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Arithmetic of Shimura Varieties

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Abstract. The aim of this workshop was to discuss recent developments on the arithmetic of Shimura varieties and on related topics within the Langlands program, and to initiate and support further research in this direction. The talks presented new methods and results covering topics ranging from geometric questions on the reduction of Shimura varieties to representations in their cohomology, automorphic forms, and questions on the geometry and arithmetic of moduli spaces of bundles on the Fargues-Fontaine curve.

Mathematics Subject Classification (2020): 11xx, 14xx.

Introduction by the Organizers

The workshop *Arithmetic of Shimura varieties* had 47 on-site and 13 online participants from all over the world, including 13 women and several young participants. We had 22 talks of 60 minutes each. The time between lunch and coffee was reserved to continue ongoing collaborations among the participants and to start new ones.

The arithmetic properties of Shimura varieties and of related moduli spaces of shtukas are of central importance in arithmetic algebraic geometry, specifically in the Langlands program, which aims to establish deep connections between number theory, algebraic geometry and representation theory. Some of the classical origins of the theory are the quadratic reciprocity law and the analytic and algebraic aspects of modular curves and modular forms. Later, people studied moduli spaces of abelian varieties with additional structure. In the past years, building on and combining new methods such as the Fargues-Fontaine curve and Scholze's notion of perfectoid space, the field has seen a rapid development. In addition to the above-mentioned moduli spaces, these methods allow a better handle on moduli spaces of local and global shtukas and of bundles over the Fargues-Fontaine curve. In this way, this research field has contributed to several spectacular results in arithmetic geometry of the past years.

The topics of the talks covered the whole subject of the current arithmetic theory of Shimura varieties, ranging from geometric and cohomological properties of Shimura varieties or moduli spaces of shtukas over the study of automorpic bundles and automorphic forms to important related theories such as Rapoport-Zink spaces or representation-theoretic questions. Many of them covered very recent or ongoing work and thus led to inspiring discussions about current and future projects.

Reductions of Shimura varieties and affine Deligne-Lusztig varieties. To study arithmetic properties of Shimura varieties, it is desirable to understand the geometry of the special fiber of a suitable integral model, and to describe it, if possible, in group-theoretic terms. Several talks were dedicated to this study of the so-called reduction of Shimura varieties.

The talks of Stefania Trentin and Naoki Imai were concerned with the geometry of the reduction of unitary Shimura varieties for signature $(2, 4)$, which is one of the first cases where the general theory for fully Hodge-Newton decomposable cases does not apply any more. Stefania Trentin talked about the results of her thesis for the basic locus in the reduction modulo a ramified prime and about flatness of the associated models, Naoki Imai presented joint work with Maria Fox for inert primes.

Xuhua He gave a talk relating affine Deligne-Lusztig varieties to so-called affine Lusztig varieties, a generalization of affine Springer fibres.

Pol van Hoften presented his joined work with Marco D'Addezio proving the Hecke orbit conjecture for all Shimura varieties of Hodge type. Shimura varieties of Hodge type have a description as moduli space of abelian varieties with additional structure, and they prove that under a mild assumption on the characteristic the isogeny class of any $\bar{\mathbb{F}}_p$-valued point of the Shimura variety is Zariski-dense in the corresponding Newton stratum.

The last talk of the workshop was given by João Lourenço who gave an overview of several of his articles on local models of Shimura varieties and their singularities and outlined a strategy towards proving the Bezrukavnikov equivalence in mixed characteristic.

Cycles on Shimura varieties and modular forms. Intersection numbers of certain cycles on Shimura varieties carry deep arithmetic information that can often be related to modular forms or similar functions. While this surprising fact is almost classical by now, as it goes back at least to the work of Hirzebruch and Zagier (and in some sense can be traced back even further), its further generalizations remain an interesting and very active area of research.

Keerthi Madapusi explained his uniform construction of integral special cycles using new methods from derived algebraic geometry. More precisely, he considers the Kisin-Pappas integral model of a Shimura variety of Hodge type. To any suitable lattice he assigns an element in a rational Chow group of the integral model such that these elements restrict to the previously considered special cycles in the generic fiber and such they are compatible with two basic operations on the lattices.

Jan Bruinier talked about joint work with Shaul Zemel on special cycles on toroidal compactifications on orthogonal Shimura varieties of signature $(n, 2)$. Introducing suitable multiplicities they define a special divisor and prove that an associated generating series yields a modular form with values in the first Chow group of the toroidal compactification of the Shimura variety.

Wei Zhang's talk reported on ongoing joint work with Daniel Disegni relating p-adic heights of arithmetic diagonal cycles on certain unitary Shimura varieties to derivatives of p-adic L-functions. He stated a general conjecture in this direction and explained the proof of several special cases.

Andreas Mihatsch gave a talk on joint work with Li on the arithmetic transfer conjecture relating intersection numbers on certain Rapoport-Zink spaces and derivatives of orbital integrals. He explained the general Arithmetic Transfer Conjecture for GL_{2n} and a recent proof for GL_4.

Galois representations in the cohomology of Shimura varieties. One of the core ingredients of the Langlands program are Galois representations, so from this point of view Shimura varieties are mainly a tool to produce such representations, by passing to cohomology groups. In this fashion, one naturally obtains relations between Galois representations and automorphic forms or representations. Understanding the precise relationship is usually quite subtle, however.

Si Ying Lee gave a talk showing that some mod p Galois representation coming from the Hecke eigensystem associated with a suitable Hilbert modular variety at an inert prime is unramified.

In her talk about joint work with Matteo Tamiozzo, Ana Caraiani explained ongoing work on the cohomology with torsion coefficients of Hilbert modular varieties and quaternionic Shimura varieties.

Teruhisa Koshikawa explained his current work proving a vanishing result for the generic part of the cohomology of Shimura varieties associated to quasi-split classical groups, and formulated a general conjecture in this direction.

In his talk on the cohomology of Shimura varieties, Sug Woo Shin presented joint work in progress with Mark Kisin and Yihang Zhu towards computing the Hasse-Weil ζ-function of Shimura varieties of abelian type in terms of automorphic L-functions and understanding the representations arising in the etale cohomology of a Shimura variety.

In an online talk Yifeng Liu explained his joint results with Yichao Tian and Liang Xiao on arithmetic level raising via unitary Shimura varieties with good reduction.

In his talk with the title 'Undercover p-adic modular forms', Valentin Hernandez explained his joint work with Hellmann and Schraen proving the existence of pairs of a classical and a non-classical modular form for GL_3 sharing the same Galois representation, weight and Hecke eigenvalues.

The Fargues-Fontaine curve and shtukas. As mentioned above, the "curve" introduced by Fargues and Fontaine is by now a fundamental object of the theory (and of p-adic Hodge theory).

Sebastian Bartling gave a talk on the cohomology of the Fargues-Fontaine curve. He explained a conjecture of Fargues on the vanishing of étale cohomology in degree greater than 2 and on a comparison between the cohomology of the algebraic and the adic Fargues-Fontaine curve and proved several partial results in this direction.

Linus Hamann associated with any L-parameter induced from a maximal torus a perverse sheaf on Bun_G, the stack of G-bundles on the Fargues-Fontaine curve. Under a genericity assumption he proved that this perverse sheaf in an eigensheaf for the prescribed eigenvalue.

Ian Gleason presented a new approach to study affine Deligne-Lusztig varieties using local Shimura varieties and moduli spaces of p-adic shtukas. In joint work with Lim, Lourenço and Xu, he generalized in this way the known results on connected components of affine Deligne-Lusztig varieties to all remaining cases, and also provided a new proof for the previously known cases.

Further related topics. We also had a small number of talks on topics that complement current developments on the Arithmetic of Shimura varieties.

Lucas Mann gave a talk with the title 'p-adic sheaves on classifying stacks and the p-adic Jacquet-Langlands correspondence'. He explained joint work with David Hansen on dualizability of sheaves and cohomological smoothness in the very general context of small stacks on the v-site of perfectoid spaces in characteristic p.

Thibaud van den Hove presented results of joint work with Cass and Scholbach on an integral motivic version of the Satake equivalence, generalizing the rational motivic Satake equivalence of Richarz and Scholbach.

Rong Zhou explained recent and ongoing work with Mark Kisin proving independence of ℓ of Frobenius conjugacy classes in the Mumford-Tate group of an abelian variety over a number field and of Weil-Deligne representations associated with abelian varieties.

Ananth Shankar discussed in his talk (on joint work with Abhishek Oswal, Xinwen Zhu and Anand Patel) how to prove a p-adic analogue of Borel's theorem that any holomorphic map from an affine complex algebraic variety to a Shimura variety over $\mathbb{C}$ with sufficient level structure must be algebraic.

Workshop: Arithmetic of Shimura Varieties

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Abstracts

Unramifiedness of Galois Representations associated to Hilbert modular varieties

Si Ying Lee

This will be a talk about a paper in preparation. [10]

Let p be prime number. The Serre weight conjecture, specialised to the case of weight one, says the following:

Conjecture 1. *Let*

$$\rho : \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_p)$$

be a continuous, irreducible, and odd Galois representation, which is moreover unramified at p. Then, there exists a mod p modular eigenform f such ρ_f, the Galois representation attached to f by Deligne-Serre [3] *is $\rho_f \simeq \rho$. Moreover, this induces an equivalence between weight 1 mod p modular eigenforms and odd irreducible unramified mod p Galois representations.*

This correspondence (at least when $p > 2$) was proved in the works of Gross, Edixhoven, Coleman-Voloch and Wiese. The Serre weight conjectures have a natural generalization to the case of Hilbert modular varieties associated to a totally real field F. This was stated in the case of regular weight by Buzzard-Diamond-Jarvis [1], and proved (assuming modualarity of the residual representation) by Gee-Liu-Savitt [8]. In the case of partial weight one, the same weight receipe in [1] also gives us a conjectural relation between mod p Hilbert modular forms which are of partial weight 1 and mod p representations of $\mathrm{Gal}(\bar{F}/F)$ which are unramified at some prime $\mathfrak{p}$ of F above p.

More precisely, let Sh be the Hilbert modular variety defined using a totally real field F, and let p be a prime unramified in F, with toroidal compactification $\mathrm{Sh}^{\mathrm{tor}}$. Let Σ be the set of embeddings $\tau : F \hookrightarrow \mathbb{R}$. Choose $\mathfrak{p}|p$ a prime of F. Let $\bar{\rho}$ be a mod p Galois representation associated to a Hecke eigensystem appearing in $H^i(\mathrm{Sh}^{\mathrm{tor}}_{\mathbb{F}_p}, \omega^\kappa)$ where $\kappa = ((k_\tau)_\tau, w)$ is a paritious weight. This Galois representation was constructed by Emerton-Reduzzi-Xiao [7].

Now, we assume that $k_\tau = 1$ for τ inducing $\mathfrak{p}$, and $w = -1$. For simplicity, I will assume that p is inert, so $\mathfrak{p} = p$, and we are in situation of parallel weight one. However, the methods of proof should work even in the partial weight one situation, however still assuming that p is unramified.

The theorem I will talk about is the following:

Theorem 1. [10] *The representation $\bar{\rho}$ appearing in $H^i(\mathrm{Sh}^{\mathrm{tor}}_{\mathbb{F}_p}, \omega^\kappa)$ is unramified at $\mathfrak{p}$.*

This confirms (part of) a conjecture of Emerton-Reduzzi-Xiao [6]. Note that in the case of degree zero, unramifiedness of the associated Galois representation was already known from the work of [6], Dimitrov-Wiese [5], and De Maria [2].

However coherent cohomology in partial weight one is expected to have torsion in many consecutive degrees, and this result applies to all degrees of cohomology.

The key insight in allowing me to prove this theorem is to work directly with the mod p algebraic correspondences that induce the action on coherent cohomology. In particular, I apply previous work on the Eichler-Shimura relation [9] to show that the T_p operator acting on weight one cohomology satisfies a polynomial relation, and use this to deduce that the representations $\bar{\rho}$ are ordinary. From here, one can adapt the arguments of [6] and [5] to conclude the theorem.

This result only deals with the mod p reduction of the Galois representation. One can naturally hope to obtain such a result for the mod p^n Galois pseudorepresentation, for which some results in parallel weight one, degree zero are know, by work of Deo, Dimitrov and Wiese [4].

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On Ihara's lemma for Hilbert modular varieties

Ana Caraiani

(joint work with Matteo Tamiozzo)

In this talk, I discussed recent joint work [2] and joint work in progress [3] with Matteo Tamiozzo, whose aim is to understand the cohomology with torsion coefficients of Hilbert modular varieties and quaternionic Shimura varieties.

Let F be a totally real field and D/F be a quaternion algebra, possibly split everywhere. We set $G := D^\times$ and we let $K \subset (D \otimes_F \mathbb{A}_{F,f})^\times$ denote a neat compact open subgroup. Associated to $\mathrm{Res}_{F/\mathbb{Q}} G$ and K we have a Shimura variety X_K

defined over the reflex field E (which is contained in the Galois closure of F) and of dimension $d \leq [F : \mathbb{Q}]$.

(1) For example, if D is split everywhere, we have $G = \mathrm{GL}_2/F$ and the X_K are Hilbert modular varieties with reflex field $E = \mathbb{Q}$ and dimension $d = [F : \mathbb{Q}]$. They are non-compact Shimura varieties of abelian type.

(2) If D is ramified at some (necessarily even) set of places of F, the associated X_K are compact quaternionic modular varieties. They do not admit a direct moduli interpretation, but can be related to certain auxiliary unitary Shimura varieties of abelian type which do admit a moduli interpretation. This depends on a choice of a CM extension of F, see [7] for more details. This connection with auxiliary unitary Shimura varieties will be used implicitly to discuss integral models and special fibres in a unified way for the Hilbert and quaternionic cases.

Tian and Xiao [7] obtained numerous explicit geometric relationships between the special fibres of Shimura varieties attached to different choices of D – these relationships are known under the umbrella term *geometric Jacquet–Langlands correspondences*. (One should also mention here the work of Deuring and Serre who first noticed the phenomenon and the work of Xiao–Zhu who provided a vast generalisation to higher-dimensional Shimura varieties.)

We now let ℓ be an odd prime and $\mathbb{T}$ denote an abstract spherical Hecke algebra which acts by correspondences on $H^*(X_K(\mathbb{C}), \mathbb{F}_\ell)$. Let $\mathfrak{m} \subset \mathbb{T}$ denote a maximal ideal in the support of $H^*(X_K(\mathbb{C}), \mathbb{F}_\ell)$. Then there exists a unique semi-simple Galois representation

$$\bar{\rho}_{\mathfrak{m}} : \mathrm{Gal}(\overline{F}/F) \to \mathrm{GL}_2(\overline{\mathbb{F}}_\ell)$$

such that the characteristic polynomial of Frobenius on $\bar{\rho}_{\mathfrak{m}}$ is determined by the Satake parameters of $\mathfrak{m}$ at (all but finitely many) unramified places. In the Hilbert case, this is essentially a special case of Scholze's theorem on Galois representations associated to torsion in the cohomology of locally symmetric spaces for GL_n over CM fields. In the quaternionic case, this is deduced in [2] using the fact that the X_K are proper.

Conjecture 1. *If $\mathfrak{m}$ is non-Eisenstein, i.e. $\bar{\rho}_{\mathfrak{m}}$ is (absolutely) irreducible, then the localisation $H^*(X_K(\mathbb{C}), \mathbb{F}_\ell)_{\mathfrak{m}}$ is concentrated in the middle degree d.*

This is a special case of a general folkore conjecture on the cohomology of locally symmetric spaces, with many potential applications to the Langlands program. However, one is often interested in understanding the structure of these mod ℓ cohomology groups, not just the degree (or range of degrees) in which they are supported. As we will see below, a generalisation of Ihara's lemma leads to a conjectural description, at least away from the prime ℓ.

Let $\mathfrak{p} \mid p$ be a prime of F such that $D_{\mathfrak{p}}$ is split and such that the level K satisfies $K = K^{\mathfrak{p}} K_{\mathfrak{p}}$ with $K_{\mathfrak{p}} = \mathrm{GL}_2(\mathcal{O}_{F,\mathfrak{p}})$. Define a new compact open subgroup

$$K_0(\mathfrak{p}) := \left\{ k = (k^{\mathfrak{p}}, k_{\mathfrak{p}}) \in K \mid k_{\mathfrak{p}} \equiv \left(\begin{smallmatrix} * & * \\ 0 & * \end{smallmatrix}\right) \pmod{\mathfrak{p}} \right\},$$

which has Iwahori level at $\mathfrak{p}$. We have two degeneracy maps $\pi_1, \pi_2 : X_{K_0(\mathfrak{p})} \to X_K$, used to define the Hecke correspondence for $T_{\mathfrak{p}}$.

Conjecture 2. *If $\mathfrak{m}$ is non-Eisenstein, then the map*

$$\pi_1^* + \pi_2^* : H^d(X_K, \mathbb{F}_\ell)_{\mathfrak{m}}^{\oplus 2} \to H^d(X_{K_0(\mathfrak{p})}, \mathbb{F}_\ell)_{\mathfrak{m}}$$

is injective.

This is (a generalisation of) Ihara's lemma. When $d = 0$, it can be proved using strong approximation. When $d = 1$ it was recently proved by Manning and Shotton [6] under a mild technical assumption, using an ingenious argument to reduce it to the case $d = 0$. In arbitrary dimension, results are known roughly in the case when the mod ℓ étale cohomology is Fontaine–Laffaille, which in practice can be a strong restriction on ℓ.

Ihara's lemma has applications to level-raising, going back to work of Ribet, and to congruences between special values of L-functions. It also has a representation-theoretic reformulation, which was used by Clozel–Harris–Taylor [4] to formulate generalisations of Ihara's lemma to higher rank groups, with intended applications to non-minimal modularity lifting. More precisely, consider the colimit

$$\widetilde{H}^d_{\mathfrak{m}} := \varinjlim_{K'_{\mathfrak{p}}} H^d(X_{K^{\mathfrak{p}} K'_{\mathfrak{p}}}, \mathbb{F}_\ell)_{\mathfrak{m}},$$

which is an admissible smooth mod ℓ representation of $\mathrm{GL}_2(F_{\mathfrak{p}})$. Ihara's lemma is equivalent (at least after assuming Conjecture 1) to asking that all the irreducible *subrepresentations* of $\widetilde{H}^d_{\mathfrak{m}}$ be generic when $\mathfrak{m}$ is non-Eisenstein.

Remark. Assume that $\ell \neq p$, $F = \mathbb{Q}$ and $G = \mathrm{GL}_2$. Together with a mod ℓ multiplicity one result, Ihara's lemma is the crucial ingredient used in Emerton's work to show that ℓ-adically completed cohomology satisfies compatibility with the local Langlands correspondence in families of Emerton and Helm [5] at the prime $\mathfrak{p}$. The representation-theoretic reformulation of Ihara's lemma is equally important for establishing the compatibility with categorical local Langlands, as in upcoming work of Emerton–Gee–Zhu. Note also that the explicit description of the mod ℓ local Langlands in families for $\ell \neq p$ shows that non-generic, i.e. one-dimensional, representations of $\mathrm{GL}_2(F_{\mathfrak{p}})$ do occur as *subquotients* of $\widetilde{H}^d_{\mathfrak{m}}$.

Theorem 1. *Assume that ℓ is an odd prime and that the image of $\bar{\rho}_{\mathfrak{m}}$ is non-solvable. Then Conjecture 1 holds.*

This is established in [2] using a modification of the strategy for unitary Shimura varieties developed in [1]. When the image of $\bar{\rho}_{\mathfrak{m}}$ is non-solvable, one can choose an auxiliary prime p which is split completely in F and such that $\bar{\rho}_{\mathfrak{m}}$ is *generic* (in an appropriate sense) at each prime $v \mid p$ of F. As in [1], we want to compute the complex of sheaves $(R\pi_{\mathrm{HT}*}\mathbb{F}_\ell)_{\mathfrak{m}}$ on the corresponding Hodge–Tate period domain. The goal is to show that this complex is only supported on the μ-ordinary Newton stratum. For each $b \in B((\mathrm{Res}_{F/\mathbb{Q}} G)_{\mathbb{Q}_p}, \mu)$ which is not μ-ordinary, there is a geometric Jacquet–Langlands isomorphism between the corresponding Igusa variety (which is closely related to the fibre of π_{HT} over the Newton stratum determined by b) and a μ-ordinary Igusa variety on a smaller Shimura variety. This fact, which is related to the results of [7], but in some sense is more natural, is played against the genericity of $\bar{\rho}_{\mathfrak{m}}$ to prove the desired result.

Theorem 2. *Assume that $\ell \geq 5$, that the image of $\bar{\rho}_{\mathfrak{m}}$ contains $\mathrm{SL}_2(\mathbb{F}_\ell)$ and, in the case $\ell = 5$, that $\bar{\rho}_{\mathfrak{m}}$ satisfies an additional Taylor–Wiles assumption. Then Conjecture 2 holds.*

This is work in progress [3]. The proof goes via induction on the dimension d of the quaternionic Shimura variety, using a generalisation of the technique of Manning and Shotton. One of the geometric Jacquet–Langlands correspondences of [7], together with the weight spectral sequence (which simplifies by Theorem 1), can be used to relate the cohomology of a quaternionic Shimura variety of dimension d to the cohomology of a well-chosen quaternionic Shimura variety of dimension $d-1$. Furthermore, once Theorem 1 is known, this relationship can be patched via the usual Taylor–Wiles method. The patched modules over the relevant local deformation rings behave just as in [6], allowing us to transfer information about Ihara's lemma between the two settings.

