Mathematisches Forschungsinstitut Oberwolfach

Report No. 12/2023

DOI: 10.4171/OWR/2023/12

MATRIX-MFO Tandem Workshop: Stochastic Reinforcement Processes and Graphs

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5 March – 10 March 2023

Abstract. Stochastic processes with reinforcement are the central theme of the present tandem workshop. We assembled a diverse group of international experts that worked on reinforcement dynamics from several different perspectives. We discussed progress and future strategies around a number of key open problems in the area of interacting urns with graph based interaction, preferential attachment, and reinforced random walks.

Mathematics Subject Classification (2020): 05Cxx, 60Gxx, 60Jxx.

Introduction by the Organizers

The MATRIX-MFO Tandem Workshop "Stochastic Reinforcement Processes and Graphs" was held synchronously at two venue: MFO in Oberwolfach and MATRIX near Melbourne. The organisers were Markus Heydenreich (LMU Munich), Mark Holmes (University of Melbourne), Victor Klepstyn (CNRS, Rennes), and Cécile Mailler (University of Bath), with Markus Heydenreich and Cécile Mailler based at MFO, Mark Holmes and Victor Klepstyn based at MATRIX. The workshop was well attended, with approximately 20 participants at MFO, 12 at MATRIX, and 5 online.

Reinforcement in a nut-shell. Pólya-type urn models are random processes where balls are repeatedly sampled from an urn, and additional balls are added depending on the colour of the sampled ball. Since their introduction in 1931, generalisations of Pólya urn models have spurred a rich variety of mathematical research activity. They are basic building blocks of competition-type probabilistic models in the fields of economics, biology and neuroscience. A single urn is often insufficient to capture the complexity inherent in real-world applications, and consequently systems of interacting urns have gained popularity. In the field of neuroscience, when a neuron fires, only synapses that are connected to this neuron can be chosen to transmit the signal. Hence, Pólya models with graph-based interactions are a natural starting point for addressing one of the mechanisms of neuroplasticity: synapses that have been identified as useful in the past are more likely to be chosen in the future.

In recent decades, various random walk models involving reinforcement have become central objects of study in the probability literature. This includes (linearly)-reinforced random walks (linearly-RRW), as introduced by Coppersmith and Diaconis, and once-RRW as introduced by Davis (this was introduced as a simpler model, but in many ways this is harder to study). Other variants have included strongly RRW, non-backtracking RRW, and RRW with finite or decaying memory. Often such models are well-understood on certain graphs with special properties such as trees or complete graphs. Connections with other fields of mathematics, such as random Schrödinger operators and stochastic dynamical systems have been successfully exploited, but there remain many important and elegant open problems, as well as some (apparently) embarrassingly simple ones.

Summary of the workshop. The workshop was organised around six overview talks with accompanying open-problem sessions:

- "The ant random walk", by Daniel Kious and Bruno Schapira,
- "Preferential attachment", by Mia Deijfen and Remco van der Hofstad,
- "Condensation", by Steffen Dereich and Peter Mörters,
- "Graph-based interacting Pólya urns", by Viktor Kleptsyn
- "Interacting urns", by Giacomo Aletti and Andrea Ghiglietti,
- "Self-reinforced random walks", by Silke Rolles and Christophe Sabot.

Each couple of participant gave a 1-hour talk presenting their model and some important results in the existing literature. They then chaired a 1-hour discussion session during which the participants had the opportunity to ask questions about the model and discuss open problems. Plenty of time was left free in the program for participants to discuss the open problems informally in smaller groups during the week. We are hoping that these open problem sessions gave the opportunity to the participants to start new collaborations on these important open problems on reinforcement.

In order to identify and highlight various open problems that arise from applications of probability to other areas of science, such as cognition, we had, early on in the week, a talk by Lucy Palmer from the Florey Institute of Neuroscience and Mental Health (Melbourne), followed by a discussion of the potential mathematical problems raised by her talk.

We had also planned a few "standard" talks on recent results, with the aim to span the whole width of the research area of reinforcement. We had the following speakers:

- Jean Bertoin on "Pólya urns with innovation"
- Stefan Grosskinsky on "Asymptotics of generalized Pólya urns with non-linear feedback"
- Christian Hirsch on "Extremal linkage networks"
- Lucile Laulin on "The superdiffusive limit of the elephant random walk"
- Bas Lodewijks on "Super-linear preferential attachment with fitness"
- Pierre Tarrès on "$\star$-Reinforced Random Walks, Bayesian Statistics and Statistical Physics"
- Stanislav Volkov on "Linear competition processes and generalized Pólya urns with removals"

Although all participants work on reinforcement, it was the first time most of them attended a workshop centered around this topic. We had excellent feedback about this, with participants saying that they enjoyed seeing how other participants also worked on reinforcement, but from different perspectives.

MATRIX-MFO Tandem Workshop: Stochastic Reinforcement Processes and Graphs

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Abstracts

Finding geodesics on graphs using reinforcement learning algorithms

Daniel Kious and Bruno Schapira
(joint work with C. Mailler)

We present different models, inspired by reinforcement learning, for the behavior of ants searching for foods. More precisely, our models consist in launching sequentially random walks on a finite graphs with two marked nodes, say N (for nest) and F (for food), all starting from N and stopped once they reach F. Now the rule is that, after reaching F, each random walk decides to increase the weights of some of the edges that it has crossed, according to some suitably chosen algorithm, with the goal of minimizing the time needed for the future random walks to reach the food.

In our first paper [1], we consider two possible models. In the first one, each random walk selects uniformly at random one of the shortest paths going from N to F within the subgraph visited by the walk, and add one to the weights of all the edges on this path. In the second model, once it has reached F, the walker performs a loop erasure on its reversed path that it took for going from N to F. This way it produces a simple path from F to N, and again one increases the weights of all the edges on this path by one. Our main conjecture is that on *any* finite graph, with two marked nodes as above, the only edges whose weight will grow linearly are those lying on a geodesic between N and F.

We prove this result, for the second model, for a class of graphs called Series-Parallel (SP), which are built recursively as follows. By definition a single edge, with one of its end vertices called N and the other one called F, is a SP graph. Furthermore, given any two SP graphs with marked nodes N_1, F_1 and N_2, F_2 respectively, one obtains a new SP graph by gluing them in parallel, i.e. identifying N_1 with N_2 and F_1 with F_2, or by gluing them in series, i.e. identifying F_1 and N_2, and calling N_1 the new nest and F_2 the new food. In particular this class of graphs includes the set of finite trees, where the root is the nest, and all leaves are identified to a unique vertex, the food.

Furthermore, we also prove the conjecture in the case of the first model, for one of the simplest possible graph which is not in the class SP, which we call the losange graph. It is made of 4 vertices, say u_1, u_2, u_3, u_4, with u_1 the nest, u_4 the food, u_1 and u_4 are linked to u_2 and u_3 by an edge, and u_2 and u_3 are also linked by an edge. The proof for this graph, despite looking very simple, is quite complicated, as it involves various couplings between different types of urn processes.

In our second paper, we consider a different algorithm, which looks much simpler, but is actually quite difficult to analyze. This algorithm simply consists in adding one to the weights of all the edges crossed by the random walk in its way from N to F. In this case we show on a variety of examples, that the algorithm no longer finds geodesics, but on the other hand it has the interesting feature that all weight sequences normalized by the number of random walks, converge almost surely to some *deterministic* constants, and we conjecture that this is a general fact which should hold on any finite graph. Thus we found a new model with a linear reinforcement mechanism, for which deterministic limits appear, a bit like for Pólya-urns with irreducible replacement matrices. In terms of stochastic approximation, this means that the limiting ODE has a unique attractor.

Several problems on this model remain open, such as proving our main conjecture on general finite graphs, or extending the model in order to be able to treat the case with several sources of food in a meaningful manner.

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Recent developments in preferential attachment models

Maria Deijfen and Remco van der Hofstad

We discuss recent developments in preferential attachment models, focussing on their degree distribution, local convergence and small-world structure. We follow [18, Chapter 8] and [19, Chapters 5 and 8].

Model definition. Preferential attachment graphs are dynamical and grow with time. Specifically, at each integer time, a vertex with m edges is added to the graph. The edges are attached to vertices choosen proportionally to degree plus some parameter δ. The graph at time n is denoted by $\mathrm{PA}_n^{(m,\delta)}$ and its vertices by $v_1^{(m)}, \ldots, v_n^{(m)}$. We start by defining the model for $m = 1$, for which the graph consists of a collection of trees. In this case, $\mathrm{PA}_1^{(1,\delta)}$ consists of a single vertex with a single self-loop.

Fix $\delta \geq -1$. We denote the degree of vertex $v_i^{(1)}$ in $\mathrm{PA}_n^{(1,\delta)}$ by $D_i(n)$, where, by convention, a self-loop increases the degree by 2. Conditionally on $\mathrm{PA}_n^{(1,\delta)}$, the edge of vertex $v_{n+1}^{(1)}$ is connected to a second vertex (including itself), according to the probabilities

$$\mathbb{P}\big(v_{n+1}^{(1)} \to v_i^{(1)} \mid \mathrm{PA}_n^{(1,\delta)}\big) = \begin{cases} \dfrac{1+\delta}{n(2+\delta)+(1+\delta)} & \text{for } i = n+1, \\ \dfrac{D_i(n)+\delta}{n(2+\delta)+(1+\delta)} & \text{for } i \in [n]. \end{cases}$$

This model was first defined by Barabási and Albert [2] for $\delta = 0$, and formalised by Bollobás and Riordan in [5], again for $\delta = 0$. See e.g. [11] for a version with random out-degrees and general δ, and [18, Section 8.9] for more background on the history of the model.

