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Relative to the Anti-Canonical
Divisor and their Refinement

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Real enumerative invariants relative to the anti-canonical divisor and their refinement

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Abstract

We introduce new invariants of the projective plane (and, more generally, of certain toric surfaces) that arise from appropriate enumeration of real elliptic curves. These invariants admit a refinement (according to the quantum index) similar to the one introduced by Grigory Mikhalkin in the rational case. We also construct tropical counterparts of the refined elliptic invariants under consideration and establish a tropical algorithm allowing one to compute, *via* a suitable version of the correspondence theorem, the above invariants.

MSC-2010 classification: Primary 14N10, Secondary 14J26, 14P05, 14T90

1 Introduction

Refined enumerative geometry, initiated in [2, 10], became one of the central topics in enumerative geometry with important links to closed and open Gromov-Witten invariants and to Donaldson-Thomas invariants. In a big part of known examples, refined invariants appear as one-parameter deformations of complex enumerative invariants (see, for example, [2, 6, 9, 10]). In his ground-breaking paper [22] G. Mikhalkin proposed a refined invariant provided by enumeration of real rational curves and related this invariant to the refined tropical invariants of F. Block and L. Göttsche [2]. Namely, he introduced an integer-valued *quantum index* for real algebraic curves in toric surfaces. To have a quantum index, a real curve should satisfy certain assumptions: it has to intersect toric divisors only at real points and to be irreducible and *separating*; the latter condition means that, in the complex point set of the normalization of the curve, the complement of the real part is disconnected, *i.e.*, formed by two halves exchanged by the complex conjugation (in fact, the quantum index is associated to a half of a separating real curve, while the other half has the opposite quantum index; for detailed definitions, see Section 2.2). Mikhalkin [22] showed that the Welschinger-type enumeration of real rational curves (*cf.* [33]) in a given divisor class and with a given quantum index can be directly related to the numerator of a Block-Göttsche refined tropical invariant (represented as a fraction with the standard denominator).

The main goal of the present paper is to extend Mikhalkin's results to the case of elliptic curves. We follow the ideas of [29] and choose constraints so that every counted real elliptic curve appears to be a *maximal* one (*i.e.*, it has two global real branches), and hence is separating. More precisely, given a toric surface with the tautological real structure and a very ample divisor class, we fix maximally many real points on the toric boundary of the positive quadrant, where elliptic curves from the given linear system must be tangent to toric divisors with prescribed even intersection multiplicities, and we fix one more real point inside a non-positive quadrant as an extra constraint. There are finitely many real elliptic curves matching the constraints and all these curves are separating. Their halves have quantum index, and we equip each curve with a certain Welschinger-type sign.

The **first main result** of the paper is as follows. For some toric surfaces (including the projective plane), we prove that the signed enumeration of (halves of) real elliptic curves that match given constraints, belong to a given linear system, and have a prescribed quantum index does not depend on the choice of a (generic) position of the constraints (the precise statement can be found in Section 2.3, Theorem 2.11). The resulting invariants are said to be *refined elliptic*. In particular, we get new real enumerative invariants (without prescribing values for quantum index) in genus one.

The **second main result** of the paper concerns tropical counterparts of the above invariants. We introduce tropical invariants arising from enumeration of certain elliptic plane tropical curves counted with multiplicities depending on one parameter. Using an appropriate version of Mikhalkin's correspondence theorem (see [21] and Theorem 3.21 in Section 3.6), we prove that these tropical invariants give rise to generating functions for refined elliptic invariants described above.

The introduced tropical elliptic invariants have two interesting features. The multiplicity of each tropical curve under enumeration is not a product of multiplicities of vertices (contrary to many previously considered tropical invariants); see Theorem 4.28. Another particularity of these invariants is their *semi-local invariance*; see Section 4.5.

The paper is organized as follows. In Section 2, we define refined invariants arising in enumeration of real rational curves (slightly generalizing Mikhalkin's refined rational invariants) and of real elliptic curves in toric surfaces with constraints described above. In Section 3, we present a version of Mikhalkin's correspondence theorem adapted to our purposes. Section 4 is devoted to the tropical counterparts of refined elliptic invariants; it contains, in particular, a tropical formula for these invariants. In Section 5, we suggest a combinatorial algorithm for computation of the tropical invariants introduced in Section 4 (and, thus, of refined elliptic invariants). The algorithm is similar to the one in [3], used for a tropical calculation of Mikhalkin's refined rational invariants.

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2 Real refined invariants of toric surfaces

2.1 Preparation

2.1.1 Convex lattice polygons and real toric surfaces

Consider the lattice \mathbb{Z}^2 and its ambient plane $\mathbb{R}^2 = \mathbb{Z}^2 \otimes \mathbb{R}$. Let $P \subset \mathbb{R}^2$ be a non-degenerate convex lattice polygon. For a lattice segment $\sigma \subset \mathbb{R}^2$ (respectively, a vector $\mathbf{a} \in \mathbb{Z}^2 \setminus \{(0,0)\}$), denote by $\|\sigma\|_{\mathbb{Z}}$ (respectively, $\|\mathbf{a}\|_{\mathbb{Z}}$) its *lattice length*, *i.e.*, the ratio of the Euclidean length and the minimal length of a non-zero parallel integral vector.

Denote by $\text{Tor}(P)$ the complex toric surface associated with P ¹. Let $\text{Tor}(P)^\times \simeq (\mathbb{C}^2)^\times$ (respectively, $\text{Tor}(\partial P)$) be the dense orbit (respectively, the union of all toric divisors) of $\text{Tor}(P)$. The toric surface $\text{Tor}(P)$ has the tautological real structure, and the real part $\text{Tor}_{\mathbb{R}}(P)$ of $\text{Tor}(P)$ contains the positive quadrant $\text{Tor}_{\mathbb{R}}(P)^+ \simeq (\mathbb{R}_{>0})^2$. The closure $\text{Tor}_{\mathbb{R}}^+(P)$ of $\text{Tor}_{\mathbb{R}}(P)^+$ (with respect to the Euclidean topology) is diffeomorphically taken onto P by the moment map. We pull back the standard orientation and the metric of $P \subset \mathbb{R}^2$ to $\text{Tor}_{\mathbb{R}}^+(P)$ and induce an orientation and a metric on the boundary $\partial \text{Tor}_{\mathbb{R}}^+(P)$ of $\text{Tor}_{\mathbb{R}}^+(P)$; in particular, we get a cyclic order on the sides of P ; we call this order *positive*. Denote by \mathcal{L}_P the tautological line bundle over $\text{Tor}(P)$; the global sections of \mathcal{L}_P are spanned by the monomials z^ω , $z = (z_1, z_2)$, $\omega \in P \cap \mathbb{Z}^2$.

For each edge $\sigma \subset \partial P$, we consider the toric curve $\text{Tor}(\sigma)$, its dense orbit $\text{Tor}(\sigma)^\times$, the real part $\text{Tor}_{\mathbb{R}}(\sigma)$ of $\text{Tor}(\sigma)$, and the positive half-axis $\text{Tor}_{\mathbb{R}}(\sigma)^+ \subset \text{Tor}(\sigma)^\times \cap \text{Tor}_{\mathbb{R}}(\sigma)$. The closure $\text{Tor}_{\mathbb{R}}^+(\sigma)$ of $\text{Tor}_{\mathbb{R}}(\sigma)^+$ coincides with $\text{Tor}(\sigma) \cap \partial \text{Tor}_{\mathbb{R}}^+(P)$.

2.1.2 Toric degree and Menelaus condition

A multi-set $\Delta \subset \mathbb{Z}^2 \setminus \{(0,0)\}$ is said to be

- *balanced* if the sum of vectors in Δ is equal to 0,
- *non-degenerate* if the vectors of Δ span \mathbb{R}^2 ,

¹We always skip the subindex \mathbb{C} when speaking about complex surfaces, while for toric surfaces over other fields \mathbb{F} , we use the notation $\text{Tor}_{\mathbb{F}}$.

- even if $\Delta \subset (2\mathbb{Z})^2$.

A balanced non-degenerate multi-set $\Delta \subset \mathbb{Z}^2 \setminus \{(0,0)\}$ is called a (*toric*) *degree*. Any vector \mathbf{a} in a degree Δ is of the form $\mathbf{a} = \|\mathbf{a}\|_{\mathbb{Z}} \cdot u$, where u a primitive integral vector, that is, a non-zero integral vector whose coordinates are relatively prime.

Let $\Delta \subset \mathbb{Z}^2 \setminus \{(0,0)\}$ be a non-degenerate balanced multi-set. For each vector $\mathbf{a} \in \Delta$, denote by $\check{\mathbf{a}}$ the vector obtained from \mathbf{a} by the counter-clockwise rotation by $\pi/2$. The vectors $\check{\mathbf{a}}$ can be attached to each other so that the next vector starts at the end of the preceding one in order to form a simple broken line bounding a convex lattice polygon. The latter polygon is denoted by P_{Δ} ; it is determined by Δ up to translation. For a convex lattice polygon δ we set $\mathcal{A}(\delta)$ to be the Euclidean area and $\mathcal{I}(\delta)$ to be the number of integer points in the interior of δ . In the case of P_{Δ} , we shortly write $\mathcal{A}(\Delta)$ and $\mathcal{I}(\Delta)$.

The elements of Δ are denoted by

$$\mathbf{a}_i^{\sigma}, \quad i = 1, \dots, n^{\sigma},$$

where σ ranges over the set P_{Δ}^1 of sides of the polygon P_{Δ} , and n^{σ} is the number of vectors in Δ that are outer normals to σ .

From now on, assume that Δ is even; for each $\sigma \in P_{\Delta}^1$ and each $i = 1, \dots, n^{\sigma}$, put $2k_i^{\sigma} = \|\mathbf{a}_i^{\sigma}\|_{\mathbb{Z}}$. The notation $\frac{1}{2}\Delta$ stands for the degree obtained from Δ by dividing all its vectors by 2.

Definition 2.1 For each $\sigma \in P_{\Delta}^1$, consider a sequence \mathbf{w}^{σ} of n^{σ} points in $\text{Tor}(\sigma)^{\times}$, and denote by \mathbf{w} the double index sequence (the upper index being $\sigma \in P_{\Delta}^1$ and the lower index being $i = 1, \dots, n^{\sigma}$) formed by the sequences \mathbf{w}^{σ} . We say that the sequence \mathbf{w} satisfies the Menelaus Δ -condition (cf. [22, Section 5.1]) if there exists a curve $C \in |\mathcal{L}_{P_{\Delta}}|$ such that C does not contain toric divisors as components and, for each $\sigma \in P_{\Delta}^1$, the scheme-theoretic intersection of C and $\text{Tor}(\sigma)$ coincides with $\sum_{i=1}^{n^{\sigma}} 2k_i^{\sigma} w_i^{\sigma}$.

Lemma 2.2 The sequences \mathbf{w} satisfying the Menelaus Δ -condition form an algebraic hypersurface $M(\Delta) \subset \prod_{\sigma \in P_{\Delta}^1} \text{Tor}(\sigma)^{n^{\sigma}}$.

Proof. We present an explicit equation of $M(\Delta)$. Consider a linear functional $\lambda : \mathbb{R}^2 \rightarrow \mathbb{R}$ which is injective on \mathbb{Z}^2 . The maximal and the minimal points of λ on P_{Δ} divide the boundary ∂P_{Δ} of P_{Δ} into two broken lines \mathfrak{P}' , \mathfrak{P}'' , and the maximal and minimal points of λ on each side σ define its orientation. An automorphism of \mathbb{Z}^2 , which takes an oriented side σ of P onto the naturally oriented segment $[0, \|\sigma\|_{\mathbb{Z}}]$ of the first axis of \mathbb{R}^2 , defines an isomorphism of $\text{Tor}(\sigma)^{\times}$ with \mathbb{C}^{\times} . Denote by $\xi^{\sigma} = (\xi_i^{\sigma})_{i=1}^{n^{\sigma}}$ the sequence of images in \mathbb{C}^{\times} of the points of \mathbf{w}^{σ} . Then, $M(\Delta)$ is given by the equation

$$\prod_{\sigma \in \mathfrak{P}'} \prod_{i=1}^{n^{\sigma}} (\xi_i^{\sigma})^{2k_i^{\sigma}} = \prod_{\sigma \in \mathfrak{P}''} \prod_{i=1}^{n^{\sigma}} (\xi_i^{\sigma})^{2k_i^{\sigma}} .$$

□

The hypersurface $M(\Delta)$ is said to be *Menelaus*. Notice that $M(\Delta)$ is reducible. More precisely, $M(\Delta)$ splits into $2k_0$ components, where

$$k_0 = \gcd\{k_i^\sigma : i = 1, \dots, n^\sigma, \sigma \subset \partial P_\Delta\}.$$

Denote by $M(\Delta)_{red}$ the component given by the equation

$$\prod_{\sigma \subset \mathfrak{P}'} \prod_{i=1}^{n^\sigma} (\xi_i^\sigma)^{k_i^\sigma/k_0} = \prod_{\sigma \subset \mathfrak{P}''} \prod_{i=1}^{n^\sigma} (\xi_i^\sigma)^{k_i^\sigma/k_0}. \quad (1)$$

Note that the above isomorphism $\text{Tor}(\sigma)^\times \simeq \mathbb{C}^\times$ takes $\text{Tor}_\mathbb{R}(\sigma)^+$ onto $\mathbb{R}_{>0} \subset \mathbb{C}^\times$. For any $\tau > 0$, denote by $M_\mathbb{R}^\tau(\Delta)$ the part of $M(\Delta)_{red}$ specified by the condition that, for each $\sigma \in P_\Delta^1$, all points of the subsequence \mathbf{w}^σ belong to $\text{Tor}_\mathbb{R}^+(\sigma)$ and lie at the distance $\geq \tau$ from the endpoints of $\text{Tor}_\mathbb{R}^+(\sigma)$.

Lemma 2.3 *For a sufficiently small $\tau > 0$, the (metric) closure $\overline{M}_\mathbb{R}^\tau(\Delta) \subset \prod_{\sigma \subset \partial P} (\text{Tor}_\mathbb{R}^+(\sigma))^{n^\sigma}$ is diffeomorphic to a convex polytope.*

Proof. Since $M_\mathbb{R}^\tau(\Delta)$ is given by equation (1) with positive variables ξ_i^σ satisfying restrictions of the form $0 < \text{const}_1 \leq \xi_i^\sigma \leq \text{const}_2 < \infty$, the coordinate-wise logarithm takes $\overline{M}_\mathbb{R}^\tau(\Delta)$ onto a hyperplane section of a convex polytope. □

2.1.3 Curves on toric surfaces

Given a morphism $\Phi : \widehat{C} \rightarrow \text{Tor}(P_\Delta)$ of a curve \widehat{C} to $\text{Tor}(P_\Delta)$, we denote by

- $\Phi_*(\widehat{C})$ the scheme-theoretic push-forward, *i.e.*, the one-dimensional part of the image, whose components are taken with the corresponding multiplicities;
- $\Phi(\widehat{C})$ the reduced model of $\Phi_*(\widehat{C})$, where all components are taken with multiplicity one;
- $\Phi^*(D)$, where $D \subset \text{Tor}(P_\Delta)$ is a divisor intersecting $\Phi(\widehat{C})$ in a finitely many points, the divisor on \widehat{C} which is the pull-back of the scheme-theoretic intersection $D \cap \Phi_*(\widehat{C})$.

Denote by $\mathcal{M}_{g,m}(\text{Tor}(P_\Delta), \mathcal{L}_{P_\Delta})$ the space of isomorphism classes of maps $\Phi : (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)$, where \widehat{C} is a smooth curve of genus g and $\mathbf{p} \subset \widehat{C}$ is a sequence of m distinct points, such that $\Phi_*(\widehat{C}) \in |\mathcal{L}_{P_\Delta}|$. Correspondingly, by $\overline{\mathcal{M}}_{g,m}(\text{Tor}(P_\Delta), \mathcal{L}_{P_\Delta})$ we denote the compactification of the above moduli space obtained by adding isomorphism classes of stable maps $\Phi : (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)$, where \widehat{C} is a connected nodal curve of arithmetic genus g and $\mathbf{p} \subset \widehat{C} \setminus \text{Sing}(\widehat{C})$ is a sequence of m distinct points, such that $\Phi_*(\widehat{C}) \in |\mathcal{L}_{P_\Delta}|$. In what follows, we work with certain

subspaces of these moduli spaces specified for genus $g = 0$ in Section 2.2, and for genus $g = 1$ in Section 2.3. For any subset $\mathcal{M} \subset \mathcal{M}_{g,m}(\mathrm{Tor}(P_\Delta), \mathcal{L}_{P_\Delta})$, denote by $\overline{\mathcal{M}}$ the closure of \mathcal{M} in the compactified moduli space $\overline{\mathcal{M}}_{g,m}(\mathrm{Tor}(P_\Delta), \mathcal{L}_{P_\Delta})$.

We denote by $(C_1 \cdot C_2)_z$ the intersection multiplicity of curves $C_1, C_2 \subset \mathrm{Tor}(P_\Delta)$ at a smooth point z of $\mathrm{Tor}(P_\Delta)$. By $C_1 C_2$ we mean the total intersection multiplicity of the curves $C_1, C_2 \subset \mathrm{Tor}(P_\Delta)$. Given a local branch B of a curve germ $(C, z) \subset \mathrm{Tor}(P_\Delta)$, we denote by $\mathrm{ord} B$ the intersection multiplicity of B with a generic smooth curve through z .

Recall that deformations of a morphism $\Phi : \widehat{C} \rightarrow \mathrm{Tor}(P_\Delta)$ of a smooth curve \widehat{C} are encoded by the normal sheaf on \widehat{C} (see, for instance, [11]):

$$\mathcal{N}_\Phi := \Phi^* \mathcal{T} \mathrm{Tor}(P_\Delta) / \mathcal{T} \widehat{C}$$

(where \mathcal{T} denotes the tangent bundle). In the case of an immersion, \mathcal{N}_Φ is a line bundle of degree

$$\mathrm{deg} \mathcal{N}_\Phi = c_1(\mathrm{Tor}(P_\Delta)) c_1(\mathcal{L}_{P_\Delta}) - 2. \quad (2)$$

Let $\Delta = (\mathbf{a}_i^\sigma, i = 1, \dots, n^\sigma, \sigma \in P_\Delta^1)$ be a toric degree as introduced in Section 2.1.2. Fix a non-negative integer $g \leq \mathcal{I}(\Delta)$ and a non-negative integer $n(\sigma) \leq n^\sigma$ for each $\sigma \in P_\Delta^1$ such that $\sum_{\sigma \in P_\Delta^1} n(\sigma) < \sum_{\sigma \in P_\Delta^1} n^\sigma$. Put $n_\partial = \sum_{\sigma \in P_\Delta^1} n(\sigma)$ and $n_{\mathrm{in}} = n - n_\partial$, where

$$n = \sum_{\sigma \in P_\Delta^1} n^\sigma + g - 1.$$

Choose a sequence \mathbf{w} of n distinct points in $\mathrm{Tor}(P_\Delta)$ splitting into two subsequences:

- \mathbf{w}_∂ consisting of n_∂ points in general position on $\mathrm{Tor}(\partial P_\Delta)$ so that, for each side $\sigma \in P_\Delta^1$, exactly $n(\sigma)$ points lie on $\mathrm{Tor}(\sigma)$;
- \mathbf{w}_{in} consisting of n_{in} points in general position in $(\mathbb{C}^\times)^2 \subset \mathrm{Tor}(P_\Delta)$.

Introduce the subset $\mathcal{M}_{g,n}(\Delta, \mathbf{w}) \subset \mathcal{M}_{g,n}(\mathrm{Tor}(P_\Delta), \mathcal{L}_{P_\Delta})$ consisting of the elements $\left[\Phi : (\widehat{C}, \mathbf{p}) \rightarrow \mathrm{Tor}(P_\Delta) \right]$ such that

- the sequence \mathbf{p} is split into disjoint subsequences \mathbf{p}_∂ and \mathbf{p}_{in} containing n_∂ and n_{in} points, respectively, and $\Phi(\mathbf{p}_\partial) = \mathbf{w}_\partial$, $\Phi(\mathbf{p}_{\mathrm{in}}) = \mathbf{w}_{\mathrm{in}}$;
- for each $\sigma \in P_\Delta^1$, one has

$$\Phi^*(\mathrm{Tor}_\mathbb{K}(\sigma)) = 2 \sum_{i=1}^{n^\sigma} k_i^\sigma p_i^\sigma \in \mathrm{Div}(\widehat{C}),$$

where $\Phi(p_i^\sigma) = w_i^\sigma$ for all $p_i^\sigma \in \mathbf{p}_\partial$ and $w_i^\sigma \in \mathbf{w}_\partial$, $\sigma \in P_\Delta^1$, $1 \leq i \leq n(\sigma)$, and p_i^σ , $n(\sigma) < i \leq n^\sigma$, are any points of \widehat{C} .

The proof of the following lemma is found in Section 3 after Theorem 3.9.

Lemma 2.4 *Suppose that the sequence \mathbf{w} is in general position subject to the location with respect to the toric divisors as indicated above. Then, the space $\mathcal{M}_{g,n}(\Delta, \mathbf{w})$ is finite. Moreover, for each element $[\Phi : (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)] \in \mathcal{M}_{g,n}(\Delta, \mathbf{w})$, the map Φ takes \widehat{C} birationally onto an immersed curve $C = \Phi(\widehat{C})$ that, for each $\sigma \in P_\Delta^1$, intersects the toric divisor $\text{Tor}(\sigma)$ at n^σ distinct points and C is smooth at each of these intersection points.*

A map $\Phi : (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)$ such that $[\Phi : (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)] \in \mathcal{M}_{g,n}(\Delta, \mathbf{w})$ is said to be *real* if

- (i) the sequence \mathbf{w} is invariant with respect to the tautological real structure on $\text{Tor}(P_\Delta)$,
- (ii) $(\widehat{C}, \mathbf{p})$ is equipped with a real structure, and
- (iii) $\Phi : (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)$ commutes with the real structures in the source and in the target.

The set of equivalence classes of such real maps $\Phi : (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)$ taken up to equivariant isomorphism is denoted by $\mathcal{M}_{g,n}^{\mathbb{R}}(\Delta, \mathbf{w})$. We say that an element $[\Phi : (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)] \in \mathcal{M}_{g,n}^{\mathbb{R}}(\Delta, \mathbf{w})$ is *separating* if the complement in \widehat{C} to the real points set $\mathbb{R}\widehat{C}$ is disconnected, *i.e.*, consists of two connected components. The choice of one of these halves induces the so-called *complex orientation* on $\mathbb{R}\widehat{C}$ as well as on $\Phi(\mathbb{R}\widehat{C})$ (in case of Φ birational onto its image). Denote by $\overrightarrow{\mathcal{M}}_{g,n}^{\mathbb{R}}(\Delta, \mathbf{w})$ the set of separating elements $[\Phi : (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)] \in \mathcal{M}_{g,n}^{\mathbb{R}}(\Delta, \mathbf{w})$ equipped with a choice of a half \widehat{C}_+ of $\widehat{C} \setminus \mathbb{R}\widehat{C}$.

Following [22], we assign to each element $\xi = ([\Phi : (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)], \widehat{C}_+) \in \overrightarrow{\mathcal{M}}_{g,n}^{\mathbb{R}}(\Delta, \mathbf{w})$ its *quantum index*

$$\text{QI}(\xi) = \frac{1}{\pi^2} \int_{\widehat{C}_+} \Phi^* \left(\frac{dx_1 \wedge dx_2}{x_1 x_2} \right), \quad x_1 = |z_1|, \quad x_2 = |z_2|, \quad (3)$$

with z_1, z_2 coordinates in the torus $(\mathbb{C}^\times)^2 \simeq \text{Tor}(P_\Delta)^\times$ such that the form $dx_1 \wedge dx_2$ agrees with the orientation of $\text{Tor}_{\mathbb{R}}^+(P_\Delta)$ defined in Section 2.1.1. By [22, Theorem 3.1], if all intersection points of $\Phi(\widehat{C})$ with the toric divisors are real, then $\text{QI}(\xi) \in \frac{1}{2}\mathbb{Z}$ and $|\text{QI}(\xi)| \leq \mathcal{A}(\Delta)$.

2.2 Refined rational invariants

In the notation of Section 2.1.3, assume that

$$g = 0, \quad n_{in} = 0, \quad n_\partial = n = \sum_{\sigma \in P_\Delta^1} n^\sigma - 1, \quad \text{and} \quad \mathbf{w} = \mathbf{w}_\partial \subset \partial \text{Tor}_{\mathbb{R}}^+(P_\Delta).$$

In particular, for all $\sigma \in P_\Delta^1$ but one, $n(\sigma) = n^\sigma$, and for the remaining edge τ , we have $n(\tau) = n^\tau - 1$. Note that for a given \mathbf{w} , there exists a unique point $w_{n^\tau}^\tau \in \partial\mathrm{Tor}_\mathbb{R}^+(P_\Delta)$ such that the sequence $\widehat{\mathbf{w}} := \mathbf{w} \cup \{w_{n^\tau}^\tau\}$ belongs to the Menelaus hypersurface $M(\Delta)$.

Denote by M_0 the subset of $M(\Delta)$ formed by the above sequences $\widehat{\mathbf{w}}$ such that $\mathcal{M}_{0,n}(\Delta, \mathbf{w})$ satisfies the conclusions of Lemma 2.4. The closure

$$\overline{\bigcup_{\widehat{\mathbf{w}} \in M_0} \mathcal{M}_{0,n}^\mathbb{R}(\Delta, \mathbf{w})}$$

is naturally fibered over the space of sequences $\mathbf{w} \subset \partial\mathrm{Tor}_\mathbb{R}^+(P_\Delta)$, and for the sake of notation, we denote the fibers by $\overline{\mathcal{M}}_{0,n}^\mathbb{R}(\Delta, \mathbf{w})$.

Consider the subset $\mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}) \subset \mathcal{M}_{0,n}^\mathbb{R}(\Delta, \mathbf{w})$, formed by the elements $[\Phi : (\mathbb{P}^1, \mathbf{p}) \rightarrow \mathrm{Tor}(P_\Delta)]$ such that $\Phi(\mathbb{R}\mathbb{P}^1) \subset \mathrm{Tor}_\mathbb{R}^+(P_\Delta)$ ^{II}. For every $[\Phi : (\mathbb{P}^1, \mathbf{p}) \rightarrow \mathrm{Tor}(P_\Delta)] \in \mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w})$, the curve $C = \Phi(\mathbb{P}^1)$ intersects each toric divisor $\mathrm{Tor}(\sigma)$ in points w_i^σ , $1 \leq i \leq n^\sigma$. Hence, each element ξ of the set

$$\begin{aligned} \overline{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}) = \{ & ([\Phi : (\mathbb{P}^1, \mathbf{p}) \rightarrow \mathrm{Tor}(P_\Delta)], \mathbb{P}_+^1) \in \overline{\mathcal{M}}_{0,n}^\mathbb{R}(\Delta, \mathbf{w}) : \\ & [\Phi : (\mathbb{P}^1, \mathbf{p}) \rightarrow \mathrm{Tor}(P_\Delta)] \in \mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}) \} \end{aligned}$$

possesses a quantum index $\mathrm{QI}(\xi) \in \frac{1}{2}\mathbb{Z}$.

Furthermore, if $\widehat{\mathbf{w}} \in M_0$, then (see [16, Section 1.1]), there is a well-defined *Welschinger sign*

$$W_0(\xi) = (-1)^{e_+(C)} \cdot \prod_{\substack{\sigma \in P_\Delta^1, 1 \leq i \leq n^\sigma \\ k_i^\sigma \equiv 0 \pmod{2}}} \varepsilon(\xi, w_i^\sigma), \quad (4)$$

where $e_+(C)$ is the number of elliptic nodes of $C = \Phi(\mathbb{P}^1)$ in the positive quadrant $\mathrm{Tor}_\mathbb{R}^+(P_\Delta)$ that arise from a real nodal equigeneric deformation of all singular points of C in $\mathrm{Tor}_\mathbb{R}^+(P_\Delta)$, and $\varepsilon(\xi, w_i^\sigma)$ equals 1 or -1 according as the complex orientation of $\mathbb{R}C = \Phi(\mathbb{R}\mathbb{P}^1)$ at w_i^σ agrees or not with the fixed orientation of $\partial\mathrm{Tor}_\mathbb{R}^+(P_\Delta)$.

For each $\kappa \in \frac{1}{2}\mathbb{Z}$ such that $|\kappa| \leq \mathcal{A}(\Delta)$, put

$$W_0^\kappa(\Delta, \mathbf{w}) = \sum_{\substack{\xi \in \overline{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}) \\ \mathrm{QI}(\xi) = \kappa}} W_0(\xi).$$

Theorem 2.5 *For each $\kappa \in \frac{1}{2}\mathbb{Z}$ such that $|\kappa| \leq \mathcal{A}(\Delta)$, the value $W_0^\kappa(\Delta, \mathbf{w})$ does not depend on the choice of a generic sequence $\mathbf{w} \subset \mathrm{Tor}_\mathbb{R}^+(P_\Delta)$.*

We prove this theorem in Section 2.4.

Remark 2.6 *Theorem 2.5 slightly generalizes the statement of [22, Theorem 5].*

^{II}Note that $\mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}) \subsetneq \mathcal{M}_{0,n}^\mathbb{R}(\Delta, \mathbf{w})$ when there exists a connected component of $\mathrm{Tor}_\mathbb{R}(P_\Delta)^\times \setminus \mathrm{Tor}_\mathbb{R}(P_\Delta)^+$ such that the boundary of the component coincides with $\partial\mathrm{Tor}_\mathbb{R}^+(P_\Delta)$.

Definition 2.7 *Following Theorem 2.5, we introduce the numerical invariants*

$$W_0^\kappa(\Delta) := (-1)^{\mathcal{I}(\frac{1}{2}\Delta)} (-1)^{(\mathcal{A}(\Delta) - \kappa)/4} W_0^\kappa(\Delta, \mathbf{w}), \quad \text{where } \kappa \in \frac{1}{2}\mathbb{Z}, |\kappa| \leq \mathcal{A}(\Delta),$$

and the refined invariant

$$\mathfrak{G}_0(\Delta) := \sum_{\kappa \in \frac{1}{2}\mathbb{Z}, |\kappa| \leq \mathcal{A}(\Delta)} W_0^\kappa(\Delta) q^\kappa.$$

Theorem 2.8 *Invariants $W_0^\kappa(\Delta)$ vanish for all $\kappa \notin 2\mathbb{Z}$ and for all $\kappa \in 2\mathbb{Z}$ such that $\kappa \not\equiv \mathcal{A}(\Delta) \pmod{4}$.*

The proof is found in Section 4 after Theorem 4.28. The invariant $\mathfrak{G}_0(\Delta)$ can be computed via the formula in [3, Theorem 3.4].

2.3 Refined elliptic invariants

In the notation of Section 2.1.3, assume that

$$g = 1, \quad n_{in} = 1, \quad n_\partial = \sum_{\sigma \in P_\Delta^1} n^\sigma - 1, \quad \mathbf{w}_\partial \subset \partial \text{Tor}_\mathbb{R}^+(P_\Delta), \quad \mathbf{w}_{in} = \{w_0\} \subset \mathfrak{Q},$$

where \mathfrak{Q} is an open quadrant in $\text{Tor}_\mathbb{R}(P_\Delta)^\times \setminus \text{Tor}_\mathbb{R}^+(P_\Delta)$. As in the preceding section, for all $\sigma \in P_\Delta^1$ but one, $n(\sigma) = n^\sigma$, and for the remaining edge τ , we have $n(\tau) = n^\tau - 1$. Note that for a given \mathbf{w} , there exists a unique point $w_{n^\tau}^\tau \in \partial \text{Tor}_\mathbb{R}^+(P_\Delta)$ such that the sequence $\widehat{\mathbf{w}}_\partial := \mathbf{w}_\partial \cup \{w_{n^\tau}^\tau\}$ belongs to the Menelaus hypersurface $M(\Delta)$.

Denote by M_1 the set of pairs $(\widehat{\mathbf{w}}_\partial, w_0) \in M(\Delta) \times \mathfrak{Q}$ such that the set $\mathcal{M}_{1,n}(\Delta, \mathbf{w})$ satisfies the conclusions of Lemma 2.4. The closure

$$\overline{\bigcup_{\widehat{\mathbf{w}} \in M_1} \mathcal{M}_{1,n}^\mathbb{R}(\Delta, \mathbf{w})}$$

is naturally fibered over the space of sequences $\mathbf{w} \subset \partial \text{Tor}_\mathbb{R}^+(P_\Delta) \times \mathfrak{Q}$, and we denote the fibers by $\overline{\mathcal{M}}_{1,n}^\mathbb{R}(\Delta, \mathbf{w})$.

Consider the subset $\mathcal{M}_{1,n}^{\mathbb{R},+}(\Delta, \mathbf{w}) \subset \mathcal{M}_{1,n}^\mathbb{R}(\Delta, \mathbf{w})$, formed by the elements

$$[\Phi : (\mathbb{E}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)], \quad \text{where } \mathbb{E} \text{ is a smooth elliptic curve,}$$

defining real curves $C = \Phi(\mathbb{E}) \subset \text{Tor}(P_\Delta)$ that have a one-dimensional real branch in $\text{Tor}_\mathbb{R}^+(P_\Delta)$ and that intersect the toric divisor $\text{Tor}(\sigma)$ in w_i^σ , $1 \leq i \leq n^\sigma$, for all $\sigma \in P_\Delta^1$. It follows from Lemma 2.4 that for $(\widehat{\mathbf{w}}_\partial, w_0) \in M_1$, the curve C has two one-dimensional branches, one in $\text{Tor}_\mathbb{R}^+(P_\Delta)$ and the other in the closed quadrant containing the point w_0 . This defines a unique real structure on \mathbb{E} whose fixed point set $\mathbb{R}\mathbb{E}$ consists of two ovals, making $\Phi : \mathcal{E} \rightarrow C$ a separating real curve. The choice of a component of $\mathbb{E} \setminus \mathbb{R}\mathbb{E}$ defines a complex orientation on the one-dimensional branches of $\mathbb{R}C$.

We impose a further restriction to the choice of the quadrant $\mathfrak{Q} \supset \{w_0\}$ dictated by our wish to avoid specific wall-crossing events that may occur in the proof of the invariance of the count to be defined below: first, the escape of a real one-dimensional branch of counted curves out of the positive quadrant and, second, degenerations in which the union of the toric divisors splits off.

Definition 2.9 *We say that the quadrant $\mathfrak{Q} \subset \text{Tor}_{\mathbb{R}}(P_{\Delta})^{\times}$ satisfies the admissible quadrant condition (AQC) if the closure $\overline{\mathfrak{Q}}$ of \mathfrak{Q} shares with $\partial\text{Tor}_{\mathbb{R}}^+(P_{\Delta})$ at most one side.*

Example 2.10 *Let d, d_1 and d_2 be positive integers. Condition (AQC) imposes no restriction on the choice of \mathfrak{Q} in the case of the triangle $\text{conv}\{(0, 0), (d, 0), (0, d)\}$ (the convex hull of the points $(0, 0), (d, 0), (0, d)$) having the projective plane \mathbb{P}^2 as associated toric surface, while for the rectangle $\text{conv}\{(0, 0), (d_1, 0), (d_1, d_2), (0, d_2)\}$ which has $\mathbb{P}^1 \times \mathbb{P}^1$ as associated toric surface, the condition autorizes only the quadrant \mathfrak{Q} defined by the inequalities $x_1 < 0$ and $x_2 < 0$.*

Denote by $\overrightarrow{\mathcal{M}}_{1,n}^{\mathbb{R},+}(\Delta, \mathbf{w})$ the subset of $\overrightarrow{\mathcal{M}}_{1,n}^{\mathbb{R}}(\Delta, \mathbf{w})$ formed by the elements

$$([\Phi : (\mathbb{E}, \mathbf{p}) \rightarrow \text{Tor}(P_{\Delta})], \mathbb{E}_+)$$

such that $[\Phi : (\mathbb{E}, \mathbf{p}) \rightarrow \text{Tor}(P_{\Delta})] \in \mathcal{M}_{1,n}^{\mathbb{R},+}(\Delta, \mathbf{w})$ and \mathbb{E}_+ is a half of $\mathbb{E} \setminus \mathbb{R}\mathbb{E}$. According to [22, Theorem 1], each element ξ of $\overrightarrow{\mathcal{M}}_{1,n}^{\mathbb{R},+}(\Delta, \mathbf{w})$ has a quantum index $\text{QI}(\xi) \in \frac{1}{2}\mathbb{Z}$, $|\text{QI}(\xi)| \leq \mathcal{A}(\Delta)$ (defined by formula (3) with the integration domain \mathbb{E}_+), and for \mathbf{w} such that $(\widehat{\mathbf{w}}_{\partial}, w_0) \in M_1$, according to the argument in [16, Section 1.1], there is a well-defined *Welschinger sign*

$$W_1(\xi) = (-1)^{e_+(C)} \cdot (-1)^{h(C, \mathfrak{Q})} \cdot \prod_{\substack{\sigma \in P_{\Delta}^1, 1 \leq i \leq n^{\sigma} \\ k_i^{\sigma} \equiv 0 \pmod{2}}} \varepsilon(\xi, w_i^{\sigma}), \quad (5)$$

where $h(C, \mathfrak{Q})$ is the number of hyperbolic nodes of $C = \Phi(\mathbb{E})$ in the quadrant \mathfrak{Q} that arise in any real nodal equigeneric deformation of all singularities of C in \mathfrak{Q} (here $e_0(C)$ and $\varepsilon(\xi, w_i^{\sigma})$ are straightforward analogs of the corresponding ingredients in formula (4)).

For each $\kappa \in \frac{1}{2}\mathbb{Z}$ such that $|\kappa| \leq \mathcal{A}(\Delta)$, put

$$W_1^{\kappa}(\Delta, \mathbf{w}) = \sum_{\substack{\xi \in \overrightarrow{\mathcal{M}}_{1,n}^{\mathbb{R},+}(\Delta, \mathbf{w}) \\ \text{QI}(\xi) = \kappa}} W_1(\xi). \quad (6)$$

Theorem 2.11 *The value $W_1^{\kappa}(\Delta, \mathbf{w})$ does not depend on the choice of a generic \mathbf{w} subject to the following conditions: $\mathbf{w}_{\partial} \subset \partial\text{Tor}_{\mathbb{R}}^+(P_{\Delta})$ and $w_0 \in \mathfrak{Q}$, where \mathfrak{Q} is a fixed non-positive quadrant satisfying (AQC).*

We prove this theorem in Section 2.5.

Definition 2.12 *Following Theorem 2.5, we introduce the numerical invariants*

$$W_1^\kappa(\Delta, (\alpha, \beta)) := (-1)^{\mathcal{I}(\frac{1}{2}\Delta)} (-1)^{(\mathcal{A}(\Delta) - \kappa)/4} W_1^\kappa(\Delta, \mathbf{w}), \quad \text{where } \kappa \in \frac{1}{2}\mathbb{Z}, |\kappa| \leq \mathcal{A}(\Delta),$$

and the couple $(\alpha, \beta) \in (\mathbb{Z}/2\mathbb{Z})^2$ is defined via

$$\Omega = \{(x, y) \in \mathbb{R}^2 \mid (-1)^\alpha x > 0, (-1)^\beta y > 0\}. \quad (7)$$

We introduce also the refined invariant

$$\mathfrak{G}_1(\Delta, (\alpha, \beta)) := \sum_{\kappa \in \frac{1}{2}\mathbb{Z}, |\kappa| \leq \mathcal{A}(\Delta)} W_1^\kappa(\Delta, (\alpha, \beta)) q^\kappa.$$

Theorem 2.13 *Invariants $W_1^\kappa(\Delta, (\alpha, \beta))$ vanish for all $\kappa \notin 2\mathbb{Z}$ and for all $\kappa \in 2\mathbb{Z}$ such that $\kappa \not\equiv \mathcal{A}(\Delta) \pmod{4}$.*

The proof is found in Section 4 after Theorem 4.28.

2.4 Proof of Theorem 2.5

Consider two sequences $\mathbf{w}(0)$ and $\mathbf{w}(1)$ satisfying the hypotheses of Theorem 2.5. We may assume that $\widehat{\mathbf{w}}(0), \widehat{\mathbf{w}}(1) \in M_{\mathbb{R}}^\tau(\Delta)$ for some $\tau > 0$, and we can join these sequences by a generic path $\{\widehat{\mathbf{w}}(t)\}_{0 \leq t \leq 1}$, entirely lying in $M_{\mathbb{R}}^\tau(\Delta)$. To prove Theorem 2.5, we need to verify the constancy of $W_0^\kappa(\Delta, \mathbf{w}(t))$ in all possible wall-crossing events along the chosen path, for all κ .

First, we verify that $W_0^\kappa(\Delta, \mathbf{w}(t))$ does not change along intervals $t' < t < t''$ such that

$$\overline{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t)) = \mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t)) \quad \text{and} \quad \widehat{\mathbf{w}}(t) \in M_0, \quad \text{for all } t' < t < t''. \quad (8)$$

To this end, we show that the projection of $\mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))_{t' < t < t''}$ to the interval (t', t'') does not have critical points. In such a case, the family $\mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))_{t' < t < t''}$ is the union of intervals trivially covering (t', t'') , and hence it lifts to a trivial family of complex oriented curves and the quantum index persists along each of the components of the latter family.

It is convenient to look at the elements $\widehat{\mathbf{w}} \in M_{\mathbb{R}}^\tau(\Delta)$ as follows: pick some pair (σ_0, i_0) , where $\sigma_0 \in P_\Delta^1$ and $1 \leq i_0 \leq n^{\sigma_0}$, and think of $w_{i_0}^{\sigma_0}$ as a mobile point, whose position is determined by the other points $w_i^\sigma \in \widehat{\mathbf{w}}$ via the Menelaus condition. Put $C = \Phi_*(\mathbb{P}^1)$. Observe that the tangent space at C to the family of curves $C' \in |\mathcal{L}_{P_\Delta}|$ intersecting $\text{Tor}(\sigma)$ at w_i^σ (where $(\sigma, i) \neq (\sigma_0, i_0)$) with multiplicity at least $2k_i^\sigma$ can be identified with the linear system $\{C' \in |\mathcal{L}_{P_\Delta}| : (C' \cdot C)_{w_i^\sigma} \geq 2k_i^\sigma\}$. In turn, the tangent space at C to the family of curves $C' \in |\mathcal{L}_{P_\Delta}|$ intersecting $\text{Tor}(\sigma_0)$ at a point close to $w_{i_0}^{\sigma_0}$ with multiplicity at least $2k_{i_0}^{\sigma_0}$ can be identified with the linear system $\{C' \in |\mathcal{L}_{P_\Delta}| : (C' \cdot C)_{w_{i_0}^{\sigma_0}} \geq 2k_{i_0}^{\sigma_0} - 1\}$. Then, the required transversality amounts to the following statement.

Lemma 2.14 *Let $\xi = [\Phi : (\mathbb{P}^1, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)] \in \mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))$ with some $t \in [0, 1]$. Then,*

$$\begin{aligned} & H^0 \left(\mathbb{P}^1, \mathcal{N}_\Phi \left(- \sum_{\sigma \neq \sigma_0, i \neq i_0} 2k_i^\sigma p_i^\sigma - (2k_{i_0}^{\sigma_0} - 1)p_{i_0}^{\sigma_0} \right) \right) \\ &= H^1 \left(\mathbb{P}^1, \mathcal{N}_\Phi \left(- \sum_{\sigma \neq \sigma_0, i \neq i_0} 2k_i^\sigma p_i^\sigma - (2k_{i_0}^{\sigma_0} - 1)p_{i_0}^{\sigma_0} \right) \right) = 0. \end{aligned}$$

Proof. We have

$$\deg \mathcal{N}_\Phi \left(- \sum_{\sigma \neq \sigma_0, i \neq i_0} 2k_i^\sigma p_i^\sigma - (2k_{i_0}^{\sigma_0} - 1)p_{i_0}^{\sigma_0} \right) \stackrel{(2)}{=} -1 > -2.$$

Thus, the claim of the lemma follows from the Riemann-Roch theorem. \square

The set of elements $\widehat{\mathbf{w}} \in M_{\mathbb{R}}^\tau(\Delta)$ satisfying (8) is a dense semialgebraic subset of full dimension $\dim M_{\mathbb{R}}^\tau(\Delta) = n$. The complement is the union of finitely many semialgebraic strata of codimension ≥ 1 . Since the path $\{\widehat{\mathbf{w}}(t)\}_{0 \leq t \leq 1}$ is generic, it avoids strata of $M_{\mathbb{R}}^\tau(\Delta)$ of codimension ≥ 2 and intersects strata of codimension one only in their generic points.

Now we characterize elements $\xi \in \overline{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))_{0 \leq t \leq 1} \setminus \mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))_{0 \leq t \leq 1}$ and specify those of them which are generic elements of strata of dimension $n - 1$.

Lemma 2.15 (1) *The following elements $\xi = [\Phi : (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)]$ cannot occur in $\overline{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))_{0 \leq t \leq 1} \setminus \mathcal{M}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))_{0 \leq t \leq 1}$:*

- (1i) \widehat{C} is a reducible connected curve of arithmetic genus 0 with a component mapped onto a toric divisor;
- (1ii) \widehat{C} is a reducible connected curve of arithmetic genus 0 with at least three irreducible components.

