pizarro [1], show that the only bounded automorphism of a κ-saturated model M are the identity and powers of the Frobenius; so the only bounded automorphism which fixes $\mathrm{cl}(M)$ is the identity.

Both results are then direct consequences of

Theorem 3. *Let $\mathcal{U}$ be κ-prime over $\mathrm{cl}_{\mathcal{U}}(\varnothing)$, with T as in (1-3). Let $\tau \in \mathrm{Aut}(\mathcal{U}/\mathrm{cl}(\varnothing))$ be unbounded. Then every $\nu \in \mathrm{Aut}(\mathcal{U}/\mathrm{cl}(\varnothing))$ can be written as the product of four conjugates of τ and τ^{-1}.*

Question. I believe that Theorem 3 should extend to the theory of separably closed fields with infinite degree of imperfection, with countably many new constants for p-independent elements. The stability of the theory should help, as will the fact that any algebraically closed set is a model (so that we don't have problems with imaginaries). However the non-regularity of the generic type might be challenging.

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Around Zilber's quasiminimality conjecture

Jonathan Kirby

This is an extended abstract for a survey talk given in Oberwolfach on 1st December 2022.

About 25 years ago, Zilber stated:

Conjecture 1. [Zil97] *The complex field with the exponential function, $\langle \mathbb{C};+,\cdot,\exp\rangle$, is quasiminimal (QM): every definable subset is countable or co-countable. (Definable here means definable with parameters.)*

This conjecture has sparked a lot of activity over that time. For example, Zilber's part of the Zilber-Pink conjecture and the related work on functional transcendence came out of his early work towards the quasiminimality conjecture. Recently there has been significant progress towards proving the conjecture itself.

In this talk I surveyed some of the work around the conjecture, including the recent result of Gallinaro and myself that the complex field equipped with complex power functions is quasiminimal.

Theorem 2 (Gallinaro, Kirby, forthcoming)**.**
For $\lambda \in \mathbb{C}$*, let* $\Gamma_\lambda = \{(\exp(z), \exp(\lambda z)) \mid z \in \mathbb{C}\}$*, the graph of the multivalued map* $w \mapsto w^\lambda$*. Then the structure*

$$\langle \mathbb{C}; +, \cdot, -, 0, 1, (\Gamma_\lambda)_{\lambda \in \mathbb{C}} \rangle$$

is quasiminimal.

1. Quasiminimality

The complex field $\mathbb{C}_{\text{field}}$ is well-known to be *minimal*: every definable subset of $\mathbb{C}$ is finite or co-finite, and indeed *strongly minimal*: the same is true of every model of its first-order theory, which is ACF_0.

It is well-known that strongly minimal theories are uncountably categorical, (model-theoretic) algebraic closure is a pregeometry, and all models are prime over a basis of the pregeometry.

Quasiminimality as defined in Conjecture 1 corresponds to minimality. There is a stronger property:

Definition 3. An uncountable structure $\mathcal{M}$ is *quasiminimal in the sense of automorphisms* ($\mathrm{QM}_{\mathrm{Aut}}$) if for all reducts $\mathcal{M}_0$ of $\mathcal{M}$ to countable languages, and all countable subsets A of (the underlying set of) $\mathcal{M}_0$, there is a co-countable orbit of $\mathrm{Aut}(\mathcal{M}_0/A)$.

This still corresponds to minimality, not strong minimality. There is a yet stronger property, called Quasiminimal Excellence (QME), defined by Zilber [Zil05a], improved in [Kir10] and [Bal09] and then substantially simplified in [BHH+14]. I refer the reader to the last of these for the simplest definition. Putting the work of these papers together we get

Theorem 4. *If* $\mathcal{M}$ *is uncountable and QME then its* $L_{\omega,\omega_1}(Q)$*-theory is uncountably categorical, the "countable closure" operator is a pregeometry, and models of that theory are prime over bases of this pregeometry.*

So we have QME $\implies$ $\mathrm{QM}_{\mathrm{Aut}}$ $\implies$ QM. A natural question was asked during the Oberwolfach meeting:

Question 5. Suppose $|\mathcal{M}| \geqslant \aleph_2$ and $\mathcal{M}$ is QM. Must it be QME?