The model with $m > 1$ is defined in terms of the model for $m = 1$ as follows. Fix $\delta \geq -m$. We start with $\mathrm{PA}_{nm}^{(1,\delta/m)}$, and denote the vertices in $\mathrm{PA}_{nm}^{(1,\delta/m)}$ by $v_1^{(1)}, \ldots, v_{mn}^{(1)}$. Then we merge the m vertices $v_1^{(1)}, \ldots, v_m^{(1)}$ in $\mathrm{PA}_n^{(1,\delta/m)}$ to become vertex $v_1^{(m)}$ in $\mathrm{PA}_n^{(m,\delta)}$. In doing so, we let all the edges that are incident to any of the vertices in $v_1^{(1)}, \ldots, v_m^{(1)}$ be incident to the new vertex $v_1^{(m)}$ in $\mathrm{PA}_n^{(m,\delta)}$. Then, we merge the m vertices $v_{m+1}^{(1)}, \ldots, v_{2m}^{(1)}$ in $\mathrm{PA}_{nm}^{(1,\delta/m)}$ to become vertex $v_2^{(m)}$ in $\mathrm{PA}_n^{(m,\delta)}$, etc. This defines the model for general $m \geq 1$. The resulting graph $\mathrm{PA}_n^{(m,\delta)}$ is a multi-graph with precisely n vertices and mn edges, so that the total degree is equal to $2mn$.

Degree structure. We start by describing the degree structure of the preferential attachment model, starting with the degree evolution of fixed vertices:

Theorem 1 (Degrees of fixed vertices)**.** *Consider* $\mathrm{PA}_n^{(m,\delta)}$ *with* $m \geq 1$ *and* $\delta > -m$. *Then,* $D_i(n)/n^{1/(2+\delta/m)}$ *converges almost surely to a proper random variable* ξ_i *as* $n \to \infty$.

See e.g. [22] for the degrees of fixed vertices, including the maximal degree. It is also known that $\mathbb{P}(\xi_i > 0) = 1$, see e.g. [18, Chapter 8].

We next investigate the degree distribution in the graph. We write

$$P_k(n) = \frac{1}{n} \sum_{i \in [n]} \mathbb{1}_{\{D_i(n)=k\}}$$

for the (random) proportion of vertices with degree k at time n. For $m \geq 1$ and $\delta > -m$, we define $(p_k)_{k\geq 0}$ by $p_k = 0$ for $k = 0, \ldots, m-1$ and, for $k \geq m$,

$$p_k = (2 + \delta/m) \frac{\Gamma(k+\delta)\Gamma(m+2+\delta+\delta/m)}{\Gamma(m+\delta)\Gamma(k+3+\delta+\delta/m)}.$$

Note that $p_k \sim k^{-(2+\delta/m)}$ as $k \to \infty$. The probability mass function $(p_k)_{k\geq 0}$ arises as the limiting degree distribution for $\mathrm{PA}_n^{(m,\delta)}$:

Theorem 2 (Degree sequence in preferential attachment model)**.** *Consider* $\mathrm{PA}_n^{(m,\delta)}$ *with* $m \geq 1$ *and* $\delta > -m$. *There exists a constant* $C = C(m, \delta) > 0$ *such that, as* $n \to \infty$,

$$\mathbb{P}\Big(\max_k |P_k(n) - p_k| \geq C\sqrt{\frac{\log n}{n}}\Big) = o(1).$$

This result was derived non-rigorously for $\delta = 0$ in [2], rigorously in [6] for $\delta = 0$, and for general $\delta > -m$ in various places, see e.g. [11] as well as [18, Section 8.9] for further references.

Local convergence. Local convergence of finite graphs was first introduced in [3] and, in a different context, independently in [1]. It describes the intuitive notion that a finite graph, seen from the perspective of a uniformly chosen vertex, looks like a certain limiting graph.

A *rooted graph* is a pair (G, o), where $G = (V(G), E(G))$ is a graph with vertex set $V(G)$, edge set $E(G)$, and root vertex $o \in V(G)$. Further, a rooted or non-rooted graph is called *locally finite* when each of its vertices has finite degree (though not necessarily uniformly bounded). For a rooted graph (G, o), we let $B_r^{(G)}(o)$ denote the (rooted) subgraph of (G, o) of all vertices at graph distance at most r away from o. Two rooted (finite or infinite) graphs (G_1, o_1) and (G_2, o_2)

are called *isomorphic*, abbreviated as $(G_1, o_1) \simeq (G_2, o_2)$, when there exists a bijection $\phi\colon V(G_1) \mapsto V(G_2)$ such that $\phi(o_1) = o_2$ and $\{u, v\} \in E(G_1)$ precisely when $\{\phi(u), \phi(v)\} \in E(G_2)$. We let $\mathscr{G}_\star$ denote the space of rooted graphs modulo isomorphisms.

We say that G_n *converges locally in probability* to $(G, o) \sim \mu$ precisely when, for every rooted graph $H_\star \in \mathscr{G}_\star$ and all integers $r \geq 0$,

$$p^{(G_n)}(H_\star) := \frac{1}{|V(G_n)|} \sum_{v \in V(G_n)} \mathbb{1}_{\{B_r^{(G_n)}(v) \simeq H_\star\}} \xrightarrow{\mathbb{P}} \mu(B_r^{(G)}(o) \simeq H_\star).$$

Theorem 3 (Local convergence of preferential attachment models)**.** *Fix $m \geq 1$ and $\delta > -m$. The preferential attachment model $\mathrm{PA}_n^{(m,\delta)}$ converges locally in probability to the Pólya point tree.*

This result was proved in [4], see also [17] for the most general result applying to settings where vertices attach an i.i.d. number of edges upon arrival as proposed in [11], as well as [15, 21, 23] for related results. The Pólya point tree is a multi-type branching process with continuous type space. Rather than giving its precise definition, we state the main ingredient of the proof of local convergence. This key ingredient concerns a finite-graph Pólya urn description, which is very much in line with the topic of the workshop.

We start by introducing the necessary notation. Let $(\psi_j)_{j\geq 1}$ be independent Beta random variables with parameters $\alpha = m + \delta, \beta_j = (2j-3)m + \delta(j-1)$, i.e.,

$$\psi_j \sim \mathsf{Beta}\big(m + \delta, (2j-3)m + \delta(j-1)\big).$$

Define

$$\varphi_j^{(n)} = \psi_j \prod_{i=j+1}^{n} (1 - \psi_i), \qquad S_k^{(n)} = \prod_{i=k+1}^{n} (1 - \psi_i).$$

We now construct a graph as follows:

- ▷ Conditionally on $\psi_1, \ldots, \psi_n$, choose $(U_{k,i})_{k\in[n], i\in[m]}$ as a sequence of independent random variables, with $U_{k,i}$ chosen uniformly at random from the (random) interval $[0, S_{k-1}^{(n)}]$.
- ▷ For $k \in [n]$ and $j < n$, join two vertices j and k if $j < k$ and $U_{k,i} \in I_j^{(n)}$ for some $i \in [m]$ (with multiple edges between j and k if there are several such i).

Call the resulting random multi-graph on $[n+1]$ the *finite-size Pólya graph of size n*. The main result is then as follows:

Theorem 4 (Finite-graph Pólya version of preferential attachment models)**.** *Fix $m \geq 1$ and $\delta > -m$. Then, the distribution of the preferential attachment model with out-degree m, no self-loops and intermediate degree updates is the same as that of the finite-size Pólya graph of size n.*

The importance of Theorem 4 is that the edges in the finite-size Pólya graph are *independent* conditionally on the Beta variables $(\psi_k)_{k\geq 1}$.

Graph distances. We next investigate the graph distances in the preferential attachment model. These graph distances turn out to depend sensitively on the precise value of δ, where, for a graph G and two vertices $u, v \in V(G)$, we let $\mathrm{dist}_G(u,v)$ denote the graph distance in G between u and v, and we let $\mathrm{diam}(G)$ denote the diameter of G:

Theorem 5 (Typical distances preferential attachment models). *Consider* $\mathrm{PA}_n^{(m,\delta)}$ *with* $m \geq 2$. *Let* o_1, o_2 *be chosen independently and uniformly at random from* $[n]$.
(a) Fix $\delta > 0$. *There exist* $0 < c_1 \leq c_2 < \infty$ *such that, as* $n \to \infty$ *and whp,*

$$c_1 \log n \leq \mathrm{dist}_{\mathrm{PA}_n^{(m,\delta)}}(o_1, o_2) \leq c_2 \log n.$$

(b) Fix $\delta = 0$. *As* $n \to \infty$,

$$\mathrm{dist}_{\mathrm{PA}_n^{(m,\delta)}}(o_1, o_2) \frac{\log\log n}{\log n} \xrightarrow{\mathbb{P}} 1, \qquad \mathrm{diam}(\mathrm{PA}_n^{(m,\delta)}) \frac{\log\log n}{\log n} \xrightarrow{\mathbb{P}} 1.$$

(c) Fix $\delta \in (-m, 0)$. *As* $n \to \infty$,

$$\frac{\mathrm{dist}_{\mathrm{PA}_n^{(m,\delta)}}(o_1, o_2)}{\log\log n} \xrightarrow{\mathbb{P}} \frac{4}{|\log(\tau-2)|}, \qquad \frac{\mathrm{diam}(\mathrm{PA}_n^{(m,\delta)})}{\log\log n} \xrightarrow{\mathbb{P}} \frac{4}{|\log(\tau-2)|} + \frac{2}{\log m}.$$

These results are proved in [16] for $\delta > 0$, [5] for $\delta = 0$ and [8, 12, 16] for $\delta \in (-m, 0)$. See [19, Chapter 8] for an extensive overview of such results.

Open problems. Here, we discuss some open problems for preferential attachment models.

(a) Preferential attachment models with removal. Preferential attachment models are usually formulated as pure growth models, that is, once a vertex or an edge has been added it cannot be removed. However, in many application vertices and/or edges can also disappear from the network and it is therefore natural to analyse versions of the model type that allow for this possibility. Attempts have been made in [9] and [7] for discrete-time models with random removal. The expected degree sequence can then be analysed but concentration results exist only in weak versions. In [23], a continuous-time embedding of the basic preferential attachment tree is introduced, which allows the model to be analysed using theory for general branching processes. In [10], this embedding is extended to a version of the model with various types of vertex removals by working with a birth and death process instead of a pure birth process. In this setting, also concentration results can be obtained. It turns out that, if vertices die randomly, the tail exponent is not affected, while if vertices die proportionally to degree, then the power-law distribution is lost. It would be interesting to analyse the effect of vertex and/or edge removal on other quantities, such as the component structure and the vertex of maximal degree. For a given choice of removal mechanism, does the network contain a giant component? Is the vertex with maximal degree persistent, or does it change with time?