(2) *If $\xi = [\Phi : (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)] \in \overline{\mathcal{M}}_{0,n}(\Delta, \mathbf{w}(t^*))$, where $\widehat{\mathbf{w}}(t^*)$ is a generic element in an $(n - 1)$ -dimensional stratum in $M_{\mathbb{R}}^\tau(\Delta) \setminus M_0$, then ξ is of one of the following types:*

- (2i) $\widehat{\mathbf{w}}(t^*)$ consists of $n + 1$ distinct points, $\widehat{C} \simeq \mathbb{P}^1$, the map Φ is birational onto its image and satisfies the following: either it is smooth at $\widehat{\mathbf{w}}(t^*)$, but has singular branches in $\text{Tor}(P_\Delta)^\times$ or it is an immersion everywhere but at one point $w_i^\sigma(t^*)$, where it has a singularity of type A_{2m} , $m \geq k_i^\sigma$, and, furthermore, $C = \Phi(\widehat{C})$ is smooth at each point of $\widehat{\mathbf{w}}(t^*) \setminus \{w_i^\sigma(t^*)\}$;
- (2ii) two points of the sequence $\widehat{\mathbf{w}}(t^*)$ coincide ($w_i^\sigma(t^*) = w_j^\sigma(t^*)$ for some $\sigma \in P_\Delta^1$ and $i \neq j$) and $\widehat{C} \simeq \mathbb{P}^1$, the map Φ being an immersion such that the point $w_i^\sigma(t^*) = w_j^\sigma(t^*)$ is a center of one or two smooth branches;

(2iii) $\widehat{\mathbf{w}}(t^*)$ consists of $n + 1$ distinct points, $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2$, where $\widehat{C}_1 \simeq \widehat{C}_2 \simeq \mathbb{P}^1$ and $\widehat{C}_1 \cap \widehat{C}_2$ is one point p , each map $\Phi : \widehat{C}_i \rightarrow \text{Tor}(P_\Delta)$, $i = 1, 2$, is either an immersion smooth along $\text{Tor}(\partial P_\Delta)$, or a multiple covering of a line intersecting only two toric divisors, while these divisors correspond to opposite parallel sides of P_Δ and the intersection points with these divisors are ramification points of the covering; furthermore, either $\Phi(p) \in \text{Tor}(P_\Delta)^\times$ and then all intersection points of the curves $C_1 = \Phi(\widehat{C}_1)$, $C_2 = \Phi(\widehat{C}_2)$ are ordinary nodes, or $w = \Phi(p) \in \mathbf{w}(t^*)$, and in the latter case at least one of the maps $\Phi : \widehat{C}_i \rightarrow \text{Tor}(P_\Delta)$ is birational onto its image and the curves $C_1 = \Phi(\widehat{C}_1)$, $C_2 = \Phi(\widehat{C}_2)$ do not have common point in $\widehat{\mathbf{w}}(t^*) \setminus \{w\}$;

(2iv) P_Δ is a triangle, $n = 3$, two points of the sequence $\widehat{\mathbf{w}}(t^*)$ coincide, the curve $C = \Phi(\mathbb{P}^1)$ is rational and smooth at $\widehat{\mathbf{w}}(t^*)$, and $\Phi : \mathbb{P}^1 \rightarrow C$ is a double covering ramified at two distinct points of $\widehat{\mathbf{w}}(t^*)$.

Proof. (1i) A toric divisor $\text{Tor}(\sigma)$ cannot split off alone, since, otherwise, in the deformation along the path $\{\widehat{\mathbf{w}}(t)\}_{0 \leq t \leq 1}$, the intersection points with the neighboring toric divisors would yield points on these toric divisors on the distance less than τ from the corners of $\text{Tor}_{\mathbb{R}}^+(P_\Delta)$. For the same reason we obtain that only all toric divisors together may split off, while their intersection points must smooth out in the deformation. However, this contradicts the rationality of the considered curves.

(1ii) First, note that $\Phi(\widehat{C})$ intersects $\text{Tor}(\partial P_\Delta)$ only at $\widehat{\mathbf{w}}(t^*)$. Furthermore, if the images of two irreducible components $\widehat{C}_1, \widehat{C}_2$ of \widehat{C} contain the same point $w_i^\sigma(t^*)$ that is different from any other point $w_j^\sigma(t^*)$, $j \neq i$, then $w_i^\sigma(t^*) = \Phi(\widehat{C}_1 \cap \widehat{C}_2)$ (cf. Lemma 2.4). Due to the genus restriction, it follows that the sequence $\widehat{\mathbf{w}}(t^*)$ satisfies at least three Menelaus conditions, which cuts off $M_{\mathbb{R}}^T(\Delta)$ a polytope of dimension $\leq n - 2$, contrary to the dimension $n - 1$ assumption.

Let ξ satisfy the hypotheses of item (2).

(2i) Let $\widehat{C} \simeq \mathbb{P}^1$ and $\widehat{\mathbf{w}}(t^*)$ consist of $n + 1$ distinct points. If all points $w_i^\sigma(t^*)$ are smooth, then we get the case (2i).

Suppose that $C = \Phi(\mathbb{P}^1)$ is singular at some point $w_i^\sigma(t^*)$. Note that ξ must be unibranch at each point of $\widehat{\mathbf{w}}(t^*)$. Since ξ is a genetic element of an $(n - 1)$ -dimensional family in $\mathcal{M}_{0,n}(\Delta)$, fixing the position of w_i^σ , we obtain a family of dimension $\geq n - 2$. Applying [16, Inequality (5) in Lemma 2.1], we obtain

$$\begin{aligned} \sum_{\sigma \subset \partial P} \|\sigma\|_{\mathbb{Z}} &\geq 2 + \left(\sum_{\sigma \subset \partial P} \|\sigma\|_{\mathbb{Z}} - n \right) + (n - 3) + \sum_B (\text{ord} B - 1) \\ &= \sum_{\sigma \subset \partial P} \|\sigma\|_{\mathbb{Z}} - 1 + \sum_B (\text{ord} B - 1), \end{aligned}$$

where B ranges over all singular branches of ξ , and hence ξ has a unique singular branch, and this branch is centered at $w_i^\sigma(t^*)$ and has multiplicity 2. Thus, the type of the singularity must be A_{2m} , $m \geq k_i^\sigma$.

(2ii) Suppose that $\widehat{C} \simeq \mathbb{P}^1$ and some of the points of the sequence $\widehat{\mathbf{w}}(t^*)$ coincide. For the dimension reason, the number $\#(\widehat{\mathbf{w}}(t^*))$ of points in $\widehat{\mathbf{w}}(t^*)$ is equal to $n - 1$ (i.e., $w_i^\sigma(t^*) = w_j^\sigma(t^*) = w$ for some $\sigma \subset \partial P$ and $i \neq j$), and all these $n - 1 \geq 2$ points are in general position subject to the Menelaus relation (1). If $n - 1 = 2$, then the claim of item (2ii) is fulfilled by [28, Lemma 3.5]. Assume that $n - 1 \geq 3$. If $C = \Phi(\widehat{C})$ is unibranch at each point of $\mathbf{w}(t^*)$, then, by Lemma 2.4, the curve C is immersed and smooth along the toric divisors. If C is not unibranch at w , then by Lemma 2.4 it has two local branches at w ; furthermore, C must be unibranch at each point $w_i^{\sigma'}(t^*) \neq w$ and, in addition, smooth if $k_i^{\sigma'} \geq 2$ due to Lemma 2.4 and claim (1ii). Let us show that C is immersed. Fixing the position of w and the position of one more point $w' \in \mathbf{w}(t^*) \setminus \{w\}$, we obtain a family of dimension at least $n - 3 \geq 1$; thus, [16, Inequality (5) in Lemma 2.1] applies:

$$\begin{aligned} c_1(\mathrm{Tor}(P_\Delta))c_1(\mathcal{L}_{P_\Delta}) &\geq 2 + (c_1(\mathrm{Tor}(P_\Delta))c_1(\mathcal{L}_{P_\Delta}) - n + 2) + \sum_B (\mathrm{ord}B - 1) + (n - 4) \\ &= c_1(\mathrm{Tor}(P_\Delta))c_1(\mathcal{L}_{P_\Delta}) + \sum_B (\mathrm{ord}B - 1) , \end{aligned}$$

where B runs over all singular local branches of C in $\mathrm{Tor}(P_\Delta)^\times \cup \{w, w'\}$, and hence C is immersed.

(2iii) We are left with the case of $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2$, where $\widehat{C}_1 \simeq \widehat{C}_2 \simeq \mathbb{P}^1$ and $\widehat{C}_1 \cap \widehat{C}_2 = \{p\}$ is one point. For the dimension reason, the points of $\widehat{\mathbf{w}}(t^*)$ are in general position subject to exactly two Menelaus conditions (induced by the components of \widehat{C}), and they are all distinct. Each of the curves C_1, C_2 , which passes through at least three points of $\widehat{\mathbf{w}}(t^*)$, is immersed and smooth along the toric divisors and has even intersection multiplicity with $\mathrm{Tor}(\partial P_\Delta)$ at any point, by Lemma 2.4. If C_1 or C_2 passes through exactly two points of $\widehat{\mathbf{w}}(t^*)$, then it is a multiple covering of a line as described in item (2iv). Let $\Phi(p) \in \mathrm{Tor}(P_\Delta)^\times$. Then, by Lemma 2.4, the curves C_1 and C_2 do not share points in $\widehat{\mathbf{w}}(t^*)$, and we claim that all their intersection points are ordinary nodes. Due to the genericity assumptions for $\widehat{\mathbf{w}}(t^*)$, we have to study the only case of both C_1 and C_2 immersed. If we freely move the points of $C_1 \cap \widehat{\mathbf{w}}(t^*)$ so that the corresponding Menelaus condition induced by C_1 retains, and fix the curve C_2 , then the persisting tangency condition of a germ $\Phi : (\widehat{C}_1, q) \rightarrow \mathrm{Tor}(P_\Delta)$ to the curve C_2 would yield that the tangent space to the considered family of curves in the linear system $|C_1|$ would be contained in $H^0(\widehat{C}_1, \mathcal{O}_{\widehat{C}_1}(\mathbf{d}_1))$, where

$$\begin{aligned} \deg \mathbf{d}_1 &= C_1^2 - (C_1^2 - c_1(\mathrm{Tor}(P_\Delta))[C_1] + 2) - (c_1(\mathrm{Tor}(P_\Delta))[C_1] - \#(\mathbf{w}(t^*) \cap C_1)) - 1 \\ &= \#(\widehat{\mathbf{w}}(t^*) \cap C_1) - 3 > -2 , \end{aligned}$$

and hence, by the Riemann-Roch theorem,

$$h^0(\widehat{C}_1, \mathcal{O}_{\widehat{C}_1}(\mathbf{d}_1)) = \#(\widehat{\mathbf{w}}(t^*) \cap C_1) - 3 + 1 = \#(\widehat{\mathbf{w}}(t^*) \cap C_1) - 2 < \#(\widehat{\mathbf{w}}(t^*) \cap C_1) - 1 ,$$

which is a contradiction.

(2iv) Suppose that $\Phi : \mathbb{P}^1 \rightarrow \text{Tor}(P_\Delta)$ is an s -multiple covering onto its image, $s \geq 2$. For the dimension reason, the sequence $\widehat{\mathbf{w}}(t^*)$ contains at least n distinct points. If $\widehat{\mathbf{w}}(t^*)$ consists of $n - 1$ distinct points, then at each point of $\widehat{\mathbf{w}}(t^*)$ we have the ramification index s ; hence, by the Riemann-Hurwitz formula, we get

$$2 \leq 2s - (s - 1)(n + 1), \quad (9)$$

which gives $n \leq 1$; the latter inequality contradicts the fact that $n + 1$ is bounded from below by the number of sides of P_Δ . If the sequence $\widehat{\mathbf{w}}(t^*)$ contains only n distinct points, we have $n \geq 3$. Note that Φ has an irreducible preimage in at least $n - 1$ points of $\widehat{\mathbf{w}}(t^*)$ with ramification index s , and hence $n = 3$ (cf. (9)). It also follows that there are no other ramifications and that $s = 2$, since the remaining point of $\widehat{\mathbf{w}}(t^*)$, where Φ is not ramified, lifts to at most two points in \mathbb{P}^1 . Thus, we are left with the case described in item (2iv) of the lemma. \square

We complete the proof of Theorem 2.5 with the following lemma.

Lemma 2.16 *Let $\{\widehat{\mathbf{w}}(t)\}_{0 \leq t \leq 1}$ be a generic path in $M_{\mathbb{R}}^\tau(\Delta)$, and let $t^* \in (0, 1)$ be such that $\overline{\mathcal{M}}_{0,n}^{\mathbb{R}}(\Delta, \mathbf{w}(t^*))$ contains an element ξ as described in one of the items of Lemma 2.15(2). Then, for each $\kappa \in \frac{1}{2}\mathbb{Z}$ such that $|\kappa| \leq \mathcal{A}(\Delta)$, the numbers $W_0^\kappa(t) = W_0^\kappa(\Delta, \mathbf{w}(t))$ do not change as t varies in a neighborhood of t^* .*

Proof. We always can assume that in a neighborhood of t^* , the path $\{\widehat{\mathbf{w}}(t)\}_{0 \leq t \leq 1}$ is defined by fixing the position of some $n - 1$ points of $\mathbf{w}(t^*)$, while the other two points remain mobile (the choice of the two mobile points may depend on the considered degeneration). We also notice that, in the degenerations as in Lemma 2.15(2i,2ii), the source curve and its real structure remain fixed, which implies that the quantum index is constant in these wall-crossings. Except for the case of Lemma 2.15(2iii) describing reducible degenerations, we work with families of curves which are trivially covered by families of complex oriented curves so that the quantum index persists along each component of the family of oriented curves.

(1) Suppose that $\xi \in \overline{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t^*))$ is as in Lemma 2.15(2i) and, moreover, if it has a singular branch at $w_i^\sigma(t^*)$, then $k_i^\sigma = 1$. We derive the constancy of $W_0^\kappa(t)$, $|t - t^*| < \delta$ from [15, Lemma 15] and [16, Lemma 2.4(1)]. To this end, we have to establish the following transversality statement (cf. [15, Lemma 13]). Choose a sufficiently large integer s . For each point $z \in \text{Sing}(C) \cap \text{Tor}(P_\Delta)^\times$, we set $I_z = I^{\text{cond}}(C, z)/\mathfrak{m}_z^s \subset \mathcal{O}_{C,z}/\mathfrak{m}_z^s$, the quotient of the conductor ideal by the power of the maximal ideal, which can be viewed as the tangent cone to the stratum parameterizing equigeneric deformations (see [5, Theorem 4.15]). For each point $w = w_i^\sigma(t^*)$, where C is smooth and which is fixed, respectively, mobile, we set

$$I_w = \{\varphi \in \mathcal{O}_{C,w} : (\varphi \cdot C)_w \geq 2k_i^\sigma\}/\mathfrak{m}_w^s \subset \mathcal{O}_{C,w}/\mathfrak{m}_w^s,$$

$$\text{respectively, } I_w = \{\varphi \in \mathcal{O}_{C,w} : (\varphi \cdot C)_w \geq 2k_i^\sigma - 1\}/\mathfrak{m}_w^s \subset \mathcal{O}_{C,w}/\mathfrak{m}_w^s,$$

which can be viewed as the tangent space to the stratum parameterizing deformations that keep the intersection number with $\text{Tor}(\sigma)$ at w , respectively, in a nearby

point on $\text{Tor}(\sigma)$. For each point $w = w_i^\sigma(t^*)$ with $k_i^\sigma = 1$, where C is singular (*i.e.*, has a singular branch of type A_{2m} , $m \geq 1$) and which is fixed, respectively, mobile, we set

$$I_w = \{\varphi \in \mathcal{O}_{C,w} : (\varphi \cdot C)_w \geq 2 + 2m\} / \mathfrak{m}_w^s \subset \mathcal{O}_{C,w} / \mathfrak{m}_w^s ,$$

resp. $I_w = \{\varphi \in \mathcal{O}_{C,w} : (\varphi \cdot C)_w \geq 1 + 2m\} / \mathfrak{m}_w^s \subset \mathcal{O}_{C,w} / \mathfrak{m}_w^s ,$

which can be viewed as the tangent cone to the stratum parameterizing equigeneric deformations with the fixed intersection multiplicity 2 with $\text{Tor}(\sigma)$ at w , respectively, at a nearby point on $\text{Tor}(\sigma)$ (see [16, Lemma 2.4(1)] and [30, Lemma 3(1)]). The required transversality is as follows: the natural image of the germ of the linear system $|\mathcal{L}_{P_\Delta}|$ at C to $\prod_{z \in \text{Sing}(C) \cap \text{Tor}(P_\Delta)^\times} \mathcal{O}_{C,z} / \mathfrak{m}_z^s \times \prod_{w \in \mathbf{w}(t^*)} \mathcal{O}_{C,w} / \mathfrak{m}_w^s$ intersects there transversally with $\prod_{z \in \text{Sing}(C) \cap \text{Tor}(P_\Delta)^\times} I_z \times \prod_{w \in \mathbf{w}(t^*)} I_w$. The cohomological reformulation amounts to

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathbf{d})) = 0 , \quad (10)$$

which holds in view of

$$\deg \mathbf{d} = C^2 - \sum_{z \in \text{Sing}(C) \cap \text{Tor}(P_\Delta)^\times} \dim \mathcal{O}_{C,z} / I_z - \sum_{w \in \mathbf{w}(t^*)} \dim \mathcal{O}_{C,w} / I_w$$

$$= C^2 - (C^2 - c_1(\text{Tor}(P_\Delta))c_1(\mathcal{L}_{P_\Delta}) + 2) - (c_1(\text{Tor}(P_\Delta))c_1(\mathcal{L}_{P_\Delta}) - 2) = 0 > -2 .$$

The established transversality reduces the constancy of $W_0^\kappa(t)$ to the constancy of the count of Welschinger signs when varying germs (C, z) and (C, w) for $z \in \text{Sing}(C) \cap \text{Tor}(P_\Delta)^\times$ and $w \in \text{Sing}(C) \cap \widehat{\mathbf{w}}(t^*)$. Thus, the constancy in the former variation follows from [15, Lemma 15] (see also [15, Lemma 13]), while the constancy in the latter variation follows from the fact that the equigeneric stratum, which we consider in $\mathcal{O}_{C,w} / \mathfrak{m}_w^s$, is smooth (see [30, Lemma 3(1)] and [16, Lemma 2.4(2)]), and the local count of Welschinger signs is invariant by [16, Lemma 2.4(1)].

(2) Suppose that $\xi \in \overline{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t^*))$ is as in Lemma 2.15(2i) with a singularity at a point $w_i^\sigma(t^*)$ of type A_{2m} , $m \geq k_i^\sigma$. We can assume that the path $\{\widehat{\mathbf{w}}(t)\}_{|t-t^*| < \varepsilon}$ is such that $n - 1$ points of $\widehat{\mathbf{w}}(t)$, including $w_i^\sigma(t)$, stay fixed, whereas the other two points move keeping the Menelaus condition. We claim that the germ at ξ of the family $\{\overline{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))\}_{|t-t^*| < \varepsilon}$ is smooth and regularly parameterized by $t \in (t^* - \varepsilon, t^* + \varepsilon)$. By [5, Theorem 4.15 and Proposition 4.17(2)] and [30, Lemma 3(1)], it is sufficient to prove that

$$H^1(C, \mathcal{J}_{Z/C}(C)) = 0 , \quad (11)$$

where $C = \Phi(\mathbb{P}^1)$ and $\mathcal{J}_{Z/C} \subset \mathcal{O}_C$ is the ideal sheaf of the zero-dimensional scheme $Z \subset C$ concentrated at $\widehat{\mathbf{w}}(t^*) \cup \text{Sing}(C)$ and given (cf. Step (1) of the proof)

- by the ideal $I_w = \{\varphi \in \mathcal{O}_{C,w} : \text{ord} \varphi|_w \geq 2k_i^{\sigma'}\}$ at each point $w = w_i^{\sigma'}(t^*) \in \widehat{\mathbf{w}}(t^*) \setminus \{w_i^\sigma(t^*)\}$, which is fixed as $|t - t^*| < \varepsilon$,
- by the ideal $I_w = \{\varphi \in \mathcal{O}_{C,w} : \text{ord} \varphi|_w \geq 2k_i^{\sigma'} - 1\}$ at each point $w = w_i^{\sigma'}(t^*) \in \widehat{\mathbf{w}}(t^*) \setminus \{w_i^\sigma(t^*)\}$, which moves as $|t - t^*| < \varepsilon$,

- by the ideal $I_w = \{\varphi \in \mathcal{O}_{C,w} : \text{ord}\varphi|_w \geq 2k_i^\sigma + 2m\}$ at the point $w = w_i^\sigma(t^*)$ (here, in the notations of [30, Lemma 3(1)], $s = 2k_i^\sigma$ and $\delta(C, w) = m$),
- by the conductor ideal $I_{C,z}^{\text{cond}}$ at each point $z \in \text{Sing}(C) \setminus \{w_i^\sigma(t^*)\}$ (for the definition of the conductor ideal see, for instance, [5, Section 1, item (iii)] and [12, Section I.3.4, item “Semigroup and Conductor”]).

Thus, the desired relation (11) turns into

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(\mathbf{d})) = 0 ,$$

(cf. (10)), where the divisor \mathbf{d} has degree

$$\begin{aligned} \deg \mathbf{d} &= C^2 - \sum_{z \in \text{Sing}(C) \setminus \mathbf{w}(t^*)} \dim \mathcal{O}_{C,z}/I_z - \sum_{w \in \mathbf{w}(t^*)} \dim \mathcal{O}_{C,w}/I_w \\ &= C^2 - (C^2 - c_1(\text{Tor}(P_\Delta))c_1(\mathcal{L}_{P_\Delta}) + 2) - (c_1(\text{Tor}(P_\Delta))c_1(\mathcal{L}_{P_\Delta}) - 2) = 0 > -2 , \end{aligned}$$

which, finally, confirms (11).

In suitable conjugation-invariant local coordinates x, y in a neighborhood $U \subset \text{Tor}(P_\Delta)$ of $w := w_i^\sigma(t^*)$, we have $w = (0, 0)$, $\text{Tor}(\sigma) = \{y = 0\}$, $\text{Tor}_{\mathbb{R}}^+(P_\Delta) = \{y \geq 0\}$. Since the real part of the germ (C, w) lies in $\text{Tor}_{\mathbb{R}}^+(P_\Delta)$, without loss of generality we can suppose that $(C, w) = \{F(x, y) := y^2 - 2x^k y + x^{2k} + \text{h.o.t.} = 0\}$, where $k := k_i^\sigma$. Set $t_1 = t - t^*$. Then, the germs $[\Phi_t(\mathbb{P}^1), w]$, where $\xi_t = [\Phi_t : (\mathbb{P}^1, \mathbf{p}_t) \rightarrow \text{Tor}(P_\Delta)]$ ranges over the germ at ξ of the family $\{\overline{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))\}_{|t-t^*| < \varepsilon}$, are given by an equation

$$F_{t_1}(x, y) := F(x, y) + \sum_{j=0}^{k-1} t_1^{(k-j)\gamma} (a_{j1} + O(t_1)) x^j y + \sum_{p+kq \geq 2k} b_{pq}(t_1) x^p y^q = 0 , \quad (12)$$

where $a_{01} \neq 0$ and $b_{pq}(0) = 0$ for all p, q in the range. The tropical limit of the germs $[\Phi_t(\mathbb{P}^1), w]$ as $t \rightarrow t^*$ or, equivalently, $t_1 \rightarrow 0$ (for the detailed description of the tropical limit, see Section 3.2 below) defines the subdivision of the Newton polygon $P(F_t)$ of $F_t(x, y)$ into the triangle $T_1 = \text{conv}\{(0, 1), (0, 2), (2k, 0)\}$ and the Newton polygon $P(F)$ of F (see Figure 1(a)). Note that, since the curves $\Phi_t(\mathbb{P}^1)$ are rational, the triangle T_1 is not subdivided further, and the corresponding limit curve

$$C_1 := \left\{ \sum_{j=0}^{k-1} a_{j1} x^j y + y^2 - 2x^k y + x^{2k} = 0 \right\} \quad (13)$$

is rational. Moreover, since the curve C is unibranch at w , the limit curve C_F defined by the polynomial F in the surface $\text{Tor}(P(F))$ is unibranch at the intersection point with the toric divisor $\text{Tor}([(0, 2), (2k, 0)])$. Hence, C_1 is unibranch at its intersection point with the toric divisor $\text{Tor}([(0, 2), (2k, 0)])$, because otherwise, the union of the limit curves C_1 and C_F would deform into a non-rational curve contrary to the fact that the curves $\Phi_t(\mathbb{P}^1)$ are rational. By [28, Lemma 3.5], the curve C_1 is smooth at the intersection points with the toric divisors.

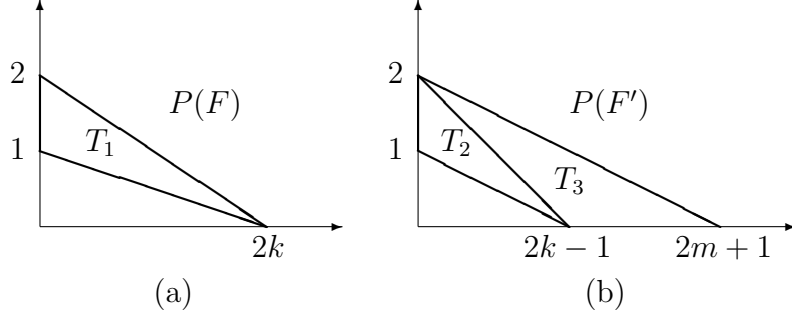


Figure 1: Proof of Lemma 2.16, part (2), I

We perform the modification of the above tropicalization along the toric divisor $\text{Tor}([(0,2), (2k,0)])$ (see the details in Section 3.3). Recall that the curve C has singularity of type A_{2m} at w , and hence there exists a coordinate change

$$x = x', \quad y = y' + (x')^k + \sum_{k < j \leq m} c_j (x')^j, \quad (14)$$

which converts the polynomial $F(x, y)$ into a polynomial

$$F'(x', y') = (y')^2 + a_{2m+1,0} (x')^{2m+1} + \text{h.o.t.}, \quad a_{2m+1,0} \neq 0.$$

There exists a deformation of the coordinate change (14)

$$x = x', \quad y = y' + (1 + O(t_1))x^k + \sum_{k < j < m} (c_j + O(t_1))x^j, \quad (15)$$

which turns the family of polynomials $F_{t_1}(x, y)$, $|t_1| < \varepsilon$, into the family

$$\begin{aligned} F'_{t_1}(x', y') &= \sum_{j=0}^{k-1} t_1^{(k-j)\alpha} (a_{j1} + O(t_1)) (x')^j y' + \sum_{j=2k-1}^{2m} t_1^{\nu(j)} (a_{j0} + O(t_1)) (x')^j \\ &\quad + (y')^2 (1 + O(t_1)) + (x')^{2m+1} (a_{2m+1,0} + O(t_1)) + \text{h.o.t.} \end{aligned} \quad (16)$$

The tropical limit of the latter family of polynomials defines the subdivision of the Newton polygon $P(F_{t_1})$ into the triangles

$$T_1 = \text{conv}\{(0,2), (0,1), (2k-1,0)\}, \quad T_3 = \text{conv}\{(0,2), (2k-1,0), (2m+1,0)\},$$

and the Newton polygon $P(F')$ (see Figure 1(b)). Moreover, in principle, the triangle T_3 can be subdivided further. Taking into account that all limit curves associated with the considered tropical limit must be rational and that the coefficients of F'_{t_1} at the interior integral points of T_3 vanish, the triangle T_3 can be subdivided only into triangles of type $\text{conv}\{(0,2), (2j'-1,0), (2j''+1,0)\}$ with some $k \leq j' \leq j'' \leq m$. The function $\nu : \{2k-1, 2k, \dots, 2m+1\} \rightarrow \mathbb{Z}$ in the exponents of t_1 in (16) (with the extra value $\nu(2m+1) = 0$) extends to a convex piecewise-linear function on the whole segment $[(2k-1,0), (2m+1,0)]$ with linearity domains induced by the subdivision of the triangle T_3 . We make two remarks.

- Since the germ at ξ of the family $\{\overline{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))\}_{|t-t^*|<\varepsilon}$ is smooth and regularly parameterized by t_1 (see the beginning of Step (2)), we derive that $\mu(2m) = 1$.
- Since the linearity domains of ν are segments of even length, the value $\nu(2k-1)$ is even. On the other hand, formulas (15) yield that $\nu(2k-1) = \alpha$, and hence all exponents of t_1 in the leading terms of the coefficients of F_{t_1} at $x^j y$, $0 \leq j \leq 2k-1$, are even (see (12)). This means, in particular, that the limit curve C_1 given by (13), contributes to the curves $\Phi_t(\mathbb{P}^1)$ the same collection of real nodes with the same location with respect to $\text{Tor}(\sigma)$ both for $t < t^*$ and for $t > t^*$. Hence, these singularities contribute the same factor to $W_0(\xi_t)$ for $t < t^*$ and for $t > t^*$ (cf. formula (4)).

Now we analyze the contribution of the limit curves associated with the fragment T_3 , and we consider the only case when T_3 is not subdivided. The other cases can be treated similarly. So, we have the limit curve

$$C_3 = \left\{ (y')^2 + \sum_{j=2k-1}^{2m+1} a_{j0}(x')^j = 0 \right\},$$

which must be rational, and this yields that

$$\sum_{j=2k-1}^{2m+1} a_{j0}(x')^j = a_{2m+1,0}(x')^{2k-1}Q(x')^2, \quad \deg Q = r := m - k + 1,$$

while the nodes of C_3 are $(\lambda_j, 0)$, $j = 1, \dots, r$, with the numbers λ_j being the roots of Q . Denote by r_+ (respectively, r_-) the number of positive (respectively, negative) roots of Q . Without loss of generality we can assume that $a_{2m+1,0} < 0$. Then, the curve C_3 has r_+ hyperbolic nodes and r_- elliptic nodes. In view of formulas (15), the nodes of C_3 yield the nodes

$$(t_1(\lambda_j + O(t_1)), t_1^k(\lambda_j^k + O(t_1))), \quad j = 1, \dots, r,$$

of the curve $\Phi_t(\mathbb{P}^1)$, where $t_1 = t - t^*$: for $t_1\lambda_j > 0$ a hyperbolic node, for $t_1\lambda_j < 0$ an elliptic one.

Thus, if k is odd, then, for $t > t^*$ (respectively, $t < t^*$), the curve $\Phi_t(\mathbb{P}^1)$ gets r_+ (respectively, r_-) hyperbolic nodes in $\text{Tor}(P_\Delta)^+$ and r_- (respectively, r_+) elliptic nodes outside $\text{Tor}(P_\Delta)^+$. None of these nodes contributes to the right-hand side of (4), and hence $W_0(\xi_t)$ remains constant as $0 < |t - t^*| < \varepsilon$.

If k is even, then the curve $\Phi_t(\mathbb{P}^1)$ gets r_+ hyperbolic and r_- elliptic nodes in $\text{Tor}(P_\Delta)^+$ if $t_1 > 0$, and gets r_- hyperbolic and r_+ elliptic nodes in $\text{Tor}(P_\Delta)^+$ if $t_1 < 0$. Suppose that r is even. Then $r_+ \equiv r_- \pmod{2}$, and hence the first product in the right-hand side of formula (4) for $W_0(\xi_t)$ remains constant. In turn, the orientation of the real local branch of $\Phi_t(\mathbb{P}^1)$ at w remains the same (see Figure 2(a,b)), and hence the second product in the right-hand side of (4) remains constant as well. Suppose that r is odd. Then $r_+ \equiv r_- + 1 \pmod{2}$, and hence the first

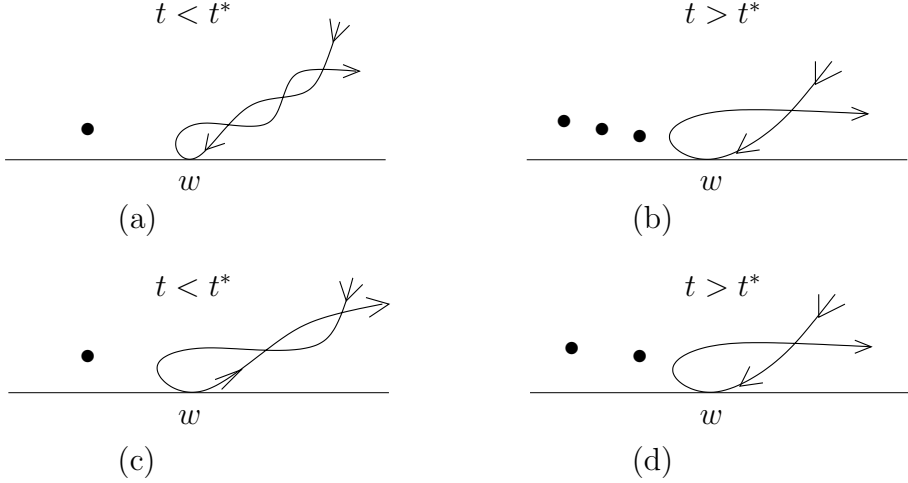


Figure 2: Proof of Lemma 2.16, part (2), II

product in the right-hand side of formula (4) for $W_0(\xi_t)$ changes sign as t passes through t^* . However, the orientation of the real local branch of $\Phi_t(\mathbb{P}^1)$ at w changes (see Figure 2(c,d)), which causes the change of sign in the second product in the right-hand side of (4).

Thus, $W_0^\kappa(t)$ remains constant in the considered bifurcation.

(3) Suppose that ξ is as in Lemma 2.15(2ii). Here we choose a path $\{\widehat{\mathbf{w}}(t)\}_{0 \leq t \leq 1}$ defined in a neighborhood of t^* so that the point $w_i^\sigma(t)$ is fixed and the point $w_j(t^*)$ is mobile (together with some other point of $\widehat{\mathbf{w}}(t)$). Since C is immersed, we extract the required local constancy of $W_0^\kappa(t)$ from the transversality relation (10) and the smoothness of the two following strata in $\mathcal{O}_{C,w}/\mathfrak{m}_w^s$:

- one stratum parameterizes deformations of a smooth branch intersecting $\text{Tor}(\sigma)$ at w with multiplicity $2(k_i^\sigma + k_j^\sigma)$ into a smooth branch intersecting $\text{Tor}(\sigma)$ at w with multiplicity $2k_i^\sigma$ and in a nearby point with multiplicity $2k_j^\sigma$,
- the other stratum parameterizes deformations of a couple of smooth branches intersecting $\text{Tor}(\sigma)$ at w with multiplicities $2k_i^\sigma$ and $2k_j^\sigma$, respectively, into a couple of smooth branches, one intersecting $\text{Tor}(\sigma)$ at w with multiplicity $2k_i^\sigma$ and the other intersecting $\text{Tor}(\sigma)$ in a nearby point with multiplicity $2k_j^\sigma$.

Both smoothness statements are straightforward.

(4) Suppose that ξ is as in Lemma 2.15(2iii) and $\Phi(p) \in \text{Tor}_{\mathbb{R}}(P_\Delta)_+^\times$. We closely follow the proof of [22, Theorem 5], providing here details for the reader's convenience. Consider the path $\{\widehat{\mathbf{w}}(t)\}_{0 \leq t \leq 1}$ locally obtained from $\widehat{\mathbf{w}}(t^*)$ by moving one point of $\mathbf{w}(t^*) \cap C_1$ and one point of $\mathbf{w}(t^*)$, while fixing the position of the other points of $\mathbf{w}(t^*)$. Then, we obtain a smooth one dimensional germ in $\overline{\mathcal{M}}_{0,n}(\Delta)$, which locally induces a deformation of the hyperbolic node $\Phi(\widehat{C}, p)$ equivalent in suitable local coordinates in a neighborhood of $z = \Phi(p)$ to $uv = \lambda(t)$ with λ a smooth

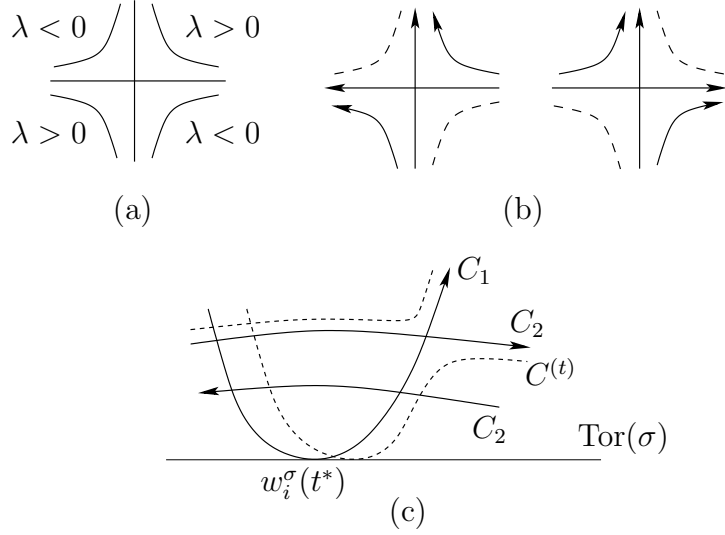


Figure 3: Proof of Lemma 2.16, part (4)

function in a neighborhood of t^* such that $\lambda(t^*) = 0$ and $\lambda'(t^*) > 0$ (see Figure 3(a)). The smoothness and the local deformation claims are straightforward from the (standard) considerations in the proof of [15, Lemma 11(2)], which reduce both claims to the following h^1 -vanishing (cf. [15, Formula (16) and computations in the first paragraph in page 251]):

$$h^1(\widehat{C}_s, \mathcal{O}_{\widehat{C}_s}(\mathbf{d}_s)) = 0, \quad s = 1, 2, \quad (17)$$

where

$$\deg \mathbf{d}_s = C_s^2 - (C_s^2 - c_1(\text{Tor}(P_\Delta))[C_s] + 2) - (c_1(\text{Tor}(P_\Delta))[C_s] - 1) = -1 > -2,$$

and hence (17) follows.

Now we analyze the change of quantum and pass to consideration of complex oriented curves. Denote by \widehat{C}_1^\pm and \widehat{C}_2^\pm the connected components of $\widehat{C}_1 \setminus \mathbb{R}\widehat{C}_1$ and $\widehat{C}_2 \setminus \mathbb{R}\widehat{C}_2$, respectively. When t varies around t^* , the curve $\widehat{C}_1 \cup \widehat{C}_2$ turns into \mathbb{P}^1 , and we encounter two types of deformations: either $\widehat{C}_1^+, \widehat{C}_1^-$ glue up with $\widehat{C}_2^+, \widehat{C}_2^-$, respectively, or $\widehat{C}_1^+, \widehat{C}_1^-$ glue up with $\widehat{C}_2^-, \widehat{C}_2^+$, respectively. The type of the deformation agrees with the local complex orientation of the central curve as shown in Figure 3(b). This means that if for $t < t^*$ one encounters a deformation of the first type, then for $t > t^*$ it is of the second type, and *vice versa*. In terms of the quantum indices, this means that, on one side of the path $\{\widehat{\mathbf{w}}(t)\}_{|t-t^*|<\varepsilon}$, where $0 < \varepsilon \ll 1$, we have two elements of $\overrightarrow{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))$ with quantum indices $\kappa_1 + \kappa_2, -\kappa_1 - \kappa_2$, while on the other side, $\kappa_1 - \kappa_2, \kappa_2 - \kappa_1$, where $\kappa_s = \text{QI}(\Phi : (\widehat{C}_s, \widehat{C}_s^+) \rightarrow \text{Tor}(\Sigma))$, $s = 1, 2$. We intend to show that, along the path $\{\widehat{\mathbf{w}}(t)\}_{|t-t^*|<\varepsilon}$, we simultaneously observe several bifurcations of the elements of $\overrightarrow{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))$ on each side of the path, each value of the quantum index $\kappa_1 + \kappa_2, -\kappa_1 - \kappa_2, \kappa_1 - \kappa_2, \kappa_2 - \kappa_1$ appears exactly r times, where r is either the half of the number of hyperbolic nodes in $C_1 \cap C_2$ if

$\Phi|_{\widehat{C}_1}, \Phi|_{\widehat{C}_2}$ both are immersions, or the number of hyperbolic nodes in $C_1 \cap C_2$ if one of $\Phi|_{\widehat{C}_1}, \Phi|_{\widehat{C}_2}$ is a multiple covering, or 2 if $\Phi|_{\widehat{C}_1}, \Phi|_{\widehat{C}_2}$ both are multiple coverings.

Note that, for the given C_1, C_2 and multiplicities of the covering (if any), we have $2r$ distinct maps $\Phi : \widehat{C}_1 \cup \widehat{C}_2 \rightarrow \text{Tor}(P_\Delta)$ with the images $\Phi(\widehat{C}_s) = C_s, s = 1, 2$, that are distinguished by the choice of a hyperbolic node in $C_1 \cap C_2$ which is the image of $p = \widehat{C}_1 \cap \widehat{C}_2$. In case of both $\Phi|_{\widehat{C}_1}, \Phi|_{\widehat{C}_2}$ immersions, with the induced by \widehat{C}_1^+ and \widehat{C}_2^+ complex orientations, we can compute the intersection multiplicity $\mathbb{R}C_1 \circ \mathbb{R}C_2 = 0$ by summing up the local multiplicities at the hyperbolic nodes in $C_1 \cap C_2$. This yields, first, that the number of these hyperbolic nodes is even and that the numbers of positive and negative intersections of $\mathbb{R}C_1, \mathbb{R}C_2$ equal r . Let $\Phi : \widehat{C}_1 \cup \widehat{C}_2 \rightarrow \text{Tor}(P_\Delta)$ deforms into $\Phi^{(t)} : \mathbb{P}^1 \rightarrow \text{Tor}(P_\Delta)$ as $t^* < t < t^* + \varepsilon$. Then for any point $p' \neq p = \widehat{C}_1 \cap \widehat{C}_2$ the germ $\Phi(\widehat{C}, p')$ never intersects with the corresponding deformed germ $\Phi^{(t)}(\mathbb{P}^1, p'_t)$ except for the two cases when p' is mapped to one of the mobile points of $w(t^*)$. Further on, the curves $C_1 \cup C_2$ and $C^{(t)} = \Phi^{(t)}(\mathbb{P}^1)$ do not intersect in a neighborhood of the hyperbolic node $z = \Phi(p)$. This is a consequence of the Bézout's bound and [13, Theorem 2] (see also [12, Lemma II.2.18]): in a neighborhood of each singular point z' of $C_1 \cup C_2$ except for z , we have at least $2\delta(C_1 \cup C_2, z')$ intersections (which is twice the sum of pairwise intersection multiplicities of distinct local branches at z'), at each fixed point $w_i^\sigma(t^*)$ at least $2k_i^\sigma$ intersections, and in a neighborhood of a mobile point $w_i^\sigma(t^*)$ at least $2k_i^\sigma - 1$ intersections, which altogether amounts to $c_1(\mathcal{L}_{P_\Delta})^2$. That is, the geometry of the deformation $\Phi^{(t)} : \mathbb{P}^1 \rightarrow \text{Tor}(P_\Delta), 0 < t < t + \varepsilon$, is as follows: starting in a neighborhood of the mobile point $w_i^\sigma(t^*) \in C_1$, the immersed circle $\mathbb{R}C^{(t)}$ goes in a neighborhood of $\mathbb{R}C_1$ always on the same side of $\mathbb{R}C_1$ (with respect to the coorientation of $\mathbb{R}C_1$ induced by the complex orientation), say, in the positive side as shown in Figure 3(c) (where $\mathbb{R}C^{(t)}$ is designated by the dashed line), until it arrives to a neighborhood of the smoothed out node $z \in C_1 \cap C_2$, where we observe a deformation as shown in Figure 3(c). It follows that if z carries the positive intersection multiplicity, then the complex orientations of $C^{(t)}$ correspond to a pair of quantum indices $\kappa_1 - \kappa_2, \kappa_2 - \kappa_1$, and if z carries the negative intersection multiplicity, then the complex orientations of $C^{(t)}$ correspond to a pair of quantum indices $\kappa_1 + \kappa_2, -\kappa_1 - \kappa_2$. This completes the proof of the constancy of the numbers $W_0^\kappa(t)$ along the path $\{\widehat{w}(t)\}_{|t-t^*|<\varepsilon}$.

The cases when one or both maps $\Phi|_{\widehat{C}_1}, \Phi|_{\widehat{C}_2}$ are multiple coverings of their images can be treated in the same way, we leave the details to the reader.

(5) Suppose that ξ is as in Lemma 2.15(2iii) and $\Phi(p) = w_i^\sigma(t^*) =: w$. Observe that $(\Phi|_{\widehat{C}_s})^*(w) = 2l_s p, s = 1, 2$, where $l_1 + l_2 = k_i^\sigma$. Suppose that $\Phi|_{\widehat{C}_1}, \Phi|_{\widehat{C}_2}$ both are immersions and that $l_1 \leq l_2$.