A counterexample for $|\mathcal{M}| = \aleph_1$ is given by the dense linear order $\omega_1 \times_{\text{lex}} \mathbb{Q}_<$, which is $\mathrm{QM}_{\mathrm{Aut}}$ but not QME. This structure is approximated by countable substructures in a fundamentally different way, via the linear order ω_1, whereas in the excellent case one approximates by the directed partial order of all countable subsets of some uncountable basis. Work of Pillay and Tanovic [PT11] may be relevant here.

2. Variant conjectures

In Zilber's conjecture, one can replace the complex exponential function with a Weierstrass $\wp$-function, or equivalently by the exponential map of an elliptic curve. More generally, one could consider the exponential map of an abelian variety, or a semiabelian variety, or indeed any commutative complex algebraic group.

One could even consider the exponential maps of all such groups together in one structure. All these conjectures seem fairly equally plausible and equally difficult. The Ax–Schanuel theorem [Ax71, Ax72] gives a strong structural result in all these cases.

While we now have an Ax–Schanuel theorem for the modular j-function [PT16], the domain of that is the upper half-plane so $\langle \mathbb{C}; +, \cdot, j \rangle$ and similar examples are far from quasiminimal.

Koiran asked if the expansion of the complex field by all unary entire complex functions is QM, and Wilkie noted that we have no counterexample. This appears to be a much harder conjecture than the group case.

The work of Fatou and Bieberbach from the 1920s mentioned in [Zil97] gives examples of holomorphic $f : \mathbb{C}^2 \to \mathbb{C}^2$ whose image is open but not dense, and they cannot be definable in a QM structure.

3. Some results

It is immediate that $\mathbb{C}_{\mathbb{Z}} \coloneqq \langle \mathbb{C}; +, \cdot, \mathbb{Z} \rangle$, and even $\mathbb{C}_{\mathbb{Z}\mathrm{IP}} \coloneqq \langle \mathbb{C}; +, \cdot, \mathbb{Z}, (z, n) \mapsto z^n \rangle$, the complex field with a predicate for $\mathbb{Z}$ and the function of raising complex numbers to integer powers, are $\mathrm{QM}_{\mathrm{Aut}}$, because they have the same automorphisms as the pure field, and indeed they are QME.

Zilber [Zil02b] gave a theory of a generic function on a field, which turns out to be the theory of an ultrapower of polynomials of degree tending to infinity [Koi05]. Any such entire function is QM, for topological reasons. Wilkie [Wil05] used power series with sparse rational coefficients to construct entire functions, which he called Liouville functions after Liouville's construction of transcendental numbers, and partly showed they are generic in Zilber's sense, before Koiran [Koi03] finished the proof.

Towards Conjecture 1, Wilkie made progress around an Oberwolfach meeting in 2003, which led to an announcement of the proof of quasiminimality of raising to the power i in 2008, although that is still unpublished. During a long visit to Oxford in 2004, Macintyre was working both on ideas to proof the conjecture and to provide a counterexample.

Boxall [Box20] showed that certain existential formulas in the language of exponential rings must define countable or co-countable sets in $\mathbb{C}_{\exp}$.

4. Zilber's exponential field $\mathbb{B}_{\exp}$

Zilber [Zil05b] constructed a QME exponential field, $\mathbb{B}_{\exp}$, using the Hrushovski–Fraïssé amalgamation-with-predimension method together with the excellence technique. It is the unique model of cardinality continuum of some $L_{\omega,\omega_1}(Q)$-expressible axioms:

(1) Algebraic properties of $\mathbb{C}_{\exp}$: ACF_0, exp is a surjective homomorphism from the additive to the multiplicative group of the field, and the kernel is $\tau\mathbb{Z}$ for a transcendental τ.
(2) Schanuel's conjecture: for any tuple $\bar{z}$ we have $\mathrm{td}(\bar{z}, \exp(\bar{z})) \geqslant \mathrm{ldim}_{\mathbb{Q}}(\bar{z})$.
(3) Strong Exponential-Algebraic Closedness (SEAC): For any algebraic subvariety $V \subseteq \mathbb{G}_a^n \times \mathbb{G}_m^n$ which is free and rotund, and defined over a finitely generated subfield A, there is $(\bar{z}, \exp(\bar{z})) \in V$, generic in V over A. If we drop the genericity clause, this is called Exponential-Algebraic Closedness (EAC).
(4) The natural pregeometry, exponential-algebraic closure, has the countable closure property: the closure of a finite set is countable. (This is true in $\mathbb{C}_{\exp}$, proved using the Ax–Schanuel property.)