(b) Relation local limits preferential attachment models and Bernoulli model.
Dereich and Mörters [15] investigate a model where vertices come in and connect *conditionally independently* to all vertices of the graph with a probability that is a

function of the in-degree of that vertex. Indeed, fix a function $f\colon \mathbb{N}\to[0,\infty)$, and start with $\mathrm{BPA}_1^{(f)}$ being a graph containing one vertex v_1 and no edges. At each time $n\geq 2$, we add a vertex v_n. Conditionally on $\mathrm{BPA}_{n-1}^{(f)}$, and independently for every $v\in[n-1]$, we connect this vertex to v by a directed edge with probability

$$\frac{f(D_v^{(\mathrm{in})}(n-1))}{n-1}, \tag{1}$$

where $D_v^{(\mathrm{in})}(n-1)$ is the in-degree of vertex v at time $n-1$.

Dereich and Mörters [15], see also [13, 14], consider *concave functions* $f\colon \mathbb{N}\mapsto (0,\infty)$ that satisfy that $f(k+1)-f(k)<1$ for every $k\geq 0$, and describe the degree evolution and giant in this model. They also show that the out-degree distribution is close to Poisson with a certain limiting parameter. Finally, in the course of the proof for the limiting size of the giant, they identify the local limit (even though they do not state local convergence explicitly). What is the relation between the local limits of the two models?

It can be expected that these agree, provided we appropriately adapt the preferential attachment model with random out-degrees studied in [17]. Indeed, we should use preferential attachment based only on the in-degree (note that for fixed degrees this makes no difference, and only changes δ to $\delta+m$), and in the setting where each vertex attaches an i.i.d. Poisson number of edges with the appropriate parameter. This, however, has not been proved, partly due to the rather different descriptions of the local limit in the model in [15] compared to those in[4] and [17]. It would be highly useful to make this connection clearer.

(c) Local limits preferential attachment models beyond the affine case.
Following up on this, for their model, Dereich and Mörters [15] are able to consider preferential attachment models with fairly general attachment functions f. The local limits for such models, but now with a fixed number of out-edges per vertex, has not been considered beyond the tree case for which $m=1$ (see [23]). For the tree case, and as described above, Rudas, Tóth and Valkó rely on the continuous-time embedding of the process to relate it to a continuous-time branching process that can be studied using the classical work of Jagers and Nerman [20]. However, for the graph setting, this embedding does not directly extend, and the Pólya urn representations that are so powerful for the affine case are no longer available.

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Condensation

Steffen Dereich and Peter Mörters

The talk is an overview over selected aspects of the mathematical research into the phenomenon of condensation.

1. What is condensation?

As temperature decreases in a gas, drops of liquid form. These drops are characterised by a higher particle density than elsewhere in the gas.

We explain the simplest stochastic model in which such phenomenon can be observed, which is the *balls-in-boxes model*. In this model the gas is contained in a container, which we partition into n boxes. We denote by $X_1, \dots, X_n$ the number of particles in each box. The total number of particles is then fixed as

$$m = \sum_{i=1}^{n} X_i \qquad \text{with } n, m \to \infty \text{ such that } \frac{m}{n} \to \rho.$$

Subject to this constraint the probability of a configuration $X_1, \dots, X_n$ is

$$\frac{1}{Z} \prod_{i=1}^{n} q_{X_i}.$$

If $q_k \sim ck^{-\beta}$ for some $\beta > 2$ the model arises

- as stationary distribution of the zero range process,
- by conditioning independent random variables X_i with $\mathbb{P}(X_i = k) \sim ck^{-\beta}$ on the large deviation event $\{X_1 + \ldots + X_n = m\}$.

In this context it was shown in [4] that if $\mu_\beta = \mathbb{E}X_i$ denotes the (unconditional) particle density then, for $\rho > \mu_\beta$,

$$Z \sim \big(\rho - \mu_\beta\big)^{-\beta} n^{1-\beta}$$

and the particle numbers in the largest and second largest box satisfy

$$X^{(1)} = (\rho - \mu_\beta)n + o(n), \quad X^{(2)} = o(n).$$

This means that all excess mass condenses in a single box.

2. Condensation and large deviations

We show two ways in which condensation arises in the context of upper large deviations for random geometric graphs.

Take a torus $\mathbb{T}_n$ of volume n and form a graph taking

- a Poisson process of intensity one as vertices,
- make each vertex v the centre of a ball with independent random radius R_v,
- form an edge $\{v, w\}$ if the balls $B(v, R_v)$ and $B(w, R_w)$ intersect,
- let E_n be the number of edges and suppose $\mu := \lim_{n\to\infty} \mathbb{E}\big[\frac{E_n}{n}\big] < \infty$.

First, in the scenario of heavy-tailed radius distributions $\mathbb{P}(R_v > x) \sim x^{-\beta+1}$ forthcoming work [5] (partly based on [6]) shows that for $\rho > 0$ non-integer, and k the unique integer such that $k - 1 < \rho < k$,

$$\mathbb{P}\{|E_n| \geq n(\rho + \mu)\} = (F(\rho) + o(1)) \binom{n}{k} n^{k(1-\beta)},$$

where $F(\rho)$ is an explicit functional of the excess edge density ρ. The underlying effect is a condensation of the degree distribution in exactly k vertices, whose associated balls cover a nonvanishing proportion of the torus volume.

Second, in the contrasting scenario of constant radii $R_v = 1$ it is shown in [1] that

$$\frac{1}{\sqrt{n}\log n} \log \mathbb{P}\big(|E_n| \geq n(\rho + \mu)\big) = -\sqrt{\frac{\rho}{2}}$$

Here the underlying effect is that $\sqrt{2\rho n}$ vertices condense in a set of diameter one and therefore form a drop of higher density.

3. Condensation and growth processes

We consider a continuous time branching processes with reinforcement, in which the emergence of condensation can be studied in a stochastic context, see [3].

We start with a single individual with a genetic fitness chosen according to a probability measure μ on $(0,1)$. Individuals never die and give birth to new individuals with a rate equal to their genetic fitness, the different reproduction rates causing a selection effect. When a new individual is born, it is a mutant with probability β, in which case it gets a fitness drawn independently of everything else from μ. If the new individual is not a mutant, it inherits the fitness of its parent. Under the assumption

$$\beta \int \frac{\mu(dx)}{1-x} < 1,$$

a positive fraction of the mass in the empirical fitness distribution condenses, as time goes to infinity, in the essential supremum of the fitness distribution μ (here set to be one). We pose some open questions on the nature of the condensation process and show how these are solved in [2] for a simplified variant of the model.

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Graph-based interacting Pólya urns

Victor Kleptsyn

(joint work with Christian Hirsch, Mark Holmes)

1. Introduction

This talk is devoted to presenting open problems in the study of the so-called (W, α)-reinforcement models, or WARMs for short, that were introduced in [6] (see also [1]). These are graph-based models of interacting Pólya urns, defined in the following way. Assume that a graph $G = (V, E)$ is given, as well as a bounded function $p : V \to (0, \infty)$. Also, assume that we are given the initial counts $(N_e(0))_{e \in E}$. Then, we take for every vertex $v \in V$ a Poisson process (times between points of the process are i.i.d. exponentials) with intensity $p(v)$, the processes for all the vertices being independent. Whenever at some t a clock at some vertex $v \in V$ rings, we consider the adjacent edges $e \sim v$, and reinforce one of these edges ξ using the generalized Pólya urn with the weight function $W(x) = x^\alpha$:

$$N_\xi(t+) = N_\xi(t-) + 1, \quad N_e(t+0) = N_e(t-0),\, e \neq \xi,$$

where for every $e_0 \sim v$, the probability of selecting e_0 from among those edges incident to v is

$$\frac{N_{e_0}^\alpha}{\sum_{e' \sim v} N_{e'}^\alpha}.$$

Hereafter we will refer to $p(v)$ as to the *firing rate* at v due to the motivation coming from a network of neurons: vertices are neurons, and edges are axons joining them, that are reinforced by the signal passing through them.

Provided that the function p is bounded and the degrees of vertices do not grow too much (for instance, it suffices to assume that the degrees are bounded), this process is well-defined [3]. Then, one can ask to determine the asymptotic behaviour of the scaled weights $X_e(t) := \frac{N_e(t)}{t}$ as $t \to \infty$.

Note that for the case of a finite graph, this continuous-time process can be transformed into a discrete-time one by considering at the process at the times at which the clocks ring. In this case, on each step the reinforcing vertex is chosen independently with the probability $p'(v) := \frac{p(v)}{\sum_u p(u)}$.

This random process admits a stochastic approximation by the differential equation

$$\frac{d}{dt} x_e = \frac{1}{t}(-x_e + f_e(x)),$$

where $f_e(x)$ is the rate of reinforcement of the edge e:

$$f_e(x) = \sum_{v \sim e} p_v \frac{x_e^\alpha}{\sum_{e' \sim v} x_{e'}^\alpha}.$$

The $\frac{1}{t}$ factor can be removed by the exponential time change, $t = e^\tau$. Also, for the finite graph where this expression is well-defined, the resulting differential equation is gradient-like (see e.g. [7]) for the function

$$L(x) = -\sum_{e \in E} x_e + \frac{1}{\alpha} \sum_{v \in V} \log \left(\sum_{e \sim v} x_e^\alpha \right).$$

Namely, one has

$$\frac{d}{d\tau} x_e = x_e \frac{\partial L(x)}{\partial x_e}.$$

The properties of this flow highly depend on whether α is smaller than 1, equal to 1 or larger than 1, in the same way as they do for one generalized Pólya urn. We say that $x = (x_e)_{e \in E}$ is an equilibrium for the process if $x_e = f_e(x)$ for every $e \in E$.

Unless otherwise stated, below we will assume that the firing rates are bounded above, and that the graph (V, E) has bounded degrees.

2. Subcritical case: $\alpha < 1$

For a finite graph, the function L is concave, which implies the convergence of the vector $(x_e(t))_{e\in E}$ to the *unique* equilibrium (which maximises L on the simplex $x_e \geq 0, \quad \sum_e x_e = \sum_v p(v)$). It is thus natural to ask if such a convergence takes place for general graphs:

Conjecture 1. *On infinite graphs there is a unique equilibrium x and $X_e(t) \to x_e$ for every $e \in E$ almost surely.*

This was partially established by Y. Couzinié, C. Hirsch, in [2, Theorem 2.2]. Assuming that $p(v) \equiv 1$ they have shown the convergence for $\alpha < \frac{1}{2}$, and for $\alpha < 1$ for the particular case $G = \mathbb{Z}$.