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p-Adic Sheaves on Classifying Stacks and the p-Adic Jacquet-Langlands Correspondence

Lucas Mann

(joint work with David Hansen)

Fix a prime p. Our goal is to provide a general framework that can be applied to the following situations:

(1) Given a p-adic Lie group G (e.g. $\mathrm{GL}_n(\mathbb{Q}_p)$), we want to study the (derived) category $\mathcal{D}(G, \mathbb{F}_p)$ of smooth G-representations on $\mathbb{F}_p$-vector spaces. Among others, by classical results of Lazard the G-group cohomology satisfies a form of Poincaré duality.

(2) Given a rigid variety over some non-archimedean field extension K of $\mathbb{Q}_p$ we want to study the (derived) category $\mathcal{D}_{\mathrm{et}}(X, \mathbb{F}_p)$ of étale $\mathbb{F}_p$-sheaves on X. If X is proper and smooth then the étale $\mathbb{F}_p$-cohomology of X satisfies a form of Poincaré duality.

In the following we explain how to generalize both of the above situations and then apply this general machine to the p-adic Jacquet-Langlands correspondence.

1. A 6-Functor Formalism for p-Adic Sheaves

We work with Scholze's category vStack of small stacks on the v-site of perfectoid spaces in characteristic p. Among others, this category can model rigid varieties over a non-archimedean field and classifying stacks $*/G$ of locally profinite groups G (see [4] for details). In [2] we prove the following result:

Theorem 1.

(1) *There is a unique hypercomplete v-sheaf of ∞-categories*

$$\mathrm{vStack}^{\mathrm{op}} \to \mathrm{Cat}_\infty, \qquad X \mapsto \mathcal{D}(X, \mathbb{F}_p)$$

such that for every "nice" affinoid perfectoid space $X = \mathrm{Spa}(A, A^+)$ with pseudouniformizer π, $\mathcal{D}(X, \mathbb{F}_p)$ is the (derived) ∞-category of solid[1] *almost A^+/π-modules equipped with Frobenius-semilinear isomorphism.*

(2) *For every $X \in$ vStack there is a natural fully faithful embedding*

$$\mathcal{D}_{\mathrm{et}}(X, \mathbb{F}_p)^{\mathrm{oc}} \hookrightarrow \mathcal{D}(X, \mathbb{F}_p)$$

which restricts to an equivalence of perfect objects. Here the left-hand side denotes the ∞-category of étale overconvergent $\mathbb{F}_p$-sheaves on X.

In this talk we do not care much about the actual shape of the ∞-category $\mathcal{D}(X, \mathbb{F}_p)$ and rather view it as a black box. Note that for any locally profinite group G with associated classifying stack $X = (\mathrm{Spa}\, \mathbb{C}_p)/G$, we have $\mathcal{D}_{\mathrm{et}}(X, \mathbb{F}_p)^{\mathrm{oc}} = \mathcal{D}(G, \mathbb{F}_p)$, the ∞-category of smooth G-representations.

Having defined a good category of sheaves, we will now introduce six operations on them. The following constructions were introduced in [2]:

Definition 1.

(1) For every small v-stack X we equip $\mathcal{D}(X, \mathbb{F}_p)$ with the tensor product $\otimes$ induced by the tensor product on solid modules. We further introduce the internal hom $\underline{\mathrm{Hom}}$ via the usual hom-tensor adjunction.

(2) For a map $f\colon Y \to X$ of small v-stacks we denote by $f^*\colon \mathcal{D}(X, \mathbb{F}_p) \to \mathcal{D}(Y, \mathbb{F}_p)$ the "restriction map" of the sheaf $\mathcal{D}(-, \mathbb{F}_p)$. We denote by f_* the right adjoint of f^*.

(3) Let $f\colon Y \to X$ be a map of small v-stacks. If f is étale then we define $f_!$ to be the left adjoint of f^*; if f is proper and p-bounded[2] then we define $f_! := f_*$. In general, if f can v-locally on X and étale locally on Y be factored into étale and p-bounded proper maps, then we define $f_!\colon \mathcal{D}(Y, \mathbb{F}_p) \to \mathcal{D}(X, \mathbb{F}_p)$ by composing the previous two cases. In this case we also denote by $f^!$ the right adjoint of $f_!$.

[1] The term "solid" was introduced by Clausen-Scholze in their condensed mathematics and roughly means "equipped with a complete topology".

[2] Roughly, "p-bounded" means "of finite p-cohomological dimension"

The maps f in part (3) of the above definition will be called *bdcs* in the following. We now explain how to extend the definition of $f_!$ and $f^!$ to a more general class of maps f, including many "stacky" maps (see [1]):

Definition 2. A map $f\colon Y \to X$ of small v-stacks is called *p-fine* if there is a map $g\colon Z \to Y$ such that both g and $f \circ g$ are bdcs and g admits "universal p-codescent".[3]

Theorem 2. *The above definition for $f_!$ extends uniquely to all p-fine maps f. The thus defined collection of functors $\otimes$, $\underline{\mathrm{Hom}}$, f^*, f_*, $f_!$, $f^!$ forms a 6-functor formalism, i.e. satisfies the usual compatibilities like proper base-change and projection formula.*

One checks easily that p-fine maps are stable under composition and base-change and that they satisfy the usual "2-out-of-3" property. Furthermore, any smooth map of rigid varieties admits universal p-codescent, which provides us with a large amount of examples for p-fine maps. Here is another important example:

Theorem 3. *Let G be a locally profinite group having an open subgroup of finite p-cohomological dimension (e.g. a p-adic Lie group). Then $*/G \to *$ is p-fine.*

In particular for any map $H \to G$ of p-adic Lie groups the associated map $f\colon */H \to */G$ of classifying stacks is p-fine. The pullback functor f^* corresponds to the restriction/inflation of representations and the pushforward functor f_* corresponds to the induction/cohomology of representations. The shriek functor $f_!$ is somewhat more mysterious and roughly corresponds to compact induction/homology, but only up to a twist and shift. This shift and twist is not visible in the ℓ-adic theory, so we propose that $f_!$ is the correct p-adic replacement for compact induction and group homology.

2. Admissibility and Smoothness

Given a p-fine map $f\colon Y \to X$ of small v-stacks and an object $\mathcal{P} \in \mathcal{D}(Y, \mathbb{F}_p)$, we say that $\mathcal{P}$ is f-dualizable if the natural map

$$\pi_1^* \underline{\mathrm{Hom}}(\mathcal{P}, f^! \mathbb{F}_p) \otimes \pi_2^* \mathcal{P} \to \underline{\mathrm{Hom}}(\pi_1^* \mathcal{P}, \pi_2^! \mathcal{P})$$

is an isomorphism, where $\pi_i\colon Y \times_X Y \to Y$ denote the two projections. This is a well-behaved notion in any 6-functor formalism; in the case of ℓ-adic sheaves it recovers the notion of f-universally locally acyclic sheaves. We have:

Theorem 4. *Let $S = \operatorname{Spa} \mathbb{C}_p$, G a p-adic Lie group and $f\colon S/G \to S$ the projection. Then $V \in \mathcal{D}(S/G, \mathbb{F}_p)$ is f-dualizable if and only if it lies in $\mathcal{D}(G, \mathbb{F}_p)$ and can be represented by a bounded complex of admissible smooth G-representations.*

Many of the basic properties of admissible representations can be recovered in a completely formal way from abstract properties of relatively dualizable sheaves.

Definition 3. A p-fine map $f\colon Y \to X$ is called *p-cohomologically smooth* if the constant sheaf $\mathbb{F}_p$ is f-dualizable and $f^! \mathbb{F}_p$ is invertible.

[3] We omit the precise definition of universal p-codescent for space reasons.

By [2], if f is a smooth map of rigid varieties of pure dimension d then f is p-cohomologically smooth with $f^!\mathbb{F}_p = \mathbb{F}_p[2d](d)$. By results of Lazard, if G is a p-adic Lie group of dimension d then $f\colon * /G \to *$ is p-cohomologically smooth with $f^!\mathbb{F}_p$ in degree $-d$.

3. The p-Adic Jacquet-Langlands Correspondence

We now sketch how to apply the above formalism to get new insights into the p-adic Jacquet-Langlands functor. Fix an integer $n > 0$ and a finite extension $F/\mathbb{Q}_p$, let $G = \mathrm{GL}_n(F)$ and let D be the central division algebra of invariant $1/n$ over F. Let $\mathcal{M}_\infty$ denote the infinite-level Lubin-Tate tower. It admits commuting actions of G and $D^\times$ and two period maps $\mathcal{M}_\infty \to \mathbb{P}^{n-1}$, $\mathcal{M}_\infty \to \Omega^{n-1}$ which are a G-torsor and a $D^\times$-torsor, respectively. Thus if we denote $X := \mathcal{M}_\infty/(G \times D^\times)$ then we have canonical identifications $X = \Omega^{n-1}/G = \mathbb{P}^{n-1}/D^\times$ and therefore get canonical maps $S/G \xleftarrow{f} X \xrightarrow{g} S/D^\times$, where $S = \operatorname{Spa}\mathbb{C}_p$. The *Jacquet-Langlands functor* [3] is the functor $\mathcal{J} := g_* f^*\colon \mathcal{D}(S/G, \mathbb{F}_p) \to \mathcal{D}(S/D^\times, \mathbb{F}_p)$. Using the smoothness of f and g and the properness of g, we get:

Theorem 5. *$\mathcal{J}$ restricts to a functor from smooth admissible G-representations to (derived) smooth admissible $D^\times$-representations. For $V \in \mathcal{D}(G, \mathbb{F}_p)$ we have*

$$\mathcal{J}(V)^\vee = \mathcal{J}(V^\vee)[2n-2](n-1).$$

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Special cycles on toroidal compactifications of orthogonal Shimura varieties

Jan Hendrik Bruinier
(joint work with Shaul Zemel)

Orthogonal Shimura varieties that arise from even lattices of signature $(n, 2)$ carry a natural family of divisors that are called special divisors or Heegner divisors. In his seminal paper [2], Borcherds constructed meromorphic modular forms on these varieties, and then used them in the sequel [3] to establish the modularity of the generating series of these divisors in the first Chow group of such a Shimura variety. This provided a new proof of the Gross-Kohnen-Zagier theorem [9] about the modularity of Heegner divisors on modular curves and a generalization to orthogonal Shimura varieties of arbitrary dimension and level. Taking Chern classes one obtains as a corollary also the modularity of the generating series of the cohomology classes of special divisors, such as the Hirzebruch-Zagier theorem for Hilbert modular surfaces [10] and particular cases of its generalization in [14].

The modularity of generating series of special cycles plays a central role in the Kudla program, see, e.g. [12], [13]. Once one knows modularity, the generating series can be used to construct arithmetic analogues of the theta correspondence, relating Siegel modular forms to classes of special cycles in (arithmetic) Chow groups. For applications to intersection and height pairings one is interested in special cycles on smooth *toroidal compactifications* and corresponding modularity results, cf. Problem 3 of [13]. In our talk we reported on our recent joint work with S. Zemel [7], which is a contribution in this direction.

Let (V,Q) be a quadratic space over $\mathbb{Q}$ of signature $(n,2)$, and write $(\cdot,\cdot)$ for the bilinear form corresponding to Q. The hermitian symmetric space associated with the special orthogonal group $\mathrm{SO}(V)$ of V can be realized as

$$D = \{z \in V \otimes_{\mathbb{Q}} \mathbb{C} \mid \ (z,z) = 0 \text{ and } (z,\bar{z}) < 0\}/\mathbb{C}^\times.$$

This domain has two connected components. We fix one of them and denote it by D^+. Let $L \subset V$ be an even lattice. For simplicity we assume throughout this exposition that L is unimodular. This simplifies several technical aspects. For the general case we refer to [7]. Let $\Gamma \subset \mathrm{SO}(L)$ be a subgroup of finite index which takes D^+ to itself. The quotient

$$X_\Gamma = \Gamma \backslash D^+$$

has a structure as a quasi-projective algebraic variety of dimension n. It is projective if and only if V is anisotropic over $\mathbb{Q}$, hence in particular when $n > 2$. It has a canonical model defined over a cyclotomic extension of $\mathbb{Q}$.

There is a vast supply of algebraic cycles on X_Γ arising from embedded quadratic spaces $V' \subset V$ of smaller dimension. Let $1 \leq r \leq n$. For any $x = (x_1, \dots, x_r) \in L^r$ with positive semi-definite inner product matrix $Q(x) = \frac{1}{2}((x_i,x_j))_{i,j}$ there is a special cycle

$$x^\perp = \{z \in D^+ \mid \ (z,x_1) = \dots = (z,x_r) = 0\}$$

on D^+, whose codimension is equal to the rank of $Q(x)$. Its image in X_Γ defines an algebraic cycle, which we also denote by $x^\perp$. If $T \in \mathrm{Sym}_r(\mathbb{Q})$ is positive semi-definite, we consider the special cycle on X_Γ given by

$$Z(T) = \sum_{\substack{x \in L^r/\Gamma \\ Q(x) = T}} x^\perp,$$

see [12], [13]. Its codimension is equal to the rank $\mathrm{rk}(T)$ of T. We obtain a class in the Chow group $\mathrm{CH}^r(X_\Gamma)$ of codimension r cycles by taking the intersection pairing

$$\mathcal{Z}(T) = [Z(T)] \cdot [\mathcal{L}^\vee]^{r - \mathrm{rk}(T)}$$

with a power of the dual of the tautological bundle $\mathcal{L}$ on X_Γ.

The following modularity result shows that there are many non-trivial relations among these classes in the Chow group.

Theorem 1. *The generating series*

$$A_r(\tau) = \sum_{\substack{T\in \mathrm{Sym}_r(\mathbb{Q})\\ T\geq 0}} \mathcal{Z}(T)\cdot q^T$$

is a Siegel modular form of weight $1+n/2$ *for* $\mathrm{Sp}_r(\mathbb{Z})$ *with values in* $\mathrm{CH}^r(X_\Gamma)_\mathbb{Q}$*. Here* $\tau\in\mathbb{H}_r$ *is in the Siegel upper half plane of genus* r *and* $q^T = e^{2\pi i\,\mathrm{tr}(T\tau)}$*.*

This theorem was conjectured by Kudla [13] motivated by the corresponding result in [14] for the cohomology classes. It was proved in [6] building on the results of [15]. A generalization to integral models of orthogonal Shimura varieties was recently given in [11].

It is our goal to establish analogues results for suitable extensions of special cycles to toroidal compactifications. Here we only consider the case of divisors, that is, we assume $r=1$ from now on. We fix a smooth toroidal compactification $X_\Gamma^{\mathrm{tor}} = X_{\Gamma,\Sigma}^{\mathrm{tor}}$ of X_Γ determined by an admissible collection of fans Σ, see e.g. [1].

It turns out that Theorem 1 does not extend to X_Γ^{tor} if one works with the closures of the special divisors on X_Γ. Instead, one has to add the irredicible components of the boundary divisor $\partial X_\Gamma^{\mathrm{tor}} = X_\Gamma^{\mathrm{tor}}\setminus X_\Gamma$ with appropriate multiplicities, a phenomenon which was already observed for Hilbert modular surfaces in [10].

The idea of our approach is the following. For every every positive integer $m\in\mathbb{Z}_{>0}$ there is a harmonic Mass form f_m of weight $1-\frac{n}{2}$ whose principal part is given by $q^{-m}+O(1)$. Its regularized theta lift

$$\Phi_m^L(z) = \int_{\mathrm{SL}_2(\mathbb{Z})\backslash\mathbb{H}}^{\mathrm{reg}} f_m(\tau)\theta_L(\tau,z)\,d\mu(\tau)$$

against the Siegel theta function θ_L of L determines an automorphic Green function on X_Γ with a logarithmic singularity along the special divisor $Z(m)$, see [4] and [5] for details. We analyze the behavior of this Green function towards each boundary component of X_Γ^{tor}. It turns out that the Green function has logarithmic growth along certain boundary divisors. Here the corresponding multiplicities are determined by regularized theta lifts of f_m associated to lattices of signature $(n-2,0)$ and $(n-1,1)$ which are obtained as quotients of L.

Every primitive rank 2 isotropic sublattice $J\subset L$ determines a 1-dimensional Baily-Borel boundary component and also a canonical boundary divisor B_J of X_Γ^{tor}. On the other hand, the lattice $D=(J^\perp\cap L)/J$ is positive definite of signature $(n-2,0)$. We show that the Green function Φ_m^L has a logarithmic growth along the divisor $\mathrm{mult}_J(m)\cdot B_J$, where the multiplicity $\mathrm{mult}_J(m)$ is determined by the regularized theta lift Φ_m^D of f_m against the theta function θ_D of the lattice D, which in turn is given by

$$\mathrm{mult}_J(m) = \frac{2m}{n-2}\cdot r_D(m),$$

where $r_D(m)$ denotes the representation number of m by the lattice D.

Every primitive rank 1 isotropic sublattice $I\subset L$ determines a 0-dimensional Baily-Borel boundary component, as well as an even lattice $K=(I^\perp\cap L)/I$

of signature $(n-1,1)$. In contrast to the canonical boundary divisor B_J from the previous paragraph, the inverse image in X_Γ^{tor} depends on the choice of an admissible cone decomposition of the rational closure of a cone C of negative norm vectors in $K_\mathbb{R}$. The lattice K and an oriented primitive negative norm vector $\omega \in K \cap C$ that spans an inner ray in the associated cone decomposition determine a boundary divisor $B_{I,\omega}$ of X_Γ^{tor}. We prove that Φ_m^L has logarithmic growth along the divisor $\mathrm{mult}_{I,\omega}(m) \cdot B_{I,\omega}$, where the multiplicity $\mathrm{mult}_{I,\omega}(m)$ is given by the regularized theta lift Φ_m^K of f_m against the Siegel theta function θ_K of K, evaluated at the special point associated with ω in the Grassmannian of K.

We define the special divisor $Z^{\mathrm{tor}}(m) \in \mathrm{Div}(X_\Gamma^{\mathrm{tor}})_\mathbb{R}$ by the formula

$$Z^{\mathrm{tor}}(m) = Z(m) + \sum_{J \subset L} \mathrm{mult}_J(m) \cdot B_J + \sum_{I \subset L} \sum_{\mathbb{R}_{\geq 0}\omega} \mathrm{mult}_{I,\omega}(m) \cdot B_{I,\omega}. \tag{1}$$

Here J (resp. I) runs over a set of representatives of rank 2 (resp. rank 1) primitive isotropic sublattices of L modulo Γ, and given I, the index $\mathbb{R}_{\geq 0}\omega$ runs over representatives for the inner rays of the chosen admissible cone decomposition. Note that while the multiplicities $\mathrm{mult}_J(m)$ are easily seen to be rational because of their description as representation numbers, the multiplicities $\mathrm{mult}_{I,\omega}(m)$ seem to be real in general. Note that the above definition of $Z^{\mathrm{tor}}(m)$ is compatible with respect to pushforward and pullback for morphisms of toroidal compactifications $X_{\Gamma,\Sigma_1}^{\mathrm{tor}} \to X_{\Gamma,\Sigma_2}^{\mathrm{tor}}$ induced by admissible refinements of fans.

Our first main result states that Φ_m^L is a logarithmic Green function for the divisor $Z^{\mathrm{tor}}(m)$ on X_Γ^{tor} with an additional term that is pre-log-log along the whole boundary in the sense of [8]. In particular it satifies the current equation

$$dd^c[\Phi_m^L] + \delta_{Z^{\mathrm{tor}}(m)} = [\eta^{\mathrm{tor}}(m)],$$

where $\eta^{\mathrm{tor}}(m)$ is the sum of a smooth $(1,1)$-form on X_Γ^{tor} and the dd^c-image of the pre-log-log term. Our second main result describes the generating series of these special divisors as follows.

Theorem 2. *Assume that n is larger than the Witt rank of L and write $[Z^{\mathrm{tor}}(m)]$ for the class of $Z^{\mathrm{tor}}(m)$ in $\mathrm{CH}^1(X_\Gamma^{\mathrm{tor}})_\mathbb{R}$. In addition set $[Z^{\mathrm{tor}}(0)] = [\mathcal{L}^\vee]$. Then the formal power series*

$$A_1^{\mathrm{tor}}(\tau) = \sum_{m \in \mathbb{Z}_{\geq 0}} [Z^{\mathrm{tor}}(m)] \cdot q^m$$

is a modular form of weight $1 + n/2$ for $\mathrm{SL}_2(\mathbb{Z})$ with coefficients in $\mathrm{CH}^1(X_\Gamma^{\mathrm{tor}})_\mathbb{R}$.

Taking advantage of our analysis of the Green functions Φ_m^L, this result can now be proved in a similar way as Theorem 1 in the $r = 1$ case, that is, by the approach of [3]. For every weakly holomorphic modular form f of weight $1 - n/2$ with integral principal part, we consider the associated Borcherds product Ψ_f, but now as a section of the tautological line bundle over X_Γ^{tor}. Since f can be (essentially) uniquely written as a linear combination of the harmonic Maass forms f_m, the logarithm of the Petersson metric of Ψ_f decomposes as a corresponding linear combination of the Green functions Φ_m^L. This implies that the divisor of Ψ_f

on X_Γ^{tor} includes the boundary components of X_Γ^{tor} with multiplicities compatible with (1). Hence f gives rise to a relation among the divisors $Z^{\mathrm{tor}}(m)$, and the theorem follows by Serre duality.

In fact, as we construct the divisors $Z^{\mathrm{tor}}(m)$ via their Green functions, our argument also shows modularity in Arakelov Chow groups arising in arithmetic settings.