For more details about the axioms and the logic needed to express them, see [Kir13].

A strengthening of Conjecture 1 is:

Conjecture 6. $\mathbb{C}_{\exp} \cong \mathbb{B}_{\exp}$. *More precisely,* $\mathbb{C}_{\exp}$ *is a model of the axioms above. Equivalently, Schanuel's conjecture is true in* $\mathbb{C}_{\exp}$ *and* $\mathbb{C}_{\exp}$ *is Strongly Exponentially-Algebraically Closed.*

D'Aquino, Macintyre and Terzo [DMT10, DMT14, DMT16] have done some analysis of what known properties of $\mathbb{C}_{\exp}$ with analytic proofs can also be proved algebraically for $\mathbb{B}_{\exp}$.

Although Schanuel's Conjecture seems well out of reach, proving EAC, or proving SEAC assuming Schanuel's conjecture to get the genericity of solutions, seems more plausible. Several people have made progress:

Marker [Mar06] proved the $n = 1$ case. In unpublished work, Mantova and Masser proved the $n = 2$ case. Brownawell and Masser [BM17] proved the case of EAC where the projection of V onto $\mathbb{G}_a^n$ is dominant. See also [DFT18]. Another proof of this was given in [AKM22] where a similar result for the exponential maps of abelian varieties was also given.

While working on the construction of $\mathbb{B}_{\exp}$, Zilber was also trying to understand issues of uniformity needed to give first-order axioms, and this led him to his Conjecture on Intersections of subvarieties with Tori, or CIT, now known as the multiplicative part of the Zilber–Pink Conjecture [Zil02a]. Conditional on this CIT, [KZ14] gives an axiomatisation of the first-order theory of $\mathbb{B}_{\exp}$.

In another direction, in 2008 I started a project with Martin Bays and Juan Diego Caycedo to do an analogous construction to $\mathbb{B}_{\exp}$ but of a Weierstrass $\wp$-function. While Caycedo eventually left the project, this work led Martin and me to the papers previously mentioned on QME classes and eventually to [BK18] where we rewrote and generalised the construction of $\mathbb{B}_{\exp}$ to give our $\mathbb{B}_{\wp}$ and many other similar constructions. In that paper we also gave a strategy to prove Conjecture 1 by showing:

Theorem 7. *If* $\mathbb{C}_{\exp}$ *satisfies EAC then it is QME.*

Using this method, and the density of the rationals in the reals, I was able to prove:

Theorem 8. [Kir19] *Let* $\Gamma = \{(z, \exp(z + q + 2\pi i r)) \mid z \in \mathbb{C}, q, r \in \mathbb{Q}\}$. *Then the "blurred exponential field"* $\langle \mathbb{C}; +, \cdot, \Gamma \rangle$ *is QME.*

5. Complex Powers

As part of his PhD thesis [Gal22b], see also [Gal22a], Gallinaro recently proved that the part of EAC which relates to complex powers is true. Specifically, if V is of the form $L \times W$ where $L \subseteq \mathbb{G}_a^n$ is given by $\mathbb{C}$-linear equations and $W \subseteq \mathbb{G}_m^n$ is any algebraic variety, and $L \times W$ is free and rotund, then there is $\bar{z} \in L(\mathbb{C})$ such that $\exp(\bar{z}) \in W(\mathbb{C})$.

The proof of Theorem 2 goes by using this EAC result together with the analogue of Theorem 7 for the powers setting, which is forthcoming.