3. Critical case: $\alpha = 1$

In the case $\alpha = 1$, the function L is non-strictly concave, and in some situations this leads to the non-uniqueness of an equilibrium. In particular, this happens on even-length cycle [8]. For instance, if the firing rates are uniform, $p(v) \equiv 1$, the equilibria on the cycle of length $2k$ are of the form

$$\{(a, b, a, b, \ldots, a, b) \mid a + b = 2\}.$$

However, on any odd cycle the equilibrium is unique. Moreover, as the length of the even-length cycle increases, the law of (a, b) converges to the Dirac measure in $(1, 1)$. Considering the graph $\mathbb{Z}$ as an "infinite cycle", this then motivates the following conjecture.

Conjecture 2. *For $G = \mathbb{Z}$ with firing rates $p(v) \equiv 1$, one has almost surely*

$$\forall v \quad x_v(t) \to 1, \qquad \textit{as } t \to \infty.$$

4. Supercritical case: $\alpha > 1$

On finite graphs, it is known that due to the gradient-likeness of the approximating flow, the weights vector $X(t)$ almost surely converges to an equilibrium, and that this equilibrium cannot be linearly unstable [4]. For $\alpha > 2$ such an equilibrium should be supported on a forest: every connected component is a tree [7]. For $\alpha > 25$ and $p(v) \equiv 1$ it is known that each such tree is a *whisker*, a tree graph of diameter at most 3 [7].

Open Problem 1. *Improve the lower bound $\alpha > 2$ for which every connected component is a tree.*

Open Problem 2. *Improve the lower bound $\alpha > 25$ for which (when $p(v) \equiv 1$) every connected component is a whisker.*

Note that for the triangle graph with $p(v) \equiv 1$ an edge dies out (i.e. some $X_e(t) \to 0$) almost surely if $\alpha > 4/3$, but this does not hold if $\alpha < 4/3$ [6]. This means that the lower bounds $\alpha > \alpha_0$ (with $\alpha_0 = 2$ and $\alpha_0 = 25$) in the open problems above are not true in general if $\alpha_0 < 4/3$. Perhaps $\alpha > 4/3$ is sharp for both results.

For infinite graphs the Lyapunov function cannot be applied (the sum becomes infinite), and much less is known. However it is known (by different arguments: comparing to a percolation-type model called the *corrupted compass* model) that for any graph with bounded degrees of vertices, for a sufficiently large α almost surely the limit of $x(t)$ exists, and the edges on which this limit is supported decompose into finite connected components [3].

On the other hand there are examples of regular trees with firing rates decreasing to 0 sufficiently quickly with the distance to the root, in which almost surely there is an infinite number of infinite connected components, formed by surviving edges [5].

Conjecture 3. *Assume that firing rates $p(v)$ are bounded away from zero and from infinity, and that $\alpha > 1$. Then the limit $\lim_{t\to\infty} X_e(t)$ exists almost surely for each e, and the surviving edges form finite connected components. In particular, for $\alpha > 2$ they are trees.*

Also, it is interesting to know whether the example on the regular tree can be modified for the case of $\mathbb{Z}^d$ for some $d \geq 2$.

Open Problem 3. *Do there exist a bounded (but not bounded away from 0) function p, $d \geq 2$, and $\alpha > 1$ for which the WARM on the graph $G = \mathbb{Z}^d$ almost surely (or indeed with positive probability) contains an infinite connected component of surviving edges? If such a component exists, is it unique?*

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Networks of reinforced stochastic processes

Andrea Ghiglietti
(joint work with Giacomo Aletti, Irene Crimaldi)

The model we introduced in [1] consists in a collection of very general stochastic processes with reinforcement that are connected among each others through a given underlying network. These processes are located on the vertices of the graph and the strength of the interactions is represented by weights associated to the edges. In [1] the complete synchronization of the system and some CLTs have been derived under some crucial assumptions on the adjacency matrix of the network and on the form of the reinformcement sequence. Then, under the same setting in [2, 3] the processes of the empirical and weighted means computed on the vertices have been investigated. Here we present the main results reported in our recent work [4] that provide the complete characterization of the first-order asymptotic behavior of this general class of stochastic models removing any assumption on the topology of the graph and on the reinforcement sequence. This very general framework allowed us to find some surprising conditions and limiting behaviors, such as the partial synchronization of the system or the periodic clockwise dynamics.

The reinforcement mechanism is a key feature governing the dynamics of many biological, economic and social systems. In paricular, we consider the framework in which there is a collection of $N \geq 2$ agents that at each discrete time $t \in \mathbb{N}$ can take one of two possible actions $S = \{0, 1\}$, so that the variable $X_{t,j} \in \{0, 1\}$ will indicate which action has been taken at time t by the agent j. The reinforcement mechanism would suggest that, if the agent j takes action 1 at time t, then the probability of taking action 1 at time $t+1$ should be greater. This can be modeled by a time-dependent Pólya urn having balls of color 0 or 1, in which the color of the sampled balls represent the action taken. In this case, the dynamics of the urn proportion $Z_{t,j}$ can be written as $Z_{t+1,j} = (1 - r_t)Z_{t,j} + r_t X_{t+1,j}$, with $E[X_{t+1,j}|\mathcal{F}_t] = Z_{t,j}$ and an appropriate deterministic reinforcement sequence $(r_t)_{t\in\mathbb{N}}$, where $r_t \in (0, 1)$ depends on the quantity of balls added to the urn at time t (e.g. for classic Pólya urn $r_t \sim 1/t$). We call this class of processes with self-reinforcement "*Reinforced Stochastic Processes* (RSPs)" and they can be put in connection with the notion of time-dependent Pólya urn.

Let us now consider a system where there are interactions, and so the decision of each agent could be influenced by the other agents' opinion; in other words, the probabilty of taking an action by a person j could depend also on the propensity towards that action of the agents connected with the person j. In order to model this phenomenon of interaction, we introduce a finite weighted direct graph (V, E, A) where $V = \{1, \ldots, N\}$ is the set of vertices (agents of the system), E the set of edges and A the weighted adjacency matrix (*interaction matrix*), with A non-negative ($A \geq 0$) and normalized ($A^\top \mathbf{1} = \mathbf{1}$). Then, using the compact notation $\boldsymbol{Z}_t = (Z_{t,1}, \ldots, Z_{t,N})^\top$ and $\boldsymbol{X}_t = (X_{t,1}, \ldots, X_{t,N})^\top$, we can write the dynamics of a system of interacting RSPs as

$$\boldsymbol{Z}_{t+1} = (1 - r_t)\boldsymbol{Z}_t + r_t \boldsymbol{X}_{t+1}, \qquad \mathbb{E}[\boldsymbol{X}_{t+1}|\mathcal{F}_t] = A^\top \boldsymbol{Z}_t. \tag{1}$$

Notice that $a_{h,k} = 0$ means that agent h has no "direct influence" on agent k, as $a_{h,k}$ quantified how much the decision $X_{k,t+1}$ depends on the propensity $Z_{h,t+1}$.

From now on, let us assume A irreducible (for extensions to A reducible see [4]) and exclude trival initial conditions ($\mathbf{Z}_0 \neq \mathbf{0}$ and $\mathbf{Z}_0 \neq \mathbf{1}$). Then, we will focus on the following goals:

(i) sufficient and necessary conditions on the reinforcement sequence $(r_t)_t$ and on the interaction matrix A for the *complete almost sure asymptotic synchronization* of the personal inclinations towards a certain random variable Z, i.e.

$$\mathbf{Z}_t \xrightarrow{a.s.} Z\mathbf{1}; \tag{2}$$

(ii) the behaviour of the system when the complete almost sure asymptotic synchronization does not hold; in particular, sufficient and necessary conditions on the reinforcement sequence and on the interaction matrix for a *partial almost sure asymptotic synchronization* of the personal inclinations;

(iii) in the case of complete almost sure asymptotic synchronization towards Z, the probability that Z takes the extreme values, 0 or 1, i.e. the probability of *asymptotic polarization* of then personal inclinations.

Starting with point (i), we have derived the following sufficient and necessary conditions:

(a) When $A^\top$ is aperiodic (period $d = 1$), (2) holds true if and only if $\sum_t r_t = +\infty$;

(b) When $A^\top$ is periodic (period $d \geq 2$), (2) holds true if and only if $\sum_t r_t(1 - r_t) = +\infty$.

Regarding point (ii), let us now focus on the behavior of the system when neither (a) nor (b) hold true. It is quite easy to prove that when $\sum_t r_t < +\infty$, all the processes $(Z_{t,h})_t$, $h = 1, \ldots, N$, converge almost surely, but, for any pair of distinct nodes, there exists a strictly positive probability that the corresponding processes do not synchronize. It is much harder instead to characterize the behavior of the system when $A^\top$ is periodic, $\sum_t r_t = +\infty$, $\sum_t r_t(1 - r_t) < +\infty$; indeed, in this case only a *partial almost sure asymptotic synchronization* takes place:

$$(Z_{t,h_1} - Z_{t,h_2}) \xrightarrow{a.s.} 0 \quad \forall h_1, h_2 \in \text{same cyclic class} \Leftrightarrow \sum_t r_t = +\infty.$$

What is even more surprising in this setting is that there exists a strictly positive probability that the entire process $(\boldsymbol{Z}_t)_t$ does not converge. Specifically, denote as $\boldsymbol{Z}_t^{(C)}$ the projection of $\boldsymbol{Z}_t$ on the eigen-space associated with the d eigenvalues with $|\lambda| = 1$, and construct the d-dimensional random vector $\boldsymbol{Z}_t^{(c)}$ taking one element for each cyclic class. Then, we have:

(1) $Z_{t,h} \overset{a.s.}{\sim} Z_{n,\ell}^{(c)}$, with ℓ being the cyclic class containing h;

(2) $Z_{t,\ell}^{(c)}(1 - Z_{t,\ell}^{(c)}) \xrightarrow{a.s.} 0$ for each $\ell = 0, \ldots, d-1$ (i.e. the limit set of each cyclic class is given by the barrier-set $\{0, 1\}$);

(3) $\|\mathbf{Z}_t^{(c)}\|$ almost surely converges;

(4) $N_\infty = a.s. - \lim_t \|\mathbf{Z}_t^{(c)}\|^2$ is a random variable taking values in $\{0, \dots, d\}$ with $P(N_\infty = 0) + P(N_\infty = d) < 1$ and it represents the limit of the number of cyclic classes that are near to 1.

