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Real Analysis, Harmonic Analysis and Applications

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Abstract. The workshop focused on important developments within the last few years in real and harmonic analysis including nonlinear Carleson theorems and singular integral theory, Fourier restriction theory and spherical maximal functions as well as concurrent progress in the application of these for example to partial differential equations.

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Introduction by the Organizers

This workshop, which continued the triennial series at Oberwolfach on Real and Harmonic Analysis that started in 1986, has brought together experts and young scientists working in harmonic analysis and its applications such as linear and non-linear PDE, number theory, and complex analysis, with the objective of furthering the important interactions between these fields.

Major areas and results represented at the workshop are:

(1) Applications of nonlinear Fourier analysis include explicit solutions to non-linear partial differential equations in the form of the inverse scattering method. This serves as model for new results on conserved quantities for the Gross-Pitaevski equation. It is a natural challenge to prove analogs of results of linear Fourier analysis in the nonlinear setting. There has been a recent breakthrough on a nonlinear variant of Carleson's theorem on almost everywhere convergence of Fourier series.

(2) Local smoothing theory encompasses a range of estimates from Fourier re-striction to decoupling. In Fourier restriction theory, there are similarities

as well as differences between restriction to submanifolds and restriction to thin arithmetic sets. New results include Fourier restriction bounds for surfaces with non-vanishing Gaussian curvature and principal curvatures of different signs, as well as sharp small cap decoupling estimates for the moment curve. Extremizers for L^2 based Fourier extension estimates on quadratic surfaces fall mostly into a predictable family. Exceptional phenomena occur for Agmon Hörmander type estimates and for cones, which are studied with the help of the Penrose transform. Extremizers for Fourier restriction to the moment curves are shown to exist.

(3) Singular Brascamp–Lieb inequalities are much less understood than the classical non-singular counterparts. A family of singular Brascamp-Lieb inequalities with cubical structure has emerged as a critical case and plays a role in applications to ergodic theory and enumerative combinatorics. Refined phase space localized operators are a useful tool in the theory of singular Brascamp Lieb inequalities. The study of generalized Brascamp Lieb inequalities on varieties has led to the understanding of the local geometry of model Radon-like transforms.

(4) Heisenberg groups arise as boundaries of Siegel domains, and boundary behaviour of holomorphic functions on strictly pseudoconvex domains has been studied using the Heisenberg groups. As it turns out, all nilpotent Lie groups avoiding the obvious obstructions arise as model boundaries of complex domains.

(5) Spherical maximal functions and their variants appear in many contexts. An improvement of an L^p bound for Wolff's circular maximal function towards cinematic curves using lens cutting has led to the solution of a conjecture of Fässler and Orponen for the Hausdorff dimension of projections. The sharp L^p bound for the helical maximal function in Euclidean space $\mathbb{R}^3$ has been established. Surprisingly, spherical maximal operators over fractal sets of dilations exhibit quite general convex sets as exponent regions of boundedness. There are new results on spherical maximal operators on the Heisenberg group for horizontal and Korányi spheres.

(6) Fine spectral analysis of one dimensional Schrödinger operators leads to improved multiplier bounds for Grushin operators. Refined analysis of Morawetz estimates leads to non-perturbative global solutions to nonlinear Schrödinger equations with cubical nonlinearity. Pointwise convergence to intitial data for Schrödinger operators has long been studied, new results concern fractal dimensions of exceptional sets.

(7) Further topics of the workshop include a non-archimedean variant of Littlewood Paley theory for space curves, the Hardy Littlewood majorant property in higher dimension, and new results on multi-parameter Carleson embedding. Improved eigenfunction bounds on manifolds imply improved eigenfunction bounds on corresponding product manifolds.

The meeting took place in a lively and active atmosphere, and greatly benefited from the ideal environment at Oberwolfach. After more than two years of the

Covid 19 pandemic, it was refreshing to many to return to a workshop in presence, aided by good measures at the Institute. This appears to have added considerably to the positive spirits at the conference. The meeting was attended in presence by 42 participants, a small number of invitees participated online. The program consisted of 23 lectures of 40 minutes. The organisers made an effort to include young mathematicians, and greatly appreciate the support through the Oberwolfach Leibniz Graduate Students Program, which allowed to invite several outstanding young scientists.

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Workshop: Real Analysis, Harmonic Analysis and Applications

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Abstracts

Sharp L^p bounds for the Helical maximal function

David Beltran

(joint work with Shaoming Guo, Jonathan Hickman, Andreas Seeger)

Let $\gamma\colon I \to \mathbb{R}^n$ be a smooth curve, where $I \subset \mathbb{R}$ is a compact interval, and $\chi \in C^\infty(\mathbb{R})$ be a bump function supported on the interior of I. Given $t > 0$, consider the averaging operator

$$A_t f(x) := \int_{\mathbb{R}} f(x - t\gamma(s))\,\chi(s)\,\mathrm{d}s$$

and define the associated maximal function

$$M_\gamma f(x) := \sup_{t>0} |A_t f(x)|.$$

If γ is a *non-degenerate* curve, in the sense that there is a constant $c_0 > 0$ such that

$$|\det(\gamma'(s), \cdots, \gamma^{(n)}(s))| \geq c_0 \qquad \text{for all } s \in I,$$

it is conjectured that the maximal function M_γ is bounded on $L^p(\mathbb{R}^n)$ if and only if $p > n$. The two-dimensional case was settled by Bourgain [1] in 1986. In this talk, we present a positive answer for $n = 3$. Note that in 3 dimensions, non-degeneracy amounts to non-vanishing curvature and torsion.

Theorem 1 ([5]). *If $\gamma : I \to \mathbb{R}^3$ is a smooth, non-degenerate space curve, then M_γ is bounded on $L^p(\mathbb{R}^3)$ if and only if $p > 3$.*

The same result was simultaneously and independently obtained by Ko, Lee and Oh [6]. This result improves the previously known range $p > 4$ obtained by the work of Pramanik and Seeger [4] in conjunction with the sharp ℓ^p-decoupling inequalities for the cone of Bourgain and Demeter [3]. Recently, there have been further developments in higher dimensions [7], and the partial range $p > 2(n-1)$ has been verified for all $n \geq 4$.

Standard reductions allow to deduce a maximal function estimate from a *local smoothing* estimate. If we set $\mathfrak{A}_\gamma f(x,t) := \rho(t) \cdot A_t f(x)$ for some $\rho \in C_c^\infty(\mathbb{R})$ with $\operatorname{supp} \rho \subseteq [1,2]$, the key inequality that proves Theorem 1 is the following.

Theorem 2. *Suppose $\gamma : I \to \mathbb{R}^3$ is a smooth, non-degenerate space curve and let $3 \leq p \leq 4$ and $\sigma < \sigma(p)$ where $\sigma(p) := \frac{1}{5}\big(1 + \frac{2}{p}\big)$. Then $\mathfrak{A}_\gamma$ maps $L^p(\mathbb{R}^3)$ boundedly into $L^p_\sigma(\mathbb{R}^4)$.*

In proving Theorem 2, the main new ingredient is a family of *microlocal* smoothing estimates for pieces of the operator $\mathfrak{A}_\gamma$ for which their associated Fourier multiplier (in both space and time) has slowest (or fairly slow) decay. On Fourier side, those directions of slow decay concentrate near a 2-dimensional cone Γ in $\mathbb{R}^4$, generated by a non-degenerate curve in $\mathbb{R}^3$. In order to establish those microlocal smoothing estimates we use the square-function approach introduced by Mockenhaupt, Seeger and Sogge [2] in the context of local smoothing estimates associated to the light cone in $\mathbb{R}^3$.

On an abstract level, given an operator T mapping functions from $\mathbb{R}^n$ to $\mathbb{R}^{n+1}$ the approach in [2] consists in a good understanding of T on L^4 via the following 4 key steps:

(i) Identify a decomposition $Tf = \sum_{\nu=1}^{N} T_\nu f_\nu$ so that the pieces T_ν satisfy, via elementary integration-by-parts arguments,

$$|T_\nu f_\nu(x,t)| \leq C(T) K_\nu(\cdot, t) * f_\nu(x). \tag{1}$$

Here $C(T)$ is typically a gain in the scale at which the operator T is frequency localized, and the kernel $K_\nu(\cdot, t)$ is L^1-normalised.

(ii) Prove an L^4 reverse square function estimate of the type

$$\|Tf\|_{L^4(\mathbb{R}^{n+1})} \leq C(N) \Big\| \Big(\sum_{\nu=1}^{N} |T_\nu f_\nu|\Big)^{1/2} \Big\|_{L^4(\mathbb{R}^{n+1})} \tag{2}$$

for the best possible constant $C(N)$. Of course here the goal is to improve, whenever possible, over the trivial estimate $C(N) \leq CN^{1/2}$ implied by the Cauchy–Schwarz inequality.

After using duality on the right-hand side of (2), the pointwise estimate (1), a standard Cauchy–Schwarz and Fubini argument, and Hölder's inequaltiy, one is left with proving two more estimates.

(iii) An L^4 forward square function (in $\mathbb{R}^n$) of the type

$$\Big\| \Big(\sum_{\nu=1}^{N} |f_\nu|\Big)^{1/2} \Big\|_{L^4(\mathbb{R}^n)} \leq C(N) \|f\|_{L^4(\mathbb{R}^n)} \tag{3}$$

for the best possible $C(N)$.

(iv) An $L^2(\mathbb{R}^4) \to L^2(\mathbb{R}^3)$ estimate for the maximal function

$$\mathcal{M}g(x) := \sup_{\nu=1,\dots,N} \int_1^2 |K_\nu(\cdot, t)| * g(\cdot, t)](x)\, \mathrm{d}t, \tag{4}$$

with sharp operator norm in terms of N.

As mentioned above, we follow this strategy when T is a piece of $\mathfrak{A}_\gamma$ whose (x,t) Fourier transform is microlocalised in a neighbourhood (or at a certain distance) of a 2-dimensional cone Γ. Using the geometry of Γ, we successfully obtain versions of the estimates (1), (2), (3) and (4) in our context, which provide a satisfactory L^4 bound. The desired L^p bounds for $p > 3$ featuring in Theorems 1 and 2 follow from interpolation with elementary L^2 estimates for the microlocalised pieces of $\mathfrak{A}_\gamma$.

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Fourier restriction for smooth hyperbolic 2-surfaces

Stefan Buschenhenke

(joint work with Ana Vargas and Detlef Müller)

Let $S \subset \mathbb{R}^n$ be a sufficiently smooth hypersurface. The Fourier restriction problem, introduced by E. M. Stein in the seventies (for general submanifolds), asks for the range of exponents $\tilde{p}$ and $\tilde{q}$ for which an a priori estimate of the form

$$\left(\int_S |\widehat{f}|^{\tilde{q}}\, d\sigma\right)^{1/\tilde{q}} \leq C\|f\|_{L^{\tilde{p}}(\mathbb{R}^n)}$$

holds true for every Schwartz function $f \in \mathcal{S}(\mathbb{R}^n)$, with a constant C independent of f. Here, $d\sigma$ denotes the Riemannian surface measure on S.

The sharp range in dimension $n = 2$ for curves with non-vanishing curvature was determined through work by C. Fefferman, E. M. Stein and A. Zygmund [F70], [Z74]. In higher dimension, the sharp $L^{\tilde{p}} - L^2$ result for hypersurfaces with non-vanishing Gaussian curvature was obtained by E. M. Stein and P. A. Tomas [To75], [St86] (see also Strichartz [Str77]). Some more general classes of surfaces were treated by A. Greenleaf [Gr81]. In work by I. Ikromov, M. Kempe and D. Müller [IKM10] and Ikromov and Müller [IM11], [IM15], the sharp range of Stein-Tomas type $L^{\tilde{p}} - L^2$ restriction estimates has been determined for a large class of smooth, finite-type hypersurfaces, including all analytic hypersurfaces.

The question about general $L^{\tilde{p}} - L^{\tilde{q}}$ restriction estimates is nevertheless still wide open. Fourier restriction to hypersurfaces with non-negative principal curvatures has been studied intensively by many authors. Major progress was due to J. Bourgain in the nineties ([Bo91], [Bo95a], [Bo95b]). At the end of that decade the bilinear method was introduced ([MVV96], [MVV99], [TVV98] [TVI00], [TVII00], [W01], [T03], [LV10]). A new impulse to the problem has been given with the multilinear method ([BCT06], [BoG11]). The best results up to date have been obtained with the polynomial partitioning method, developed by L. Guth ([Gu16], [Gu18]) (see also [HR19] and [Wa18] for recent improvements).

For the case of hypersurfaces of non-vanishing Gaussian curvature but principal curvatures of different signs, besides Tomas-Stein type Fourier restriction estimates, until a few years ago the only case which had been studied successfully was the case of the hyperbolic paraboloid (or "saddle") in $\mathbb{R}^3$: in 2005, independently S. Lee [L05] and A. Vargas [V05] established results analogous to Tao's theorem [T03] on elliptic surfaces (such as the 2 -sphere), with the exception of the end-point, by means of the bilinear method.

First results based on the bilinear approach for particular one-variate perturbations of the saddle were eventually proved by the authors in [BMV20], [BMVp19] and [BMVp20a]. Furthermore, B. Stovall [Sto17b] was able to include also the end-point case for the hyperbolic paraboloid. Building on the papers [L05], [V05] and [Sto17b], and by strongly making use of Lorentzian symmetries, even global restriction estimates for one-sheeted hyperboloids have been established recently by B. Bruce, D. Oliveira e Silva and B. Stovall [BrOS20], with extensions to higher dimensions by Bruce [Br20b]. Results on higher dimensional hyperbolic paraboloids have been reported by A. Barron [Ba20]. All these results are in the bilinear range given by [T03].

Improvements over the results for the saddle by means of an adaptation of the polynomial partitioning method from Guth's articles [Gu16] were achieved by C. H. Cho and J. Lee [ChL17], and J. Kim [K17]. Moreover, for a particular class of one-variate perturbations of the hyperbolic paraboloid, an analogue of Guth's result had been proved by the authors in [BMVp20b], and more lately by making use of Lorentzian symmetries, B. Bruce [Br20a] has established analogous results for compact subsets of the one-sheeted hyperboloid.

In this article, we shall obtain the analogous result to [Gu16] for compact subsets of any sufficiently smooth hyperbolic surface.

More precisely, we shall study embedded C^m- hypersurfaces S in $\mathbb{R}^3$ of sufficiently high degree of regularity $m \geq 3$ which are *hyperbolic* in the sense that the Gaussian curvature is strictly negative at every point, i.e., that at every point of S one principal curvature is strictly positive, and the other one is strictly negative.

A result comparable to the one of the authors was reported by S. Guo and C. Oh [GO20], though the initial approach is different and is based on approximation of arbitrary compact hypersurfaces with negative curvature in $\mathbb{R}^3$ by polynomial surfaces.

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Which nilpotent Lie groups arise as model boundaries of complex domains?

Michael G. Cowling

(joint work with M. Ganji, A. Ottazzi, G. Schmalz)

On the one hand, it is well known that the Heisenberg groups arise as boundaries of Siegel domains in $\mathbb{C}^n$, and boundary behaviour of holomorphic functions on strictly pseudoconvex domains has been studied using the Heisenberg groups as local models for the boundary. More complicated nilpotent Lie groups appear in the study of boundary behaviour for other model domains. On the other hand, geometers have studied more complicated domains whose boundaries may be identified with more complicated nilpotent Lie groups. See, for example, [3, 5, 6, 7]. However, I am not aware of any systematic and complete study that answers the question of the title of this proposed talk. As we will see, the answer to the question, at least in one interpretation, is all.

A joint paper (in the final stages of preparation) considers a nilpotent Lie group G with a left-invariant integrable horizontal CR structure. By this we mean that the Lie algebra $\mathfrak{g}$ of G may be written as a direct sum $\mathfrak{h} \oplus \mathfrak{n}$, where the subspace (not a subalgebra) $\mathfrak{h}$, of real dimension $2n$, carries an almost complex structure J and generates the Lie algebra $\mathfrak{g}$, while $\mathfrak{n}$ is an ideal of real dimension k. The horizontal bundle obtained by left-translating $\mathfrak{h}$ carries an almost complex structure, and we assume that the CR structure so defined is integrable. We can then show that G is the "edge of the wedge" boundary of a domain D in $\mathbb{C}^{n+k}$ defined by inequalities. The additional dimensions arise by complexifying $\mathfrak{n}$.

In our proof, we first appeal to results of Baouendi and Rothschild [1] and of Baouendi, Rothschild and Treves [2], which are based on the Newlander–Nirenberg theorem [8], to show that G has a local CR embedding in $\mathbb{C}^{n+k}$ as a graph of a smooth function. We then show that this graph satisfies a family of polynomial differential equations, following [4], and that the solutions of such polynomial differential equations in G are polynomials. At the present time, we are trying to see whether the particular structure that we are dealing with enables us to give a simpler proof of the Baouendi–Rothschild–Treves embedding theorem.