In fact we prove QME for a slightly more expressive structure which also turns out to be easier to work with: the exponential sums language of Zilber from [Zil03, Zil11]. Given a countable subfield K of $\mathbb{C}$, he considers the two-sorted structure

$$\mathbb{C}^K \quad := \quad \left(\mathbb{C}_{K\text{-VS}} \xrightarrow{\exp} \mathbb{C}_{\text{field}}\right)$$

where the left sort has just the K-vector space structure and the right sort has the field structure. The induced structure on the right sort is that of $\langle \mathbb{C}; +, \cdot, (\Gamma_\lambda)_{\lambda \in K} \rangle$, but not all automorphisms of that sort lift to the covering sort. Terms in variables from the first sort naturally correspond to complex exponential sums with exponents in K.

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Exponential-Algebraic Closedness

Francesco Paolo Gallinaro

This talk focuses on a case of the *Exponential-Algebraic Closedness Conjecture*. This conjecture, due to Boris Zilber, predicts sufficient conditions for systems of exponential sums equations to have solutions in the complex numbers.

Definition 1. Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be an algebraic subvariety.

V is *free* if the projection of V to $\mathbb{C}^n$ is not contained in any translate of a linear subspace of $\mathbb{C}^n$ defined over $\mathbb{Q}$ and the projection of V to $(\mathbb{C}^\times)^n$ is not contained in any translate of an algebraic subgroup of $(\mathbb{C}^\times)^n$.

For a subspace $Q \leq \mathbb{C}^n$ defined over $\mathbb{Q}$, denote by $\pi_Q : \mathbb{C}^n \times (\mathbb{C}^\times)^n \to \mathbb{C}^n/Q \times (\mathbb{C}^\times)^n/\exp(Q)$ the algebraic quotient map.

V is *rotund* if for every subspace $Q \leq \mathbb{C}^n$, $\dim \pi_Q(V) \geq n - \dim Q$.

Conjecture 2 (Exponential-Algebraic Closedness Conjecture)**.** *Let $V \subseteq \mathbb{C}^n \times (\mathbb{C}^\times)^n$ be an algebraic subvariety.*

If V is free and rotund, then there is a point in V of the form $(z, \exp(z))$.

The model-theoretic relevance of the conjecture is that by work of Bays and Kirby it implies the *Quasiminimality Conjecture*, another conjecture due to Zilber, which predicts that every subset of $\mathbb{C}$ that is definable in the language of rings expanded by a symbol for the exponential function is countable or cocountable; for more details see [1] and [3]. This conjecture has been discussed in the talk of Jonathan Kirby.

Many cases of the Exponential-Algebraic Closedness Conjecture have now been solved. This talk will focus on the following.

Theorem 3. *Let $L \leq \mathbb{C}^n$ be a linear subspace, $W \subseteq (\mathbb{C}^\times)^n$ an algebraic subvariety.*

If $L \times W$ is free and rotund, then there is a point in $L \times W$ of the form $(z, \exp(z))$.

The systems of equations which correspond to varieties of this form are systems of *exponential sums equations*. For example, we may consider the equation $\exp(z) + \exp(iz) + 1 = 0$. Solving this equation is the same as finding points of the form $(z, \exp(z))$ in the variety $L \times W$, where

$$L := \{(z, iz) \mid z \in \mathbb{C}\}$$

and

$$W := \{(w_1, w_2) \in (\mathbb{C}^\times)^2 \mid w_1 + w_2 + 1 = 0\}.$$

1. Tropical Geometry

The method of proof relies on *tropical geometry*, a realtively young branch of mathematics which studies combinatorial features of certain objects from algebraic geometry; for example, it has many connections to toric geometry. We review some material from the textbook [2].

Tropical geometry allows us to encode the behaviour of a subvariety of $(\mathbb{C}^\times)^n$ into finitely many real semilinear objects. Let us make this more precise through the notion of *initial form*.

Definition 4. Let $f = \sum_{u \in S} c_u z^u$ be a complex Laurent polynomial (so S is a finite subset of $\mathbb{Z}^n$ and $c_u \in \mathbb{C}$ for each $u \in S$), $w \in \mathbb{R}^n$.

The *initial form* of f with respect to w is the polynomial

$$\mathrm{in}_w(f) = \sum_{u \in S'} c_u z^u$$

where $S' := \{u \in S \mid u \cdot w \geq u' \cdot w \text{ for all } u' \in S\}$.