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Genericity and cohomology of locally symmetric spaces

Teruhisa Koshikawa

The generic part of the cohomology of Shimura varieties has been studied, and it has seen several applications; see Caraiani's article [3], for instance. We formulate a general conjecture on the vanishing of the generic part of cohomology of locally symmetric spaces, and state one new result.

Our notation is as follows. Let G be a reductive group over $\mathbb{Q}$. Set $G_\infty = G(\mathbb{R})$, and let K_∞ be a maximal compact subgroup of G_∞. Write G_∞° and K_∞° for the identity components of G_∞ and K_∞ respectively. Let Z_G denote the center of G and $A_G \subset Z_G$ the maximal split subtorus. Write A_∞^+ for the identity component of $A_G(\mathbb{R})$. Further, we set

$$d = \dim G_\infty/K_\infty, \quad l_0 = \operatorname{rank} G_\infty - \operatorname{rank} A_\infty^+ K_\infty, \quad q_0 = \frac{d-l_0}{2} \in \mathbb{Z}.$$

Take a sufficiently small open compact subgroup $K_f \subset G(\mathbb{A}_f)$ and set $K = K_\infty^\circ K_f$. We consider the following locally symmetric space

$$X_K = G(\mathbb{Q})\backslash G(\mathbb{A})/A_\infty^+ K.$$

For $R = \overline{\mathbb{F}_\ell}, \overline{\mathbb{Q}_\ell}, \mathbb{C}$, where ℓ is a prime, we consider cohomology groups $H^i(X_K, R)$ and $H_c^i(X_K, R)$, and they are equipped with the Hecke action. More specifically, let S denote a set of bad primes so that G_p is unramified if $p \notin S$ and K has the form $K_S \prod_{p \notin S} K_p$, where K_p is a hyperspecial maximal open compact subgroup of $G(\mathbb{Q}_p)$. For $p \notin S$, we consider the local Hecke algebra at p

$$\mathbb{T}_p = C_c(K_p\backslash G(\mathbb{Q}_p)/K_p, R)$$

acting on $H^i(X_K, R)$ and $H_c^i(X_K, R)$.

Let $\mathfrak{m}_p \subset \mathbb{T}_p$ be a maximal ideal. For any maximal split torus $T_p \subset B_p \subset G_p$, it gives rise to an orbit $W \cdot \chi$, where W is the Weyl group and χ is an unramified character $T_p(\mathbb{Q}_p) \to R^\times$.

Definition 1. Let $p \notin S \cup \{p\}$ and $\mathfrak{m}_p \subset \mathbb{T}_p$ a maximal ideal. We say that $\mathfrak{m}_p$ is *generic* if the normalized parabolic induction $\text{n-Ind}_{T_p(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(w \cdot \chi)$ is irreducible for all $T_p \subset B_p$ and $w \in W$ as above. This irreducibility is known to be held on a Zariski dense open subset.

For example, suppose $G = \mathrm{GL}_n$ for an integer $n > 1$. Then $\mathfrak{m}_p$ gives rise to the Satake parameter $\{\alpha_1, \dots, \alpha_n\} \in (R^\times)^n$ (up to permutation), and it is known that $\mathfrak{m}_p$ is generic if and only if $\alpha_i/\alpha_j \neq p$ in R for all $i, j, i \neq j$. This condition (or a slightly stronger one asking $\alpha_i \neq \alpha_j$) has been used by Boyer [1], Caraiani–Scholze [4, 5], Caraiani–Tamiozzo [6], Hamann [8], and the author [9] in the context of global and local Shimura varieties.

Remark. The definition above is certainly not optimal, and there were some discussions during the workshop (see also [8]). If $R = \mathbb{C}$, it would be natural to allow all $\mathfrak{m}_\mathfrak{p}$ with $\text{n-Ind}_{T_p(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(w \cdot \chi)$ being completely reducible. However, the torsion case seems more subtle, and we keep the original definition here.

Note also that, with the definition above, $\text{n-Ind}_{T_p(\mathbb{Q}_p)}^{G(\mathbb{Q}_p)}(w \cdot \chi)$ has a Whittaker model for *any* choice of Whittaker datum at least if $R = \mathbb{C}, \overline{\mathbb{Q}_\ell}$.

Our main conjecture is the following.

Conjecture 1. *Assume $\mathfrak{m}_p \subset \mathbb{T}_p$ is generic. If $H^i(X_K, R)_{\mathfrak{m}_p} \neq 0$, then $i \geq q_0$. Dually, if $H_c^i(X_K, R)_{\mathfrak{m}_p} \neq 0$, then $i \leq q_0 + l_0$.*

Note that vanishing conjectures based on "non-Eisensetein" condition, which is a conjectural notion in general and global in nature, have been made by Calegari–Geraghty [2], and Emerton [7] (see also [3]). The statement of our conjecture is somewhat elementary and only uses the local condition, while the range of vanishing in the conclusion is different.

If $R = \overline{\mathbb{F}_\ell}$, this conjecture has been shown for some unitary Shimura varieties, where G is the Weil restriction of a similitude unitary group, under several assumptions by Boyer, Caraiani–Scholze, and the author. The case of Hilbert modular varieties is studied by Caraiani–Tamiozzo.

One of our original motivation is to clarify the situation for $R = \mathbb{C}, \overline{\mathbb{Q}_\ell}$. The following result is in preparation.

Theorem 1. *Let G be the Weil restriction of a quasi-split classical group (symplectic, orthogonal, unitary) over a number field. The endoscopic classification of Arthur and Mok implies Conjecture for G and $R = \mathbb{C}, \overline{\mathbb{Q}_\ell}$.*

Let us give an outline of proof. The cohomology $H^i(X_K, \mathbb{C})$ can be described using a certain Lie algebra cohomology of the space of automorphic forms by the work of Franke. Recall that the endoscopic classification classifies discrete automorphic representations using A-parameters ψ and their A-packets Π_ψ. By a careful analysis of the Lie algebra cohomology, inspired by works on Eisenstein cohomology, the problem reduces, after passing to Levi subgroups, to show the following: the component π'_∞ at ∞ of $\pi' \in \Pi_\psi$ is tempered if the component π_p at p of a given $\pi \in \Pi_\psi$ is unramified and generic in the usual sense that π_p has a Whittaker model. As a crucial ingredient, we use (a slightly generalized form of) the so-called enhanced Shahidi conjecture, which claims that the local A-parameter ψ_p is trivial on Arthur SL_2 if the local A-packet Π_{ψ_p} contains a representation with Whittaker model. The enhanced Shahidi conjecture is not known in full generality, but can be verified in our unramified setup as pointed out to us by Hiraku Atobe.

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The connected components of affine Deligne–Lusztig varieties and p-adic period domains.

Ian Gleason
(joint work with João Lourenço, Dong Gyu Lim, Yujie Xu)

The purpose of this talk is to explain the p-adic approach to compute the connected components of affine Deligne–Lusztig varieties (ADLV). Although interesting in its own right, understanding the space of connected components of ADLV is a key ingredient to prove mod p and p-adic uniformization theorems of Newton strata of integral models of Shimura varieties. The advantage of the p-adic approach, in contrast to previous approaches [16], [3], [11], [12], [9], is that the complexity of the combinatorial considerations necessary to tackle the problem do not increase with the complexity of the group G. This allow us to compute connected components of ADLV for all groups, including even anisotropic groups.

Let G be a reductive group over $\mathbb{Q}_p$ with parahoric integral model $\mathcal{G}$ over $\mathbb{Z}_p$. Let μ denote a conjugacy class of geometric cocharacters of G with field of definition $E := E(\mu)$ and let $b \in B(G, \mu)$. To the triple (G, b, μ) Scholze–Weinstein [15] attach a tower of diamonds $\mathrm{Sht}_{(G,b,\mu,K)}$ over $\mathrm{Spd}(\breve{E})$ indexed by open compact subgroups $K \subseteq G(\mathbf{Q}_p)$ called *moduli spaces of p-adic shtukas*. When μ is minuscule this tower is the so called local Shimura variety (LSV) whose existence in full generality was predicted by Rapoport–Viehmann [13]. Local Shimura varieties are group theoretic generalizations of the Drinfeld and Lubin–Tate towers. The systematic study of these group theoretic analogues starts with Rapoport–Zink's seminal work [14]. An important gadget in the theory of local Shimura varieties and moduli spaces of shtukas is the Grothendieck–Messing period map.

$$\pi_{\mathrm{GM}} : \mathrm{Sht}_{(G,b,\mu,K)} \to \mathrm{Gr}_G^{\leq\mu}.$$

This is an étale morphism whose target is a flag variety (or more generally a Schubert variety inside a B_{dR}-Grassmannian when μ is not minuscule).

One way to define *p-adic period domains*, $\mathrm{Gr}_G^{b,\mu} \subseteq \mathrm{Gr}_G^{\leq\mu}$, is to take the image of π_{GM} which is an open subdiamond of $\mathrm{Gr}_G^{\leq\mu}$. The following theorem confirms and generalizes a conjecture of Hartl [10].

Theorem 1 (G., Lourenço, [6])**.** *The diamond* $\mathrm{Gr}_G^{b,\mu}$ *is geometrically connected.*

The relation between ADLV and LSV is through integral models of the latter. Whenever $K = \mathcal{G}(\mathbb{Z}_p)$, Scholze–Weinstein construct a v-sheaf $\mathrm{Sht}^{\mathcal{G}}_{\mu}(b)$ over $\mathrm{Spd}(O_{\breve{E}})$ and prove that for local Shimura data of PEL type the v-sheaf is represented by a Rapoport–Zink space. In particular, the $\overline{\mathbb{F}}_p$-points of $\mathrm{Sht}^{\mathcal{G}}_{\mu}(b)$ can be indentified, using Dieudonné theory, with an ADLV.

$$X^{\mathcal{G}}_{\mu}(b) := \{g\mathcal{G}(\breve{\mathbb{Z}}_p) \mid g^{-1}b\sigma(g) \in \mathrm{Adm}(\mu)\} \subseteq G(\breve{\mathbb{Q}}_p)/\mathcal{G}(\breve{\mathbb{Z}}_p)$$

Through recent advances in perfect geometry [2] [17], one can endow general ADLV with the structure of a perfect scheme even when μ is not minuscule. Moreover, for general tuples $(G, b, \mu, \mathcal{G})$ the v-sheaf $\mathrm{Sht}^{\mathcal{G}}_{\mu}(b)$ behaves like a formal scheme.

Theorem 2 (G. [4])**.** *The v-sheaf $\mathrm{Sht}^{\mathcal{G}}_{\mu}(b)$ is a prekimberlite with generic fiber $\mathrm{Sht}_{(G,b,\mu,\mathcal{G}(\mathbb{Z}_p)}$ and reduced special fiber $X^{\mathcal{G}}_{\mu}(b)$. In particular, we get a continuous specialization map* $\mathrm{sp} : |\mathrm{Sht}_{(G,b,\mu,\mathcal{G}(\mathbb{Z}_p))}| \to |X^{\mathcal{G}}_{\mu}(b)|$.

Through a local model diagram comparing $\mathrm{Sht}^{\mathcal{G}}_{\mu}(b)$ to the local models $\mathcal{M}_{\mathcal{G},\mu}$ studied in [1] we obtained the following result.

Theorem 3 (G., Lourenço [7])**.** *The induced map $\pi_0(\mathrm{sp}) : \pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{G}(\mathbb{Z}_p))}) \to \pi_0(X^{\mathcal{G}}_{\mu}(b))$ is bijective.*

This theorem shows that to compute $\pi_0(X^{\mathcal{G}}_{\mu}(b))$ it suffices to compute instead $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{G}(\mathbb{Z}_p))})$, in this way we can exploit the geometry of π_{GM}.

Part of the flexibility of the theory of diamonds is that limits exist. In particular, one can consider the *infinite level moduli space of p-adic shtukas*

$$\mathrm{Sht}_{(G,b,\mu,\infty)} = \varprojlim_{K \subseteq G(\mathbb{Q}_p)} \mathrm{Sht}_{(G,b,\mu,K)}.$$

The advantage of working with this space is that it receives commuting actions by $\underline{G(\mathbb{Q}_p)}$ and $\underline{J_b(\mathbb{Q})}$. One easily shows that

$$\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{G}(\mathbb{Z}_p))}) \cong \pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)})/\mathcal{G}(\mathbb{Z}_p).$$

This observation reduces the computation of $\pi_0(\mathrm{Sht}_{(G,b,\mu,\mathcal{G}(\mathbb{Z}_p))})$ to the computation of $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)})$. In turn, this computation can be tackled using crystalline representations and classical p-adic Hodge theory.

Let $G^{\mathrm{der}}(\mathbb{Q}_p)$ denote the derived subgroup of G and let G^{sc} its simply connected cover. Let $G^{\circ} = G(\mathbb{Q}_p)/G^{\mathrm{sc}}(\mathbb{Q}_p)$. Let $\pi_1(G)$ Borovoi's fundamental group, it receives an action by I the inertia group of $\mathbb{Q}_p$. We have the following result.

Theorem 4 (G., Lim, Xu [5])**.** *Suppose that G^{sc} has no anisotropic factors, then the following are equivalent.*

(1) *The pair (b, μ) is HN-irreducible.*
(2) *The $G(\mathbb{Q}_p)$ action on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathbb{C}_p)$ is trivial when restricted to $G^{\mathrm{sc}}(\mathbb{Q}_p)$ and the induced action of G° on $\pi_0(\mathrm{Sht}_{(G,b,\mu,\infty)} \times \mathbb{C}_p)$ is transitive.*
(3) *The Kottwitz map $\kappa : X^{\mathcal{G}}_{\mu}(b) \to \pi_1(G)_I$ induces an injective morphism $\omega : \pi_0(X^{\mathcal{G}}_{\mu}(b)) \to \pi_1(G)_I$.*

(4) *There exist a finite field extension $[F : \breve{E}] < \infty$ and a crystalline representation $\rho : \Gamma_F \to G(\mathbb{Q}_p)$ with invariants (b, μ) such that $\rho(\Gamma_F) \cap G^{\mathrm{der}}(\mathbb{Q}_p)$ is open in $G^{\mathrm{der}}(\mathbb{Q}_p)$.*

Through a reduction process using the Hodge–Newton decomposition [8], the computation of $\pi_0(X_\mu^{\mathcal{G}}(b))$ can be reduced to the HN-irreducible case. Moreover, it is also known that the map $\kappa : X_\mu^{\mathcal{G}}(b) \to \pi_1(G)_I$ factors through a unique coset $c_{b,\mu} \cdot \pi_1(G)_I^{\varphi} \subseteq \pi_1(G)_I$, where $\pi_1(G)_I^{\varphi}$ is the subgroup of Frobenius invariant elements. The following corollary finishes the computation of $\pi_0(X_\mu^{\mathcal{G}}(b))$.

Corollary 1 (G., Lim, Xu [5]). If (b, μ) is HN-irreducible then $\omega : \pi_0(X_\mu^{\mathcal{G}}(b)) \to c_{b,\mu}\pi_1(G)_I^{\varphi}$ is bijective.

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Derived special cycles on Shimura varieties

Keerthi Madapusi

We consider a Shimura datum (G, X) and a compact open subgroup $K \subset G(\mathbb{A}_f)$ satisfying the following conditions:

- There is a $\mathbb{Q}$-algebra D equipped with a positive involution ι acting G-equivariantly on W such that W is *free* over D and such that there is an ι-Hermitian form H on W, preserved by G up to $\mathbb{Q}$-multiples.
- Every point $x \in X$ will determine a Hodge structure on W polarized by the alternating form underlying H, and we will require that this structure is pure of weights $(-1, 1), (0, 0), (1, -1)$.
- There is an order $D_{\mathbb{Z}} \subset D$ and a $D_{\mathbb{Z}}$-stable lattice $W_{\mathbb{Z}} \subset W$ whose base-change over $\widehat{\mathbb{Z}}$ is stabilized by K.

Examples of such data include those Hermitian symmetric domains appearing in the work of Kudla-Millson [4], but also—as observed during the talk by Rapoport—other interesting cases such as the adjoint representation of *any* Shimura datum (where $D = \mathbb{Q}$).

For any such datum, the Shimura variety $\mathrm{Sh}_K = \mathrm{Sh}_K(G, X)$ admits a family of *special cycles* $Z(\Lambda)$ indexed by positive semi-definite ι-Hermitian lattices Λ over $D_{\mathbb{Z}}$: over the complex numbers, they parameterize isometric maps from Λ to the weight $(0, 0)$ part of the Hodge structure induced on $W_{\mathbb{Z}}$. They give rise to cycle classes $[Z(\Lambda)]$ in $\mathrm{CH}^{d_+ m(\Lambda)}(\mathrm{Sh}_K)$, where d_+ is the rank of the $(-1, 1)$ part of the Hodge structure induced on $W_{\mathbb{Z}}$, and $m(\Lambda)$ is the rank of the Hermitian form on Λ. Following Kudla, one can *correct* this by multiplying by a suitable multiple of the top Chern class of a certain *tautological* bundle to get classes $C(\Lambda) \in \mathrm{CH}^{d_+ r(\Lambda)}(\mathrm{Sh}_K)$, where $r(\Lambda)$ is now the rank of Λ. In the case of orthogonal Shimura varieties (see [3]), it is known that these corrected classes satisy the following properties:

(1) They are *linearly invariant*: That is the class $C(\Lambda)$ depends only on the isometry class of the Hermitian lattice Λ.

(2) They satisfy a *product formula*:

$$C(\Lambda_1) \cdot C(\Lambda_2) = \sum_{\Lambda \in \mathrm{H}(\Lambda_1, \Lambda_2)} C(\Lambda),$$

where the sum is indexed by Hermitian lattices Λ with underlying $D_{\mathbb{Z}}$-module $\Lambda_1 \oplus \Lambda_2$, and where the Hermitian form restricts to the given ones on Λ_1 and Λ_2 separately.

(3) They interact well with pullbacks along natural maps of Shimura varieties.

We now move on to *integral models*: If the Shimura datum is of Hodge type, then Sh_K extends to a smooth integral canonical model $\mathcal{S}_K$ over $\mathbb{Z}[D_K^{-1}]$ where D_K is the product of primes p at which K_p is not hyperspecial.

In [1], Howard and I showed that in the case of orthogonal Shimura data associated with quadratic spaces over $\mathbb{Q}$, one can extend the classes $C(\Lambda)$ to classes $\mathcal{C}(\Lambda) \in \mathrm{CH}^{d_+ r(\Lambda)}(\mathcal{S}_K)_{\mathbb{Q}}$ that satisfy all the properties listed above.

The goal of this talk was to summarize some results from [5] where I used derived algebraic geometry to give a *uniform, geometric* construction of these corrected cycle classes over the integral model $\mathcal{S}_K$ in the generality laid out at the beginning. These results should be viewed as evidence for a general modularity conjecture for generating series of cycles obtained from these corrected classes, which includes the conjectures of Kudla, but also some other interesting cases, which arise essentially from pullback of Kudla-Millson type generating series from *non*-Hermitian symmetric spaces.

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Hecke orbits on Shimura varieties of Hodge type

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(joint work with Marco D'Addezio)

1. Introduction

1.1. The Hecke orbit conjecture. Let p be a prime number and g a positive integer. Problem 15 on Oort's 1995 list of open problems in algebraic geometry, [14], is the following conjecture.

Conjecture 1. *Let $x = (A_x, \lambda)$ be an $\overline{\mathbb{F}}_p$-point of the moduli space $\mathcal{A}_g$ of principally polarised abelian varieties of dimension g over $\overline{\mathbb{F}}_p$. The Hecke orbit of x, consisting of all points $y \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ corresponding to principally polarised abelian varieties related to (A_x, λ) by symplectic isogenies, is Zariski dense in the Newton stratum of $\mathcal{A}_g$ containing x.*

More generally, for the special fibre of a Shimura variety of Hodge type at a prime of good reduction one expects that the isogeny classes are Zariski dense in the Newton strata containing them. This article contains a proof of this expectation under the assumption that p is not too small with respect to the given Shimura datum; for $\mathcal{A}_g$ this comes down to the assumption that $p \geq g$.

1.1.1. There is a refined version of Conjecture 1, also due to Oort, which considers instead the prime-to-p Hecke-orbit of x, consisting of all $y \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ related to x by prime-to-p symplectic isogenies. In this case, the quasi-polarised p-divisible group $(A_x[p^\infty], \lambda)$ is constant on prime-to-p Hecke orbits (not just constant up to isogeny). Therefore, the prime-to-p Hecke orbit of x is contained in the central leaf

$$C(x) = \left\{ y \in \mathcal{A}_g(\overline{\mathbb{F}}_p) \mid A_y[p^\infty] \simeq_\lambda A_x[p^\infty] \right\}, \tag{1}$$

where $\simeq_\lambda$ denotes a symplectic isomorphism. Oort proved in [13] that $C(x)$ is a smooth closed subvariety of the Newton stratum of $\mathcal{A}_g$ containing x. He also conjectured that the prime-to-p Hecke orbit of x was Zariski dense in the central leaf $C(x)$. This conjecture is known as the *Hecke orbit conjecture* (for $\mathcal{A}_g$). Thanks to Mantovan–Oort product formula, [13, 11], the Hecke orbit conjecture implies Conjecture 1.

Central leaves and prime-to-p Hecke orbits can also be defined for the special fibres of Shimura varieties of Hodge type at primes of good reduction by work of Hamacher and Kim [5, 8]. The Hecke orbit conjecture for Shimura varieties of Hodge type then predicts that the prime-to-p Hecke orbits of points are Zariski dense in the central leaves containing them (see question 8.2.1 of [10] and Conjecture 3.2 of [2]).