Define the path $\{\widehat{w}(t)\}_{|t-t^*|<\varepsilon}$ by picking one mobile point in $C_1 \cap \widehat{w}(t^*) \setminus \{w_i^\sigma(t^*)\}$ and the other in $C_2 \cap \widehat{w}(t^*) \setminus \{w_i^\sigma(t^*)\}$. It defines a one-dimensional subvariety of $\overline{\mathcal{M}}_{0,n}(\Delta)$, and let $[\Phi_t : (\widehat{C}^{(t)}, \mathbf{p}_t) \rightarrow \text{Tor}(P_\Delta)], |t - t^*| < \varepsilon$, represents one of the irreducible germs of that variety at ξ , where $\widehat{C}^{(t)} \simeq \mathbb{P}^1$ as $t \neq 0$. The one-dimensional family \mathcal{F} of curves $C^{(t)} = \Phi_t(\widehat{C}^{(t)}) \in |\mathcal{L}_{P_\Delta}|, |t - t^*| < \varepsilon$, has a tangent cone at $C_1 \cup C_2$ spanned by $C_1 \cup C_2$ and some curve $C^* \in |\mathcal{L}_{P_\Delta}| \setminus \{C_1 \cup C_2\}$. We show that \mathcal{F} is

smoothly parameterized by t .

We start with describing the local behavior of C^* at the point w . Choose local equivariant coordinates x, y in a neighborhood of w so that $w = (0, 0)$, $\text{Tor}(\sigma) = \{y = 0\}$, $\text{Tor}_{\mathbb{R}}^+(P_{\Delta}) = \{y \geq 0\}$, $C_s = \{y = \eta_s x^{l_s} + \text{h.o.t.}\}$ with $\eta_s > 0$, $s = 1, 2$, where without loss of generality we assume $\eta_1 \neq \eta_2$. Thus, the equation of $C_1 \cup C_2$ has the Newton diagram ∇ at w as shown in Figures 4(a,b) by fat lines. We claim that C^* has under ∇ the monomial $x^{2l_1-1}y$ with a nonzero coefficient and no other monomials. Since C^* can be taken close to $C_1 \cup C_2$, its Newton diagram either coincides or lies below ∇ . The coincidence is not possible for the following reason. The curve C^* passes through $C_1 \cap C_2 \cap \text{Tor}(P_{\Delta})^{\times}$ (which consists of only nodes by Lemma 2.15(2iii)) and, in case C^* has Newton diagram ∇ at w , intersects C_1 at w with multiplicity $\geq 2l_1 = (C_1 \cdot C_2)_w$. Hence, by the Noether's fundamental theorem, C^* lies in the subspace of $|\mathcal{L}_{P_{\Delta}}|$ spanned by the curves of type $C_1 \cup C'_2$, $C'_2 \in |C_2|$, and $C'_1 \cup C_2$, $C'_1 \in |C_1|$. However, the Bézout type restriction dictates that $C'_1 = C_1$ and $C'_2 = C_2$: for example, at each point $z \in \text{Sing}(C_1)$, the curve C'_1 must induce in $\mathcal{O}_{C_1, z}$ an element of the conductor ideal (cf. [5, Theorem 4.15]), and hence $(C'_1 \cdot C_1)_z \geq 2\delta(C_1, z)$, at each fixed point $w_{i'}^{\sigma'}(t^*)$ of $\widehat{w}(t^*) \cap C_1$ (including w), we have $(C'_1 \cdot C_1)_{w_{i'}^{\sigma'}(t^*)} \geq 2k_{i'}^{\sigma'}$, and at last, at the mobile point $w_{i'}^{\sigma'}(t^*) \in \widehat{w}(t^*) \cap C_1$, we have $(C'_1 \cdot C_1)_{w_{i'}^{\sigma'}(t^*)} \geq 2k_{i'}^{\sigma'} - 1$, which altogether amounts to

$$\sum_{z \in \text{Sing}(C_1)} \delta(C_1, z) + c_1(\mathcal{L}_{P_{\Delta}})[C_1] - 1 = C_1^2 + 1.$$

Further on, a monomial in an equation of C^* below ∇ cannot be x^s , $0 \leq s < 2(l_1 + l_2)$, since $(C^* \cdot \text{Tor}(\Sigma))_w \geq 2(l_1 + l_2)$, and it cannot be $x^s y$ with $0 \leq s \leq 2l_1 - 2$. For the latter claim, we observe that each curve $C^{(t)}$, $t \neq 0$, has $2l_1 - 1$ nodes in a neighborhood of w and satisfies $C^{(t)} \cdot \text{Tor}(\sigma)_w = 2(l_1 + l_2)$. Thus, if the tangent line at $C^{(t)}$ to the considered family of curves is spanned by $C^{(t)}$ and $(C^*)^{(t)}$, then $C^{(t)}$ and $(C^*)^{(t)}$ intersect in a neighborhood of w with multiplicity $\geq 2(2l_1 - 1) + 2(l_1 + l_2) = 4l_1 + 2l_2 - 2$. Hence, C^* and $C_1 \cup C_2$ intersect at w with multiplicity $\geq 4l_1 + 2l_2 - 2$, which leaves the only possibility of the extra monomial $x^{2l_1-1}y$.

If $l_1 = l_2 = l$, we can write down an equation of $C^{(t)}$ in a neighborhood of w in the form

$$y^2(1 + O(t_1^{>0})) - (\eta_1 + \eta_2 + O(t_1^{>0}))x^{2l}y + x^{4l}(\eta_1\eta_2 + O(t_1^{>0})) + \text{h.o.t.} + \sum_{r=0}^{2l-1} a_{r1}(t_1)x^r y = 0,$$

where h.o.t. designates the terms above the Newton diagram, and

$$t_1 = t - t^*, \quad a_{r1}(0) = 0 \quad \text{for all } r = 0, \dots, 2l - 1.$$

Consider the tropical limit at $t_1 \rightarrow 0$ (see Section 3.2). It includes a subdivision of the triangle $T = \text{conv}\{(0, 1), (4l, 0), (0, 2)\}$ and certain limit curves in the toric surfaces associated with the pieces of the subdivision. Since locally $C^{(t)}$ is an immersed cylinder, we obtain that the union of the limit curves is a curve of arithmetic genus

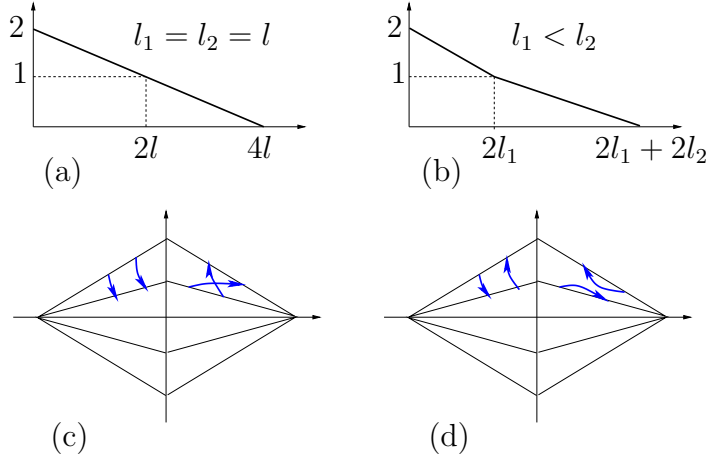


Figure 4: Proof of Lemma 2.16, part (5), I

zero, which finally allows the only following tropical limit: the triangle T is the unique piece of the subdivision, and

$$a_{r1}(t_1) = t_1^{\lambda(2l-r)}(a_{r1}^0 + O(t_1^{>0})), \quad r = 0, \dots, 2l - 1 ,$$

where $a_{2l-1,1}^0 \neq 0$ in view of the above conclusion on the Newton diagram of C^* , and in addition, the limit curve with Newton triangle T given by

$$y^2 - (\eta_1 + \eta_2)x^{2l}y + \eta_1\eta_2x^{4l} + \sum_{r=0}^{2r-1} a_{r1}^0 x^r y = 0 \quad (18)$$

is rational. It also follows, that \mathcal{F} is smooth, regularly parameterized by t_1 (*i.e.*, $\lambda = 1$), and its real part submersively projects onto the path $\{\widehat{\mathbf{w}}(t)\}_{|t-t^*|<\varepsilon}$. By the patchworking theorem [27, Theorems 3.1 and 4.2], we uniquely restore \mathcal{F} as long as we compute the coefficients a_{r1}^0 , $0 \leq r \leq 2l_1 - 2$ (here $a_{2l-1,1}^0$ is determined by C^*). The rationality of the curve (18) can be expressed as follows. Write equation (18) in the form $y^2 - 2P(x)y + \eta_1\eta_2x^{4l} = 0$, then resolve with respect to y :

$$y = P(x) \pm \sqrt{P(x)^2 - \eta_1\eta_2x^{4l}} = P(x) \pm x^{2l} \sqrt{Q(1/x)^2 - \eta_1\eta_2}, \quad \deg Q = 2l .$$

The rationality means that the expression under the radical has $2l - 1$ double roots (corresponding to the nodes of the curve). This means that Q is a modified Chebyshev polynomial:

$$Q(u) = \lambda_1 \text{Cheb}_{2l}(u + \lambda_2) + \lambda_3, \quad \text{Cheb}_{2l}(u) = \cos(2l \cdot \arccos u) ,$$

where $\lambda_1, \lambda_2, \lambda_3$ can be computed out of η_1, η_2 , and $a_{2l-1,1}^0$. As noticed in [28, Proof of Proposition 6.1], there are exactly two real solutions, one corresponding to a curve with the real part, shown in Figure 4(c) by lines inside the four Newton triangles designating four real quadrants, and this curve has one hyperbolic node and $2l - 2$ non-real nodes, while the other curve, shown in Figure 4(d), has $2l - 1$ elliptic nodes. Bringing the complex orientations to the play, we see that the

former solution appears when the local real branches $(\mathbb{R}C_1, w)$ and $(\mathbb{R}C_2, w)$ are cooriented, while the latter solution corresponds to the case when these real branches have opposite orientations, see Figures 4(c,d). That is, on both sides of the path $\{\mathbf{w}(t)\}_{|t-t^*|<\varepsilon}$ we encounter four elements of $\overrightarrow{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))$ having quantum indices $\kappa_1 + \kappa_2, -\kappa_1 - \kappa_2, \kappa_1 - \kappa_2, \kappa_2 - \kappa_1$. It remains to notice that the Welschinger signs (4) of the elements with quantum indices $\kappa_1 + \kappa_2, -\kappa_1 - \kappa_2$ are the same on both sides, since the modified limit curves (see lemma 3.4(3)) shown in Figure 4(c) for $t > t^*$ and $t < t^*$ are obtained from each other by reflection with respect to the vertical axis. The same holds for the elements with quantum indices $\kappa_1 - \kappa_2, \kappa_2 - \kappa_1$. This completes the proof of the constancy of the numbers $W_0^\kappa(t)$ in the considered case $l_1 = l_2$.

In the case $l_1 < l_2$, an equation of $C^{(t)}$ in a neighborhood of w takes the form

$$(1+O(t_1^{>0}))y^2 - (\eta_1+O(t_1^{>0}))x^{2l_1}y + (\eta_1\eta_2+O(t_1^{>0}))x^{2(l_1+l_2)} + \sum_{r=0}^{2l_1-1} a_{r1}(t_1)x^r y + \text{h.o.t.} = 0,$$

where $t_1 = t - t^*$, $a_{r1}(0) = 0$ for all $r = 0, \dots, 2l_1 - 1$, and h.o.t. stands for the terms above the Newton diagram. In this situation, due to the condition of arithmetic genus zero, the tropical limit is defined uniquely: the area under the Newton diagram is divided into two triangles

$$\text{conv}\{(0, 1), (2l_1, 1), (2(l_1 + l_2), 0)\} \quad \text{and} \quad \text{conv}\{(0, 1), (2l_1, 0), (0, 2)\}$$

(see Figure 5(a)), while the monomials on the segment $[(0, 1), (2l_1, 1)]$ sum up to $\eta_1(x + \lambda t_1)^{2l_1}y$, where λ is uniquely determined by the coefficient $a_{2l_1-1,1}^0$ coming from C^* . The two limit curves have branches intersecting each other with multiplicity $2l_1$ along the toric divisor $\text{Tor}([(0, 1), (2l_1, 0)])$ (see Figure 5(b)). The genuine geometry of $C^{(t)}$ can be recovered when we deform that intersection point into $2l_1 - 1$ nodes. The modification (see Section 3.3) describes such a deformation as a replacement of the intersection point by one of the $2l_1$ modified limit curves in the sense of Lemma 3.4(3), among which exactly two are real, and their real parts are shown in Figure 5(c,d) (cf. Figures 4(c,d)): one of them has a hyperbolic node and $2l_1 - 2$ non-real nodes, and the other $2l_1 - 1$ elliptic nodes. As in the preceding paragraph, the former modified limit curve fits the case when the local real branches $(\mathbb{R}C_1, w), (\mathbb{R}C_2, w)$ are cooriented, which means that on each side of the path $\{\widehat{\mathbf{w}}(t)\}_{|t-t^*|<\varepsilon}$ we have two elements of $\overrightarrow{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))$ with quantum indices $\kappa_1 + \kappa_2, -\kappa_1 - \kappa_2$, while all four elements have the same Welschinger sign (4) since the constructions for $t > t^*$ and $t < t^*$ are symmetric with respect to the vertical axis. Hence, the constancy of the numbers $W_0^\kappa(t)$. The same holds in the case of the opposite orientation of the branches $(\mathbb{R}C_1, w), (\mathbb{R}C_2, w)$ with the use of the second modified limit curve.

The remaining case is as follows: $\Phi : \widehat{C}_1 \rightarrow \text{Tor}(P_\Delta)$ is an $2l_1$ -multiple covering of a line through w and $\Phi : \widehat{C}_2 \rightarrow \text{Tor}(\Sigma)$ is an immersion onto a rational curve having a smooth branch at w intersecting $\text{Tor}(\sigma)$ with multiplicity $2l_2$. Here we remove the restriction $l_1 \leq l_2$. We extend the range of the coordinate system x, y to a neighborhood of the line $C_1 = \Phi(\widehat{C}_1)$, assuming that the axis $\{x = 0\}$ is the

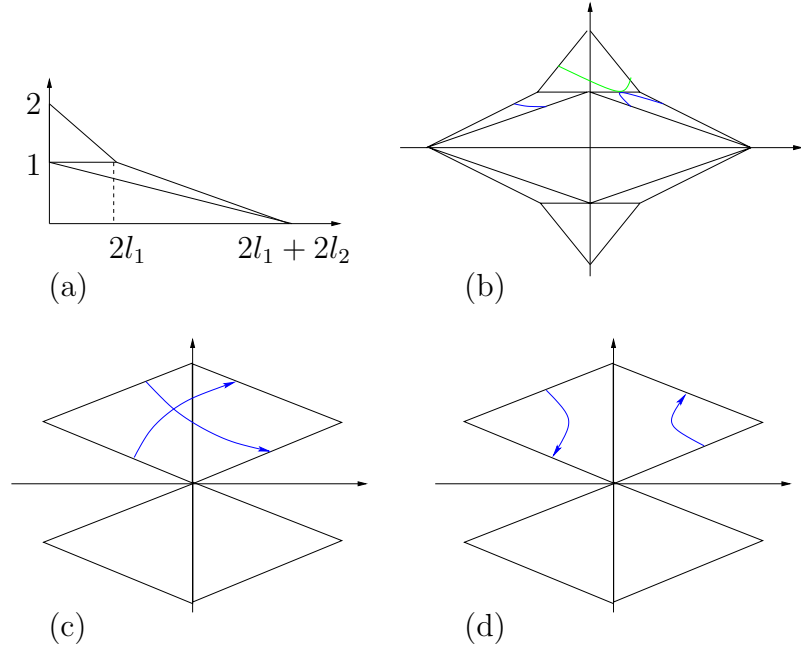


Figure 5: Proof of Lemma 2.16, part (5), II

whole line C_1 . Then the Newton diagram of $\Phi_*(\widehat{C})$ in this neighborhood of C_1 is as shown in Figure 6(a) by fat lines, where $s = C_1C_2$. Following the recipe used in the preceding cases, we obtain that the tropical limit of the family $\{C^{(t)}\}_{|t-t^*|<\varepsilon}$ as $t_1 := t - t^* \rightarrow 0$ includes the triangle $\text{conv}\{(0, 1), (2l_1, 1), (2l_1 + l_2), 0\}$ and the rectangle $\text{conv}\{(0, 1), (0, s + 1), (2l_1, 1), (2l_1, s + 1)\}$ (see Figure 6(b)), and, in addition, the part of the equation of $C^{(t)}$ restricted to the union of these polygons is as follows:

$$y \cdot ((x - \lambda_1 t_1)^{2l_1} - a(x - \lambda_2 t_1)^{2l_1} y) \prod_{j=1}^{s-1} (y - y_j) - \eta x^{2(l_1+l_2)} = 0 ,$$

where y_1, \dots, y_{s-1} are the coordinates of the points $C_1 \cap C_2 \setminus \{w\}$, $\lambda_1 \neq 0$ is determined by C^* , $\lambda_2 t_1 \neq 0$ is the current coordinate of the mobile point from $C_1 \cap \widehat{w}(t^*) \setminus \{w\}$, and $a, \eta > 0$ are determined by C_2 . Again the two limit curves have intersection of multiplicity $2l_1$ in a point on the toric divisor $\text{Tor}([(0, 1), (2l_1, 1)])$. The genuine geometry of $C^{(t)}$ in a neighborhood of w is obtained by deforming this tangency point by means of the two real modified limit curves (see Lemma 3.4(3)), which we used in the preceding paragraph. Then, the argument from the preceding paragraph in the same manner completes the proof of the constancy of the numbers $W_0^\kappa(t)$ in the considered wall-crossing bifurcation.

(6) Suppose that ξ is as in Lemma 2.15(2iv). Without loss of generality we can suppose that

$$P = \text{conv}\{(0, 2m), (2p, 0), (2q, 0)\}, \quad 0 \leq p < q, \quad m < q ,$$

and that Φ is ramified at the points of $\widehat{w}(t^*)$ on the toric divisors associated with the segments $[(0, 2m), (2p, 0)]$ and $[(0, 2m), (2q, 0)]$, while the two points $w_1^\sigma(t), w_2^\sigma(t) \in \mathbf{w}(t)$ on the toric divisor $\text{Tor}(\sigma)$, where $\sigma = [(2p, 0), (2q, 0)]$, merge to one point as

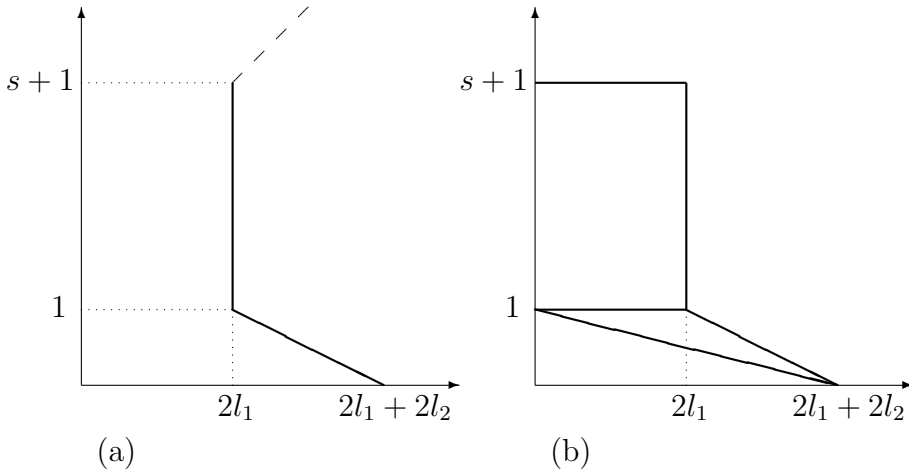


Figure 6: Proof of Lemma 2.16, part (5), III

$t \rightarrow t^*$. Then $C = \Phi(\mathbb{P}^1)$ admits a parametrization

$$x = a\theta^m, \quad y = b\theta^p(\theta - 1)^{q-p}, \quad \theta \in \mathbb{C},$$

and correspondingly Φ is given by

$$x = a\theta^{2m}, \quad y = b\theta^{2p}(\theta^2 - 1)^{q-p}, \quad \theta \in \mathbb{C}.$$

Assuming that the path $\{\widehat{\mathbf{w}}(t)\}_{|t-t^*|<\varepsilon}$, is defined by fixing the point w_1^σ , associated with $\theta = 1$, and the point of $\widehat{\mathbf{w}}(t^*)$ on $\text{Tor}([(0, 2m), (2q, 0)])$, we obtain a one-parameter deformation of Φ

$$x = a\theta^{2m}, \quad y = b\theta^{2p}(\theta - 1)^{q-p}(\theta + 1 + \lambda)^{q-p}, \quad |\lambda| \ll 1. \quad (19)$$

This deformation is regularly parameterized by the difference of the x -coordinates of $w_2^\sigma(t)$ and $w_1^\sigma(t)$ equal to $2ma\lambda + O(\lambda^2)$. It follows from formulas (19) that the corresponding element $\xi(t) \in \overline{\mathcal{M}}_{0,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))$ has the same orientations at a point on the toric divisor $\text{Tor}([(0, 2m), (2p, 0)])$ for $\lambda < 0$ and for $\lambda > 0$, and so does for the point on the toric divisor $\text{Tor}([(0, 2m), (2q, 0)])$, while the local branches at $w_1^\sigma(t)$, $w_2^\sigma(t)$ have opposite orientations with respect to the orientation of $\text{Tor}_{\mathbb{R}}(\sigma)$. Furthermore, $\xi(t)$ has exactly two elliptic nodes in a neighborhood of each elliptic node of C . Thus, the constancy of $W_0^\kappa(t)$ follows. \square

2.5 Proof of Theorem 2.11

Take two sequences $\mathbf{w}(0) = (\mathbf{w}_\partial(0), w_0(0))$ and $\mathbf{w}(1) = (\mathbf{w}_\partial(1), w_0(1))$, satisfying the conditions of Theorem 2.11. We may assume that $\widehat{\mathbf{w}}_\partial(0), \widehat{\mathbf{w}}_\partial(1) \in M_{\mathbb{R}}^\tau(\Delta)$ for some $\tau > 0$. Then, we join these sequences by a generic path $\{\widehat{\mathbf{w}}_\partial(t)\}_{0 \leq t \leq 1}$, in $M_{\mathbb{R}}^\tau(\Delta)$ and the points $w_0(0), w_0(1)$ by a generic path $\{w_0(t)\}_{0 \leq t \leq 1}$ in \mathfrak{Q} , and verify the constancy of $W_1^\kappa(\Delta, \mathbf{w}(t))$ in all possible wall-crossing events, for all κ .

Similarly to the proof of Theorem 2.5, we start with the verification of the fact that $W_1^\kappa(\Delta, \mathbf{w}(t))$ does not change along intervals $t' < t < t''$ such that $\overline{\mathcal{M}}_{1,n}^{\mathbb{R}}(\Delta, \mathbf{w}(t)) = \mathcal{M}_{1,n}^{\mathbb{R}}(\Delta, \mathbf{w}(t))$ for all $t' < t < t''$ (that is, the elements of $\mathcal{M}_{1,n}^{\mathbb{R}}(\Delta, \mathbf{w}(t))$ do not degenerate as long as $t' < t < t''$). To this end, it is enough to show that the projection $\mathcal{M}_{1,n}^{\mathbb{R}}(\Delta, \mathbf{w}(t))_{t' < t < t''} \rightarrow (t', t'')$ has no critical points (cf. the proof of Theorem 2.5). This requirement amounts to the following statement (cf., Lemma 2.14 and [29, Lemma 2.3]).

Lemma 2.17 *Let $\xi = [\Phi : (\mathbb{E}, \mathbf{p}) \rightarrow \text{Tor}(P_\Delta)] \in \mathcal{M}_{1,n}^{\mathbb{R}}(\Delta, \mathbf{w}(t))$. Then*

$$H^0 \left(\mathbb{E}, \mathcal{N}_\Phi \left(-p_0 - \sum_{\sigma \neq \sigma_0, i \neq i_0} 2k_i^\sigma p_i^\sigma - (2k_{i_0}^{\sigma_0} - 1)p_{i_0}^{\sigma_0} \right) \right) = 0. \quad (20)$$

Proof. We proceed along the lines of the proof of [29, Lemma 2.3]. Assume that $H^0 \neq 0$ in (20). Then, there exists a real curve $C' \in |\mathcal{L}_{P_\Delta}| \setminus \{C\}$, where $C = \Phi_*(\mathbb{E})$, which intersects C

- at each point $q \in \text{Sing}(C)$ with multiplicity $\geq 2\delta(C, q)$,
- at each point w_i^σ , $(\sigma, i) \neq (\sigma_0, i_0)$ with multiplicity $\geq 2k_i^\sigma$,
- at $w_{i_0}^{\sigma_0}$ with multiplicity $\geq 2k_{i_0}^{\sigma_0} - 1$,
- and at w_0 .

Since C has in $\overline{\mathfrak{Q}}$ a null-homologous immersed circle S , there must be an additional intersection point $w' \in C' \cap S$, not mentioned in the above list. However, then

$$\begin{aligned} C' C &\geq 2 \sum_{q \in \text{Sing}(C)} \delta(C, q) + \sum_{\sigma \in P_\Delta^1} \sum_{i=1}^{n^\sigma} 2k_i^\sigma - 1 + 2 \\ &= (C^2 - c_1(\text{Tor}(P_\Delta)[C])) + c_1(\text{Tor}(P_\Delta)[C] - 1 + 2) = C^2 + 1, \end{aligned}$$

which is a contradiction. \square

The set of elements $(\widehat{\mathbf{w}}_\partial, w_0) \in M_{\mathbb{R}}^{\mathbb{I}}(\Delta) \times \mathfrak{Q}$ satisfying

$$\overline{\mathcal{M}}_{1,n}^{\mathbb{R}}(\Delta, \mathbf{w}) = \mathcal{M}_{1,n}^{\mathbb{R}}(\Delta, \mathbf{w}) \quad \text{and} \quad (\widehat{\mathbf{w}}_\partial, w_0) \in M_1,$$

is a dense semialgebraic subset of full dimension $\dim M_{\mathbb{R}}^{\mathbb{I}}(\Delta) \times \mathfrak{Q} = n + 1$. The complement is the union of finitely many semialgebraic strata of codimension ≥ 1 . Since the path $\{(\widehat{\mathbf{w}}_\partial(t), w_0(t))\}_{0 \leq t \leq 1}$ is generic, it avoids strata of $M_{\mathbb{R}}^{\mathbb{I}}(\Delta) \times \mathfrak{Q}$ of codimension ≥ 2 and intersects strata of codimension one only in their generic points.

Now, we study the strata S of the dimension n in $(M_{\mathbb{R}}^{\mathbb{I}} \times \mathfrak{Q}) \setminus M_1$ in terms of the geometry of elements $\xi \in \overline{\mathcal{M}}_{1,n}^{\mathbb{R}}(\Delta, \mathbf{w}(t)) \setminus \mathcal{M}_{1,n}^{\mathbb{R}}(\Delta, \mathbf{w}(t))$, $0 \leq t \leq 1$.

Lemma 2.18 (1) The following elements $\xi = [\Phi : (\widehat{C}, \widehat{\mathbf{p}}) \rightarrow \text{Tor}(P_\Delta)]$ cannot occur in $\overline{\mathcal{M}}_{1,n}^{\mathbb{R}}(\Delta, \mathbf{w}(t)) \setminus \mathcal{M}_{1,n}^{\mathbb{R}}(\Delta, \mathbf{w}(t))$, $0 \leq t \leq 1$, as $(\widehat{\mathbf{w}}_\partial(t), w_0(t)) \in S$:

- (1i) \widehat{C} is a reducible curve of arithmetic genus 1 with a component mapped onto a toric divisor;
- (1ii) \widehat{C} is a connected curve of arithmetic genus 1 either with at least three irreducible components, or with two rational irreducible components and no other components.

(2) If $\xi = [\Phi : (\widehat{C}, \widehat{\mathbf{p}}) \rightarrow \text{Tor}(P_\Delta)] \in \overline{\mathcal{M}}_{1,n}(\Delta, \mathbf{w}(t^*))$, where $(\widehat{\mathbf{w}}_\partial(t^*), w_0(t^*))$ is a generic element in an n -dimensional stratum in $(M_{\mathbb{R}}^r \times \Omega) \setminus M_1$, then ξ is of one of the following types:

(2a) either $w_0(t^*)$ is a singular point of the curve $C = \Phi_*(\widehat{C})$, and the following holds:

- (2i) $\widehat{\mathbf{w}}_\partial(t^*)$ consists of n distinct points, the curve \widehat{C} is smooth elliptic, Φ is an immersion onto a curve C that is smooth along the toric divisors, while $w_0(t^*)$ is a center of at least one real local branch;

(2b) or $w_0(t^*)$ is a smooth point of the curve $C = \Phi(\widehat{C})$, and one of the following holds:

- (2ii) $\widehat{\mathbf{w}}_\partial(t^*)$ consists of n distinct points, the curve \widehat{C} is smooth elliptic, Φ is birational onto its image, but not an immersion; furthermore, $C = \Phi(\widehat{C})$ is smooth at $w_0(t^*)$ and at each point $w_i^\sigma(t^*)$ with $k_i^\sigma \geq 2$, and is unibranch at each point $w_i^\sigma(t^*)$ with $k_i^\sigma = 1$;
- (2iii) $w_i^\sigma(t^*) = w_j^\sigma(t^*)$ for some $\sigma \in P_\Delta^1$ and $i \neq j$, the curve \widehat{C} is smooth elliptic, and the map Φ is an immersion such that the point $w_i^\sigma(t^*) = w_j^\sigma(t^*)$ is a center of one or two smooth branches;
- (2iv) $\widehat{\mathbf{w}}_\partial(t^*)$ consists of n distinct points, $\widehat{C} = \widehat{C}_1 \cup \widehat{C}_2$, where $\widehat{C}_1 \simeq \mathbb{P}^1$ and \widehat{C}_2 is a smooth elliptic curve, the intersection $\widehat{C}_1 \cap \widehat{C}_2$ consisting of one point p ; the map $\Phi : \widehat{C}_1 \rightarrow \text{Tor}(P_\Delta)$, $s = 1, 2$, is either an immersion, smooth along $\text{Tor}(\partial P_\Delta)$, or a multiple covering of a line intersecting only two toric divisors, while these divisors correspond to opposite parallel sides of P_Δ and the intersection points with these divisors are ramification points of the covering; the map $\Phi : \widehat{C}_2 \rightarrow \text{Tor}(P_\Delta)$ is an immersion, smooth along $\text{Tor}(\partial P_\Delta)$; the point p either is mapped to $\text{Tor}(P_\Delta)^\times$, and then the curves $C_1 = \Phi(\widehat{C}_1)$, $C_2 = \Phi(\widehat{C}_2)$ intersect only in $\text{Tor}(P_\Delta)^\times$ and each of their intersection point is an ordinary node, or p is mapped to some point $w_i^\sigma(t^*)$, and the curves $C_i = \Phi(\widehat{C}_i)$, $i = 1, 2$, do not have other common point in $\widehat{\mathbf{w}}_\partial(t^*)$;
- (2v) $\widehat{\mathbf{w}}_\partial(t^*)$ consists of n distinct points, \widehat{C} is an irreducible rational curve with one node p , the map Φ is an immersion that sends p to some point $w_i^\sigma(t^*)$,

center of two smooth branches intersecting $\text{Tor}(\sigma)$ with odd multiplicities $2l_1 + 1, 2l_2 + 1$, where $l_1 + l_2 + 1 = k_i^\sigma$, and intersecting each other with multiplicity $\min\{2l_1 + 1, 2l_2 + 1\}$, furthermore, each of the other points of $\widehat{\mathbf{w}}_\partial(t^*)$ is a center of one smooth branch;

(2vi) \widehat{C} is a smooth elliptic curve, $\widehat{\mathbf{w}}_\partial(t^*)$ consists of $n = 4$ points, the curve $C = \Phi(\mathbb{P}^1)$ is rational and smooth at $\widehat{\mathbf{w}}_\partial(t^*)$, and $\Phi : \widehat{C} \rightarrow C$ is a double covering ramified at $\widehat{\mathbf{w}}_\partial(t^*)$.

Proof. (1i) In such a case, we must have $\widehat{C} = \widehat{C}' \cup \widehat{C}''$, where \widehat{C}' is a connected curve of arithmetic genus one such that \widehat{C}' is mapped onto the union of all toric divisors, and \widehat{C}'' is a non-empty (due to $w_0 \in \text{Tor}(P_\Delta)^\times$) union of connected curves of arithmetic genus zero, each one joined with \widehat{C}' in one point (cf. the proof of Lemma 2.15(1i)). Let \widehat{C}''_0 be a connected component of \widehat{C}'' such that $w_0(t^*) \in \Phi(\widehat{C}''_0)$. Note that all local branches of $\Phi : \widehat{C}''_0 \rightarrow \text{Tor}(P_\Delta)$ centered on toric divisors, except for $\Phi : (\widehat{C}''_0, p) \rightarrow \text{Tor}(P_\Delta)$, where $p = \widehat{C}' \cap \widehat{C}''_0$, are, in fact, centered at some points of $\widehat{\mathbf{w}}_\partial(t^*)$. Observe that either \widehat{C}''_0 intersects at least three toric divisors of $\text{Tor}(P_\Delta)$, or it intersects two toric divisors corresponding to parallel sides of P_Δ . However, in the both cases we reach a contradiction with (AQC).

(1ii) In the case of at least 3 irreducible components, we follow the lines of the proof of Lemma 2.15(1ii). If \widehat{C} contains an elliptic component and at least 2 rational components, then we similarly obtain at least three independent Menelaus type conditions for $\widehat{\mathbf{w}}_\partial(t^*)$, which bounds from above the dimension of the considered stratum by $(n - 3) + 2 = n - 1 < n$, a contradiction. If all irreducible components of \widehat{C} are rational, then we encounter at least two independent Menelaus conditions on $\widehat{\mathbf{w}}_\partial(t^*)$ as well as a condition on the position of the point $w_0(t^*)$ due to the finiteness statement of Lemma 2.4, which altogether bounds from above the dimension of the considered stratum by $(n - 2) + 1 = n - 1 < n$.

(2a) Let $w_0(t^*)$ be a singular point of C . Then, for the dimension reason, $\widehat{\mathbf{w}}_\partial(t^*)$ must be a generic element of $M_{\mathbb{R}}^r$. This implies, in particular, that \widehat{C} is irreducible and $\widehat{\mathbf{w}}_\partial(t^*)$ consists of n distinct points. Furthermore, \widehat{C} cannot be a rational curve with a node. Thus, \widehat{C} is a smooth elliptic curve. Then (see Lemma 2.4) we derive that $\Phi : \widehat{C} \rightarrow \text{Tor}(P_\Delta)$ is an immersion and C is smooth along the toric divisors. Since $w_0(t^*)$ turns into a real smooth point in the deformation along the path $\{(\widehat{\mathbf{w}}_\partial(t), w_0(t))\}_{0 \leq t \leq 1}$, it must be a center of at least one real local branch.

(2b) From now on we can assume that $w_0(t^*)$ is a smooth point of $C = \Phi_*(\widehat{C})$.

Suppose that \widehat{C} is a smooth elliptic curve and $\widehat{\mathbf{w}}_\partial(t^*)$ consists of n distinct points. Then, we obtain the claim (2ii) due to Lemma 2.4 and the claim (1ii) above.

Suppose that some of the points of the sequence $\widehat{\mathbf{w}}_\partial(t^*)$ coincide. For the dimension reason, we immediately get that $\widehat{\mathbf{w}}_\partial(t^*)$ consists of $n - 1$ distinct points, and all these points must be in general position subject to the unique Menelaus relation (1). Hence, \widehat{C} must be irreducible. Moreover, \widehat{C} cannot be rational, since otherwise, by Lemma 2.4, one would get the total dimension of the considered stratum

$\leq (n - 2) + 1 = n - 1 < n$, a contradiction. Thus, \widehat{C} is a smooth elliptic curve. We have $w_i^\sigma(t^*) = w_j^\sigma(t^*) = w$ for some $\sigma \subset \partial P$ and $i \neq j$. If C is unibranch at each point of $\widehat{\mathbf{w}}_\partial(t^*)$, then, by Lemma 2.4, the curve C is immersed and smooth along the toric divisors. Otherwise, the curve C has two local branches at w . Moreover, it must be unibranch at each point $w_i^{\sigma'}(t^*) \neq w$ and, in addition, smooth if $k_i^{\sigma'} \geq 2$ due to Lemma 2.4 and the claim (1ii). Let us show that C is immersed. Fixing the position of $w_0(t^*)$ and the position of one more point $w' \in \widehat{\mathbf{w}}_\partial(t^*) \setminus \{w\}$, we obtain a family of dimension $\geq n - 2 \geq 2$; hence, [16, Inequality (5) in Lemma 2.1] applies:

$$\begin{aligned} c_1(\mathrm{Tor}(P_\Delta))c_1(\mathcal{L}_{P_\Delta}) &\geq (c_1(\mathrm{Tor}(P_\Delta))c_1(\mathcal{L}_{P_\Delta}) - n + 2) + 1 + \sum_B (\mathrm{ord}B - 1) + (n - 3) \\ &= c_1(\mathrm{Tor}(P_\Delta))c_1(\mathcal{L}_{P_\Delta}) + \sum (\mathrm{ord}B - 1) , \end{aligned}$$

where B runs over all singular local branches of C in $\mathrm{Tor}(P_\Delta)^\times \cup \{w, w'\}$. That is, C is immersed, and we finally fit the requirements of claim (2iii).

Thus, we are left with the case of $\widehat{\mathbf{w}}_\partial(t^*)$ consisting of n distinct points and \widehat{C} either consisting of two components, or being a rational curve with a node. Observe that the case of two rational components is not possible, since by Lemma 2.4, the dimension of such a stratum would not exceed $(n - 2) + 1 = n - 1 < n$.

Suppose that \widehat{C} is as in item (2iv). For the dimension reason, the sequence $\widehat{\mathbf{w}}_\partial(t^*)$ is in general position subject to exactly two Menelaus type relations, while $w_0(t^*)$ is in general position in \mathfrak{Q} . Thus, by Lemma 2.4, we obtain the immersion and smoothness statements as required. The rest of the argument literally coincides with the corresponding part of the proof of Lemma 2.15(2iii).

Suppose that \widehat{C} is a rational curve with a node p . Then, this node cannot be mapped to $\mathrm{Tor}(P_\Delta)^\times$, since otherwise one would encounter a real rational curve with two one-dimensional branches, one in $\mathrm{Tor}_\mathbb{R}^+(P_\Delta)$ and the other in $\overline{\mathfrak{Q}}$. Hence the node is mapped to $w \in \widehat{\mathbf{w}}_\partial(t^*)$, and for the above reason, the intersection multiplicities of the two branches of $C = \Phi(\widehat{C})$ at w are odd. Observe that $\Phi' : \mathbb{P}^1 \rightarrow \widehat{C} \rightarrow \mathrm{Tor}(P_\Delta)$ is not a multiple covering of the image. Indeed, in such a case, one would have that P is a triangle, $l_1 = l_2 = l$, and $C = \Phi(\widehat{C})$ is a rational curve intersecting at least two of the toric divisors with odd multiplicity, the double covering being ramified at the two points of $\widehat{\mathbf{w}}_\partial(t^*) \setminus \{w\}$. With an automorphism of \mathbb{Z}^2 , we can turn the Newton triangle of C (which is $\frac{1}{2}P_\Delta$) into the triangle

$$\mathrm{conv}\{(0, s), (r, 0), (r + 2l + 1, 0)\}, \quad \mathrm{gcd}(s, r) \equiv 1 \pmod{2} ,$$

with $w \in \mathrm{Tor}([(r, 0), (r + 2l + 1, 0)])$ and C given by a parametrization (in some affine coordinates x, y)

$$x = a\theta^s, \quad y = b\theta^r(\theta - 1)^{2l+1}, \quad a, b \in \mathbb{R}^\times .$$

Since $x(1) > 0$, we get $a > 0$, and hence $x > 0$ as $0 < \theta \ll 1$. Then $y > 0$ for $0 < \theta \ll 1$, and hence $b < 0$. Thus, if r and s are odd, we obtain $x > 0, y < 0$ as $\theta \rightarrow +\infty$, and $x < 0, y < 0$ as $\theta \rightarrow -\infty$. If r is even and s is odd, we get

$x > 0, y < 0$ as $\theta \rightarrow +\infty$, and $x < 0, y > 0$ as $\theta \rightarrow -\infty$. In both cases, it follows that the intersection point with the toric divisor $\text{Tor}([(0, s), (r + 2l + 1, 0)])$ is out of $\partial\text{Tor}_{\mathbb{R}}^+(P_{\Delta})$ against the initial assumption. If r is odd and s is even, we get that the segment of $\mathbb{R}C$ in the quadrant $\{x > 0, y < 0\}$ touches all three toric divisors, which contradicts the condition (AQC). Thus, a multiple covering is excluded. The rest of claims can be derived in the same way as the statement of Lemma 2.15(2ii) with one exception: we shall show that the local branches of C at w intersect each other with multiplicity $\min\{2l_1 + 1, 2l_2 + 1\}$. For $l_1 \neq l_2$, this is immediate. Suppose that $l_1 = l_2 = l$. If $n = 3$, then P is a triangle and, in the above setting, we get the parametrization of C in the form

$$x = a\theta^{2s}, \quad y = b\theta^{2r}(\theta - 1)^{2l+1}(\theta - \theta_0)^{2l+1}, \quad \theta_0 \neq 1, \quad a, b \in \mathbb{R}^{\times}.$$

Since $x(1) = x(\theta_0)$, we obtain the only real solution $\theta_0 = -1$, and hence $y = b\theta^{2r}(\theta^2 - 1)^{2l-1}$ is a function of θ^2 as well as x is, which means that this is a double covering, forbidden above.

Suppose now that \widehat{C} is a smooth elliptic curve and $\Phi : \widehat{C} \rightarrow \text{Tor}(P_{\Delta})$ is a multiple covering of its image. Since the preimages of at least $n - 2$ points of $\widehat{\mathbf{w}}_{\partial}(t^*)$ in \widehat{C} are irreducible, the map $\Phi : \widehat{C} \rightarrow C \subset \text{Tor}(P_{\Delta})$ is ramified, and hence C cannot be elliptic by the Riemann-Hurwitz formula. Thus, C is rational, which for dimension reason implies that $\widehat{\mathbf{w}}_{\partial}(t^*)$ consists of n points in general position subject to one Menelaus condition. In particular, C is smooth at each point of $\widehat{\mathbf{w}}_{\partial}(t^*)$, and Φ has ramification index s at all points of $\widehat{\mathbf{w}}_{\partial}(t^*)$. By the Riemann-Hurwitz formula,

$$0 \leq 2s - (s - 1)n.$$

Hence, $n \leq \frac{2s}{s-1}$, that is, $s = n = 3$ or $s = 2, 3 \leq n \leq 4$. If $n = s = 3$, the rational curve C intersects toric divisors with even multiplicity at each point of $\widehat{\mathbf{w}}_{\partial}(t^*)$; thus, the one-dimensional real branch of C entirely lies in the quadrant $\text{Tor}_{\mathbb{R}}^+(P_{\Delta})$ and cannot hit the point $w_0(t^*)$, which is a contradiction. The case of $s = 2$ and $n = 3$ is not possible either. Indeed, then P must be a triangle, and $\mathbb{R}C$ contains an immersed segment in \mathfrak{Q} joining points on two sides of $\partial\mathfrak{Q}$, but this contradicts the condition (AQC). The remaining option is a stated in item (2vi). \square

We complete the proof of Theorem 2.11 with the following lemma.

Lemma 2.19 *Let $\{(\widehat{\mathbf{w}}_{\partial}(t), w_0(t))\}_{0 \leq t \leq 1}$ be a generic path in $M_{\mathbb{R}}^{\tau} \times \mathfrak{Q}$, and let $t^* \in (0, 1)$ be such that $\overline{\mathcal{M}}_{1,n}^{\mathbb{R}}(\Delta, \mathbf{w}(t^*))$ contains an element ξ as described in one of the items of Lemma 2.18(2). Then, for each $\kappa \in \frac{1}{2}\mathbb{Z}$ such that $|\kappa| \leq \mathcal{A}(\Delta)$, the numbers $W_1^{\kappa}(t) := W_1^{\kappa}(\Delta, \mathbf{w}(t))$ do not change as t varies in a neighborhood of t^* .*

Proof. We always can assume that, in a neighborhood of t^* , the path $\{(\widehat{\mathbf{w}}_{\partial}(t), w_0(t))\}_{0 \leq t \leq 1}$ is defined by fixing the position of $w_0(t^*)$ and some $n - 2$ points of $\widehat{\mathbf{w}}_{\partial}(t^*)$, while the other two points remain mobile. Except for the case of Lemma 2.18(2iv) describing reducible degenerations, we work with families of curves which are trivially covered by families of complex oriented curves so that the quantum index persists along each component of the family of oriented curves.

(1) Suppose that ξ is as in Lemma 2.18(2i). Consider the lifts of $w_0(t^*)$ upon \mathbb{E} that correspond to real branches of C centered at $w_0(t^*)$. Since C is immersed, we only have to show that each of the lifts induces a germ of a smooth one-dimensional subfamily in $\mathcal{M}_{1,n}(\text{Tor}(P_\Delta), \mathcal{L}_{P_\Delta})$. This follows from the smoothness statements in [5, Proposition 4.17(2)] and in [30, Lemma 3(1)], [16, Lemma 2.4(2)].

(2) Suppose that ξ is as in Lemma 2.18(2ii). The argument used in step (1) of the proof of Lemma 2.16 applies, provided we establish a relevant transversality statement. Here, the transversality condition is similar to (10):

$$H^1(\mathbb{E}, \mathcal{O}_{\mathbb{E}}(\mathbf{d})) = 0 , \quad (21)$$

where

$$\deg \mathbf{d} = C^2 - (C^2 + c_1(\text{Tor}(P_\Delta))c_1(\mathcal{L}_{P_\Delta})) - (c_1(\text{Tor}(P_\Delta))c_1(\mathcal{L}_{P_\Delta}) - 2) - 1 = 1 > 0 .$$

The latter inequality yields (21) by the Riemann-Roch formula.