In simple terms, the initial form of a complex polynomial "forgets" all the monomials in f except for the ones whose exponents have maximal scalar product with w.

Given an ideal I in the ring of Laurent polynomials in n variables and $w \in \mathbb{R}^n$, we denote by $\text{in}_w(I)$ the ideal generated by the initial forms with respect to w of the polynomials in I. This is called the *initial ideal* of I with respect to w.

Definition 5. Let $W \subseteq (\mathbb{C}^\times)^n$ be an algebraic subvariety, I the ideal of Laurent polynomials which vanish on W.

The *tropicalization* of W is the set

$$\{w \in \mathbb{R}^n \mid \text{in}_w(I) \neq \langle 1 \rangle\}.$$

Since in the ring of Laurent polynomials the units are the monomials, this corresponds to saying that for no polynomial $f \in I$ we have that $\text{in}_w(f)$ is a monomial.

Fundamental results in tropical geometry give the following facts.

1. The tropicalization of W is a finite union of semilinear sets.
2. To each of this semilinear sets corresponds an algebraic subvariety of $(\mathbb{C}^\times)^n$ which is invariant under multiplication by a positive-dimensional algebraic subgroup of $(\mathbb{C}^\times)^n$. These are called the *initial varieties* of W.

The initial varieties are the varieties that W "looks like" as its points approach 0 or ∞. As an example, consider the variety W cut out in $(\mathbb{C}^\times)^2$ by $f(w_1, w_2) = w_1 + w_2 + 1 = 0$. The point $(1,1)$ is in its tropicalization, as the initial form of f with respect to it is $w_1 + w_2$: the initial variety associated to $(1,1)$ is the subvariety of $(\mathbb{C}^\times)^2$ defined by $w_1 + w_2$. As a matter of fact, when points in W have very large absolute value, then the monomial 1 is negligible in the polynomials f, and therefore the points have to be "close" to points which satisfy $w_1 + w_2 = 0$.

2. Outline of the Proof

The proof distinguishes between two cases: the case in which L is defined over the reals and the case in which it is not.

If L is defined over $\mathbb{R}$, then (under freeness) we have that $\exp(L)$ is dense in $\exp(L) \cdot \mathbb{S}_1^n$ ($\mathbb{S}_1$ denotes the unit circle in $\mathbb{C}^\times$). Rotundity of $L \times W$ implies that the function $\delta : L \times W \to (\mathbb{C}^\times)^n$ which maps $(l_1, ..., l_n, w_1, ..., w_n)$ to $\left(\frac{w_1}{\exp(l_1)}, ..., \frac{w_n}{\exp(l_n)}\right)$ can be assumed to be open without loss of generality. These facts imply that it is sufficient to solve the system "up to absolute value", that is, we just need to find a point $(l, w) \in L \times W$ such that $\delta(l, w) \in \mathbb{S}_1^n$. This is achieved through a theorem of Khovanskii.

If L is not defined over $\mathbb{R}$, then we use the theory of *tropical compactifications*: we can find an initial variety W_τ of W such that $L \times W_\tau$ is still rotund, and a sequence of points in $\exp(L) \cap W_\tau$ which diverges in $(\mathbb{C}^\times)^n$, in a suitable toric variety has a limit which is also a limit of points on W. We can then show that this sequence gives approximations of points in $W \cap \exp(L)$, which turn out to exist.

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A definable (p, q)-theorem for NIP theories

Itay Kaplan

Definition 1. Say that a set system $(X, \mathcal{F})$ has the *(p, q)-property* for $q \le p < \omega$ if for any $F \subseteq \mathcal{F}$ of size $|F| = p$ there is some $F_0 \subseteq F$ such that $|F_0| = q$ and $\bigcap F_0 \neq \varnothing$.