The Hecke orbit conjecture naturally splits up into a discrete part and a continuous part. The discrete part states that the prime-to-p Hecke orbit of x intersects each connected component of $C(x)$, whereas the continuous part states that the Zariski closure of the prime-to-p Hecke orbit of x is of the same dimension as $C(x)$. The discrete part of the conjecture is Theorem C of [10] (see [7] for related results). In this paper, we will focus instead on the continuous part of the conjecture.

1.2. **Main result.** Let (G, X) be a Shimura datum of Hodge type with reflex field E, and assume for simplicity that G^{ad} is $\mathbb{Q}$-simple throughout this introduction. Let p be a prime such that $G_{\mathbb{Q}_p}$ is quasi-split and split over an unramified extension, let $U_p \subseteq G(\mathbb{Q}_p)$ be a hyperspecial subgroup, and let $U^p \subseteq G(\mathbb{A}_f^p)$ be a sufficiently small compact open subgroup. We choose a place v of E dividing p and we write $\mathrm{Sh}_{G,U}$ for the geometric special fibre of the canonical integral model over $\mathcal{O}_{E,v}$ of the Shimura variety $\mathbf{Sh}_{G,U}$ over E of level $U := U^p U_p$; the canonical integral model exists by [9]. Let $C \subseteq \mathrm{Sh}_{G,U}$ be a central leaf as constructed in [5] (cf. [8]) and let $h(G)$ be the Coxeter number of G.

Theorem 1. *Let $Z \subseteq C$ be a nonempty reduced closed subvariety that is stable under the prime-to-p Hecke operators. If $p \geq h(G)$, then $Z = C$.*

When $\mathrm{Sh}_{G,U}$ is a Siegel modular variety, this result (without any conditions on p) is due to Chai–Oort, see their forthcoming book [4] for the continuous part and [3] for the discrete part. Their proofs do not generalise to more general Shimura varieties because they rely on the existence of hypersymmetric points in Newton strata, which is usually false for Shimura varieties of Hodge type. Moreover, their proof of the continuous part of the conjecture relies on the fact that any point

$x \in \mathcal{A}_g(\overline{\mathbb{F}}_p)$ is contained in a large Hilbert modular variety, and they use work of Chai–Oort–Yu on the Hecke orbit conjecture for Hilbert modular varieties at (possibly ramified) primes. There are many other partial results e.g. for prime-to-p Hecke orbits of hypersymmetric points in the PEL case, [16], or for prime-to-p Hecke orbits of (μ)-ordinary points, [1, 17, 15, 12, 6].

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Independence of ℓ for G-valued Weil–Deligne representations associated to abelian varieties

Rong Zhou
(joint work with M. Kisin)

In a recent work with Kisin [3], we proved the independence of ℓ for Frobenius conjugacy classes inside the Mumford–Tate group of an abelian variety defined over a number field. In a work in progress [4], we extend our ℓ-independence results to some places of bad reduction. This involves an independence of ℓ statement for the corresponding Weil–Deligne representation, and should be thought of as the motivic version of Fontaine's C_{WD}-conjecture [2, §2.4.3].

To fix ideas, let A be an abelian variety of dimension g over a number field E which we assume is equipped with a fixed embedding $E \subset \mathbb{C}$. We let G denote the Mumford–Tate group of $A_{\mathbb{C}}$, which is a reductive group over $\mathbb{Q}$. Deligne's theorem on absolute Hodge cycles [1] implies that upon replacing E by a finite extension, we may assume that for any prime ℓ, the action of the absolute Galois group $\Gamma_E := \mathrm{Gal}(\overline{E}/E)$ on the ℓ-adic Tate module of A factors through $G(\mathbb{Q}_\ell)$ under the Betti-étale comparison isomorphism. We write $\rho_\ell^G : \Gamma_E \to G(\mathbb{Q}_\ell)$ for the induced representation.

Let v be a place of E dividing a rational prime p and let E_v denote the completion. Then for $\ell \neq p$, restricting ρ_ℓ^G to the decomposition group at p gives a representation of the local Galois group $\Gamma_{E_v} = \mathrm{Gal}(\overline{E}_v/E_v)$. Upon fixing an isomorphism $\overline{\mathbb{Q}}_\ell \cong \mathbb{C}$, we obtain a $G(\mathbb{C})$-valued Weil–Deligne representation $\rho_{\ell,v}^{\mathrm{WD},G}$ via the ℓ-adic monodromy theorem.

Theorem 1 (Kisin, Z.)**.** *Let $p > 2$ and $v|p$ a place where A has semistable reduction. Then the $G(\mathbb{C})$-conjugacy class of $\rho_{\ell,v}^{\mathrm{WD},G}$ is defined over $\mathbb{Q}$ and independent of ℓ.*

Explicitly, this says that for any $\theta \in \mathrm{Aut}(\mathbb{C}/\mathbb{Q})$, $\theta \circ \rho_{\ell,v}^{\mathrm{WD},G}$ is $G(\mathbb{C})$-conjugate to $\rho_{\ell,v}^{\mathrm{WD},G}$ (defined over $\mathbb{Q}$), and that $\rho_{\ell,v}^{\mathrm{WD},G}$ and $\rho_{\ell',v}^{\mathrm{WD},G}$ are $G(\mathbb{C})$-conjugate for any primes $\ell, \ell' \neq p$ (independent of ℓ).

At places of good reduction, the Weil–Deligne representation is unramified and this theorem reduces to our previous result [3]. The ℓ-independence inside GL_{2g} was proved by Raynaud; in [8] he gives an explicit description of the character of the Weil group acting on the graded pieces of the monodromy filtration in terms of the reduction type of the abelian variety. Noot [7] has also obtained some results towards this theorem; his result includes some extra assumptions on the Frobenius as well as a more coarse notion of conjugacy.

The idea of the proof is to use the global Langlands correspondence for GL_n to compare the representations ρ_ℓ^G as ℓ-varies. The point is that such a correspondence should imply the existence of *strongly compatible* systems of Γ_E-representations (i.e. even compatible at ramified places) which would follow from knowing local-global compatibility of the correspondence. Assuming the existence of such systems, a simple argument using the Chebotarev density/Brauer–Nesbitt theorems and our previous result at unramified places then shows that for any algebraic representation $i : G_{\overline{\mathbb{Q}}} \to \mathrm{GL}_n$, the conjugacy classes of the $\mathrm{GL}_n(\mathbb{C})$-valued Weil–Deligne representations $i \circ \rho_{\ell,v}^{\mathrm{WD},G}$ are independent of ℓ. A representation theoretic argument then proves ℓ-independence of the conjugacy classes inside $G(\mathbb{C})$. The proof of this last step uses the semistable assumption, which implies that our Weil–Deligne representations are unipotently ramified. Another application of Chebotarev density implies that these conjugacy classes are in fact defined over $\mathbb{Q}$.

Of course, such an argument doesn't work due to our lack of knowledge of the Langlands correspondence over number fields. The main idea then, is to realize the Weil–Deligne representation $\rho_{\ell,v}^{\mathrm{WD},G}$ as the monodromy of some $G(\mathbb{Q}_\ell)$-local systems over a smooth curve $X/\mathbb{F}_q$ at a point on its boundary. Again the semistable assumption is essential to allow us to compare Weil–Deligne representations over local fields of different characteristic. The key input to realizing $\rho_{\ell,v}^{\mathrm{WD},G}$ as the monodromy over some curve in characteristic p is to prove some geometric results about the boundary of the toroidal compactifications of integral models of Shimura varieties $\mathcal{S}_{\mathrm{K}}(G,X)^{\Sigma}$ constructed by Madapusi Pera [6].

It then suffices to prove independence of ℓ for the local monodromy of these $G(\mathbb{Q}_\ell)$-local systems. By construction, the $G(\mathbb{Q}_\ell)$ local systems on these curves are pulled back from the Shimura variety, and by the main result concerning Shimura varieties proved in [3], these local systems are G-compatible. But now since we are working over function fields, we know everything about the Langlands correspondence for GL_n due to work of L. Lafforgue [5], including the existence of strongly compatible systems. Thus the argument of the previous paragraph can be applied unconditionally to prove the ℓ-independence statement that we want.

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p-adic heights of arithmetic diagonal cycles and derivatives of *p*-adic *L*-functions

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(joint work with Daniel Disegni)

Gan, Gross, and Prasad have formulated a vast generalization of the Gross–Zagier formula, relating the Beilinson–Bloch height pairings of the so-called arithmetic diagonal cycle to the central value of the first derivative of certain Rankin–Selberg convolution L-function. We formulate a p-adic analogue of the arithmetic Gan–Gross–Prasad (GGP) conjecture for unitary Shimura varieties, generalizing various previous work on the p-adic Gross–Zagier formula, notably the work of Perrin–Riou. We have proved our conjecture under certain local restrictions.

Let F_0 be a totally real field and F a CM quadratic extension of F_0. Let $n \geq 1$ be an integer and p a fixed prime number. Let W_{n+1} be a non-degenerate Hermitian space with F-dimension $n+1$. Let $W_n \subset W_{n+1}$ be a non-degenerate subspace of codimension one. Let $\mathrm{U}(W_i)$ be the unitary groups for $i = n, n+1$, and $\mathrm{U}(W_n) \hookrightarrow \mathrm{U}(W_{n+1})$ the induced embedding. Consider

$$\mathrm{G} = \mathrm{U}(W_n) \times \mathrm{U}(W_{n+1}), \qquad \mathrm{H} = \mathrm{U}(W_n),$$

with the "diagonal" embedding $\mathrm{H} \hookrightarrow \mathrm{G}$. Further assume that the signatures of W_n at the archimedean places are $(n,1), (n+1,0), \cdots, (n+1,0)$ and that the orthogonal complement of W_n in W_{n+1} is totally negatively definite. Then there are Shimura varieties $Sh_{K_{\mathrm{H}}}(\mathrm{H})$ and $Sh_K(\mathrm{G})$ (for compact open subgroups $K_{\mathrm{H}} \subset \mathrm{H}(\mathbb{A}_f), K \subset \mathrm{G}(\mathbb{A}_f)$), where the dimensions are $n-1, 2n-1$ respectively, both with reflex field F.

The étale cohomology of the Shimura variety $Sh_K(\mathrm{G})$ in its middle degree is expected by Kottwitz's conjecture to decompose as

$$H^{2n-1}(Sh_K(\mathrm{G})_{\overline{F}}, \overline{\mathbb{Q}}_p(n)) = \bigoplus_{\pi} \pi_f^K \boxtimes \rho_\pi^\vee, \tag{1}$$

where the sum runs over all cohomological (with respect to the trivial coefficient system) automorphic representations π of G appearing in the discrete spectum (we are implicitly identifying $\overline{\mathbb{Q}}_p \simeq \mathbb{C}$), and when π is stable and tempered, ρ_π is the representation of the absolute Galois group $\mathrm{Gal}(\overline{F}/F)$ associated to π.

Let π be tempered cuspidal as in (1). We have the *arithmetic diagonal cycle*

$$Sh_{K_{\mathrm{H}}}(\mathrm{H}) \to Sh_{K_{\mathrm{G}}}(\mathrm{G})$$

(for subgroups $K_{\mathrm{H}} \subset K_{\mathrm{G}}$). Its (absolute) cycle class lies in $H^{2n}(Sh_K(\mathrm{G}), \overline{\mathbb{Q}}_p(n))$, which admits a Hecke action. Take the projection to the π-part:

$$Z_{K,\pi} \in \pi_f^K \boxtimes H^1(F, \rho_\pi^\vee)$$

which will be called the GGP cycle (for π). Let $\phi^\vee \in \pi_f^\vee$ be a K-invariant vector. Then we define

$$\int_{Sh(\mathrm{H})} \phi^\vee := \phi^\vee(Z_{K,\pi}) \in H^1(F, \rho_\pi^\vee).$$

In other words, $\int_{Sh_{\mathrm{H}}}$ defines a "(vector valued) linear functional",

$$\int_{Sh(\mathrm{H})} : \pi_f \to H^1(F, \rho_\pi).$$

It is $\mathrm{H}(\mathbb{A}_f)$-invariant. In fact the image lies in the Bloch–Kato Selmer group

$$\int_{Sh(\mathrm{H})} : \pi_f \to H^1_f(F, \rho_\pi).$$

This may be viewed as an arithmetic analog of the automorphic period integral.

Nekovář has defined a p-adic height pairing[1]

$$(-,-)_p\colon H^1_f(F,\rho_\pi)\times H^1_f(F,\rho_\pi^\vee(1))\to \overline{\mathbb{Q}}_p$$

depending on some auxiliary choice (a certain splitting of Hodge filtration) that can be made canonically in the ordinary case.

Conjecture 1. *Assume that π is tempered, and ordinary at all p-adic places. Let $\phi\in\pi_f,\phi^\vee\in\pi_f^\vee$. Then, up to a certain non-zero explicit factor,*

$$\left(\int_{Sh(\mathrm{H})}\phi,\int_{Sh(\mathrm{H})}\phi^\vee\right)_p = e_p(\pi)^{-1}L_p'(\frac{1}{2},\pi,R)\prod_v \alpha_v(\phi_v,\phi_v^\vee),$$

where $L_p(s,\pi,R)$ is a p-adic Rankin–Selberg L-function (interpolating the central values of L-functions twisted by finite order characters), and $e_p(\pi)$ is a certain partial Euler factor at p.

Here the local factor $\alpha_v\in\mathrm{Hom}_{\mathrm{H}(F_{0,v})\times\mathrm{H}(F_{0,v})}(\pi_v\boxtimes\pi_v^\vee,\mathbb{C})$ is the Ichino–Ikeda canonical invariant linear functional:

$$\alpha_v(\phi_v,\phi_v^\vee)=\int_{\mathrm{H}(F_{0,v})}\langle\pi_v(h)\phi_v,\phi_v^\vee\rangle_v\,dh.$$

Note that α_v is non-zero if and only if $\mathrm{Hom}_{\mathrm{H}(F_{0,v})}(\pi_v,\mathbb{C})\neq 0$.

In [1] we prove some cases of the conjecture.

Theorem 1. *The conjecture holds when*

(1) $p>2n$.
(2) *F/F_0 is unramified everywhere and all 2-adic places are split.*
(3) *All $v|p$ are split in F and π_v are unramified and ordinary.*
(4) *π is stable cuspidal such that*
    - *If v is split, then one of the two factors in $\pi_v=\pi_{n,v}\boxtimes\pi_{n+1,v}$ is unramified.*
    - *If v is inert, then each factor in $\pi_v=\pi_{n,v}\boxtimes\pi_{n+1,v}$ is either unramified or almost unramified (namely, admitting an invariant under a parahoric defined by a vertex lattice of type 1).*

We remark that in the case $n=1$, much more is essentially known due to the work of many authors: Perrin–Riou, Nekovář, Kobayashi, Shnidman, and Disegni.

The proof is based on the comparison of a pair of relative-trace formulas with p-adic coefficients, analogously to the approach over archimedean coefficients.

On the p-adic height side, we study the arithmetic "relative trace"

$$I(f)=(f*[Sh_{\mathrm{H}}],\ [Sh_{\mathrm{H}}])_p,$$

[1]Assuming the weight–monodromy conjecture for ρ_π, which is known in our case.

which computes the p-adic height pairing, for f in the Hecke algebra of $\mathrm{G}(\mathbb{A}_f)$. With a mild condition on the support of f, this decomposes into a sum of local heights

$$I(f) = \sum_{v<\infty} I_v(f).$$

By previous work of Li–Liu and Disegni–Liu, the local height at a place w away from p can be related to an arithmetic intersection number on a suitable regular integral model for the RSZ Shimura variety. At $v|p$, we rely on a crucial result of Disegni–Liu to reduce the crystalline property of the mixed extension to disjointness of the cycles on a smooth integral model. Then one follows the well-known strategy similar to Perrin–Riou's proof in the modular curve case to show the limit

$$\lim_{N\to\infty} I_p(f^p \xi U_p^{N!-1}) = 0,$$

where U_p is the analog of the Hida U_p-operator for G, and $\xi = \xi_1$ is the first of a "miraculous" sequence of explicit elements ξ_m appearing in Januszewski's p-adic interpolation of the Rankin–Selberg L-function.

On the analytic side, we p-adically interpolate the Jacquet–Rallis relative-trace formula for $G' = \mathrm{Res}_{F/F_0}(\mathrm{GL}_n \times \mathrm{GL}_{n+1})$. We let f'_∞ be a "(relative) pseudo-coefficient" (called the "Gaussian" test function), which has the property

$$\mathrm{Orb}(\gamma, f'_\infty) = \begin{cases} 1, & \gamma \text{ is "compact"} \\ 0, & \text{otherwise.} \end{cases}$$

The existence is proved in the work of Beuzart-Plessis–Liu–Zhang–Zhu. Let $f'_{p,N,m}$ be a family of test functions, essentially of the form

$$\xi_m U_p^{N!-m}$$

where U_p is the Hida operator for $G'(F_{0,p})$. Then we show that the twisted orbital integral $\mathrm{Orb}(\gamma, f'^p f'_{p,N,m}, \chi)$ is always integral and independent of the twisting character χ of conductor $\leq m \leq N$. The limit as $N \to \infty$ of the sum of orbital integrals gives a "p-adic RTF", expressing

$$J_p(s, f') := \sum_{\substack{\pi \\ ordinary}} L_p(s,\pi) J_{\pi^p}(f'^p)$$

in terms of measures defined by orbital integrals. Using the now proven Arithmetic Fundamental Lemma at hyperspecial level (Zhang, Mihatsch–Zhang), and the Arithmetic Transfer for the almost self-dual case (Thesis of Zhiyu Zhang), the derivatives of orbital integrals are related to arithmetic intersections.

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Arithmetic level raising via unitary Shimura varieties with good reduction

Yifeng Liu
(joint work with Yichao Tian, Liang Xiao)

We start by recalling a classical theorem of Ribet on level raising of modular forms via the arithmetic of modular curves. Consider a positive integer Σ, a cusp newform $f = q + a_2q^2 + a_3q^3 + \cdots \in \mathbb{Z}[[q]]$ of weight 2 and level $\Gamma_0(\Sigma)$, and a prime number ℓ. Take a prime number $p \nmid \Sigma\ell$ and denote by $\mathbb{T}^{\Sigma p}$ the unramified Hecke algebra away-from-Σp, so that f gives a maximal ideal $\mathfrak{m}_f$ of $\mathbb{T}^{\Sigma p}$ with residue field $\mathbb{F}_\ell$. Denote by $Y_0(\Sigma)$ the modular curve of level $\Gamma_0(\Sigma)$ over $\mathbb{F}_p$, so that its supersingular locus is a disjoint union of $\operatorname{Spec}\mathbb{F}_{p^2}$ indexed by a finite set $S_0(\Sigma)$. We then have the absolute cycle class map

$$\alpha\colon \mathbb{Z}_\ell[S_0(\Sigma)] \to \mathrm{H}^2(Y_0(\Sigma) \otimes \mathbb{F}_{p^2}, \mathbb{Z}_\ell(1)).$$

Ribet shows that if $\mathfrak{m}_f$ is a non-Eisenstein ideal, then $\alpha_{\mathfrak{m}_f}$ is surjective. In particular, when p is a *level-raising prime* for $f \mod \ell$, that is, $a_p^2 \equiv (p+1)^2 \mod \ell$, the target of $\alpha_{\mathfrak{m}_f}$ is nontrivial. By Ribet's theorem, $\mathbb{Z}_\ell[S_0(\Sigma)]_{\mathfrak{m}_f}$ is nontrivial. By the well-known description of $S_0(\Sigma)$ as a double coset from the definite quaternion algebra ramified at p and the Jacquet–Langlands correspondence, we obtain a new modular form of weight 2 that is congruent to $f \mod \ell$ and semistable at p.

In our ongoing work [3], we study the analogue of Ribet's theorem for certain unitary Shimura varieties. Consider a CM extension F/F^+ contained in $\mathbb{C}$, a positive integer N, a hermitian space V over F/F^+ that has signature $(N-1,1)$ at the default real place and signature $(N,0)$ at all other real places, and a p-adic place $\mathfrak{p}$ of F^+ *inert* in F such that $G := \operatorname{Res}_{F^+/\mathbb{Q}} \mathrm{U}(V)$ is unramified at p. Fix an isomorphism $\mathbb{C} \simeq \overline{\mathbb{Q}_p}$ that induces the place $\mathfrak{p}$ and a hyperspecial maximal subgroup K_p of $G(\mathbb{Q}_p)$. Let q be the residue cardinality of $F^+_\mathfrak{p}$.

We may construct, for every neat open compact subgroup $K^p \subseteq G(\mathbb{A}^{\infty,p})$, a moduli scheme $X(K^p)$ over $\mathbb{Z}_{q^2}$ whose complex fiber is (essentially) the Shimura variety associated with G of level K^pK_p. Put $Y(K^p) := X(K^p) \otimes_{\mathbb{Z}_{q^2}} \mathbb{F}_{q^2}$ and denote by $Y(K^p)^b$ its basic locus. In [2], we construct a natural correspondence

$$S(K^p) \xleftarrow{\pi} B(K^p) \xrightarrow{\iota} Y(K^p)$$

over $\mathbb{F}_{q^2}$, satisfying

(1) The correspondence is equivariant with the obvious Hecke correspondences away from p.
(2) The morphism π is projective smooth with fibers being certain irreducible Deligne–Lusztig varieties of dimension $r := \lfloor \frac{N-1}{2} \rfloor$.
(3) The morphism ι is locally a closed embedding, whose image is exactly $Y(K^p)^b$.
(4) $S(K^p)$ is a finite copy of $\operatorname{Spec}\mathbb{F}_{q^2}$ naturally indexed by the following double coset: Let V' be the totally positive definite hermitian space over F/F^+

such that $V' \otimes_{F^+} \mathbb{A}_{F^+}^{\infty;\mathfrak{p}} \simeq V \otimes_{F^+} \mathbb{A}_{F^+}^{\infty;\mathfrak{p}}$ (and fix such an isometry). Then the index set is

$$G'(\mathbb{Q})\backslash G'(\mathbb{A}^\infty)/K^p K'_p,$$

where $G' := \mathrm{Res}_{F^+/\mathbb{Q}} \mathrm{U}(V')$ and K'_p is a fixed maximal special subgroup of $G'(\mathbb{Q}_p)$.