(3) The treatment of the case where ξ is as in Lemma 2.18(2iii) literally coincides with that in step (2) of the proof of Lemma 2.16.

(4) Suppose that ξ is as in Lemma 2.18(2iv). Due to the condition (AQC), the real part of the rational curve $C_1 = \Phi(\widehat{C}_1)$ must lie in $\text{Tor}_{\mathbb{R}}^+(P_\Delta)$, and the real part of the elliptic curve $C_2 = \Phi(\widehat{C}_2)$ has two one-dimensional components, one in $\text{Tor}_{\mathbb{R}}^+(P_\Delta)$ and the other in $\overline{\mathcal{Q}}$. Then, the argument proving the constancy of the numbers $W_1^\kappa(t)$ literally coincides with that in steps (3) and (4) of the proof of Lemma 2.16, when we restrict our attention to the real branches of the considered curves located in $\text{Tor}_{\mathbb{R}}^+(P_\Delta)$. We only comment on the h^1 -vanishing conditions analogous to (17): in our situation, for the rational curve $\Phi : \widehat{C}_1 \rightarrow \text{Tor}(P_\Delta)$, it simply coincides with (17), and for the elliptic curve $\Phi : \widehat{C}_2 \rightarrow \text{Tor}(P_\Delta)$, it reads $h^1(\widehat{C}_2, \mathcal{O}_{\widehat{C}_2}(\mathbf{d}_2)) = 0$, where

$$\deg \mathbf{d}_2 = C_2^2 - (C_2^2 - c_1(\mathcal{L}_{P_\Delta})[C_2]) - (c_1(\mathcal{L}_{P_\Delta})[C_2] - 1) - 1 = 0 .$$

However, geometrically, $\mathbf{d}_2 = p_i^\sigma - p_0$, where $\Phi(p_i^\sigma) = w_i^\sigma(t^*)$, the mobile point in $C_2 \cap \widehat{\mathbf{w}}_\partial(t^*)$. Since $p_i^\sigma \neq p_0$ and \widehat{C}_2 is elliptic, we have $h^0(\widehat{C}_2, \mathcal{O}_{\widehat{C}_2}(\mathbf{d}_2)) = 0$, and hence the required vanishing $h^1(\widehat{C}_2, \mathcal{O}_{\widehat{C}_2}(\mathbf{d}_2)) = 0$ follows by the Riemann-Roch formula.

(5) Suppose that ξ is as in Lemma 2.18(2v). Since \widehat{C} is rational, both local branches B_1, B_2 of $C = \Phi(\widehat{C})$ at $w = w_i^\sigma(t^*)$ are real and each one intersects $\text{Tor}(\sigma)$ with odd multiplicities $2l_1 + 1, 2l_2 + 1$, respectively, where $2l_1 + 2l_2 + 2 = 2k_i^\sigma$, $l_1 \leq l_2$. Choose two mobile points of $\widehat{\mathbf{w}}_\partial(t^*)$ different from w . The one-dimensional stratum $\{\overline{\mathcal{M}}_{1,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))\}_{|t-t^*|<\varepsilon}$ projects to the germ at C of a one-dimensional variety in $|\mathcal{L}_{P_\Delta}|$, and denote by \mathcal{F} one of its real irreducible components. Let the tangent line to \mathcal{F} at C be spanned by C and $C^* \in |\mathcal{L}_{P_\Delta}| \setminus \{C\}$. Fix equavariant coordinates x, y in a neighborhood of w so that $w = (0, 0)$ and $\text{Tor}(\sigma) = \{y = 0\}$. Then, C in a neighborhood of w is given by an equation

$$(y + \eta_1 x^{2l_1+1})(y + \eta_2 x^{2l_2+1}) + \text{h.o.t.} = 0, \quad \eta_1, \eta_2 \in \mathbb{R}^\times ,$$

with the Newton diagram

$$\overline{\Gamma} = [(0, 2), (2l_1 + 1, 1)] \cup [(2l + 1, 1), (2l_1 + 2l + 2 + 2, 0)]$$

(cf. Figures 5(a,b)). As observed in step (4) of the proof of Lemma 2.16, an equation of C^* may have at most the monomial $x^{2l_1}y$ below the Newton diagram $\overline{\Gamma}$. Let us show that this monomial is present with a nonzero coefficient. Indeed, otherwise, the intersection of C and $C^* \neq C$ would violate the Bézout's bound:

$$\begin{aligned} C^2 &\geq 2 \sum_{z \in \text{Sing}(C) \setminus \{w\}} \delta(C, z) + (c_1(\mathcal{L}_{P_\Delta})[C] - 2k_i^\sigma - 2) + (6l_1 + 2 + 2l_2 + 2) + 1 \\ &= (C^2 - c_1(\mathcal{L}_{P_\Delta})[C] + 2 - 4l_1 - 2) + (c_1(\mathcal{L}_{P_\Delta})[C] - 2l_1 - 2l_2 - 4) + (6l_1 + 2l_2 + 4) + 1 = C^2 + 1 \end{aligned}$$

(the latter summand 1 in each expression comes from $(C \cdot C^*)_{w_0(t^*)}$).

If $l_1 = l_2 = l$, then as in step (4) of the proof of Lemma 2.16, we obtain that the equation of the germ of $C^{(t)} \in \mathcal{F}$ at $w = (0, 0)$ is as follows:

$$y^2 + P(x)y + \eta_1\eta_2x^{4l+2} + \text{h.o.t.} = 0, \quad P(x) = (\eta_1 + \eta_2)x^{2l+1} + \sum_{j=2l}^0 a_j t_1^{2l+1-j} x^j, \quad (22)$$

$$\text{where } t_1 = t - t^*, \quad \text{and } x^{2l+1}P(1/x) = \lambda_1 \text{Cheb}_{2l+1}(t_1x + \lambda_2) + \lambda_3$$

with real numbers $\lambda_1, \lambda_2, \lambda_3$ uniquely determined by η_1, η_2 and the coefficient of $x^{2l}y$ in the equation of C^* , and h.o.t. includes higher powers of t_1 in the present monomials and the sum of monomials above $\overline{\Gamma}$. If $\eta_1\eta_2 < 0$, then in (22) in a neighborhood of w we get

$$y = \frac{1}{2} \left(-P(x) \pm \sqrt{P(x)^2 - 4\eta_1\eta_2x^{4l+2}} \right) + \text{h.o.t.},$$

which means that $C^{(t)}$ has only non-real nodes in a neighborhood of w . If $\eta_1\eta_2 > 0$, the latter formula yields $2l$ elliptic nodes in a neighborhood of w . Observe that the change of sign of t_1 yields the symmetry with respect to the origin for the real part of $C^{(t)}$ (up to h.o.t.), and hence the Welschinger sign (5) persists along the path $\{(\widehat{w}_\partial(t), w_0(t))\}_{|t-t^*|<\varepsilon}$.

If $l_1 < l_2$, then again as in step (4) of the proof of Lemma 2.16, we obtain that the equation of the germ of $C^{(t)} \in \mathcal{F}$ at $w = (0, 0)$ is as follows:

$$y^2 + \eta_1(x - \eta_0 t_1)^{2l_1+1}y + \eta_1\eta_2x^{2l_1+2l_2+2} + \text{h.o.t.} = 0, \quad (23)$$

where η_0 is determined by C^* , and h.o.t. includes higher powers of t_1 in the present monomials and the sum of monomials above $\overline{\Gamma}$. It follows that \mathcal{F} is smooth and its real part submersively projects onto the path $\{(\widehat{w}_\partial(t), w_0(t))\}_{|t-t^*|<\varepsilon}$. The tropical limit at $t_1 \rightarrow 0$ (see Section 3.2 for details) includes two triangles

$$\text{conv}\{((0, 1), (2l + 1, 1), (2l_1 + 2l + 2 + 2, 0))\} \quad \text{and} \quad \text{conv}\{(0, 1), (2l_1, 1), (0, 2)\},$$

and two limit curves each having a smooth branch intersecting the toric divisor $\text{Tor}([(0, 1), (2l_1 + 1, 1)])$ with multiplicity $2l_1 + 1$ at the same point. The genuine

geometry of $C^{(t)}$ is obtained by a deformation of this singular point into $2l_1$ nodes, which is described via the modification (see Section 3.3) with the unique modified limit curve that has $2l_1$ elliptic nodes (see Lemma 3.4(3)). Since the construction for $t_1 > 0$ and $t_1 < 0$ is symmetric with respect to the coordinate change $(x, y) \rightarrow (-x, -y)$, we derive the constancy of the Welschinger sign (5) in the move along \mathcal{F} .

Remark 2.20 *In equations (22) and (23), the sign of the coefficient of $x^{2l_1+2l_2+2}$ remains constant, and the sign of the coefficient of y changes as t_1 changes its sign. Geometrically, this means that, for $t_1 < 0$, the real branch of $C^{(t)}$, which is tangent to $\text{Tor}(\sigma)$ at w , lies in $\text{Tor}_{\mathbb{R}}^+(P_{\Delta})$, while, for $t_1 > 0$, this real branch lies in $\overline{\mathfrak{D}}$, or vice versa.*

(6) Suppose that ξ is as in Lemma 2.18(2vi) and that the path $\{(\widehat{w}_{\partial}(t), w_0(t))\}_{|t-t^*|<\varepsilon}$ has the point $w_0(t)$ and two points $w_1(t), w_2(t) \in \widehat{w}_{\partial}(t)$ in a fixed position, while the other two points $w_3(t), w_4(t) \in \widehat{w}_{\partial}(t)$ are mobile. Also, for the dimension reason, we can assume that $w_0 = w_0(t^*)$ is in general position with respect to $\widehat{w}_{\partial}(t^*)$ on the rational curve C , which, in particular, yields the following: the point $p_0 \in \widehat{C}$ is in general position with respect to the remaining points $p_j = \Phi^{-1}(w_j) \in \widehat{p}_{\partial} \subset \widehat{C}$, $j = 1, 2, 3, 4$. By Lemma 2.4, the curve C is immersed and smooth at $\widehat{w}_{\partial}(t^*)$ and $w_0(t^*)$. We can assume, in addition, that C is nodal (the treatment of more complicated immersed singularities is the same with a bit more complicated notations).

Let $\xi(t) = [\Phi_t : (\widehat{C}_t, \mathbf{p}_t) \rightarrow \text{Tor}(P_{\Delta})] \in \overline{\mathcal{M}}_{1,n}^{\mathbb{R}}(\Delta, \mathbf{w}(t))$, $|t - t^*| < \varepsilon$, run over the germ of a real one-dimensional subfamily of $\{\overline{\mathcal{M}}_{1,n}^{\mathbb{R},+}(\Delta, \mathbf{w}(t))\}_{|t-t^*|<\varepsilon}$. It induces a one-dimensional family $\mathcal{V} \subset |\mathcal{L}_{P_{\Delta}}|$ of curves $C^{(t)} = \Phi_{t,*}(\widehat{C}_t)$ on $\text{Tor}(P_{\Delta})$, where $C^{(t^*)} = \Phi_*(\widehat{C}) = 2C$. Any curve $C^{(t)}$, $t \neq t^*$, has four nodes in a neighborhood of each of the $(C^2 - c_1(\text{Tor}(P_{\Delta})[C])/2 + 1)$ nodes of C . We show that the remaining $2c_1(\text{Tor}(P_{\Delta})[C]) - 4$ nodes of $C^{(t)}$ are located in a neighborhood of $\widehat{w}_{\partial}(t^*)$. The intersection of $C^{(t)}$ with a neighborhood of a point $w_i(t^*)$, $1 \leq i \leq 4$, is an immersed disc. Following the lines of part (2) of the proof of Lemma 2.16, introduce local conjugation-invariant coordinates x, y in a neighborhood of $w_j(t^*)$ such that $w_i(t) = (0, 0)$ (which means that, if $w_i(t)$ is mobile, then the coordinate system moves too), the toric divisor containing $w_i(t)$ is $\text{Tor}(\sigma) = \{y = 0\}$, and $\text{Tor}_{\mathbb{R}}^+(P_{\Delta}) = \{y \geq 0\}$. Then, the equation $F_{t_1}(x) = 0$ of C_t , where $t_1 = t - t^*$, is given by formula (12), where we set $k = k_i$, and correspondingly, the geometry of $C^{(t)}$ in a neighborhood of $w_i(t^*)$ is described by the real rational affine limit curve C_1 given by (13). Observe that C_1 has $k_i - 1$ nodes and a global real one-dimensional branch having one vertical tangent. We then conclude that

- the nodes of $C^{(t)}$ are located in a neighborhood of $\text{Sing}(C) \cup \widehat{w}_{\partial}(t^*)$;
- the parameter γ in (12) must be even, since otherwise, the change of sign of t_1 would lead to the change of the position of that global real branch with respect to the vertical tangent, but this position is determined by the double covering $\Phi : \widehat{C} \rightarrow C$; in particular, it follows that the change of sign of t_1 does

not change the contribution of the part of $C^{(t)}$ in a neighborhood of $w_i(t^*)$ to formula (5) for $W_1(\xi_t)$.

Let $\text{Span}\{2C, C'\}$ and $\text{Span}\{C^{(t)}, C_t'\}$ be the lines in $|\mathcal{L}_{P_\Delta}|$ tangent to \mathcal{V} at $2C$ and $C^{(t)}$, respectively. For $t \neq t^*$, the curve C_t' intersects $C^{(t)}$ at the point w_0 , at each node, at each point $w_j^\sigma(t)$ with multiplicity $2k_j^\sigma$ or $2k_j^\sigma - 1$ according as w_j^σ is fixed or mobile, and at one more point w' (follows from the Bézout theorem). Lifting these points to $\widehat{C}^{(t)}$ and passing to the limit as $t \rightarrow t^*$, we obtain on \widehat{C} the divisor

$$D = p_0 + 4\Phi^*(\text{Sing}(C)) + (4k_1 - 2)p_1 + (4k_2 - 2)p_2 + (4k_3 - 3)p_3 + (4k_4 - 3)p_4 + p' .$$

Due to the general position of w_0 on C , we get $p' \notin \widehat{\mathcal{P}}_\partial \cup \{p_0\}$. We use this to show that for $t' < t^* < t''$, the points $w_i(t')$ and $w_i(t'')$ are separated on the toric divisor by the point $w_i(t^*)$, $i = 3, 4$. Indeed, otherwise, we would get $w_i(t') = w_i(t'')$ for, say, $i = 3$ and for $t' < t^* < t''$ depending on each other and converging to t^* . Lifting intersection $C^{(t')} \cap C^{(t'')}$ upon $\widehat{C}_{t'}$, we would obtain on \widehat{C} the divisor

$$D' = p_0 + 4\Phi^*(\text{Sing}(C)) + (4k_1 - 2)p_1 + (4k_2 - 2)p_2 + (4k_3 - 2)p_3 + (4k_4 - 3)p_4 ,$$

not linearly equivalent to D , which is a contradiction. Thus, the real part of the family \mathcal{V} homeomorphically projects onto the interval $|t - t^*| < \varepsilon$, and the sign $W_1(\xi(t))$ remains constant as $0 < |t - t^*| < \varepsilon$. \square

3 Tropical limits of real nodal curves

In this section, we shortly recall tropical limits of nodal curves in toric surfaces mainly following [21, 28] (see also [18, Chapter 2]), and explain in details the structure of the modified tropical limit with emphasis on separating real curves and the distribution of their nodes in the quadrants of the real torus $(\mathbb{R}^\times)^2$.

3.1 Plane tropical curves

An abstract tropical curve is a finite connected graph $\overline{\Gamma}$ such that the complement $\Gamma = \overline{\Gamma} \setminus \Gamma_\infty^0$ in $\overline{\Gamma}$ to the set Γ_∞^0 of univalent vertices contains at least one vertex of $\overline{\Gamma}$ and is endowed with a metric satisfying the following property: the compact edges of Γ are isometric to closed intervals and the non-compact edges of Γ are isometric to the closed ray $[0, \infty)$. The latter edges are called *ends* of Γ , and their set is denoted by Γ_∞^1 . We denote by Γ^1 (respectively, Γ^0) the set of edges (respectively, vertices) of Γ . The *genus* of an abstract tropical curve $\overline{\Gamma}$ is the first Betti number $b_1(\overline{\Gamma}) = b_1(\Gamma)$.

A *parameterized plane tropical curve* is a couple $(\overline{\Gamma}, h)$, where $\overline{\Gamma}$ is an abstract tropical curve and $h : \Gamma \rightarrow \mathbb{R}^2$ is a non-constant continuous proper map such that

- for each edge $e \in \Gamma^1$, the restriction of h to e is affine (in the length coordinate) such that the image of a unit tangent vector of e under the differential $D(h|_e)$ is a vector with integer coordinates;

- for each vertex $v \in \Gamma^0$, one has the *balancing condition*:

$$\sum_{e \in \Gamma^1, v \in \partial e} \mathbf{a}_v(e) = 0 ,$$

where ∂e is the set of vertices in Γ that are adjacent to e and $\mathbf{a}_v(e)$ is the $D(h|_e)$ -image of the unit tangent vector pointing out from v .

The multiset $\Delta(\bar{\Gamma}, h)$ constituted by the non-zero vectors of the form $\mathbf{a}_{v_e}(e)$, where $e \in \Gamma^1_\infty$ and v_e is the vertex of Γ such that e is adjacent to v , is called the *degree* of $(\bar{\Gamma}, h)$. This multiset is balanced. The vectors $\mathbf{a} \in \Delta(\bar{\Gamma}, h)$, appropriately ordered and counter-clockwise rotated by $\frac{\pi}{2}$ form a convex lattice polygon $P = P(\bar{\Gamma}, h)$, called *Newton polygon* of $(\bar{\Gamma}, h)$. Each non-contracted by h edge e of Γ possesses a positive integral weight $\text{wt}(e) := \|\mathbf{a}_v(e)\|_{\mathbb{Z}}$, where v is a vertex adjacent to e .

We consider parameterized plane tropical curves up to isomorphism (an *isomorphism* between $(\bar{\Gamma}, h)$ and $(\bar{\Gamma}', h')$ is an isometry $\rho : \Gamma \rightarrow \Gamma'$ such that $h = h' \circ \rho$). By abuse of language, the isomorphism classes of parameterized plane tropical curves we simply call *plane tropical curves*. Two isomorphic parameterized plane tropical curves have the same degree and the same genus (this is the genus of the underlying abstract tropical curves), so we can speak about the degree and the genus of a plane tropical curve. A plane tropical curve is said to be *embedded* if it is represented by a couple $(\bar{\Gamma}, h)$, where h is injective.

Two plane tropical curves represented by $(\bar{\Gamma}, h)$ and $(\bar{\Gamma}', h')$ are said to be in *mutually general position* if for any two points $p \in \Gamma$ and $p' \in \Gamma'$ the equality $h(p) = h'(p')$ implies that none of the points p and p' is a vertex. For such plane tropical curves and any couple of points $p \in \Gamma$, $p' \in \Gamma'$ with the property $h(p) = h'(p')$, we define the *tropical intersection multiplicity* $[(\Gamma, h)_p \cdot (\Gamma', h')_{p'}]$ at p and p' putting

$$[(\Gamma, h)_p \cdot (\Gamma', h')_{p'}] = |D(h|_e)_p \wedge D(h'|_{e'})_{p'}|, \quad \text{where } (\eta_1, \eta_2) \wedge (\zeta_1, \zeta_2) = \begin{vmatrix} \eta_1 & \eta_2 \\ \zeta_1 & \zeta_2 \end{vmatrix},$$

and $e \in \Gamma^1$ (respectively, $e' \in \Gamma'^1$) is the edge containing p (respectively, p'). The sum

$$\sum_{\substack{p \in \Gamma, p' \in \Gamma' \\ h(p) = h'(p')}} [(\Gamma, h)_p \cdot (\Gamma', h')_{p'}]$$

is called the *tropical intersection* of the considered plane tropical curves. By the Bernstein-Kouchnirenko theorem (see, for instance, [19, Theorem 4.6.8]), one has

$$\sum_{\substack{p \in \Gamma, p' \in \Gamma' \\ h(p) = h'(p')}} [(\Gamma, h)_p \cdot (\Gamma', h')_{p'}] = \mathcal{A}(P(\Gamma, h), P(\Gamma', h')),$$

where $\mathcal{A}(P(\Gamma, h), P(\Gamma', h'))$ is the mixed area of the Newton polygons $P(\Gamma, h)$ and $P(\Gamma', h')$.

In this paper, we are particularly interested in *rational* and *elliptic* plane tropical curves, that is, plane tropical curves of genus 0 and 1, respectively.

If the polygon $P = P(\bar{\Gamma}, h)$ is non-degenerate, one can define a tropical toric surface $\mathbb{T}P$ associated with the Newton polygon P (see [19, Section 6.2 and 6.4] and [23, Section 3.3]). As a (stratified) topological space, $\mathbb{T}P$ can be identified with P . The *interior* $\text{Int}(\mathbb{T}P)$ of $\mathbb{T}P$ (identified with the interior $\text{Int}(P)$ of P) is a copy of \mathbb{R}^2 equipped with its standard integer affine structure. The complement $\partial\mathbb{T}P = \mathbb{T}P \setminus \text{Int}(\mathbb{T}P)$ is called the *tropical boundary* (or the *tropical divisor at infinity*) of $\mathbb{T}P$. It is naturally stratified according to the faces of P . Each oriented affine straight line L in $\text{Int}(\mathbb{T}P)$ can be completed by a point in $\partial\mathbb{T}P$; this point belongs to the strata corresponding to the face $\mathfrak{F} \subset P$ whose open outer normal cone contains the direction of L . In particular, there is a continuous extension $\bar{h} : \bar{\Gamma} \rightarrow \mathbb{T}P$ such that each univalent vertex $v_\infty \in \Gamma_\infty^0$, which is incident to a non-contracted by h edge $e \in \Gamma_\infty^1$, is mapped to the interior of the strata corresponding to the face $\mathfrak{F} \subset P$ for which $\mathbf{a}_{v_e}(e) \neq 0$ (where v_e is the only vertex in ∂e) is an outer normal vector.

A *marked parameterized plane tropical curve* is a triple $(\bar{\Gamma}, h, \mathbf{p}_{tr})$, where $(\bar{\Gamma}, h)$ is a parameterized plane tropical curve and \mathbf{p}_{tr} is a finite sequence of distinct points of the graph $\bar{\Gamma}$.

Let $(\bar{\Gamma}, h)$ be a parameterized plane tropical curve. The image $h(\Gamma) \subset \mathbb{R}^2$ is a connected closed finite one-dimensional polyhedral complex without univalent vertices. Every edge of this polyhedral complex is of rational slope. We simplify this polyhedral complex removing all its bivalent vertices. Each edge $E \subset h(\Gamma)$ is equipped with the weight $\text{wt}(E)$ which is the sum of the weights of the (non-contracted) edges of Γ intersecting $h^{-1}(x)$, where $x \in E$ is a generic point. The resulting weighted polyhedral complex is called the *image* of $(\bar{\Gamma}, h)$. We can consider an embedded parameterized plane tropical curve $(\bar{\Gamma}_h, \hat{h})$ having the same image as $(\bar{\Gamma}, h)$; this embedded parameterized plane tropical curve is denoted by $h_*\bar{\Gamma}$.

3.2 Parameterized tropical limit of a nodal curve over the field of Puiseux series

To compute the invariants introduced in Section 2, we include the constraints \mathbf{w} , as well as the counted curves, into one-parameter families over the punctured disk in the complex case and over a small interval $(0, \eta)$, $\eta \ll 1$, in the real case. Under some conditions, it is possible to define limits of these families which allow one to reduce the enumeration of algebraic curves to enumeration of tropical curves. The families under consideration can be regarded as objects over the field \mathbb{K} of locally convergent complex Puiseux series, or over its subfield $\mathbb{K}_{\mathbb{R}}$ of the series with real coefficients. The field \mathbb{K} possesses a non-Archimedean valuation.

$$\text{val} : \mathbb{K} \rightarrow \mathbb{Q} \cup \{-\infty\}, \quad \text{val} \left(\sum_r a_r t^r \right) = -\min\{r, a_r \neq 0\}, \quad \text{val}(0) = -\infty.$$

. For the field \mathbb{K} we use notations similar to the ones introduced in the complex case in Sections 2.1-2.2.

Let $\left[\Phi : (\widehat{C}, \mathbf{p}) \rightarrow \text{Tor}_{\mathbb{K}}(P_{\Delta}) \right] \in \mathcal{M}_{g,n}^{\mathbb{K}}(\Delta, \mathbf{w})$. The curve $C = \Phi(\widehat{C}) \in |\mathcal{L}_{P_{\Delta}}|$ is given by a polynomial

$$F(z_1, z_2) = \sum_{(i,j) \in P_{\Delta} \cap \mathbb{Z}^2} a_{ij}(t) z_1^i z_2^j = \sum_{(i,j) \in P_{\Delta} \cap \mathbb{Z}^2} (a_{ij}^0 + O(t^{>0})) t^{\nu(i,j)} z_1^i z_2^j,$$

where $\nu : P_{\Delta} \rightarrow \mathbb{R}$ is a convex piecewise-linear function, whose graph defines a subdivision $\Sigma_{\nu} : P_{\Delta} = \delta_1 \cup \dots \cup \delta_N$ into linearity domains $\delta_1, \dots, \delta_N$ (which are convex lattice polygons), and the coefficients $a_{ij}(t) \in \mathbb{K}$ verify the property $a_{ij}^0 \neq 0$ if (i, j) is a vertex of some δ_k , $1 \leq k \leq N$. By a suitable parameter change $t \mapsto t^{m_1}$ we make all exponents of t in $a_{ij}(t)$, $(i, j) \in P_{\Delta} \cap \mathbb{Z}^2$, integral, and make $\nu(P_{\Delta} \cap \mathbb{Z}^2) \subset \mathbb{Z}$. In this setting, the curve $C = \Phi(\widehat{C})$ defines a germ of a flat analytic family of complex curves

$$\begin{array}{ccc} (C, \mathbf{w}) & \hookrightarrow & (\mathfrak{X}, \mathbf{w}) \\ \downarrow & & \downarrow \\ (\mathbb{C}, 0) & \xlongequal{\quad} & (\mathbb{C}, 0) \end{array} \quad (24)$$

where

- $\mathfrak{X} = \text{Tor}(\text{OG}(\nu))$ with the three-dimensional lattice polytope

$$\text{OG}(\nu) = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : (x_1, x_2) \in P_{\Delta}, \quad x_3 \geq \nu(x_1, x_2)\};$$

here, $\mathfrak{X} \setminus \mathfrak{X}_0 = \text{Tor}(P_{\Delta}) \times ((\mathbb{C}, 0) \setminus \{0\})$, while \mathfrak{X}_0 is the union of the toric surfaces $\text{Tor}(\delta_k)$, $k = 1, \dots, N$, intersecting each other according to intersections of polygons δ_k , $k = 1, \dots, N$;

- the family $\{C^{(t)}\}_{t \neq 0}$ consists of equisingular irreducible curves of genus g , while $C^{(0)} \subset \mathfrak{X}_0$ is the union of curves $C_k^{(0)} \subset \text{Tor}(\delta_k)$, $C_k^{(0)} \in |\mathcal{L}_{\delta_k}|$, $k = 1, \dots, N$ (called *limit curves*).

By [31, Theorem 1, page 73] or [4, Proposition 3.3], after another parameter change $t \mapsto t^{m_2}$, the family $C \setminus C^{(0)} \rightarrow (\mathbb{C}, 0)$ can be simultaneously normalized

$$\begin{array}{ccccc} (\widehat{C}, \mathbf{p})|_{t \neq 0} & \xrightarrow{\Phi} & (C, \mathbf{w})|_{t \neq 0} & \hookrightarrow & (\mathfrak{X}, \mathbf{w})|_{t \neq 0} \\ \downarrow & & \downarrow & & \downarrow \\ (\mathbb{C}, 0) \setminus \{0\} & \xlongequal{\quad} & (\mathbb{C}, 0) \setminus \{0\} & \xlongequal{\quad} & (\mathbb{C}, 0) \setminus \{0\} \end{array} \quad (25)$$

and, furthermore, the latter family extends in a flat manner to the central point:

$$\begin{array}{ccccc} (\widehat{C}, \mathbf{p}) & \xrightarrow{\Phi} & (C, \mathbf{w}) & \hookrightarrow & (\mathfrak{X}, \mathbf{w}) \\ \downarrow & & \downarrow & & \downarrow \\ (\mathbb{C}, 0) & \xlongequal{\quad} & (\mathbb{C}, 0) & \xlongequal{\quad} & (\mathbb{C}, 0) \end{array} \quad (26)$$

where $\widehat{C}^{(0)}$ is a connected nodal curve of arithmetic genus g (see, for instance [1, Theorem 1.4.1]), whose non-contracted components are mapped onto the limit curves $C_i^{(0)}$, $i = 1, \dots, N$, and no component is entirely mapped to $\text{Tor}(\sigma)$, $\sigma \in P_\Delta^1$.

The central fiber $(\widehat{C}^{(0)}, \mathbf{p}^{(0)}) \xrightarrow{\Phi} (C^{(0)}, \mathbf{w}^{(0)}) \hookrightarrow (\mathbf{x}_0, \mathbf{w}^{(0)})$ together with the associated marked parameterized plane tropical curve $(\overline{\Gamma}, h, \mathbf{p}_{tr})$ (cf. [32, Section 2]) form the *parameterized tropical limit* of the given marked parameterized curve $(\widehat{C}, \mathbf{p}) \xrightarrow{\Phi} (C, \mathbf{w}) \hookrightarrow \text{Tor}_{\mathbb{K}}(P_\Delta)$. By abuse of language, we use the term *tropicalization* for the tropical limit or for its parts (*e.g.*, marked points etc.). Recall that the marked parameterized plane tropical curve $(\Gamma, h, \mathbf{p}_{tr})$ possesses the following properties:

- the vertices of Γ bijectively correspond to the components of $\widehat{C}^{(0)}$, while the univalent vertices in Γ_∞^0 correspond to the points of $\widehat{C}^{(0)}$ mapped to the toric divisors $\text{Tor}(\sigma)$, $\sigma \in P_\Delta^1$;
- the finite edges of Γ bijectively correspond to the intersection points of distinct components of $\widehat{C}^{(0)}$, while the infinite edges are incident to the vertices in Γ_∞^0 on one side and to components of $\widehat{C}^{(0)}$ having points mapped to toric divisors $\text{Tor}(\sigma)$, $\sigma \in P_\Delta^1$, on the other side;
- the sequence of tropical marked points $\mathbf{p}_{tr} \subset \overline{\Gamma}$ is in the bijective correspondence with the sequence of marked points \mathbf{p} , and q can uniquely be restored from \mathbf{p} via the relations $\Phi(\mathbf{p}) = \mathbf{w}$ and $\text{val}(\mathbf{w}) = \mathbf{x} = h(\mathbf{p}_{tr})$ (here the valuation is applied coordinate-wise to each point of the sequence \mathbf{w}), provided that sequence \mathbf{x} is in (tropical) general position subject to the distribution of the points of \mathbf{x} on $\partial \mathbb{T}P_\Delta$ and $\text{Int}(\mathbb{T}P_\Delta)$; for details on the notion of tropical general position, see [21].

The obtained tropical curve $(\overline{\Gamma}, h, \mathbf{p}_{tr})$ has genus g and degree Δ . Furthermore, the embedded plane tropical curve $h_*\Gamma$ is supported at the corner locus of the tropical polynomial

$$N_F : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad N_F(x) = \max_{\omega \in P_\Delta \cap \mathbb{Z}^2} (\langle \omega, x \rangle + \text{val}(a_\omega)), \quad x \in \mathbb{R}^2.$$

The convex piecewise-linear function $\nu : P_\Delta \rightarrow \mathbb{R}$ is Legendre dual to $N_F(x)$, and, moreover, it is determined by the embedded plane tropical curve $h_*\Gamma$ uniquely up to adding an affine function. This yields a geometric duality: the edges of $h_*\Gamma$ are in bijective correspondence with the edges of the subdivision Σ_ν , and the weight $\text{wt}(e)$ of an edge e of $h_*\Gamma$ is the lattice length of the dual edge of the subdivision Σ_ν .

In what follows, whenever we are given a plane tropical curve T of degree Δ , we denote the corresponding subdivision Σ_ν of P_Δ by $\Sigma(T)$.

For any marked parameterized plane tropical curve $(\overline{\Gamma}, h, \mathbf{p}_{tr})$, denote by $(\overline{\Gamma}_{red}, h_{red}, \mathbf{p}_{trred})$ the marked parameterized plane tropical curve obtained from $(\overline{\Gamma}, h, \mathbf{p}_{tr})$ by contracting the edges of $\overline{\Gamma}$ along which h is constant and removing bivalent vertices V such that $h_{red}(V)$ is the coordinatewise val-image of a contracted

component of $\widehat{C}^{(0)}$. Clearly, $h_*\Gamma = h_{red,*}\Gamma_{red}$. A marked parameterized plane tropical curve $(\bar{\Gamma}, h, \mathbf{p}_{tr})$ which coincides with $(\bar{\Gamma}_{red}, h_{red}, \mathbf{p}_{trred})$ is said to be *reduced*.

Denote by $\mathcal{T}_{g,n}^{red}(\Delta, \mathbf{x})$ the set of reduced marked parameterized plane tropical curves $(\bar{\Gamma}, h, \mathbf{p}_{tr})$ of degree Δ and genus g and such that $h(\mathbf{p}_{tr}) = \mathbf{x}$.

3.3 Geometry of the modified parameterized tropical limit of a nodal curve

The following genericity statement is a corollary of the completely standard dimension count (see [21, Lemmas 4.5 and 4.20 and Theorem 7.1], [28, Theorem 3], [18, Lemma 2.46], and [8, Remark 4.10]).

Lemma 3.1 *In the notation and setting of Section 3.2, suppose that the configuration $\mathbf{x} \subset \mathbb{TP}_\Delta$ is in tropical general position (subject to the condition that the subconfiguration \mathbf{x}_∂ is distributed along $\partial\mathbb{TP}_\Delta$). Then, for any plane tropical curve $T = [(\bar{\Gamma}_{red}, h, \mathbf{p}_{tr})]$, where $(\bar{\Gamma}_{red}, h, \mathbf{p}_{tr}) \in \mathcal{T}_{g,n}^{red}(\Delta, \mathbf{x})$, the graph Γ_{red} is trivalent. Furthermore, the inverse image $h^{-1}(p)$ of any point $p \in h_*\Gamma_{red}$ contains at most two points, and if $h^{-1}(p)$ is formed by two points, then none of them is a vertex of Γ_{red} . \square*

Lemma 3.1 implies that

- the subdivision $\Sigma(T)$ consists of triangles and parallelograms; the curve Γ_{red} has genus g , the trivalent vertices are mapped to the trivalent vertices of $h_*\Gamma_{red}$,
- each connected component $K \subset \bar{\Gamma}_{red} \setminus \mathbf{p}_{tr}$ is a tree containing exactly one univalent vertex, and it possesses a unique orientation of the edges such that the edges incident to marked points are oriented outwards, and at each vertex of K exactly one edge is outgoing.

Under the assumptions of Lemma 3.1, the embedded tropical curve $h_*\Gamma_{red}$ and the weights of its edges uniquely determine the map $h : \Gamma_{red} \rightarrow h_*\Gamma_{red}$ and the metric on Γ_{red} . Furthermore, the reduced curve $(\bar{\Gamma}_{red}, h, \mathbf{p}_{tr})$ uniquely determines the (nonreduced) marked parameterized plane tropical curve $(\bar{\Gamma}, h, \mathbf{p}_{tr})$ obtained in the following operation (inverse to the reduction operation introduced above): for each point $x \in h_*\Gamma_{red}$ which has two preimages, we make these preimages the bivalent vertices. Denote by $\mathcal{T}_{g,n}(\Delta, \mathbf{x})$ the set of these nonreduced curves $(\bar{\Gamma}, h, \mathbf{p}_{tr})$ obtained from the reduced curves $(\bar{\Gamma}_{red}, h, \mathbf{p}_{tr}) \in \mathcal{T}_{g,n}^{red}(\Delta, \mathbf{x})$; we call them *nonreduced models*.

We call a rational curve $\Phi' : \mathbb{P}^1 \rightarrow \text{Tor}(\delta')$, where δ' is a lattice triangle and Φ' is nonconstant, *peripherally unibranch*, if $\Phi'(\mathbb{P}^1)$ does not hit the intersection points of the toric divisors, and $(\Phi')^*(\text{Tor}(\sigma)) \subset \mathbb{P}^1$ is either empty, or concentrated at one point for each toric divisor $\text{Tor}(\sigma) \subset \text{Tor}(\delta')$.

Lemma 3.2 (1) For any triangle δ_k of the subdivision $\Sigma(T)$, the limit curve $C_k \subset \text{Tor}(\delta_k)$ is a nodal, rational, peripherally unibranch curve in the linear system $|\mathcal{L}_{\delta_k}|$ which is smooth along the toric divisors of $\text{Tor}(\delta_k)$; the corresponding component of $\Phi : \widehat{C}^{(0)} \rightarrow \mathfrak{X}_0$ is an immersion $\Phi : \widehat{C}_k^{(0)} = \mathbb{P}^1 \rightarrow C_k \hookrightarrow \text{Tor}(\delta_k)$.

(2) For a parallelogram δ_k of the subdivision $\Sigma(T)$, the reduced limit curve $(C_k)_{\text{red}} \subset \text{Tor}(\delta_k)$ consists of two smooth rational, peripherally unibranch curves $C_{k,1}, C_{k,2}$ transversally intersecting each other; each of the components transversally intersects two disjoint toric divisors of $\text{Tor}(\delta_k)$; the corresponding two components of $\Phi : \widehat{C}^{(0)} \rightarrow \mathfrak{X}_0$ are multiple covers

$$\Phi : \widehat{C}_{k,1}^{(0)} = \mathbb{P}^1 \rightarrow C_{k,1} \subset \text{Tor}(\delta_k), \quad \Phi : \widehat{C}_{k,2}^{(0)} = \mathbb{P}^1 \rightarrow C_{k,2} \subset \text{Tor}(\delta_k),$$

ramified at the intersection points with the toric divisors of $\text{Tor}(\delta_k)$.

Remark 3.3 Let $(\bar{\Gamma}_{\text{red}}, h, \mathbf{p}_{\text{tr}}) \in \mathcal{T}_{g,n}^{\text{red}}(\Delta, \mathbf{x})$ be a reduced marked parameterized plane tropical curve, and let $(\bar{\Gamma}, h, \mathbf{p}_{\text{tr}}) \in \mathcal{T}_{g,n}(\Delta, \mathbf{x})$ be the nonreduced model (in the sense of Section 3.3) of $(\bar{\Gamma}_{\text{red}}, h, \mathbf{p}_{\text{tr}})$. The parallelograms of the subdivision $\Sigma(T)$, where $T = h_*\Gamma$, correspond to pairs of bivalent vertices of Γ . Let $e \in \bar{\Gamma}_{\text{red}}^1$ contain bivalent vertices of Γ . The edges $E_i = h(e_i)$, $i = 1, \dots, s$, of the embedded plane tropical curve T such that $E_i \subsetneq h(e)$, $i = 1, \dots, s$, are dual to the segments σ_i , $i = 1, \dots, s$, of the subdivision $\Sigma(T)$ that are parallel sides of a sequence of parallelograms δ_i , $i = 1, \dots, s-1$, so that $\sigma_i, \sigma_{i+1} \subset \delta_i$, for all $i = 1, \dots, s-1$. Observe that the toric divisors $\text{Tor}(\sigma_i)$, $i = 1, \dots, s$, are canonically isomorphic, and we call the corresponding points on these toric divisors canonically identified points.

An additional information, in particular, contracted components of $\widehat{C}^{(0)}$, can be recovered *via modifications* (see details in [28, Sections 3.5 and 3.6], [18, Section 2.5.8], and [23, Chapter 5]^{III}). Tropical modifications are performed along the edges $e \in \bar{\Gamma}_{\text{red}}^1$ which have weight $\ell > 1$, either bounded without marked point, or bounded with an interior marked point, or unbounded with an interior marked point. The modification procedure starts with a torus automorphism that induces an integral-affine automorphism of \mathbb{R}^2 bringing the edge $E = h(e)$ to a horizontal position. Assume that e does not contain bivalent vertices of the nonreduced curve Γ . The edge σ of the subdivision $\Sigma(T)$ dual to E becomes vertical, and the corresponding divisor $\text{Tor}(\sigma) \subset \mathfrak{X}_0$ contains the intersection point z with limit curves associated with the (trivalent) endpoints of e (cf. Lemma 3.2). We then perform the coordinate change

$$(x_1, x_2) = (x'_1, x'_2 + b), \tag{27}$$

where b is the coordinate of z on $\text{Tor}(\sigma)$. This coordinate change is not toric, and it transforms both the Newton polygon and its subdivision into a new one that contains a specific fragment described in Lemmas 3.4, 3.5, and 3.6 below. This fragment reflects a birational transformation $\mathfrak{X}_{\text{mod}} \rightarrow \mathfrak{X}$ of the family (26) with the appearance of new components of the central surface and the new limit

^{III}The term “refinement” used in the two former sources is replaced by “modifications” in order to unify the terminology and to distinguish from the refined invariants discussed in the paper.

curves in these components (called *modified limit curves*). The modified limit curves correspond to the components of \widehat{C}_0 that were contracted in the original family (26).

Tropically, we replace the original plane \mathbb{R}^2 with the tropical plane in \mathbb{R}^3 , consisting of three half-planes glued along the line through the edge E . One of the half-planes contains the new fragment of the tropical curve which is dual to the fragment of the new subdivision (cf. [23, Chapter 5]).

The three next lemmas present the aforementioned new subdivision fragments and the new modified limit curves. The proofs and detailed explanations can be found in [28, Section 3.5], [18, Section 2.5.8], and [7, Section 2.2].

We fix a common initial data for the three lemmas. Let an edge e of Γ_{red} of weight $\ell > 1$ do not contain bivalent vertices of Γ . Assume that the edge $E = h(e) \subset \mathbb{R}^2$ is horizontal, and the dual edge σ of the subdivision $\Sigma(T)$ lies on the vertical coordinate axis. Denote by $z \in \text{Tor}(\sigma) \subset \mathfrak{X}_0$ the unique intersection point $\text{Tor}(\sigma) \cap C^{(0)}$ (which is the intersection point of $\text{Tor}(\sigma)$ with the limit curves attached to the trivalent endpoints of E). At last, denote by b the coordinate of z on $\text{Tor}(\sigma)$.

Lemma 3.4 *Given the above initial data, let e be a bounded edge. Then, the coordinate change (27) yields the new subdivision fragment as shown in Figure 7(a,b) dual to the fragment of the tropical curve on the modified tropical plane shown in Figure 7(d,e). The trivalent vertex of the latter fragment is dual to the triangle $\delta_{mod}(E) = \text{conv}\{(-1, 0), (1, 0), (0, \ell)\}$ (see Figure 7(c)). The surface $\text{Tor}(\delta_{mod}(E))$ is the exceptional divisor of the weighted blow-up $\mathfrak{X}_{mod} \rightarrow \mathfrak{X}$. The component of $\widehat{C}^{(0)}$ which was contracted to the point z by the map Φ in the family (26), becomes immersively mapped onto the nodal rational curve by the modified map $\Phi_{mod} : \mathbb{P}^1 \rightarrow C_{mod}(E) \hookrightarrow \text{Tor}(\delta_{mod}) \subset \mathfrak{X}_{0,mod}$. The curve $C_{mod}(E)$ can be recovered in ℓ ways using the intersection points with the toric divisors $\text{Tor}([-1, 0], (0, \ell])$, $\text{Tor}([(1, 0), (0, \ell)])$ fixed by the limit curves $C_1 \subset \text{Tor}(\delta_1)$, $C_2 \subset \text{Tor}(\delta_2)$ (see Figure 7(a)) and the condition of the vanishing coefficient of $y^{\ell-1}$ in the defining polynomial.*

Lemma 3.5 *Given the above initial data, let e be a bounded edge with an interior marked point p_{tr} . Let $v = h(p_{tr}) = (-c, 0)$, and let the point $w \in \mathbf{w}_{in}$ that tropicalizes to v have coordinates*

$$w = (a_1 t^c (1 + O(t^d)), a_2 + O(t^d)), \quad \text{where } a_1 a_2 \neq 0, d \gg 0.$$

(The requirement $d \gg 0$ here and later ensures that the marked point appears in the tropical modification always on the vertical end directed downward.) The coordinate change (27) yields a tropical modification shown in Figure 7(f,g) with a pair of trivalent vertices dual to the pair of triangles

$$\delta_{mod,1}(E) = \text{conv}\{(1, 0), (0, 0), (0, \ell)\}, \quad \delta_{mod,2}(E) = \text{conv}\{(0, 0), (-1, 0), (0, \ell)\},$$

(see Figure 7(h)). Two components of $\widehat{C}^{(0)}$ mapped to the point $z \in \text{Tor}(\sigma)$ after modification, become isomorphically mapped onto smooth rational curves $C_{mod,1}(E) \subset \text{Tor}(\delta_{mod,1}(E))$, $C_{mod,2}(E) \subset \text{Tor}(\delta_{mod,2}(E))$ that intersect the common toric divisor $\text{Tor}([(0, 0), (0, \ell)])$ in one point z_0 with multiplicity ℓ . The curve $C_{mod,2}(E)$

can be recovered in ℓ ways using the intersection point with the toric divisor $\text{Tor}([-1, 0], (0, \ell))$ fixed by the limit curve C_2 and the intersection point with the toric divisor $\text{Tor}([(0, 0), (-1, 0)])$ fixed by w . In turn, the curve $C_{\text{mod},1}(E)$ is uniquely defined by $z_0 \in \text{Tor}([(0, 0), (0, \ell)])$ and the intersection point with $\text{Tor}([(1, 0), (0, \ell)])$ fixed by the limit curve C_1 . The second tropical modification along the edge mapped to the horizontal edge between two new trivalent vertices (as shown in Figure 7(h)) is performed as described in part (1). It recovers one more previously contracted component of $\widehat{C}^{(0)}$, which then becomes immersed into the exceptional divisor $\text{Tor}(\delta_{\text{mod}}(E))$ of the blow-up of the point z_0 .