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Singular Brascamp-Lieb forms with cubical structure

Polona Durcik

(joint work with Lenka Slavíková, Christoph Thiele)

Brascamp-Lieb inequalities are L^p estimates for forms on functions on Euclidean spaces. They generalize several classical inequalities such as Hölder's inequality, Young's convolution inequality, and the Loomis-Whitney inequality. Brascamp-Lieb inequalities take the form

$$\Big| \int_{\mathbb{R}^m} \Big(\prod_{i=1}^{n} F_i(\Pi_i x) \Big) dx \Big| \le C \prod_{i=1}^{n} \|F_i\|_{L^{p_i}(\mathbb{R}^{k_i})},$$

where $\Pi_i \colon \mathbb{R}^m \to \mathbb{R}^{k_i}$ are linear surjections and the constant C is independent of the measurable functions $F_i : \mathbb{R}^{k_i} \to \mathbb{C}$. A necessary and sufficient condition for the Brascamp-Lieb inequality to hold is that

$$\dim(V) \leq \sum_{i=1}^{n} p_i^{-1} \dim(\Pi_i V)$$

for all subspaces V of $\mathbb{R}^m$, with equality if $V = \mathbb{R}^m$. This was established by Bennet, Carbery, Christ, and Tao [1].

Singular Brascamp-Lieb integrals arise when we replace one of the functions in the classical Brascamp-Lieb integral by a singular integral kernel. We consider Calderón-Zygmund kernels on $\mathbb{R}^k$, i.e. temepered distributions K, whose Fourier transform $\widehat{K}$ is a smooth function on $\mathbb{R}^k \setminus \{0\}$ and satisfies

$$|\partial^\alpha \widehat{K}(\xi)| \leq c_0 |\xi|^{-|\alpha|} \tag{1}$$

for some constant c_0, all $\xi \neq 0$, and all multi-indices α up to a large finite order. Singular Brascamp-Lieb inequalities then take the form

$$\Big| \int_{\mathbb{R}^m} \Big(\prod_{i=1}^{n} F_i(\Pi_i x) \Big) K(\Pi x) dx \Big| \leq C \prod_{i=1}^{n} \|F_i\|_{L^{p_i}(\mathbb{R}^{k_i})} \tag{2}$$

where on the left-hand side, $\Pi \colon \mathbb{R}^m \to \mathbb{R}^k$ is a surjective linear map and the constant C is independent of the test functions F_j. The constant C is allowed to depend on K only through the constant c_0 in (1) and the bound on the order of derivatives in (1). As we want the inequality (2) to hold for all kernels satisfying the above symbol estimates, a necessary condition for (2) can be obtained by specifying K to be the Dirac delta, which turns the left-hand side into a non-singular Brascamp-Lieb integral. At present, there is no general necessary and sufficient condition for the inequality (2) to hold and the theory of singular Brascamp-Lieb integrals remains a case by case study.

We mention two examples of singular Brascamp-Lieb integrals. In the extremal case when the dimension of the kernel is as large as possible (while fixing the number and the dimensions of the functions and avoiding trivial examples and counterexamples), one obtains the Coifman-Meyer multipliers, whose L^p inequalities are well-known [3]. On the other extreme, when the dimensions of the kernel is the smallest possible, one finds the simplex Hilbert transform

$$\text{p.v.} \int_{\mathbb{R}^n} \Big(\prod_{j=1}^{n} F_j(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_n) \Big) \frac{1}{x_1 + \cdots + x_n} dx_1 \ldots dx_n.$$

Its boundedness is one of the major open problems in harmonic analysis. Case $n = 2$ specializes to the dual of the classical Hilbert transform. Choosing the functions F_j suitably, the simplex Hilbert transform specializes to the multilinear Hilbert transform. Bounds for the bilinear Hilbert transform, which would also follow from the case $n = 3$ of the simplex Hilbert transform, were shown by Lacey and Thiele [12, 13]. On the other hand, the multilinear Hilbert transform in the case $n \geq 3$ is another major open problem.

We turn our focus to an intermediate case which features a particular cubical structure. Let $m \geq 1$. For $j \in \{0,1\}^m =: \mathcal{C}$, let $\Pi_j : \mathbb{R}^{2m} \to \mathbb{R}^m$ be given by

$$\Pi_j(x) = (x_1^{j_1}, \dots, x_m^{j_m}),$$

where $x = (x_1^0, \dots, x_m^0, x_1^1, \dots x_m^1) \in \mathbb{R}^{2m}$. Let the linear surjection $\Pi : \mathbb{R}^{2m} \to \mathbb{R}^m$ be arbitrary. The singular Brascamp-Lieb integral that we consider is

$$\text{p.v.} \int_{\mathbb{R}^{2m}} \Big(\prod_{j \in \mathcal{C}} F_j(\Pi_j x) \Big) K(\Pi x) dx.$$

Several instances of this form have been studied previously. The case $m = 1$ specializes to the dual of a classical linear Calderón-Zygmund operator. The case $m = 2$ and when one of the functions F_j is the constant function 1 is the so-called twisted paraproduct, which is a special case of the two-dimensional bilinear Hilbert transform by Demeter and Thiele [4]. First L^p bounds for the twisted paraproduct are due to Kovač [10], after a conditional result by Bernicot [2]. In the dimension $m = 2$, forms with two particular choices of Π were further studied in [5] and [6], respectively. The latter article deals with an application of such singular Brascamp-Lieb integrals to quantitative norm convergence of ergodic averages with respect to two commuting transformations.

The higher-dimensional case $m \geq 2$ was addressed in [8] for the particular choice of exponents $p_j = 2^m$ for each j. This choice of Lebesgue exponents allows for global arguments. The paper [8] also establishes a necessary and sufficient condition on the surjection Π for the singular Brascamp-Lieb inequality in [8] to hold. Bounds in dimensions $m \geq 2$ and in a larger range of exponents were investigated in [7]. The main result of [7] is the following theorem.

Theorem. *Let Π be a real $m \times 2m$ matrix such that for each $j \in \mathcal{C}$, the composition $\Pi_j \Pi^T$ is regular. For each $j \in \mathcal{C}$ let*

$$2^{m-1} < p_j \leq \infty$$

and assume $\sum_{j \in \mathcal{C}} {p_j}^{-1} = 1$. Then there exists a constant C such that for all Calderón-Zygmund kernels K on $\mathbb{R}^m$ of order 2^{6m} and all tuples $(F_j)_{j \in \mathcal{C}}$ of Schwartz functions on $\mathbb{R}^m$,

$$\Big|\text{p.v.} \int_{\mathbb{R}^{2m}} \Big(\prod_{j \in \mathcal{C}} F_j(\Pi_j x) \Big) K(\Pi x) dx \Big| \leq C \prod_{j \in \mathcal{C}} \|F_j\|_{L^{p_j}(\mathbb{R}^m)}. \tag{3}$$

A variant of this theorem, in which the lower bound 2^{m-1} is replaced by a smaller number, is false. However, we do not know the sharp range of exponents for the inequality (3) to hold.

The key step in the proof of (3) is to prove an estimate for a localized form. The latter is proved by an iteration of Fourier expansion, the Cauchy-Schwarz inequality, and telescoping, all developed in a suitable localized setting. The local estimate proven in [7] also implies sparse bounds. Sparse bounds can be used to show certain weighted and vector-valued estimates by a standard procedure.

The results in [7] are continuous variants of some of the results in [9] in the dyadic setting. Other results in the dyadic setting include [11, 14], which investigate non-translation invariant kernels. It would be desirable to have a more complete understanding of the continuous analogues of [9, 11, 14].

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The Mass Transference Principle and the fractal pointwise convergence problem for the Schrödinger equation

Daniel Eceizabarrena

(joint work with Felipe Ponce-Vanegas)

The pointwise convergence problem for the Schrödinger equation, introduced by Carleson in [5] in 1980, asks for the minimal Sobolev regularity s such that

$$\lim_{t\to 0} e^{it\Delta} f(x) = f(x) \quad \text{for almost every } x \in \mathbb{R}^d, \qquad \forall f \in H^s(\mathbb{R}^d).$$

The problem was solved quickly in $d = 1$ by Carleson himself and Dahlberg-Kenig [6], who proved that $s = 1/4$ is sufficient and necessary. However, in higher dimensions the problem remained open until 2019, when Bourgain [4], Du-Guth-Li [7] and Du-Zhang [9] proved that the critical exponent, up to the endpoint, was

$$s = \frac{d}{2(d+1)}.$$

The focus of this talk is the fractal refinement of this problem. For a fixed $0 \leq \alpha \leq d$, we want to compute

(1)
$$s(\alpha) = \inf\{\, s \,:\, \lim_{t\to 0} e^{it\Delta} f(x) = f(x) \quad \mathcal{H}^\alpha\text{-almost everywhere, } \forall f \in H^s(\mathbb{R}^d)\}.$$

When $\alpha \leq n/2$, the problem was solved and the exponent is $(n-\alpha)/2$ [2, 15]. However, the question is still open when $\alpha > n/2$. Before the work presented in this talk, the best result was

$$\frac{d}{2(d+1)} + \frac{d-1}{2(d+1)}(d-\alpha) \leq s(\alpha) \leq \frac{d}{2(d+1)} + \frac{d}{2(d+1)}(d-\alpha). \tag{2}$$

The upper bound follows from the positive results by Du-Zhang [9], while the best counterexample giving the lower bound is due to Lucà–Rogers [13] and Lucà–Ponce-Vanegas [12].

To give positive results, the standard method of maximal estimates can be adapted to the fractal setting. However, disproving a *fractal* maximal estimate is not enough to disprove fractal convergence, and therefore counterexamples must be constructed directly. Typically, these are built via a dyadic sum of functions f_R such that

- f_R is Fourier localized in an annulus of radius R, and
- $|e^{it\Delta} f_R|$ is large in a *bad* set F_R.

The corresponding set of divergence is then given by the limit superior set

$$\limsup_{R\to\infty} F_R = \bigcap_{R\geq 1} \bigcup_{M \geq R} F_M.$$

Thus, to give a fractal counterexample, one of the main challenges is to compute the Hausdorff dimension of these sets. It is well known that computing the Hausdorff dimension of a particular given set is often a difficult task.

To deal with this problem, we use the Mass Transference Principle introduced by Beresnevich-Velani [3] in the context of Diophantine approximation and the Duffin-Schaeffer conjecture. It reads as follows:

Theorem 1 (Mass Transference Principle [3])**.** *Let $B(x_i, r_i) \subset \mathbb{R}^d$ be a sequence of balls such that $\lim_{i\to\infty} r_i = 0$. Suppose that there exists $0 \leq \alpha \leq d$ such that $\limsup_{i\to\infty} B(x_i, r_i^{\alpha/d})$ has full Lebesgue measure. Then,*

$$\dim_{\mathcal{H}} \left(\limsup_{i\to\infty} B(x_i, r_i) \right) \geq \alpha. \tag{3}$$

The moral of this method is that we can translate a Hausdorff measure fractal problem to a Lebesgue measure problem, which will typically be easier to deal with.

As a token of the power of this method, as well as to exemplify how it works, let me prove in the following few lines the Jarník-Besicovitch theorem. This classical theorem states that the set

$$S_\tau = \Big\{ x \in (0,1) \, : \, \big|x - \frac{p}{q}\big| < \frac{1}{q^\tau} \text{ for infinitely many } p/q \Big\}, \quad \tau \geq 2,$$

where all fractions are considered to be irreducible, has $\dim_{\mathcal{H}} S_\tau = 2/\tau$. By the Dirichlet approximation theorem, or using continued fractions, one immediately gets $(0,1) \setminus \mathbb{Q} \subset S_2$. Therefore, $|S_2| = 1$ and $\dim_{\mathcal{H}} S_2 = 1$. For $\tau > 2$, write the set as

$$S_\tau = \limsup_{q\to\infty} B\left(\frac{p}{q}, \frac{1}{q^\tau}\right).$$

Choose the dilation by α such that $\tau\alpha = 2$, so that

$$\limsup_{q\to\infty} B\left(\frac{p}{q}, \left(\frac{1}{q^\tau}\right)^\alpha\right) = \limsup_{q\to\infty} B\left(\frac{p}{q}, \frac{1}{q^{\tau\alpha}}\right) = \limsup_{q\to\infty} B\left(\frac{p}{q}, \frac{1}{q^2}\right) = S_2.$$

Since S_2 has full Lebesgue measure, the Mass Transference Principle implies that $\dim_{\mathcal{H}} S_\tau \geq 2/\tau$. The upper bound follows from covering the set in the standard way.

In our works [10, 11], we apply the Mass Transference Principle similarly to the limsup sets that arise from Bourgain's counterexample. A technical difficulty is that in this case we do not get a limsup of balls, but of rectangles. To tackle this, we use a generalization of the Mass Transference Principle given by Wang-Wu [14], which allows dilations from rectangles to rectangles.

The results that I present in this talk are two:

- in [10], we improve the current best lower bound in (2) by combining Bourgain's counterexample and the intermediate space trick of Du-Kim-Wang-Zhang [8], and using the Mass Transference Principle as in the proof of the Jarnik-Besicovitch theorem above.
- in [11], we consider the alternative symbols

  $$P_k(\xi) = \xi_1^k + \xi_2^k + \ldots + \xi_d^k, \qquad k \in \mathbb{N}, \quad k \geq 2,$$

  studied by An-Chu-Pierce [1] in the Lebesgue case $\alpha = d$. Using the Mass Transference Principle, we generalize their result to the fractal case.

The precise results that we obtain depend on several parameters and are too long to write explicitly here. Instead, we refer the reader to the statements and visual diagrams that we included in our articles [10, 11].

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Phase space projections

Marco Fraccaroli

(joint work with Olli Saari, Christoph Thiele, Gennady Uraltsev)

Let $\mathcal{P}$ be a partition into dyadic intervals of the unit interval $[0,1)$ in $\mathbb{R}$, and let $\Sigma(\mathcal{P})$ be the σ-algebra generated by $\mathcal{P}$. Given a function f on $\mathbb{R}$, by averaging it on each element of $\mathcal{P}$ we define an orthogonal projection from $L^2([0,1))$ onto the subspace $L^2([0,1)) \cap \mathcal{M}(\Sigma(\mathcal{P}))$ of functions that are measurable with respect to $\Sigma(\mathcal{P})$. This projection is at the core of the Calderón-Zygmund decomposition and it can be generalized to the case of the unit cube $[0,1)^d$ in $\mathbb{R}^d$ for every dimension $d \in \mathbb{N}$. In particular, it is a cornerstone in the analysis of forms satisfying translation and dilation symmetries, for example those associated with Calderón-Zygmund kernels, see [7]. However, this projection does not preserve well the localization properties of the Fourier transform $\hat{f}$ of the function f. As a consequence, it is not well-suited for the time-frequency analysis of forms satisfying additional modulation symmetries, for example those associated with the bilinear Hilbert transform and the Carleson operator, see [8].

In this talk we describe a substitute construction, the phase space projection, that is more sensitive to the localization of both f and $\hat{f}$ at the same time. The construction is inspired by that of Muscalu, Tao, and Thiele [6] in the case of dimension $d = 1$. The novelty of our upcoming work in [2] is the description of such a construction in the multidimensional case $d > 1$. Moreover, the estimates we obtain for the phase space projection are useful in the proof of uniform bounds for families of multilinear forms satisfying translation, dilation, and modulation symmetries. A first class of examples is the one-parameter family of forms associated with the bilinear Hilbert transforms in the case of dimension $d = 1$. Bounds for these forms in terms of the L^p norms of the functions and uniformly in the parameter were proved in [9], [6], [4], [5], and finally settled in [10]. A second class of examples is the two-parameter family of forms associated with the complex bilinear Hilbert transforms (or bilinear Beurling-Ahlfors transforms) and more generally with a certain subfamily of two-dimensional bilinear Hilbert transforms (2D-BHTs) in the case of dimension $d = 2$. For these forms we obtain bounds in the local L^2 range uniformly in the parameters in our upcoming work [3]. The question about uniform bounds for the forms associated with the whole family of 2D-BHTs remains a difficult open problem. We refer to [1] and [11] for classifications of the different subfamilies of 2D-BHTs.

This talk is based on joint work in progress with Olli Saari, Christoph Thiele, and Gennady Uraltsev.

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Local Geometry of Model Radon-Like Transforms

Philip T. Gressman

We discuss a local characterization of the geometry in all dimensions and codimensions of so-called model Radon-like transforms, which are those that locally map L^p to L^q for a maximal collection of pairs (p^{-1}, q^{-1}). While there have been several explorations of this class of operators over the past two decades, most notably by Ricci [4] and D. Oberlin [3] (see also [1]), there has until now been no coherent way to describe the local geometry of such objects.

The precise formulation is as follows. Let d_1, n, and k be positive integers such that $n > k$. Suppose that U is an open subset of $\mathbb{R}^n \times \mathbb{R}^{d_1}$, that $\phi(x,t)$ is a smooth function of $(x,t) \in U$ with values in $\mathbb{R}^k$ and that $\gamma_t(x) := (t, \phi(x,t))$. Let $n_1 := d_1 + k$ and $d := n - k$ and consider the Radon-like transform

$$Tf(x) := \int_{\mathbb{R}^{d_1}} f(\gamma_t(x))\eta(x,\gamma_t(x))dt, \tag{1}$$

which is well-defined *a priori* for all nonnegative Borel-measurable functions f on $\mathbb{R}^{n_1}$. Here $\eta(x,y)$ is a cutoff function which will typically be restricted to have small compact support containing some distinguished point $(x_*, \gamma_{t_*}(x_*))$ for $(x_*, t_*) \in U$. It is assumed that the Jacobian matrix $D_x \phi$ (i.e., the matrix of first partial derivatives of ϕ with respect to the x-variables) has full rank k for every (x,t) belonging to the support of $\eta(x, \gamma_t(x))$.