In particular, the absolute cycle classes give a map

$$\iota_! \circ \pi^* \colon \mathrm{H}^0(S(K^p), \mathbb{Z}_\ell) \to \mathrm{H}^{2(N-1-r)}(Y(K^p), \mathbb{Z}_\ell(N-1-r))$$

for every prime $\ell \neq p$. Denote by $\mathrm{H}^0(S(K^p), \mathbb{Z}_\ell)^\diamondsuit$ the kernel of the composition of α with the restriction map $\mathrm{H}^\bullet(Y(K^p), -) \to \mathrm{H}^\bullet(Y(K^p)_{\overline{\mathbb{F}_p}}, -)$.

From now on, we assume $N = 2r$ even. In particular, we have the induced Abel–Jacobi map

$$\alpha_N \colon \mathrm{H}^0(S(K^p), \mathbb{Z}_\ell)^\diamondsuit \to \mathrm{H}^1(\mathbb{F}_{q^2}, \mathrm{H}^{2r-1}(Y(K^p)_{\overline{\mathbb{F}_p}}, \mathbb{Z}_\ell(r))).$$

We would like to study the surjectivity of the above map after localization at Hecke ideas, which is the analogue of Ribet's theorem in this context.

Recall that Ribet deduced his theorem from Ihara's lemma for modular curves. In our case, there is also a certain version of Ihara's lemma that is responsible for the surjectivity of α_N. From now on, we further assume that $\ell \nmid q \prod_{i=1}^N (1-(-q)^i)$.

Let $\mathcal{K}$ be the $\mathfrak{p}$-component of K_p, which is a hyperspecial maximal subgroup of $\mathrm{U}(V)(F_\mathfrak{p}^+)$. Fix a hermitian Siegel parahoric subgroup $\mathcal{P} \subseteq \mathcal{K}$. Let $\mathcal{Q}$ be the double coset in $\mathcal{P}\backslash\mathcal{K}/\mathcal{P}$ that parameterizes a pair of Lagrangian subspaces with intersection of codimension 1. In [2], it is shown that there a canonical decomposition

$$\mathbb{Z}_\ell[\mathcal{P}\backslash\mathcal{K}] = \bigoplus_{j=0}^{r} \Omega_{N,\ell}^j$$

of $(\mathbb{Z}_\ell[\mathcal{P}\backslash\mathcal{K}/\mathcal{P}], \mathbb{Z}_\ell[\mathcal{K}])$-bimodules in which $\Omega_{N,\ell}^j$ is the eigenspace of $\mathcal{Q}$ with eigenvalue $\frac{-(-q)^{N+1-j}-(-q)^j-q+1}{q^2-1}$. Note that the differences of these eigenvalues are all invertible in $\mathbb{Z}_\ell$.

Let $f\colon \widetilde{X}(K^p) \to X(K^p)_{\mathbb{Q}_{q^2}}$ be the natural finite cover over $\mathbb{Q}_{q^2}$ corresponding to the inclusion $\mathcal{P} \subseteq \mathcal{K}$. Then $\widetilde{X}(K^p)$ admits a natural involution i. We have the composite map

$$\begin{aligned}\beta_N \colon \mathrm{H}^{N-1}(X(K^p)_{\overline{\mathbb{Q}_p}}, \Omega_{N,\ell}^1) &\hookrightarrow \mathrm{H}^{N-1}(X(K^p)_{\overline{\mathbb{Q}_p}}, \mathrm{Ind}_\mathcal{P}^\mathcal{K} \mathbb{Z}_\ell) \\ &= \mathrm{H}^{N-1}(\widetilde{X}(K^p)_{\overline{\mathbb{Q}_p}}, \mathbb{Z}_\ell) \xrightarrow{f_* \circ i_*} \mathrm{H}^{N-1}(X(K^p)_{\overline{\mathbb{Q}_p}}, \mathbb{Z}_\ell).\end{aligned}$$

Denote by $\mathbb{T}_N^\square$ the abstract spherical unitary Hecke algebra over F/F^+ of rank N away from $\square$. Fix a finite set Σ of prime numbers not containing p, away from which K^p is hyperspecial. Then $\mathbb{T}_N^{\Sigma\cup\{p\}}$ acts on $X(K^p)$ via Hecke correspondences which are finite étale. The following conjecture is a generalization of Ihara's lemma in the current case.

Conjecture 1. *Let $\mathfrak{m}$ be a maximal ideal of $\mathbb{T}_N^{\Sigma}$ of residue characteristic ℓ that is "non-Eisenstein" (which can be made precise) such that the Satake parameters modulo $\mathfrak{m}$ at $\mathfrak{p}$ contain q at most once. Then the map β_N is surjective after localizing at $\mathfrak{m} \cap \mathbb{T}_N^{\Sigma\cup\{p\}}$.*

We have the following theorem proved in [3], which generalizes Ribet's argument toward level raising.

Theorem 1. *Suppose that p is odd and $q = p$. Then for every maximal ideal $\mathfrak{m}$ of $\mathbb{T}_N^{\Sigma\cup\{p\}}$ of residue characteristic ℓ, the surjectivity of $(\beta_N)_{\mathfrak{m}}$ implies the surjectivity of $(\alpha_N)_{\mathfrak{m}}$.*

Combining results from [2] and [3], we have the following theorem, providing examples for which the above conjecture is known.

Theorem 2. *Consider a prime $\mathfrak{p}^\dagger$ of F^+ inert in F and a maximal ideal $\mathfrak{m}^\dagger$ of $\mathbb{T}_N^{\Sigma\setminus\{p^\dagger\}}$ of residue characteristic ℓ satisfying*

- *$F^+_{\mathfrak{p}^\dagger} = \mathbb{Q}_{p^\dagger}$ for an odd prime number $p^\dagger$ unramified in F;*
- *V is not split at $\mathfrak{p}^\dagger$ ($\Rightarrow p^\dagger \in \Sigma$) but splits at other $p^\dagger$-adic places of F^+;*
- *$\mathfrak{m}^\dagger$ is "non-Eisenstein";*
- *the Satake parameters modulo $\mathfrak{m}^\dagger$ at $\mathfrak{p}$ contain p at most once and do not contain -1;*
- *the Satake parameters modulo $\mathfrak{m}^\dagger$ at $\mathfrak{p}^\dagger$ contain $p^\dagger$ exactly once and do not contain -1;*
- *some other technical conditions ...*

Put $\mathfrak{m} := \mathfrak{m}^\dagger \cap \mathbb{T}_N^{\Sigma\cup\{p\}}$. Then $(\beta_N)_{\mathfrak{m}}$ is surjective; hence $(\alpha_N)_{\mathfrak{m}}$ is surjective as well.

The above conjecture or rather the surjectivity of α_N after localization has important application in the study of Selmer groups of motives appearing in the cohomology of $X(K^p)$. For example, it is a key ingredient to establish the (strong) second explicit reciprocity law for the diagonal cycle in the Rankin–Selberg product of unitary Shimura varieties, that is, to turn the inequality in Theorem 7.3.4 of [2] into an equality. Using such explicit reciprocity law, one can further prove part of Iwasawa's main conjecture for certain motives of Rankin–Selberg type, following the argument in [1] for the GL_2-case – this is the main number-theoretical goal of [3].

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Cohomology of Shimura varieties

SUG WOO SHIN
(joint work with Mark Kisin, Yihang Zhu)

A long-term goal in the study of Shimura varieties is to prove that the Hasse–Weil ζ-functions of Shimura varieties are equal to alternating products of automorphic L-functions. In representation-theoretic formulation, the key problem is to understand the ℓ-adic cohomology of Shimura varieties equipped with two commuting actions by a finite adelic group ("Hecke action") and a Galois group.

The Langlands–Kottwitz–Rapoport (LKR) method consists in four steps:

(i) construct canonical integral models for Shimura varieties,
(ii) understand the $\overline{\mathbb{F}}_p$-points of Shimura varieties,
(iii) prove a stabilized point-counting formula for the Hecke and Galois actions,
(iv) apply the trace formula techniques to arrive at a desired description of the cohomology.

This talk was mainly concerned with (ii), which is precisely formulated as the Langlands–Rapoport (LR) conjecture [1]. In fact (i) is mostly known for Shimura varieties of abelian type thanks to Kisin, Pappas, Rapoport, and others. While experts have known how to go from (ii) to (iii) to (iv) conjecturally, the LR conjecture has been a major obstacle as it remains wide open to this day. Kisin (2017) proved a weaker form of the LR conjecture but it was not enough to imply (iii).

The purpose of the talk was to report on the recent theorem by the speaker with Kisin and Y. Zhu [2], where they proved the so-called Langlands–Rapoport-τ (LR-τ) conjecture and deduced (iii) from it. In ongoing work, the authors are going to accomplish (iv) in interesting new cases. Therefore the proof of the LR-τ conjecture should suffice for most applications although the LR-τ conjecture is not as strong as the original LR conjecture.

The remainder of the talk was devoted to stating the LR and LR-τ conjectures in detail and explaining the strategy and ideas of proof by constructing a "special points correspondence" and establishing an extra compatibility in terms of "amicable pairs".

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Affine Deligne-Lusztig varieties with and without Frobenius

Xuhua He

Affine Deligne-Lusztig is the affine analog of the (classical) Deligne-Lusztig variety. It was introduced by Rapoport in [6]. It serves as the group-theoretic model of the reduction of Shimura varieties and plays an important role in the arithmetic geometry and Langlands program. We recall the definition here.

Let F be a non-archimedean local field and $\breve{F}$ be the completion of the maximal unramified extension of F. Let $\mathbb{G}$ be a connected reductive group over F, $\breve{G} = \mathbb{G}(\breve{F})$ and σ be the Frobenius morphism on $\breve{G}$. Let $\mathcal{I}$ be a σ-stable Iwahori subgroup of $\breve{G}$. Let $b \in \breve{G}$ and w be an element in the Iwahori–Weyl group $\tilde{W}$ of $\breve{G}$. The affine Deligne–Lusztig variety $X_w(b)$ (in the affine flag variety $\breve{G}/\mathcal{I}$) is defined by

$$X_w(b) = \{g\mathcal{I} \in \breve{G}/\mathcal{I}; g^{-1}b\sigma(g) \in \mathcal{I}\dot{w}\mathcal{I}\}.$$

When $\mathbb{G}$ is quasi-split over F, one may also define the affine Deligne-Lusztig variety $X_\mu(b)$ inside the affine Grassmannian of $\breve{G}$.

Lusztig variety was first introduced in [4]. It plays a central role in Lusztig's theory of character sheaves. Affine Lusztig variety is the affine analog of the (classical) Lusztig variety. It was first studied in [5] and [3]. The definition is similar to affine Deligne-Lusztig varieties but without Frobenius. In particular, the affine Lusztig variety can also be defined for the groups over $\mathbb{C}((t))$. Let γ be an element in $\breve{G}$ and w be an element in $\tilde{W}$. The affine Lusztig variety $Y_w(\gamma)$ (in the affine flag variety $\breve{G}/\mathcal{I}$) is defined by

$$Y_w(\gamma) = \{g\mathcal{I} \in \breve{G}/\mathcal{I}; g^{-1}\gamma g \in \mathcal{I}\dot{w}\mathcal{I}\}.$$

One may also consider the affine Lusztig variety $Y_\mu(\gamma)$ inside the affine Grassmannian of $\breve{G}$. The affine Lusztig variety encodes the information on the orbital integral and also serves as a key ingredient in the conjectural theory of affine character sheaves.

Up to now, we have good knowledge of affine Deligne-Lusztig varieties. The nonemptiness pattern and the dimension formula are completely known for affine Deligne-Lusztig varieties in the affine Grassmannian and are known in most cases for affine Deligne-Lusztig varieties in the affine flag variety. However, much less is known for the affine Lusztig varieties. For affine Lusztig varieties in the affine Grassmannian, the nonemptiness pattern is known, and the dimension formula is known for split groups in the equal characteristic case (under a mild assumption on the residue characteristic). See the work of Chi [1]. Little is known for affine Lusztig varieties in the affine flag variety.

In this talk, I announce the following result in [2]. Suppose that the σ-conjugacy class of b is associated with the (ordinary) conjugacy class of γ. Let Y_γ be the affine Springer fiber associated with the element γ. Then we have

$$(*) \qquad \dim Y_w(\gamma) = \dim X_w(b) + \dim Y_\gamma.$$

The matching data between γ and b and the definition of the affine Springer fiber for non-compact elements are a bit involved and I will not define them in this report. The precise definitions can be found in [2, §1.6 & §5.3].

By convention, the dimension of the empty set is $-\infty$ and $-\infty + n = -\infty$ for any n. In particular, $Y_w(\gamma) \neq \emptyset$ if and only if $X_w(b) \neq \emptyset$. As a consequence of the equality $(*)$ and the known results on the affine Deligne-Lusztig varieties, we establish a lot of new results on the affine Lusztig varieties: the dimension formula of affine Lusztig varieties in the affine Grassmannian of ramified groups in both equal and mixed characteristics (under a mild assumption on the residue characteristic); and the nonemptiness pattern and dimension formula in most cases for affine Lusztig varieties in the affine flag variety.

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The integral motivic Satake equivalence

Thibaud van den Hove

(joint work with Robert Cass, Jakob Scholbach)

1. Introduction

Let G be a reductive group over an algebraically closed field k, and $\mathrm{Gr}_G := LG/L^+G$ its (power series) affine Grassmannian. The geometric Satake equivalence, first proved in [5], is a symmetric monoidal equivalence between L^+G-equivariant perverse sheaves on Gr_G, and representations of the Langlands dual group $\hat{G}$. This equivalence is a cornerstone of modern algebra, having a wide range of applications such as in the geometric Langlands program and modular representation theory.

The geometric Satake equivalence can be stated in a variety of ways, depending on the chosen cohomology theory. For example, depending on k, one can use Betti cohomology, étale cohomology, algebraic or arithmetic D-modules, and so on. On the other hand, the representation category of $\hat{G}$ does not depend much on the chosen cohomology theory, only on the coefficients. Thus, it is a natural question to ask whether one can formulate a version of the geometric Satake equivalence which is independent of the base field and cohomology theory, i.e., using motives. For rational coefficients, this was accomplished by Richarz-Scholbach [6]. The case of integral (and hence also modular) coefficients is the subject of the present talk, based on [2].

Namely, let S be a base ring satisfying the Beilinson-Soulé vanishing conjecture. Examples include $S = \mathbb{Z}, \mathbb{F}_q, K, \mathcal{O}_K$, where $\mathcal{O}_K \subset K$ is the ring of integers in a global field. Let G/S be a split reductive group, and $\mathrm{Gr}_G := LG/L^+G$ its power series affine Grassmannian. Finally, let $\Lambda \in \{\mathbb{Z}, \mathbb{F}_p, \mathbb{Q}\}$ be a coefficient ring. Our main theorem is the following.

Theorem 1. *There is a symmetric monoidal equivalence*

$$(\mathrm{MTM}(\mathrm{Gr}_G, \Lambda), {}^{\mathrm{p}}\star) \cong \left(\mathrm{Rep}_{\hat{G}}\left(\mathrm{MTM}(S, \Lambda)\right), \otimes\right),$$

where ${}^{\mathrm{p}}\star$ *is the truncated convolution product.*

In the next section, we will explain what these terms mean. Let us already mention that the categories MTM are abelian categories arising as the heart of a t-structure. Moreover, graded Λ-modules are monoidally embedded as $\mathrm{gr}\text{-}\Lambda\text{-}\mathrm{Mod} \subseteq \mathrm{MTM}(S, \Lambda)$, where the grading corresponds to the Tate twists. It is also possible to equip the Hopf algebra $\Lambda[\hat{G}]$ with a grading, so that we may consider it as a Hopf algebra in $\mathrm{MTM}(S, \Lambda)$. The category $\mathrm{Rep}_{\hat{G}}(\mathrm{MTM}(S, \Lambda))$ then denotes the category of comodules in $\mathrm{MTM}(S, \Lambda)$ under this Hopf algebra.

We note that this is the first version of a Satake equivalence where both the base S and the coefficient ring Λ are allowed to have equal characteristic $p > 0$.

2. Motives on affine Grassmannians

Let S be as above, and $\Lambda = \mathbb{Z}$ for simplicity. We use the integral motivic cohomology spectrum in the motivic stable homotopy category SH as constructed in [9]. Considering modules under this ring spectrum gives rise to a functor

$$\mathrm{DM}\colon (\mathrm{Sch}_S^{\mathrm{ft}})^{\mathrm{op}} \to \mathrm{Pr}_{\mathbb{Z}}^{\mathrm{St}}$$

to the ∞-category of stable $\mathbb{Z}$-linear presentable ∞-categories, which is equipped with a six-functor-formalism. And while it does satisfy Nisnevich descent, it does not satisfy étale descent by [1, Proposition A.3.1].

Let $\mathrm{PreStk} := \mathrm{Fun}(\mathrm{Sch}_S^{\mathrm{op}}, \infty\text{-}\mathrm{Gpd})$ be the ∞-category of prestacks. In particular, it contains all ind-schemes and algebraic stacks. Then we can Kan extend DM to a functor

$$\mathrm{DM}\colon \mathrm{PreStk} \to \mathrm{Pr}_{\mathbb{Z}}^{\mathrm{St}},$$

although this functor does not satisfy the six-functor-formalism anymore.

As the existence of motivic t-structures is part of the standard conjectures on algebraic cycles, we need to restrict our categories of motives. For a scheme X, one can consider the category of *Tate motives* on X, defined as the full stable subcategory $\mathrm{DTM}(X) \subseteq \mathrm{DM}(X)$ generated under colimits and extensions by the Tate twists $\mathbb{Z}(k)$, for $k \in \mathbb{Z}$. More generally, for a stratified ind-scheme $X = \sqcup_w X_w$, we have the category of *stratified Tate motives*, consisting of those motives whose pullback to each stratum X_w is Tate. We denote it by $\mathrm{DTM}(X, \coprod_w X_w) \subseteq \mathrm{DM}(X)$, or even by $\mathrm{DTM}(X)$ for simplicity. Under certain assumptions on $X = \sqcup_w X_w$, we can combine results of [4, 8, 10] to equip $\mathrm{DTM}(X)$ with a t-structure, for which various realisation functors $\mathrm{DTM}(X) \to \mathrm{D}_{\mathrm{Betti}}(X), \mathrm{D}_{\text{ét}}(X)$ are t-exact for the perverse t-structure on the target. The heart of this t-structure are the *mixed Tate motives* $\mathrm{MTM}(X)$. For such a stratified ind-scheme X equipped with a stratified action from a pro-algebraic group H, let $H\backslash X$ be the prestack quotient. Then the full subcategory $\mathrm{DTM}_H(X) \subseteq \mathrm{DM}(H\backslash X)$ of objects whose underlying non-equivariant motive in $\mathrm{DM}(X)$ is stratified Tate, admits a t-structure with heart denoted by $\mathrm{MTM}_H(X)$. While the required assumptions on X are fairly strong, they are satisfied for the stratification of Gr_G by L^+G-orbits by [6].

In fact, in [2] we show that for any finite set I, the Beilinson-Drinfeld affine Grassmannian $\mathrm{Gr}_{G,I}$ living over $\mathbb{A}^I_S$ also admits a stratification satisfying the required assumptions. This gives the abelian category $\mathrm{MTM}_{L^+_I G}(\mathrm{Gr}_{G,I})$, and for a certain subcategory $\mathrm{Sat}^{G,I} \subseteq \mathrm{MTM}_{L^+_I G}(\mathrm{Gr}_{G,I})$, we can then show

Theorem 2. *There is a symmetric monoidal equivalence*

$$\left(\mathrm{Sat}^{G,I}, {}^{\mathrm{p}}\star\right) \cong \left(\mathrm{Rep}_{\hat{G}^I}\left(\mathrm{MTM}(S,\Lambda)\right), \otimes\right).$$

Along the way, we show that the fusion product preserves Tate motives, which allows us to construct the commutativity constraints for the convolution product even for integral and modular coefficients. Another essential part of the proof are the geometric constant term functors. To show that these preserve Tate motives, we show that the intersections of Schubert cells and semi-infinite orbits are paved by products of $\mathbb{A}^1_S$'s and $\mathbb{G}_{m,S}$'s.

3. Hecke algebras

Recall that the geometric Satake equivalence is a categorification of the Satake isomorphism from [7]. To decategorify the equivalence from Theorem 1, it is easier to work with categories of *reduced motives* DM_r from [3], for which $\mathrm{MTM}_r(S) \cong$ gr - Ab. Theorem 1 also holds for these reduced motives, and we get

$$\mathrm{MTM}_{r,L^+G}(\mathrm{Gr}_G) \cong \mathrm{Rep}_{\hat{G}}(\mathrm{gr}\text{-}\mathrm{Ab}) \cong \mathrm{Rep}_{\hat{G}\rtimes\mathbb{G}_m}(\mathrm{Ab}), \tag{1}$$

where $\hat{G} \rtimes \mathbb{G}_m$ is Deligne's modification of the Langlands dual group.