As a consequence:

(5) On $\{N_\infty = 0\} \cup \{N_\infty = d\}$, we have that all the $Z_{t,\ell}^{(c)}$ converges towards the same barrier (hence, we have complete synchronization of the system toward the same barrier).

(6) On $\{1 \leq N_\infty \leq d-1\}$, we have an asymptotic periodic behaviour of the components of $\mathbf{Z}_t^{(c)}$: $\exists$ an integer-valued increasing sequence $(\sigma_t)_t$ s.t. $Z_{\sigma_t,\ell}^{(c)} - Z_{\sigma_{t+1},\ell-1}^{(c)} \xrightarrow{a.s.} 0$.
In particular, when $\sum_t (1-r_t) < +\infty$, we have $\sigma_{t+1} = \sigma_t + 1$ eventually, so that $\mathbf{Z}_{t+d} - \mathbf{Z}_t \xrightarrow{a.s.} \mathbf{0}$ and $P(\mathbf{X}_{t+d} = \mathbf{X}_t \text{ eventually}) = 1$.

Finally, regarding point (iii), we established several results in which the probability of asymptotic polarition has been proved to be strongly related with the asympotic behavior of the reinforcement sequence $(r_t)_t$. Indeed, r_t and $(1-r_t)$ indicate in (1) the weights associated to the "new" and the "past" information, respectively. Hence, since the new information $X_{j,t+1}$ is an element of the barrier set $\{0,1\}$, we can heuristically see that the process $Z_{j,t}$ will tend to polarize when r_t is relevant compared to $(1-r_t)$ or $\prod_n^t (1-r_n)$, while $Z_{j,t}$ will hardly reach the barriers when r_t quickly vanishes. Formally, given the complete synchronization and denoting $Q = P(Z=0) + P(Z=1)$, we proved the following:

(a) if $r_t = O(\prod_{n=0}^{t-1}(1-r_n))$ and $\sum_t \frac{r_t^2}{[\prod_{n=0}^{t-1}(1-r_n)]^{1-\delta}} < +\infty$ for some $\delta \in (0,1)$, then $Q = 0$ *(Non-trivial asymptotic polarization is negligible)*;

(b) if $\sum_t \prod_{n=0}^t (1-r_n) < +\infty$, then $Q > 0$ *(non-trivial asymptotic polarization with a strictly positive probability)*;

(c) if $\sum_t r_t^2 < +\infty$, then $Q < 1$ *(non-almost sure asymptotic polarization)*;

(d) if $\sum_t r_t^2 = +\infty$, then $Q = 1$ *(almost sure asymptotic polarization)*.

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On some aspects of linearly edge-reinforced random walks and vertex-reinforced jump processes

Silke Rolles and Christophe Sabot

Linearly edge-reinforced random walk (errw) was introduced in 1986 by Coppersmith and Diaconis [5] and first investigated by Pemantle [18] on trees. A crucial property of the errw is partial exchangeability. The process is a mixture of reversible Markov chains. The mixing measure was computed by Coppersmith-Diaconis [5] and later by Keane-Rolles [10]. The remarkable shape of this mixing measure remained mysterious for quite some time. However, bounds for hitting probabilities in $\mathbb{Z}^2$ [15], recurrence on strips at all reinforcements [14] and on a modification of $\mathbb{Z}^2$ at strong reinforcement [16] were shown based on this mixing measure. In the last 15 years many progress has been made on errw leading to a rather precise description of its asymptotic behavior and its companion process, the vertex-reinforced jump process (vrjp).

From a different perspective, in 2010, Disertori, Spencer, and Zirnbauer investigated a non-linear hyperbolic supersymmetric sigma model, called $H^{2|2}$-model, motivated by localisation/delocalisation properties of the Anderson model [9, 8]. A key step in the understanding of the errw has been the understanding of the relation between the errw, the vrjp, and the $H^{2|2}$-model [21]. This led to a proof of localisation at strong reinforcement in any dimension [21, 1] and existence of a transient regime at weak reinfocement in $\mathbb{Z}^d$, $d \geq 3$, [21, 7].

More recently, a representation of the vrjp in terms of random Schrödinger operators [23] has proved useful to show recurrence in dimension 2 for any reinforcement, both for errw and vrjp [24, 20, 11]. In [19] a convex monotonicity property of the vrjp, generalising Rayleigh's monotonicity, was understood and used to prove uniqueness of the phase transition between recurrence and transience in dimension $d \geq 3$. Recently, a better understanding of the density of states of this Schrödinger operator was obtained [6].

The vrjp and the $H^{2|2}$-model have shown very rich connections with different topics. Let us mention the deep connections with Dynkin's or Ray-Knight's theorems about local times, starting with [22, 13], culminating in the deep understanding of the relation between geometry of the spin space and Ray-Knight type theorems of different self-interacting processes [4, 3]. Besides, in the same family of supersymmetric models, the $H^{2|4}$-model was related to percolation conditioned to have no loops. Mermin-Wagner's estimates proved for the $H^{2|2}$-model were generalised to the $H^{2|4}$-model to prove absence of phase transition in dimension 2 for this model [2]. In different directions, a relation with the vrjp/$H^{2|2}$-model and interlacement was exhibited [17] and relations with stochastic calculus [12, 25].

In the overview talk, we presented the models and

- the relation between errw, vrjp, and the $H^{2|2}$-model,
- the representation with random Schrödinger operators,
- the application to the asymptotic behavior, namely recurrent phase, transient phase, exponential localisation, and
- some of the many open questions which remain in the area.

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Asymptotics of generalized Pólya urns with non-linear feedback

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(joint work with Thomas Gottfried)

Generalized Pólya urns with non-linear feedback are an established probabilistic model to describe the dynamics of growth processes with reinforcement, a generic example being competition of agents in evolving markets [1]. Depending on the feedback function, it is well known that the model may exhibit monopoly, where a single agent achieves full market share. Besides this general result, various further results for particular feedback mechanisms have been derived from different perspectives, see [3] for more details and [6] for a review. The purpose of this paper is to provide a comprehensive account of the possible asymptotic behaviour for a large general class of feedback functions.

1. The model

Let $A \geq 2$ be the number of agents and $F_i\colon \mathbb{N} \to (0,\infty)$ the feedback function of agent $i \in [A] := \{1,\dots,A\}$. We define a homogeneous, discrete-time Markov process $(X(n))_{n\in\mathbb{N}_0} = ((X_1(n),\dots,X_A(n))_{n\in\mathbb{N}_0}$ on the state space $\mathbb{N}^A$ with initial condition $X(0) = (X_1(0),\dots,X_A(0)) \in \mathbb{N}^A$ such that $X_i(0) \geq 1$ for all $i \in [A]$, and transition probabilities

$$\mathbb{P}\Big(X(n+1) = X(n) + e^{(i)} \,\Big|\, X(n)\Big) = \frac{F_i(X_i(n))}{F_1(X_1(n)) + ... + F_A(X_A(n))},\ i = 1,\dots,A,$$

where $e^{(i)} = (\delta_{i,j})_{j=1}^A$ is the i-th unit vector. We denote by $N := X_1(0) + ... + X_A(0) \geq A$ the initial market size.

In addition, we define the corresponding time-inhomogeneous Markov process $(\chi(n))_{n\in\mathbb{N}_0}$ of market shares

$$\chi_i(n) := \frac{X_i(n)}{N+n} \in (0,1)\ , \quad i = 1,\dots,A,\ n \in \mathbb{N}_0\ ,$$

with $\chi(n) = (\chi_1(n),\dots,\chi_A(n)) \in \Delta^o_{A-1}$, where Δ^o_{A-1} is the interior of the unit simplex $\Delta_{A-1} := \{(x_1,\dots,x_A) \in [0,1]^A \colon x_1 + ... + x_A = 1\}$. Of particular interest is the event of strong monopoly of an agent $i \in [A]$

$$sMon_i(\chi(0), N) := \mathbb{P}\Big\{\lim_{n\to\infty} \sum_{j\neq i} X_j(n) < \infty\Big\},$$

i.e. only one agent wins in infinitely many steps.

This model was essentiall introduced in [4]. A useful alternative construction of the process is provided by the so-called exponential embedding (see e.g. [6] and references therein). Dynamic results with stochastic approximation (see e.g. [7] and references therein) are also covered in [3] but not included in this report.

2. Monopoly case

It is generally known (see e.g. [5]) that strong monopoly occurs with probability one, i.e. $\mathbb{P}\left(\bigcup_{i=1}^{A} sMon_i(\chi(0), N)\right) = 1$, if and only if

$$\text{(M)} \qquad \sum_{k=1}^{\infty} \frac{1}{F_i(k)} < \infty \quad \text{for at least one } i,$$

otherwise this probability is zero. We distinguish two main types of feedback.

Definition: Let agent i fulfill (M). We call i *of type P* (for polynomial) if

$$\lim_{k\to\infty} F_i(k) \sum_{l=k}^{\infty} \frac{1}{F_i(l)} = \infty$$

and *of type E* (for exponential) if

$$\limsup_{k\to\infty} F_i(k) \sum_{l=k}^{\infty} \frac{1}{F_i(l)} < \infty.$$

The *(asymptotic) attraction domain* of an agent $i \in [A]$ is defined as

$$D_i := \left\{\chi(0) \in \Delta^o_{A-1} \colon \lim_{N\to\infty} \mathbb{P}(sMon_i(\chi(0), N)) = 1\right\} \subset \Delta^o_{A-1}.$$

Theorem 1: Let at least one agent satisfy (M) and all agents satisfying (M) are either of type P or type E. Moreover, assume one of the following conditions:

(1) At least one agent is of type E and for all $\chi(0) \in \Delta^o_{A-1}$, $i, j \in [A]$

$$\liminf_{N\to\infty} \frac{F_i(\chi_i(0)N)}{F_j(\chi_j(0)N)} = 0, \ \limsup_{N\to\infty} \frac{F_i(\chi_i(0)N)}{F_j(\chi_j(0)N)} = \infty \quad \text{do not hold simultaneously .}$$

(2) No agent is of type E and all agents of type P (there is at least one) fulfill

$$\limsup_{k\to\infty} \frac{1}{k} F_i(k) \sum_{l=k}^{\infty} \frac{1}{F_i(l)} < \infty.$$

In addition, suppose that $\lim_{N\to\infty} \frac{F_i(\chi_i(0)N)}{F_j(\chi_j(0)N)} \in [0, \infty]$ exists for all $\chi(0) \in \Delta^o_{A-1}, i, j \in [A]$.