When studying the $L^p \to L^q$ mapping properties of such Radon-like transforms, a particular pair of exponents arise via Knapp-type examples as the best-possible p and q for any specific values of n, k, and d_1, namely:

$$p_b = \frac{kd}{nd_1} + 1 \text{ and } q_b = \frac{n_1 d}{kd_1} + 1. \tag{2}$$

In the special case when $n = n_1$ and $k = 1$, $p_b = (n+1)/n$ and $q_b = n+1$; it has long been understood that the operator (1) maps $L^{(n+1)/n}(\mathbb{R}^n)$ to $L^{n+1}(\mathbb{R}^n)$ precisely when the family of submanifolds indexed by $x \in \mathbb{R}^n$ and parametrized by $\gamma_t(x)$ for $t \in \mathbb{R}^{d_1}$ for each fixed x exhibits nonzero rotational curvature in the sense of Phong and Stein. However, when $k > 1$, nonvanishing rotational curvature is sufficient but not generally necessary.

The fundamental geometric condition governing nondegeneracy of T can be phrased in terms of a Newton-diagram-like construction for an associated trilinear curvature form, and the proof establishes both necessity and sufficiency up to restricted strong-type endpoint estimates. Suppose as noted above that the Jacobian matrix $D_x\phi$ (arranged so that rows correspond to the coordinates of ϕ and columns to the coordinates of x) is rank k at (x,t). Let $w_1, \ldots, w_d$ be any orthonormal vectors in $\mathbb{R}^n$ which span the kernel of $D_x\phi$ at the point (x,t). For $i \in \{1,\ldots,d_1\}$, $i' \in \{1,\ldots,k\}$, and $i'' \in \{1,\ldots,d\}$, let

$$Q_{ii'i''} := \sum_{\ell=1}^{n} w^{\ell}_{i''} \frac{\partial^2 \phi^{i'}}{\partial t^i \partial x^{\ell}}(x,t)$$

where upper indices as in $p^{i'}$ and $w^{\ell}_{i''}$ indicate the coordinates in the standard bases, e.g, $\phi := (\phi^1, \ldots, \phi^k)$ and $w_{i'} := (w^1_{i'}, \ldots, w^n_{i'})$. From these coefficients, we build a trilinear functional $Q : \mathbb{R}^{d_1} \times \mathbb{R}^k \times \mathbb{R}^d \to \mathbb{R}$ by means of the formula

$$Q(u,v,w) := \sum_{i=1}^{d_1} \sum_{i'=1}^{k} \sum_{i''=1}^{d} Q_{ii'i''} u^i v^{i'} w^{i''} \tag{3}$$

for all $x \in \mathbb{R}^{d_1}$, $y \in \mathbb{R}^k$, and $z \in \mathbb{R}^d$.

Given a multiindex $\beta \in \mathbb{Z}^k_{\geq 0}$ and a sequence $\mathcal{I} := \{i_1, \ldots, i_s\}$ of integers belonging to $\{1, \ldots, k\}$, it will be said that β counts $\mathcal{I}$ when for each $\ell \in \{1, \ldots, k\}$, there are exactly β_ℓ values of $j \in \{1, \ldots, s\}$ such that $i_j = \ell$; in other words, β_ℓ is number of elements of the sequence $\mathcal{I}$ that equal ℓ. Given Q as in (3), let $N(Q)$ denote the convex hull in $[0,\infty)^{d_1+k+d}$ of the collection of all triples $(\alpha, \beta, \gamma) \in \mathbb{Z}^{d_1}_{\geq 0} \times \mathbb{Z}^k_{\geq 0} \times \mathbb{Z}^d_{\geq 0}$ with $|\alpha| = |\beta| = |\gamma| \leq \min\{d,k\}$ (where α, β, and γ are regarded as multiindices) for which either $(\alpha,\beta,\gamma) = (0,0,0)$ or for which there exist $\mathcal{I} := \{i_1, \ldots, i_s\} \subset \{1, \ldots, k\}$ and $\mathcal{J} := \{j_1, \ldots, j_s\} \subset \{1, \ldots, d\}$ such that β counts $\mathcal{I}$, γ counts $\mathcal{J}$, and

$$\partial^\alpha_\tau|_{\tau=0} \det \begin{bmatrix} Q(\tau, e_{i_1}, e_{j_1}) & \cdots & Q(\tau, e_{i_1}, e_{j_s}) \\ \vdots & \ddots & \vdots \\ Q(\tau, e_{i_s}, e_{j_1}) & \cdots & Q(\tau, e_{i_s}, e_{j_s}) \end{bmatrix} \neq 0, \tag{4}$$

where $\{e_i\}_{i=1}^k$ is the standard basis of $\mathbb{R}^k$, $\{e_j\}_{j=1}^d$ is the standard basis of $\mathbb{R}^d$, and $\tau \in \mathbb{R}^{d_1}$. Then let

$$\begin{aligned}\mathcal{N}_{\mathcal{R}}(Q) := \bigcap \Big\{ & N(Q') \mid Q'(x,y,z) = Q(O_1 x, O_2 y, O_3 z) \text{ for orthogonal} \\ & \text{matrices } O_1, O_2, O_3 \text{ and all } x \in \mathbb{R}^{d_1}, y \in \mathbb{R}^k, z \in \mathbb{R}^d \Big\}\end{aligned} \tag{5}$$

(i.e., $\mathcal{N}_{\mathcal{R}}(Q)$ is the intersection of all such $N(Q')$). The functional Q will be called nondegenerate when the point

$$\Big(\overbrace{\frac{dk}{d_1 n}, \ldots, \frac{dk}{d_1 n}}^{d_1 \text{ copies}}, \overbrace{\frac{d}{n}, \ldots, \frac{d}{n}}^{k \text{ copies}}, \overbrace{\frac{k}{n}, \ldots, \frac{k}{n}}^{d \text{ copies}} \Big) \tag{6}$$

belongs to $\mathcal{N}_{\mathcal{R}}(Q)$. Any Q for which (6) does not belong to $\mathcal{N}_{\mathcal{R}}(Q)$ is called degenerate. The main theorem is the following

Theorem 1. *Consider the transform T given by* (1)*. Let $(x,y) \in \mathbb{R}^n \times \mathbb{R}^{n_1}$ have the property that $y = \gamma_t(x)$ for some $(x,t) \in U$, and suppose that the Jacobian matrix $D_x\phi$ is rank k at (x,t). Let Q be the trilinear functional given by* (3) *at the point (x,t). Let $\Delta \subset [0,1]^2$ be the closed triangle with vertices $(0,0), (1,1)$ and $(1/p_b, 1/q_b)$.*

(1) *If Q is nondegenerate and ϕ is a polynomial in x and t, then there exists an η of compact support which is nonvanishing at (x,y) such that* (1) *satisfies a restricted strong-type (p_b, q_b) inequality. By interpolation, T maps $L^p(\mathbb{R}^{n_1})$ to $L^q(\mathbb{R}^n)$ for all points $(1/p, 1/q)$ belonging to the triangle in Δ with the possible exception of the endpoint $(1/p_b, 1/q_b)$.*

(2) *If Q is nondegenerate and ϕ is merely a smooth function of x and t, then there exists an η of compact support which is nonvanishing at (x, y) such that* (1) *maps $L^p(\mathbb{R}^{n_1})$ to $L^q(\mathbb{R}^n)$ for all pairs $(1/p, 1/q)$ belonging to the interior of the triangle Δ (note that T also trivially maps L^p to itself for all $p \in [1, \infty]$).*

(3) *If Q is degenerate and $\eta(x, y) \neq 0$, then* (1) *fails to be bounded from $L^p(\mathbb{R}^{n_1})$ to $L^q(\mathbb{R}^n)$ for all pairs $(1/p, 1/q)$ belonging to some neighborhood of $(1/p_b, 1/q_b)$. This neighborhood may be taken to depend only on Q.*

In short, for smooth ϕ, nondegeneracy of Q is necessary and sufficient for $L^p \to L^q$ boundedness for some set of pairs $(1/p, 1/q)$ having $(1/p_b, 1/q_b)$ in its closure.

The proof is based on a recent testing condition characterization of the norm of Radon-like transforms [2] and a series of results previously used to characterize affine Hausdorff measure on general submanifolds. While it seems likely that nondegeneracy of Q is both necessary and sufficient for full $L^{p_b} \to L^{q_b}$ boundedness of (1), the methods to be used here just fail to answer this question completely in the polynomial case. The robust characterization of boundedness of (1) that appears in [2] leaves open the possibility that this discrepancy may ultimately be resolved with only minor adaptations of the current proof, but resolving the endpoint question for the smooth case would likely require a substantially different approach.

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On Littlewood–Paley theory for space curves

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(joint work with James Wright)

In a variety of situations in harmonic analysis, PDE and analytic number theory, one encounters functions with Fourier transform supported in a neighbourhood of some submanifold of the frequency domain. Square function or Littlewood–Paley inequalities are powerful tools for analysing such functions, allowing one to break up the function into pieces which are Fourier support in smaller regions and are, consequently, more easily understood.

Classically, Littlewood–Paley theory forms part of Euclidean harmonic analysis, but here we describe recent work from [4] which explores these questions in other setting such the rings $\mathbb{Z}/N\mathbb{Z}$ of congruence classes modulo some integer N. To introduce the topic, however, it is worthwhile reviewing the basic setup in the classical Euclidean setting.

The Euclidean case. Let $n \geq 2$ and consider the compact portion of the moment curve parameterised by the moment map

$$\gamma\colon [0,1] \to \mathbb{R}^n, \quad \gamma\colon t \mapsto (t, t^2, \dots, t^n).$$

We define the *Fourier extension operator* by

$$(1) \qquad Ef(x) := \int_0^1 e^{2\pi i x\cdot\gamma(t)} f(t)\, dt \qquad \text{for all } f \in L^1([0,1]) \text{ and } x \in \mathbb{R}^n.$$

Note that the (distributional) Fourier support of Ef lies inside the image of γ. According to the philosophy described above, we wish to break up the Ef into pieces which have smaller Fourier support. To do this, let $0 < \delta < 1$ be a small dyadic number and let $\mathcal{I}(\delta)$ denote the family of dyadic subintervals of $[0,1]$ of length δ. We may then write

$$(2) \qquad Ef = \sum_{I \in \mathcal{I}(\delta)} E_I f \qquad \text{where } E_I f := E\big(f\chi_I\big) \text{ for all } I \in \mathcal{I}(\delta).$$

Each $E_I f$ has Fourier support lying in $\gamma(I)$, the image of the interval I under the moment map.

The problem is now to compare the size of Ef with $E_I f$. One easy way to do this is via the triangle inequality:

$$|Ef(x)| \leq \sum_{I \in \mathcal{I}(\delta)} |E_I f(x)|.$$

This estimate is typically very poor, however. Indeed, by definition the functions $E_I f$ all oscillate with different frequencies and we therefore expect significant cancellation between the terms in the sum (2). When taking into account cancellation, the best one can reasonably hope for is square-root cancellation:

$$(3) \qquad |Ef(x)| \leq C\Big(\sum_{I \in \mathcal{I}(\delta)} |E_I f(x)|^2\Big)^{1/2},$$

where here C denotes a constant which does not depend on the parameter δ. It is not difficult to see that the pointwise estimate (3) fails in general. Nevertheless, the inequality does hold on average.

Theorem 1 ([2])**.** *For all $n \geq 2$ and $1 \leq m \leq n$, there exists a constant $C_m \geq 1$ such that*

$$(4) \qquad \|Ef\|_{L^{2m}(B_{\delta^{-n}})} \leq C_m \Big\| \Big(\sum_{I \in \mathcal{I}(\delta)} |E_I f|^2\Big)^{1/2} \Big\|_{L^{2m}(w_{B_{\delta^{-n}}})}$$

holds for all $f \in L^1([0,1])$ and $0 < \delta < 1$ dyadic.

Here $B_{\delta^{-n}}$ denotes a Euclidean ball of radius δ^{-n} and arbitrary centre, and $w_{B_{\delta^{-n}}}$ is a rapidly decaying weight function concentrated on $B_{\delta^{-n}}$; we refer to [2] for the precise definitions. The inequality in the $n = 2$ case goes back to work of Fefferman [1]. The general case is implicit in works of Prestini [8, 7], albeit the arguments of these papers are somewhat lacking in detail. More recently, the inequality was rediscovered in [2], which includes a complete proof and contextualises the result in relation to recent developments in harmonic analysis and analytic number theory. It is remarked that a reverse form of (4) holds as a simple consequence of a classical and elementary square function estimate due to Carleson (see, for instance, [9]).

Open Problem. *There is no standard interpolation theory for square function inequalities such as* (4)*. In particular, for $n = 2$ one cannot directly deduce from Theorem 1 that*

$$\|Ef\|_{L^p(B_{\delta^{-2}})} \leq C_p \Big\| \Big(\sum_{I \in \mathcal{I}(\delta)} |E_I f|^2 \Big)^{1/2} \Big\|_{L^p(w_{B_{\delta^{-2}}})} \tag{5}$$

holds for all $2 \leq p \leq 4$. Alternative methods can be used to prove (5) *for $2 \leq p \leq 4$, but these arguments come with logarithmic losses in the δ parameter. It would be interesting to establish* (5) *for some value of $2 < p < 4$ with C_p truly uniform in δ.*

By a well-known $2n$-*orthogonality argument* due to Córdoba and Fefferman, the proof of Theorem 1 reduces to establishing the following number-theoretic proposition.

Proposition 2 ([2])**.** *For all $n \in \mathbb{N}$ there exists a constant $C_n \geq 1$ such that the following holds. Let $0 < \delta < 1$ and suppose $(x_1, \dots, x_n)$, $(y_1, \dots, y_n) \in [0,1]^n$ satisfy*

$$|x_1^j + \cdots + x_n^j - y_1^j - \cdots - y_n^j| \leq \delta^n \qquad \text{for } 1 \leq j \leq n.$$

Then there exists a permutation σ on $\{1, \cdots, n\}$ such that $|x_j - y_{\sigma(j)}| \leq C_n \delta$ for all $1 \leq j \leq n$.

Proposition 2 examines the structure of 'almost solutions' to a Vinogradov-type system of equations. In particular, it can be roughly interpreted as saying that every 'almost solution' to the system $x_1^j + \cdots + x_n^j = y_1^j + \cdots + y_n^j$ for $1 \leq j \leq n$ is 'almost trivial', in the sense that the y_j are close to some permutation of the x_j.

The passage from Proposition 2 to Theorem 1 is elementary. By translation invariance, we may assume $B_{\delta^{-n}}$ is centred at 0. Fix φ a non-negative Schwartz function which dominates (up to a constant factor) the characteristic function of the ball $B_{\delta^{-n}}$ and is Fourier supported in $B(0, \delta^n)$. We may then write

$$|Ef|^{2m} \cdot \varphi = \sum_{\substack{I_j, J_j \in \mathcal{I}(\delta) \\ 1 \leq j \leq m}} \prod_{j=1}^m Ef_{I_j} \cdot \varphi \, \overline{\prod_{j=1}^m Ef_{J_j} \cdot \varphi}. \tag{6}$$

Thus, by Parseval's theorem,

$$\|Ef\|_{L^{2m}(B_{\delta^{-n}})}^{2m} \leq C \sum_{\substack{I_j, J_j \in \mathcal{I}(\delta) \\ 1 \leq j \leq m}} \int_{\mathbb{R}^n} \Big(\prod_{j=1}^m Ef_{I_j} \cdot \varphi \Big)\hat{}\,(\xi) \, \overline{\Big(\prod_{j=1}^m Ef_{J_j} \cdot \varphi \Big)\hat{}\,(\xi)} \, d\xi. \tag{7}$$

It is easy to see that each function $\big(Ef_I \cdot \varphi\big)\widehat{\ }$ is supported in a δ^n-neighbourhood of $\gamma(I)$. From this observation and the basic properties of the Fourier transform, a given term in the above sum is non-zero if and only if there exist $x_j \in I_j$ and $y_j \in J_j$ for $1 \le j \le m$ such that

$$|\gamma(x_1) + \cdots + \gamma(x_n) - \gamma(y_1) - \cdots - \gamma(y_n)| \le C\delta^n.$$

One may then apply Proposition 2 to conclude that essentially the only terms which contribute to the sum are those where the $J_1, \dots, J_m$ are a permutation of $I_1, \dots, I_m$. Using this observation and then reversing the initial steps (6) and (7) concludes the argument.

The discrete setting. Having given an overview of the Euclidean theory, we turn to the problem in the discrete setting $\mathbb{Z}/N\mathbb{Z}$, which was investigated in the recent paper [4]. Since $\mathbb{Z}/N\mathbb{Z}$ is a locally compact abelian group, it admits a Fourier analysis and tools such as Plancherel's theorem are available. In particular, it is possible to make sense of an analogue of Ef in this setting and also of square function inequalities such as (4).[1] We will not detail precisely the formulation of the square function problem over $\mathbb{Z}/N\mathbb{Z}$, but simply remark that a variant of the argument described above reduces matters to proving an analogue of Proposition 2 involving congruence equations.