Fact 2 (The (p, q)-theorem)**.** [Mat04] *There exists a function $N_{\mathrm{pq}} : \mathbb{N}^2 \to \mathbb{N}$ such that for any $q \le p < \omega$, if $(X, \mathcal{F})$ is a finite set system with the (p, q)-property such that every $s \in \mathcal{F}$ is nonempty and $\mathrm{VC}^*(\mathcal{F}) < q$, then there is $X_0 \subseteq X$ of size $|X_0| = N_{\mathrm{pq}}(p, q)$ such that $X_0 \cap s \neq \varnothing$ for all $s \in \mathcal{F}$.*

Model theoretically, this implies that if $\phi(x, y)$ is NIP then for any $\mathrm{VC}^*(\phi) < q \le p$ and $n := N_{\mathrm{pq}}(p, q)$, if B is a finite set of y-tuples such that $\{\phi(x, b) \mid b \in B\}$ has the (p, q)-property then there are n elements $a_0, \dots, a_{n-1}$ such that for all $b \in B$ there is some $i < n$ for which $\phi(a_i, b)$ holds.

This theorem turned out to be tremendously useful in the model-theoretic study of NIP. For instance, it was instrumental in the proof of the uniform definability of types over finite sets (UDTFS) in NIP theories by Chernikov and Simon [CS15], in their study of definably amenable NIP groups [CS18] and more recently in the proof that honest definitions exist uniformly for NIP formulas [BKS22].

In order to phrase a definable version of the (p, q)-theorem, we use the following definition.

Definition 3. Let M be a structure. Say that a pair of formulas $(\phi(x, y), \psi(y))$ over M has the *(p, q)-property* if $\mathcal{F} := \{\phi(x, b) \mid b \in \psi(M)\}$ is a family of nonempty sets with the (p, q)-property: for every choice of distinct p elements $\mathcal{F}$, some q of them have a nonempty intersection.

The following is a corollary of the (p, q)-theorem, formulated in [CS15, Proposition 25] (see also [Sim14, beginning of Section 2]).

Fact 4. *Suppose that T is NIP and that $M \models T$. Assume that $\phi(x, y)$ and $\psi(y)$ are formulas over M and that (ϕ, ψ) has the (p, q)-property for $\mathrm{VC}^*(\phi) < q \le p$. Then there are sets $W_0, \dots, W_{n-1} \subseteq S^y(M)$ for $n := N_{\mathrm{pq}}(p, q)$ such that $\bigcup_{i<n} W_i = \{p \in S^y(M) \mid \psi(y) \in p\}$ and for each $i < n$, $\{\phi(x, b) \mid \mathrm{tp}(b/M) \in W_i\}$ is consistent.*

For example consider the family $\mathcal{F}$ of rays in DLO (i.e., $\mathrm{Th}(\mathbb{Q}, <)$): sets defined by $x > a$ or $x < a$. It is easy to see that the dual VC-dimension of $\mathcal{F}$ is 1 (given any two rays, if they intersect, then their union is everything). In the context of

Fact 4, $\mathcal{F}$ can be formalized by setting $\phi(x,y,z_1,z_2) = ((z_1 = z_2 \to x > y) \wedge (z_1 \neq z_2 \to x < y))$ and $\psi(y,z_1,z_2) = (y = y)$. Then (ϕ,ψ) has the $(3,2)$ property: every three rays must intersect. Given any model M, let W_0 be the set of types of pairs (a,a) over M and W_1 be the set of types of pairs (a,b) over M where $a \neq b$. In other words, we cover $\mathcal{F}$ by positive and negative rays.

Note that in this case, W_0, W_1 are clopen: they come from definable sets over M. Our main theorem says this is not an accident.

Theorem 5. *Suppose that T is NIP, $M \models T$ and that $\phi(x,y), \psi(y)$ are formulas over M. Assume that (ϕ,ψ) has the (p,q)-property for $\mathrm{VC}^*(\phi) < q \leq p$. Then there are formulas $\psi_0(y), \ldots, \psi_{n-1}(y)$ over M such that $\psi(y)$ is equivalent to the disjunction $\bigvee_{i<n} \psi_i(y)$ and for each $i < n$, $\{\phi(x,b) \mid b \in \psi_i(M)\}$ is consistent.*

We also consider a uniform version of Theorem 5, i.e., varying the model M.