Then the asymptotic attraction domains are polytopes that dissect the simplex up to boundaries, i.e. $\bigcup_{i=1}^{A} \overline{D_i} = \Delta_{A-1}$, where $\overline{(\cdot)}$ is the topological closure. If agent i does not satisfy (M) then $D_i = \emptyset$.

Moreover, in the situation of assumption (1), the monopoly is even *total* with high probability for large N, i.e. one agent wins in all steps. This does in general not hold in the situation of assumption (2). In addition, [3] provides results allowing an explicit calculation of the attraction domains in generic examples.

3. Non-monopoly case

Now, assume that no agent fulfills (M) and define $a_i(t) := \int_1^t \frac{1}{F_i(s)} ds$ for an appropriate extension of F_i.

Theorem 2: Let all agents fulfill

$$\limsup_{k\to\infty} \frac{1}{k} F_i(k) \sum_{l=1}^{k} \frac{1}{F_i(l)} < \infty \quad \text{and} \quad \liminf_{k\to\infty} \frac{1}{k^p} F_i(n) \sum_{l=1}^{k} \frac{1}{F_i(l)} > 0$$

for some $p > \frac{1}{2}$. If the limit

$$\chi_i(\infty) := \lim_{t\to\infty} \frac{a_i^{-1}(t)}{a_1^{-1}(t) + ... + a_A^{-1}(t)} \in [0,1] \tag{1}$$

exists for an $i \in [A]$, then

$$\chi_i(n) \xrightarrow{n\to\infty} \chi_i(\infty) \quad \text{almost surely .}$$

If the limit in (1) does not exist, then $\chi_i(n)$ does not converge for $n \to \infty$.

The assumptions of Theorem 2 are fulfilled e.g. for $F_i(k) = k^\beta$, $\beta < 1$ or $F_i(k) = \log k$, but almost linear feedback such as $F_i(k) = k \log k$ is not included. Nevertheless, using results in [2], we also cover the almost linear case in [3], which reveals *weak monopoly*, i.e. all agents win in infinitely many steps, but $\lim_{n\to\infty} \chi_i(n) = 1$ for a random agent i.

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Super-linear preferential attachment with fitness

Bas Lodewijks
(joint work with Tejas Iyer)

We consider a model of randomly growing trees called super-linear preferential attachment with fitness, which features both super-linear reinforcement and fitness. In this model, we have an *attachment function* $f : \mathbb{N}_0 \times \mathbb{R}_+ \to \mathbb{R}_+$, a sequence of random i.i.d. vertex-weights $(W_i)_{i\in\mathbb{N}}$, and we initially start with a tree T_1, which consists of a root vertex labelled 1 with weight W_1. At each step $n \geq 2$ we construct T_n from T_{n-1} as follows. A new vertex n with weight W_n is introduced and connected to one vertex already present in the tree. Conditionally on T_n, vertex n connects to vertex $i \in \{1, \dots, n-1\}$ with probability

$$\frac{f(\deg_{n-1}(i), W_i)}{\sum_{j=1}^{n-1} f(\deg_{n-1}(j), W_j)},$$

where $\deg_{n-1}(i)$ denotes the degree of vertex i in the tree of size $n-1$ created so far. We focus on the case where f grows super-linear in its first argument. In particular, we discuss the two examples $f(k, W) = Wk^p$ and $f(k, W) = k^p + W$ (multiplicative fitness and additive fitness, respectively), where $p > 1$ is a constant called the super-linear exponent.

Super-linear preferential attachment models with fitness are a combination of two different types of preferential attachment models. In the first, super-linear preferential attachment models, independently introduced by Krapivsky and Redner [7] and Drinea et al. [3], vertices are introduced one at a time and a new vertex connects to an existing vertex, sampled with probability proportional to k^p when the degree of the vertex is k, for some $p > 1$. The other class of models is preferential attachment with fitness. In these kind of models, each vertex is assigned a (random) weight and the probability to connect to a vertex depends on both its weight and degree. Most commonly studied are two variants: multiplicative fitness and additive fitness as introduced by Barabasi and Bianconi [1] and Ergün and Rodgers [4], respectively. Here, the probability to connect to a vertex is proportional to the product, respectively the sum, of its degree and weight.

Super-linear preferential attachment models exhibit *winner-takes-all* behaviour, where there almost surely exists a unique vertex that attains an infinite degree, whilst all other vertices have finite degree, as shown by Oliviera and Spencer [9]. Preferential attachment models with fitness, on the other hand, can exhibit *fitter-take-all* behaviour, where new and fitter vertices with higher weights are able to outcompete older vertices and attain the largest degrees, see e.g. [2, 5, 8].

We are interested in the effect of the competition induced by the vertex-weights and how this competition manifests itself in the presence of super-linear reinforcement of degrees. For both the multiplicative and additive fitness setting, we identify a phase transition in the structure of the infinite limiting tree $T_\infty = \lim_{n\to\infty} T_n$. In particular, we identify whether the limiting infinite tree contains a unique vertex with infinite degree or a unique infinite path almost surely, based on work of Iyer [6]. We quantify the phase-transition in terms of the tail-behaviour of the fitness distribution and the super-linear exponent p.

Let us assume that the vertex-weight satisfy

$$\mathbb{P}(W_1 \geq x) = \ell(x)x^{-(\alpha-1)}, \qquad x > 0,$$

for some $\alpha > 1$ and ℓ a slowly-varying function (i.e. $\lim_{x\to\infty} \ell(cx)/\ell(x) = 1$ for any $c > 0$). Then, in the multiplicative setting:

- When $(p-1)(\alpha-1) > 1$, T_∞ contains a unique vertex with infinite degree.
- When $(p-1)(\alpha-1) < 1$, T_∞ contains a unique infinite path and all vertices have finite degree.

On the other hand, in the additive setting:

- When $p(\alpha - 1) > 1$, T_∞ contains a unique vertex with infinite degree.
- When $p(\alpha - 1) < 1$, T_∞ contains a unique infinite path and all vertices have finite degree.

The analysis of the phase transition and the behaviour of the tree is based on embedding the preferential attachment tree into an explosive Crump-Mode-Jagers processes. Here, each vertex gives birth according to an associated explosive point process, and explosion denotes the concept that the process produces an infinite total progeny in finite time. The analysis of these CMJ processes uses a novel idea based on the number of individuals in such a process that explode before all their ancestors. When this number is almost surely finite, this implies that the time at which explosion occurs, coincides with with the explosion time of the random point process associated with a single individual. As a result, this unique individual attains an infinite degree almost surely. On the other hand, when instead every individual almost surely has a child that explodes before itself, it follows that a unique infinite path appears and all vertices have finite degree almost surely. This is due to the fact that the explosion time of the process is then strictly smaller than the explosion time of the point process associated with any individual.

This novel approach allows us to recover the results of Oliviera and Spencer [9] and extend them from preferential attachment models with only super-linear reinforcement to models with both super-linear reinforcement and fitness. Our approach also allows us to deal with settings without fitness for which the analysis used by Oliviera and Spencer breaks down.

We also obtain results in a more general setting, where fewer assumptions on the attachment function f are required. Here we provide conditions on f and the weight distribution that imply the almost surely existence of a unique vertex with infinite degree or a unique infinite path. Furthermore, we can prove results related to the number of subtrees in the infinite tree T_∞ in the infinite degree regime.

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Linear competition processes and generalized Pólya urns with removals

Stanislav Volkov

(joint work with Serguei Popov, Vadim Shcherbakov)

A competition process is a continuous time Markov chain that can be interpreted as a system of interacting birth-and-death processes, the components of which evolve subject to some sort of interactions. This process is, probably, the most known example of such Markov chains: for instance, competition processes with *non-linear interaction* (e.g., of the Lotka-Volterra type) were originally proposed to model competition between populations; please see [1], [3], [6], [7] and references therein.

During the talk, we analyse the *linear version* of the above process, namely the case where a component of the process increases with a linear birth rate and decreases with a rate given by some linear function of other components, and a zero is an absorbing state for each component (when a certain component becomes zero, it stays zero forever, i.e., becomes extinct). We show that, with probability one, eventually only a random subset of non-interacting components of the process survives. A similar result also holds for the relevant generalized Pólya urn model with removals.

Formally, fix an integer $N \geq 1$. An $N \times N$ matrix $A = (a_{ij})$ with non-negative elements and zeros on the main diagonal is called *an interaction matrix.* Given a number $\alpha > 0$ and an interaction matrix $A = (a_{ij})$ consider a continuous-time Markov chain $X(t) = (X_1(t), \dots, X_N(t)) \in \mathbb{Z}_+^N$, $t \in \mathbb{R}_+$, with the following transition rates

$$q_{xy} = \begin{cases} \alpha x_i, & y = x + e_i; \\ \left(\sum_{j=1}^N a_{ij} x_j\right) 1_{\{x_i > 0\}}, & y = x - e_i, \end{cases} \tag{1}$$

where $x, y \in \mathbb{Z}_+^N$, and $e_i = (0, 0, \dots, 0, 1, 0, \dots, 0)$ is the i-th unit vector in $\mathbb{Z}_+^N$. We call the process $X(t)$ with transition rates (1) a *linear competition process.* The quantity $a_{ij} \geq 0$ indicates how much component i is affected by component j

(in biological terms, the fact that $a_{ij} > 0$ can be interpreted as a predator j hunting prey i). Note that the birth rate α is the same for all the components; unfortunately, the question about the long-term behaviour of the process remains open for the situation when it is not the case.