Lemma 3.6 *Given the above initial data, let e be an unbounded edge with an interior marked point p_{tr} . Let $v = h(p_{\text{tr}}) = (-c, 0)$, and let the point $w \in \mathbf{w}_{\text{in}}$ that tropicalizes to v have coordinates*

$$w = (a_1 t^c (1 + O(t^d)), a_2 + O(t^d)), \quad \text{where } a_1 a_2 \neq 0, d \gg 0.$$

The modification similar to that in part (2), see Figure 7(i,j), contains a trivalent vertex dual to the triangle $\delta_{\text{mod}}(E) = \text{conv}\{(-1, 0), (0, 0), (0, \ell)\}$, see Figure 7(k). The component of $\widehat{C}^{(0)}$ contracted to the point z turns into a smooth rational curve $\Phi_{\text{mod}} : \mathbb{P}^1 \rightarrow C_{\text{mod}}(E) \hookrightarrow \text{Tor}(\delta_{\text{mod}}(E))$, which can be recovered in ℓ ways using the intersection point with the toric divisor $\text{Tor}([-1, 0], (0, \ell))$ fixed by the limit curve $C_1 \subset \text{Tor}(\delta_1)$ and using the point in $\text{Tor}([(0, 0), (-1, 0)])$ fixed by w .

Remark 3.7 *If the edge e in the above lemmas contains bivalent vertices of Γ , then we perform almost the same modification procedure: the algebraic modification is just the same, while the tropical modification contains extra vertical rays (cf. [7, Figure 2(d,e,f,g)]). The construction of the modified limit curves is not affected by the presence of bivalent vertices. The reader can find all the details in [28, Section 3.6], [18, Section 2.5.8], and [7, Section 2.2].*

3.4 The correspondence theorem

We recall here Mikhalkin's correspondence theorem [21, Section 7] (see also [28, §3] and [18, Section 2.5]) adapted to the setting and notation of Sections 3.1-3.3. Then, we analyze its real version.

Definition 3.8 *Let us be given a sequence $\mathbf{w} \subset \text{Tor}_{\mathbb{K}}(P_{\Delta})$ of n distinct points defined as in Section 2.1.3 and a reduced tropical curve $(\overline{\Gamma}_{\text{red}}, h, \mathbf{p}_{\text{tr}}) \in \mathcal{T}_{g,n}^{\text{red}}(\Delta, \mathbf{x})$, where $\mathbf{x} = \text{val}(\mathbf{w}) = h(\mathbf{p}_{\text{tr}})$ (see Section 3.2, part (2)). Assume that \mathbf{x} is in tropical general position subject to the restriction $\mathbf{x}_{\partial} \subset \partial \mathbb{T}P_{\Delta}$. Denote by $\delta_1, \dots, \delta_N$ all triangles of the subdivision $\Sigma(T)$, where $T = h_*\Gamma$. We define an admissible collection of limit curves and an extended admissible collection of limit curves associated with \mathbf{w} and with the nonreduced model $(\overline{\Gamma}, h, \mathbf{p}_{\text{tr}})$ of $(\overline{\Gamma}_{\text{red}}, h, \mathbf{p}_{\text{tr}})$ (see Section 3.3).*

- (1) *An admissible collection is a sequence of nodal rational curves $C_i \in |\mathcal{L}_{\delta_i}|$, $i = 1, \dots, N$, intersecting each toric divisor in one smooth point; all the curves C_1, \dots, C_N are subject to the conditions:*

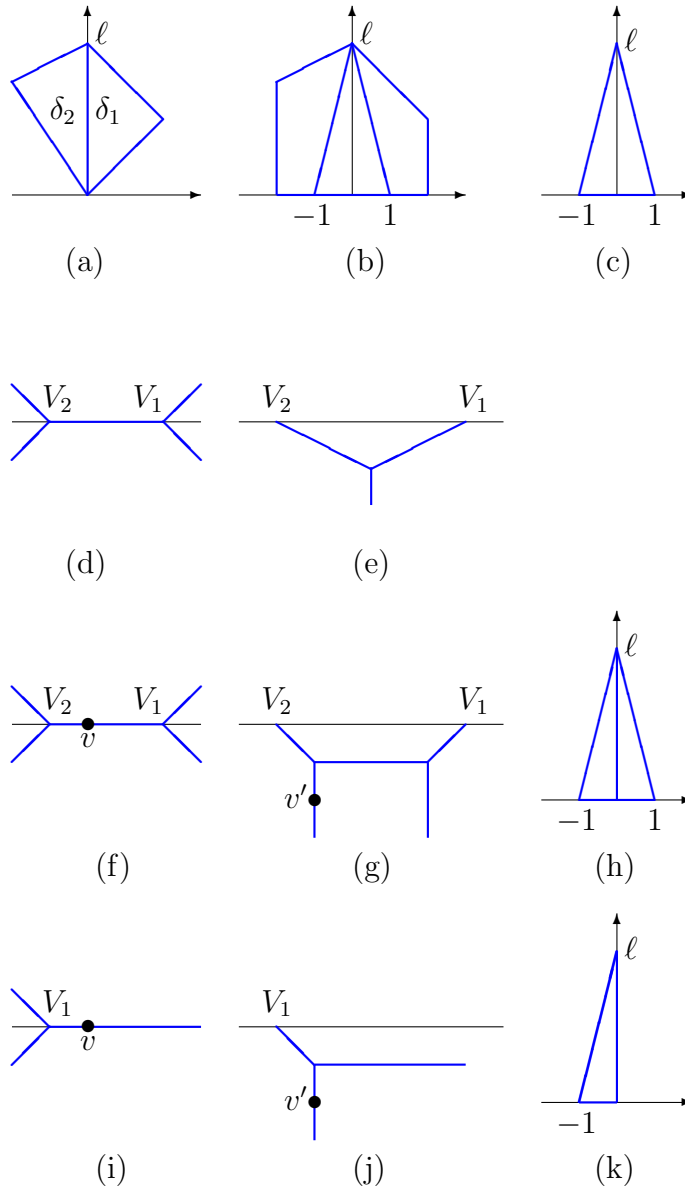


Figure 7: Modification of a multiple edge with and without marked point

- each curve C_i contains all the points of $\mathbf{w}^{(0)}$ belonging to $\text{Tor}(\delta_i)$,
- for each pair of triangles δ_i, δ_j with a common side $\sigma = \delta_i \cap \delta_j$, the limit curves $C_i \subset \text{Tor}(\delta_i)$, $C_j \subset \text{Tor}(\delta_j)$ intersect $\text{Tor}(\sigma)$ at the same point,
- for each pair of triangles δ_i, δ_j with the sides $\sigma_i \subset \partial\delta_i$, $\sigma_j \subset \partial\delta_j$, dual to the edges $E_i = h(e_i) \neq E_j = h(e_j)$, $e_i, e_j \in \Gamma^1$, such that e_i, e_j lie on the same bounded edge $e \in \Gamma_{red}^1$, the curves C_i, C_j intersect the toric divisors $\text{Tor}(\sigma_i)$, $\text{Tor}(\sigma_j)$ at the canonically identified points (see Remark 3.3).

(2) An extended admissible collection includes an admissible collection (see item (1)) and a sequence of modified limit curves attached to edges $e \in \Gamma_{red}^1$ of weight $\ell > 1$ and satisfying the conclusions of Lemmas 3.4, 3.5, and 3.6; namely, one takes

- either a rational curve $C_{mod} \in |\mathcal{L}_{\delta_{mod}(E)}|$, which meets the toric divisors $\text{Tor}([(-1, 0), (0, \ell)])$ and $\text{Tor}([(1, 0), (0, \ell)])$ at the points fixed by the curves C_1, \dots, C_N , and is defined by a polynomial with vanishing coefficient of $y^{\ell-1}$,
- or a triple of rational curves $C_{i,mod} \in |\mathcal{L}_{\delta_{i,mod}(E)}|$, $i = 1, 2$, $C_{mod} \in |\mathcal{L}_{\delta_{mod}(E)}|$, which meet the toric divisors $\text{Tor}([(-1, 0), (0, \ell)])$ and $\text{Tor}([(1, 0), (0, \ell)])$ at the points fixed by the curves C_1, \dots, C_N , and such that $C_{1,mod}, C_{2,mod}$ are peripherally unibranch, intersect the toric divisor $\text{Tor}([(0, 0), (0, \ell)])$ at the same point and satisfy $w_{mod} \in C_{1,mod} \cup C_{2,mod}$, while C_{mod} is as in the preceding item,
- or a rational, peripherally unibranch curve $C_{mod} \in |\mathcal{L}_{\delta_{mod}(E)}|$ that meets the toric divisor $\text{Tor}([(-1, 0), (0, \ell)])$ at the point fixed by the curves C_1, \dots, C_N and passes through w_{mod} , where $\delta_{mod}(E) = \text{conv}\{(-1, 0), (0, 0), (0, \ell)\}$.

Denote by $\text{Ad}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ (respectively, by $\text{EAd}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$) the set of admissible (respectively, extended admissible) collections associated with a tropical curve $(\bar{\Gamma}, h, \mathbf{p}_{tr}) \in \mathcal{T}_{g,n}^{red}(\Delta, \mathbf{x})$ and a sequence of n points $\mathbf{w} \subset \text{Tor}_{\mathbb{K}}(P_{\Delta})$. Define $\mathcal{M}_{g,n}^{\mathbb{K}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w}) \subset \mathcal{M}_{g,n}^{\mathbb{K}}(\Delta, \mathbf{w})$ to be the set of elements of $\mathcal{M}_{g,n}^{\mathbb{K}}(\Delta, \mathbf{w})$ tropicalizing to $(\bar{\Gamma}, h, \mathbf{p}_{tr}) \in \mathcal{T}_{g,n}^{red}(\Delta, \mathbf{x})$. In the case $\mathbf{w} \subset \text{Tor}_{\mathbb{K}_{\mathbb{R}}}(P_{\Delta})$, the symbols $\mathcal{M}_{g,n}^{\mathbb{K}_{\mathbb{R}}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ and $\text{EAd}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ denote the sets of real elements.

The following theorem is a reformulation of Mikhalkin's correspondence theorem [21], Section 7] (see also [28, Section 3] for a closer version of the correspondence theorem).

Theorem 3.9 *The natural maps*

$$\mathcal{M}_{g,n}^{\mathbb{K}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w}) \rightarrow \text{EAd}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$$

and (in the case $\mathbf{w} \subset \text{Tor}_{\mathbb{K}_{\mathbb{R}}}(P_{\Delta})$)

$$\mathcal{M}_{g,n}^{\mathbb{K}_{\mathbb{R}}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w}) \rightarrow \text{EAd}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$$

are bijective, for each $(\bar{\Gamma}, h, \mathbf{p}_{tr}) \in \mathcal{T}_{g,n}^{red}(\Delta, \mathbf{x})$. Moreover, the map $\mathbf{z} \in (\mathbb{K}^{\times})^2 \mapsto \mathbf{z}^{(0)} \in \mathfrak{X}_0$ defines a bijection between the set of nodes of an element $[\Phi : (\hat{C}, \mathbf{p}) \rightarrow \text{Tor}_{\mathbb{K}}(P_{\Delta})] \in \mathcal{M}_{g,n}^{\mathbb{K}}(\Delta, \mathbf{w})$ and the disjoint union of sets of nodes of the curves in the corresponding extended admissible collection.

Proof of Lemma 2.4. The finiteness of $\mathcal{M}_{g,n}^{\mathbb{K}}(\Delta, \mathbf{w})$ follows from Theorem 3.9 and from the finiteness of $\mathcal{T}_{g,n}(\Delta, \mathbf{x})$ (cf. [21, Proposition 4.13]). The smoothness of $C = \Phi(\widehat{C})$ at each intersection point with any toric divisor $\text{Tor}_K(\sigma)$, $\sigma \in P_{\Delta}^1$, follows from (a) the fact that all the n^{σ} ends of Γ directed by the vectors $\mathbf{a} \in \Delta$ that are outer normals to $\sigma \subset \partial P_{\Delta}$ are disjoint from each other, and from (b) the fact that all limit curves in $\text{EAd}((\overline{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ are smooth along the toric divisors. The singularities of C come from the nodes of the limit curves in $\text{EAd}((\overline{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ and from the intersection points of the components of the limit curves in the parallelograms of the subdivision; note that the latter intersection points are centers of smooth local branches which do not glue up in the deformation of $\Phi_0 : \widehat{C}^{(0)} \rightarrow \mathfrak{X}_0$ into $\Phi_t : \widehat{C}^{(t)} \rightarrow \mathfrak{X}_t$, $t \neq 0$, in the family (26).

3.5 Real limit curves and their nodes

In this section we specify the statement of Theorem 3.9 considering real separating curves. Recall that a reduced, irreducible curve over \mathbb{R} is said to be *separating* if the set of the non-real points in its normalization is disconnected. The choice of a connected component of this complement defines the so-called *complex orientation* of the one-dimensional branches of the real point set. In this connection, we observe that, for a sequence $\mathbf{w} \subset \text{Tor}_{\mathbb{K}_{\mathbb{R}}}(P_{\Delta})$, each curve in a real extended admissible collection has a one-dimensional global real branch, and hence is separating.

A reduced, irreducible curve over the field $\mathbb{K}_{\mathbb{R}}$ is said to be *separating*, if each curve $\Phi_t : \widehat{C}^{(t)} \rightarrow C_t \hookrightarrow \text{Tor}(P_{\Delta}) = \mathfrak{X}_t$, $0 < t \ll 1$, in the family (25) is separating. Choosing a continuous subfamily of halves $\Phi_t : \widehat{C}_+^{(t)} \rightarrow \text{Tor}(P_{\Delta})$, $0 < t \ll 1$, we define a complex orientation on the given curve over $\mathbb{K}_{\mathbb{R}}$.

Lemma 3.10 *Let δ be a nondegenerate lattice triangle with at least one side of even lattice length, $\Phi : \mathbb{P}^1 \rightarrow C \hookrightarrow \text{Tor}(\delta)$ a real rational, peripherally unibranch curve, where $C \in |\mathcal{L}_{\delta}|$. Then the closure of exactly one quadrant contains an arc of $\mathbb{R}C$ that passes through intersection points with all toric divisors.*

Proof. Straightforward. □

Definition 3.11 *Let δ be a nondegenerate lattice triangle with at least one side of even lattice length, $\Phi : \mathbb{P}^1 \rightarrow C \hookrightarrow \text{Tor}(\delta)$ a real rational, peripherally unibranch curve, where $C \in |\mathcal{L}_{\delta}|$. The arc of $\mathbb{R}C$ mentioned in Lemma 3.10 is called 3-arc and is denoted $\mathfrak{N}(C, \delta)$. Furthermore, if $\mathfrak{N}(C, \delta) \subset \text{Tor}_{\mathbb{R}}^+(\delta)$, then the curve is called Harnack rational curve. In the case of a Harnack rational curve $C \subset \text{Tor}(\delta)$, where*

$$\delta = \text{conv}\{(0, 0), (2i, 0), (k, l)\} \subset \mathbb{R}^2, \quad i, j, k \in \mathbb{Z}, \quad i, l > 0, \quad (28)$$

denote by $h_{++}(C)$ and $h_{+-}(C)$ the numbers of elliptic nodes in the open positive quadrant $\text{Tor}_{\mathbb{R}}(\delta)^+$ and in the quadrant $\{x > 0, y < 0\} \subset \text{Tor}_{\mathbb{R}}(\delta)^{\times}$, respectively.

Lemma 3.12 *Under the assumptions of Definition 3.11, we have*

$$h_{++}(C) = \#(\text{Int}(\delta) \cap 2\mathbb{Z}^2),$$

$$h_{+-}(C) = \#\{(a, b) \in \text{Int}(\delta) \cap \mathbb{Z}^2 : a \equiv 1 \pmod{2}, b \equiv 0 \pmod{2}\}. \quad (29)$$

Proof. By [28, Lemma 3.5] the curve C is smooth at the intersection points with the toric divisors, and is nodal; furthermore, by [21, The last paragraph of Section 8.4], the curve C cannot have hyperbolic nodes. We claim that all the nodes of C are elliptic. Indeed, the curve C admits a real parametrization

$$x = \lambda(\tau - 1)^l, \quad y = \mu\tau^{2i}(\tau - 1)^{-k}, \quad \tau \in \mathbb{C}, \lambda, \mu \in \mathbb{R}^\times. \quad (30)$$

The nodes appear as solutions to the system

$$\lambda(\tau_1 - 1)^l = \lambda(\tau_2 - 1)^l, \quad \mu\tau_1^{2i}(\tau_1 - 1)^{-k} = \mu\tau_2^{2i}(\tau_2 - 1)^{-k}, \quad \tau_1 \neq \tau_2,$$

from which we immediately derive that

$$|\tau_1| = |\tau_2|, \quad |\tau_1 - 1| = |\tau_2 - 1|, \quad \tau_1 \neq \tau_2,$$

and hence

$$\tau_1 = \bar{\tau}_2 \in \mathbb{C} \setminus \mathbb{R}.$$

That is, all the nodes of C are elliptic.

Next, we notice that C has in $\text{Tor}_{\mathbb{R}}^+(\delta)$ an arc passing through all three intersection points of C with the toric divisors. There exists a small deformation of C in the real part of the linear system $|\mathcal{L}_\delta|$ which turns each intersection point with a toric divisor into a bunch of real transversal intersections, and turns each elliptic node into an oval (this follows, for instance, from [26, Theorem]). The obtained curve C' is a smooth M -curve with respect to the triangle δ in the sense of [20, Definition 2]. By [20, Theorem 3] the isotopy type of the real part $\mathbb{R}C'$ with respect to the toric divisors of $\text{Tor}(\delta)$ is uniquely determined, and, moreover, relations (29) follow from [20, Lemma 11]. \square

Lemma 3.13 (1) *Let δ be the lattice triangle (28), and*

$$z_1 \in \text{Tor}_{\mathbb{R}}^+([(0, 0), (2i, 0)]), \quad z_2 \in \text{Tor}_{\mathbb{R}}([(0, 0), (k, l)])$$

be fixed points disjoint from the intersection points of toric divisors.

- (i) *A real rational, peripherally unibranch curve $C \in |\mathcal{L}_\delta|$ passing through z_1, z_2 exists if and only if either $l_1 := l/\text{gcd}(k, l)$ is odd, or l_1 is even and $z_2 \in \text{Tor}_{\mathbb{R}}^+([(0, 0), (k, l)])$.*
- (ii) *If the local real branch of C at z_1 lies in $\text{Tor}_{\mathbb{R}}^+(\delta)$, then C is a Harnack rational curve, and in such a case one has $z_2 \in \text{Tor}_{\mathbb{R}}^+([(0, 0), (k, l)])$.*
- (iii) *If l_1 is odd, then C is defined uniquely.*

(iv) If l_1 is even and $z_2 \in \text{Tor}_{\mathbb{R}}^+([(0,0), (k,l)])$, there exist two suitable curves C : one is a Harnack rational curve and the other is obtained from a Harnack rational curve by the coordinate change $y \mapsto -y$.

(2) If additionally we require that the real local branch centered at z_1 lies in the positive quadrant, then C as above exists if and only if $z_2 \in \text{Tor}_{\mathbb{R}}^+([(0,0), (k,l)])$. Moreover, such a curve C is unique, and its real local branch centered at z_2 intersects the positive quadrant.

Proof. (1) To prove claim (i), we need to find conditions of the existence of parametrization (30) with real λ, μ . Under the given data, (λ, μ) is a solution of the system

$$\lambda(-1)^l = \xi, \quad \lambda^{k_1} \mu^{l_1} = \eta, \quad \text{where } k_1 = \frac{k}{\gcd(k,l)}, \quad \xi > 0, \quad \eta \in \mathbb{R}^\times.$$

Then $\lambda = \xi(-1)^l$ is always real, whereas $\mu^{l_1} = (-1)^{k_1 l} \eta / \xi^{k_1}$. A real root μ exists precisely under the hypotheses of item (i).

The same computation yields claims (ii), (iii), and (iv) as well.

(2) The additional requirement means that $\mu(-1)^k > 0$. Hence, $\eta = \xi^{k_1} \mu^{l_1} (-1)^{k_1 l} = \xi^{k_1} (\mu(-1)^k)^{l_1} > 0$, that is, $z_2 \in \text{Tor}_{\mathbb{R}}^+([(0,0), (k,l)])$, and, in such a case, the corresponding real solution (λ, μ) is unique. For the topological reason, the real arc of C in the positive quadrant touches all toric divisors. \square

Lemma 3.14 *Given a family (26) representing a real nodal curve over the field \mathbb{K} , suppose that the parameterized tropical limit yields a lattice parallelogram δ of the subdivision $\Sigma(T)$ and real limit curves $C_1, C_2 \subset \text{Tor}(\delta)$ as in Lemma 3.2(2). Suppose that the intersection points of $C_1 \cup C_2$ with the toric divisors lies in $\partial \text{Tor}_{\mathbb{R}}^+(\delta)$. Denote by $C^{(t)} \subset \text{Tor}(P_\Delta)$ the fiber of the family $(\mathcal{C}, \mathbf{w})$ in (26) for $t \in (\mathbb{C}, 0)$. Denote also by σ_1, σ_2 two non-parallel sides of δ . Then, one has the following statements.*

- (i) *The intersection point $C_1 \cap C_2$ develops into real hyperbolic nodes and complex conjugate nodes along the deformation $\{C^{(t)}\}_{0 \leq t \ll 1}$.*
- (ii) *If $\|\sigma_1\|_{\mathbb{Z}} \cdot \|\sigma_2\|_{\mathbb{Z}} \equiv 0 \pmod{2}$, then the numbers of real hyperbolic nodes designated in item (i) are even, both in $\text{Tor}_{\mathbb{R}}^+(P_\Delta)$ and outside $\text{Tor}_{\mathbb{R}}^+(P_\Delta)$.*
- (iii) *If $\|\sigma_1\|_{\mathbb{Z}} \cdot \|\sigma_2\|_{\mathbb{Z}} \equiv 1 \pmod{2}$ and $\mathcal{A}(\delta) \equiv 0 \pmod{2}$, then the numbers of real hyperbolic nodes designated in item (i) are odd, both in $\text{Tor}_{\mathbb{R}}^+(P_\Delta)$ and outside $\text{Tor}_{\mathbb{R}}^+(P_\Delta)$.*

Proof. Claim (i) is evident. If, for instance, $\|\sigma_1\|_{\mathbb{Z}}$ is even, then the real arc of C_1 in each quadrant develops into an even number of local arcs along the deformation $\{C^{(t)}\}_{0 \leq t \ll 1}$, and hence the statement (ii) holds. Under the hypotheses of item (iii), the intersection $C_1 \cap C_2$ contains one real point in $\text{Tor}_{\mathbb{R}}^+(\delta)$ and one real point outside $\text{Tor}_{\mathbb{R}}^+(\delta)$. Then, the required statement follows from the fact that each real arc of

C_1 and C_2 in each quadrant develops an odd number of real local arcs along the deformation $\{C^{(t)}\}_{0 \leq t \ll 1}$. \square

Lemma 3.15 *Let $\delta_\ell = \text{conv}\{(-1, 0), (1, 0), (0, \ell)\}$ (see Figure 7(c)), and let $a_{00}, a_{20}, a_{1\ell} \in \mathbb{R}^\times$.*

(1) *Let ℓ be even. A real polynomial with Newton triangle δ_ℓ , defining a real rational curve and having coefficients $a_{-1,0}, a_{1,0}, a_{0,\ell}$ at the vertices of δ_ℓ , exists if and only if $a_{-1,0}a_{1,0} > 0$. Assuming, furthermore, that $a_{-1,0}, a_{1,0} > 0$, $a_{0,\ell} < 0$, and $a_{0,\ell-1} = 0$, one obtains exactly two polynomials as above. These two polynomials come from the following ones*

$$\psi_{1,\ell}(x, y) = x + x^{-1} - 2 \cdot \text{Cheb}_\ell(y), \quad \psi_{2,\ell}(x, y) := \psi_1(x(-1)^{\ell/2}, y\sqrt{-1}), \quad (31)$$

where $\text{Cheb}_\ell(\tau) = \cos(\ell \cdot \arccos \tau)$ is the ℓ -th Chebyshev polynomial, by a suitable transformation $F(x, y) \mapsto \gamma_1 F(\gamma_2 x, \gamma_3 y)$ with $\gamma_1, \gamma_2, \gamma_3 > 0$. The curve $\psi_{1,\ell} = 0$ has $\frac{\ell}{2} - 1$ elliptic nodes in the half-plane $x > 0$ and $\frac{\ell}{2}$ elliptic nodes in the half-plane $x < 0$, while the curve $\psi_{2,\ell} = 0$ has a hyperbolic node in the half-plane $x > 0$ as its only real singularity.

(2) *Let ℓ be odd. For any real nonzero $a_{-1,0}, a_{1,0}, a_{0,\ell}$ and $a_{0,\ell-1} = 0$, there exists a unique real polynomial with Newton triangle T_ℓ , defining a real rational curve and having coefficients $a_{-1,0}, a_{1,0}, a_{0,\ell}, a_{0,\ell-1}$ as specified. In particular, if $a_{1,0} > 0$ and $a_{0,\ell} < 0$, each polynomial as above is obtained from one of the following ones*

$$\psi_{1,\ell}(x, y), \quad \psi_{2,\ell}(x, y) := -\psi_{1,\ell}(x\sqrt{-1}^l, y\sqrt{-1}), \quad (32)$$

by a suitable transformation $F(x, y) \mapsto \gamma_1 F(\gamma_2 x, \gamma_3 y)$ with $\gamma_1, \gamma_2, \gamma_3 > 0$: namely, those with $a_{-1,0} > 0$ come from $\psi_{1,\ell}$, and those with $a_{-1,0} < 0$ come from $\psi_{2,\ell}$. The curve $\psi_{1,\ell} = 0$ has the same number $\frac{\ell-1}{2}$ of elliptic nodes in the half-plane $x > 0$ and in the half-plane $x < 0$, while the curve $\psi_{2,\ell} = 0$ has no real singularities at all.

Proof. Both statements are well known and can easily be derived from [28, Lemma 3.9 and its proof]. In particular, all real polynomials having Newton triangle δ_ℓ and defining rational curves form orbits of the $(\mathbb{R}^\times)^3$ -action

$$F(x, y) \mapsto \gamma_1 F(\gamma_2 x, \gamma_3 y), \quad \gamma_1, \gamma_2, \gamma_3 \in \mathbb{R}^\times, \quad (33)$$

generated by polynomials (31) (ℓ even), or (32) (ℓ odd). The elliptic nodes of the curve $\psi_{1,\ell} = 0$ correspond to the maxima and minima of Cheb_ℓ . In turn, the required properties of the curves $\psi_{2,\ell} = 0$ can be derived from the fact that $(\sqrt{-1})^\ell \text{Cheb}_\ell(\tau\sqrt{-1}) = (-1)^\ell \cosh(\ell \cdot \text{arccosh } \tau)$. (See the curves given by (31) and (32) shown in red color in Figure 8(a-d).) \square

3.6 Correspondence theorem for real separating curves

Throughout this section, we assume that $\mathbf{w} \subset \text{Tor}_{\mathbb{K}}(P_\Delta)$.

Note that all real limit curves and real modified limit curves in the modified parameterized tropical limit described in Section 3.3 have an infinite real point set and the complement to the real point set consists of two connected components, *i.e.*, they are separating.

Denote by $\overrightarrow{\mathcal{M}}_{g,n}^{\mathbb{K}_{\mathbb{R}}}((\overline{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ the set of complex oriented elements of $\mathcal{M}_{g,n}^{\mathbb{K}_{\mathbb{R}}}((\overline{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ (see Section 3.5). An element of $\overrightarrow{\mathcal{M}}_{g,n}^{\mathbb{K}_{\mathbb{R}}}((\overline{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ induces a complex orientation on each limit and modified limit curve of the tropical limit of that element. Denote by $\overrightarrow{\text{Ad}}^{\mathbb{R}}((\overline{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ (respectively, $\overrightarrow{\text{EAd}}^{\mathbb{R}}((\overline{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$) the set of the induced complex oriented admissible (respectively, extended admissible) collections, associated with a curve $(\overline{\Gamma}, h, \mathbf{p}_{tr}) \in \mathcal{T}_{g,n}^{\text{red}}(\Delta, \mathbf{x})$.

In what follows, we use notations of Definition 3.8.

Lemma 3.16 *Given a curve $(\overline{\Gamma}, h, \mathbf{p}_{tr}) \in \mathcal{T}_{g,n}^{\text{red}}(\Delta, \mathbf{x})$ and the polygons $\delta_1, \dots, \delta_N$ forming the dual subdivision of P_{Δ} , a sequence of real rational curves $C_k \in |\mathcal{L}_{\delta_k}|$ satisfying the conditions of Definition 3.8(1) yields an element of $\overrightarrow{\text{Ad}}^{\mathbb{R}}((\overline{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ if and only if the following holds:*

- (i) *for each bounded edge $e = [v_i, v_j] \in \Gamma_{\text{red}}^1$ of even weight, the real local branches of the curves $C_i \subset \text{Tor}(\delta_i)$, $C_j \subset \text{Tor}(\delta_j)$ (where δ_i, δ_j are dual to $h(v_i)$, $h(v_j)$, respectively) centered at the points $z_i \simeq z_j$ of the canonically isomorphic (see Remark 3.3) toric divisors of $\text{Tor}(\delta_i)$, $\text{Tor}(\delta_j)$ lie in the same closed quadrant in $\text{Tor}_{\mathbb{R}}(\delta_i)$, $\text{Tor}_{\mathbb{R}}(\delta_j)$, respectively; if, in addition, e contains a marked point p_{tr} such that $h(p_{tr}) = \text{val}(w)$, $w = (t^{c_1}(a_1 + O(t^d)), t^{c_2}(a_2 + O(t^d))) \in \mathbf{w}$, $d \gg 0$, where $a_1, a_2 \in \mathbb{R}^{\times}$, then the point (a_1, a_2) lies in the same quadrant as the aforementioned real local branches of C_i, C_j ;*
- (ii) *for each unbounded edge $e \in \Gamma^1$ of an even weight, incident to a trivalent vertex v_i and containing an interior marked point p_{tr} such that $h(p_{tr}) = \text{val}(w)$, $w = (t^{c_1}(a_1 + O(t^d)), t^{c_2}(a_2 + O(t^d)))$, where $a_1, a_2 \in \mathbb{R}^{\times}$, the point (a_1, a_2) and the real local branch of $C_i \subset \text{Tor}(\delta_i)$ centered on the toric divisor $\text{Tor}(\sigma)$, where σ is orthogonal to $h(e)$, lie in the same quadrant.*

Furthermore, under the above conditions, there are $2^{m_1+m_2}$ elements of $\overrightarrow{\text{EAd}}^{\mathbb{R}}((\overline{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ containing the sequence C_1, \dots, C_N , where m_1 is the number of marked points on edges of $\overline{\Gamma}_{\text{red}}$ of even weight, and m_2 is the total number of edges of even weight.

Proof. We have to show that the condition stated in the lemma is necessary and sufficient for the completion of the sequence C_1, \dots, C_N to a real admissible collection.

First, observe that there is no restriction to find suitable real modified limit curves in the case of an edge e of odd weight ℓ . One can easily see that this fact reduces to the following statements.

- (1) For any prescribed real nonzero coefficients at the vertices of the triangle $\delta_{\ell} = \text{conv}\{(-1, 0), (1, 0), (0, \ell)\}$ (Figure 7(c)), there exists a real polynomial $F(x, y)$ with Newton triangle δ_{ℓ} , having the above prescribed coefficients, having the vanishing coefficient of $y^{\ell-1}$, and defining a real rational curve.

- (2) For any prescribed real nonzero coefficients at the vertices of the triangle $\delta'_\ell = \text{conv}\{(-1, 0), (0, 0), (0, \ell)\}$ (Figure 7(k)), there exists a real Laurent polynomial with the Newton triangle δ'_ℓ having the above prescribed coefficients and defining a peripherally unibranch curve in $\text{Tor}(\delta'_\ell)$.

Claim (2) is evident. Claim (1) follows from Lemma 3.15(2).

For ℓ even, the necessary and sufficient condition for the existence of a suitable real modified limit curve reads $a_{-1,0}a_{1,0} > 0$ (see Lemma 3.15(1)). Geometrically, the latter condition means that the local real branches of C_i, C_j centered at z_i, z_j , respectively, lie in the same quadrant. Note that there are two suitable modified limit curves (see Lemma 3.15(1) and Figures 8(a,b)).

Furthermore, if e contains an interior marked point, then we have an additional restriction for the coefficients at the vertices of the triangle δ'_ℓ (see Figure 7(h,k)): $b_{00}b_{0\ell} > 0$, which geometrically means that the local real branch of C_i centered on the toric divisor $\text{Tor}(\sigma)$ and the point $\text{ini}(w)$ lie in the same quadrant. Note that in view of $b_{00}b_{0\ell} > 0$, there are two real polynomials $b_{0\ell}(y + \lambda)^\ell$ matching the condition $b_{0\ell}\lambda^\ell = b_{00}$. For each of the choices of λ , there are two suitable real modifications as mentioned in the preceding paragraph.

The results of the two last paragraphs yield $2^{m_1+m_2}$ possible extensions of the collection of limit curves C_1, \dots, C_N . \square

Definition 3.17 (1) Let $(C_1, \dots, C_N) \in \text{Ad}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$, where $C_i \in |\mathcal{L}_{\delta_i}|$, $i = 1, \dots, N$. A complex orientation of (C_1, \dots, C_N) is a complex orientation of each of the curves C_1, \dots, C_N . We say that a complex orientation of (C_1, \dots, C_N) is coherent if (in the notation of Lemma 3.16) for each bounded edge $e = [v_i, v_j] \in \Gamma^1$ of odd weight, the real local arcs of the curves C_i, C_j , centered at the points $z_i \simeq z_j$, respectively, and lying in the same quadrant, are oriented so that one of the arcs is incoming and the other is outgoing. An element $(C_1, \dots, C_N) \in \text{Ad}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ equipped with a coherent complex orientation will be called oriented real admissible collection.

(2) Let $(C_1, \dots, C_N) \in \text{Ad}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ be equipped with a coherent complex orientation, and let an element $\xi \in \text{EAd}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ extend (C_1, \dots, C_N) . Choose a complex orientation of each of the modified limit curves in ξ . We say that a complex orientation of ξ is coherent if the complex orientation of each modified limit curve is coordinated with the complex orientations of the corresponding curves C_i, C_j in the way shown in Figure 8.

Lemma 3.18 (1) The complex orientation of each element of $\overrightarrow{\text{EAd}}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ is coherent. Each element of $\text{EAd}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ possessing a coherent complex orientation belongs to $\overrightarrow{\text{EAd}}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$.

(2) Each element of $\text{Ad}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ possessing a coherent complex orientation belongs to $\overrightarrow{\text{Ad}}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$. Further on, it can be extended to an element of $\overrightarrow{\text{EAd}}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ in 2^m ways, where m is the number of edges $e \in \Gamma^1$ having even weight and containing an interior marked point.

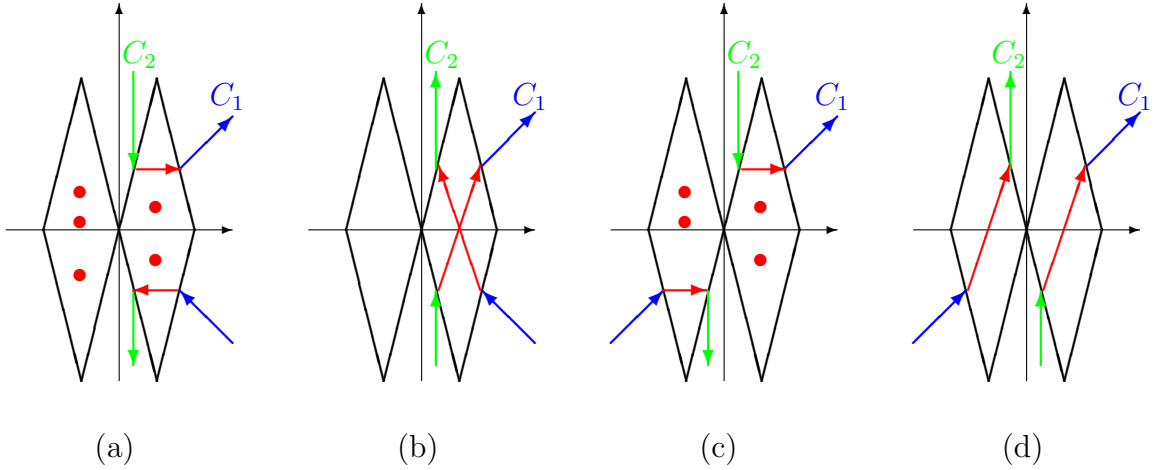


Figure 8: Complex orientations and modifications

Remark 3.19 *Under the assumptions of Sections 2.2 and 2.3 stated over the field $\mathbb{K}_{\mathbb{R}}$, the tropical curves appearing in the tropical limits do not have edges of even weight with an interior marked point, and hence the statement (2) of lemma 3.18 implies that each element of $\overline{\text{Ad}}^{\mathbb{R}}((\overline{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ admits a unique extension to an element of $\overline{\text{EAd}}^{\mathbb{R}}((\overline{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$.*

Proof of Lemma 3.18. (1) Straightforward from Theorem 3.9.

(2) Let a bounded edge $e = [v_1, v_2] \in \Gamma^1$ do not contain a marked point. Then, there exists a unique real modified limit curve $C_{mod} \subset \text{Tor}(\delta_{mod})$. where

$$\delta_{mod} = \text{conv}\{(0, 0), (2, 0), (1, \ell)\} \quad (\text{cf. Figure 7(c)}).$$

whose complex orientation makes the whole modified fragment coherent: see Figure 8(a,b,c,d) showing in red the curves

$$C_{mod} = \{\psi_{1,\ell} = 0\} \quad \text{and} \quad C_{mod} = \{\psi_{2,\ell} = 0\} \quad (\text{see formulas (31)}),$$

for ℓ even or odd, respectively. The same holds if e contains a marked point, and (as we mentioned above in Lemma 3.16) the two choices come from the fact that there are two real modified limit curves $C_{2,mod}(E)$ with Newton polygon $\delta_{2,mod}$ (the left triangle in Figure 7(h)) and prescribed coefficients at the vertices of $\delta_{2,mod}$ (cf. the proof of Lemma 3.16). \square

Lemma 3.20 (1) *Let δ_k be a triangle of the subdivision $\Sigma(T)$, and let $C_k \in |\mathcal{L}_{\delta_k}|$ be a real rational, peripherally unibranch curve equipped with a complex orientation. Then,*

- *the intersection of $\mathbb{R}C_k$ with any closed quadrant is either finite, or contains a unique one-dimensional arc; the complex orientation of such an arc defines a cyclic order of the sides of δ_k , and all these cyclic orders coincide (see an example of an oriented line in Figure 9);*

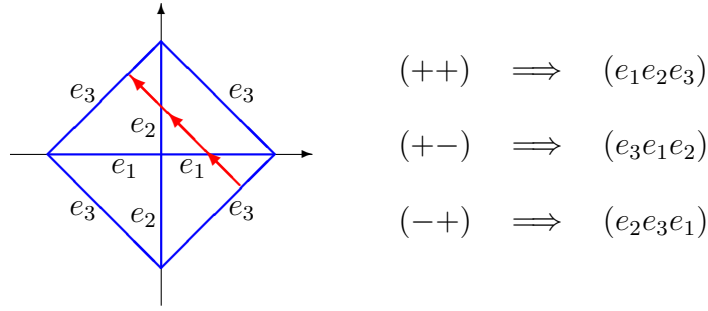


Figure 9: Complex orientation and orientation of a triangle

- $\text{QI}(C_k) = \varepsilon \cdot \|\delta_k\|_{\mathbb{Z}}$, where $\varepsilon = 1$ if the complex orientation of $\mathbb{R}C_k$ defines the positive cyclic order of the sides of δ_k , and $\varepsilon = -1$ otherwise.

(2) Let δ_k be a parallelogram of the subdivision $\Sigma(T)$, and let $C_k \in |\mathcal{L}_{\delta_k}|$ be a real limit curve as in Lemma 3.2(2). Then $\text{QI}(C_k) = 0$.

Proof. Straightforward from [22, Theorem 3.4 and Example 3.5]. \square

Now, we state a version of the correspondence Theorem 3.9 taking into account real separating curves equipped with the complex orientation.

Theorem 3.21 *Under the hypotheses of Theorem 3.9, assume that $\mathbf{w} \subset \text{Tor}_{\mathbb{K}_{\mathbb{R}}}(P_{\Delta})$. Then, for each curve $(\bar{\Gamma}, h, \mathbf{p}_{tr}) \in \mathcal{T}_{g,n}^{\text{red}}(\Delta, \mathbf{x})$, the natural map*

$$\vec{\mathcal{M}}_{g,n}^{\mathbb{K}_{\mathbb{R}}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w}) \rightarrow \overline{\text{EAd}}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$$

is bijective. Furthermore, in the setting of Section 2.2 or 2.3, the natural map

$$\vec{\mathcal{M}}_{g,n}^{\mathbb{K}_{\mathbb{R}}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w}) \rightarrow \overline{\text{EAd}}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w}) \rightarrow \overline{\text{Ad}}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$$

is bijective.

For each element $\xi \in \vec{\mathcal{M}}_{g,n}^{\mathbb{K}_{\mathbb{R}}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$, one has

$$\text{QI}(C) = \sum_{k=1}^N \text{QI}(C_k),$$

where $(C_1, \dots, C_N) \in \overline{\text{Ad}}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ is the image of ξ .

Proof. Straightforward from Theorem 3.9 and Lemmas 3.16, 3.18, and 3.20. \square

Theorem 3.22 *In the setting of Section 2.2 or 2.3 and under the hypotheses of Theorem 3.9, suppose, in addition, that the curves $C_i \subset \text{Tor}(\delta_i)$ are Harnack when δ_i is a triangle with all sides of even length. Then, the Welschinger sign $W_1(\xi)$ (as defined by formula (5)) can be computed by summing up the contributions of the curves C_1, \dots, C_N and of the finite edges of the underlying tropical curve $(\bar{\Gamma}, h, \mathbf{p}_{tr})$ to the exponent of (-1) in formula (5). The non-trivial contributions are as follows.*

- The curve $C_j \subset \text{Tor}(\delta_j)$, where δ_j is a triangle with at least one side σ of even length, contributes either $h_{++}(C_j)$ or $h_{+-}(C_j)$ according as the local real arc of C_j centered on $\text{Tor}(\sigma)$ lies in the positive quadrant, or not; furthermore, C_j additionally contributes 1 each time when
 - δ_j is dual to a trivalent vertex $V = h(v)$, where v is the trivalent endpoint of an end $e \in \Gamma_{red}^1$,
 - the side $\sigma \subset \partial\delta_j$ orthogonal to $E = h(e)$ satisfies $\|\delta_j\|_{\mathbb{Z}} \equiv 0 \pmod{4}$,
 - the orientation of $\mathbb{R}C_j$ at the intersection point with $\text{Tor}(\sigma)$ is opposite to the orientation of $\text{Tor}_{\mathbb{R}}(\sigma)$ induced by the given orientation of $\text{Tor}_{\mathbb{R}}^+(\delta_j)$ (which in turn, comes from the a priori fixed orientation of $\text{Tor}_{\mathbb{R}}^+(P_{\Delta})$).
- The curve $C_j \subset \text{Tor}(\delta_j)$, where δ_j is a parallelogram with both sides of odd length and of area $\|\delta_j\|_{\mathbb{Z}} \equiv 0 \pmod{4}$, contributes 1.
- A pair of triangles δ_i, δ_j of the subdivision $\Sigma(T)$ which are dual to trivalent vertices $V_1 = h(v_1), V_2 = h(v_2)$ such that $v_1, v_2 \in \Gamma_{red}^0$ are endpoints of the same edge of Γ_{red} of weight $\ell > 1$ contributes $\lfloor \frac{\ell-1}{2} \rfloor$, if
 - the local arc of $\mathbb{R}C_i$ touching the toric divisor $\text{Tor}(\sigma_i) \subset \text{Tor}(\delta_i)$ and the local arc of $\mathbb{R}C_j$ touching the toric divisor $\text{Tor}(\sigma_j) \subset \text{Tor}(\delta_j)$ lie in $\text{Tor}_{\mathbb{R}}^+(\delta_i), \text{Tor}_{\mathbb{R}}^+(\delta_j)$, respectively, where the toric divisors $\text{Tor}(\sigma_i), \text{Tor}(\sigma_j)$ either coincide, or are canonically isomorphic in the sense of Remark 3.3,
 - the complex orientations of above arcs of $\mathbb{R}C_i, \mathbb{R}C_j$ induce opposite orientations of $\text{Tor}_{\mathbb{R}}(\sigma_i), \text{Tor}_{\mathbb{R}}(\sigma_j)$, respectively.