Theorem 6. *Suppose that T is NIP, and that $\phi'(x,y,z)$, $\psi'(y,z)$ are two formulas without parameters. Then for any $q \leq p < \omega$ there is $n < \omega$ and formulas $\psi_0(y,w), \ldots, \psi_{n-1}(y,w)$ such that the following hold.*

Suppose that $M \models T$ and $c \in M^z$. Let $\phi(x,y) = \phi'(x,y,c)$ and $\psi(y) = \psi'(y,c)$. If (ϕ,ψ) has the (p,q)-property and $\mathrm{VC}^*(\phi(x,y)) < q$ then for some $d_0, \ldots, d_{n-1} \in M^w$, $\psi(y,c)$ is equivalent to the disjunction $\bigvee_{i<n} \psi_i(y,d_i)$ and for each $i < n$, the set $\{\phi(x,b) \mid b \in \psi_i(M,d_i)\}$ is consistent.

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Domination in hesenlian valued fields of equicharacteristic zero

Mariana Vicaría

Henselian valued fields are structures at the heart of *algebra and number theory*, and during the last decades there has been a significant interplay between them and model theory. One of the most striking results in the model theory of henselian valued fields is the *Ax-Kochen/ Ershov theorem* which roughly states that the first order theory of a henselian valued field of equicharacteristic zero or a henselian valued field of mixed characteristic, unramified [1] and perfect residue field is completely determined by the first order theory of its residue field and its value group. A natural philosophy follows from this theorem: model theoretic questions about

[1] A valued field is said to be unramified if its value group has a least positive element.

the valued field itself can be understood by reducing them to its residue field, its value group and their interaction in the field.

Model theory is a branch of mathematical logic that studies *structures* (that is sets equipped with relations, functions and constants) and their *definable sets*, that is the subsets of various cartesian powers that can be defined in terms of these distinguished constants, relations and functions via the logical connectives and quantifiers. For instance, in an algebraically closed field we distinguish the constants 0 and 1 and symbols for the multiplication and the addition. This particular case is well understood: the definable sets are exactly the constructible sets, which are fundamental objects of algebraic geometry.

A fruitful application of the Ax-Kochen/ Ershov principle is illustrated by several relative quantifier elimination results, that describe the class of definable sets in a henselian valued field. For example, in [4] J. Pas proved field quantifier elimination relative to the residue field and the valued group once angular component maps are added to the language in the equicharacteristic zero case. Further studies for the case where no angular map is added were done by S.A. Basarab and F.V. Kuhlmann in [5], [6] (relative to the RV_n sorts).

Modern model theory has been heavily influenced by S. Shelah's remarkable work in classification theory [13]. In the 1970s S. Shelah developed a tremendously profound structure theory for the class of stable theories, in which no first order formula can totally order arbitrarily large sets of tuples. The study of stable theories initiated by S. Shelah, and later refined by many others, brought to the picture tools and ideas that have been the key to solve many problems in other branches of mathematics, such as the Mordell-Lang conjecture for function fields proved by E. Hrushovski. The *neostability* program seeks to generalize Shelah's work to other dividing lines beyond stability, by identifying and studying boundaries between *tame* and *wild* first-order theories. In other words, model theory studies dividing lines between prototypical tame structures like vector spaces and the field of complex numbers, in which the definable sets are well understood, and wilder structures for which there is no control, such as the ring of integers. This program has been tremendously fruitful for several classes of theories, most notably the simple theories, dependent theories and o-minimal theories . These developments have enriched the applications of model theory; the Pila-Wilkie theorem in diophantine geometry is a prime example.

It is therefore very natural to attempt to establish which is the combinatorical complexity of a henselian valued field. A very successful path of research addresses this question by following the Ax-Kochen/ Ershov principle: tameness conditions in the residue field and the value group transfer into the same tameness in the entire henselian valued field. The first result in the literature is due to Delon: *a henselian valued field of equicharacteristic zero is dependent if and only if its residue field and its value group are dependent.* It is a well known result by Gurevich and Schmitt that any pure ordered abelian group is dependent, so Delon's translates to: *a henselian valued field of equicharacteristic zero is dependent if and only if its residue field is dependent.* Further results were deeply understood later on: the dp-minimal case in [9], the strongly dependent case in [8], the distal case in [10], the NTP_2 case in [11] and [12], among many others.