Proposition 3. *Let n, $a \in \mathbb{N}$ and p be a rational prime such that $p > n \ge 2$. Suppose $(x_1, \dots, x_n)$, $(y_1, \dots, y_n) \in \mathbb{Z}^n$ satisfy the congruence equations*

$$x_1^j + \cdots + x_n^j \equiv y_1^j + \cdots + y_n^j \mod p^{na} \qquad \text{for } 1 \le j \le n. \tag{8}$$

Then there exists a permutation σ on $\{1, \cdots, n\}$ such that $x_j \equiv y_{\sigma(j)} \mod p^a$ for all $1 \le j \le n$.

Proposition 3 can also be interpreted as saying almost solutions to a Vinogradov system of equations are almost trivial. Here, however, the notion of 'almost' is p-adic: we assume the integers $(x_1, \dots, x_n)$, $(y_1, \dots, y_n) \in \mathbb{Z}^n$ solve the system modulo p^{na} (in this sense, they form an 'almost solution') and conclude that, up to permutation, each y_j is close to x_j in the sense that the two integers agree modulo p^a (the solution is 'almost trivial'). The statement is therefore directly analogous to that of Proposition 2. Despite this, the method of proof of [2] breaks down completely in the discrete setting. In particular, the arguments of [2] rely on order properties of the real line and tools from calculus which are simply not available when working over $\mathbb{Z}/N\mathbb{Z}$. Consequently, in order to prove Proposition 3, a new method was developed in [4] which is based on the careful analysis of sublevel sets of univariate polynomials. Central to the proof of Proposition 3

[1] One key consideration, however, is that the function spaces $\ell^p([\mathbb{Z}/N\mathbb{Z}]^n)$ are finite dimensional and so all ℓ^p-norms are equivalent. Thus, it is possible to prove a square function inequality for *any* ℓ^p by factoring through ℓ^2 using equivalence of norms and using the fact that the ℓ^2 square function estimate is a trivial consequence of Plancherel's theorem. However, the equivalence of norms involves a constant which depends on N, the cardinality of the underlying ring. To make the problem non-trivial, one must stipulate that the constants in the norm inequalities are independent of N (or have some sub-polynomial dependence).

is a non-archimedean structural decomposition for sublevel sets of univariate real polynomials due to Phong–Stein–Sturm [6] (see also [10]).

Proposition 3 constitutes the main result presented in the talk. Although the result was motivated by the square function inequality, the authors believe that the almost solution count is of interest in its own right, both in terms of the statement and the method of proof. This work forms part of a wider programme to study classical topics in harmonic analysis over the discrete rings $\mathbb{Z}/N\mathbb{Z}$ and the authors hope that Proposition 3 will lead to progress on more difficult problems concerning factorisation of polynomials over $\mathbb{Z}/N\mathbb{Z}$. Such questions arise naturally when studying Fourier restriction in this setting, see [3].

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The Gross-Pitaevskii equation: Phase space and energies

Herbert Koch

(joint work with Xian Liao)

The Gross-Pitaveskii equation is essentially the defocusing nonlinear Schrödinger equation with nonvanishing conditions at ∞,

$$iq_t + q_{xx} - 2(|q|^2 - 1)q = 0 \quad \text{on } \mathbb{R} \times \mathbb{R} \ni (t,x),$$
$$\lim_{x\to\pm\infty} |q(t,x)| = 1.$$

The energy

$$\mathcal{E}(q) = \int |q_x|^2 + (|q|^2-1)^2 dx$$

is conserved. It motivates to consider the equation in

$$X^s = \Big\{ q \in H^s_{loc} : E^s(u) := \big(\|q_x\|^2_{H^{s-1}} + \||q|^2 - 1\|^2_{H^{s-1}} \big)^{1/2} < \infty \Big\} / \mathbb{S}^1$$

for $s \geq 0$ where we consider functions modulo a multiplicative constant of size 1. Special solutions are the dark resp. black solitons $Q_c(x - 2ct)$ where $|c| \leq 1$ and

$$Q_c(x) = ic + \sqrt{1-c^2} \tanh(\sqrt{1-c^2} x),$$

with $Q_{-1} = Q_1$ due to the identification in X^s. The soliton resolution conjecture amounts to an asymptotic decomposition of solutions into solitons with velocity $<$ 2, KdV waves moving with velocity ± 2, and dispersive waves with group velocities outside $[-2, 2]$.

We equipp X^s with the metric

$$d^s(p,q) = \Big(\int_{-\infty}^{\infty} \inf_{|\mu|=1} \| \operatorname{sech}(x-y)(\mu p(x) - q(x)) \|^2_{H^s_x} dy \Big)^{1/2}. \tag{1}$$

The metric, topological and analytic structure of the metric space (X^s, d^s) can be described as follows.

Theorem 1. *The space X^s with the metric d^s is a complete metric space. The set $1 + C_c^\infty(\mathbb{R})$ is dense. The metric is compatible with E^s:*

$$c^{-1} d^s(1,p) \leq E^s(p) \leq E^s(q) + c(1 + E^s(q))^{1/2}) d^s(p,q) + c(d^s(p,q))^2.$$

There is an analytic structure compatible with this metric. The set $\{Q_c\}$ is a strong deformation retract.

It is instructive to verify that

$$\lim_{\varepsilon \to 0} d^s(\exp(i\varepsilon \log(2 + |x|^2)), 1) = 0.$$

The last statement of the theorem says that there is a continuous map

$$\Psi : X^s \times [0,1] \to X^s$$

so that $\Psi(q, 0) = q$, $\Psi(Q_c, t) = Q_c$, $\Psi(q, 1) \in \{Q_c\}$. In particular X^s has the topology of a circle.

There is simple reasoning which shows that the topology of X^s is non trivial. Both the momentum

$$\mathcal{P} = \frac{1}{i} \int q \bar{q}_x dx$$

and the asymptotic phase change

$$\Xi(q) = \frac{1}{i} \log(\lim_{x \to \infty} \bar{q}(x) q(-x)) \in \mathbb{R}/2\pi\mathbb{Z}$$

are defined on $\{q \in X^0 : q' \in L^1\}$. One calculates

$$\Xi(Q_c) = 2 \arccos(c).$$

The difference

$$H_1(q) = \mathcal{P} - \Xi \in \mathbb{R}/2\pi\mathbb{Z}$$

is a continuous function on $X^{1/2}$, which again cannot be lifted to map to $\mathbb{R}$, not even the restriction to $\{Q_c\} \in X^s$, hence X^s has nontrivial topology.

The Gross-Pitaevskii equation has a Lax-pair with the Lax operator

$$L\psi := \begin{pmatrix} i\partial & -iq \\ i\bar{q} & -i\partial \end{pmatrix} \psi$$

and the Gross-Pitaevskii equation is equivalent to a commutator identity

$$[\partial_t + P, L] = 0.$$

The Lax operator is self adjoint with essential spectrum $(-\infty, -1] \cup [1, \infty)$ and isolated simple eigenvalue in $(-1, 1)$ which correspond to solitons. Let $q \in 1 + \mathcal{S}$ and $\operatorname{Im}\lambda > 0$. The equation

$$L\psi = z\psi$$

has a two dimensional space of solutions spanned by left and right Jost solutions ψ_l, ψ_r. One defines the transmission coefficient by

$$T(\lambda) = \det(\psi_l, \psi_r)^{-1}.$$

Due to the commutator identity the transmission coefficient is conserved under the evolution of the Schrödinger equation. Faddeev and Takhtajan [1] define conserved Hamiltonians as coefficients of the asymptotic series

$$-i \log T(\lambda) \sim \sum_{n=1}^{\infty} \mathcal{H}_n (2z)^{-1-n}$$

where $z = \sqrt{1 - \lambda^2}$ with positive imaginary part,

$$\mathcal{H}_0 = \mathcal{M} = \int |q|^2 - 1 \, dx, \quad \mathcal{H}_1 = \mathcal{P} = \frac{1}{i} \int q\bar{q}_x dx, \quad \mathcal{H}_2 = \mathcal{E} = \int |q_x|^2 + (|q|^2 - 1)^2 dx,$$

$$\mathcal{H}_3 = \frac{1}{i} \int q_x \bar{q}_{xx} + 3(|q|^2 - 1) q\bar{q}_x dx - \mathcal{P}.$$

The energy $\mathcal{E}$ is defined on X^1. Unfortunately, it is the only quantity of the four above which can be defined on any X^s space. In [3] we prove (with a nontrivial adaptation of the techniques of [2, 5]) that

$$-i \log \underline{T_c} := -i \log T - \mathcal{M}(2z)^{-1} - i\mathcal{P}(2z(\lambda + z))^{-1}$$

defines a smooth map from X^s with $s > \frac{1}{2}$ to holomorphic functions on $\{z = x + iy, y > 1\}$. We use it to define conserved energies on X^s controlling E^s for $s > \frac{1}{2}$. In [4] we prove that

$$-i \log T_c := -i \log T - \mathcal{M}(2z)^{-1} - i\Xi(2z(\lambda + z))^{-1}$$

defines a smooth map on the universal covering space of X^0 to holomorphic function which allows to define conserved energies as above for $s \geq 0$. The coefficients of its asymptotic series define Hamiltonians H_n on $X^{n/2}$ for all n, with quadratic part

$$H_{2n}(q) = \int |q^{(n)}|^2 dx + O((E^n(q))^3,\ H_{2n+1}(q) = \frac{1}{i}\int q^{(n)}\bar{q}^{(n+1)}dx + O((E^{n+\frac{1}{2}}(q))^3)$$

for $n \geq 1$.

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Small cap decoupling for the moment curve in $\mathbb{R}^3$

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(joint work with Larry Guth)

We prove sharp small cap decoupling estimates for the moment curve $\mathcal{M}^3 = \{(t,t^2,t^3) : t \in [0,1]\}$ in $\mathbb{R}^3$. Begin by describing the problem and our results in terms of exponential sums. One of our motivations for this work was the conjecture of Demeter, Guth, and Wang [5] that for each $N \geq 1$, $0 \leq \sigma \leq 2$, and $s \geq 1$,

$$(1) \qquad \int_{[0,1]^2\times[0,\frac{1}{N^\sigma}]} |\sum_{k=1}^{N} e(kx_1 + k^2x_2 + k^3x_3)|^{2s}dx \leq C_\epsilon N^\epsilon \Big[N^{s-\sigma} + N^{2s-6}\Big],$$

where $e(t) = e^{2\pi it}$. The $s = 1$ and $s = \infty$ versions of this conjecture are easily verified using L^2-orthogonality and the triangle inequality, respectively. When $\sigma = 0$, this is Vinogradov's mean value theorem, solved in three dimensions by Wooley [8] and using decoupling for the moment curve by Bourgain, Demeter, and Guth [1]. The case of $\sigma = 2$ was proven by Bombieri and Iwaniec [2] and by Bourgain [3] using a different argument. In [5], they prove (1) in the range $0 \leq \sigma \leq \frac{3}{2}$. Our general small cap decoupling theorem has the following exponential sum corollary, which implies (1) in the full range $0 \leq \sigma \leq 2$: for each $R \geq 1$, $\frac{1}{3} \leq \beta \leq 1$, $2 \leq p \leq 6 + \frac{2}{\beta}$, and $r \geq R^{\max(2\beta,1)}$,

$$(2) \qquad |Q_r|^{-1}\int_{Q_r} |\sum_{\xi\in\Xi} a_\xi e(x\cdot(\xi,\xi^2,\xi^3))|^p dx \leq C_\epsilon R^\epsilon R^{\beta\frac{p}{2}}$$

for any r-cube Q_r and any collection $\Xi \subset [0,1]$ with $|\Xi| \sim R^\beta$ consisting of $\sim R^{-\beta}$-separated points and $a_\xi \in \mathbb{C}$ with $|a_\xi| \leq 1$. For comparison, the canonical decoupling of $\mathcal{M}^3$ from [1] only implies (2) in the range $r \geq R^{3\beta}$. We obtain such general exponential sum estimates (without requiring frequency points with extra structure, e.g. $\{(n/N, n^2/N^2, n^3/N^3)\}$ for $N \sim R^\beta$) since decoupling techniques do not rely on counting arguments from number theory.

Now we describe the general set-up which we use to prove (2). Small cap decoupling is an estimate for functions with Fourier support in a certain neighborhood of $\mathcal{M}^3$ in terms of its Fourier projections onto small caps γ which partition the neighborhood. Let $\beta \in [\frac{1}{3}, 1]$ and $R \geq 1$ and assume for simplicity that R^β is an integer. For each $l \in \{0, \dots, R^\beta - 1\}$, a small cap γ is roughly a block centered at $(lR^{-\beta}, (lR^{-\beta})^2, (lR^{-\beta})^3)$ that goes $R^{-\beta}$ in the $(1, 2lR^{-\beta}, 3(lR^{-\beta})^2)$ direction, $R^{-2\beta}$ in the $(0, 2, 6lR^{-\beta})$ direction, and R^{-1} in the $(0, 0, 1)$ direction. Small cap decoupling gives sharp estimates for $\|f\|_p^p / \sum_\gamma \|f_\gamma\|_p^p$ where $f : \mathbb{R}^3 \to \mathbb{C}$ is a Schwartz function with Fourier transform supported in $\sqcup\gamma$ and f_γ is the Fourier projection onto γ. Our strategy follows the basic structure of a high/low proof of decoupling first established for canonical decoupling of the parabola in [7]. Very roughly, the high/low proof of decoupling is a way to interpolate between multilinear restriction estimates (an L^6 inequality for $\mathcal{M}^3$) and L^∞ estimates. The multilinear restriction inequality essentially gives the square function estimate

$$\int |f|^6 \leq C \int |\sum_\gamma |f_\gamma|^2|^3.$$

In order to obtain a sharp estimate for the right hand side, we split the integrand into a high-frequency and low-frequency portion. The low-frequency portion is fed into an iterative process. We identify the Fourier support of the high-frequency portion as related to a small cap tiling of a cone. Applying the recently proved small cap decoupling estimates for the cone [6] to estimate the right hand side above leads to a sharp $\mathcal{M}^3$ small cap inequality.

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Sharp multiplier theorems for Grushin operators

Alessio Martini
(joint work with Gian Maria Dall'Ara)

Let $\mathcal{L} = -\Delta_x - V(x)\Delta_y$ be a Grushin-type operator on $\mathbb{R}^{n_1}_x \times \mathbb{R}^{n_2}_y$, where the coefficient $V : \mathbb{R}^{n_1} \to [0,\infty)$ is comparable to a power law, i.e., $V(x) \simeq |x|^D$ for some exponent $D \in (0,\infty)$. Techniques based on heat-kernel estimates [8] allow one to prove an L^p spectral multiplier theorem of Mihlin–Hörmander type for $\mathcal{L}$, so that an operator of the form $F(\mathcal{L})$ is of weak-type $(1,1)$ and bounded on $L^p(\mathbb{R}^{n_1}_x \times \mathbb{R}^{n_2}_y)$ whenever the spectral multiplier $F : \mathbb{R} \to \mathbb{C}$ satisfies a scale-invariant local Sobolev condition

$$\sup_{t>0} \|F(t\cdot)\chi\|_{L^q_s} < \infty \tag{1}$$

with $q = \infty$ and order $s > Q/2$. Here $\chi \in C^\infty_c((0,\infty))$ is any nontrivial cutoff, while $Q = n_1 + (1 + D/2)n_2$ is the homogeneous dimension associated with the degenerate Riemannian metric determined by $\mathcal{L}$. Clearly the value of Q grows with the degree D, and consequently does the smoothness requirement.

A natural question is whether the above multiplier theorem is sharp, or if instead the smoothness assumption on F can be weakened. As it turns out, improvements are possible in particular cases. Indeed, when $V(x) = |x|^2$, in [7, 5] it was proved that the condition (1) with $q = 2$ and $s > (n_1 + n_2)/2$ is enough; an analogous improvement was obtained in [1] in the case $V(x) = \sum_j |x_j|$ and $n_1 \geq n_2/2$. As $\mathcal{L}$ is elliptic away from $x = 0$, one can see via transplantation that the smoothness condition $s > (n_1+n_2)/2$ in these results is sharp, i.e., it cannot be further pushed down. However, the aforementioned improvements are for very particular choices of the coefficient V (e.g., in these cases V is homogeneous of degree $D \leq 2$), and it is natural to ask whether these restrictions on V are really needed, or instead one can push down the smoothness condition to $s > (n_1 + n_2)/2$ in greater generality.

In joint work with G. M. Dall'Ara [2, 3, 4], we show that neither the homogeneity of V nor the constraint $D \leq 2$ are needed for this improvement. In particular, in the case $n_1 = n_2 = 1$, we can prove that, if $V : \mathbb{R} \to [0,\infty)$ is C^2 off the origin and satisfies the estimates

$$|x^2 V''(x)| \lesssim xV'(x) \simeq V(x) \simeq V(-x), \tag{2}$$

then the condition (1) with $q = 2$ and $s > 2/2$ is enough for the weak-type $(1,1)$ and L^p boundedness $(1 < p < \infty)$ of $F(\mathcal{L})$. In other words, in contrast to the result in [8], where the smoothness assumption depends on V and may be arbitrarily large for fixed dimension of the underlying space, our assumption is independent of V and matches the corresponding assumption in the classical Mihlin–Hörmander theorem for the Euclidean Laplacian on $\mathbb{R}^2$. The estimates (2) are satisfied, among others, by $V(x) = |x|^D$ for any $D \in (0,\infty)$, but also by $V(x) = |x|^d + |x|^D$ and $V(x) = 1/(|x|^{-d} + |x|^{-D})$ for $d, D \in (0,\infty)$, thus showing that comparability with a specific polynomial law is not even needed.