Proof. Note that, in the rational case (setting of Section 2.2), all triangles δ_i have sides of even length, and then the claim follows from Lemma 3.12, Lemma 3.15(1), and Lemma 3.18 (cf. Figure 8(a,b)).

In the elliptic case (setting of Section 2.3), each triangle δ_i has at least one side of even length (otherwise, the corresponding limit curve C_i would have real arcs in three quadrants, while the considered elliptic curves may have real arcs only in two quadrants. By our assumption, the limit curves in associated with the triangles having all sides of even length are Harnack, *i.e.*, have a real arc lying in the positive quadrant. It easily follows that all intersection points of C_1, \dots, C_N with toric divisors lie in the positive halves of these divisors. Thus, the count of elliptic nodes in the positive quadrant and hyperbolic nodes outside it, asserted in the lemma, follows from Lemmas 3.12, 3.14, 3.15, and the coherence condition (cf. Figure 8). \square

Remark 3.23 Recall that the curves in Figure 8 are presented in the coordinates x_1, y_1 linked with the main coordinates x, y by formulas $(x, y) = (x_1, y_1 + b)$, where b is the second coordinate of the intersection point of C_1, C_2 with the common toric divisor of the toric surfaces $\text{Tor}(\delta_1), \text{Tor}(\delta_2)$. That is, the singularities shown in the half-planes $\{x_1 > 0\}$ and $\{x_1 < 0\}$ in Figure 8 correspond to singularities of the curves $C^{(t)}$, $0 < t \ll 1$, in the quadrants

$$\{x > 0, y \text{ sign}(b) > 0\} \quad \text{or} \quad \{x < 0, y \text{ sign}(b) > 0\},$$

respectively.

4 Tropical elliptic invariants

The main purpose of this section is to describe tropical analogs of the refined elliptic invariants introduced above and to provide a tropical calculation of these invariants.

4.1 Tropical enumerative problem

4.1.1 Tropical constraints

For each vector $\mathbf{a} \in \mathbb{Z}^2 \subset \mathbb{R}^2$, consider the linear form $\lambda_{\mathbf{a}} : \mathbb{R}^2 \rightarrow \mathbb{R}$ whose gradient $\check{\mathbf{a}}$ is obtained from \mathbf{a} by the clockwise rotation by $\pi/2$. If $L_{\mathbf{a}} \subset \mathbb{R}^2$ is an affine straight line directed by the vector \mathbf{a} , then $\lambda_{\mathbf{a}}$ takes the same value at all points of $L_{\mathbf{a}}$; we denote this value by $\lambda_{\mathbf{a}}(L_{\mathbf{a}})$.

Let $\Delta \subset \mathbb{Z}^2$ be a non-degenerate balanced multi-set. Any collection of affine straight lines $\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}$ in \mathbb{R}^2 such that, for each $\mathbf{a} \in \Delta$, the vector \mathbf{a} is contained in the direction of $L_{\mathbf{a}}$ is called a *tropical Δ -constraint*.

For a plane tropical curve T represented by a parameterized plane tropical curve $(\bar{\Gamma}, h)$, we say that a given point of \mathbb{R}^2 belongs to T if this point belongs to the image of h . We also speak about bounded edges and ends of T : these are images under h of the bounded edges and the ends of Γ .

Definition 4.1 *A tropical Δ -constraint $\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}$ satisfies the tropical Menelaus condition if there exists a plane tropical curve T of degree Δ such that for every $\mathbf{a} \in \Delta$ the end of T directed by the vector $\mathbf{a} \in \Delta$ is contained in the line $L_{\mathbf{a}}$.*

The following lemma is an easy well-known statement (cf., [22]).

Lemma 4.2 *A tropical Δ -constraint $\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}$ satisfies the tropical Menelaus condition if and only if*

$$\sum_{\mathbf{a} \in \Delta} \lambda_{\mathbf{a}}(L_{\mathbf{a}}) = 0 . \quad (34)$$

Proof. The ‘only if’ part of the statement immediately follows from the balancing condition for tropical curves. The ‘if’ part can be proved by a straightforward inductive construction. \square

An *elliptic plane tropical curve* is a plane tropical curve of genus 1. Any representative $(\bar{\Gamma}, h)$ of such an elliptic plane tropical curve T has a unique simple cycle. We call it (respectively, its image under h) the *cycle* of Γ (respectively, of T).

Assume that a tropical Δ -constraint $\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}$ satisfies the tropical Menelaus condition, and consider a point $x_0 \in \mathbb{R}^2$. The pair $(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$ is called an *extended tropical Δ -constraint*. An elliptic plane tropical curve T is said to satisfy the extended tropical Δ -constraint $(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$ if

- T is of degree Δ ,
- for every $\mathbf{a} \in \Delta$ the end of T directed by the vector $\mathbf{a} \in \Delta$ is contained in the line $L_{\mathbf{a}}$,
- and $x_0 \in T$.

Lemma 4.3 *Let $(\bar{\Gamma}, h)$ be a representative of an elliptic plane tropical curve T satisfying the extended tropical Δ -constraint $(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$. Assume that Δ is even and Γ is 3-valent. Then,*

- each edge e of Γ such that e does not belong to the cycle of Γ is of even weight;
- either each edge of the cycle of Γ is of even weight, or each edge of the cycle of Γ is of odd weight and the primitive vectors in the directions of the images under h of these edges all have the same parity (by the parity of $(a, b) \in \mathbb{Z}^2$ we mean $(a \bmod 2, b \bmod 2) \in (\mathbb{Z}/2\mathbb{Z})^2$).

Proof. The statement of the lemma is a straightforward corollary of the balancing condition. \square

The condition that an elliptic plane tropical curve T , represented by a parameterized plane tropical curve $(\bar{\Gamma}, h)$, satisfies an extended tropical Δ -constraint $(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$ can be reformulated in terms of the tropical toric surface $\mathbb{T}P_{\Delta}$ (cf. Section 3.1). For any vector $\mathbf{a} \in \Delta$, the line $L_{\mathbf{a}}$ oriented by \mathbf{a} is completed by a point $x_{\mathbf{a}}$ in the (open) strata $\mathcal{Z} \subset \partial \mathbb{T}P_{\Delta}$ corresponding to the side $\sigma \subset P_{\Delta}$ having \mathbf{a} as outer normal vector; we equip $x_{\mathbf{a}}$ with the weight $\text{wt}(\mathbf{a})$ (the weight of \mathbf{a}). A parameterized plane tropical curve $(\bar{\Gamma}, h)$ that represents an elliptic plane tropical curve T satisfying an extended tropical Δ -constraint $(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$ gives rise to a marked parameterized elliptic plane tropical curve $(\bar{\Gamma}, h, \mathbf{p}_{tr})$ of degree Δ such that (cf., Section 3.1)

- a point $p_{tr} \in \mathbf{p}_{tr}$ belongs to Γ , the other points of \mathbf{p}_{tr} being the univalent vertices of $\bar{\Gamma}$;
- for any vector $\mathbf{a} \in \Delta$, one of the univalent vertices in $\mathbf{p}_{tr} \setminus p_{tr}$ is sent to $x_{\mathbf{a}}$ by the extension $\bar{h} : \bar{\Gamma} \rightarrow \mathbb{T}P_{\Delta}$ of h , the weight of the end adjacent to this vertex being equal to $\text{wt}(\mathbf{a})$;
- $h(p_{tr}) = x_0$.

The other way around, forgetting the marking of such a marked parameterized elliptic plane tropical curve $(\bar{\Gamma}, h, \mathbf{p}_{tr})$, we get a parameterized plane tropical curve $(\bar{\Gamma}, h)$ that represents an elliptic plane tropical curve satisfying an extended tropical Δ -constraint $(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$.

4.1.2 Elliptic plane tropical curves of given parity

Fix an even non-degenerate balanced multi-set $\Delta \subset \mathbb{Z}^2$ and a pair $(\alpha, \beta) \in (\mathbb{Z}/2\mathbb{Z})^2 \setminus \{(0, 0)\}$. We say that Δ and (α, β) satisfy the *admissibility condition* if P_Δ has at most one side whose primitive normal vectors are of parity (α, β) . This condition is a reformulation of the admissible quadrant condition (AQC) on the quadrant

$$\mathfrak{Q} = \{(x, y) \in \mathbb{R}^2 \mid (-1)^\alpha x > 0, (-1)^\beta y > 0\}$$

(see Section 2.3).

An elliptic plane tropical curve T , represented by $(\bar{\Gamma}, h)$, of degree Δ , is said to be *of parity* (α, β) if Γ has edges of odd weight, the restriction of h on any such edge is non-constant, and the primitive vectors in the directions of the images under h of these edges have the parity (α, β) . A plane tropical curve, represented by $(\bar{\Gamma}, h)$, is said to be *simple* if Γ is 3-valent and the inverse image $h^{-1}(x)$ of any point $x \in \mathbb{R}^2$ contains at most two points; if $h^{-1}(x)$ is formed by two points, none of them is a vertex of Γ .

Let T be a simple elliptic plane tropical curve of degree Δ and parity (α, β) , and let $(\bar{\Gamma}, h)$ be a parameterized plane tropical curve representing T . Denote by c the cycle of Γ . Lemma 4.3 implies that the edges of Γ that have an odd weight are exactly the edges of the cycle c . The graph Γ can be represented as the union of c and a collection $\{\Gamma_v\}$ of trees formed by edges of even weight, the latter collection being indexed by the set of vertices of c .

Let $\Sigma(T)$ be the subdivision of P_Δ that is dual to T . Any polygon of $\Sigma(T)$ is either a triangle, or a parallelogram. A triangle of $\Sigma(T)$ (or, equivalently, a vertex of T) is called *even* if its sides are even, that is, have even lattice lengths, and is called *odd* otherwise. An odd triangle of $\Sigma(T)$ is said to be *mobile* if the tree Γ_v adjacent to the corresponding odd vertex v of Γ satisfies the following property: for each edge of Γ_v that is an end of Γ , the primitive integer vectors in the direction of this edge are of parity (α, β) ; in this case, the odd vertex v is also said to be *mobile*. If Δ and (α, β) satisfy the admissibility condition, then any mobile odd vertex of Γ is adjacent to an end such that the primitive integer vectors in its direction are of parity (α, β) .

Example 4.4 *Figure 10 shows an example of a subdivision dual to an elliptic plane tropical curve of parity $(\alpha, \beta) = (0, 1)$. The subdivision contains one parallelogram, two even triangles (indicated by the black marks), and five odd triangles. One of the odd triangles is mobile (it is indicated by the blue mark), the other four odd triangles are non-mobile (they are indicated by the red marks).*

Choose an orientation \mathfrak{o} of the cycle c . For each vertex v of c , the orientation \mathfrak{o} provides a local orientation of \mathbb{R}^2 at $h(v)$: if e is the even edge adjacent to v and if e_1 and e_2 are the edges of c that are adjacent to v and such that e_1 precedes e_2 in \mathfrak{o} , then the above local orientation is given by the following cyclic order: e_1, e, e_2 . In the dual way, the orientation \mathfrak{o} provides also an orientation of the triangle dual to v .

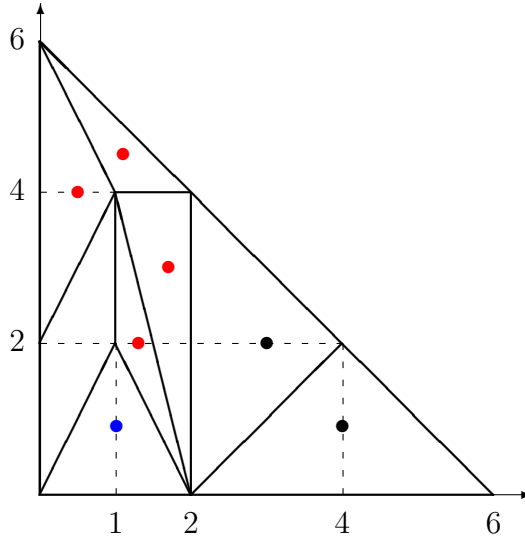


Figure 10: Example in degree 6

An *orientation kit* \mathcal{R} of T is a choice of an orientation $\mathfrak{o}_{\mathcal{R}}$ of the cycle c and a choice, for each vertex v of Γ , of a local orientation of \mathbb{R}^2 at $h(v)$ in such a way that the local orientations associated to all non-mobile odd vertices of T coincide with those provided by $\mathfrak{o}_{\mathcal{R}}$. Alternatively, an *orientation kit* \mathcal{R} of $\Sigma(T)$ is a choice of an orientation $\mathfrak{o}_{\mathcal{R}}$ of c and a choice of an orientation for each triangle of $\Sigma(T)$ in such a way that the orientations of all non-mobile odd triangles of $\Sigma(T)$ coincide with those provided by $\mathfrak{o}_{\mathcal{R}}$. The collections of orientation kits of T and $\Sigma(T)$ are in a natural one-to-one correspondence. In what follows, we use the both settings and freely translate notions concerning orientation kits from one setting to the other without additional explanations. Notice that orientation kits of T can be seen as a way of encoding of ribbon structures of T .

If Δ and (α, β) satisfy the admissibility condition (*i.e.*, P_{Δ} has at most one side whose primitive normal vectors are of parity (α, β)), then the cycle c contains at least one non-mobile odd vertex. Thus, in this situation, while describing an orientation kit, we do not specify the choice of an orientation of c , since this orientation is uniquely restored from the other data in the orientation kit.

For a given orientation kit of T , if the local orientation at a vertex $h(v)$ of T coincides with the canonical orientation of \mathbb{R}^2 , we say that the vertex v of Γ is *positive*; otherwise, we say that v is *negative*. The switch of the chosen orientation of the cycle c makes each positive (respectively, negative) non-mobile vertex of c negative (respectively, positive).

An orientation kit is said to be *maximal* if all triangles are oriented positively in this orientation kit. If T admits a maximal orientation kit, such an orientation kit is unique.

If \mathcal{R} is an orientation kit of T , a mobile odd vertex of Γ is said to be \mathcal{R} -*compatible* (respectively, \mathcal{R} -*noncompatible*) if \mathcal{R} and $\mathfrak{o}_{\mathcal{R}}$ provide the same local orientation (respectively, opposite local orientations) for this vertex. An even vertex of Γ is said

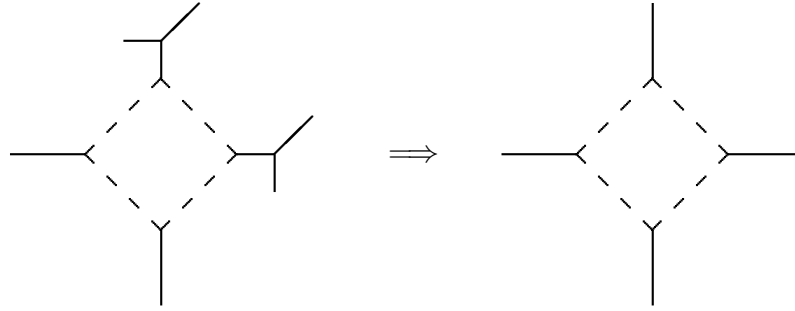


Figure 11: Simplification

to be \mathcal{R} -noncompatible if it belongs to an even tree Γ_v attached to a \mathcal{R} -noncompatible mobile odd vertex v of Γ ; otherwise, the even vertex is said to be \mathcal{R} -compatible. Under the admissibility condition on Δ and (α, β) , all even vertices of Γ are \mathcal{R} -compatible.

The plane tropical curve T may admit a *rectification*: this is an elliptic plane tropical curve represented by a parameterized plane tropical curve $(\bar{\Gamma}', h')$, where the graph $\bar{\Gamma}'$ is obtained from $\bar{\Gamma}$ by replacing each tree Γ_v of the collection $\{\Gamma_v\}$ by an edge connecting v to a new one-valent vertex and the map $h' : \Gamma' \rightarrow \mathbb{R}^2$ is obtained by extending the restriction of h to $\Gamma \setminus \Gamma_v$ in such a way that the balancing condition at v is satisfied (see Figure 11). A rectification of T has the same parity as T , but, if the rectification is nontrivial, not the same (even) degree.

A *partial rectification* of T at an even vertex u_1 adjacent to two one-valent vertices and a vertex u_2 of valency bigger than 1 is an elliptic plane tropical curve represented by a parameterized plane tropical curve $(\bar{\Gamma}'', h'')$, where the graph $\bar{\Gamma}''$ is obtained from $\bar{\Gamma}$ by replacing the union U of three edges adjacent to u_1 by an edge connecting u_2 to a new one-valent vertex and the map $h'' : \Gamma'' \rightarrow \mathbb{R}^2$ is obtained by extending the restriction of h to $\Gamma \setminus U$ in such a way that the balancing condition at u_2 is satisfied. Again, a partial rectification of T has the same parity as T , but not the same (even) degree. A rectification of T can be represented as a result of a sequence of partial rectifications of elliptic plane tropical curves, the first of these curves being T .

4.1.3 Quantum indices and Welschinger signs of elliptic plane tropical curves

Fix again a couple $(\alpha, \beta) \in (\mathbb{Z}/2\mathbb{Z})^2 \setminus \{(0, 0)\}$. Let T be a simple elliptic plane tropical curve of parity (α, β) and of degree $\Delta = \Delta(T)$, and let $(\bar{\Gamma}, h)$ be a parameterized plane tropical curve representing T . To each orientation kit \mathcal{R} of T we associate a *quantum index* and a *Welschinger sign*. The *quantum index* $\kappa(\mathcal{R})$ of \mathcal{R} is $\mathcal{A}_+(\mathcal{R}) - \mathcal{A}_-(\mathcal{R})$, where $\mathcal{A}_+(\mathcal{R})$ (respectively, $\mathcal{A}_-(\mathcal{R})$) is the total Euclidean area of positive (respectively, negative) triangles in $\Sigma(T)$.

Proposition 4.5 *The difference $\mathcal{A}(\Delta) - \kappa(\mathcal{R})$ is an integer divisible by 4.*

Proof. Denote by $\Pi(T)$ the total area of parallelograms in $\Sigma(T)$. The difference $\mathcal{A}(\Delta) - \kappa(\mathcal{R})$ is equal to $2\mathcal{A}_-(\mathcal{R}) + \Pi(T)$. The Euclidean area of each triangle in $\Sigma(T)$ is an integer and the Euclidean area of each parallelogram in $\Sigma(T)$ is an even integer. So, $2\mathcal{A}_-(\mathcal{R}) + \Pi(T)$ is an even integer. Furthermore, the Euclidean area of any even triangle in $\Sigma(T)$ is an even integer and the Euclidean area of any parallelogram in $\Sigma(T)$ with all sides even is an integer divisible by 4. Denote by $\mathcal{A}'_-(\mathcal{R})$ the total area of negative odd triangles in $\Sigma(T)$, and denote by $\Pi'(T)$ the total area of parallelograms in $\Sigma(T)$ that have an odd side. Our purpose is to prove that the even integer number $2\mathcal{A}'_-(\mathcal{R}) + \Pi'(T)$ is divisible by 4.

Denote by \mathcal{R}^{opp} the orientation kit obtained from \mathcal{R} by reversing the orientation of c and the orientations of all triangles of $\Sigma(T)$.

Lemma 4.6 *The integer $2\mathcal{A}'_-(\mathcal{R}) - 2\mathcal{A}'_-(\mathcal{R}^{\text{opp}})$ is divisible by 4.*

Proof. The statement of the lemma is equivalent to the fact that the integer $\mathcal{A}'_-(\mathcal{R}) + \mathcal{A}'_-(\mathcal{R}^{\text{opp}})$ is even. The latter integer is equal to the total Euclidean area of odd triangles in $\Sigma(T)$. Thus, this integer is even, because it is equal to the difference between $\mathcal{A}(\Delta)$ (which is even) and the total Euclidean area of even triangles and parallelograms in $\Sigma(T)$. \square

Lemma 4.7 *Let $L \subset \mathbb{R}^2$ be a straight line with rational slope. Assume that L intersects the cycle $h(c)$ of T only at interior points of edges. For each intersection point of L with an edge $h(e)$ of $h(c)$, consider the integer equal to the Euclidean area of the parallelogram formed by a primitive vector in the direction of L and a primitive vector in the direction of $h(e)$. Then, all the integers obtained in this way from the intersection points of L and $h(c)$ have the same parity.*

Proof. This is an immediate corollary of the fact that all primitive vectors in the directions of edges of the cycle $h(c)$ have the same parity. \square

We prove the statement of the proposition using an induction on the number of self-intersections of the cycle of T , that is, the number of parallelograms in $\Sigma(T)$ that have all sides odd. Assume that the restriction of h on the cycle of Γ is injective, that is, the cycle $h(c)$ of T does not have any self-intersection. In this case, the complement of the cycle $h(c) \subset \mathbb{R}^2$ has two connected components: the *interior* $\text{In}(c)$, homeomorphic to an open 2-disc, and the *exterior* $\text{Ex}(c)$.

For any vertex v of c , let Γ_v be the tree (formed by edges of even weight) adjacent to v . Exactly one edge of Γ_v is adjacent to v ; denote this edge by e_v and denote by \mathbf{L}_v the straight line containing $h(e_v)$. The line \mathbf{L}_v is divided by $h(v)$ into two rays. Denote by \mathbf{L}_v^+ the ray containing $h(e_v)$, and denote by \mathbf{L}_v^- the other ray. Due to Lemma 4.6, we can reverse, if necessary, the orientation of c and assume that a vertex v of c is positive if and only if at $h(v)$ the ray \mathbf{L}_v^+ points towards $\text{Ex}(c)$.

For a vertex v of c , let $\tilde{v} \in c$ be a generic point sufficiently close to v . Denote by $\mathbf{L}_{\tilde{v}}$ the straight line parallel to \mathbf{L}_v and passing through $h(\tilde{v})$; note that the lines \mathbf{L}_v

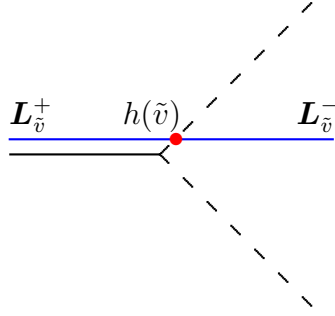


Figure 12: Rays $L_{\tilde{v}}^+$ and $L_{\tilde{v}}^-$

and $L_{\tilde{v}}$ are different. Denote by $L_{\tilde{v}}^+$ and $L_{\tilde{v}}^-$ the rays of $L_{\tilde{v}}$ having $h(\tilde{v})$ as vertex and pointing in the same directions as L_v^+ and L_v^- , respectively (see Figure 12).

The balancing condition implies that the total area Π'_v of the parallelograms corresponding to the intersection points (different from $h(v)$) of $h(\Gamma_v)$ and $h(c)$ is congruent modulo 4 to the total area, multiplied by the weight of e_v , of the parallelograms corresponding to the intersection points (different from $h(\tilde{v})$) of $L_{\tilde{v}}^+$ and $h(c)$. Thus, Lemma 4.7 implies that, if the vertex v is negative (respectively, positive), then the integer $2A(v) + \Pi'_v$ (respectively, Π'_v) is divisible by 4, where $A(v)$ is the Euclidean area of the triangle dual to v . This proves the required statement in the case where the cycle $h(c)$ of T does not have any self-intersection.

Assume now that the cycle $h(c)$ of T has a self-intersection ι . Then, we can consider the *surgery* of T at ι . This is a pair of elliptic plane tropical curves T_1 and T_2 defined as follows. Denote by e and f the two edges of Γ whose images contain ι , and denote by ι_e and ι_f the inverse images of ι in e and f , respectively. Cut the edge e at ι_e into two segments \vec{e} (oriented towards ι_e) and \overleftarrow{e} (oriented from ι_e). In a similar way, cut the edge f at ι_f into two segments \vec{f} and \overleftarrow{f} . Each of the segments \vec{e} , \overleftarrow{e} , \vec{f} , and \overleftarrow{f} has a distinguished vertex sent by h to ι . Identify the distinguished vertices of \vec{e} and \vec{f} , and attach to the resulting vertex v_1 a new edge g_1 . Similarly, identify the distinguished vertices of \overleftarrow{f} and \overleftarrow{e} , and attach to the resulting vertex v_2 a new edge g_2 (see Figure 13). This procedure gives rise to two respective graphs $\bar{\Gamma}_1$ and $\bar{\Gamma}_2$. Denote by Γ_1 (respectively, Γ_2) the complement in $\bar{\Gamma}_1$ (respectively, $\bar{\Gamma}_2$) of univalent vertices. The restriction of h to $\Gamma_1 \setminus g_1$ admits a unique extension h_1 to Γ_1 such that (Γ_1, h_1) is a parameterized plane tropical curve. Denote by T_1 the corresponding plane tropical curve. In a completely similar way, we define a parameterized plane tropical curve (Γ_2, h_2) and the corresponding plane tropical curve T_2 . Let L be the straight line containing the rays $h_1(g_1)$ and $h_2(g_2)$. Slightly modifying, if necessary, the elliptic plane tropical curves T_i , $i = 1, 2$, we can assume that the inverse image under h of any point of L is not a vertex of Γ and the only point of L having two inverse images under h is ι . Thus, T_1 and T_2 satisfy all conditions for simple elliptic plane tropical curves of parity (α, β) . The orientation kit \mathcal{R} induces orientation kits of T_1 and T_2 , so we can apply to these curves the induction hypothesis.

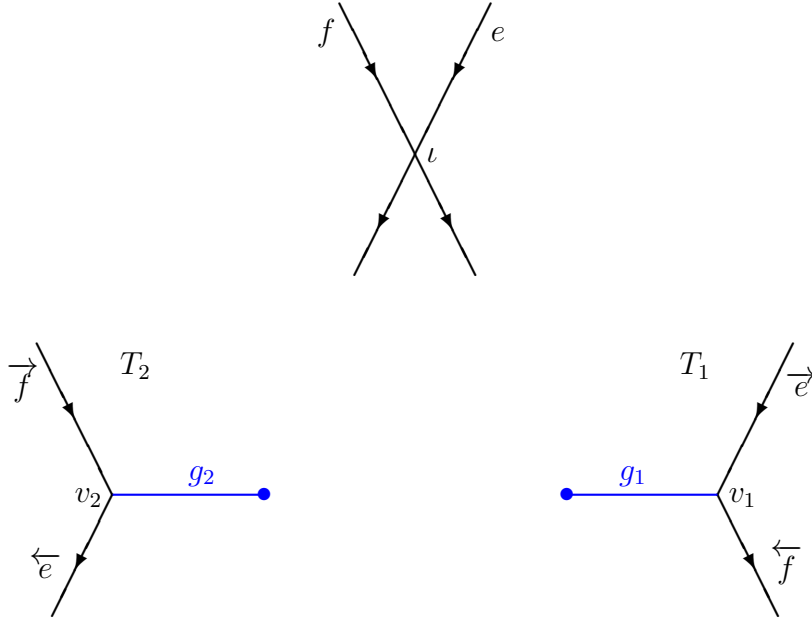


Figure 13: Surgery of T at ι

It remains to notice that the quantity $2\mathcal{A}'_-(\mathcal{R}) + \Pi'(T)$ is congruent modulo 4 to the sum of the corresponding quantities for T_1 and T_2 . Indeed, exactly one of the vertices v_1 and v_2 is negative, and for any vertex v_i , $i = 1, 2$, the Euclidean area of the triangle dual to v_i in the subdivision $\Sigma(T_i)$ dual to T_i is equal to the half of the Euclidean area of the parallelogram dual to ι in the subdivision $\Sigma(T)$. In addition, the tropical intersection number of T_1 and T_2 is divisible by 4, and the tropical intersection number of T and \mathbf{L} is divisible by 2 (for the definition of tropical intersection number, see, for example, [19, Definition 3.6.5]). \square

In order to define the Welschinger sign $\mathfrak{w}(\mathcal{R})$ of \mathcal{R} , let us introduce the following notions and notations. For any vertex p of Γ , denote by $(\theta_1(p), \theta_2(p))$ the parity of each of the vertices of the triangle δ_p dual to $h(p)$ if p is even and the parity of each of the vertices adjacent to the even edge of δ_p if p is odd. The *Harnack number* $H(p)$ (respectively, the *twisted Harnack number* $H_{(\alpha, \beta)}(p)$) of a vertex p of Γ is the number of integer points in the interior of δ_p that have the parity $(\theta_1(p), \theta_2(p))$ (respectively, the parity $(\theta_1(p) + \beta, \theta_2(p) + \alpha)$). The *content* $\ell(e)$ of an edge e of Γ is the integer part of $(\text{wt}(e) - 1)/2$, where $\text{wt}(e)$ is the weight of e . A bounded edge of Γ is said to be \mathcal{R} -*coherent* (respectively, \mathcal{R} -*noncoherent*) if it connects vertices having the same sign (respectively, opposite signs) in \mathcal{R} . An end of Γ is said to be \mathcal{R} -*coherent* (respectively, \mathcal{R} -*noncoherent*) if the adjacent vertex is positive (respectively, negative) in \mathcal{R} .

Consider the following quantities:

- the number of self-intersections $\pi(T)$ of the cycle of T , that is, the number of parallelograms in $\Sigma(T)$ that have all sides odd,
- the sum $\zeta_{\text{even, com}}(\mathcal{R})$ of Harnack numbers of \mathcal{R} -compatible even vertices of Γ ,

- the sum $\zeta_{\text{even,ncom}}(\mathcal{R})$ of twisted Harnack numbers of \mathcal{R} -noncompatible even vertices of Γ ,
- the sum $\zeta_{\text{odd,nmob}}(T)$ of Harnack numbers of non-mobile odd vertices of Γ ,
- the sum $\zeta_{\text{odd,com}}(\mathcal{R})$ of Harnack numbers of \mathcal{R} -compatible mobile odd vertices of Γ ,
- the sum $\zeta_{\text{odd,ncom}}(\mathcal{R})$ of twisted Harnack numbers of \mathcal{R} -noncompatible mobile odd vertices of Γ ,
- the sum $\tau_{\text{bound}}(\mathcal{R})$ of contents of \mathcal{R} -coherent bounded edges of Γ ,
- the sum $\tau_{\text{ends}}(\mathcal{R})$ of contents of \mathcal{R} -noncoherent ends of Γ .

Note that $\pi(T)$ and $\zeta_{\text{odd,nmob}}(T)$ depend only on T (and not on the choice of \mathcal{R}).

The *Welschinger sign* $\mathfrak{w}(\mathcal{R})$ of \mathcal{R} is $(-1)^{s(\mathcal{R})}$, where

$$s(\mathcal{R}) = \mathcal{I}\left(\frac{1}{2}\Delta\right) + \frac{\mathcal{A}(\Delta) - \kappa(\mathcal{R})}{4} + \pi(T) + \zeta_{\text{even,com}}(\mathcal{R}) + \zeta_{\text{even,ncom}}(\mathcal{R}) \\ + \zeta_{\text{odd,nmob}}(T) + \zeta_{\text{odd,com}}(\mathcal{R}) + \zeta_{\text{odd,ncom}}(\mathcal{R}) + \tau_{\text{bound}}(\mathcal{R}) + \tau_{\text{ends}}(\mathcal{R}).$$

Lemma 4.8 *One has*

$$\mathcal{I}\left(\frac{1}{2}\Delta\right) = \frac{\mathcal{A}(\Delta)}{4} - \frac{\mathcal{P}(\Delta)}{4} + 1,$$

where $\mathcal{P}(\Delta)$ is the lattice perimeter of P_Δ , i.e., the sum of lattice lengths of the sides of P_Δ .

Proof. This is an immediate consequence of Pick's formula. \square

Due to Lemma 4.8, the expression for $s(\mathcal{R})$, appearing in the definition of the Welschinger sign $\mathfrak{w}(\mathcal{R})$, can be rewritten in the following form:

$$s(\mathcal{R}) = \frac{\mathcal{A}(\Delta)}{2} - \frac{\mathcal{P}(\Delta) + \kappa(\mathcal{R})}{4} + \pi(T) + \zeta_{\text{even,com}}(\mathcal{R}) + \zeta_{\text{even,ncom}}(\mathcal{R}) \\ + \zeta_{\text{odd,nmob}}(T) + \zeta_{\text{odd,com}}(\mathcal{R}) + \zeta_{\text{odd,ncom}}(\mathcal{R}) + \tau_{\text{bound}}(\mathcal{R}) + \tau_{\text{ends}}(\mathcal{R}) + 1.$$

4.1.4 Correspondence

Fix an even non-degenerate multi-set Δ and a couple $(\alpha, \beta) \in (\mathbb{Z}/2)^2 \setminus \{(0, 0)\}$ satisfying the admissibility condition (that is, P_Δ has at most one side whose primitive normal vectors are of parity (α, β)). Choose in a generic way a tropical Δ -constraint $\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}$ satisfying the tropical Menelaus condition and a point $x_0 \in \mathbb{R}^2$. Denote by $\mathfrak{T}_{\alpha, \beta}(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$ the set of simple elliptic plane tropical curves satisfying the extended tropical Δ -constraint $(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$ and having the parity (α, β) . For each

tropical curve $T \in \mathfrak{T}_{\alpha,\beta}(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$, denote by $\mathcal{OK}(T)$ the set of orientation kits of T , and put

$$\mathcal{OK}_{\alpha,\beta}(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0) = \coprod_{T \in \mathfrak{T}_{\alpha,\beta}(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)} \mathcal{OK}(T).$$

For each vector $\mathbf{a} \in \Delta$, consider the point $x_{\mathbf{a}} \in \partial \mathbb{T}P_{\Delta}$ corresponding to the line $L_{\mathbf{a}}$ oriented by \mathbf{a} , and attribute to the point $x_{\mathbf{a}}$ the lattice length $\|\mathbf{a}\|_{\mathbb{Z}}$ of \mathbf{a} .

For any couple $(\varepsilon_1, \varepsilon_2) \in (\mathbb{Z}/2\mathbb{Z})^2$, consider the quadrant $\mathbb{K}_{\varepsilon_1, \varepsilon_2}^2$ of $(\mathbb{K}^{\times})^2$; this quadrant is formed by the elements $(t^{\mu_1}(x + O(t^d)), t^{\mu_2}(y + O(t^d))) \in \mathbb{K}^2 \setminus \{(0, 0)\}$, where $x, y \in \mathbb{R}^{\times}$ and $d > 0$, such that

$$(-1)^{\varepsilon_1}x > 0, \quad (-1)^{\varepsilon_2}y > 0.$$

The quadrant associated with the zero of $(\mathbb{Z}/2\mathbb{Z})^2$ is said to be *positive*.

Choose a lifting of $\{x_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0$ to a configuration $\widehat{\mathbf{w}}$ of $n + 1$ real points in $\text{Tor}_{\mathbb{K}}(P_{\Delta})$, where n is the number of elements in Δ , such that the liftings of all points $\{x_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}$ belong to the boundary of the positive part of $\text{Tor}_{\mathbb{K}}(P_{\Delta})$ and satisfy the Menelaus condition, and the lifting of x_0 belongs to the quadrant $\mathbb{K}_{\alpha,\beta}^2$. The admissibility condition for Δ and (α, β) implies that, for any real curve in $\mathcal{M}_{1,n+1}^{\mathbb{K}}(\Delta, \mathbf{w})$, each real curve $C^{(t)}$ in the corresponding family $\{C^{(t)}\}_{t \neq 0}$ (see Section 3.2) has one-dimensional real part formed by two connected components: one in the closure of the positive quadrant $\mathbb{R}_+^2 = \{(x, y) \in \mathbb{R}^2 \mid x > 0, y > 0\}$ and the other in the closure of the quadrant $\mathbb{R}_{\alpha,\beta}^2 = \{(x, y) \in \mathbb{R}^2 \mid (-1)^{\alpha}x > 0, (-1)^{\beta}y > 0\}$ (cf. Section 2.3). Thus, the real elliptic curves $C^{(t)}$ are separating and can be equipped with complex orientations in a continuous way.

Fix a real curve $\mathcal{C} \in \mathcal{M}_{1,n+1}^{\mathbb{K}}(\Delta, \mathbf{w})$ and a complex orientation \mathcal{O} of this curve. Denote by $(\overline{\Gamma}, h, \mathbf{p}_{tr})$ the tropicalization of \mathcal{C} , and denote by T the corresponding elliptic plane tropical curve. Due to Theorem 3.21, the couple $(\mathcal{C}, \mathcal{O})$ gives rise to a complex oriented real admissible collection \mathcal{D} of limit curves associated with $(\overline{\Gamma}, h, \mathbf{p}_{tr})$ and $\widehat{\mathbf{w}}$.

Lemma 4.9 *The elliptic plane tropical curve T corresponding to $(\overline{\Gamma}, h, \mathbf{p}_{tr})$ satisfies the extended tropical Δ -constraint $(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$. In addition, T is simple of parity (α, β) .*

Proof. The first part of the statement is immediate. To prove the second part, note that there are exactly two quadrants in \mathbb{R}^2 whose closures contain one-dimensional parts of the real point sets of limit curves in the main part of \mathcal{D} : the positive quadrant \mathbb{R}_+^2 and the quadrant $\mathbb{R}_{\alpha,\beta}^2$. Thus, Lemma 3.16 implies that T has edges of odd weight and they are of parity (α, β) . It remains to apply Lemmas 3.1 and 4.3. \square

The observation (concerning the one-dimensional parts of the real point sets of limit curves in the main part of \mathcal{D}) used in the proof of Lemma 4.9 can be made more precise.

Lemma 4.10 *Let δ be a triangle in $\Sigma(T)$, and let C_δ be the corresponding limit real rational curve belonging to the real admissible collection \mathcal{D} .*

- *If δ is even, then C_δ is Harnack.*
- *If δ is odd, then C_δ has two arcs in $(\mathbb{R}^\times)^2$: one in the positive quadrant and one in the quadrant $\mathbb{R}_{\alpha,\beta}^2$.*
- *If δ is odd and non-mobile, then C_δ is Harnack.*

Proof. Recall that the graph Γ can be represented as the union of the cycle c and the collection $\{\Gamma_v\}$ of trees formed by edges of even weight, the latter collection being indexed by the set of vertices of c . Since for any even triangle δ' in $\Sigma(T)$ the one-dimensional part of the curve $C_{\delta'}$ is contained in the closure of one quadrant of \mathbb{R}^2 , Lemma 3.16 implies that, for any vertex v of c , the one-dimensional parts of all limit curves corresponding to the even vertices of the tree Γ_v (i.e., the vertices of Γ_v that are different from v) are contained in the closure of the same quadrant. If v is non-mobile, this quadrant is necessarily the positive one. This proves the first statement of the lemma.

The second statement is immediate. To prove the third statement, it remains to apply again Lemma 3.16. \square

For each triangle δ of the dual subdivision $\Sigma(T)$, the complex orientation of the corresponding real rational curve C_δ (belonging to the main part of the real admissible collection \mathcal{D}) defines an orientation $\mathfrak{o}(\delta)$ of δ (cf. Lemma 3.20).

Lemma 4.11 *The resulting collection of orientations $\{\mathfrak{o}(\delta)\}$, indexed by the collection of triangles of $\Sigma(T)$, forms an orientation kit \mathcal{R} of T . Furthermore, if δ is a mobile odd triangle in $\Sigma(T)$, then C_δ is Harnack if and only if δ is \mathcal{R} -coherent.*

Proof. Consider the limit curves in \mathcal{D} that correspond to odd triangles in $\Sigma(T)$. For each of these curves, take its arc in the closure of the positive quadrant of $(\mathbb{R}^\times)^2$, and denote by \mathcal{U} the circle which is the union of the taken arcs. A choice of an orientation of the cycle c of Γ is equivalent to a choice of an orientation of \mathcal{U} .

If δ is a non-mobile odd triangle in $\Sigma(T)$, Lemma 4.10 implies that the 3-arc $\aleph(C_\delta) \subset C_\delta$ (see Definition 3.11) is contained in the closure of the positive quadrant of $(\mathbb{R}^\times)^2$, and thus, $\aleph(C_\delta) \subset \mathcal{U}$. The oriented arc $\aleph(C_\delta)$ gives rise to an orientation $\mathfrak{o}(\delta)$ of δ . Consider the orientation \mathfrak{o} of c such that \mathfrak{o} induces the orientation $\mathfrak{o}(\delta)$ on δ . The orientation \mathfrak{o} gives an orientation of \mathcal{U} , and the coherency of the complex orientations of the limit curves guarantees that, for any other non-mobile odd triangle δ' in $\Sigma(T)$, the orientation \mathfrak{o} induces the orientation $\mathfrak{o}(\delta')$ on δ' .

If δ is a mobile odd triangle in $\Sigma(T)$, the oriented 3-arc $\aleph_\delta \subset C_\delta$ is contained in \mathcal{U} (in other words, C_δ is Harnack) if and only if δ is \mathcal{R} -coherent. \square

The above considerations define a map $\mathcal{B}_T : \overrightarrow{\text{Ad}}^{\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w}) \rightarrow \mathcal{OK}(T)$, and thus, using Theorem 3.21, a map $\mathcal{B}'_T : \overrightarrow{\mathcal{M}}_{g,n}^{\mathbb{K}\mathbb{R}}((\bar{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w}) \rightarrow \mathcal{OK}(T)$. We also

obtain a map $\mathcal{B}' : \overrightarrow{\mathcal{M}}_{g,n}^{\mathbb{K}_{\mathbb{R}}}(\Delta, \mathbf{w}) \rightarrow \mathcal{OK}_{\alpha,\beta}(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$, where

$$\overrightarrow{\mathcal{M}}_{g,n}^{\mathbb{K}_{\mathbb{R}}}(\Delta, \mathbf{w}) = \coprod_{[(\overline{\Gamma}, h, \mathbf{p}_{tr})] \in \mathfrak{T}_{\alpha,\beta}(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)} \overrightarrow{\mathcal{M}}_{g,n}^{\mathbb{K}_{\mathbb{R}}}((\overline{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w}).$$

Theorem 4.12 *For each tropical curve $T \in \mathfrak{T}_{\alpha,\beta}(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$, represented by $(\overline{\Gamma}, h, \mathbf{p}_{tr})$, the map $\mathcal{B}'_T : \overrightarrow{\text{Ad}}^{\mathbb{R}}((\overline{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w}) \rightarrow \mathcal{OK}(T)$ is a one-to-one correspondence preserving the quantum indices and the Welschinger signs. In particular, the map $\mathcal{B}' : \overrightarrow{\mathcal{M}}_{g,n}^{\mathbb{K}_{\mathbb{R}}}(\Delta, \mathbf{w}) \rightarrow \mathcal{OK}_{\alpha,\beta}(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$ is a one-to-one correspondence preserving the quantum indices and the Welschinger signs.*

Proof. Consider a tropical curve $T \in \mathfrak{T}_{\alpha,\beta}(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$, represented by $(\overline{\Gamma}, h, \mathbf{p}_{tr})$, and fix an orientation kit $\mathcal{R} \in \mathcal{OK}(T)$ of T . Let $\mathcal{D} \in \overrightarrow{\text{Ad}}^{\mathbb{R}}((\overline{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ be an oriented a real admissible collection such that $\mathcal{B}_T(\mathcal{D}) = \mathcal{R}$. Lemmas 4.10 and 4.11 imply that, for each point $w \in \widehat{\mathbf{w}}$, the local branch at $\text{ini}(w)$ of the real rational curve $C_{\delta} \in \mathcal{D}$ incident to $\text{ini}(w)$ is not contained in the closure of the first quadrant if and only if the Newton triangle δ of C_{δ} is odd, mobile and \mathcal{R} -noncoherent. So, for each unbounded edge e of T we associate a quadrant $Q(e) \subset \mathbb{R}^2$, where $Q(e)$ is the positive quadrant unless e is adjacent to an \mathcal{R} -noncoherent mobile odd vertex of T ; in the latter case, $Q(e)$ is the quadrant $\mathbb{R}_{\alpha,\beta}^2$. The edge of T containing the point x_0 gets a pair of quadrants: the positive one and $\mathbb{R}_{\alpha,\beta}^2$. Now, each time we have a triangle δ in $\Sigma(T)$ such that two sides of δ have already quadrants associated to them, Lemma 3.13 uniquely reconstructs a real rational curve with the Newton polygon δ (and, thus, quadrant(s) for the third side of δ) such that this curve can be an element of an oriented real admissible collection whose image under \mathcal{B}_T coincides with \mathcal{R} . This proves the uniqueness of an oriented real admissible collection $\mathcal{D} \in \overrightarrow{\text{Ad}}^{\mathbb{R}}((\overline{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w})$ such that $\mathcal{B}_T(\mathcal{D}) = \mathcal{R}$.

The recursive procedure mentioned above provides also a construction of an oriented real admissible collection $\mathcal{D} \in \mathcal{B}_T^{-1}(\mathcal{R})$. Indeed, the procedure ends with an odd triangle whose three sides have quadrants associated to them. Again, Lemma 3.13 (applied to the even side of the triangle and to any of its two odd sides) provides a suitable real rational curve, the result being independent of the choice of an odd side, since the configuration $\widehat{\mathbf{w}}$ satisfies the Menelaus condition.