The model theoretic study of henselian valued fields took a leap forward with the sequence of papers of Haskell, Hurshovski and Macpherson in [2] and [3], where they develop the theory of *stable domination*, an abstract theory that formalizes how a structure is governed by its stable part. Their prime example is the theory of algebraically closed valued fields (ACVF), which is a dependent theory due to Delon's theorem; but where classical machinery of stability has been lifted to give a broader insight of the behavior of these structures and represented the initial grounds to develop tame geometry in ACVF. Applications of stable domination include the classification of definable abelian groups in ACVF by Hrushovski and Rideau-Kikuchi in [7] and the deep theory of Berkovich spaces of Hurshovski and Loeser in [15].

In [2] Haskell, Hurshovski and Macpherson provide a complete characterization of the stable part of the structure, and prove that it coincides with the *linear sorts*, which carry a k-vector space structure, where k is the residue field. We recall some of the definitions and results for sake of completeness.

Definition 1 (Linear sorts)**.** For any parameter set C, let $VS_{k,C}$ be the many sorted structure whose sorts are the k-vector spaces $\mathrm{red}(s)$ where $s \in \mathrm{dcl}(C) \cap \mathcal{S}$.

- Each sort $\mathrm{red}(s)$ is equipped with its k-vector space structure.
- In addition, $VS_{k,C}$ has as $\varnothing$-definable relations any C-definable relation on products of the sorts.

Proposition 2. *Let $K \vDash ACVF$ and D a C-definable subset of K^{eq}. Then:*

- *D is k-internal if and only if D is stable and stably embedded,*
- *If D is k-internal, $D \subseteq \mathrm{dcl}(C \cup VS_{k,C})$.*

In later work of Haskell, Ealy and Maricova (see [1]) the following notion of domination, which generalizes the one present in [2]:

Definition 3. Let T be a complete first order theory and let S and Γ be stably embedded sorts, and $C \subseteq A, B$ be sets of parameters in the monster model $\mathfrak{C}$.

(1) the type $\mathrm{tp}(A/C)$ is said to be *dominated by the sort* S, if whenever $S(B)$ is *independent* from $S(A)$ over $S(C)$ then that $\mathrm{tp}(A/CS(B)) \vdash \mathrm{tp}(A/CB)$.
(2) the type $\mathrm{tp}(A/C)$ is said to be dominated by the sort S over Γ if the type $\mathrm{tp}(A/C\Gamma(A))$ is dominated by the sort S.

In [1] domination results for the setting of real closed convexly valued fields are obtained, which suggests that the presence of a stable part of the structure is not fundamental to achieve domination results and indicates that the right notion should be residue field domination or domination by the internal sorts to the residue field in broader classes of henselian valued fields. Later in [14], Haskell, Ealy and Simon generalized the residue field domination results for henselian valued fields of equicharacteristic zero with bounded Galois group. In their work, it becomes clear that the key ingredients to obtain domination results are the existence of separated basis and a relative quantifier elimination statement.

Our main motivation arises from the natural question of how much further a notion of residue field domination could be extended to henselian valued fields of equicharacteristic zero. This is the first step towards obtaining a unifying model theoretic theory of henselian valued fields, by lifting tameness conditions on the residue field to the entire structure generalizing the theory of stable domination applied to ACVF to simple domination, $NSOP_1$-domination,etc.. to henselian valued fields.

We fix some complete extension of the theory of henselian valued of equicharacteristic zero in the languange introduced by Achembrenner, Chernikov,Ziegler and Gehther in [10]. We write $\mathfrak{C}$ to indicate the monster model and we obtain the following domination results for elments in the home-sort:

Theorem 4 (Vicaria). *Let $C \subseteq L$ be substructures of $\mathfrak{C}$ with C a maximal model of T. Then* $\mathrm{tp}(L/C)$ *is dominated by the value group and the residue sorts.*

Theorem 5 (Vicaria). *Let L be an elementary substructure of $\mathfrak{C}$ and let $C \subseteq L$ be a maximal model of T. Then the type* $\mathrm{tp}(L/C)$ *is dominated over its value group by the sorts internal to the residue field.*

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