Defining the category $\mathrm{MTM}_{\mathrm{r},L^+G}(\mathrm{Gr}_G)^{\mathrm{anti}} \subseteq \mathrm{MTM}_{\mathrm{r},L^+G}(\mathrm{Gr}_G)$ of *anti-effective* stratified mixed Tate motives as the category of motives which are stratawise generated by the nonpositive Tate twists $\mathbb{Z}(k)$ for $k \leq 0$ (in contrast to all Tate twists), (1) restricts to a symmetric monoidal equivalence

$$\mathrm{MTM}_{\mathrm{r},L^+G}(\mathrm{Gr}_G)^{\mathrm{anti}} \cong \mathrm{Rep}_{V_{\hat{G},\rho_{\mathrm{adj}}}}(\mathrm{Ab}), \tag{2}$$

where $V_{\hat{G},\rho_{\mathrm{adj}}}$ is a modified form of Vinberg's universal monoid as in [11], containing $\hat{G} \rtimes \mathbb{G}_m$ as its open dense group of units.

Taking Grothendieck rings of (2) then gives an isomorphism

$$\mathcal{H}^{\mathrm{sph}}_G(\mathbf{q}) \cong R(V_{\hat{G},\rho_{\mathrm{adj}}})$$

between the *generic spherical Hecke algebra* of G, and the representation ring of $V_{\hat{G},\rho_{\mathrm{adj}}}$. This generalizes the integral Satake isomorphism from [11], and follows naturally from the motivic formalism.

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On the Rapoport-Zink space for GU(2, 4) over a ramified prime

Stefania Trentin

In this talk I discussed work in progress on the supersingular locus of the Shimura variety for $\mathrm{GU}(2,4)$ over a ramified prime. By the Uniformization Theorem of [7] this can be reduced to the problem of describing the undelying reduced scheme of the associated Rapoport-Zink space. In [6] Rapoport, Terstiege and Wilson give an explicit description of the RZ space for ramified unitary groups in signature one in terms of Deligne-Lusztig varieties (DLVs). In particular, the irreducible components of the RZ space are isomorphic to generalized Deligne-Lusztig varieties of Coxeter type for the symplectic group. Moreover, there is a canonical stratification indexed over the Bruhat-Tits' building for the corresponding algebraic group J over $\mathbb{Q}_p$. It turns out that these properties characterize a larger family of Shimura data, called Fully Hodge-Newton decomposable, which were studied and classified by Görtz, He and Nie in [2]. The Shimura varieties associated to ramified unitary groups of signatures $(1, n-1), (2,2)$ and $(2,3)$ are fully Hodge-Newton decomposable. The signature $(2,4)$ is the smallest one in which this is not the case, which makes it already an interesting example.

We briefly recall the definition of the RZ space corresponding to ramified $\mathrm{GU}(2,4)$, for more details see for example [5]. Let $p \neq 2$ be a prime, E a quadratic ramified extensions of $\mathbb{Q}_p$ and π a uniformizer of the ring of intergers $\mathcal{O}_E$ of E. Denote by $\breve{E} = E \otimes_{\mathbb{Q}_p} \mathrm{Quot}(\mathrm{W}(\bar{\mathbb{F}}_p)$ and consider the category Nilp of $\mathcal{O}_{\breve{E}}$-schemes S such that π gives a locally nilpotent ideal sheaf on S. Fix a six-dimensional supersingular p-divisible group $\mathbb{X}$ over $\bar{\mathbb{F}}_p$ endowed with a principal polarization

$\lambda_{\mathbb{X}}$ and an action $\iota_{\mathbb{X}}$ of $\mathcal{O}_E$. Then the RZ functor $\mathcal{N}$ is defined as the set-valued functor on Nilp which associates to a schemes S the set of isomorphism classes of quadruples $(X, \lambda, \rho, \iota)$. Here X is a p-divisible group over S and λ is a principal quasi-polarization $X \to X^\vee$ over $\mathbb{Q}_p$ whose Rosati involution induces on $\mathcal{O}_E$ the non-trivial automorphism over $\mathbb{Q}_p$. Let $\bar{S}$ be the reduction modulo π of S. Then $\rho : X \times_{\bar{S}} \operatorname{Spec} \bar{\mathbb{F}}_p \to \mathbb{X} \times_{\bar{\mathbb{F}}_p} \bar{S}$ is an $\mathcal{O}_E$-linear quasi-isogeny. Last, ι is a homomorphism $\mathcal{O}_E \to \operatorname{End}(X)$ satisfying the following two conditions

$$\operatorname{charpoly}(\iota(a)|\operatorname{Lie}(X)) = (T-a)^2(T-\bar{a})^4, \ \forall a \in \mathcal{O}_E$$
$$\bigwedge^3(\iota(\pi) - \pi|\operatorname{Lie}(X)) = 0, \ \bigwedge^5(\iota(\pi) + \pi|\operatorname{Lie}(X)) = 0.$$

The condition on the characteristic polynomial is the *Kottwitz condition*, the condition on the wedge product is the *Pappas condition*. It is proved in [7] that the functor $\mathcal{N}$ is representable over $\operatorname{Spf} \mathcal{O}_{\breve{E}}$. The analogue functor defined for $\mathrm{GU}(1, n-1)$ if flat over $\mathcal{O}_E$ as proved by Pappas in [5]. It is conjectured that flatness holds in any dimension. Using computational methods we are able to prove the following.

Proposition 1. *For* $\mathrm{GU}(2,4)$ *over a ramified prime the RZ space defined as above is flat over* $\mathcal{O}_E$*.*

Proof. We give a rough idea of the proof. In [5] the flatness conjecture is related to a conjecture in commutative algebra concerning certain polynomial ideals. In particular, let X be a symmetric matrix whose entries are the variables in the polynomial ring $\mathbb{F}_p[x_{ij}, 1 \leq i \leq j \leq 6]$ and let J be the ideal generated by the entries of X^2, the 3 by 3 minors of X and the non-leading coefficients of the characteristic polynomial of X. Then the proof of flatness is reduced by Pappas to proving that the ideal J is radical. We design an algorithm for radicality testing via elimination theory analogue to the primality testing algorithm by Gianni, Trager and Zacharias [3]. The advantage is that our algorithm can be performed exclusively using Gröbner basis. A result by Winkler [8] in computational algebra states that the Gröbner basis of a polynomial ideal with coefficients in $\mathbb{F}_p$ coincides with the image modulo p of the Gröbner basis for a lift of the ideal to $\mathbb{Q}$, for almost all primes p. Those primes for which the two basis may differ can be computed using the coefficients of the rational basis. This makes our proof almost independent from the characteristic, so that we can run our algorithm over $\mathbb{Q}$, where we obtain a positive answer, and then compute the set of bad primes, which turns out to be only 2. □

Once we have flatness of the model we can turn to the description of the irreducible components of the underlying reduced scheme. We still expect the irreducible components to be related to Deligne-Lusztig varieties. Following the strategy used for signature one in [6], by means of Dieudonné theory we associate to the datum $(\mathbb{X}, \lambda_{\mathbb{X}}, \iota_{\mathbb{X}})$ an E-vector space C, which is the rational module of $\mathbb{X}$, endowed with an Hermitian form h and a σ-semilinear map on $C \otimes \breve{E}$. For this talk we assumed h split, the other case being analogous but involving more notation.

As in *loc.cit.* we also associate to any $\bar{\mathbb{F}}_p$-point in the reduction modulo π of the reduced scheme $\bar{N}_{\mathrm{red}}$ its Dieudonné module, thus obtaining a bijection with a set $\mathcal{V}$ consisting of $\mathcal{O}_{\breve{E}}$-lattices in $C \otimes \breve{E}$ satisfying some properties that essentially ensure that the lattices come from p-divisible groups.

To any lattice in $\mathcal{V}$ we can associate a unique lattice Λ in C in the same way as in [6]. The main difference with the case of signature one is that two types of lattices can appear in our case: either vertex lattices, as defined in *loc.cit.*, or π^{-2}-modular lattices, first introduced in [4]. Roughly speaking, to these two types of lattices correspond two different types of irreducible components of the RZ space as follows.

Theorem 1. *The reduction modulo π of the reduced underlying scheme $\bar{\mathcal{N}}^{\circ}_{\mathrm{red}}$ of the subscheme $\mathcal{N}^{\circ}$ consisting of the points of the RZ space where the height of the isogeny ρ is zero has two types of irreducible components:*

- *Components $\mathcal{N}_{\mathcal{L}}$, indexed over the set of vertex lattices $\mathcal{L}$ in C and isomorphic to generalized Deligne-Lusztig varieties for the symplectic group* Sp_6*. These components have dimension five.*
- *Components $\mathcal{N}_{\Lambda}$, indexed over the set of π^{-2}-modular lattices Λ in C and isomorphic to line bundles over generalized Deligne-Lusztig varieties of Coxeter type for the orthogonal group* SO_6*. These components have dimension* 4*.*

Conjecture 1. *Some computations in higher dimensions suggest that similar results hold for the RZ space associated to* $\mathrm{GU}(2, n-2)$*. In particular we expect again two types of irreducible components, indexed over vertex lattices and π^{-2}-modular lattices, the former type again isomorphic to DLVs for the symplectic group, the second to line bundles over orthogonal DLVs.*

Last, we consider the associated group theoretical datum in the sense of [2]. It consists of a quadruple (W_a, μ, b, J) where W_a is an affine Weyl group, μ is a cocharacter, b is an element of the Kottwitz set $B(G)$ and J is a parahoric subgroup of W_a. In our case, the affine Weyl group is of type $\widetilde{BC}_3$, the cocharacter encodes the information of the signature and is $(1, 1, 0)$, the choice of a split Hermitian form h corresponds to the trivial element in $B(G)$, while the parahoric subgroup comes from the level structure and is the stabilizer of a selfdual lattice with respect to the form h.

We know that the closed points of the reduced scheme $\bar{\mathcal{N}}^{\circ}_{\mathrm{red}}$ are related to the points in the union

$$\bigcup_{w \in \mathrm{Adm}(\mu)_J} X_{J,w}(1)$$

where $\mathrm{Adm}(\mu)_J$ is the J-admissible set as defined in [2] and $X_{J,w}(1)$ are fine affine Deligne-lusztig varieties in the sense of [1].

The admissible set of μ has the interesting property that all its elements are either contained in a finite Weyl subgroup of W_a or can be reduced to such in exactly in one step of Deligne and Lusztig's reduction method. More precisely, we obtain some elements in a finite Weyl subgroup of type C_3, which correspond to the symplectic group and are exactly the same elements as in the first part of our main theorem. By [1] the corresponding affine Deligne-Lusztig variety can be decomposed as a union of copies of classical Deligne-Lusztig varieties. The index set can be set in bijection with the set of vertex lattices. For the remaining elements in the admissible set the reduction method decomposes the corresponding affine Deligne-Lusztig variety into the union of an open and closed set. The open set turns out to be empty while the closed set is isomorphic a line bundle over a union of classical Deligne-Lusztig varieties of Coxeter type for the orthogonal group. In other words we have obtained the both types of irreducible components in Theorem 1 above.

Conjecture 2. *Computational evidence suggests that this is the case in higher dimension, too. We expect the admissible set to contain elements that reduce in at most one step to elements in a finite Weyl subgroup of W_a and whose corresponding affine Deligne-Lusztig varieties are at most line bundles over classical DLVs.*

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On the étale cohomology of the Fargues-Fontaine curve

Sebastian Bartling

The Fargues-Fontaine curve has played on centre court in arithmetic geometry in recent years with diverse applications to p-adic Hodge theory, the theory of local Shimura varieties and in the monumental work of Fargues-Scholze on the geometrization of the local Langlands correspondence.

One of the reasons (besides others) that this object is important is the fact that the étale cohomology of torsion local systems on the Fargues-Fontaine curve allows one to think geometrically about torsion Galois representations of a p-adic field and their Galois cohmology. The question that was considered in the talk was the following: What can be said about the étale cohomology of constructible sheaves?

Let us quickly recall the Fargues-Fontaine curve in the situation of interest to us. It is associated to the datum of a pair E, F, where E is a finite extension of $\mathbb{Q}_p$ and F is a complete non-archimedean and algebraically closed extension of the residue field of E. Then the curve comes in two incarnations: as a regular noetherian 1-dimensional scheme over $\mathrm{Spec}(E)$, which admits a cover by spectra of principal ideal domains and as a qcqs analytic adic space over $\mathrm{Spa}(E)$, which is covered by adic spectra of Banach E-algebras which are principal ideal domains and which are also strongly noetherian (a result of Kedlaya [9]). The algebraic version was initially studied by Fargues-Fontaine in [4], the adic version was also considered by Fargues [2] and studied by Kedlaya-Liu [10] and the basic results were reproven by Fargues-Scholze in [3]. Denote these two versions by $X^{\mathrm{alg}}_{E,F}$ resp. $X^{\mathrm{ad}}_{E,F}$.

These versions are related by a morphism of locally ringed spaces

$$u\colon X^{\mathrm{ad}}_{E,F} \to X^{\mathrm{alg}}_{E,F},$$

such that pullback under u induces an equivalence on the category of vector bundles on both sides.

Let $\mathcal{L}$ be a torsion étale local system on $X^{\mathrm{alg}}_{E,F}$. Since $X^{\mathrm{alg}}_{E,F}$ is geometrically simply connected by a result of Fargues-Fontaine, the local system comes from the base, i.e. $\mathcal{L} \simeq f^*(M)$, where $f\colon X^{\mathrm{alg}}_{E,F} \to \mathrm{Spec}(E)$ is the structure morphism and M is a torsion $\mathrm{G}_E = \mathrm{Gal}(\overline{E}/E)$-representation. Then Fargues showed ([3]) that

$$H^i(X^{\mathrm{alg}}_{E,F}, \mathcal{L}) = H^i(\mathrm{G}_E, M),$$

for $i = 0, 1, 2$, using that $\mathrm{Br}(X^{\mathrm{alg}}_{E,F}) = 0$. Motivated by these results and the guiding analogy between the curve and a smooth projective curve over an algebraically closed field resp. a compact Riemann-surface, he made the following conjecture (also in [3]).

Conjecture 1. *(Fargues)*

(1): *Let $\mathcal{F}$ be an étale abelian torsion sheaf on $X^{\mathrm{alg}}_{E,F}$. Then*

$$H^i(X^{\mathrm{alg}}_{E,F}, \mathcal{F}) = 0$$

for $i \geq 3$.

(2): *The morphism of locally ringed spaces $u\colon X^{\mathrm{ad}}_{E,F} \to X^{\mathrm{alg}}_{E,F}$ induces a morphism of étale sites $u\colon (X^{\mathrm{ad}}_{E,F})_{et} \to (X^{\mathrm{alg}}_{E,F})_{et}$ and for $\mathcal{F}$ as in (1), let $\mathcal{F}^{\mathrm{ad}} = u^*(\mathcal{F})$, then we have*

$$H^i(X^{\mathrm{alg}}_{E,F}, \mathcal{F}) \simeq H^i(X^{\mathrm{ad}}_{E,F}, \mathcal{F}^{\mathrm{ad}})$$

for all $i \geq 0$.

(3): *Let $E(X^{\mathrm{alg}}_{E,F})$ be the function field of $X^{\mathrm{alg}}_{E,F}$. Then $E(X^{\mathrm{alg}}_{E,F})$ is $(C1)$.*

Remark. Recall that a $(C1)$ field is in particular of cohomological dimension less or equal than 1. Using absolute purity and the méthode de la trace, one can show that part (3) implies part (1). As a consequence of the results discussed in the talk one can show that part (1) also implies (2).

The main result ([1]) presented in the talk was the following:

Theorem 1. *Let $\mathcal{F}$ be an étale abelian torsion sheaf on $X^{\mathrm{alg}}_{E,F}$.*

(a): *If $\mathcal{F}$ is of torsion prime to p, then Conjectures (1) and (2) are true. Furthermore, $\mathrm{cd}_\ell(E(X^{\mathrm{alg}}_{E,F})) \leq 1$, where $\ell \neq p$ is a prime.*

(b): *Let $\mathcal{F}$ be a constructible $\mathbb{F}_p$-module, then*

$$H^i(X^{\mathrm{ad}}_{E,F}, \mathcal{F}^{\mathrm{ad}}) = 0$$

for $i \geq 3$ if the following statement is true:

(Stein): Let U^{ad} be the adification of an open $U \subset X^{\mathrm{ad}}_{E,F}$ and E_∞ the cyclotomic $\mathbb{Z}_p$-extension of E. Then the tilt of the perfectoid space $U^{\mathrm{ad}}_\infty := U^{\mathrm{ad}} \times_{\mathrm{Spa}(E)} \mathrm{Spa}(\widehat{E}_\infty)$ is a perfectoid Stein space, i.e.

$$(U^{\mathrm{ad}}_\infty)^\flat = \bigcup_{n \in \mathbb{N}} V_n,$$

where V_n are affinoid perfectoid opens, $V_n \subset V_{n+1}$ and res: $\mathcal{O}(V_{n+1}) \to \mathcal{O}(V_n)$ has dense image.

Then the proof of these results was sketched. Part (a) has two steps: first show that $\mathrm{cd}_\ell(X^{\mathrm{ad}}_{E,F}) \leq 2$ and $\mathrm{cd}_\ell(X^{\mathrm{alg}}_{E,F}) \leq 2$ and then deduce the comparison from an adic version of a result of Gabber [6] (by adapting the proof given by Lieblich [11] using twisted sheaves) comparing the Azumaya-Brauer group and the cohomological Brauer group of separated unions of two affinoids. Part (b) uses a perfectoid $\mathbb{Z}_p$-cover of a ramified cover of the curve, vanishing results coming from the vanishing of coherent cohomology of Stein spaces and a trick to kill $\mathbb{Z}_p$-cohomology. Here a ramified cover of the algebraic curve is a 1-dimensional noetherian regular scheme X' with a finite locally free morphism $\pi\colon X' \to X^{\mathrm{alg}}_{E,F}$, which is finite étale over a dense open of $X^{\mathrm{alg}}_{E,F}$. Using a result of Huber [8] one can define its adification X'^{ad}. Let us end with a couple of speculative questions which might be interesting to think about, but I do not know whether the answer is positive to any of those:

(a): Are diamonds of adifications of ramified covers of the curve ℓ cohomologically smooth over $\mathrm{Spd}(E)$? Can one describe the dualizing complex? This would have applications to show Verdier-duality in the prime to p torsion case. What happens in the p-torsion case? Maybe here the results of Lucas Mann ([12]) might be helpful.

(b): Does the Riemann-extension theorem hold for the adic curve, i.e. do finite étale covers of the adic curve minus a finite set of classical points extend to finite covers of the adic curve? This would help showing that Zariski-constructible sheaves (in the sense of Hansen [7]) are equivalent to constructible sheaves on the algebraic curve.

(c): There is a proof of Tsen's theorem for classical curves which proceeds roughly along the following lines: if X is a smooth projective curve over an algebraically closed field k, F the function field and $P \in F[X_1, ..., X_n]$ a non-constant homogenouse polynomial with $\deg(P) < n$. Then P induces a morphism

$$P\colon H^0(X, \mathcal{O}(eD))^n \to H^0(X, \mathcal{O}((\deg(P)e + 1)D)),$$

where D is an ample divisor on X, so that P has coeffients in $H^0(X, \mathcal{O}(D))$. Using asymptotic Riemann-Roch one can compute the dimensions of these spaces of global sections and then one may deduce using that our assumption $\deg(P) < n$ implies that we have more variabels than equations in $P(\alpha) = 0$, so that we find a non-trivial solution.

In view of the analogy of affine spaces and Banach-Colmez spaces in perfectoid geometry one could be tempted to ask the following question: Can one adapt this proof using Banach-Colmez spaces?

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Geometric Eisenstein Series and the Fargues-Fontaine Curve

Linus Hamann

Let $\ell \neq p$ be distinct primes and $G/\mathbb{Q}_p$ a split (for simplicity) connected reductive group over the p-adic numbers. We consider Bun_G the moduli stack of G-bundles on the Fargues-Fontaine curve X, and $\mathrm{D}(\mathrm{Bun}_G) := \mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G, \overline{\mathbb{Q}}_\ell)$ the derived category of lisse-étale $\overline{\mathbb{Q}}_\ell$-sheaves on it [3, Section VII.7]. For all dominant cocharacters $\mu \in \mathbb{X}_*(T_{\overline{\mathbb{Q}}_p})^+$, this category comes equipped with an action by Hecke operators

$$T_\mu : \mathrm{D}(\mathrm{Bun}_G) \to \mathrm{D}(\mathrm{Bun}_G)^{BW_{\mathbb{Q}_p}}$$

taking a sheaf on Bun_G to a sheaf with continuous $W_{\mathbb{Q}_p}$-action. For $b \in B(G)$ varying in the Kottiwtz set of G, the stack Bun_G has a locally closed stratification $j_b : \mathrm{Bun}_G^b \hookrightarrow \mathrm{Bun}_G$ by Harder-Narasimhan strata. This stratification induces a semi-orthogonal decomposition of $\mathrm{D}(\mathrm{Bun}_G)$ into $\mathrm{D}_{\mathrm{lis}}(\mathrm{Bun}_G^b, \overline{\mathbb{Q}}_\ell) \simeq \mathrm{D}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$, the unbounded derived category of smooth $\overline{\mathbb{Q}}_\ell$-representations of the σ-centralizer $J_b/\mathbb{Q}_p$ of b, for $b \in B(G)$ varying.