The proof of our result hinges on the analysis of one-dimensional Schrödinger operators of the form

$$\mathcal{H} = -\partial_x^2 + V(x),$$

where V satisfies the estimates (2), which imply that V is a single-well potential diverging at infinity. In particular, we obtain universal estimates for eigenvalues, eigenvalue gaps and eigenfunctions of $\mathcal{H}$, as well as for the matrix elements of the potential V in the basis of the eigenfunctions of $\mathcal{H}$. The universality of these estimates, which may be of independent interest, lies in the fact that they hold for any potential satisfying (2), with implicit constants only depending on those in (2).

The above results for Grushin operators can be thought of as part of a more general research programme aimed at understanding sharp conditions in L^p multiplier theorems for sub-elliptic, non-elliptic operators, where a number of fundamental questions remain open [6].

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Maximizers for Fourier adjoint restriction estimates to cones

Giuseppe Negro

1. Introduction: Maximizers

Let $d \geq 2$. In 1977, Strichartz [9] proved that solutions $u = u(t,x)$ to the wave equation $u_{tt} = \Delta u$ on $\mathbb{R}^{1+d}$, with initial data $u(0,x) = u_0(x)$ and $u_t(0,x) = \dot{u}_0(x)$, satisfy the estimate

$$(1) \qquad \|u\|^2_{L^q(\mathbb{R}^{1+d})} \leq C_d^2 \int_{\mathbb{R}^d} \left(|\xi||\widehat{u_0}(\xi)|^2 + |\xi|^{-1}|\widehat{\dot{u}_0}(\xi)|^2 \right) d\xi, \quad q = 2\tfrac{d+1}{d-1}.$$

Here the multiplicative constant $C_d > 0$ is finite, but it is not explicit.

In 2007, Foschi [2] found that, when $d = 3$, the optimal value for such constant is $C_3 = (3/(16\pi))^{1/4}$. Moreover, he characterized all the *maximizers*, that is, the functions that attain this optimal constant. One of these maximizers is the solution with initial data $(u_0(x), \dot{u}_0(x)) = ((1 + |x|^2)^{-\frac{d-1}{2}}, 0)$, for $d = 3$. Foschi conjectured that this should be a maximizer for arbitrary $d \geq 2$.

In the next section, we will observe that (1) is intimately connected to a more general family of estimates, known as the *Fourier adjoint restriction* estimates on the cone, and extend the conjecture of Foschi to this general setting. In Section 3 we will present the current state of this conjecture. Finally, in Section 4 we will give some ideas of proof.

2. The Fourier adjoint restriction estimates

Consider the measure $\mu_+ := \delta(\tau^2 - |\xi|^2)\mathbf{1}_{\tau>0} = \delta(\tau - |\xi|)/(2|\xi|)$, supported on the one-sheeted cone $\mathbb{K}^d_+ := \{\tau = |\xi|\} \subset \mathbb{R}^{1+d}$; see Figure 1. The *restriction conjecture* for $\mathbb{K}^d_+$ asks whether there is a $C^+_{d,p} > 0$ such that

(2) $$\|\widetilde{f\mu_+}\|_{L^q(\mathbb{R}^{1+d})} \leq C^+_{d,p}\|f\|_{L^p(\mu_+)}, \quad \text{provided } \tfrac{1}{p} = 1 + \tfrac{1+d}{1-d}\tfrac{1}{q} \text{ and } q > \tfrac{2d}{d-1},$$

and $\widetilde{\cdot}$ denotes the Fourier transform on $\mathbb{R}^{1+d}$. It is obvious that (2) holds for μ_+ if and only if it holds, perhaps with a different constant $C_{d,p}$, for the measure $\mu := \delta(\tau^2 - |\xi|^2)$, which is supported on the two-sheeted cone $\mathbb{K}^d := \{\tau^2 = |\xi|^2\}$.

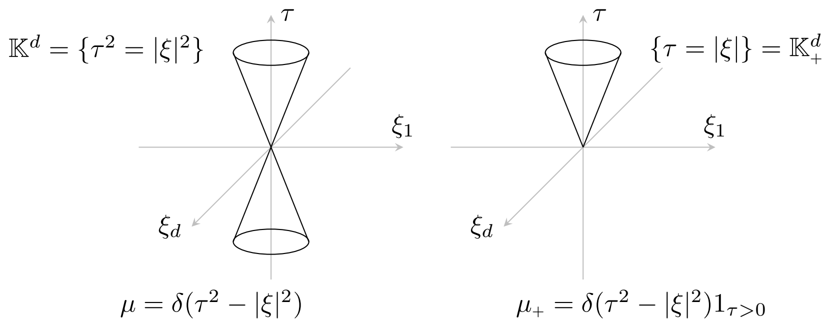

Figure 1. The two-sheeted and the one-sheeted cone.

Here we will make no attempt whatsoever to enlarge the range of parameters for which (2) is proved; see [7] for more information on that. Our task will be to give information on the maximizers to (2), for those cases for which it is known to hold.

Note that (2) surely holds on $\mathbb{K}^d$, hence also on $\mathbb{K}^d_+$, for $p = 2$. Indeed, letting $u(t, x) = \widetilde{f\mu}(t, x)$, we see that $u_{tt} = \Delta u$, with initial data

$$\widehat{u_0}(\xi) = \frac{1}{2|\xi|}\left(f(|\xi|, \xi) + f(-|\xi|, \xi)\right), \quad \widehat{\dot{u}_0}(\xi) = \frac{i}{2}\left(f(|\xi|, \xi) - f(-|\xi|, \xi)\right),$$

where $\widehat{\cdot}$ denotes the Fourier transform on $\mathbb{R}^d$. It follows that the Strichartz estimate (1) is exactly the $p = 2$ case of (2) on $\mathbb{K}^d$, and the conjectured maximizer of

Foschi corresponds to $f_\star(\tau,\xi) := \exp(-|\tau|)$. It is natural to ask whether this $f_\star$ could be a maximizer for (2) for a wider range of parameters, and also on $\mathbb{K}^d_+$; we will see that the answer is generically negative. See also [6] for a survey on this and related questions.

3. Results

We say that $f_\star(\tau,\xi) = \exp(-|\tau|)$ is a *critical point* for (2) on $\mathbb{K}^d_+$ if

$$\frac{\partial}{\partial \epsilon}\left(\frac{\|\widetilde{(f_\star+\epsilon f)\mu_+}\|_{L^q(\mathbb{R}^{1+d})}}{\|f_\star+\epsilon f\|_{L^p(\mu_+)}}\right)^q\Bigg|_{\epsilon=0} = 0, \quad \forall f \in L^p(\mu_+).$$

Theorem 1 ([5])**.** *The function $f_\star$ is a critical point for* (2) *on $\mathbb{K}^d_+$ if and only if $p = 2$.*

In particular, it is clear that $f_\star$ is never a maximizer for the restriction estimate (2) on $\mathbb{K}^d_+$ when $p \neq 2$. The scenario is very similar on the paraboloid, with the gaussian functions playing the role of $f_\star$; see [1].

In the case $p = 2$, hence $q = 2\frac{d+1}{d-1}$, it is possible to give a great deal more information for the maximizers to (2), both on $\mathbb{K}^d_+$ and on $\mathbb{K}^d$; the latter corresponds to the Strichartz inequality (1) as already noted in the previous section. We say that $f_\star$ is a *local maximizer* for (2) on $\mathbb{K}^d_+$ if

$$\frac{\|\widetilde{f\mu_+}\|_{L^q(\mathbb{R}^{1+d})}}{\|f\|_{L^2(\mu_+)}} \le \frac{\|\widetilde{f_\star\mu_+}\|_{L^q(\mathbb{R}^{1+d})}}{\|f_\star\|_{L^2(\mu_+)}}, \quad \forall f \text{ in some } L^2(\mu_+)\text{-neighborhood of } f_\star.$$

The corresponding definitions of *critical point* and *local maximizer* on $\mathbb{K}^d$ are entirely analogous to the ones for $\mathbb{K}^d_+$ and we omit them.

Theorem 2 ([2, 3, 4])**.** *Let $p = 2$. The function $f_\star$ is a maximizer for* (2) *on $\mathbb{K}^d_+$ for $d = 2, 3$. It is a local maximizer for $d \geq 3$. The function $f_\star$ is a maximizer for* (2) *on $\mathbb{K}^d$ for $d = 3$. It is a local maximizer for all odd d, and it is not even a critical point for even d.*

Spatial dimension d	on $\mathbb{K}^d$ with μ	on $\mathbb{K}^d_+$ with μ_+
2	NO	YES
3	YES	YES
4, 6, 8, ...	NO	Local
5, 7, 9, ...	Local	Local

Table 1. Is $f_\star(\tau,\xi) = \exp(-|\tau|)$ a maximizer for the Fourier adjoint restriction estimate (2) with $p = 2$?

The results of Theorem 2 are summarized in Table 1.

4. Idea of proof: the Penrose transform

All results mentioned in Theorem 1 and Theorem 2, with the exception of the original ones of Foschi [2], rely on conformally mapping the Minkowski spacetime $\mathbb{R}^{1+d}$ to a compact submanifold $\mathbb{D}^{1+d}$ of $[-\pi, \pi] \times \mathbb{S}^d$. This $\mathbb{D}^{1+d}$ is known as the *Penrose diamond*, and originally appeared in [8]. See Figure 2.

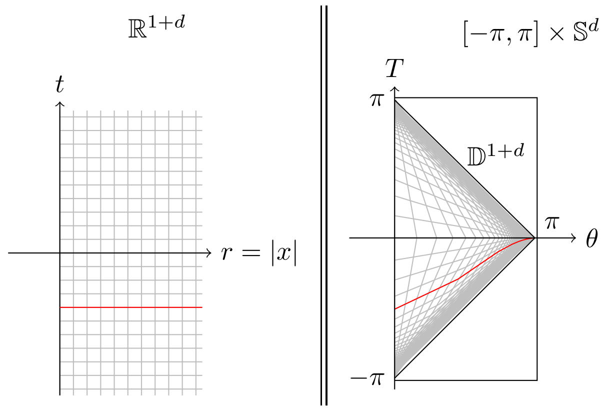

Figure 2. The Penrose map of $\mathbb{R}^{1+d}$ onto the Penrose diamond $\mathbb{D}^{1+d} \subset [-\pi, \pi] \times \mathbb{S}^d$. Here $r \geq 0$ and $\theta \in [0, \pi]$ respectively denote the distance from the origin of $\mathbb{R}^d$ and the geodesic distance from the North Pole of $\mathbb{S}^d$.

Under this map, the right-hand side of (2) reads

$$\|\widetilde{f\mu_+}\|^q_{L^q(\mathbb{R}^{1+d})} = \int_{\mathbb{D}^{1+d}} |U|^q (\cos T + \cos\theta)^{\frac{d+1}{2(p-1)}(2-p)}, \tag{3}$$

where U is a solution of the spherical wave equation $\partial_T^2 U = (\Delta_{\mathbb{S}^d} - \frac{(d-1)^2}{4})U$. More precisely, there is a 1:1 correspondence between $f \in L^p(\mu_+)$ and the initial data of the spherical wave equation. Analogous results hold for $\mathbb{K}^d$, in which case $f \in L^p(\mu)$.

Crucially, both for $\mathbb{K}^d$ and for $\mathbb{K}^d_+$ the conjectured maximizer $f_\star$ corresponds to initial data that are constant functions on $\mathbb{S}^d$; this enables much more precise computations.

We conclude with a remark. As formula (3) shows, the case $p = 2$ is especially simpler, since the right-hand side enjoys a greater degree of symmetry. Thus, our application of the Penrose transform gives a qualitative reason why we can obtain much more precise results in this $p = 2$ case than in the general one.

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Majorant properties in arbitrary dimensions

Lillian B. Pierce
(joint work with Philip T. Gressman, Shaoming Guo, Joris Roos, Po-Lam Yung)

This talk introduced the systematic study of majorant properties on $L^p([0,1]^d)$ in arbitrary dimensions d, motivated by a well-known circle of ideas that is nearly 100 years old. This abstract reprises some motivating questions and new theorems recently developed in a joint paper by Gressman, Guo, Roos, Yung, and the author [GGPRY22].

In 1935, Hardy and Littlewood [HL35] wrote a brief paper on one-dimensional majorant inequalities of the form

$$\Big\| \sum_{n\in\Gamma} a_n e(n\cdot x)\Big\|_{L^p([0,1])} \le \Big\| \sum_{n\in\Gamma} A_n e(n\cdot x)\Big\|_{L^p([0,1])} \tag{1}$$

where $\Gamma \subset \mathbb{Z}$ is a finite set of frequencies. Here as usual, $e(\theta) := e^{2\pi i\theta}$ for $\theta \in \mathbb{R}$. Given a set of frequencies $\Gamma \subset \mathbb{Z}$ and an exponent p, if this inequality holds for all choices of coefficients a_n, A_n with $|a_n| \le A_n$ for each $n \in \Gamma$, then we say the *strict majorant property* holds for Γ, p. For any finite set $\Gamma \subset \mathbb{Z}$, the strict majorant property holds for all $p \in 2\mathbb{N}$ by a simple expansion of the integral, as Hardy and Littlewood point out.

Does it also hold for all $p \notin 2\mathbb{N}$? Hardy and Littlewood write: "This is untrue and, since it is the falsity of (1) which first reveals the difficulties of our problem, we prove it at once..." for $p = 3$. The falsity was verified for all $p \ge 1, p \notin 2\mathbb{N}$ by Boas [Boa63]. Next one can ask whether a weaker *majorant property* holds, for example the property that there is some constant C_p such that (1) holds for all $\Gamma \subset \mathbb{Z}$ upon taking $|a_n| \le 1$ and $A_n = 1$ for $n \in \Gamma$, if the right-hand side is enlarged by C_p. Work of Bachelis [Bac73], Mockenhaupt and Schlag [MS09], and Green and Ruzsa [GR04] dramatically confirmed that the majorant property is violated for every $p > 2, p \notin 2\mathbb{N}$. Majorant properties and possible violations of these properties continue to inspire interest, also because of their close relationship to the local restriction conjecture for the sphere and the Kakeya conjecture [Moc96].

In this talk we introduced the systematic study of strict majorant properties in arbitrarily high dimensions. Let $\Gamma \subset \mathbb{Z}^d$ be a fixed set of d-tuples of integers. We say that Γ satisfies the *strict majorant property* on $L^p([0,1]^d)$ if for all choices of real coefficients $(a_n)_{n\in\Gamma}$, $(A_n)_{n\in\Gamma}$ with $|a_n| \leq A_n$,

$$\Big\| \sum_{n\in\Gamma} a_n e(n\cdot x)\Big\|_{L^p([0,1]^d)} \leq \Big\| \sum_{n\in\Gamma} A_n e(n\cdot x)\Big\|_{L^p([0,1]^d)}. \tag{2}$$

For any set $\Gamma \subset \mathbb{Z}^d$, this statement is true for all $p \in 2\mathbb{N}$. When $p \notin 2\mathbb{N}$, for which Γ is it true?

Our first main result characterizes the sets $\Gamma \subset \mathbb{Z}^d$ for which the strict majorant property holds for all $p > 0$. We recall that a set $\Gamma \subset \mathbb{Z}^d$ is affinely independent if for any $n_0 \in \Gamma$, $\{n - n_0 \in \mathbb{Z}^d : n \in \Gamma, n \neq n_0\}$ is linearly independent.

Fix an integer $d \geq 1$. We prove that a non-empty set $\Gamma \subset \mathbb{Z}^d$ satisfies the strict majorant property on $L^p([0,1]^d)$ for all $p > 0$ if and only if Γ is affinely independent. Furthermore, whenever Γ is not affinely independent, then there exists an integer $m \geq 0$, and real coefficients $(a_n)_{n\in\Gamma}$, such that for every $p \in (2m, 2m+2)$,

$$\left\|\sum_{n\in\Gamma} |a_n| e(n\cdot x)\right\|_{L^p([0,1]^d)} < \left\|\sum_{n\in\Gamma} a_n e(n\cdot x)\right\|_{L^p([0,1]^d)}.$$

In particular, this holds for every set $\Gamma \subset \mathbb{Z}^d$ of cardinality at least $d+2$.

Second, if $\Gamma \subset \mathbb{Z}^d$ is infinite, we prove that for infinitely many positive integers m, there exist real coefficients $(a_n)_{n\in\Gamma}$ such that for every $p \in (2m, 2m+2)$,

$$\left\|\sum_{n\in\Gamma} |a_n| e(n\cdot x)\right\|_{L^p([0,1]^d)} < \left\|\sum_{n\in\Gamma} a_n e(n\cdot x)\right\|_{L^p([0,1]^d)}.$$

The length of these intervals of p is tight, since the strict majorant property holds for all $p \in 2\mathbb{N}$.