Let $\zeta(n) = (\zeta_1(n), \dots, \zeta_N(n)) \in \mathbb{Z}_+^N$, $n \in \mathbb{Z}_+$, be the embedded Markov chain corresponding to $X(t)$. Denote $\mathcal{F}_n = \sigma(\zeta(1), \dots, \zeta(n))$, and

$$R(\zeta) = \sum_{i=1}^{N} \left(\alpha\zeta_i + 1_{\{\zeta_i > 0\}} \sum_{j=1}^{N} a_{ij}\zeta_j \right), \qquad \zeta = (\zeta_1, \dots, \zeta_N),$$

then the transition probabilities of Markov chain ζ are given by

$$\mathbb{P}\left(\zeta(n+1) = \zeta(n) + e_i \,\middle|\, \mathcal{F}_n\right) = \frac{\alpha\zeta_i(n)}{R(\zeta(n))},$$

$$\mathbb{P}\left(\zeta(n+1) = \zeta(n) - e_i \,\middle|\, \mathcal{F}_n\right) = \begin{cases} \frac{\sum_{j=1}^{N} a_{ij}\zeta_j(n)}{R(\zeta(n))} & \text{if } \zeta_i(n) \geq 1; \\ 0 & \text{if } \zeta_i(n) = 0. \end{cases}$$

Our main result is the following:

Theorem. Let $X(t)$ be a linear competition process with transition rates (1) specified by a parameter $\alpha > 0$ and an interaction matrix A and $\zeta(n)$ be the corresponding embedded Markov chain. Suppose that $X_i(0) = \zeta_i(0) \geq 0$, $i = 1, 2, \dots, N$, and for every subset $\mathcal{I} = \{i_1, i_2, \dots, i_K\} \subseteq \{1, \dots, N\}$ denote

$$\mathcal{E}_{\mathcal{I}} = \left\{ \lim_{t\to\infty} X_i(t) = \lim_{n\to\infty} \zeta_i(n) = \infty \text{ if } i \in \mathcal{I}; \quad .... = 0, \text{ if } i \notin \mathcal{I} \right\}.$$

Let us call a non-empty $\mathcal{I}$ *disjoint in* A, if $a_{ij} = a_{ji} = 0$ for all $i, j \in \mathcal{I}$. Then for every disjoint in A subset $\mathcal{I}$ we have $\mathbb{P}(\mathcal{E}_{\mathcal{I}}) > 0$. Moreover, no other limiting behaviour is possible, i.e.

$$\mathbb{P}\left(\bigcup_{\mathcal{I}:\ \mathcal{I} \text{ disjoint in } A} \mathcal{E}_{\mathcal{I}} \right) = 1$$

so, with probability one, *some* random subset $\mathcal{I}$ of non-interacting components of the process $X(t)$ survives, and the surviving components behave as independent Yule processes, each with the same parameter α. As a result, for large n the process $\{\zeta_i(n),\, i \in \mathcal{I}\}$ has the same distribution as the classical Pólya urn with K different types of balls.

(A slightly more restrictive result holds if $X_i(0) \geq 1$ for all i.)

An interesting observation is that the embedded Markov chain can be also regarded as a *Pólya-type urn model with removals* (for some extensive results of multi-type Pòlya urns see e.g. [5]). Indeed, w.l.o.g. assume that α and all a_{ij} are integers. Consider a Markov chain $Y(n) = (Y_1(n), \dots, Y_N(n)) \in \mathbb{Z}_+^N$, $n \in \mathbb{Z}_+$, where $Y_i(n)$ represents a number of balls of type $i = 1, \dots, N$ in a urn. The dynamics of the model is as follows. Suppose an urn contains $Y_i \geq 1$ balls of type $i \in \{1, 2, \dots, N\}$. Pick a ball of type i with probability proportional to Y_i, and then return it to the urn with α additional balls of the same type; at the same time for each $j \neq i$

remove a_{ji}, Y_j balls of type j (or just *all available balls* of type j if there are less of them than this quantity). Similar processes were considered in [2] and [4].

It turns out that the above result holds for both linear competition process and the urn model with removals.

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Limits of Pólya urns with innovations

Jean Bertoin

At each step of a Pólya urn scheme, a ball is drawn uniformly at random from an urn, independently of the preceding steps. One observes its color; then the ball is returned in the urn together with a random numbers of balls of different colors according to some fixed distribution that depends only on the color of the sampled ball. Most often, it is assumed that the set of all possible colors is finite, with the notable exception of certain recent contributions that include [1, 2, 3, 5, 6, 8, 9]. The quantities of interest are the proportions of balls with given colors after a large number of steps.

We are interested in a variation of the Pólya urn scheme which incorporates innovations, in the sense that at each step, balls with new colors that have never been used before can be returned in the urn, and the space S of colors is an arbitrary measurable space. A typical replacement consists of a pair (C, ξ), where C is a random variable with values in $\{-1, 0, 1, 2, \ldots\}$ which represents the number of copies of the sampled ball which are returned in the urn (the case $C = -1$ accounts for the situation where the sampled ball is removed from the urn), and ξ a point process on S which represents the random family of new balls which are simultaneously added. The dynamics are hence fully encoded by the kernel of laws $(P_s)_{s\in S}$ that specify the distributions of the pair (C, ξ) as a function of the color s of the sampled ball.

We make two key assumptions. First, we suppose that the average number of copies that are returned at a typical step does not depend on the sampled color, viz.

$$\text{the function } s \mapsto E_s(C) \text{ is constant on } S, \tag{1}$$

where the notation E_s is used for the mathematical expectation under P_s. Second, we suppose that there exists a measurable function a on S that is bounded away from 0 and from ∞, such that for every $s \in S$, the intensity measure of the point processes ξ under P_s is given by

$$E_s(\xi(f)) = a(s)\mu(f), \qquad \text{for all } f \in \mathcal{L}^\infty(S). \tag{2}$$

In words, a is a factor which modulates the intensity of innovations as a function of the color of the sampled ball.

An important result for urn schemes with finitely many colors is that the first order asymptotic of the contain of the urn as the number of steps goes to infinity is determined by the largest eigenvalue of the mean replacement matrix and its eigenvectors. The same feature holds in the present setting; it implies the almost-sure convergence of the empirical distribution of colors towards μ. It is well-known for classical urn schemes with finitely many colors, that the fluctuations of the empirical distributions of colors in the urn depend crucially of whether the largest eigenvalue of the mean replacement matrix is larger or smaller than twice the real part of the second largest eigenvalue; see [4, 7]. This incites us to introduce

$$\lambda_1 := E_s(C) + \mu(a) \quad \text{and} \quad \lambda_2 := E_s(C), \tag{3}$$

and set

$$\rho := \frac{\lambda_2}{\lambda_1} \in (-\infty, 1). \tag{4}$$

In particular, $\rho > 1/2$ if and only if $E_s(C) > \mu(a)$, which we interpret as reinforcement being stronger than innovation. The main result presented in the talk is:

Theorem. Let $\bar{U}_n$ denote the empirical distribution of colors in the urn after n steps. Assume (1), (2), and some further mild technical conditions.

(i) If $\rho > 1/2$, then we have for every $f \in \mathcal{L}^\infty(S)$:

$$\lim_{n\to\infty} n^{1-\rho}\left(\bar{U}_n(f) - \mu(f)\right) = L_f \qquad \text{a.s.},$$

where L_f is some non degenerate random variable.

(ii) If $\rho < 1/2$, then the sequence of processes

$$\left(\lambda_1\sqrt{(1-2\rho)n}\left(\bar{U}_n(f) - \mu(f)\right) : f \in \mathcal{L}^\infty(S)\right), \qquad n \geq 1$$

converges in the sense of finite dimensional distributions to a Gaussian bridge $G^{(\mathrm{br})}$ indexed by $\mathcal{L}^\infty(S)$.

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The superdiffusive limit of the elephant random walk

Lucile Laulin

(joint work with Hélène Guérin and Kilian Raschel)

The **Elephant Random Walk (ERW)** was first introduced by Schütz and Trimper [6] in order to investigate the long-term memory effects in non-Markovian random walks. It is a one-dimensional discrete-time random walk on integers, which has a complete memory of its whole history. It was referred to as the ERW in allusion to the famous saying that elephants can remember where they have been.

Over the last decade, the ERW and other processes derivated from it have received a lot of attention from mathematicians. The one-dimensional ERW is defined as follows. The random walk starts at the origin at time zero, $S_0 = 0$. At time $n = 1$, the elephant moves to the right with probability q and to the left with probability $1 - q$ where q lies between zero and one. Hence, the position of the elephant at time $n = 1$ is given by $S_1 = X_1$ where X_1 has a Rademacher $\mathcal{R}(q)$ distribution. Afterwards, at any time $n \geq 2$, the elephant chooses uniformly at random an interger k among the previous times $1, \ldots, n$, and

$$X_{n+1} = \begin{cases} +X_k & \text{with probability} \quad p, \\ -X_k & \text{with probability} \quad 1-p, \end{cases}$$

where the parameter $p \in [0, 1]$ is the memory of the ERW. Then, the position of the ERW is given by

$$S_{n+1} = S_n + X_{n+1}.$$

In particular, when $p = 1/2$, the ERW reduces to the simple RW.

The ERW appears to have three regimes of behavior, depending on wether $p < 3/4$ (diffusive), $p = 3/4$ (critical) or $p > 3/4$ (superdiffusive), see below for the main results.

	Diffusive	**Critical**	**Superdiffusive**
LLN	$\frac{S_n}{n} \xrightarrow[n\to\infty]{\text{a.s.}} 0$	$\frac{S_n}{\sqrt{n}\log n} \xrightarrow[n\to\infty]{\text{a.s.}} 0$	$\frac{S_n}{n^{2p-1}} \xrightarrow[n\to\infty]{\text{a.s.}} L$
CLT	$\frac{S_n}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{3-4p}\Big)$	$\frac{S_n}{\sqrt{n\log n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}(0,1)$	$\frac{S_n - n^{2p-1}L}{\sqrt{n}} \xrightarrow[n\to\infty]{\mathcal{L}} \mathcal{N}\Big(0, \frac{1}{4p-3}\Big)$

There are multiple approaches to study the ERW, but two of them have been used the most. In 2016, Baur and Bertoin [1] used the **connection to Pólya-type urns** with random replacement as well as two functional limit theorems for multitype branching processes due to Janson [4]. They established almost sure and functional convergences. In 2017, Bercu [2] used **martingale theory** to prove the law of iterated logarithm and quadratic strong law, as well as the central limit theorem and law of large numbers. In 2019, Kubota and Takei [5] used martingale theory to prove that in the superdiffusive regime, the fluctuations around the almost sure limiting random variable are Gaussian. It was already known that L was not Gaussian [2].