Given a semi-simple L-parameter $\phi : W_{\mathbb{Q}_p} \to \hat{G}(\overline{\mathbb{Q}}_\ell)$ valued in the Langlands dual group, one of the goals of the geometric Langlands is to construct and describe eigensheaves $\mathcal{S}_\phi$ with eigenvalue ϕ, in the sense that, for all μ, we have isomorphisms

$$T_\mu(\mathcal{S}_\phi) \simeq \mathcal{S}_\phi \otimes r_\mu \circ \phi$$

where $r_\mu : \hat{G} \to \mathrm{GL}(V_\mu)$ is the highest weight representation attached to μ. It was first noted by Fargues [4] that, for ϕ supercupsidal (e.g does not factor through a proper Levi of $\hat{G}$), the Kottwitz conjecture [8, Conjecture 7.3] would follow from showing that

$$\mathcal{S}_\phi := \bigoplus_{b \in B(G)_{\mathrm{basic}}} \bigoplus_{\pi_b \in \Pi_\phi(J_b)} j_{b!}(\pi_b) \in \mathrm{D}(\mathrm{Bun}_G)$$

is an eigensheaf with supercuspidal eigenvalue ϕ where $\Pi_\phi(J_b)$ are the L-packets over ϕ, and $B(G)_{\mathrm{basic}}$ is the set of basic elements. The construction and description of this eigensheaf has now been carried out for various groups, by proving and using local-global compatibility of the Fargues-Scholze local Langlands correspondence.

It is a natural question to wonder if Fargues' vision can be extended to more general parameters ϕ. Assume from now on that ϕ factorizes over a toral parameter $\phi_T : W_{\mathbb{Q}_p} \to \hat{T}(\overline{\mathbb{Q}}_\ell)$. For $\mathcal{S}_\phi$ any eigensheaf with eigenvalue ϕ, the stalks $\mathcal{S}_\phi|_{\mathrm{Bun}_G^b} \in \mathrm{D}(J_b(\mathbb{Q}_p), \overline{\mathbb{Q}}_\ell)$ should be valued in representations whose semi-simplified L-parameter under the Fargues-Scholze local Langlands correspondence is ϕ. Since ϕ is induced from ϕ_T, if one assumes that the Fargues-Scholze local Langlands behaves like more classical instances of the correspondence, a naive guess is that J_b must be quasi-split with Borel B_b and that $\mathcal{S}_\phi|_{\mathrm{Bun}_G^b}$ is valued in sub-quotients of the normalized parbaolic induction $i_{B_b}^{J_b}(\chi)$, where χ is the character attached to ϕ_T via local class field theory. The elements $b \in B(G)$ for which J_b is quasi-split are the set of unramified elements $B(G)_{\mathrm{un}} := \mathrm{Im}(B(T) \to B(G))$, as in [10]. Thinking through this more carefully, one is lead to consider the perverse sheaf

$$\mathcal{S}_\phi := \bigoplus_{b \in B(G)_{\mathrm{un}}} \bigoplus_{w \in W_b} j_{b!}(\rho_{b,w})[-\langle 2\rho_G, \nu_b \rangle] \tag{1}$$

on Bun_G, where $M_b \simeq J_b$ is the centralizer of the slope homorphism of b, $W_b := W_G/W_{M_b}$ is a quotient of Weyl groups, $\rho_{b,w} := i_{B_b}^{J_b}(\chi^w) \otimes \delta_{P_b}^{1/2}$, and δ_{P_b} is the modulus character of the parabolic P_b with Levi factor M_b. We formulate the following naive conjecture.

Conjecture 1. *(Naive) The sheaf $\mathcal{S}_\phi$ is an eigensheaf with eigenvalue ϕ.*

Unfortunately, this is too naive; in particular, there can exists representations of non quasi-split groups, whose *semi-simplified* L-parameter factors through a maximal torus T (e.g the trivial representation of $D^*_{\frac{1}{2}}$, units in the quaternion division algebra). These representations should appear in the stalks of the eigensheaf $\mathcal{S}_\phi$; so, for our conjecture to have a chance of being true, we impose the following condition on ϕ_T.

Definition 1. We say ϕ_T is *generic* if, for all coroots α, the character $\alpha \circ \phi_T$ is not isomorphic to the trivial representation or the norm character $|\cdot|$; equivalently, this holds if the Galois cohomology $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ is trivial for all coroots α.

This implies that the parameter ϕ is not the semi-simplification of a parameter with non-trivial monodromy and, assuming the Fargues-Scholze local Langlands correspondence satisfies certain expected properties, one can show that the stalks of an eigensheaf with eigenvalue ϕ can only be of the form described above. Our main Theorem is as follows.

Theorem 1. [6, Theorem 1.17] *If ϕ is induced from a generic toral parameter ϕ_T then, assuming certain properties of the Fargues-Scholze local Langlands correspondence, and possible additional constraints on ϕ_T, Conjecture 1 is true.*

To see why this could be true, we consider the diagram of spaces

$$\begin{tikzcd} \mathrm{Bun}_B \arrow[r, "j"] \arrow[d, "\mathfrak{q}"] \arrow[dr, "\mathfrak{p}"] & \overline{\mathrm{Bun}}_B \arrow[d, "\overline{\mathfrak{p}}"] \\ \mathrm{Bun}_T & \mathrm{Bun}_G \end{tikzcd}$$

where Bun_B is the moduli space parametrizing parabolic B-structures on G-bundles and $\overline{\mathrm{Bun}}_B$ is a compactification of Bun_B. Using this diagram, one can define a sheaf $\mathrm{Eis}_B(\mathcal{S}_{\phi_T}) \in \mathrm{D}(\mathrm{Bun}_G)$, where $\mathcal{S}_{\phi_T}$ is the eigensheaf attached to ϕ_T.

Following work of Braverman-Gaitsgory-Laumon [2, 7] in classical geometric Langlands, the true candidate should be given as $\overline{\mathrm{Eis}}_B(\mathcal{S}_{\phi_T})$, defined by pushing-forward along $\overline{\mathfrak{p}}$ instead of $\mathfrak{p}$. Unfortunately, for this to be the correct definition one needs to tensor by a kernel sheaf $\mathrm{IC}_{\overline{\mathrm{Bun}}_B}$, the intersection cohomology of the Drinfeld compactification, and, in the geometric context we are working in, defining this sheaf and showing it has good properties is a very difficult problem.

Nevertheless, in classical geometric Langlands, there exists a map

$$\mathrm{Eis}_B(\mathcal{S}_{\phi_T}) \to \overline{\mathrm{Eis}}_B(\mathcal{S}_{\phi_T})$$

which should be an isomorphism for ϕ_T generic, essentially because the Galois cohomogy groups $R\Gamma(W_{\mathbb{Q}_p}, \alpha \circ \phi_T)$ appear in the cone of this map and are killed by the generic assumption. This suggests that the sheaf $\mathrm{Eis}_B(\mathcal{S}_{\phi_T}) \in \mathrm{D}(\mathrm{Bun}_G)$, which is computable and understandable in the Fargues-Scholze setting, should be the sought after eigensheaf, at least when ϕ_T is generic. It also suggests that, under genericity, the sheaf $\mathrm{Eis}_B(\mathcal{S}_{\phi_T})$ should satisfy the same good properties that $\overline{\mathrm{Eis}}_B(\mathcal{S}_{\phi_T})$ does classically. Namely, it should satisify a functional equation with respect to the action of the Weyl group ([2, Theorem 2.24]) and behave well under Verdier duality on Bun_G. If these two properties were to hold for $\mathrm{Eis}_B(\mathcal{S}_{\phi_T})$ it would immediately imply Conjecture 1, for ϕ_T generic. The eigensheaf property holding for $\mathcal{S}_\phi$, as in equation 1, implies the following formula for the cohomology of local Shimura varieties/shtuka spaces:

$$\bigoplus_{b\in B(G,\mu)_{\mathrm{un}}} \bigoplus_{w\in W_b} R\Gamma_c(G,b,\mu)[\rho_{b,w}][\langle 2\rho_G,\nu_b\rangle] \simeq i_B^G(\chi)\otimes r_\mu\circ\phi.$$

This formula is compatible with more classical work of Shin [9], work of Xiao-Zhu [10] on the irreducible components of affine Deligne-Lusztig varieties, and work of Boyer [1] (or rather it's generalization considered in [5]) describing the contribution coming from the μ-ordinary element in $B(G,\mu)_{\mathrm{un}}$. In particular, the latter tells us that the summands on the LHS coming from the μ-ordinary element must be of the form $i_B^G(\chi^w)$ for $w\in W_G$, and it thereby follows that one must have an intertwiner $i_B^G(\chi^w)\simeq i_B^G(\chi)$. Such an isomorphism does not always exist, but will exist under the generic hypothesis.

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A p-adic analogue of Borel's theorem

Ananth Shankar
(joint work with Abhishek Oswal, Xinwen Zhu, and an appendix by Anand Patel)

Let S be a Shimura variety with sufficient level structure. More precisely, suppose that the complex points of connected components of $S_{\mathbb{C}}$ equal X/Γ, where X is a Hermitian symmetric domain, and Γ is a *torsion-free* arithmetic group. Borel proves that any holomorphic map from an affine complex algebraic variety to $S_{\mathbb{C}}$ must be algebraic. He proves this by proving the following extension theorem. Let E denote the open unit disc, and let $\mathsf{E}^* = \mathsf{E} \setminus \{0\}$. Borel proves that any homomorphic map $f : \mathsf{E}^a \times \mathsf{E}^{*b} \to S_{\mathbb{C}}$ uniquely extends to a holomorphic map $\mathsf{E}^a \times \mathsf{E}^b \to S_{\mathbb{C}}^{\mathrm{bb}}$, the Baily-Borel compactification of S.

In joint work with Abhishek Oswal and Xinwen Zhu (and an appendix with Anand Patel), we prove a p-adic analogue of Borel's theorems for compact Shimura variety of Abelian type. Namely, let K denote a finite extension of $\mathbb{Q}_p$, or a finite extension of $W(\overline{\mathbb{F}}_p)[\frac{1}{p}]$. Let S be as above, and let D denote the rigid-analytic closed unit disc, and let $\mathsf{D}^* = \mathsf{D} \setminus \{0\}$. We prove that any rigid-analytic map $f : \mathsf{D}^* \to S$ defined over K must extend to a map $\mathsf{D} \to S$. This yields an analogous algebraicity result. Namely, let V denote a variety over K. Then, any rigid-analytic map $V \to S$ defined over K must be algebraic.

We also strongly expect the same result for arbitrary Shimura varieties of abelian type. We prove the following result, which we expect is the key step for the general case. Let $\mathcal{A}_{g,N}$ denote the moduli space of principally polarized abelian schemes with full level N structure, for N a large enough integer. Let $A \to \mathcal{A}_{g,N}^{\mathrm{good}}$ denote the universal abelian scheme defined over the good-reduction locus of $\mathcal{A}_{g,N}$. Then, the p-adic analogue of Borel's extension theorem holds for A.

On the Rapoport–Zink space for GU(2, 4) for unramified primes

Naoki Imai
(joint work with Maria Fox)

Let p be an odd prime number. Let $\mathbb{Q}_{p^2}$ and $\mathbb{Z}_{p^2}$ denote the unramified quadratic extensions of $\mathbb{Q}_p$ and $\mathbb{Z}_p$ respectively. Let V be a hermitian space of rank n over $\mathbb{Q}_{p^2}$ having a self-dual lattice Λ. Let G be the general unitary group over $\mathbb{Q}_p$ associated to V. We take a basis of Λ over $\mathbb{Z}_{p^2}$ such that the hermitian paring is given by the anti-diagonal matrix. Then we have an isomorphism $G_{\mathbb{Q}_{p^2}} \simeq \mathrm{GL}_n \times \mathbb{G}_{\mathrm{m}}$ determined by the basis. Let m be a positive integer less than or equal to n. Let μ be the cocharacter of G corresponding to $z \mapsto (\mathrm{diag}(1, \ldots, 1, z^{-1}, \ldots, z^{-1}), z^{-1})$ under the isomorphism $G_{\mathbb{Q}_{p^2}} \simeq \mathrm{GL}_n \times \mathbb{G}_{\mathrm{m}}$, where 1 and z^{-1} appear $(n-m)$ and m times respectively in $\mathrm{diag}(1, \ldots, 1, z^{-1}, \ldots, z^{-1})$. We put $b = (1_n, p^{-1}) \in (\mathrm{GL}_n \times \mathbb{G}_{\mathrm{m}})(\mathbb{Q}_{p^2}) \simeq G(\mathbb{Q}_{p^2})$. Let $\mathrm{RZ}(G, b, \mu)$ be the Rapoport–Zink formal scheme for (G, b, μ). Let $\mathrm{RZ}(G, b, \mu)^{\mathrm{red}}$ denote the underlying reduced scheme of $\mathrm{RZ}(G, b, \mu)$.

When $m = 1$, Vollaard–Wedhorn [4] studied the irreducible components of $\mathrm{RZ}(G, b, \mu)^{\mathrm{red}}$ and their intersections. In this case, the irreducible components are isomorphic to Deligne–Lusztig varieties.

Assume that $m = 2$ in the sequel. When $n = 4$, Howard–Pappas [3] obtained similar results. The irreducible components are isomorphic to Deligne–Lusztig varieties also in this case. On the other hand, (G, b, μ) is not fully Hodge–Newton decomposable in the sense of [2, Definition 3.1] if $n \geq 5$. In such a case, we can not expect that $\mathrm{RZ}(G, b, \mu)^{\mathrm{red}}$ is a union of Deligne–Lusztig varieties by [2, Theorem B].

In this talk, we explain a result for $n \geq 5$ following [1], where we study the irreducible components of $\mathrm{RZ}(G, b, \mu)^{\mathrm{red}}$ and their intersections after taking perfection.

For simplicity, we concentrate on the case where $n = 6$. Let $X_\mu(b)$ be the affine Deligne–Lusztig variety associated to (b, μ). Then we have $\mathrm{RZ}(G, b, \mu)^{\mathrm{red}} \cong X_\mu(b)$. Hence the question is reduced to studying $X_\mu(b)$. Following a construction of the irreducible components of $X_\mu(b)$ by Xiao–Zhu in [5], we can construct schemes $X_\mu^{\mathbf{a}_i}(b)$ for $1 \leq i \leq 3$ using Satake cycles. Up to translation under the action of $G(\mathbb{Q}_p)$, these 3 schemes give all the irreducible components.

First, we show the following proposition:

Proposition 2. *The schemes $X_\mu^{\mathbf{a}_1}(b)$ and $X_\mu^{\mathbf{a}_3}(b)$ are isomorphic to the perfection of Deligne–Lusztig varieties.*

Next, we study $X_\mu^{\mathbf{a}_2}(b)$. We construct a kind of Demazure resolution X of $X_\mu^{\mathbf{a}_2}(b)$. We write $\mathring{X}$ and $\mathring{X}_\mu^{\mathbf{a}_2}(b)$ for the inverse images in X and $X_\mu^{\mathbf{a}_2}(b)$ of the Schubert cell $\mathring{\mathrm{Gr}}_\nu$ of an affine Grassmannian Gr_ν under natural morphisms $X \to X_{\mu^*}^{\mathbf{b}_2}(b) \to \mathrm{Gr}_\nu$. Explicitly, we construct a vector bundle $\mathscr{V}$ of rank 3 over a perfection Y of a Deligne–Lusztig variety. We have a natural morphism

$$\phi \colon \mathscr{V} \to F(\mathscr{V}^\vee)$$

by a hermitian pairing related to the unitary group G, where $F(\mathscr{V}^\vee)$ is some Frobenius twist of $\mathscr{V}^\vee$. Let Fl_Y denote the perfection of the flag scheme parametrizing subvector bundles $\mathscr{W} \subset \mathscr{V}$ of rank 1.

Theorem 1. *The scheme X is isomorphic to the closed subscheme of Fl_Y defined by the condition $\phi(\mathscr{W}) \subset F(\mathscr{W}^\perp)$ on $\mathscr{W}$. Further $\mathring{X}$ is isomorphic to $\mathring{X}_\mu^{\mathbf{a}_2}(b)$.*

Using the above results, we study the intersections of irreducible components. In particular we show that the intersection $X_\mu^{\mathbf{a}_1}(b) \cap X_\mu^{\mathbf{a}_2}(b)$ is isomorphic to the perfection of the closed subscheme of $\mathbb{P}^5$ defined by two equations

$$\sum_{i=1}^{6} x_i^{p+1} = 0, \quad \sum_{i=1}^{6} x_i^{p^3+1} = 0.$$

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Undercover p-adic modular forms

Valentin Hernandez

(joint work with Eugen Hellmann, Benjamin Schraen)

A classical question in number theory is to be able to detect classical modular forms among p-adic modular forms. This last space is a much bigger one than the space of classical modular forms, and classical modular forms enjoys much stronger arithmetic properties than their p-adic analogs.

A first theorem in this direction is due to Coleman ([6]), and states that a p-adic modular form of classical weight which has *small slope* (compared to its weight) is actually a classical one. More recently, Kisin ([7]) proved that if f is a p-adic modular eigenform of weight $k \geq 2$, with finite slope, such that its Galois representation ρ_f is de Rham at p, then f is actually classical.

We would like to have a generalisation of Kisin's theorem for modular forms in more general groups than GL_2. Let $n \geq 1$, and G be a n-variables unitary group on a CM extension of number fields E/F. We assume G is compact at infinity, and split at (places above) p, so $G(\mathbb{Q}_p) \simeq \mathrm{GL}_n(\mathbb{Q}_p)$. For simplificity in this talk, we assume $F = \mathbb{Q}$, i.e. E quadratic imaginary and p split in E. We fix a torus and a Borel $T \subset B \subset G_{\mathbb{Q}_p} \mathrm{GL}_n(\mathbb{Q}_p)$ (seen as the upper Borel), and we moreover fix a tame level $K^p \subset G(\mathbb{A}_{\mathbb{Q}}^{\infty,p})$.

Definition 1. A *p-adic modular form* (for G) of weight $\lambda = (k_1, \dots, k_n) \in \mathbb{Z}^n$ and tame level K^p is a function

$$f : G(\mathbb{Q}) \backslash G(\mathbb{A}^\infty)/K^p \longrightarrow \overline{\mathbb{Q}_p},$$

which is locally analytic, and such that

$$\forall b \in B(\mathbb{Z}_p), \forall g \in G(\mathbb{A}^\infty), f(gb) = \chi_\lambda(b) f(g),$$

where χ_λ is the character $\begin{pmatrix} a_1 & & \star \\ & \ddots & \\ & & a_n \end{pmatrix} \in B(\mathbb{Z}_p) \mapsto a_1^{k_1} \dots a_n^{k_n}$. We denote $M_\lambda^\dagger$ the space of p-adic modular forms of weight λ. We say that $f \in M_\lambda^\dagger$ is *classical*, if f is a locally polynomial function (this implies that λ is *dominant*, i.e. $k_1 \geq \dots \geq k_n$.

There is a natural action of Hecke operators away from p (of level K^p) and at p, generated by the matrices

$$\begin{pmatrix} p^{a_1} & & \\ & \ddots & \\ & & p^{a_n} \end{pmatrix},$$

with $a_1 \leq \cdots \leq a_n$. We say that f is of *finite slope* if the eigenvalue for each of these operators at p is non-zero. To such an f, we can associate a p-adic representation $\rho_f : G_E \longrightarrow \mathrm{GL}_n(\overline{\mathbb{Q}_p})$. Assume that ρ_f is crystalline at p, and denote $D = D_{cris}(\rho_f)$. Denote $\kappa = (\kappa_1 \leq \cdots \leq \kappa_n)$ the Hodge-Tate weights of D, and $\varphi_1, \ldots, \varphi_n$ the Eigenvalues of the Frobenius φ of D. κ is related to the weight λ of f, and $(\varphi_1, \ldots, \varphi_n)$ to the Hecke eigenvalues at p. We say that f is *HT-regular* if $\kappa_1 < \cdots < \kappa_n$, and φ-generic if for all $i \neq j, \varphi_i/\varphi_j \neq 1, p$: we now assume both these conditions. To an ordering $(\varphi_1, \ldots, \varphi_n)$ is associated a (unique) full flag $F_\bullet$ of D. We denote $w = w(f)$ the relative position of $F_\bullet$ with the Hodge filtration $\mathrm{Fil}^\bullet D$, it is an element of the Weyl group W of GL_n.

Then we have the following results

Theorem 1.

1. *(Chenevier, [5]) If $w = w_0$, f is classical*

From now on we assume some usual Taylor-Wiles hypothesis.

2. *(Breuil-Hellmann-Schraen, [3]) If $w_0 w^{-1}$ is a product of distinct simple reflections, then f is classical*
3. *If $n = 3$, $w = 1$, there exists a classical form f, and a* non-classical *form f' with same Galois representation, same (classical) weight, and same Hecke eigenvalues (both at and away from p) than f.*

Because of the phenomena of *companion forms*, we know that there exists p-adic forms with non-dominant weight with the same Galois representation and Hecke eigenvalues than classical forms ([6, 1, 3]). Here we insist than the f in point 3. has dominant weight. In particular the non-classical form f' looks like a classical form while being non-classical : it is *undercover* among classical forms.