Third, we prove violations of the strict majorant property for a nice geometric example: the moment curve. Let $\gamma(t) = (t, t^2, \ldots, t^d)$ parametrize the moment curve in $\mathbb{R}^d$. For any $p > 0$ with $p \notin 2\mathbb{N}$, there exists $k \in \mathbb{N}$ and $a_0, \ldots, a_d \in \mathbb{R}$ such that

$$\Big\|1 + \sum_{i=0}^{d} |a_i| e(\gamma(k+i)\cdot x)\Big\|_{L^p([0,1]^d)} < \Big\|1 + \sum_{i=0}^{d} a_i e(\gamma(k+i)\cdot x)\Big\|_{L^p([0,1]^d)}.$$

Nevertheless, a weaker majorant property does hold: for all choices of real coefficients a_n, A_n with $|a_n| \leq A_n$ for all n,

$$\Big\| \sum_{n\in\mathbb{Z}} a_n e(\gamma(n)\cdot x)\Big\|_{L^p([0,1]^d)} \leq (d!)^{1/2d} \Big\| \sum_{n\in\mathbb{Z}} A_n e(\gamma(n)\cdot x)\Big\|_{L^p([0,1]^d)}$$

for *all* $2 \leq p \leq 2d$. Both results for the moment curve are motivated by recent work of Bennett and Bez [BB12], who introduced the study of the strict majorant property for frequencies on the parabola in the case $d = 2$.

It seems that the setting of majorant properties in arbitrary dimensions is rich with unexplored questions. For example, in the d-dimensional setting, how big a correction factor $B_p(\Gamma)$ on the right-hand side of (2) is required to make a weaker majorant property hold on $L^p([0,1]^d)$ for a particular set of frequencies $\Gamma \subset \mathbb{Z}^d$? For the moment curve, the result immediately above shows that $B_p(\Gamma) \ll_d 1$ suffices for all $2 \leq p \leq 2d$. For which sets $\Gamma \subset [1,N]^d$ does $B_p(\Gamma) \ll N^{\varepsilon}$ suffice for every $\epsilon > 0$? Can the methods of [MS09] be adapted to this setting? Such investigations could have interesting applications in recent work of Demeter and Langowski [DL21] on restriction of exponential sums to hypersurfaces (see [DL21, Conj. 1.2, Conj. 1.3, Lem. 2.1]).

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Pointwise convergence of scattering data

Alexei Poltoratski

The scattering transform for the Dirac system of differential equations is commonly viewed as a non-linear analog of the Fourier transform. This connection brings up a series of natural questions on finding analogs of key properties and estimates of the classical Fourier analysis in non-linear settings. In my talk I discuss an analog of Carleson's theorem on almost everywhere convergence of Fourier integrals for the non-linear Fourier transform.

The tools of Krein-de Branges theory of canonical Hamiltonian systems allow one to apply the methods of complex function theory to spectral and scattering problems for Dirak systems. The proof of convergence is obtained as a result of the study of the dynamics of resonances of a Dirak system. One of the main ingredients of our approach is an auxiliary family of inner functions satisfying a Riccati equation corresponding to the initial system.

Among further problems in this direction are the underlying maximal estimates conjectured in a paper by Muscalu, Tao and Thiele [2] and relations of the non-linear result with the classical Carleson's theorem.

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Spherical maximal functions and fractal dimensions of dilation sets

Joris Roos

(joint work with Andreas Seeger)

Let $d \geq 2$ and consider spherical means

$$A_t f(x) = \int_{S^{d-1}} f(x - ty)\, d\sigma(y),$$

where $f \in L^1_{\mathrm{loc}}(\mathbb{R}^d)$, $t > 0$, $x \in \mathbb{R}^d$, and σ is the normalized surface measure on S^{d-1}. Given a set $E \subset [1, 2]$ we are interested in the *local* maximal operator

$$M_E f(x) = \sup_{t \in E} |A_t f(x)|$$

and we seek to determine the closure of its *type set*

$$\mathcal{T}_E = \{(\tfrac{1}{p}, \tfrac{1}{q}) \in [0, 1]^2 \,:\, M_E \text{ bounded } L^p \to L^q\}.$$

The study of spherical maximal operators goes back to classical works of Stein [14] ($d \geq 3$) and Bourgain [2] ($d = 2$). Schlag [11] and Schlag-Sogge [12] have determined $\overline{\mathcal{T}_E}$ for the full dilation set $E = [1, 2]$, while the single average case $E = \{\text{point}\}$ goes back to earlier work of Littman [8]. One motivation to study $L^p \to L^q$ bounds originates in the connection to sparse bounds and weighted L^p estimates for associated *global* maximal operators (see Lacey [7]). However, we are interested in L^p improving properties for their own sake. The sharp range of $L^p \to L^p$ bounds for general $E \subset [1, 2]$ has been determined in [13] (up to endpoints), depending in essence on the Minkowski dimension of E. When it comes to $L^p \to L^q$ bounds however, it turns out that the sharp region of exponents depends not only on the Minkowski dimension of E, but also other fractal dimensions. For $\theta \in (0, 1]$ we define the *upper Assouad spectrum* at θ, denoted $\overline{\dim}_{\mathrm{A},\theta} E$, as the infimum over all $a > 0$ so that there exists $c > 0$ such that for every $\delta \in (0, 1)$ and every interval $I \subset [1, 2]$ with $|I| \geq \delta^\theta$,

$$N(E \cap I, \delta) \leq c\,(|I|/\delta)^a,$$

where $N(F, \delta)$ denotes the minimum number of intervals of length δ required to cover a set $F \subset [1, 2]$. This was recently introduced in work of Fraser and Yu [4, 5] (also see [3, 9]). One sees that $\overline{\dim}_{\mathrm{A},0} E$ is simply the (upper) Minkowski dimension of E. Moreover, the function $\theta \mapsto \overline{\dim}_{\mathrm{A},\theta} E$ is non-decreasing, continuous and converges to a limit as $\theta \to 1-$. That limit is called the *quasi-Assouad dimension* of E,

$$\dim_{\mathrm{qA}} E = \lim_{\theta \to 1-} \overline{\dim_{\mathrm{A},\theta}}\, E.$$

Given $0 \le \beta \le \gamma \le 1$ let $\mathcal{Q}(\beta, \gamma)$ denote the closed quadrilateral with vertices

$$Q_1 = (0,0), \quad Q_{2,\beta} = (\tfrac{d-1}{d-1+\beta}, \tfrac{d-1}{d-1+\beta}),$$

$$Q_{3,\beta} = (\tfrac{d-\beta}{d-\beta+1}, \tfrac{1}{d-\beta+1}), \quad Q_{4,\gamma} = (\tfrac{d(d-1)}{d^2+2\gamma-1}, \tfrac{d-1}{d^2+2\gamma-1}).$$

Given a set E we write

$$\beta = \overline{\dim_{\mathrm{A},0}}\, E \le \gamma = \dim_{\mathrm{qA}} E.$$

In earlier work with T. Anderson and K. Hughes [1] we showed that for $d \ge 3$ (or $d = 2$ and $\gamma \le 1/2$) we have $\mathcal{Q}(\beta, \gamma) \subset \overline{\mathcal{T}_E}$. Near the points $Q_1, Q_{2,\beta}, Q_{3,\beta}$, it suffices to use Littlewood-Paley theory and interpolation to exploit the crucial decay estimate for the surface measure,

$$|\widehat{\sigma}(\xi)| \lesssim |\xi|^{-\frac{d-1}{2}}.$$

The quasi-Assouad dimension enters estimates near the point $Q_{4,\gamma}$, where we use a TT^* argument in the spirit of the Stein-Tomas restriction theorem. This method fails entirely in the case $d = 2$, $\gamma > 1/2$. However, in [10] it is proved that the inclusion $\mathcal{Q}(\beta, \gamma) \subset \overline{\mathcal{T}_E}$ does continue to hold when $d = 2$, $\gamma > 1/2$. The argument follows the general outline of [12] and boils down to a crucial weighted L^2 estimate, which in [12] is proved using a certain space-time estimate due to Klainerman and Machedon [6], which is not available in our setting. Instead, we exhibit a certain property of almost orthogonality between contributions from sufficiently separated points in E that involves the Assouad spectrum.

The inclusion $\mathcal{Q}(\beta, \gamma) \subset \overline{\mathcal{T}_E}$ is an equality for certain E (as already shown in [1]). Also, known examples show that $\overline{\mathcal{T}_E} \subset \mathcal{Q}(\beta, \beta)$. Surprisingly, these inclusions characterize type sets.

Theorem 1 ([10]). *Let $A \subset [0,1]^2$ be a closed convex set. There exists $E \subset [1,2]$ such that $A = \overline{\mathcal{T}_E}$ if and only if*

$$\mathcal{Q}(\beta, \gamma) \subset A \subset \mathcal{Q}(\beta, \beta)$$

for some $0 \le \beta \le \gamma \le 1$.

In particular, it is possible for the boundary of the type set to contain an arbitrary convex curve segment in the critical triangle given by $\mathcal{Q}(\beta, \beta) \setminus \mathcal{Q}(\beta, \gamma)$ (if $\beta < \gamma$). To prove this result we need to construct a set E such that $A = \overline{\mathcal{T}_E}$. Let us first consider the case when $\mathcal{Q}(\beta, \gamma) = \overline{\mathcal{T}_E}$. Call E *quasi-Assouad regular* if $\overline{\dim_{\mathrm{A},\theta}}\, E = \gamma$ for all $1 > \theta > 1 - \beta/\gamma$. Expressed in other words, this means that the Assouad spectrum attains its maximum value as soon as it can, with respect to trivial upper bounds. It turns out that if E is quasi-Assouad regular, then

$\overline{\mathcal{T}_E} = \mathcal{Q}(\beta, \gamma)$. On the other hand, if E is a finite union of quasi-Assouad regular sets E_i with Minkowski dimension β_i and quasi-Assouad dimension γ_i, then

$$\overline{\mathcal{T}_E} = \bigcap_i \mathcal{Q}(\beta_i, \gamma_i).$$

To prove the theorem we construct a specific countably infinite union of carefully chosen quasi-Assouad regular sets.

An interesting open problem is to determine whether and how for *any* given E, the set $\overline{\mathcal{T}_E}$ can be determined from the Assouad spectrum of E and subsets of E.

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Global maximizers for spherical restriction

Diogo Oliveira e Silva

1. Introduction

A consequence of the classical Hausdorff–Young inequality is that the Fourier transform $\widehat{f}$ of an L^p function $f : \mathbb{R}^d \to \mathbb{C}$ is defined almost everywhere on $\mathbb{R}^d$ if $1 \leq p \leq 2$. It is a striking observation of Stein from the late 1960s that for a special range of p's the function $\widehat{f}$ can be meaningfully defined on submanifolds of $\mathbb{R}^d$

possessing some degree of curvature. The simple yet fundamental observation that *curvature causes the Fourier transform to decay* links geometry to analysis, and lies at the base of Fourier restriction theory. The celebrated restriction conjecture predicts the validity of the estimate

$$\int_{\mathbb{S}^{d-1}} |\widehat{f}(\omega)|^q \mathrm{d}\sigma(\omega) \leq C\|f\|^q_{L^p(\mathbb{R}^d)}, \text{ for } 1 \leq p < \tfrac{2d}{d+1} \text{ and } q \leq (\tfrac{d-1}{d+1})p',$$

and is remarkable in its numerous connections and applications. It exhibits deep links to Bochner–Riesz summation methods and to decoupling phenomena for the Fourier transform, and is known to imply the Kakeya conjecture. Despite the great deal of attention that this circle of problems has received during the past four decades, the restriction conjecture remains open in dimensions $d \geq 3$.

Sharp inequalities have a rich tradition in harmonic analysis, going back to the epoch making works of Beckner and Lieb for the sharp Hausdorff–Young and Hardy–Littlewood–Sobolev inequalities, respectively. Even though the history of sharp restriction theory is considerably shorter, it is moving at an incredible pace. The exciting surge of activity from the last decade produced the following highlights. Firstly, gaussians are known to maximize the Strichartz inequality for the Schrödinger equation,

$$\|e^{it\Delta}f\|_{L^p_t(\mathbb{R};L^q_x(\mathbb{R}^d))} \leq \mathbf{S}_{p,q}\|f\|_{L^2(\mathbb{R}^d)}, \quad p,q \geq 2, \tfrac{2}{p} + \tfrac{d}{q} = \tfrac{d}{2},$$

if $(d,p,q) \in \{(1,6,6),(1,8,4),(2,4,4)\}$. Secondly, constant functions maximize the Stein–Tomas adjoint restriction (or extension) inequality to the sphere,

$$\|\widehat{f\sigma}\|_{L^q(\mathbb{R}^d)} \leq \mathbf{T}_{d,q}\|f\|_{L^2(\mathbb{S}^{d-1})}, \quad q \geq 2\tfrac{d+1}{d-1}, \tag{1}$$

in the endpoint case $(d,q) = (3,4)$. Different proofs of these facts relying on heat flow monotonicity, representation formulae, and orthogonal polynomials are available, but they all ultimately hinge on the Lebesgue exponents in question being even integers. In this case, one can invoke Plancherel's theorem in order to reduce the problem to a simpler multilinear convolution estimate.

In the remainder of this abstract, we will discuss some results from sharp restriction theory on the sphere which we recently obtained in [1, 2, 6, 7, 9, 8].

2. Two simple case studies

In this section, we discuss the sharp form of two simple inequalities due to Agmon-Hörmander (1975) and Vega (1988) for the extension operator on $\mathbb{S}^{d-1}$ which exhibit interesting properties.

2.1. **Agmon–Hörmander.** ([6]) We compute the optimal constant and characterize the maximizers at all spatial scales for the Agmon–Hörmander inequality,

$$\frac{1}{\rho}\int_{B_\rho} |\widehat{f\sigma}(x)|^2 \frac{\mathrm{d}x}{(2\pi)^d} \leq \mathbf{A}_d(\rho)\int_{\mathbb{S}^{d-1}} |f(\omega)|^2 \,\mathrm{d}\sigma(\omega).$$

The maximizers switch back and forth from being constants to being non-symmetric at the zeros of two Bessel functions. We also study the stability of this estimate and establish a sharpened version in the spirit of Bianchi–Egnell. The corresponding stability constant and maximizers again exhibit a curious intermittent behaviour.

2.2. **Vega.** ([1]) Let $d \geq 2$ be an integer and let $\frac{2d}{d-1} < q \leq \infty$. We investigate the sharp form of the mixed norm Fourier extension inequality

$$\|\widehat{f\sigma}\|_{L^q_{\text{rad}}L^2_{\text{ang}}(\mathbb{R}^d)} \leq \mathbf{V}_{d,q}\,\|f\|_{L^2(\mathbb{S}^{d-1})},$$

established by Vega in 1988. Letting $\mathcal{A}_d \subset (2d/(d-1), \infty]$ denote the set of exponents for which the constant functions on $\mathbb{S}^{d-1}$ are the unique maximizers of this inequality, we show that:

- $\mathcal{A}_d$ contains the even integers and ∞;
- $\mathcal{A}_d$ is an open set in the extended topology;
- $\mathcal{A}_d$ contains the half-line $(q_0(d), \infty]$ with $q_0(d) \leq \big(\frac{1}{2} + o(1)\big)\, d \log d$.

In low dimensions, we further show that $q_0(2) \leq 6.76$; $q_0(3) \leq 5.45$; $q_0(4) \leq 5.53$; $q_0(5) \leq 6.07$. In particular, this breaks for the first time the even exponent barrier in sharp restriction theory. Our approach relies on a hierarchy between certain weighted Bessel integrals, a question of independent interest within the theory of special functions.

3. Foschi revisited

In this section, we recall Foschi's argument for the sharp endpoint Stein–Tomas inequality on $\mathbb{S}^2$ in a form reflecting more recent insights; for complete details, see [7, §2]. Foschi's original argument from [5] consisted of three steps:

(1) a *magic identity* involving 4-tuples of unit vectors summing to zero;
(2) an ingenious application of the Cauchy–Schwarz inequality;
(3) a spectral decomposition of the relevant quadratic form.

We recently observed in [7] that Steps 1 and 3 above can be replaced by the following more robust counterparts:

(1') an application of the Helmholtz equation $u + \Delta u = 0$ satisfied by the extension $u := \widehat{f\sigma}$ followed by an integration-by-parts identity;
(3') Plancherel's theorem applied to the homogeneous distribution $h_s(\phi) := \Gamma(\frac{s+3}{2})^{-1} \int_{\mathbb{R}^3} |x|^s \phi(x)\mathrm{d}x$ for $s = 1$ acting on the convolution measure $g\sigma * g\sigma$, where $g = f^2 - 1$ has mean zero.

Step (1') has already found a novel application in [2] in the setting of sharp weighted restriction estimates. Explorations related to (3') are currently underway.

4. Bootsrapping maximizers

In this section, we discuss how to use Foschi's theorem (discussed in §3) to prove that constants continue to maximize inequality (1) in the higher dimensional setting of arbitrary even exponents.