One of the main question regarding the ERW is the law of the superdiffusive limit. In a forthcoming work, we establish that the limit L has a density supported by the whole real line, and we give a recursive relation for computing all of its moments. To do so, we use the connection with Pólya-type urns and we rely on the work of Chauvin et al. [3] for two-colors Pólya urns with deterministic replacement. Our strategy is to obtain a distributional equation satisfied by the limit.

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Extremal linkage networks

Christian Hirsch
(joint work with Markus Heydenreich)

The human brain is one of the most prototypical real-world example of a complex network. In order to process sensory information, the brain relies on a system of neuronal cells that are linked by synapses of varying strengths. Some of the neurons can be considered hubs that have the ability to connect to a large number of other neurons. Moreover, it is important to have small distances in the sense that typically neurons can be connected in a small number of hops

One of the key questions in neuroscience is to identify the mechanisms that lead to the evolution of such complex neuronal networks. This is an attractive goal as it offers the potential to gain insights on how to design effective learning architectures for artificial neuronal networks.

In this context, neuroscientists proposed the *tabula rasa hypothesis* [7]. According to this suggestion, at the beginning of the development, the brain exhibits an enormously large number of possible connections. Only through the process of perception, sensory experience and learning, this network develops into a sparse network of important functional connections.

It is at this point, where we draw a connection to the core topic of the workshop, namely *reinforced graphs.* Loosely speaking, the idea is to develop a model based on the phenomenon that when a neuron fires, it is more likely that we strengthen synapses with a high weight with a higher probability since they have been already used successfully in the passed.

One early mathematical model in this direction are Pólya urns with graph-based competition that were introduced in [6] and further studied in [1, 4, 5]. However, these typically lead to a large number of small isolated components, thereby failing to reflect the structure of the brain.

To address these shortcomings, in [2] a model was proposed that gives rise to a layered structure with short distances. However, this model relies both on a process of external node fitnesses as well as on a very particular choice of a base network and interaction of fitness and weights.

To address this issue, the *extremal linkage networks* were introduced in [3], which we now describe. The base network has nodes (neurons) of the form $V_N = (\mathbb{Z}/N) \times \mathbb{Z}_{\ge 0}$ for some $N \ge 1$. Moreover, $\{F_v\}_{v \in V_N}$ is a collection of iid node fitnesses that are Fréchet distributed with some parameter $\delta > 0$. That is, $\mathbb{P}(F_v \le r) = e^{-r^{-\delta}}$. The fitnesses have two purposes. First, they determine the *scope* of the considered vertex. That is, a vertex $v = (i, h)$ of fitness F_v can possibly connect to vertices of distance at most F_v in the next layer $h + 1$. Moreover, v connects to precisely to one vertex in the next layer, namely the one with the largest weight among all vertices in the scope.

We now discuss the key results of [3] for the parameter $\delta = 1$, where the model is most similar to the one described in [2]. For $\delta = 1$ the typical in-degrees converge in distribution as $N \to \infty$ to a random variable with exponential tails. Moreover, if

H_N denotes the layer of the most common ancestor of two typical nodes in the base layer of V_N, then $H_N/\log_\mu(N) \to 1$ in probability, where $\mu := \exp(\mathbb{E}[\log(2F)])$.

From the point of modelling, one of the disadvantages of extremal linkage is that the construction relies on externally assigned fitnesses. It would be more natural if the evolution of such weights could be explained by a mechanism of self-organization. A promising candidate in this direction could be to tie the fitness to the number of reinforcements like in the following suggestion.

(1) Start with nodes of fitness 1. That is $W_0(v) = 1$.
(2) Fire iid at rate $W_t(v)/t$. Except if v is at the bottom layer, then it fires at rate 1.
(3) In the next layer, select the vertex of maximal weight among all vertices in the range $W_t(v)/t$ and add weight 1.

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$\star$-Reinforced Random Walks, Bayesian Statistics and Statistical Physics

Pierre Tarrès

(joint work with S. Bacallado, C. Sabot)

Consider a directed graph $G = (V, E)$ endowed with an involution on the vertices denoted by $*$, and such that

$$(i,j) \in E \iff (j^*, i^*) \in E. \tag{1}$$

A discrete Markov Chain on G with transition probability $p(i,j)$ from i to j, and endowed with that involution $*$, is called Yaglom reversible if and only if there exists a probability measure π on V such that, for all $i, j \in V, i \sim j$,

$$\pi(i,j) := \pi(i)p(i,j) = \pi(j^*)p(j^*,i^*) = \pi(j^*,i^*),$$
$$\pi(i) = \pi(i^*).$$

This implies in particular that π is an invariant measure for that Markov Chain.

Consider for instance the reversible k-dependent Markov Chain, i.e. the random processes Y such that the law of Y_{n+1} depends only on $(Y_{n-k+1},\dots,Y_n)$. It induces a Markov chain (X_n) on the (directed) de Bruijn graph $G=(V^k,E)$ with

$$\mathrm{w}=(i_1,\dots,i_k)\to\tilde{\mathrm{w}}=(i_2,\dots,i_{k+1})$$

with transition rate $p(\mathrm{w},\tilde{\mathrm{w}})$, and invariant measure $\pi(\mathrm{w})$. The classic reversibility assumption of Y

$$(Y_1,\dots,Y_n)\overset{law}{=}(Y_n,\dots,Y_1),\ \text{if } (Y_1,\dots,Y_k)\sim\pi$$

corresponds to the Yaglom reversibility of X on de Bruijn graph with involution $*$ with

$$\mathrm{w}=(i_1,\dots,i_k)\mapsto\mathrm{w}^*=(i_k,\dots,i_1)\ \text{flipped } k\text{-string}.$$

Note that the dependence of the Random Walk on the past could also be chosen to be of variable-order, with context set $\mathcal{C}\subseteq S\cup S^2\cup\dots\cup S^k$ on de Bruijn graph: for all $(i_1,\dots,i_\ell)\in\mathcal{C}$, the transition probabilities out of x and y would be the same whenever x and y both end in $(i_1,\dots,i_\ell)$. A generalization of the latter is the Random Walk with amnesia, on $G=(V,E)$ defined by $V=S\cup S^2\cup\dots S^k$ with two types of edges: "forgetting" ones $(i_1,\dots,i_m)\to(i_2,\dots,i_m)$, if $m>1$, and "appending" ones $(i_1,\dots,i_m)\to((i_1,\dots,i_m,j)$, for each $j\in V$, if $m<k$.

Given initial weights $\alpha_{i,j}>0$, $(i,j)\in E$ such that $\alpha_{i,j}=\alpha_{j^*,i^*}$, a process $(X_n)_{n\in\mathbb{N}}$ on a directed graph $G=(V,E)$ endowed with an involution $*$ on the vertices is called a $\star$-Edge Reinforced Random Walk ($\star$-ERRW) if $X_0=i_0$ for some $i_0\in V$ and, for all $n\in\mathbb{N}$, $j\sim X_n$,

$$\mathbb{P}^{i_0,\alpha}(X_{n+1}=j\mid X_0,X_1,\dots X_n)=1_{X_n\to j}\frac{Z_n((X_n,j))}{\sum_{l,X_n\to l}Z_n((X_n,l))}$$

where

$$Z_n((i,j))=\alpha_{i,j}+N_{i,j}(n)+N_{j^*,i^*}(n)$$
$$N_{i,j}(n)=\sum_{k=1}^{n}1_{\{(X_{k-1},X_k)=(i,j)\}}.$$

The $\star$-ERRW can be seen as a generalisation of the Edge-Reinforced Random Walk (ERRW) introduced in the seminal work of Diaconis and Coppersmith [2], corresponding to the case where $\star$ is the identity map.

Let div be the divergence operator from $\mathbb{R}^E$ to $\mathbb{R}^V$ defined by

$$\mathrm{div}(z)(i)=\sum_{j,i\to j}z_{i,j}-\sum_{j,j\to i}z_{j,i}.$$

One can show that, for all $i_0\in V$, if $\mathrm{div}(\alpha)=\delta_{i_0^*}-\delta_{i_0}$, then the $\star$-ERRW starting from i_0 is partially exchangeable, that is, the probability of a path only depends on its starting point and on the number of crossings of directed edges.

It follows from a result of Diaconis and Freedman [3] that any partially exchangeable process can be seen as a mixture of Markov chains P^ω. That result allows one to make use of the $\star$-ERRW to do Bayesian statistics on the unknown parameter ω. Indeed, if we let $\mathcal{L}(i_0,\alpha)$ be the mixing measure on ω from $\mathbb{P}^{i_0,\alpha}$, then the distribution of ω after n first steps is given by $\mathcal{L}(X_n, (Z_n(e))_{e\in E})$: in other words, the posterior is in the same family of distributions as the prior, given by the mixing measure of the $\star$-ERRW given initial conditions, and thus they are conjugate priors.

Next, we explain a new argument that allows to guess the mixing measure (which we eventually compute explicitly), based on that Bayesian approach, see the OOPS minicourse on YouTube (2019) for more details.

Then we note that the $\star$-ERRW can be seen (at jump times) as mixture (for random weights β) of the so-called $\star$-Vertex-Reinforced Jump Process ($\star$-VRJP), which jumps from i to j at time t a rate $\beta_{ij}L_j^*(t)$, where

$$L_j(t) = 1 + \int_0^t 1_{\{Y_s=j\}}$$

is the local time (plus one) spent by the process at site j. This generalizes the result obtained with Sabot [4] on nonoriented graphs.

Contrary to the VRJP, the $\star$-VRJP is in general not exchangeable. Now, after an adequate randomization of the initial local time, it becomes partially exchangeable: we compute the mixing measure of that randomized $\star$-VRJP. We can describe the non-randomized $\star$-VRJP as a mixture of conditioned Markov Jump Processes. Finally, we show a link between the $\star$-VRJP with a random Schrödinger operator, which generalizes the one previously obtained in [6, 7].

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