Let us insist that because of Kisin's theorem, this cannot happen for GL_2. There is a related notion though, explained by Coleman. For GL_2, we have an exact sequence (for $k > 2$)

$$0 \longrightarrow (M^\dagger_{2-k)})_0(k-1) \overset{\theta^{k-1}}{\longrightarrow} (M^\dagger_k)_{k-1} \longrightarrow (S_k)_{k-1} \longrightarrow 0,$$

where the second index denotes the slope, $(k-1)$ denote a twist of the Hecke operators, and θ is Serre's operator "$q\frac{d}{dq}$". In particular the cuspforms $(S_k)_{k-1}$ sits inside $M^\dagger_k$ as a subspace and a quotient. If $f \in (S_k)_{k-1}$ has its system of Hecke eigenvalues in the image of θ^{k-1}, then this system appears twice : in particular, associated to f there is a form f_0, of weight $2-k$ with same Galois representation and Hecke eigenvalues than f (up to twist), this is the companion form associated to f, but its weight is non-dominant, and another form $f' \in M^\dagger_k$ whose image in

S_k is f. But Coleman's results assures that f' is *not* an eigenvector for the Hecke operators, only a generalized eigenvector. In the previous theorem, f' is indeed an eigenvector.

Let us give an idea of the proof of existence of such an f'. The first step is to consider Π the $G(\mathbb{Q}_p)$-representation of all continuous functions $G(\mathbb{Q})\backslash G(\mathbb{A}^\infty)/K^p \longrightarrow \overline{\mathbb{Q}_p}$, and the locally analytic functions $\Pi^{\ell a}$ inside it. Among other things, Taylor-Wiles hypothesis assure that $\overline{\rho}$ is absolutely irreducible, thus we have a deformation ring, for $\rho = \rho_f$, $R_{\overline{\rho}}$ and denote $\mathcal{X}_{\overline{\rho}}$ its rigid fiber. Results of Chenevier, Emerton associate to $\Pi^{\ell a}$ a coherent sheaf $\mathcal{M}$ on $\mathcal{X}_{\overline{\rho}} \times \mathcal{T}$ where $\mathcal{T}$ denotes the rigid space of characters $(\mathbb{Q}_p^\times)^n \longrightarrow \overline{\mathbb{Q}_p}^\times$. The support of this sheaf, denoted $\mathcal{E}$, is the *Eigenvariety*, it parametrizes system of Hecke eigenvalues of finite slope.

The second step is to "thicken" this situation using Taylor-Wiles primes. We thus use the construction of the patched Eigenvariety of [2] (building on [4]), it gives a coherent sheaf $\mathcal{M}_\infty$ on $\mathcal{X}_{\infty,\overline{\rho}} \times \mathcal{T}$, with support $\mathcal{E}_\infty$, where $\mathcal{X}_{\infty,\overline{\rho}}$ is the generic fiber of a ring R_∞, and an ideal $\mathfrak{a}$ of R_∞ such that the restriction of $\mathcal{M}_\infty$ to $V(\mathfrak{a})$ is $\mathcal{M}$. Then, we can construct an exact functor F_∞ from the category $\mathcal{O}$ of Bernstein-Gelfand-Gelfand to coherent sheaves on $\mathfrak{X}_\infty$. This functor is a locally analytic equivalent to patching functors which appeared in the proof of the Breuil-Mézard conjecture. Now focus on $n = 3$ for simplicity. In particular, for $\lambda \in \mathbb{Z}^3$ a dominant weight for $\mathfrak{gl}_3$, the BGG resolution gives an exact sequence,

$$M(s_1 \cdot \lambda) \oplus M(s_2 \cdot \lambda) \longrightarrow M(\lambda) \longrightarrow L(\lambda) \longrightarrow 0,$$

where $M(w \cdot \lambda), w \in W$ are Verma modules, s_1, s_2 are the two simple reflexions of W, and $L(\lambda)$ a simple, finite dimensional, module. Then, applying the functor F_∞ to it we get,

$$F_\infty(M(s_1 \cdot \lambda)) \oplus F_\infty(M(s_2 \cdot \lambda)) \longrightarrow F_\infty(M(\lambda)) \longrightarrow F_\infty(L(\lambda)) \longrightarrow 0. \tag{1}$$

The third step is to analyse theses sheaves and their support. The first thing we prove is that theses sheaves are Cohen-Macaulay sheaves, and we study their support. We should think of these sheaves as supported (on $\mathcal{E}_\infty$) around f and the other forms f', classical with same Galois representation but with a different refinement, i.e. ordering of the eigenvalues of φ. To analyse those supports, we use the local model of [3]. The local model is given by the Steinberg variety,

$$X := \{(gB, hB, N) \in \mathrm{GL}_3/B \times \mathrm{GL}_3/B \times \mathfrak{gl}_3 | g^{-1}Ng, h^{-1}Ng \in \mathfrak{b}\} = \bigcup_{w \in W} X_w,$$

where X_w are the connected components of X, given by the closure of the open where the two flags gB, hB have position w. We denote $Z = G/B \times G/B \subset X_{w_0}$ the locus where $N = 0$. The space X gives a local model for the image of $\mathcal{E}_\infty$ in $\mathcal{X}_\infty$ at the point f. The main point is that for $n = 3$, our choice of f corresponds to the point $x_0 := (B, B, 0) \in X_{w_0}$, and X_{w_0} is not Gorenstein at this point, while the other components are smooth (Z is also smooth). We can then identify the supports of the sheaves appearing in the exact sequence (1) : in the local model, it is the local ring of X_{w_0} at x_0 for $F_\infty(M(\lambda))$, and those of $X_{s_2 s_1}, X_{s_1 s_2}, Z$ at x_0 for the other three. Those last three being smooth, the corresponding sheaves are locally free. Assuming there is no such f' as in the theorem, $F_\infty(M(\lambda)$ is locally free too, and we then run an explicite computation to show this cannot happen.

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Arithmetic Transfer for GL_4

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(joint work with Qirui Li)

The linear Arithmetic Fundamental Lemma Conjecture (AFL), first formulated by Q. Li [2] and going back to ideas of W. Zhang [8], relates the following two types of quantities. Fix $n \geq 1$ and a p-adic local field F.

(1) Intersection numbers on moduli spaces of p-divisible groups (RZ spaces) for GL_{2n}. More precisely, the spaces of interest are those parametrizing 1-dimensional strict (p-divisible) O_F-modules of height $2n$ (Lubin–Tate spaces).

(2) Derivatives of orbital integrals on $GL_{2n}(F)$ for the standard Hecke function $1_{GL_{2n}(O_F)}$. The orbits in question here are for the action by left and right multiplication of $GL_n(F \times F) \times GL_n(F \times F)$ (Guo–Jacquet).

In our work [4], we generalize the linear AFL to a family of conjectures that relates (1) intersection numbers on RZ spaces for *inner forms of* GL_{2n} with (2) derivatives of orbital integrals for *parahoric and Iwahori Hecke functions*. This is parallel to the work of Rapoport–Smithling–Zhang [5, 6], who have similarly formulated analogs of the unitary AFL from [8] in cases of bad reduction.

Initial data. Consider an unramified quadratic extension E/F. Let D/F be a central simple algebra (CSA) of degree $2n$ together with an embedding $E \to D$; put $C = \mathrm{Cent}_D(E)$ which is a CSA of degree n over E. We fix compatible maximal orders $O_E \subseteq O_C \subset O_D$.

Denote by $\breve{F}$ the completion of a maximal unramified extension of F, with ring of integers $O_{\breve{F}}$ and residue field $\mathbb{F}$. Fix a special O_C-module $(\mathbb{Y}, \iota)$ over $\mathbb{F}$. That is,

$\mathbb{Y}$ is a strict O_E-module of height $2n^2$ and dimension n with a special (in the sense of Drinfeld) O_C-action $\iota : O_C \to \mathrm{End}(\mathbb{Y})$. Using the Serre tensor construction, we define $(\mathbb{X}, \kappa) := O_D \otimes_{O_C} (\mathbb{Y}, \iota)$ which is a special O_D-module.

Moduli Spaces. We consider the RZ spaces $\mathcal{M}_C$ and $\mathcal{M}_D$ for $(\mathbb{Y}, \iota)$ and $(\mathbb{X}, \kappa)$, cf. [7], which are certain formal schemes over $\mathrm{Spf}\, O_{\breve{F}}$. For example, the S-points $\mathcal{M}_D(S)$ are the set of isomorphism classes of triples (X, κ, ρ), where (X, κ) is a strict O_D-module over S and

$$\rho : \bar{S} \times_S (X, \kappa) \to \bar{S} \times_{\mathrm{Spec}\,\mathbb{F}} (\mathbb{X}, \kappa)$$

an O_D-linear quasi-isogeny over the special fiber of S. A similar definition exists for $\mathcal{M}_C$.

Example 1. If $D \cong M_{2n}(F)$, then $\mathcal{M}_D$ is a Lubin–Tate moduli space. If the Hasse invariant of D is $1/2n$ however, then $\mathcal{M}_D$ is the formal Drinfeld half space of dimension $2n$.

Intersection Numbers. In general, $\mathcal{M}_C$ and $\mathcal{M}_D$ are regular and of formal dimensions n resp. $2n$. The Serre tensor construction provides a closed immersion $\mathcal{M}_C \to \mathcal{M}_D$. Moreover, the groups $J_{\mathbb{Y}} = \mathrm{End}^0(\mathbb{Y}, \iota)^\times$ and $J_{\mathbb{X}} = \mathrm{End}^0(\mathbb{X}, \kappa)^\times$ act by composition in ρ. The intersection $\mathcal{M}_C \cap g\mathcal{M}_C$ only depends (as formal scheme) on the double coset $J_{\mathbb{Y}} \cdot g \cdot J_{\mathbb{Y}}$. It is acted on by the stabilizer

$$S_g := \{(h_1, h_2) \in J_{\mathbb{Y}} \times J_{\mathbb{Y}} \mid h_1 g = g h_2\}.$$

There is a group-theoretic definition for g to be *regular semi-simple* that ensures that $S_g = L_g^\times$ for some étale F-algebra L_g of degree n. In this case, write $L_g^\times = \Gamma \times O_{L_g}^\times$ for some subgroup Γ. The quotient $\Gamma \setminus (\mathcal{M}_C \cap g \cdot \mathcal{M}_C)$ is then a quasi-compact scheme.

Definition 1. For $g \in J_{\mathbb{X}}$ regular semi-simple as before, we define

$$\mathrm{Int}(g) := \big(\Gamma \backslash \mathcal{M}_C, \ \ \Gamma \backslash (g \cdot \mathcal{M}_C)\big)_{\Gamma \backslash \mathcal{M}_D} \in \mathbb{Z}.$$

Arithmetic Transfer. There is a notion of matching that relates the regular semi-simple elements in the two double quotients

$$\mathrm{GL}_n(F \times F) \backslash \mathrm{GL}_{2n}(F) / \mathrm{GL}_n(F \times F) \quad \text{and} \quad J_{\mathbb{Y}} \backslash J_{\mathbb{X}} / J_{\mathbb{Y}}.$$

Moreover, there is a notion of orbital integral $O(\gamma, \phi, s)$ for $\gamma \in \mathrm{GL}_{2n}(F)$ regular semi-simple, $\phi \in C_c^\infty(\mathrm{GL}_{2n}(F))$ and $s \in \mathbb{C}$. With γ and ϕ fixed, this is some polynomial in q^s and q^{-s} that only depends on the double coset of γ. We denote by $\partial O(\gamma, \phi, 0)$ its normalized derivative at $s = 0$. In [4], we also assign to D a specific parahoric subgroup $P(D) \subset GL_{2n}(F)$. Our main conjecture is the following.

Conjecture 1 (Arithmetic Transfer Conjecture, Li–M.)**.** *There exists a function $\phi_{\mathrm{corr}} \in C_c^\infty(\mathrm{GL}_{2n}(F))$ with the following property. Whenever $\gamma \in \mathrm{GL}_{2n}(F)$ and $g \in J_{\mathbb{X}}$ are two matching regular semi-simple elements, then*

$$O(\gamma, \phi_{\mathrm{corr}}, 0) + \partial O(\gamma, 1_{P(D)}, 0) = \mathrm{Int}(g).$$

The linear AFL from [2] corresponds to the case $D \cong M_{2n}(F)$. Then $P(D) = \mathrm{GL}_{2n}(O_F)$ and one expects $\phi_{\mathrm{corr}} = 0$. The linear AFL is known for $n \le 2$, see [3].

CDAs of degree 4**.** Assume from now on that D is a central division algebra of degree 4 and Hasse invariant $\lambda \in \{1/4,\ 3/4\}$. Our main result is a verification of the above conjecture in this case. Let $\mathrm{Iw}, \mathrm{Par} \subset GL_4(O_F)$ be the Iwahori and parahoric subgroups that reduce modulo π_F to

$$\mathrm{Iw} = \begin{pmatrix} * & * & * & * \\ & * & * & * \\ & & * & * \\ & & & * \end{pmatrix}, \qquad \mathrm{Par} = \begin{pmatrix} * & * & * & * \\ * & * & * & * \\ & & * & * \\ & & * & * \end{pmatrix}.$$

Then $P(D) = \mathrm{Iw}$ and we have:

Theorem 1 (Li–M. 2022)**.** *Let $\gamma \in \mathrm{GL}_4(F)$ and $g \in J_{\mathbb{X}}$ be two matching regular semi-simple elements. Then*

$$\mathrm{Int}(g) = \begin{cases} \partial O(\gamma, 1_{\mathrm{Iw}}, 0) - qO(\gamma, 1_{\mathrm{Par}}, 0) & \text{if } \lambda = 1/4 \\ \partial O(\gamma, 1_{\mathrm{Iw}}, 0) & \text{if } \lambda = 3/4. \end{cases}$$

An important ingredient in our proof is Drinfeld's isomorphism [1] which gives a linear algebra description of $\mathcal{M}_D$ if $\lambda = 1/4$. With its help, we can directly compute $\mathrm{Int}(g)$ for that space. Afterwards, we employ a mix of Cartier and display theory to relate the geometries for the two different Hasse invariants, and we are ultimately able to also fully treat the case $\lambda = 3/4$.

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Towards Bezrukavnikov via p-adic local models

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(joint work with Johannes Anschütz, Ian Gleason, Najmuddin Fakhruddin, Thomas Haines, Timo Richarz, Zhiyou Wu, Jize Yu)

In my talk, I explained the main contributions from [1, 2, 9, 13]. Local models are a linear algebra device meant to tame the singularities of arithmetic models of Shimura varieties and their systematic study was initiated by Rapoport–Zink [18]. It was noticed by Görtz [14] that they relate to affine Grassmannians, and since then the topic knew major developments due to Faltings [8], Pappas–Rapoport [16], Zhu [20] and Pappas–Zhu [17]. Recently, it became possible to work over p-adic fields thanks to Zhu [21], Bhatt–Scholze [7], and Scholze–Weinstein [19].

Let F be a finite extension of $\mathbb{Q}_p$ with ring of integers O and residue field k. We denote by C a complete algebraic closure of F. Let G be a connected reductive F-group with Iwahori O-model $\mathcal{I}$. Consider the Beilinson–Drinfeld Grassmannian $\mathrm{Gr}_{\mathcal{I}} = L\mathcal{I}/L^+\mathcal{I}$ over $\mathrm{Spd}\, O$ from [19]. Its generic fiber is the B^+_{dR}-Grassmannian Gr_G of [19] and it is stratified into L^+G-orbits over $\mathrm{Spd}\, C$ labelled by coweights μ of G_C up to conjugacy. Its special fiber is the v-sheaf attached to the Witt flag variety $\mathrm{Fl}_{\mathcal{I}}$ of [21, 7] which is stratified into $L^+\mathcal{I}$-orbits labelled by w in the Iwahori–Weyl group W.

We define the local models $M_{\mathcal{I},\mu}$ as the v-sheaf closure of $\mathrm{Gr}_{G,\leq\mu}$ inside $\mathrm{Gr}_{\mathcal{I},O_E}$ with E the reflex field of μ. These are proper p-adic kimberlites in the sense of Gleason [12], and thus admit a specialization map.

Theorem 1 ([1]). *If μ is minuscule, then $M_{\mathcal{I},\mu}$ is representable by a projective flat normal O_E-scheme with reduced special fiber. If $p > 2$ or the root system is reduced, then the scheme is Cohen–Macaulay.*

In [9] we adapted the constructions in [17] of certain candidates for scheme-theoretic local models that live inside power series affine Grassmannians and prove their Cohen–Macaulayness under a mild condition. Our strategy to prove Theorem 1 consists in comparing the fibers and the specialization map for the objects in [1, 9]. For the fibers, we use the Demazure resolution together with v-descent along semi-ample line bundles. For the specialization map, we apply convolution and the Iwasawa decomposition.

If μ is not minuscule, then representability fails already in the generic fiber. We can still say something about their geometry via étale sheaves. We consider the derived category $D_{\text{ét}}(\mathrm{Hk}_{\mathcal{I}})$ of $L^+\mathcal{I}$-equivariant étale $\bar{\mathbb{Q}}_\ell$-sheaves on $\mathrm{Gr}_{\mathcal{I}}$. Fargues–Scholze [10] studied the same object in the generic fiber setting, applying ULA sheaves and hyperbolic localization to prove geometric Satake.

Theorem 2 ([1]). *There are equivalences $c\colon D^b_{\text{ét},c}(L^+\mathcal{I}\backslash\mathrm{Fl}_{\mathcal{I}}) \to D^{\mathrm{ULA}}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I},k})$ and $Rj_*\colon D^{\mathrm{ULA}}_{\text{ét}}(\mathrm{Hk}_{G,C}) \to D^{\mathrm{ULA}}_{\text{ét}}(L^+\mathcal{I}\backslash\mathrm{Hk}_{\mathcal{I},O_C})$ giving rise to a nearby cycles functor $R\Psi := c^{-1}i^*Rj_*\colon D^{\mathrm{ULA}}_{\text{ét}}(\mathrm{Hk}_{G,C}) \to D^b_{\text{ét},c}(L^+\mathcal{I}\backslash\mathrm{Fl}_{\mathcal{I},\bar{k}})$.*

These results are proved by applying constant term functors in order to reduce to sheaves on 0-dimensional spaces. The composition $Z = R\Psi \circ \mathrm{Sat}$ is a p-adic counterpart to Gaitsgory's central functor [11]. As a corollary, we are able to show that the special fiber of $M_{\mathcal{I},\mu}$ is the v-sheaf attached to the μ-admissible locus $A_{\mathcal{I},\mu}$ by calculating the support of $Z(V_\mu)$ where V_μ is the irreducible $\hat{G}$-representation with highest weight μ. Here, the perfect scheme $A_{\mathcal{I},\mu}$ is the union of all Schubert varieties bounded by the translations associated with V_μ.

We haven't yet answered the question of whether $R\Psi$ preserves perversity. This holds at the level of schemes due to Artin vanishing, see Beilinson–Bernstein–Deligne–Gabber [5], but it is not available for v-sheaves. To overcome it, we construct in [2] the Wakimoto functor $J : \operatorname{Rep} \hat{T} \to D^{\mathrm{ULA}}_{\text{ét}}(\mathrm{Hk}_{\mathcal{I},\bar{k}})$ following Arkhipov–Bezrukavnikov [3] as the monoidal extension of standard sheaves on dominant weights.

Theorem 3 ([2]). *$Z(V) = R\Psi \circ \mathrm{Sat}(V)$ is a central convolution-exact perverse sheaf admitting a filtration with graded isomorphic to $J(V)$.*

Our proof differs from [3] in that we prove all the statements at once by deriving J, and using centrality of $Z(V)$ to write it as a successive extension of Wakimoto complexes. The perversity follows then by applying constant terms to recover each term in the extension.

Furthermore, this has consequences for the geometry of $M_{\mathcal{I},\mu}$. We say that it is unibranch if the tubes over a closed point of the reduction are geometrically connected. This notion is a topological avatar of normality.

Theorem 4 ([13]). *$M_{\mathcal{I},\mu}$ is unibranch.*

We prove this by first performing a cohomological calculation in codimension 1 strata of $A_{\mathcal{I},\mu}$ using the Wakimoto filtration. Then, we show that $A_{\mathcal{I},\mu}$ satisfies the S_2 condition of Serre by translating into a statement about Bruhat graphs and reducing to equicharacteristic, which is known by the normality theorem of [20]. With a bit more work, we even reproved [20] by replacing φ-splittings with a loop rotation reasoning due to Le–Le Hung–Levin–Morra [15].

Returning to the theme of [3], we now assume G is split and show in [2] that the functor $Z \times J$ extends to a functor $F\colon D^b_{\mathrm{coh}}(\hat{G}\backslash\tilde{\mathcal{N}}_{\hat{\mathfrak{g}}}) \to D^b_{\text{ét},c}(L^+\mathcal{I}\backslash\mathrm{Fl}_{\mathcal{I},\bar{k}})$. Fixing a Whittaker datum, we can consider the derived category $D^b_{\text{ét},c}(\mathcal{IW}\backslash\mathrm{Fl}_{\mathcal{I},\bar{k}})$ of Iwahori–Whittaker equivariant sheaves (despite the suggestive notation, there's no underlying stack!) equipped with an averaging map $\mathrm{av}_{\mathcal{IW}}$.

Conjecture 5 ([2]). *The composition $\mathrm{av}_{\mathcal{IW}} \circ F$ is an equivalence.*

There are two main obstacles preventing us from replicating the proof in [3]. First, they use loop rotation at crucial steps to ensure the vanishing of certain Ext groups, which is not available in the p-adic setting. Secondly, they use Gabber's proof of the local weight-monodromy conjecture in the function field case, see Beilinson–Bernstein [4]. We have some ideas on how to surpass these difficulties in most cases, which we'll report on in [2]. If we succeed, it should also be possible to generalize the full equivalence due to Bezrukavnikov [6].

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