4.1. **Smoothness.** ([8]) An instance of our result concerns the regularity of solutions of the convolution equation

$$a \cdot (f\sigma)^{*(q-1)}\big|_{\mathbb{S}^{d-1}} = f \text{ a.e. on } \mathbb{S}^{d-1},$$

where $a \in C^\infty(\mathbb{S}^{d-1})$, $q \geq 2\frac{d+1}{d-1}$ is an integer, and the only *a priori* assumption is $f \in L^2(\mathbb{S}^{d-1})$. We prove that any such solution belongs to the class $C^\infty(\mathbb{S}^{d-1})$. In particular, we show that all critical points associated to the sharp form of the corresponding extension inequality on $\mathbb{S}^{d-1}$ are C^∞-smooth. This extends previous work of Christ–Shao [4] to arbitrary dimensions and general even exponents, and plays a key role in the next and final subsection of this abstract.

4.2. **Maximizers.** ([9]) We prove that constant functions are the unique real-valued maximizers for all $L^2 - L^{2k}$ extension inequalities on $\mathbb{S}^{d-1}$, $d \in \{3, 4, 5, 6, 7\}$, where $k \geq 3$ is an integer. The proof uses tools from probability theory, Lie theory, functional analysis, and the theory of special functions. It also relies on general solutions of the underlying Euler–Lagrange equation being smooth, a fact of independent interest which we discussed in the previous subsection. We further show that complex-valued maximizers coincide with nonnegative maximizers multiplied by the character $e^{i\xi\cdot\omega}$, for some ξ, thereby extending previous work of Christ–Shao [3] to arbitrary dimensions $d \geq 2$ and general even exponents.

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Product manifolds with improved spectral cluster and Weyl remainder estimates

Christopher D. Sogge
(joint work with Xiaoqi Huang, Michael E. Taylor)

In a joint work Xiaoqi Huang, Michael Taylor and I investigated improved spectral cluster and Weyl remainder bounds in certain geometric situations. We consider product manifolds $X \times Y$, with (X, g_X) and (Y, g_y) being compact Riemannian manifolds, and, since we are considering Cartesian products we consider eigenfunctions and eigenvalues of the product Laplacian $\Delta = \Delta_{g_X} + \Delta_{g_Y}$, with Δ_{g_X} and Δ_{g_Y} being the Laplace-Beltrami operators on X and Y, respectively.

We have two types of main results. First, we show that if, say, Y has improved spectral cluster or Weyl remainder estimates versus the universal ones, then so does $X \times Y$. Second, we show that for large enough exponents one can get optimal spectral cluster estimates on products of spheres of length 5 or more and near optimal ones involving products of shorter length. Our work was inspired in part and improves in part recent work of Iosevich and Wyman [11]. We were also motivated by recent work of Canzani and Galkowski [5]–[6].

Let us recall the classical Weyl formula of Avakumovic [1] and Levitan [12]. We let (M, g) be a compact Riemannian manifold of dimension $d \geq 2$. Then if e_{λ_j} are the eigenfunctions, i.e., $-\Delta_g e_{\lambda_j} = \lambda_j^2 e_{\lambda_j}$ with eigenvalues λ_j of $\sqrt{-\Delta_g}$ labeled with respect to multiplicity, $0 = \lambda_0 < \lambda_1 \leq \lambda_2 \leq \cdots$, then the Weyl counting function $N(\lambda)$ denotes the number of the λ_j which are $\leq \lambda$. The sharp Weyl formula of Avakumovic and Levitan then says that

$$N(\lambda) = c\lambda^d + O(\lambda^{d-1}), \quad c = c_M = (2\pi)^{-d}\omega_d \cdot \mathrm{Vol}_g(M), \tag{1}$$

with ω_d denoting the volume of the unit ball in Euclidean space and $\mathrm{Vol}_g(M)$ denoting the Riemannian volume of M. These universal bounds are saturated on round spheres S^d due to the fact that spherical harmonics of degree k there each have eigenvalue $\lambda = \sqrt{k(k+n-1)}$ which repeats with multiplicity $\approx k^{d-1}$ (the dimension of spherical harmonics of degree k).

So on the standard sphere these universal bounds cannot be improved. However, there are many cases in which the bounds for the Weyl error term can be improved. For instance for the standard torus, $\mathbb{T}^d$, (a product manifold) one can get power improvements

$$c\lambda^d - N(\lambda) = O(\lambda^{d-1-\sigma_d}), \quad \text{some } \ \sigma_d > 0. \tag{2}$$

Indeed, a classical result of Walfisz [18] says that one can take $\sigma_d = 1$ (best possible) if $d \geq 5$. Also, by classical results of Bérard [2] one can obtain logarithmic improvements if the sectional curvatures of M are all nonnegative. More recently, Canzani and Galkowski [6] obtained these sorts of bounds for all product manifolds $X \times Y$ without curvature assumptions.

In addition to investigating situations in which the error bounds in (1) can be improved, we are also interested in studying cases where the universal spectral cluster bounds of one of us [13] can be improved. Recall that the following universal bounds for $L^q(M)$, $q > 2$, eigenfunctions are valid

$$\|e_\lambda\|_{L^q(M)} \lesssim \lambda^{\sigma(q)} \|e_\lambda\|_{L^2(M)}, \tag{3}$$

with

$$\begin{aligned}\sigma(q) = \sigma(q,d) &= \max\{d(\tfrac{1}{2}-\tfrac{1}{q})-\tfrac{1}{2},\ \tfrac{d-1}{2}(\tfrac{1}{2}-\tfrac{1}{q})\} \\ &= \begin{cases} d(\frac{1}{2}-\frac{1}{q})-\frac{1}{2}, & q \ge q_d = \frac{2(d+1)}{d-1} \\ \frac{d-1}{2}(\frac{1}{2}-\frac{1}{q}), & 2 < q \le q_d. \end{cases}\end{aligned} \tag{4}$$

More generally, one has the spectral cluster estimates for "quasimodes", which say that

$$\|\psi_\lambda\|_{L^q(M)} \lesssim \lambda^{\sigma(q)}, \quad \text{if Spec } (\psi_\lambda) \subset [\lambda-1,\lambda+1]. \tag{5}$$

The first estimate, (3), is also saturated on round spheres and the second one, (5) cannot be improved on *any* (M,g) (see [15]) due to its "local nature" of involving unit bands of the spectrum.

Over the years there have been many attempts to improve the bounds in (5) when one replaces the unit interval there with one of the form $[\lambda-\varepsilon(\lambda),\lambda+\varepsilon(\lambda)]$ with $\varepsilon(\lambda) \to 0$ as $\lambda \to \infty$. These of course lead to improvements of (3). For exponents $q \ge q_d$ a goal would be to show that if $\chi_{[\lambda-\lambda(\varepsilon),\lambda+\varepsilon(\lambda)]}$ denotes the projection onto the $[\lambda-\varepsilon(\lambda),\lambda+\varepsilon(\lambda)]$ part of the spectrum of $\sqrt{-\Delta_g}$ then one has bounds of the form

$$\|\chi_{[\lambda-\varepsilon(\lambda),\lambda+\varepsilon(\lambda)]} f\|_q \le C\sqrt{\varepsilon(\lambda)}\,\lambda^{\sigma(q)}\|f\|_2. \tag{6}$$

One can check that these are optimal in the sense that the factor $\sqrt{\varepsilon(\lambda)}$ cannot be replaced by a smaller factor; however, one wishes for $\varepsilon(\lambda)$ to go to zero as fast as possible for a given q. For the torus, or more generally products of spheres, one can check that one can never take $\varepsilon(\lambda) = o(\lambda^{-1})$. So bounds of the form (6) with $\varepsilon(\lambda) = \lambda^{-1}$ (the "wavelength") are optimal. These sort of bounds are natural analogs of the Stein-Tomas [17] restriction/extension theorem for $\mathbb{R}^d$.

It is implicit in Bérard [2] that one can take $\varepsilon(\lambda) = (\log\lambda)^{-1}$ if M has nonpositive curvature and $q = \infty$. More recently, Hassell and Tacy [8] showed that under this assumption one can obtain the analogous bounds for all $q \in (q_d,\infty]$. Also, Canzani and Galkowski [5] obtained these bounds for any product manifold $X \times Y$ of dimension d. Obtaining improved spectral cluster estimates for $q \in (2,q_d)$ or, more importantly, $q = q_d$, proved to be a bit more elusive; however, log-power improvements were obtained by Blair and one of us [3] for $q = q_d$ (and hence for all other exponents) if (M,g) has negative curvature. It would be interesting if this sort of result also held for arbitrary products $X \times Y$.

Let us return our focus on cases where the optimal "half power" improvement of the universal bounds may be valid:

$$\|\chi_{[\lambda-\lambda^{-1},\lambda+\lambda^{-1}]} f\|_{L^q(M)} \le C\lambda^{\sigma(q)-1/2}\,\|f\|_{L^2(M)}. \tag{7}$$

We are also interested in variants involving additional arbitrary λ^ε, $\varepsilon > 0$, in the right. Since $\sigma(q) < 1/2$ if $\sigma < \frac{2d}{d-2}$, it is easy to check that neither bound can hold (7) to be valid on any M if $q < \frac{2d}{d-2}$. In the case of the torus, an important conjecture is that these sort of bounds should be valid for this endpoint case, i.e.,

$$\|e_\lambda\|_{L^{2d/(d-2)}(\mathbb{T}^d)} \le C_\varepsilon \lambda^\varepsilon \|e_\lambda\|_{L^2(\mathbb{T}^d)}. \tag{8}$$

The best possible results to date seem to be due to Bourgain and Demeter [4] who showed that the analog of (8) is valid if $2d/(d-2)$ is replaced by the somewhat smaller exponent $2(d+1)/(d-1)$, and they also obtained variants of (7) for a range of exponents larger than the critical exponent for this problem $2d/(d-2)$.

One of the main results in our joint work with Huang and Taylor, [10], says that one can obtain the optimal bounds (7) for large exponents q if M is a product of 5 or more spheres or slightly weaker ones for products of shorter length. So we are now considering $M = S^{d_1} \times S^{d_2} \times \cdots \times S^{d_n}$ with $n \ge 2$. The special case where all the d_j equal one of course is the n-torus. To state our result, let

$$\rho(\lambda) = \#\{j : \mathbb{Z}^n : |j| = \lambda\}$$

denote the number of integer lattice points on $\lambda \cdot S^{n-1}$. Then one of the bounds from [10] says that

Theorem 1. *Let $M = S^{d_1} \times S^{d_2} \times \cdots \times S^{d_n}$ be a cartesian product of spheres of dimension $d = d_1 + \cdots + d_n$ involving n factors. Then if e_λ is an eigenfunction on M we have*

$$\|e_\lambda\|_{L^q(M)} \le C\sqrt{\rho(\lambda)}\,\Big(\prod_{j=1}^{d} \lambda^{\sigma(q,d_j)}\Big)\|e_\lambda\|_{L^2(M)}. \tag{9}$$

In particular if, if $n \ge 5$

$$\|e_\lambda\|_{L^q(M)} \le C\lambda^{\sigma(q,d)-1/2}\|e_\lambda\|_{L^2(M)}, \quad q \ge \max_{1\le j\le n} \tfrac{2(d_j+1)}{d_j-1}. \tag{10}$$

Also if $2 \le n \le 4$ the analog of (10) is valid if one includes an additional factor of λ^ε in the right with $\varepsilon > 0$ arbitrary.

One proves (9) using the universal bounds (3) and an orthogonality argument after noting that an eigenfunctions on M as above involve linear combinations of products of eigenfunctions on the S^{d_j}. One obtains (10) from (9) using the fact that the aforementioned result of Walfisz [18] implies that $\rho(\lambda) = O(\lambda^{n-2})$ if $n \ge 5$, and one obtains the variants for $2 \le n \le 4$ using the fact that classical estimates for this case say that $\rho(\lambda) = O(\lambda^{n-2+\varepsilon})$, for all $\varepsilon > 0$. Also note that, since on products of spheres the distinct eigenvalues are square roots of integers, the eigenfunction estimates in (10) are equivalent to bounds in (7) for the same range of exponents.

Our Theorem 1 was motivated by an earlier result of Iosevich and Wyman [11] which says that for M as above one has power improvements over the universal Weyl bounds in (1), namely:

$$N(\lambda) = c\lambda^d + O(\lambda^{d-1-\frac{n-1}{n+1}}), \tag{11}$$

which, of course, generalizes the classical result of Hlwaka [9] corresponding to $d_1 = \cdots = d_n = 1$. The bounds in (11) immediately imply that for such M we have that the multiplicity of a distinct eigenvalue λ, i.e., $\#\lambda_j = \lambda$ must be $O(\lambda^{d-1-\frac{n-1}{n+1}})$. Because of the aforementioned spectral properties of M this in turn implies the bounds

$$\|\chi_{[\lambda-\lambda^{-1},\lambda+\lambda^{-1}]}f\|_{L^\infty(M)} \lesssim \lambda^{\frac{d-1}{2}-\frac{n-1}{2(n+1)}}\|f\|_{L^2(M)},$$

which are a bit weaker than the ones in Theorem 1 which say that one has $O(\lambda^{\frac{d-1}{2}-\frac{1}{2}})$ if $n \geq 5$ and $O(\lambda^{\frac{d-1}{2}-\frac{1}{2}+\varepsilon})$ for all $\varepsilon > 0$ if $2 \leq n \leq 4$.

Iosevich and Wyman conjectured that when $n \geq 5$ the Weyl error term should be $O(\lambda^{d-2})$ (as in Walfisz [18] when $d_1 = \cdots = d_n = 1$). It is often the case that sup-norm estimates like the ones we just mentioned from Theorem 1 lead to these types of bounds; however, it is not clear how to use them in this case to obtain this natural conjecture.

Additionally, Iosevich and Wyman [11] also showed that if $M = X \times Y$ is an arbitrary Cartesian product of dimension $d = d_X + d_Y$, then

$$N(\lambda) = c\lambda^d + o(\lambda^{d-1}). \tag{12}$$

To do this they first showed that for such M the set of periodic geodesics through any $x \in M$ has measure zero, and so (12) follows from the Duistermaat-Guillemin [7] theorem. As a corollary to this observation of Iosevich and Wyman [11] and techniques of one of us and Zelditch [16] one can show that there exists $\varepsilon(\lambda) \to 0$ so that on $M = X \times Y$ one has

$$\|\chi_{[\lambda-\varepsilon(\lambda),\lambda+\varepsilon(\lambda)]}\|_{L^2\to L^\infty} = O(\sqrt{\varepsilon(\lambda)}\lambda^{\frac{d-1}{2}}).$$

Although one cannot replace the right side by $o(\sqrt{\varepsilon(\lambda)}\lambda^{\frac{d-1}{2}})$ bounds, a deficiency of these bounds is that one can not specify how $\varepsilon(\lambda)$ goes to zero using the above techniques. Recently, though, Canzani and Galkowski [5] showed that on arbitrary products one can take $\varepsilon(\lambda) = (\log\lambda)^{-1}$, as well as analogous results for all exponents $q > q_d$. Improvements for $q = q_d$ or $2 < q < q_d$ remain open in this case, though.

In our work with Huang and Taylor [10] we are also able to show that if Y has improved spectral projection bounds than so does $X \times Y$. We assume here that d_Y denotes the dimension of Y and d_X that of X so that $d = d_X + d_Y$ is that of $X \times Y$.

Theorem 2. *Assume that $\lambda \to \lambda\varepsilon(\lambda)$ is non-decreasing, and that, moreover,*

$$\|\chi_{[\lambda-\varepsilon(\lambda),\lambda+\varepsilon(\lambda)]}\|_{L^2(Y)\to L^q(Y)} = O(\delta(\lambda)\lambda^{\sigma(q,d_Y)}), \quad \textit{with } \delta(\lambda) \to 0. \tag{13}$$

Then

$$\|\chi_{[\lambda-\varepsilon(\lambda),\lambda+\varepsilon(\lambda)]}\|_{L^2(X\times Y)\to L^q(X\times Y)} = O(\delta(\lambda)\lambda^{\sigma(q,d_Y)}\lambda^{\sigma(q,d_X)}\lambda^{1/2}). \tag{14}$$

In particular, if $q \geq \max(q_X, q_Y)$, then

$$\|\chi_{[\lambda-\varepsilon(\lambda),\lambda+\varepsilon(\lambda)]}\|_{L^2(X\times Y)\to L^q(X\times Y)} = O(\delta(\lambda)\lambda^{\sigma(q,d)}). \tag{15}$$

As was shown in [10], one proves (14) using the assumption (13) and an orthogonality argument (which is a bit more intricate than the one used to obtain (9)). The estimate (15) is a simple consequence of (14) and the numerology of the exponents $\sigma(q,n)$.