Equip the real rational curves of the collection obtained with complex orientations provided by \mathcal{R} . Lemma 3.16 implies that this gives rise to an oriented real admissible collection $\mathcal{D} \in \mathcal{B}_T^{-1}(\mathcal{R})$.

To show that the map $\mathcal{B}'_T : \overrightarrow{\text{Ad}}^{\mathbb{R}}((\overline{\Gamma}, h, \mathbf{p}_{tr}), \mathbf{w}) \rightarrow \mathcal{OK}(T)$ is a bijection, it remains to apply Theorem 3.21. This theorem implies also that the bijection \mathcal{B}' preserves the quantum indices, and Theorem 3.22 implies that \mathcal{B}' preserves the Welschinger signs. \square

4.2 Reinterpretation of the Welschinger sign of an orientation kit

4.2.1 Statement

Fix an even non-degenerate balanced multi-set $\Delta \subset \mathbb{Z}^2$ and a couple $(\alpha, \beta) \in (\mathbb{Z}/2\mathbb{Z})^2 \setminus \{(0, 0)\}$. Choose in a generic way a tropical Δ -constraint $\{L_a\}_{a \in \Delta}$ satisfying the tropical Menelaus condition and a point $x_0 \in \mathbb{R}^2$. Consider a tropical curve $T \in \mathfrak{T}_{\alpha, \beta}(\{L_a\}_{a \in \Delta}, x_0)$.

The following theorem is a key combinatorial statement of the paper, and the current section is devoted to the proof of this theorem.

Theorem 4.13 *Let $\mathcal{R} \in \mathcal{OK}(T)$ be an orientation kit of T . Then, the Welschinger sign $\mathfrak{w}(\mathcal{R})$ of \mathcal{R} coincides with $(-1)^{N_T(\mathcal{R})}$, where $N_T(\mathcal{R})$ is the number of negative vertices of T with respect to \mathcal{R} .*

Consider the following three types of operations on orientation kits of T (respectively, on orientation kits of $\Sigma(T)$): *reversing of an even triangle, simultaneous reversing of a mobile odd triangle and all even triangles of its tree, and simultaneous reversing of all non-mobile odd triangles* (here, by ‘reversing’ we mean the change of the orientation). These operations allow one to obtain all orientation kits of T starting from one of them.

Lemma 4.14 *Let $\delta \subset \mathbb{R}^2$ be a lattice polygon (not necessarily convex). Then, the lattice area $2\mathcal{A}(\delta)$ of δ has the same parity as the number of integer points on the boundary of δ .*

Proof. The statement is an immediate corollary of Pick’s formula. □

Lemma 4.15 *Let $\mathcal{R} \in \mathcal{OK}(T)$ be an orientation kit of T , and let \mathcal{R}' be the orientation kit obtained from \mathcal{R} by reversing one even triangle of $\Sigma(T)$. Then, the Welschinger signs of \mathcal{R} and \mathcal{R}' are opposite.*

Proof. Let δ be the even triangle reversed. Without loss of generality, we can assume that the vertices of δ have even coordinates. The change of the orientation of δ multiplies the Welschinger sign of the orientation kit by $(-1)^{\mathcal{A}(\delta)/2 + \mathcal{P}(\delta)/2 + 1}$, where, as before, $\mathcal{P}(\delta)$ is the lattice perimeter of δ . Thus, the statement follows from Lemma 4.14 applied to the image of δ under the homothety with coefficient $1/2$. □

Lemma 4.16 *Let $\mathcal{R} \in \mathcal{OK}(T)$ be an orientation kit of T , and let \mathcal{R}' be the orientation kit obtained from \mathcal{R} by reversing one mobile odd triangle of $\Sigma(T)$ and all even triangles of the tree of this triangle. Then, the Welschinger signs of \mathcal{R} and \mathcal{R}' are opposite.*

Proof. Let δ be the odd mobile triangle reversed. Without loss of generality, we can assume that $(\alpha, \beta) = (0, 1) \in (\mathbb{Z}/2\mathbb{Z})^2$ and that the second coordinates of the vertices of δ are even. Denote by $v \in \Gamma$ the vertex dual to δ , and denote by Ξ the union of all triangles in $\Sigma(T)$ that are dual to the vertices of the even tree Γ_v (the vertex v included) and all parallelograms in $\Sigma(T)$ that correspond to the intersection points of images by h of edges of Γ_v . The union Ξ is a lattice polygon (not necessarily convex): it is homeomorphic to a closed 2-disk, because all but two sides of Ξ are contained in the boundary of P_Δ .

Denote by $\mathcal{A}(\Xi)$ (respectively, $\mathcal{P}(\Xi)$) the Euclidean area (respectively, the lattice perimeter) of Ξ , and denote by $\mathcal{I}_2(\Xi)$ the number of interior integer points of Ξ that have even second coordinate. Since each parallelogram appearing in Ξ has Euclidean area divisible by 4, the change of orientations of all triangles appearing in Ξ multiplies the Welschinger sign of the orientation kit by $(-1)^{\mathcal{A}(\Xi)/2 + \mathcal{P}(\Xi)/2 + \mathcal{I}_2(\Xi)}$. Thus, the statement follows from Pick's formula applied to the polygon Ξ considered in the sublattice $\mathbb{Z} \times 2\mathbb{Z} \subset \mathbb{Z}^2 \subset \mathbb{R}^2$. \square

Lemma 4.17 *Let $\mathcal{R} \in \mathcal{OK}(T)$ be an orientation kit of T , and let \mathcal{R}' be the orientation kit obtained from \mathcal{R} by simultaneously reversing all non-mobile odd triangles of $\Sigma(T)$. Then, the Welschinger sign of \mathcal{R}' is obtained from that of \mathcal{R} by the multiplication by $(-1)^a$, where a is the number of non-mobile odd triangles of $\Sigma(T)$.*

Proof. Consider the orientation kit $\mathcal{R}'' \in \mathcal{OK}(T)$ obtained from \mathcal{R} by reversing the orientations of all triangles of $\Sigma(T)$. Due to Lemmas 4.15 and 4.16, it is enough to prove that the Welschinger sign of \mathcal{R}'' is obtained from that of \mathcal{R} by the multiplication by $(-1)^b$, where b is the number of triangles of $\Sigma(T)$, that is, to prove that b has the same parity as $(\mathcal{A}_+(\mathcal{R}'') - \mathcal{A}_-(\mathcal{R}''))/2 + \Upsilon(T)$, where $\Upsilon(T)$ is the sum of contents of boundary edges of $\Sigma(T)$. The subdivision of each boundary edge of $\Sigma(T)$ into segments of integer length 2 induces a refinement of Σ , and this refinement procedure simultaneously changes the parities of b and $\Upsilon(T)$, so we can assume that all the boundary edges of $\Sigma(T)$ have integer length 2. In this case, one has $\Upsilon(T) = 0$.

The number b has the same parity as the number of boundary edges of $\Sigma(T)$, that is, the same parity as $\mathcal{P}(\Delta)/2$, where $\mathcal{P}(\Delta)$ is the lattice perimeter of P_Δ . Due to Lemma 4.14 applied the image of P_Δ under the homothety with coefficient $1/2$ (we assume that the vertices of P_Δ have even coordinates), the parity of $\mathcal{P}(\Delta)/2$ coincides with the parity of $\mathcal{A}(\Delta)/2$. It remains to apply Proposition 4.5. \square

Proposition 4.18 *Assume that, for some orientation kit $\mathcal{R} \in \mathcal{OK}(T)$ of T , the Welschinger sign $\mathfrak{w}(\mathcal{R})$ of \mathcal{R} coincides with $(-1)^{N_T(\mathcal{R})}$. Then, for any orientation kit $\mathcal{R}' \in \mathcal{OK}(T)$ of T , the Welschinger sign $\mathfrak{w}(\mathcal{R}')$ coincides with $(-1)^{N_T(\mathcal{R}')}$.*

Proof. The statement immediately follows from Lemmas 4.15, 4.16 and 4.17. \square

We say that a simple elliptic plane tropical curve T' of parity (α, β) satisfies the sign property if, for some (and, thus, for any) orientation kit \mathcal{R} of T' , the Welschinger sign $\mathfrak{w}(\mathcal{R})$ of \mathcal{R} coincides with $(-1)^{N_{T'}(\mathcal{R})}$.

4.2.2 Trifurcations

Continuously moving the tropical Δ -constraint $\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}$ (always assuming that at each moment the tropical Menelaus condition is satisfied) and a point $x_0 \in \mathbb{R}^2$, one can deform the tropical curves of the collection $\mathfrak{T}_{\alpha, \beta}(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$. If a deformation of $\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}$ and x_0 is supposed to be sufficiently generic, the types of vertices of tropical curves appearing in the deformation of $\mathfrak{T}_{\alpha, \beta}(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$ can be classified (see, for example, [8, Proposition 3.9]).

A typical event in such a deformation is a *trifurcation*: it is a sufficiently small neighbourhood (in the deformation) of a plane tropical curve having one 4-valent vertex (other vertices of the curve being 3-valent). This tropical curve is called the *central curve* of the trifurcation. If no two of the edges adjacent to the 4-valent vertex of the central curve are parallel, there are three combinatorial types of simple plane tropical curves appearing in the trifurcation (relevant fragments of their dual subdivisions are shown on Figure 14). The simple plane tropical curves appearing in a trifurcation are said to be *friends*. If among the edges adjacent to the 4-valent vertex there are no parallel edges that are either both even or both odd, such a trifurcation is said to be *non-degenerate* and the simple plane tropical curves appearing in it are called *non-degenerate friends*. The plane tropical curves having the same *combinatorial type* (i.e., same Newton polygon and same dual subdivision) are said to be *similar*. The similarity and the friendship (respectively, the similarity and non-degenerate friendship) generate an equivalence relation \mathcal{F} (respectively, \mathcal{NDF}) on the set of simple elliptic plane tropical curves of degree Δ and parity (α, β) .

A simple elliptic plane tropical curve of parity (α, β) is said to be *convex* if the cycle of this curve is a simple broken line forming the boundary of a convex polygon in \mathbb{R}^2 . We distinguish two particular types of convex simple elliptic plane tropical curves: *triangular* (this means that the cycle consists of three edges) and *parallelogramic* (this means that the cycle consists of four edges forming a parallelogram in \mathbb{R}^2).

Lemma 4.19 *Any convex simple elliptic plane tropical curve of degree Δ and parity (α, β) admits a (unique) maximal kit. The Welschinger sign of this orientation kit is positive.*

Proof. Let T be a convex simple elliptic plane tropical curve of degree Δ and parity (α, β) . Since the cycle c of T is the boundary of a convex polygon, the positive orientation of all vertices of c can be induced by an orientation of c . This proves the existence of a maximal orientation kit \mathcal{R} of T . One has

$$\pi(T) = 0, \quad \zeta_{\text{even, ncom}}(\mathcal{R}) = \zeta_{\text{odd, ncom}}(\mathcal{R}) = \tau_{\text{ends}}(\mathcal{R}) = 0, \quad \frac{\mathcal{A}(\Delta) - \kappa(\mathcal{R})}{4} = 0.$$

The sum $\zeta_{\text{even, com}}(\mathcal{R}) + \zeta_{\text{odd, nmob}}(T) + \zeta_{\text{odd, com}}(\mathcal{R}) + \tau_{\text{bound}}(\mathcal{R})$ is equal to the number of integer points in the interior of P_{Δ} that have the same parities of coordinates as the vertices of P_{Δ} , that is, equal to $\mathcal{I}(\frac{1}{2}\Delta)$. Thus, the Welschinger sign $\mathfrak{w}(\mathcal{R})$ of \mathcal{R} is positive. \square

Corollary 4.20 *Any convex simple elliptic plane tropical curve of parity (α, β) satisfies the sign property. \square*

4.2.3 Invariance of the sign condition

This subsection is devoted to the proof of the following theorem.

Theorem 4.21 *Let T_1 and T_2 be two simple elliptic plane tropical curves of parity (α, β) . Assume that T_1 and T_2 are non-degenerate friends. Then, T_1 satisfies the sign condition if and only if T_2 satisfies the sign condition.*

Proof. Suppose that T_1 and T_2 have different combinatorial types, and consider a non-degenerate trifurcation certifying that T_1 and T_2 are non-degenerate friends. Let T_0 be the elliptic plane tropical curve appearing in this trifurcation and having a 4-valent vertex v . Assume, first, that the four edges of T_0 that are adjacent to v have pairwise non-parallel directions. In this case, the polygon dual to v is a quadrangle with pairwise non-parallel sides; denote it by δ . Denote the vertices of δ by $A, B, C,$ and D (in a cyclic order). There are three combinatorial types of simple elliptic plane tropical curves appearing in the trifurcation (see, for example, [8]). Their dual subdivisions coincide with the dual subdivision of T_0 outside of δ . For one of these combinatorial types, the quadrangle δ is subdivided by the diagonal AC , for another combinatorial type, the quadrangle δ is subdivided by the diagonal BD , and for the third combinatorial type, the quadrangle δ is subdivided into a parallelogram \mathbf{P} and two triangles (see Figure 14). Assume that A is a vertex of the parallelogram \mathbf{P} , and denote by G the opposite vertex of \mathbf{P} . Denote by $\mathfrak{E}(\mathbf{P})$ (respectively, $\mathfrak{E}_{\text{int}}(\mathbf{P})$) the number of integer points of \mathbf{P} that have even coordinates and are not contained in the sides AB and AD (respectively, the number of integer points having even coordinates and belonging to the interior of \mathbf{P}). Denote by \mathfrak{E}_δ the number of integer points having even coordinates and contained in the interior of δ .

Two of the above mentioned combinatorial types are represented by T_1 and T_2 . Let T_3 be a simple elliptic plane tropical curve representing the third combinatorial type. It is convenient for us to rename the tropical curves $T_1, T_2,$ and T_3 . The curve whose dual subdivision contains the diagonal AC (respectively, BD) is now denoted by T_b (respectively, T_y). The remaining curve is denoted by T_r .

Orient the cycles of $T_b, T_y,$ and T_r coherently, *i.e.*, so that the orientations coincide outside of δ . Let $\mathcal{R}_b, \mathcal{R}_y,$ and \mathcal{R}_r be the orientation kits of $T_b, T_y,$ and T_r , respectively, such that the orientations of all even triangles are positive and the orientation of each odd triangle is induced by the chosen orientation of the cycle; in particular, each mobile odd triangle is \mathcal{R}_b - (respectively, \mathcal{R}_y -, \mathcal{R}_r -) coherent.

For the tropical curves T_b and T_y , there is a natural bijection between their vertices (respectively, edges) dual to triangles (respectively, segments) not contained in δ . The same is true for the tropical curves T_b and T_r (and the tropical curves

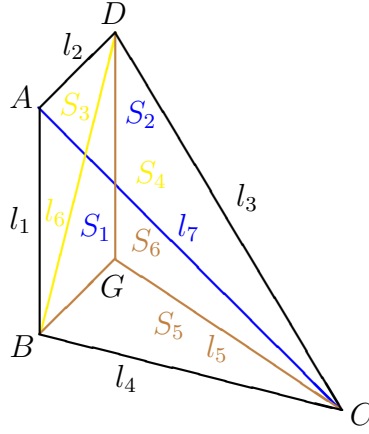


Figure 14: Non-degenerate trifurcation without parallel edges

T_y and T_r). These bijections preserve the contributions of vertices and edges to the Welschinger signs of the orientation kits \mathcal{R}_b , \mathcal{R}_y , and \mathcal{R}_r .

Denote by S_1 (respectively, S_2, S_3, S_4, S_5, S_6) the Euclidean area of the triangle ABC (respectively, ACD, ABD, BCD, BCG, CDG), and denote by l_1 (respectively, $l_2, l_3, l_4, l_5, l_6, l_7$) the lattice length of the segment AD (respectively, AB, CD, BC, CG, BD, AC); see Figure 14. For every integer $1 \leq i \leq 6$, denote by H_i the Harnack number of the triangle whose area is denoted by S_i , and denote by \mathfrak{E}_i the number of integer points having even coordinates and contained in the interior of this triangle.

Each of the sides of δ can be even or odd. There are seven possible collections of parities, they are shown on Figure 15, where even (respectively, odd) sides of δ are represented by solid (respectively, dashed) lines. The arrows show the chosen orientation of the cycle and the induced orientations of odd triangles. The situation where all the sides of δ are odd is not realisable, because in this case, the tropical curves T_1, T_2 , and T_3 would be of genus bigger than 1. We analyse each of the seven realisable cases.

Case shown on Figure 15(a). All triangles appearing in the considered subdivisions of δ are even and oriented positively. Thus, one has $N_{T_b}(\mathcal{R}_b) = N_{T_y}(\mathcal{R}_y) = N_{T_r}(\mathcal{R}_r)$. Furthermore, $s(\mathcal{R}_b) = s(\mathcal{R}_y)$. Assume that A has even coordinates. The difference $s(\mathcal{R}_r) - s(\mathcal{R}_y)$ is equal to $\mathcal{A}(\mathbf{P}) - \mathfrak{E}(\mathbf{P})$. Thus, $s(\mathcal{R}_r) = s(\mathcal{R}_y)$. We obtain that each of the tropical curves T_b, T_y , and T_r satisfies the sign condition if and only if one of these tropical curves satisfies the sign condition.

Case shown on Figure 15(b). One has $N_{T_b}(\mathcal{R}_b) = N_{T_y}(\mathcal{R}_y) = N_{T_r}(\mathcal{R}_r) - 1$. Furthermore, $s(\mathcal{R}_b) = s(\mathcal{R}_y)$. The difference $s(\mathcal{R}_r) - s(\mathcal{R}_y)$ is equal modulo 2 to

$$\left(H_5 + H_6 + \frac{l_1}{2} - 1 + \frac{l_3 - 1}{2} - \frac{\mathcal{A}(\mathbf{P}) + 2S_6}{4} \right) - \mathfrak{E}_\delta.$$

One has $H_5 = \mathfrak{E}_5$.

Lemma 4.22 *Let $\nabla \subset \mathbb{R}^2$ be a lattice triangle with two odd edges. Assume that the endpoints of the even edge of ∇ have the same parity as primitive integer vectors of*

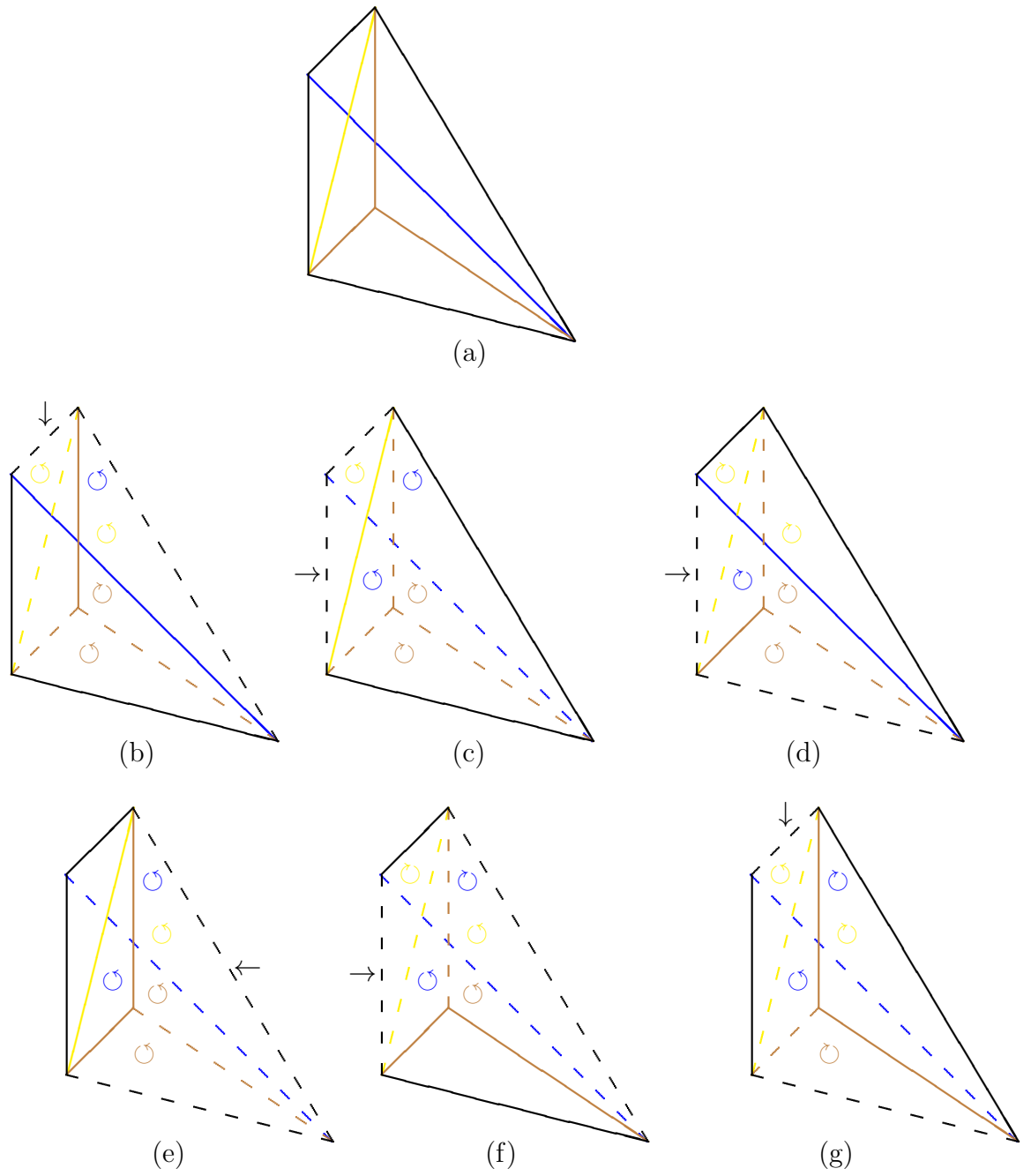


Figure 15: Even and odd edges in trifurcation

the odd edges of ∇ ; denote this parity by (β, α) . Let a be the integer length of the even edge $e(\nabla)$ of ∇ . Put $\tilde{\mathcal{A}}(\nabla) = \mathcal{A}(\nabla)$ if a primitive integer vector of the even edge of ∇ has parity (β, α) , and put $\tilde{\mathcal{A}}(\nabla) = \mathcal{A}(\nabla) + a/2$ otherwise. Then,

$$H(\nabla) = \mathfrak{E}(\nabla) + \frac{\mathcal{P}(\nabla) + \tilde{\mathcal{A}}(\nabla)}{2} + 1 \pmod{2},$$

where $H(\nabla)$ is the Harnack number of ∇ and $\mathfrak{E}(\nabla)$ is the number of integer points with even coordinates that are contained in the interior of ∇ .

Proof. Consider the sublattice of $\mathbb{Z}^2 \subset \mathbb{R}^2$ (of index 2) formed by the vectors with even coordinates and the vectors with coordinates of parity (β, α) . In both cases, the required equality follows from Pick's formula applied to the triangle ∇ in the sublattice considered. \square

In the setting of Lemma 4.22, we call $\tilde{\mathcal{A}}(\nabla)$ the *corrected Euclidean area* of ∇ .

Lemma 4.23 *Let $\rho \subset \mathbb{R}^2$ be a lattice parallelogram with two even sides and two odd sides. Assume that the endpoints of one of even sides of ρ have even coordinates; denote this side by σ and its integer length by a . Remove from ρ two neighbouring sides, and denote by \mathfrak{E} the number of integer points with even coordinates contained in the remaining figure. If a primitive integer vector of σ has the same parity as primitive vectors of odd sides of ρ , then,*

$$\mathfrak{E} = \frac{\mathcal{A}(\rho)}{4}.$$

Otherwise,

$$\mathfrak{E} = \frac{\mathcal{A}(\rho) - a}{4}$$

if the side σ was removed, and

$$\mathfrak{E} = \frac{\mathcal{A}(\rho) + a}{4},$$

if the side opposite to σ was removed.

Proof. Denote by (β, α) the parity of primitive integer vectors of odd sides of ρ . Consider again the sublattice of $\mathbb{Z}^2 \subset \mathbb{R}^2$ (of index 2) formed by the vectors with even coordinates and the vectors with coordinates of parity (β, α) . All the statements of the lemma follow from the following observation (applied to ρ in the sublattice considered): the half of the lattice area of a lattice parallelogram is equal to the number of lattice points of the parallelogram that do not belong to two given neighbouring sides. \square

Using Lemma 4.22, one can rewrite the expression modulo 2 for $s(\mathcal{R}_r) - s(\mathcal{R}_y)$ in the following form:

$$\mathfrak{E}_5 + \mathfrak{E}_6 + \frac{l_5 - 1}{2} + \frac{\tilde{S}_6}{2} + 1 - \frac{\mathcal{A}(P) + 2S_6}{4} - \mathfrak{E}_\delta.$$

where \tilde{S}_6 is the corrected Euclidean area of the triangle CDG . Now, using Lemma 4.23, we rewrite this expression modulo 2 as

$$\mathfrak{E}_5 + \mathfrak{E}_6 + \frac{l_5 - 1}{2} + 1 + \mathfrak{E} - \mathfrak{E}_\delta,$$

where \mathfrak{E} is the number of integer points having even coordinates and belonging to the complement in \mathbf{P} of the sides AB et AD . The resulting expression is 1. Thus, $\mathfrak{w}(\mathcal{R}_y) = -\mathfrak{w}(\mathcal{R}_r)$. We obtain that each of the tropical curves T_b , T_y , and T_r satisfies the sign condition if and only if one of these tropical curves satisfies the sign condition.

Case shown on Figure 15(c). One has $N_{T_b}(\mathcal{R}_b) = N_{T_y}(\mathcal{R}_y) = N_{T_r}(\mathcal{R}_r) - 2$. Furthermore, $s(\mathcal{R}_b) = s(\mathcal{R}_y)$. Assume that B has even coordinates. The difference $s(\mathcal{R}_r) - s(\mathcal{R}_y)$ is equal modulo 2 to

$$\begin{aligned} & \left(\mathfrak{E}_5 + \mathfrak{E}_6 + \frac{l_1 - 1}{2} + \frac{l_2 - 1}{2} + \frac{l_3}{2} - 1 + \frac{l_4}{2} - 1 + \frac{l_5 - 1}{2} - \frac{\mathcal{A}(\mathbf{P}) + 2S_5 + 2S_6}{4} \right) - \mathfrak{E}_\delta \\ &= \mathfrak{E}_5 + \mathfrak{E}_6 + \frac{l_1 - 1}{2} + \frac{l_2 - 1}{2} + \frac{l_3}{2} - 1 + \frac{l_4}{2} - 1 + \frac{l_5 - 1}{2} - \frac{S_4}{2} - \mathfrak{E}_\delta. \end{aligned}$$

Using Pick's formula, the latter expression for $s(\mathcal{R}_r) - s(\mathcal{R}_y)$ can be rewritten in the following form:

$$\mathfrak{E}_{\text{int}}(\mathbf{P}) + \frac{l_6}{2} + 1.$$

To show that $s(\mathcal{R}_r) - s(\mathcal{R}_y)$ is even, it remains to apply the following lemma.

Lemma 4.24 *Let $\rho \subset \mathbb{R}^2$ be a lattice parallelogram whose sides are either all even, or all odd. Assume that some vertices of ρ have even coordinates. Let \mathfrak{d} be the length of one of the diagonals of ρ if all sides of ρ are even, and let \mathfrak{d} be the length of the diagonal with even endpoints if all sides of ρ are odd. Let $\mathfrak{E}_{\text{int}}(\rho)$ be the number of integer points having even coordinates and belonging to the interior of ρ . Then, the numbers $\mathfrak{E}_{\text{int}}(\rho)$ and $\mathfrak{d}/2 - 1$ have the same parity.*

Proof. The statement follows from the fact that, for every integer point j with even coordinates in the interior of ρ , the point symmetric to j with respect to the center of ρ also has even coordinates. \square

Thus, $\mathfrak{w}(\mathcal{R}_y) = \mathfrak{w}(\mathcal{R}_r)$. We obtain that each of the tropical curves T_b , T_y , and T_r satisfies the sign condition if and only if one of these tropical curves satisfies the sign condition.

Case shown on Figure 15(d). This case is completely similar to the one shown on Figure 15(b).

Case shown on Figure 15(e). One has $N_{T_b}(\mathcal{R}_b) = N_{T_y}(\mathcal{R}_y) = N_{T_r}(\mathcal{R}_r)$. Furthermore, $s(\mathcal{R}_b) = s(\mathcal{R}_y)$. Assume that A has even coordinates. The difference

$s(\mathcal{R}_r) - s(\mathcal{R}_y)$ is equal modulo 2 to

$$\left(\mathfrak{E}_5 + \mathfrak{E}_6 + \frac{l_5 - 1}{2} - \frac{\mathcal{A}(\mathbf{P})}{4} \right) - \mathfrak{E}_\delta.$$

The number $\mathfrak{E}(\mathbf{P})$ is equal to $\mathcal{A}(\mathbf{P})/4$. Thus, $s(\mathcal{R}_r) - s(\mathcal{R}_y)$ is even and $\mathfrak{w}(\mathcal{R}_y) = \mathfrak{w}(\mathcal{R}_r)$. We obtain that each of the tropical curves T_b , T_y , and T_r satisfies the sign condition if and only if one of these tropical curves satisfies the sign condition.

Case shown on Figure 15(f). One has $N_{T_b}(\mathcal{R}_b) = N_{T_y}(\mathcal{R}_y) = N_{T_r}(\mathcal{R}_r) + 1$. Assume that B has even coordinates. The difference $s(\mathcal{R}_r) - s(\mathcal{R}_b)$ is equal modulo 2 to

$$\left(\mathfrak{E}_5 + \mathfrak{E}_6 + \frac{l_5}{2} - 1 + \frac{l_2}{2} - 1 + \frac{l_3 - 1}{2} - \frac{\mathcal{A}(\mathbf{P})}{4} \right) - \left(\mathfrak{E}_1 + H_2 - \frac{S_2}{2} \right).$$

Using Lemma 4.22, the latter expression can be rewritten modulo 2 as follows:

$$\left(\mathfrak{E}(\mathbf{P}) + \frac{l_2}{2} - 1 + \frac{l_3 - 1}{2} - \frac{\mathcal{A}(\mathbf{P})}{4} \right) - \left(\frac{l_7 - 1}{2} + \frac{P_2}{2} + \frac{\tilde{S}_2}{2} - 1 - \frac{S_2}{2} \right),$$

where P_2 (respectively, \tilde{S}_2) is the lattice perimeter (respectively, the corrected Euclidean area) of the triangle ACD . If a primitive integer vector of AD has the same parity as a primitive vector of AB , then Lemma 4.23 implies that $\mathfrak{E}(\mathbf{P}) = \mathcal{A}(\mathbf{P})/4$ and $\tilde{S}_2 = S_2$. Otherwise, Lemma 4.23 implies that $\mathfrak{E}(\mathbf{P}) = \mathcal{A}(\mathbf{P})/4 + l_2/4$ and $\tilde{S}_2 = S_2 + l_2/2$. Thus, in any case, the difference $s(\mathcal{R}_r) - s(\mathcal{R}_b)$ is equal to 1 modulo 2, *i.e.*, $\mathfrak{w}(\mathcal{R}_r) = -\mathfrak{w}(\mathcal{R}_b)$.

To prove that $\mathfrak{w}(\mathcal{R}_r) = -\mathfrak{w}(\mathcal{R}_y)$, we proceed in a similar way. The difference $s(\mathcal{R}_r) - s(\mathcal{R}_y)$ is equal modulo 2 to

$$\left(\mathfrak{E}_5 + \mathfrak{E}_6 + \frac{l_5}{2} - 1 + \frac{l_1 - 1}{2} + \frac{l_2}{2} - 1 - \frac{\mathcal{A}(\mathbf{P})}{4} \right) - \left(\mathfrak{E}_4 + H_3 - \frac{S_3}{2} \right).$$

Using Lemma 4.22, the latter expression can be rewritten modulo 2 as follows:

$$\left(\mathfrak{E}(\mathbf{P}) + \frac{l_1 - 1}{2} + \frac{l_2}{2} - 1 - \frac{\mathcal{A}(\mathbf{P})}{4} \right) - \left(\frac{l_6 - 1}{2} + \frac{P_3}{2} + \frac{\tilde{S}_3}{2} - 1 - \frac{S_3}{2} \right),$$

where P_3 (respectively, \tilde{S}_3) is the lattice perimeter (respectively, the corrected Euclidean area) of the triangle ABD . If a primitive integer vector of AD has the same parity as a primitive vector of AB , then $\tilde{S}_3 = S_3$. Otherwise, one has $\tilde{S}_3 = S_2 + l_3/2$. Thus, in any case, the difference $s(\mathcal{R}_r) - s(\mathcal{R}_y)$ is equal to 1 modulo 2, *i.e.*, $\mathfrak{w}(\mathcal{R}_r) = -\mathfrak{w}(\mathcal{R}_y)$.

We obtain that each of the tropical curves T_b , T_y , and T_r satisfies the sign condition if and only if one of these tropical curves satisfies the sign condition.

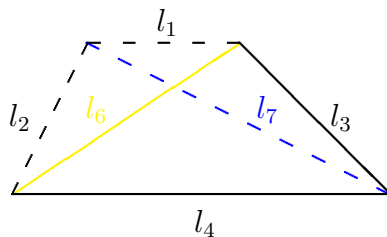


Figure 16: Trapezoid case

Case shown on Figure 15(g). This case is completely similar to the one shown on Figure 15(f).

It remains to consider the case where, among the four edges of T_0 that are adjacent to v , there exist an odd edge and an even edge that are parallel. Assume, first, that the directions of these two edges (oriented from v) are opposite. In such a situation, the polygon dual to v (in the dual subdivision of T_0) is a trapezoid (see Figure 16). that can be considered as a degenerated version of the case shown on Figure 15(c) or Figure 15(e), so the arguments presented above (either in the case shown on Figure 15(c), or in the case shown on Figure 15(e)) and concerning the tropical curves T_b and T_y apply.

Assume that the directions of two parallel edges coincide; denote these edges by e and e' . In such a situation, the polygon dual to v is a triangle; denote it by δ . Denote the vertices of δ by A , B , and C so that BC is the side orthogonal to the directions of e and e' . There are two combinatorial types of simple elliptic plane tropical curves appearing in the trifurcation; they are represented by T_1 and T_2 . In T_1 , the vertex v of T_0 is replaced with two vertices, denoted by u_1 et u'_1 . Similarly, in T_2 , the vertex v of T_0 is replaced with two vertices, denoted by u_2 and u'_2 . There is a natural bijection between the vertices of T_1 that are different from u_1 and u'_1 (respectively, the edges of T_1 that are not adjacent to u_1 and u'_1) and the vertices of T_2 that are different from u_2 and u'_2 (respectively, the edges of T_2 that are not adjacent to u_2 and u'_2). Choose orientations of the cycles of T_1 and T_2 coherently, *i.e.*, so that the bijection mentioned above preserves the orientation. Let \mathcal{R}_1 and \mathcal{R}_2 be the orientation kits of T_1 and T_2 , respectively, such that the orientations of all even triangles are positive and the orientation of each odd triangle is induced by the chosen orientation of the cycle. Then, the bijection preserves the contributions of vertices and edges to the Welschinger signs of the orientation kits \mathcal{R}_1 and \mathcal{R}_2 .

For T_1 , the triangle δ is subdivided by the segment AN_1 (into two triangles dual to u_1 and u'_1 , respectively); for T_2 , the triangle δ is subdivided by the segment AN_2 (into two triangles dual to u_2 and u'_2 , respectively); see Figure 17. Assume that the point A has even coordinates, and denote by \mathfrak{E}_δ the number of integer points having even coordinates and contained in the interior of δ .

Assume that the point N_1 does not belong to the interior of the segment BN_2 (otherwise, we can inverse the numeration of T_1 and T_2). Denote by S_1 (respectively, S_2 , S_3 , S_4) the Euclidean area of the triangle AN_1C (respectively, ABN_1 , ABN_2 ,

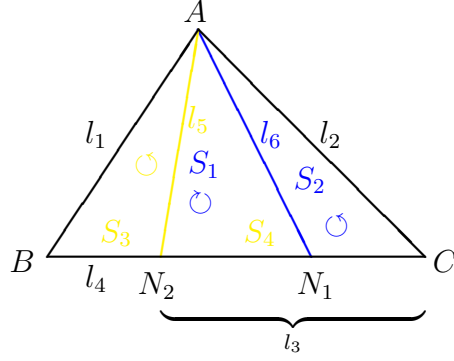


Figure 17: Triangle case

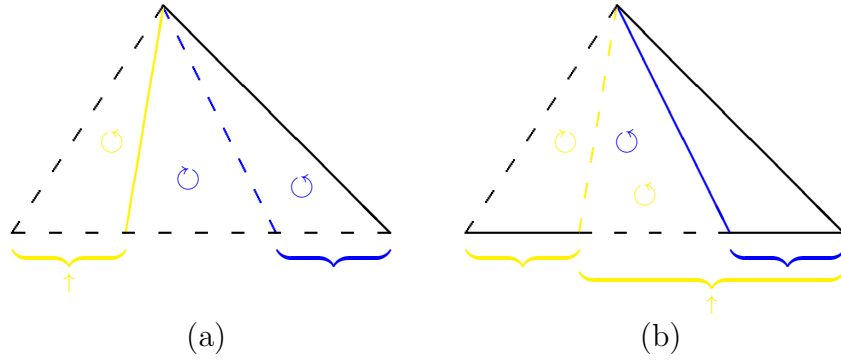


Figure 18: Even and odd edges in the triangle case

AN_2C), and denote by l_1 (respectively, l_2, l_3, l_4, l_5, l_6) the lattice length of the segment AB (respectively, $AC, CN_2, BN_2, AN_2, AN_1$); see Figure 17. The lattice length of the segment BN_1 is equal to l_3 and the lattice length of the segment CN_1 is equal to l_4 . For every integer $1 \leq i \leq 4$, denote by H_i the Harnack number of the triangle whose area is denoted by S_i , and denote by \mathfrak{E}_i the number of integer points having even coordinates and contained in the interior of this triangle.

Consider two cases: l_4 is odd (see Figure 18(a)) and l_4 is even (see Figure 18(b)). The arrows show the chosen orientation of the cycle and the induced orientations of odd triangles.

Case shown on Figure 18(a). One has $N_{T_1}(\mathcal{R}_1) = N_{T_2}(\mathcal{R}_2) + 1$. The difference $s(\mathcal{R}_2) - s(\mathcal{R}_1)$ is equal modulo 2 to

$$\mathfrak{E}_\delta - \left(H_1 + \mathfrak{E}_2 + \frac{l_1 - 1}{2} - 1 + \frac{l_3}{2} - 1 - \frac{S_1}{2} \right).$$

Using Lemma 4.22, the latter expression can be rewritten modulo 2 as follows:

$$\frac{l_6 - 1}{2} + \frac{P_1}{2} + \frac{\tilde{S}_1}{2} + 1 + \frac{l_1 - 1}{2} - 1 + \frac{l_3}{2} - 1 - \frac{S_1}{2},$$

where P_1 (respectively, \tilde{S}_1) is the lattice perimeter (respectively, the corrected Euclidean area) of the triangle ABN_1 . Since primitive integer vectors of BN_1 and CN_1 coincide, one has $\tilde{S}_1 = S_1$. Thus, the difference $s(\mathcal{R}_2) - s(\mathcal{R}_1)$ is equal to 1 modulo 2, *i.e.*, $\mathfrak{w}(\mathcal{R}_2) = -\mathfrak{w}(\mathcal{R}_1)$. We conclude that, in this case, each of the tropical curves T_1 and T_2 satisfies the sign condition if and only if one of these tropical curves satisfies the sign condition.

Case shown on Figure 18(b). The proof in this case is similar to the one presented above. One has $N_{T_1}(\mathcal{R}_1) = N_{T_2}(\mathcal{R}_2) - 1$. The difference $s(\mathcal{R}_2) - s(\mathcal{R}_1)$ is equal modulo 2 to

$$\left(H_3 + \mathfrak{E}_4 + \frac{l_1 - 1}{2} + \frac{l_4}{2} - 1 - \frac{S_3}{2} \right) - \mathfrak{E}_\delta.$$

Using Lemma 4.22, the latter expression can be rewritten modulo 2 as follows:

$$\frac{l_5 - 1}{2} + \frac{P_3}{2} + \frac{\tilde{S}_3}{2} + 1 + \frac{l_1 - 1}{2} + \frac{l_4}{2} - 1 - \frac{S_3}{2},$$

where P_3 (respectively, \tilde{S}_3) is the lattice perimeter (respectively, the corrected Euclidean area) of the triangle ABN_2 . Since primitive integer vectors of BN_2 and CN_2 coincide, one has $\tilde{S}_3 = S_3$. Thus, the difference $s(\mathcal{R}_2) - s(\mathcal{R}_1)$ is equal to 1 modulo 2, *i.e.*, $\mathfrak{w}(\mathcal{R}_2) = -\mathfrak{w}(\mathcal{R}_1)$. We conclude that each of the tropical curves T_1 and T_2 satisfies the sign condition if and only if one of these tropical curves satisfies the sign condition.

This finishes the list of cases to treat in the proof of Theorem 4.21. \square

4.3 Proof of Theorem 4.13

We start with two lemmas.

Lemma 4.25 *Let T be a simple elliptic plane tropical curve of parity (α, β) . Assume that a partial rectification of T is a simple elliptic plane tropical curve (of parity (α, β)). Then, T satisfies the sign condition if and only if this partial rectification of T satisfies the sign condition.*

Proof. Let $(\bar{\Gamma}, h)$ represents T , and let v be an even vertex of $\bar{\Gamma}$ such that v is connected by edges to two one-valent vertices v_1 and v_2 and a vertex u of valency bigger than 1. Denote by T'' the partial rectification of T at v , and assume that T'' is simple. Denote by Δ (respectively, Δ'') the degree of T (respectively, T'').

Choose an orientation kit $\mathcal{R} \in \mathcal{OK}(T)$ such that v and u are oriented positively, and restrict \mathcal{R} to an orientation kit \mathcal{R}'' of T'' (forgetting the orientation of v). One has $N_T(\mathcal{R}) = N_{T''}(\mathcal{R}'')$. Furthermore,

$$s(\mathcal{R}) - s(\mathcal{R}'') = \frac{\mathcal{A}(\Delta) - \mathcal{A}(\Delta'')}{2} - \frac{\mathcal{P}(\Delta) - \mathcal{P}(\Delta'') + \kappa(\mathcal{R}) - \kappa(\mathcal{R}'')}{4} + \zeta_{\text{even,com}}(T) - \zeta_{\text{even,com}}(T'') + \tau_{\text{bound}}(\mathcal{R}) - \tau_{\text{bound}}(\mathcal{R}'').$$

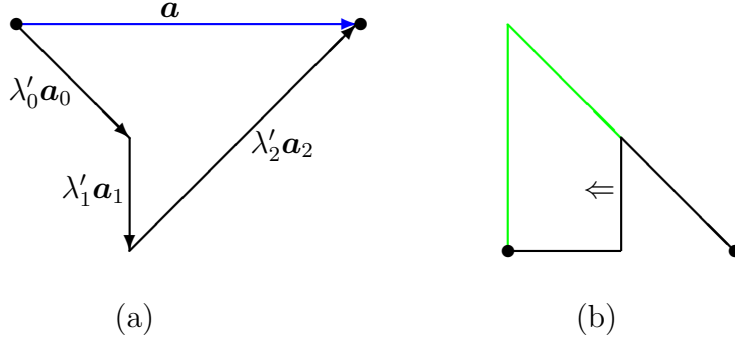


Figure 19: 3-leg broken line

The difference $\mathcal{A}(\Delta) - \mathcal{A}(\Delta'')$ is an integer divisible by 4, since this number, increased by the lattice area of the even triangle δ dual to v , can be represented as a difference of the tropical intersections of T with two tropical curves of even degrees. The difference $\mathcal{P}(\Delta) - \mathcal{P}(\Delta'')$ is equal to the lattice perimeter $\mathcal{P}(\delta)$ of δ diminished by twice the weight $\text{wt}(e)$ of the edge e connecting v and u , and $\kappa(\mathcal{R}) - \kappa(\mathcal{R}'')$ is equal to the Euclidean area $\mathcal{A}(\delta)$ of δ . The difference $\zeta_{\text{even,com}}(T) - \zeta_{\text{even,com}}(T'')$ coincides with the Harnack number $H(\delta)$ of δ , and $\tau_{\text{bound}}(\mathcal{R}) - \tau_{\text{bound}}(\mathcal{R}'')$ coincides with the content $\text{wt}(e)/2 - 1$ of e . Thus, the statement follows from Lemma 4.8. \square

Lemma 4.26 *Let A_0, A_1, A_2, A_3 be four pairwise distinct points in \mathbb{R}^2 such that the vectors $\overrightarrow{A_1 A_2}$ and $\overrightarrow{A_0 A_3}$ are not collinear and none of the points A_0 and A_3 belongs to the interior of a segment of the broken line $A_0 - A_1 - A_2 - A_3$. Put $a_i = \overrightarrow{A_i A_{i+1}}$ for each $i = 0, 1, 2$, and put $a = \overrightarrow{A_0 A_3}$; see Figure 19(a). Then, there exists two distinct elements $i, j \in \{0, 1, 2\}$ and two positive real numbers λ_i and λ_j such that $a = \lambda_i a_i + \lambda_j a_j$.*

Proof. The set

$$\{(\lambda'_1, \lambda'_2, \lambda'_3) \in (\mathbb{R}_{\geq 0})^3 \mid a = \lambda'_0 a_0 + \lambda'_1 a_1 + \lambda'_2 a_2\}$$

is a convex polyhedron (either of dimension 2, or of dimension 1) with non-empty boundary. Let $p = (\lambda'_0, \lambda'_1, \lambda'_2)$ be a point belonging to the boundary of this convex polyhedron. At least one of the coordinates of p is zero. If such a coordinate is unique, we obtain the statement required. If p has two zero coordinates, then

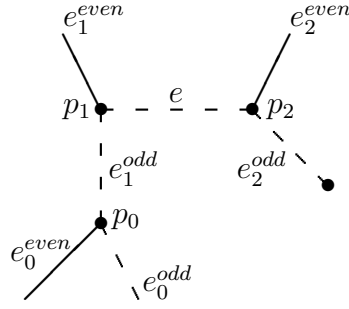


Figure 20: Contracting an odd edge

$\lambda'_1 = 0$ (since the vectors $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_0A_3}$ are not collinear), and, without loss of generality, we can assume that $\lambda'_2 = 0$. Then, $a = \gamma a_0$, where $\gamma > 1$ (since A_3 does not belong to the segment $[A_0A_1]$); see Figure 19(b). In this case, there exist positive real numbers λ_1 and λ_2 such that $a = \lambda_1 a_1 + \lambda_2 a_2$. \square

Let T be a simple elliptic plane tropical curve of parity (α, β) represented by a marked parameterized plane tropical curve $(\overline{\Gamma}, h, \mathbf{p}_{tr})$, and let e be a bounded edge of Γ . We say that e can be contracted if, continuously moving the images of points of \mathbf{p}_{tr} and modifying accordingly $(\overline{\Gamma}, h, \mathbf{p}_{tr})$, one can contract the edge e without contracting other edges of Γ .

Proof of Theorem 4.13. Let $T \in \mathfrak{T}_{\alpha, \beta}(\{L_a\}_{a \in \Delta}, x_0)$ be a tropical curve, and denote by m the number of edges of the cycle c of T . Assume that $m \geq 4$ and that T is not parallelogramic. Slightly moving $\{L_a\}_{a \in \Delta}$ and x_0 to be able to apply Lemma 4.25, we can assume that all even edges of T are unbounded.

Choose a representative $(\overline{\Gamma}, h, \mathbf{p}_{tr})$ of T and an orientation of the cycle c . Moving appropriately the configuration $\{L_a\}_{a \in \Delta}$ and x_0 , it is possible to contract one of the edges of c . Indeed, if $n \geq 5$, we can choose four consecutive vertices of c , denote them (respecting the order) by A_0, A_1, A_2 , and A_3 , and slightly moving the configuration $\{L_a\}_{a \in \Delta}$ ensure that the vectors $\overrightarrow{A_1A_2}$ and $\overrightarrow{A_0A_3}$ are not collinear in order to be able to apply Lemma 4.26. If $n = 4$, since T is not parallelogramic, we can also choose four consecutive vertices of c in order to apply Lemma 4.26.

Denote by e an edge of the cycle c such that e can be contracted, and denote by p_1 and p_2 its endpoints. Denote by e_1^{even} (respectively, e_2^{even}) the even edge adjacent to p_1 (respectively, p_2), and denote by e_1^{odd} (respectively, e_2^{odd}) the odd edge adjacent to p_1 (respectively, p_2) and different from e . Denote by p_0 the vertex adjacent to e_1^{odd} and different from p_1 , and denote by e_0^{even} (respectively, e_0^{odd}) the even edge adjacent to p_0 (respectively, the odd edge adjacent to p_0 and different from e_1^{odd}); see Figure 20.

If the images under h of e_1^{even} and e_2^{even} are not parallel and the images under h of e_1^{odd} and e_2^{odd} are not parallel, then the contraction of e leads to a non-degenerate trifurcation containing a simple elliptic plane tropical curve whose cycle has $n - 1$ edges.

Assume that either the images under h of e_1^{even} and e_2^{even} are parallel, or the images under h of e_1^{odd} and e_2^{odd} are parallel. In this case, we replace T with another simple elliptic plane tropical curve T_{corr} with the following properties:

- T and T_{corr} have the same parity;
- the cycle c_{corr} is sufficiently close to c : there is a bijection \mathfrak{B} between the edges of c and the edges of c_{corr} such that \mathfrak{B} sends each edge different from e_0^{odd} and e_1^{odd} to itself, sends e_0^{odd} to an edge contained in the same line, and sends e_1^{odd} to an edge having a slope sufficiently close to that of e_1^{odd} (but different from the slope of e_1^{odd});
- the edge $\mathfrak{B}(e)$ can be contracted;
- the even edges adjacent to the extremal points of $\mathfrak{B}(e)$ are not parallel, and the odd edges adjacent to the extremal points of $\mathfrak{B}(e)$ are not parallel,
- T satisfies the sign condition if and only if T_{corr} satisfies the sign condition.

The procedure to obtain such a tropical curve T_{corr} is as follows. First, we construct a particular simple elliptic plane tropical curve T_1 whose rectification is T . Choose a point r_0 in the interior of e_0^{even} . This point divides e_0^{even} into two segments; the segment adjacent to p_0 is now considered as an edge and the other one is replaced with two edges f_0^{even} and g_0^{even} connecting r_0 to two new one-valent vertices of the new graph $\bar{\Gamma}_1$ (the other edges of $\bar{\Gamma}$ are kept in $\bar{\Gamma}_1$). The map h_1 coincides with h on $\Gamma \setminus e_0^{\text{even}}$ and on the segment connecting p_0 and r_0 ; in addition, h_1 is chosen in such a way that the balancing condition at r_0 is satisfied, the edges f_0^{even} and g_0^{even} are even and the direction of one of them (oriented from r_0) is sufficiently close to the direction of e_1^{odd} (oriented from p_0 to p_1); see Figure 21.

The second step is the contraction of the edge connecting p_0 and r_0 (moving the images of the one-valent vertices of f_0^{even} and g_0^{even}). It gives rise to a non-degenerate trifurcation containing a simple elliptic plane tropical curve T_2 whose cycle c_2 has $m + 1$ edges and the same number of self-intersections as c ; see Figure 21.

Finally, in T_2 , the odd edge that is a small modification of e_1^{even} can be contracted. It gives rise to a non-degenerate trifurcation containing a simple elliptic plane tropical curve T_{corr} whose cycle c_{corr} has n edges; see Figure 21. The fact that the resulting tropical curve T_{corr} satisfies all the desired properties follows from the construction, Lemma 4.25 and Theorem 4.21.

Once we obtained such a tropical curve T_{corr} , the contraction of $\mathfrak{B}(e)$ leads to a non-degenerate trifurcation containing a simple elliptic plane tropical curve whose cycle has $m - 1$ edges. It remains to apply Theorem 4.21 and Corollary 4.20. \square

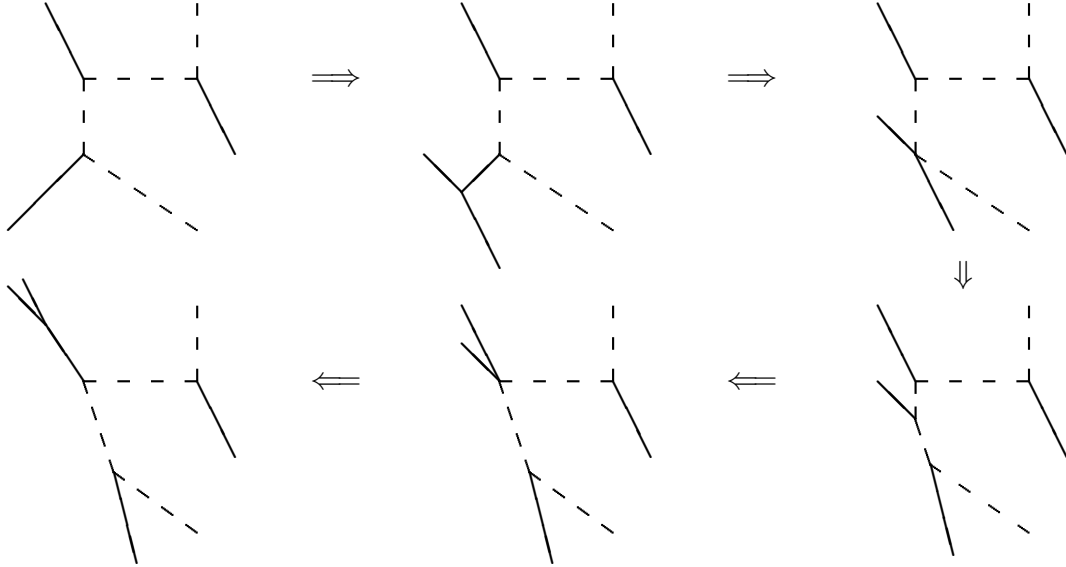


Figure 21: Deparallelization

4.4 Refined tropical multiplicity and tropical calculation of refined elliptic invariants

Fix an even non-degenerate balanced multi-set Δ and a couple $(\alpha, \beta) \in (\mathbb{Z}/2\mathbb{Z})^2 \setminus \{(0, 0)\}$ satisfying the admissibility condition. Choose in a generic way a tropical Δ -constraint $\{L_a\}_{a \in \Delta}$ satisfying the tropical Menelaus condition and a point $x_0 \in \mathbb{R}^2$. Consider a tropical curve $T \in \mathfrak{T}_{\alpha, \beta}(\{L_a\}_{a \in \Delta}, x_0)$.

Define the *refined tropical multiplicity* $\mu_T(q)$ of T as the sum $\mu_T(q) = \sum_{\mathcal{R} \in \mathcal{OK}(T)} \mathbf{w}(\mathcal{R}) q^{\kappa(\mathcal{R})}$ (here, q is a formal variable). Notice that the refined tropically multiplicity $\mu_T(q)$ is determined by the combinatorial type of T , so we can speak about the refined tropical multiplicity of such a combinatorial type.

Let \mathcal{R} be an orientation kit of T such that all even vertices and all odd mobile vertices of T are oriented positively. Denote by $\overline{\mathcal{R}}$ the orientation kit obtained from \mathcal{R} by reversing local orientations of all non-mobile odd vertices. Denote by $\mathcal{A}_T(\mathcal{R})$ (respectively, $\mathcal{A}_T(\overline{\mathcal{R}})$) the *signed Euclidean area* of non-mobile odd triangles in \mathcal{R} (respectively, $\overline{\mathcal{R}}$), that is, the difference between the total Euclidean area of positive triangles under consideration and the total Euclidean area of negative triangles.

The following statement is an immediate corollary of Theorem 4.13.

Corollary 4.27 *The refined multiplicity $\mu_T(q)$ of T is equal to*

$$((-1)^{n(\mathcal{R})} q^{\mathcal{A}_T(\mathcal{R})} + (-1)^{n(\overline{\mathcal{R}})} q^{\mathcal{A}_T(\overline{\mathcal{R}})}) \prod_{\delta \in \text{Even}(T)} (q^{\mathcal{A}(\delta)} - q^{-\mathcal{A}(\delta)}) \prod_{\delta \in \text{Odd}_m(T)} (q^{\mathcal{A}(\delta)} - q^{-\mathcal{A}(\delta)}),$$

where $n(\mathcal{R})$ and $n(\overline{\mathcal{R}})$ are the numbers of negative non-mobile odd triangles in \mathcal{R} and $\overline{\mathcal{R}}$, respectively, $\text{Even}(T)$ and $\text{Odd}_m(T)$ are the collections of even triangles and odd mobile triangles of T , respectively, and $\mathcal{A}(\delta)$, as always, stands for the Euclidean area of δ . \square

A tropical calculation of the refined elliptic invariants introduced in Section 2.3 is provided by the following theorem.

Theorem 4.28 *In the setting of Section 4.1.4, one has*

$$\mathfrak{G}_1(\Delta, (\alpha, \beta)) = \sum_{T \in \mathfrak{T}_{\alpha, \beta}(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)} \mu_T(q),$$

where the refined tropical multiplicity $\mu_T(q)$ of T is equal to

$$((-1)^{n_T(\mathcal{R})} q^{A_T(\mathcal{R})} + (-1)^{n_T(\overline{\mathcal{R}})} q^{A_T(\overline{\mathcal{R}})}) \prod_{\delta \in \text{Even}(T)} (q^{A(\delta)} - q^{-A(\delta)}) \prod_{\delta \in \text{Odd}_m(T)} (q^{A(\delta)} - q^{-A(\delta)}).$$

Proof. The statement follows from Theorem 4.12 and Corollary 4.27. \square

Proof of Theorem 2.13. Theorem 4.28 and Proposition 4.5 imply that, if the difference $\mathcal{A}(\Delta) - \kappa$ is not an integer divisible by 4, then $W_1^{\kappa}(\Delta, (\alpha, \beta)) = 0$. \square

Remark 4.29 *An analogue of Theorem 4.28 for genus zero holds too, and it can be viewed as a generalization of Mikhalkin's theorem [22, Theorem 5.9] to the case of arbitrary even intersections with the toric divisors. The precise statement is as follows. Let Δ be an arbitrary even toric degree, $\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}$ a generic tropical Δ -constraint satisfying the tropical Menelaus condition. Denote by $\mathfrak{T}_0(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta})$ the set of rational tropical curves of degree Δ matching the tropical constraints $\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}$. This set is finite, consists of rational plane tropical curves whose edges have even weights and whose dual subdivision is formed by triangles and parallelograms (parallelograms are not necessarily present in the subdivision). Define the refined multiplicity of any $T \in \mathfrak{T}_0(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta})$ to be*

$$\mu_T(q) = \prod_{\delta} (q^{A(\delta)} - q^{-A(\delta)}), \quad (35)$$

where δ ranges over all triangles in the subdivision $\Sigma(T)$. Then,

$$\mathfrak{G}_0(\Delta) = \sum_{T \in \mathfrak{T}_0(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta})} \mu_T(q). \quad (36)$$

The proof is a simplified version of the proof of Theorem 4.28 in the absence of odd vertices of the considered tropical curves. As a corollary, we obtain Theorem 2.8.

Since the tropical refined multiplicities $\mu_T(q)$ for rational curves are numerators of Block-Göttsche polynomials associated to these tropical curves (see [2]), one obtains, as another corollary, that, after an appropriate renormalization of $\mathfrak{G}_0(\Delta)$, the value at $q = 1$ corresponds to the number of complex solutions in the enumerative problem considered (cf. [22]). Unfortunately, we do not know such a connection to the number of complex solutions in the elliptic case.

4.5 Semi-local invariance

If the multiset Δ and the couple (α, β) satisfy the admissibility condition, Theorems 2.11 and 4.28 imply that the Laurent polynomial $\sum_{T \in \mathfrak{T}_{\alpha, \beta}(\{L_a\}_{a \in \Delta}, x_0)} \mu_T(q)$, where

$$((-1)^{n_T(\mathcal{R})} q^{A_T(\mathcal{R})} + (-1)^{n_T(\overline{\mathcal{R}})} q^{A_T(\overline{\mathcal{R}})}) \prod_{\delta \in \text{Even}(T)} (q^{A(\delta)} - q^{-A(\delta)}) \prod_{\delta \in \text{Odd}_m(T)} (q^{A(\delta)} - q^{-A(\delta)}),$$

does not depend on the choice of generic extended tropical constraint $(\{L_a\}_{a \in \Delta}, x_0)$. This tropical invariance statement admits also a different proof that does not use the correspondence theorem: one can prove the statement combinatorially following the tropical approach developed in [8] (and used, for example, in [14] and [17]). The combinatorial proof allows one to generalize the tropical invariance statement, namely, to adapt and to extend the statement to the situation where the admissibility condition on Δ and (α, β) is not satisfied. The approach suggests to prove *local tropical invariance* of the quantities considered, that is, the invariance for each possible trifurcation and for a *2-four-valent-vertices bifurcation*, the latter being a neighbourhood in a generic deformation of a plane tropical curve having exactly two four-valent vertices (all other vertices of the tropical curve being 3-valent) connected by exactly two edges. Contrary to the situations described in [8], [14], and [17], in our case the approach requires a small correction.

For any trifurcation, the combinatorial types of simple elliptic plane tropical curves appearing in the trifurcation are naturally divided into two groups: combinatorial types of the same group can be represented by tropical curves satisfying same tropical constraints. These groups are called the *sides* of the trifurcation. Each of the groups contains at most two combinatorial types (and at least one of the groups is formed by one combinatorial type). Denote by $\mu_+(q)$ the sum of refined tropical multiplicities of the combinatorial types of one group and denote by $\mu_-(q)$ the sum of refined tropical multiplicities of the combinatorial types of the other group.

We say that a trifurcation is *solitary* if

- the trifurcation is non-degenerate,
- or the polygon dual to the four-valent vertex of the central curve is a parallelogram,
- or the central curve contains two unbounded edges adjacent to the four-valent vertex that have the same direction (but not necessarily the same weights).

For any solitary trifurcation, one can prove the equality $\mu_+(q) = \mu_-(q)$ using the arguments similar to those used in the proof of Theorem 4.21 (taking into account the sides of the trifurcation and the quantum indices of the tropical curves under consideration). For any 2-four-valent-vertices bifurcation, there are also two sides; each of them contains one combinatorial type of simple elliptic plane tropical curves, and the refined tropical multiplicities of these combinatorial types coincide.

For a non-solitary trifurcation, the equality $\mu_+(q) = \mu_-(q)$ is not valid in general. However, non-solitary trifurcations appearing in a generic path in the space

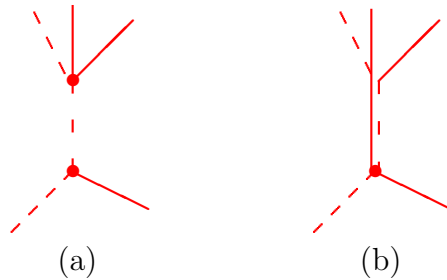


Figure 22: Paired central curves

of tropical constraints can be paired. Namely, consider a non-solitary trifurcation such that its central curve T_0 contains two parallel edges adjacent to the four-valent vertex and having opposite directions (and different weights). Then, there exists a unique non-solitary trifurcation such that its central curve T'_0 contains two parallel edges adjacent to the four-valent vertex and having the same direction (at least one of these edges being bounded), and the images of T_0 and T'_0 in \mathbb{R}^2 coincide as sets. An example of such central curves is shown on Figures 22(a) and (b). This operation establishes a bijection between the set of non-solitary trifurcations whose central curve contains two parallel edges adjacent to the four-valent vertex and having opposite directions and the set of non-solitary trifurcations whose central curve contains two parallel edges adjacent to the four-valent vertex and having same direction. Trifurcations related *via* this bijection are said to be *paired*.

Paired trifurcations can be treated together: the sides of one of them are naturally identified with the sides of the other. One can check that the differences $\mu_+(q) - \mu_-(q)$ for paired trifurcations sum up to 0. This phenomenon of *semi-local tropical invariance*, together with the local tropical invariance for the solitary trifurcations and 2-four-valent-vertices bifurcations lead to a combinatorial tropical proof of the invariance of $\sum_{T \in \mathfrak{T}_{\alpha, \beta}(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta, x_0})} \mu_T(q)$. We omit the details of this proof.

5 Computation of refined elliptic invariants

5.1 Algorithm

In this section, we provide a finite algorithm to compute the refined elliptic invariants $\mathfrak{G}_1(\Delta, (\alpha, \beta))$ *via* the tropical formula in Theorem 4.28. It follows the ideas of [3].

5.1.1 Cycle procedure

Let us be given the following data:

- a positive integer r and a sequence of vectors $\mathbf{a}_1, \dots, \mathbf{a}_r \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ which sum up to zero;
- a sequence of signs $\varepsilon_1, \dots, \varepsilon_r \in \{\pm 1\}$;

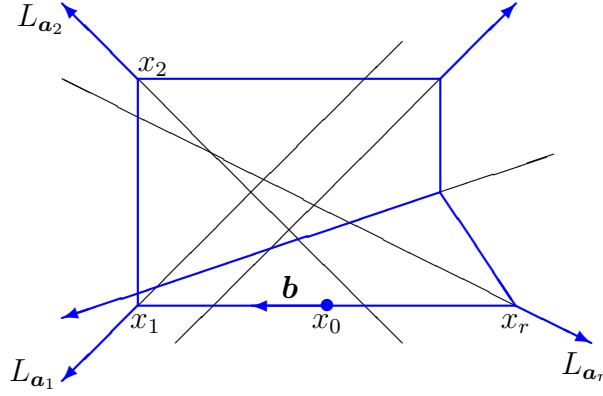


Figure 23: Cycle procedure

- a tropical constraint $L_{a_1}, \dots, L_{a_r} \subset \mathbb{R}^2$ which is in general position subject to the tropical Menelaus condition (where the collection $\mathbf{a}_1, \dots, \mathbf{a}_r$ is considered as degree; cf. Definition 4.1);
- a vector $\mathbf{b} \in \mathbb{Z}^2 \setminus \{(0, 0)\}$ such that $(\mathbf{b} - \sum_{j=1}^i \mathbf{a}_j) \wedge \mathbf{a}_{i+1} \neq 0$ for all $i = 0, \dots, r-1$;
- a generic point $x_0 \in \mathbb{R}^2 \setminus \bigcup_{i=1}^r L_{a_i}$ such that the line L through x_0 parallel to \mathbf{b} and x_0 belongs to the segment cut off on L by L_{a_1}, L_{a_r} .

The *cycle procedure* goes as follows.

- In the first step, we consider the ray $R(x_0, \mathbf{b})$ emanating from the point x_0 in the direction of vector \mathbf{b} . If either $R(x_0, \mathbf{b}) \cap L_{a_1} = \emptyset$, or $\varepsilon_1(\mathbf{b} \wedge \mathbf{a}_1) < 0$, we stop the procedure. Otherwise, we set $x_1 = R(x_0, \mathbf{b}) \cap L_{a_1}$ and $\mathbf{b}_1 = \mathbf{b} - \mathbf{a}_1$.
- Suppose we have $x_k \in L_{a_k}$ and $\mathbf{b}_k = \mathbf{b} - \sum_{j=1}^k \mathbf{a}_j$, where $1 \leq k < r$. If either $R(x_k, \mathbf{b}_k) \cap L_{a_{k+1}} = \emptyset$, or $\varepsilon_{k+1}(\mathbf{b}_k \wedge \mathbf{a}_{k+1}) < 0$, we stop the procedure. Otherwise, we set $x_{k+1} = R(x_k, \mathbf{b}_k) \cap L_{a_{k+1}}$ and $\mathbf{b}_{k+1} = \mathbf{b}_k - \mathbf{a}_{k+1}$ (see Figure 23).

Lemma 5.1 *Given the above data, suppose that the cycle procedure yields x_r and $\mathbf{b}_r = \mathbf{b}$. Then,*

- (1) $x_0 \in R(x_r, \mathbf{b})$;
- (2) *there exists a simple elliptic plane tropical curve T , represented by $(\bar{\Gamma}, h)$, of degree $\Delta = \{\mathbf{a}_1, \dots, \mathbf{a}_r\}$ and a point $p \in \Gamma$ such that Γ consists of a cycle formed by r bounded edges and of r ends, the map h sends*
 - *the point p to x_0 ,*
 - *the bounded edges $e_1, \dots, e_r \in \Gamma^1$ to the segments $[x_1, x_2], \dots, [x_{r-1}, x_r], [x_r, x_1]$, respectively, so that the unit tangent vector to e_k is taken to $\pm \mathbf{b}_k$ for each $k = 1, \dots, r$,*

- and the ends $e'_1, \dots, e'_r \in \Gamma_\infty^1$ to the rays $R(z_i, \mathbf{a}_k)$, $k = 1, \dots, r$, respectively, so that the unit tangent vector to e'_k oriented towards the corresponding one-valent vertex is taken to \mathbf{a}_k for each $k = 1, \dots, r$.

Proof. The union of the segments $[x_0, x_1], [x_1, x_2], \dots, [x_{r-1}, x_r]$ together with the rays $R(x_0, -\mathbf{b})$, $R(x_r, \mathbf{b})$, and $R(x_k, \mathbf{a}_k)$, $i = k, \dots, r$, can be seen as the image of a (rational) parameterized plane tropical curve, for which the tropical Menelaus condition (34) reads $\lambda_{\mathbf{b}}(x_0) - \lambda_{\mathbf{b}}(x_r) = 0$, whence the first claim. The second claim is straightforward by construction. \square

5.1.2 Initial data

Fix an even non-degenerate balanced multiset $\Delta \subset \mathbb{Z}^2 \setminus \{(0, 0)\}$ and a parity $(\alpha, \beta) \in (\mathbb{Z}/2\mathbb{Z})^2 \setminus \{(0, 0)\}$. The other part of the initial data, namely, an extended tropical Δ -constraint $(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$, is now chosen in the following restrictive way.

Note that the tropical Δ -constraint $\{L_{\mathbf{a}}^0\}_{\mathbf{a} \in \Delta}$ consisting of lines through the origin satisfies the tropical Menelaus condition (see Lemma 4.2). We assume that, for each $\mathbf{a} \in \Delta$, the distance between $L_{\mathbf{a}}$ and $L_{\mathbf{a}}^0$ is less than some $\rho_0 > 0$ specified below, and that $\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}$ is in tropical general position subject to the tropical Menelaus condition. Observe that, for each non-empty proper subset $\Delta' \subset \Delta$ such that $\mathbf{c}(\Delta') := \sum_{\mathbf{a} \in \Delta'} \mathbf{a} \neq 0$, there exists a unique oriented line $L_{\mathbf{c}(\Delta')}$ directed by the vector $\mathbf{c}(\Delta')$ for which the sequence of the directed lines $\{L_{-\mathbf{c}(\Delta')}\} \cup \{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta'}$ satisfies the tropical Menelaus condition. The general position condition yields that $L_{\mathbf{c}(\Delta')}$ differs from the lines $L_{\mathbf{a}}$, $\mathbf{a} \in \Delta \setminus \Delta'$, and from each of the lines $L_{\mathbf{c}(\Delta'')}$, where Δ'' is a non-empty proper subset of Δ and $\Delta'' \neq \Delta'$.

Now we specify a value of $\rho_0 > 0$ imposing the following requirements:

- (r1) all intersection points of non-parallel lines in $\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta} \cup \{L_{\mathbf{c}(\Delta')}\}_{\Delta' \subsetneq \Delta, \mathbf{c}(\Delta') \neq 0}$ lie in the (open) unit disc D_1 centered at the origin;
- (r2) for any subset $\Delta' \subsetneq \Delta$ such that $\mathbf{c}(\Delta') \neq 0$, all vertices of rational plane tropical curves of degree $\Delta' \cup \{-\mathbf{c}(\Delta')\}$ whose ends are contained in the lines $L_{\mathbf{c}(\Delta')}$ and $L_{\mathbf{a}}$, $\mathbf{a} \in \Delta'$, lie in the disc D_1 (notice that there are finitely many such tropical curves).

A choice of $\rho_0 > 0$ is possible, since all the above intersection points and vertices merge to the origin as $\rho_0 \rightarrow 0$.

Now we choose a component K of the complement in $\mathbb{R}^2 \setminus D_1$ to the tubular neighborhoods of size 1 of all the lines $\{L_{\mathbf{a}}^0\}_{\mathbf{a} \in \Delta} \cup \{L_{\mathbf{c}(\Delta')^0}\}_{\Delta' \subsetneq \Delta, \mathbf{c}(\Delta') \neq 0}$. Then, introduce the set \mathfrak{B} of all vectors \mathbf{b} obtained from $v_1 - v_2$, where $v_1, v_2 \in P_\Delta \cap \mathbb{Z}^2$, by clockwise and counterclockwise rotation by $\frac{\pi}{2}$ so that

- they all have the given parity (α, β) ,

- each line $L \subset \mathbb{R}^2 \setminus D_1$, parallel to a vector $\mathbf{b} \in \mathfrak{V}$, has a bounded intersection with K ,
- $\mathbf{b}' \wedge \mathbf{b} > 0$ for any vector $\mathbf{b}' \in \mathbb{R}^2 \setminus \{(0,0)\}$ such that $R(0, \mathbf{b}') \cap K$ is unbounded.

Let $Z(K, \rho_0, \mathfrak{V}) \subset K$ be the subset consisting of the points $x_0 \in K$ such that

- all the lines $L_{\mathbf{b}}$, $\mathbf{b} \in \mathfrak{V}$, passing through x , are disjoint from D_1 ;
- for any arbitrarily ordered subset

$$\{L_{\mathbf{a}_1}, \dots, L_{\mathbf{a}_r}\}, \quad \mathbf{a}_1, \dots, \mathbf{a}_r \in \Delta \cup \{\mathbf{c}(\Delta')\}_{\Delta' \subsetneq \Delta, \mathbf{c}(\Delta') \neq 0}$$

such that $\mathbf{a}_1 + \dots + \mathbf{a}_r = 0$, the cycle procedure starting with $x_0 \in Z(K, \rho_0, \mathfrak{V})$ and an arbitrary vector $\mathbf{b} \in \mathfrak{V}$ yields only lines $L_{\mathbf{b}_k}$ through x_k , $1 \leq k \leq r$, which are disjoint from D_1 .

Lemma 5.2 *One has $Z(K, \rho_0, \mathfrak{V}) \neq \emptyset$.*

Proof. If we shift the lines L_1, \dots, L_r so that they pass through the origin, the rays resulting from the cycle procedure would never pass through the origin, since, due to $\Delta \subset (2\mathbb{Z})^2$ and $\mathfrak{V} \cap (2\mathbb{Z})^2 = \emptyset$, the vectors $\mathbf{b} - \sum_{i=1}^k \mathbf{a}_i$, $0 \leq k \leq r$, never vanish. Then, we obtain a point in $Z(K, \rho_0, \mathfrak{V})$ when taking a point $x_0 \in K$ and shifting it in parallel to a side of K sufficiently far from the origin. \square

Fix a generic extended tropical Δ -constraint $(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$ such that $\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}$ satisfies the above restrictions and $x_0 \in Z(K, \rho_0, \mathfrak{V})$.

5.1.3 Objects to count

Given the above initial data, we define a set of objects $(\mathfrak{H}, \mathcal{D})$ to count as follows.

- (i) Choose \mathfrak{H} to be a (possibly empty) set of pairwise disjoint, proper, nonempty subsets of Δ such that $\mathbf{c}(\Delta') \neq 0$ for each $\Delta' \in \mathfrak{H}$, and $\bigcup_{\Delta' \in \mathfrak{H}} \Delta' \subsetneq \Delta$.
- (ii) Given such a set \mathfrak{H} , consider
 - the multiset of vectors $\{\mathbf{c}(\Delta')\}_{\Delta' \in \mathfrak{H}} \cup (\Delta \setminus \bigcup_{\Delta' \in \mathfrak{H}} \Delta')$,
 - the multiset of oriented lines $\{L_{\mathbf{c}(\Delta')}\}_{\Delta' \in \mathfrak{H}} \cup \{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta \setminus \bigcup_{\Delta' \in \mathfrak{H}} \Delta'}$,
 - the collection of the signs

$$\varepsilon(\mathbf{c}(\Delta')) = -1, \quad \Delta' \in \mathfrak{H}, \quad \varepsilon(\mathbf{a}) = 1, \quad \mathbf{a} \in \Delta \setminus \bigcup_{\Delta' \in \mathfrak{H}} \Delta'.$$

Let $\mathbf{a}_1, \dots, \mathbf{a}_r$ be all the above vectors, which are arranged in an arbitrary linear order, and let L_1, \dots, L_r and $\varepsilon_1, \dots, \varepsilon_r$ be the respectively ordered oriented lines and signs.

(iii) Pick a vector $\mathbf{b} \in \mathfrak{V}$ and perform the cycle procedure starting with the data

$$\mathfrak{D} := \left\{ \{\mathbf{a}_i\}_{i=1}^r, \{L_i\}_{i=1}^r, \{\varepsilon_i\}_{i=1}^r, x_0, \mathbf{b} \right\}, \quad (37)$$

where $x_0 \in Z(K, \rho_0, \mathfrak{V})$ is any chosen point.

We say that the data \mathfrak{D} is *cyclic*, if this cycle procedure results in a simple elliptic plane tropical curve $T(\mathfrak{D}) = (\bar{\Gamma}, h)$ of degree $\Delta(\mathfrak{D}) = \{\mathbf{a}_1, \dots, \mathbf{a}_r\}$.

5.1.4 Refined multiplicities of the objects in count

We define the refined tropical multiplicity of $(\mathfrak{H}, \mathfrak{D})$ as follows:

$$\mu_{\mathfrak{H}, \mathfrak{D}}(q) = \prod_{\Delta' \in \mathfrak{H}} \mathfrak{G}_0(\widehat{\Delta}') \cdot \mu_{T(\mathfrak{D})}(q),$$

where

- $\mathfrak{G}_0(\widehat{\Delta}')$ is the refined rational invariant associated with the degree $\widehat{\Delta}' = \{-\mathbf{c}(\Delta')\} \cup \Delta'$, see Definition 2.7;
- $\mu_{T(\mathfrak{D})}(q)$ is the refined tropical multiplicity of the elliptic curve $T(\mathfrak{D})$ defined by the second formula of Theorem 4.28 (notice, that, in our case, only the first and the third factors are nontrivial).

Theorem 5.3 *We have*

$$\mathfrak{G}_1(\Delta, (\alpha, \beta)) = \sum_{\mathfrak{H}, \mathfrak{D}} \mu_{\mathfrak{H}, \mathfrak{D}}(q), \quad (38)$$

where $(\mathfrak{H}, \mathfrak{D})$ ranges over all possible couples matching the fixed initial data and such that \mathfrak{D} is cyclic.

Proof. (1) Each elliptic plane tropical curve $T(\mathfrak{D})$ gives rise to a collection $\mathfrak{P}(\mathfrak{D})$ of elliptic plane tropical curves of degree Δ matching the extended tropical Δ -constraint $(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$. Namely, for each $\Delta' \in \mathfrak{H}$, there is a finite set $\mathfrak{P}_0(\Delta')$ of rational plane tropical curves of degree $\widehat{\Delta}'$, whose ends lie in the lines $L_{\mathbf{a}}$, $\mathbf{a} \in \Delta'$, and in the line $L_{-\mathbf{c}(\Delta')}$ (which is L_i in the given data \mathfrak{D} for some $1 \leq i \leq r$). By construction, all the vertices of these curves lie in the disc D_1 . Since the cycle of the curve $T(\mathfrak{D})$ is disjoint from D_1 , the vertex $x_i \in L_i$ of $T(\mathfrak{D})$ lies inside the end contained in L_i of each curve $T \in \mathfrak{P}_0(\Delta')$. It follows that we can replace the end $R(x_i, \mathbf{c}(\Delta'))$ of the curve $T(\mathfrak{D})$ with the fragment $T \setminus R(x_i, -\mathbf{c}(\Delta'))$ of any curve $T \in \mathfrak{P}_0(\Delta')$ so that the rest of $T(\mathfrak{D})$ is not affected (the operation is opposite to partial rectification defined in Section 4.1.2).

Thus, in the right-hand side of (38) we count refined tropical multiplicities of all elliptic tropical curves of degree Δ that match the extended tropical Δ -constraint $(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$ and arise from the elliptic plane tropical curves of the form $T(\mathfrak{D})$ that

are obtained from the couples $(\mathfrak{H}, \mathfrak{D})$ ranging over all possible couples matching the fixed initial data and such that \mathcal{D} is cyclic. It is also easy to see that all counted elliptic plane tropical curves are pairwise different.

(2) It remains to show that we count all curves in $\mathfrak{T}_1(\Delta, \{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, z)$, whose cycle edges are directed by vectors of the given parity (α, β) .

Pick a tropical curve $T \in \mathfrak{T}_1(\Delta, \{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, z)$. The cycle $c(T)$ contains the point x_0 . The complement to $c(T)$ in T consists of ends or fragments of rational curves matching pairwise disjoint constraints $\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta'}$ for some $\Delta' \subsetneq \Delta$ with $\mathbf{c}(\Delta') \neq 0$. Denote by T' the rectification of T .

The edge of $c(T)$ containing x_0 is dual to an integral segment inside P_{Δ} , whence is directed by a vector $\mathbf{b} = (v_1 - v_2)^\perp$, $v_1, v_2 \in P_{\Delta} \cap \mathbb{Z}^2$. Due to the choice of ρ_0 , the complement K_{ρ_0} to the ρ_0 -neighborhood of ∂K in K is disjoint from all lines $L_{\mathbf{a}}$, $\mathbf{a} \in \Delta$, and $L_{\mathbf{c}(\Delta')}$, $\Delta' \subsetneq \Delta$. Hence, the edge of $c(T)$ containing x_0 has a bounded intersection with K_{ρ_0} , which also yields that the intersection with K is bounded. Furthermore, by the definition of the set $Z(K, \rho_0, \mathfrak{D})$, the lines through the edges of $c(T)$ avoid the disc D_1 .

Number the vertices x_1, \dots, x_r of $c(T)$ as they occur when moving along the cycle in the direction of \mathbf{b} from the point x_0 . Since the ends of T' lie on the lines crossing the ρ_0 -neighborhood of the origin, we inductively obtain that $\vec{x}_k \wedge \mathbf{b}_k > 0$ for all $k = 1, \dots, r$, where \vec{x}_k is the radius-vector of the point x_k , and $\mathbf{b}_k = \mathbf{b} - \mathbf{a}_1 - \dots - \mathbf{a}_k$ with \mathbf{a}_i being the directing vector of the end of T' emanating from the vertex x_i . It then follows that $\mathbf{b}_{k-1} \wedge \mathbf{a}_k < 0$, if $\mathbf{a}_k = \mathbf{c}(\Delta')$ with $\Delta' \subsetneq \Delta$ consisting of more than one vector: indeed, in this situation, the end $R(z_k, \mathbf{a}_k)$ must cross the ρ_0 -neighborhood of the origin. Summarizing the above remarks, we conclude that T arises from some elliptic plane tropical curve $T(\mathfrak{D})$. \square

5.2 Examples

We present here two easy examples of explicit calculations of the invariant $\mathfrak{G}_1(\Delta, (\alpha, \beta))$:

5.2.1 Plane quartics

Consider the multi-set

$$\Delta = \{(-2, 0), (-2, 0), (0, -2), (0, -2), (2, 2), (2, 2)\}$$

and the parity $(0, 1) \in (\mathbb{Z}/2\mathbb{Z})^2$. The corresponding enumerative problem consists in counting real plane elliptic quartics that are quadratically tangent to each toric divisor in two fixed points and pass through some point in the quadrant

$$\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 < 0\}.$$

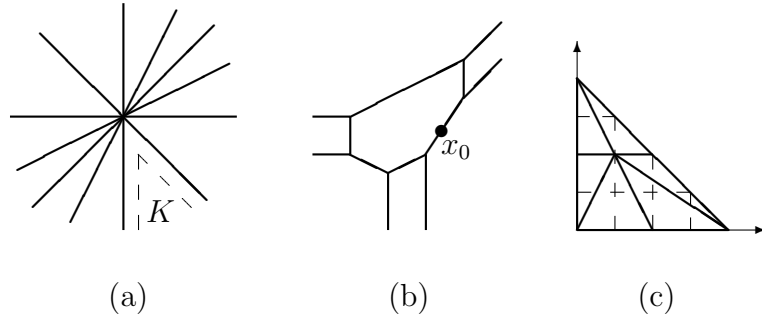


Figure 24: Counting tropical elliptic quartics

In the notation of the preceding section, all possible vectors $\mathbf{c}(\Delta') \neq 0$, $\Delta' \subsetneq \Delta$, belong to the set

$$\pm(2, 0), \pm(0, 2), \pm(2, 2), (2, -2), (4, 2), (2, 4) .$$

The union of the lines $L_{\mathbf{a}}^0$, $\mathbf{a} \in \Delta$, and $L_{\mathbf{c}(\Delta')}^0$, where $\Delta' \subsetneq \Delta$ and $\mathbf{c}(\Delta') \neq 0$, and the component K are shown in Figure 24(a). One can easily verify that the given initial conditions only allow the cyclic data \mathfrak{D} with $\mathfrak{H} = \emptyset$ and $\mathbf{b} = (-2, -1)$, which henceforth yields the unique elliptic plane tropical curve as shown in Figure 24(b) (dual to the subdivision of the Newton triangle depicted in Figure 24(c)).

By Theorem 4.28 and formula (38), one has

$$\mathfrak{G}_1(\Delta, (0, 1)) = (q^2 - q^{-2})^2(q^4 + q^{-4}) .$$

The result differs from the numerator of the corresponding Block-Göttsche refined invariant, which here is equal to

$$3(q^2 - q^{-2})^2(q - q^{-1})^4 .$$

This is in contrast to the refined count of real rational curves, where the invariant $\mathfrak{G}_0(\Delta)$ coincides with the numerator of the Block-Göttsche invariant [22, Theorem 5.9].

5.2.2 Elliptic curves in the blown-up plane

Consider the multi-set

$$\Delta = \{(-2, 0), (0, -2), m \times (2, 2), (m - 1) \times (-2, -2)\}$$

and the parity $(0, 1)$. The corresponding enumerative problem consists in counting real elliptic curves in $\text{Tor}(P_\Delta)$ that are quadratically tangent to toric divisors (the number of tangency points on each toric divisor being equal to the half of the lattice length of the corresponding edge of P_Δ ; see Figure 25(a)) and pass through some point in the quadrant

$$\{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 < 0\} .$$

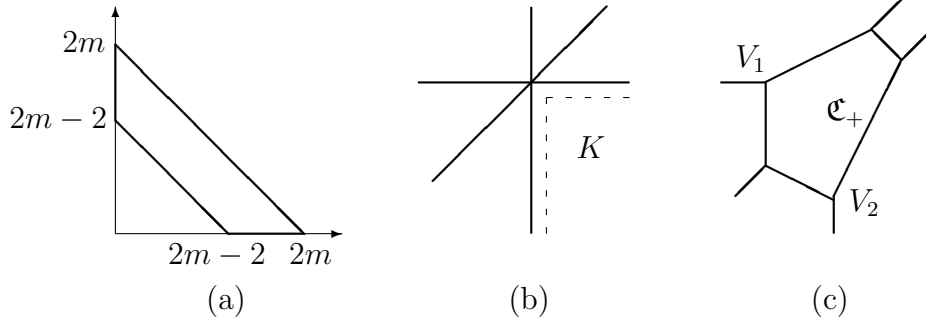


Figure 25: Counting tropical elliptic curves in the blow-up plane

The tropical Bézout theorem [25, Section 4] yields that, for each counted elliptic tropical curve, represented by $(\bar{\Gamma}, h, \mathbf{p}_{tr})$, the restriction of h on the cycle c of Γ is an embedding, and the projection of $h(c)$ to a straight line with the normal vector $(1, 1)$ is a segment whose interior points have exactly two preimages in $h(c)$. The vertices V_1 and V_2 of $h(c)$ that project to the segment endpoints are incident to the ends directed by the vectors $(-2, 0)$ and $(0, -2)$, respectively. The only possible sets $\Delta' \in \mathfrak{H}$ are either $\{(2, 2)\}$, or $\{(-2, -2)\}$. The initial configuration of lines $L_{\mathbf{a}}^0$, $\mathbf{a} \in \Delta$ or $\mathbf{a} = \mathbf{c}(\Delta')$, $\Delta' \in \mathfrak{H}$, is as shown in Figure 25(b), and the cycle is shaped as shown in Figure 25(c). Pick the domain K as indicated in Figure 25(b) and denote by \mathfrak{C}_+ the right-upper half of the cycle such that \mathfrak{C}_0 has endpoints V_1, V_2 (see Figure 25(c)). Each end attached to an interior vertex of \mathfrak{C}_+ is directed by either $(2, 2)$, or $(-2, -2)$; denote the number of such ends directed by $(2, 2)$ (respectively, $(-2, -2)$) by s_+ (respectively, s_-). Put $\mathbf{b} = (-2k, -2l - 1)$, where $k \geq 1$ and $l \geq 0$ are integers. Choose a generic extended tropical Δ -constraint $(\{L_{\mathbf{a}}\}_{\mathbf{a} \in \Delta}, x_0)$ as described in Section 5.1.2. The cycle procedure applied to a couple $(\mathfrak{H}, \mathfrak{D})$, matching the initial data, ends up with an elliptic plane tropical curve if and only if

$$2 \leq s_+ \leq m, \quad 0 \leq s_- \leq s_+ - 2, \quad 1 \leq k = l \leq s_+ - s_- - 1. \quad (39)$$

So, whenever we fix k , and pick s_+ lines $L_{\mathbf{a}}$ with $\mathbf{a} = (2, 2)$ and s_- lines $L_{\mathbf{a}}$ with $\mathbf{a} = (-2, -2)$, all these data subject to restrictions (39), we obtain an elliptic plane tropical curve, whose refined tropical multiplicity equals

$$(-1)^{m-s_++s_-} (q^{2k} - q^{-2k}) (q^{4s_+-4s_- - 2k - 2} + q^{-4s_++4s_- + 2k + 2}),$$

and, hence,

$$\begin{aligned} \mathfrak{G}_1(\Delta, (0, 1)) &= \sum_{s_+=2}^m \sum_{s_-=0}^{s_+-2} \sum_{k=1}^{s_+-s_--1} \left[\binom{m}{s_+} \binom{m-1}{s_-} (-1)^{m-s_++s_-} \right. \\ &\quad \left. \times (q^{2k} - q^{-2k}) (q^{4s_+-4s_- - 2k - 2} + q^{-4s_++4s_- + 2k + 2}) \right]. \end{aligned}$$

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