In [10] we also showed that improved Weyl error term bounds on Y carry over to the product $X \times Y$:

Theorem 3. *If*

$$N(Y;\lambda) = c_Y\lambda^{d_Y} + O(\varepsilon(\lambda)\lambda^{d_Y-1}) \tag{16}$$

with $\varepsilon(\lambda)$ as above then

$$N(X\times Y;\lambda) = c_{X\times Y}\lambda^d + O(\varepsilon(\lambda)\lambda^{d-1}). \tag{17}$$

To prove Theorem 3 we first note that if the spectrum of $-\Delta_X$ is $\{\mu^2\}$ and the spectrum of $-\Delta_Y$ is $\{\nu^2\}$ then the spectrum of $\sqrt{-\Delta_X - \Delta_Y}$ is $\{\sqrt{\mu^2+\nu^2}\}$. Therefore,

$$\begin{aligned} N(X\times Y;\lambda) &= \#\{(\mu,\nu):\ \sqrt{\mu^2+\nu^2}\le\lambda\} \\ &= \sum_{\sqrt{\mu^2+\nu^2}\le\lambda} 1 \\ &= \sum_{\mu\le\lambda}\Big(\sum_{\nu\le\sqrt{\lambda^2-\mu^2}} 1\Big) \\ &= \sum_{\nu\le\lambda} N(Y;\sqrt{\lambda^2-\mu^2}) = \sum_{\mu\le\lambda}\big[c_Y(\lambda^2-\mu^2)^{d_Y/2} + R_Y(\sqrt{\lambda^2-\mu^2})\big] \\ &= I + II. \end{aligned}$$

Since $R_Y(\sqrt{\lambda^2-\mu^2}) = O(\varepsilon(\sqrt{\lambda^2-\mu^2})\cdot(\lambda^2-\mu^2)^{(d_Y-1)/2})$, one can use our assumption on $\varepsilon(\lambda)$ along with (16) and a simple calculation to see that

$$II = O(\varepsilon(\lambda)\lambda^{d-1})$$

as desired.

Consequently, since $\lambda^{-1}\lesssim\varepsilon(\lambda)$, our proof would be complete if we could show that

$$I = \sum_{\mu\le\lambda} c_Y(\lambda^2-\mu^2)^{d_Y/2} = c_{X\times Y}\lambda^d + O(\lambda^{d-2}). \tag{18}$$

To prove this, we note that if $\{e_\mu\}$ is an orthonormal basis of eigenfunctions with eigenvalues $\{\mu\}$ then

$$\begin{aligned} I = c_Y\lambda^{d_Y}\int_X \sum_{\mu\le\lambda}\big(1-(\mu/\lambda)^2\big)^{d_Y/2} e_\mu(x)\,\overline{e_\mu(x)}\,dx \\ = c_Y\lambda^{d_Y}\int_X S_\lambda^{d_Y/2}(x,x)\,dx, \end{aligned} \tag{19}$$

if $S_\lambda^\delta(x,y)=\sum_\mu(1-(\mu/\lambda)^2)_+^\delta e_\mu(x)\,\overline{e_\mu(y)}$ is the Bochner-Riesz kernel of index δ. It is known (see e.g. [14] or [15]) that

$$\text{(20)}\quad S_\lambda^{d_Y/2}(x,x)=c_X\lambda^{d_X}+O(\lambda^{d_X-1-d_Y/2}),$$
$$c_X=(2\pi)^{-d_X}\cdot\int_{\xi\in\mathbb{R}^{d_X}}(1-|\xi|^2)^{d_Y/2}\,d\xi.$$

It is a straightforward calculation to obtain (18) from (19)–(20) along with the definition of $c_{X\times Y}$ in (1), which completes the proof.

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Two Analogues of the Euclidean Spherical Maximal Function on Heisenberg Groups

Rajula Srivastava
(joint work with Joris Roos, Andreas Seeger)

We shall discuss sharp (up to end points) $L^p - L^q$ estimates for local maximal operators associated with dilates of two different surfaces on Heisenberg groups. The first is the "horizontal sphere" of codimension two. The second is the Korányi sphere: a surface of codimension one compatible with the non-isotropic dilation structure on the group but with points of vanishing curvature. We shall examine the geometry of these surfaces in light of two different notions of curvature and compare their effect on the estimates for the corresponding maximal operators. The Heisenberg group structure will play a crucial role in our arguments. However, the theory of Oscillatory Integral Operators will be central despite the non-Euclidean setting. We shall also discuss two new counterexamples which imply the sharpness of our results (up to endpoints). Partly based on joint work with Joris Roos and Andreas Seeger.

Extremizability of Fourier restriction to the moment curve

Betsy Stovall
(joint work with Chandan Biswas)

The Fourier restriction operator associated to the moment curve,

$$\mathcal{R}f(t) := \hat{f}(\gamma(t)), \qquad \gamma(t) := (t, \dots, t^d),$$

initially defined on smooth functions with compact support, has been known since 1985 work of Drury [2] to obey the Lebesgue space inequalities

$$\|\mathcal{R}f\|_{L^s(\mathbb{R})} \lesssim A_r \|f\|_{L^r(\mathbb{R}^d)}, \tag{1}$$

for all exponent pairs $r, s \in [1, \infty]$ obeying the relations $s = \frac{2r'}{d(d+1)}$ and $1 \leq r < \frac{d(d+1)+2}{d(d+1)}$. Moreover, (1) fails for all other values of s, r. Here A_r denotes the operator norm, which is the minimal constant for which (1) holds for all $f \in C_c^\infty(\mathbb{R}^d)$, and we identify functions on the curve with their composition with the parametrization above. While the inequality (1) is an elementary consequence of Hölder's inequality when $r = 1, s = \infty$, its validity for larger values of r is a more subtle question, due to the zero Lebesgue measure of the curve γ. Here, the nonvanishing of the torsion of the curve γ plays a critical role.

This talk concerns recent work [1], in which we have studied questions relating to the reverse of the inequality (1). In particular, Do there exist nonzero functions f attaining the upper bound expressed in (1) (we call these *extremizers*)?, and, Must sequences of norm-1 functions saturating this inequality (we call these *normalized extremizing sequences*) possess convergent subsequences? Such questions have recently attracted broader interest within the harmonic analysis literature (see [4] for an extensive review and the introduction of [3] for a description of some more recent results). For linear operators acting on finite dimensional vector spaces, the answers to both questions are always yes, due to the compactness of the unit sphere. However, in the case of infinite dimensional Banach spaces, these questions are generally nontrivial. Indeed, while the answers to both questions can sometimes be yes, there also exist bounded linear operators with no extremizers, there exist bounded linear operators with extremizers for which normalized extremizing sequences need not have convergent subsequences, and there exist bounded linear operators with no extremizers (whose normalized extremizing sequences necessarily cannot converge).

The curve γ possesses a rich symmetry group, namely, the dilations:

$$\gamma(\lambda t) = \mathrm{diag}(\lambda, \ldots, \lambda^{\mathrm{d}})\gamma(\mathrm{t}),$$

translations,

$$\gamma(t + t_0) - \gamma(t_0) = [\gamma'(t_0), \ldots, \gamma^{(d)}(t_0)]\gamma(t),$$

and the compositions thereof. These symmetries, along with the modulation symmetry of the Fourier transform, induce symmetries of the operator, that is, $L^r(\mathbb{R}^d)$ isometries S for which there exist $L^s(\mathbb{R})$ isometries T such that $\mathcal{R} \circ S = T \circ \mathcal{R}$. Due to the presence of a noncompact symmetry group for $\mathcal{R}$, we know that normalized extremizing sequences need not possess subsequences that converge; this is because there exist sequences of symmetries converging to zero in the weak operator topology. However, the richness of the symmetry group leads us to expect that normalized extremizing sequences might possess convergent subsequences if we are first allowed to apply the symmetries of the operator. Our main result is that this is indeed the case.

Theorem 1 ([1]). *For every $1 \le r < \frac{d(d+1)+2}{d(d+1)}$, there exist nonzero extremizers for (1). Moreover, when $1 < r < \frac{d(d+1)+2}{d(d+1)}$, every normalized extremizing sequence has a subsequence that converges in $L^r(\mathbb{R})$, after the application of suitable symmetries.*

Our proof of Theorem 1 passes through the proof of the analogous result for its adjoint, the Fourier extension operator

$$\mathcal{E}f(x) := \int_{\mathbb{R}} e^{ix\cdot\gamma(t)} f(t)\, dt, \qquad f \in C_c^\infty(\mathbb{R}).$$

Letting $p = s'$, $q = r'$, and $B_p = A_r$ in (1), we obtain

$$\|\mathcal{E}f\|_{L^q(\mathbb{R}^d)} \le B_p \|f\|_{L^p(\mathbb{R})}, \qquad q = \tfrac{d(d+1)}{2}p',\ 1 \le p < \tfrac{d(d+1)+2}{2}, \tag{2}$$

and results about extremizers for $\mathcal{E}$ imply the analogues for $\mathbb{R}$ by duality. The argument follows an outline established by Stovall in [5] for the case of the restriction problem for paraboloid, but new difficulties are encountered in the case of the moment curve, due to both the geometry of separated sets and some inconvenient inequalities for the exponents relevant to the problem.

Open questions include both the identity of the extremizers, perhaps only in the case $p = 2$ and also analogous results for higher degree curves.

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Global solutions for 1D cubic defocusing dispersive equations

Daniel Tataru
(joint work with Mihaela Ifrim)

This work is devoted to a general class of one dimensional NLS problems with a cubic nonlinearity. The question of obtaining scattering, global in time solutions for such problems has attracted a lot of attention in recent years, and many global well-posedness results have been proved for a number of models under the assumption that the initial data is both *small* and *localized*. However, except for the completely integrable case, no such results have been known for small but non-localized initial data.

Here we consider instead the much more difficult case where the initial data which is just *small*, without any localization assumption. Then it is natural to restrict the analysis to defocusing problems, as focusing one-dimensional cubic NLS type problems typically admit small solitons and thus, generically, the solutions do not scatter at infinity. For this class of problems formulate the following broad conjecture:

Conjecture 1. *One dimensional dispersive flows with cubic defocusing nonlinearities and small initial data have global in time, scattering solutions.*

Our objective is to prove the first global in time well-posedness result of this type, assuming a Schrödinger type dispersion relation and a cubic nonlinearity with a smooth and bounded symbol. As part of our results, we also prove that our global solutions are scattering at infinity in a very precise, quantitative way, in the sense that they satisfy both L^6 Strichartz estimates and bilinear L^2 transversality bounds. This is despite the fact that the nonlinearity is non-perturbative on large time scales. Our method is based on a robust reinterpretation of the idea of interaction Morawetz estimates, developed almost 20 years ago by the I-team.

Our estimates are new even in the case of the classical cubic NLS problem; there they improve earlier estimates of Planchon and Vega.

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Dyadic Rectangles

Alex Volberg

(joint work with N. Arcozzi, I. Holmes, P. Mozolyako, P. Zorin-Kranich)

Weighted Carleson embedding (weighted paraproduct estimates in another language) lies in the core of various harmonic analysis and PDE results. Not much is known about it in multi-parameter situation, while one parameter is completely understood. I will formulate several new results on weighted multi-parameter Carleson embedding on multi-trees and their corollaries as embedding of Hilbert spaces of analytic functions on poly-discs. I will also formulate corresponding Poincaré inequalities on multi-trees and poly-discs. Some of those results are final, but even embedding of Hardy space on bi-disc is not completely described. My presentation is based on the series of joint works with various coathors: N. Arcozzi, I. Holmes, P. Mozolyako, P. Zorin-Kranich.

W. Rudin meets E.M. Stein

James Wright

(joint work with Odysseas Bakas, Valentina Ciccone)

In this talk we review the theories developed by W. Rudin from the 1950s about various thin sets (for example, Sidon sets, Paley sets, $\Lambda(q)$ sets) in classical Fourier series and show that these theories have an equivalent formulation as a problem in Fourier restriction theory which was developed by E.M. Stein in the 1970s.

For any set $\Lambda \subset \mathbb{Z}^n$, we denote by $C_\Lambda(\mathbb{T}^n)$ the closed subspace of continuous functions $f \in C(\mathbb{T}^n)$ on the n-torus $\mathbb{T}^n$ whose fourier coefficients $\widehat{f}(n) = 0$ for all $n \notin \Lambda$. Similarly we define the closed subspace L^p_Λ of $L^p(\mathbb{T}^n), p \geq 1$ functions which are fourier supported in Λ. Finally we denote by P_Λ the dense subspace of trigonometric polynomials with frequency support in Λ.

(S) A set $\Lambda \subset \mathbb{Z}^n$ is called a *Sidon* set if $C_\Lambda(\mathbb{T}^n) \subset \mathbb{A}(\mathbb{T}^n)$; in other words, every continuous function which is fourier supported in Λ automatically has an absolutely convergent fourier series. In [1], Rudin developed the theory of Sidon sets and laid out a programme to find an arithmetic characterisation of such sets. He intimated that the key to this characterisation is the following analytic *improving* property for Sidon sets: $L^2_\Lambda \subset F$ or

$$\|f\|_F \ \leq \ C\,\|f\|_{L^2} \quad \text{holds for all } f \in P_\Lambda \tag{1}$$

where $F = \exp(L^2)(\mathbb{T}^n)$ is the exponential square integrable class. In [3], Pisier showed that indeed (1) gives a characterisation of Sidon sets and he did this for any compact abelian group. In his phd thesis, O. Bakas extended Pisier's work to product spectral sets $\Lambda = \Lambda_1 \times \cdots \times \Lambda_n \subset \mathbb{Z}^n$ where each $\Lambda_j \subset \mathbb{Z}$ is a Sidon set. Such product sets are characterised by the inequality (1) where now $F = \exp(L^{2/n})(\mathbb{T}^n)$.

(P) A set $\Lambda \subset \mathbb{Z}^n$ is called a *Paley* set if $L^2_\Lambda \subset BMO(\mathbb{T}^n)$ or

$$\|f\|_F \ \le \ C\,\|f\|_{L^2} \quad \text{holds for all } f \in P_\Lambda \tag{2}$$

where $F = BMO(\mathbb{T}^n)$ is the space of functions on $\mathbb{T}^n$ with bounded mean oscillation. In his Ph.d. thesis, O.Bakas extended work of W. Rudin in [2] from $\mathbb{T}$ to $\mathbb{T}^n$ showing that Λ is a Paley set if and only if $\sup_{R\in\mathcal{R}} \#(\Lambda \cap R) < \infty$ (here the supremum is taken over all dyadic rectangles R).

(q) For $q > 2$, a set $\Lambda \subset \mathbb{Z}^n$ is called a $\Lambda(q)$ set if $L^2_\Lambda \subset L^q(\mathbb{T}^n)$ or

$$\|f\|_F \ \le \ C\,\|f\|_{L^2} \quad \text{holds for all } f \in P_\Lambda \tag{3}$$

where $F = L^q(\mathbb{T}^n)$.

A well-known fourier restriction result from the 1970s states that for $f \in L^p(\mathbb{R}^{2n})$ and $p < 4/3$, one can make sense of the restriction of the fourier transform of f to $\mathbb{T}^n$ as an L^q density on $\mathbb{T}^n$ for a certain range of q; in fact, the *a priori* bound

$$\|\widehat{f}\|_{L^q(\mathbb{T}^n)} \ \le \ C\,\|f\|_{L^p(\mathbb{R}^{2n})}$$

holds holds if and only if $q < p'/3$ and $1 \le p < 4/3$. This was established on $\mathbb{T}$ (the $n = 1$ case) by Fefferman-Stein and also Zymund in the early 1970s but their arguments extend readily to the n-dimensional case $\mathbb{T}^n$ as observed by E. Prestini and M. Christ. To express our characterisation of Sidon/Paley/$\Lambda(q)$/*etc*... sets as a fourier restriction bound, it is more convenient to use the equivalent formulation in terms of the fourier extension operator: let σ denote surface measure on $\mathbb{T}^n$ and define the extension operator

$$\mathcal{E}g(\underline{x}) \ := \ \widehat{g d\sigma}(\underline{x}) \ = \ \int_{\mathbb{T}^n} g(\omega) e^{-2\pi i \underline{x}\cdot\omega}\, d\sigma(\omega),$$

taking functions on $\mathbb{T}^n$ and extending it via the fourier transform to a function on $\mathbb{R}^{2n}$. Finally, viewing $\mathbb{R}^{2n} \simeq \mathbb{C}^n$, we denote by $B_R = \mathbb{D}_R \times \cdots \times \mathbb{D}_R$ the n-fold product of complex discs of radius R.

Our main theorem is the following.

Theorem Suppose $\Lambda \subseteq \mathbb{Z}^n$ and let $F = E^*$ be the dual of a Banach space of functions E. We assume F continuously embeds into $L^q(\mathbb{T}^n)$ for some $q > 4$. The following two bounds are equivalent.

(B1) There is a constant A_Λ such that

$$\|f\|_{F(\mathbb{T}^n)} \ \le \ A_\Lambda\,\|f\|_{L^2(\mathbb{T}^n)} \quad \text{for all } f \in P_\Lambda(\mathbb{T}^n).$$

(B2) There is a constant C_Λ such that for all $R > 0$,

$$\|\mathcal{E}g\|_{L^4(B_R)} \leq C_\Lambda (\log R)^{n/4} \|g\|_{E(\mathbb{T}^n)} \text{ for all } g \in P_\Lambda(\mathbb{T}^n).$$

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Lens counting, circular maximal functions, and restricted projections

Joshua Zahl

Planar incidence geometry studies the intersection patterns of arrangements of curves in the plane. I will discuss how ideas from this area, and specifically the theory of lens counting, can be used to prove sharp L^p bounds for rough circular maximal functions, and also resolve a conjecture of Fassler and Orponen on restricted projections from a C^2 curve. This is joint work with Malabika Pramanik and Tongou Yang.

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