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Patterns and waves in theory, experiment, and application

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In this snapshot of modern mathematics we describe some of the most prevalent waves and patterns that can arise in mathematical models and which are used to describe a number of biological, chemical, physical, and social processes. We begin by focussing on two types of patterns that do not change in time: space-filling patterns and localized patterns. We then discuss two types of waves that evolve predictably as time goes on: spreading waves and rotating waves. All our examples are motivated with real-world applications and we highlight some of the main lines of research that mathematicians pursue to better understand them.

1 Introduction

Imagine yourself sitting in your favourite coffee shop. Start by thinking about what is on the walls. Is there wallpaper or an illustration with complex geometric patterns on it? Maybe there is some artwork from a local artist that features spots or wavy lines arranged repetitively. What about the floor? You might see square or hexagonal tiles with colours that repeat themselves in a predictable way. Think about the espresso machine with its plume of steam that rises vertically at an almost constant speed. Think about stirring your coffee in its cup, creating a constant swirl that looks like a whirlpool swallowing your spoon. What's fascinating here is that even inside a coffee shop we can find many illustrative examples of patterns and waves that arise naturally throughout the physical sciences.

It is exactly the emergence of these patterns and waves in nature that has perplexed mathematicians for centuries. In this snapshot, we will focus on two kinds of patterns and two kinds of waves that continue to receive attention from the mathematical community due to their ubiquity in nature. We will begin by discussing *space-filling patterns*. These are repeated, geometric tilings that spread all over a space. Think of the floor tiles in the example above. In nature these patterns arise, for example, as a tiger's stripes or a leopard's spots. However, not all patterns completely fill the space they are given. So another type of pattern we will consider are those confined to a small region of a given space - they are *localized*. Like a stain on an otherwise spotless white shirt, localized patterns typically present themselves as small imperfections on a featureless background. Think of a lush oasis in the middle of an otherwise inhospitable desert.

As mentioned above, as well as static patterns in space, there are many patterns, such as waves, that move or change in time. We will discuss two kinds of waves: *invading/receding waves* and *rotating waves*. The former may come in the form of a cold weather front that relentlessly advances bad weather toward you. The latter rotate around some point in space, typically creating a distinctive spiral pattern, as can be seen, for instance, in this video of hurricanes filmed from above with their eye at the centre.

2 Space-Filling Patterns

The study of pattern formation in mathematics goes back at least to the seminal work of Alan Turing in the 1950s. His paper "The chemical basis for morphogenesis" provided a theoretical framework to describe the formation of stripes, hexagons, spirals, and other patterns in nature [6]. The models presented by Turing, referred to as "reaction-diffusion" equations, are typically associated with chemical reactions, but can also be used to describe biological, physical, geological, and even social processes. Such mathematical models are just as relevant today as they were when Turing first proposed his theory and they are continually being applied to describe more and more pattern-forming phenomena in the applied sciences.

To illustrate the main ideas of his theory, let us think about a chemical reaction between two or more substances taking place in a small dish or on an animal's skin, which we think of as a two-dimensional space with spatial variables x and y so that every point in such a space is described by a pair (x, y).

One might expect that these spatially-extended reactions mix in such a way that the concentration of each chemical everywhere throughout space is approximately the same. Turing showed that this need not be the case. Indeed, if the substances spread throughout space at different rates, the concentrations of each substance can be different in different regions of space. In particular, Turing showed that if the substances only vary in one spatial dimension, for example only along the x-direction, then the concentration of the substances at a point (x, y) only depends on the variable x and can be approximately estimated using a function of type $\cos(kx)$. Here k is a real number, referred to as the *wave number*, and it depends on the physical and chemical properties of the substances in the experiment, as well as on its physical set-up. Following similar mathematical analysis, we find that when the substances vary in both of the two spatial dimensions, the concentration of the substances at a point (x, y) in space depends on both x and y and is approximately estimated by the values of a sum of functions of type $\cos(k_1x + k_2y)$, where $k_1^2 + k_2^2$ always has the same value. Again, k_1 and k_2 are real numbers which describe the spatial variation of the concentration in the x- and y-directions respectively, and they are also referred to as wave numbers. These resulting spatial patterns are now referred to as *Turing patterns*, in honour of Alan Turing who first discovered them.

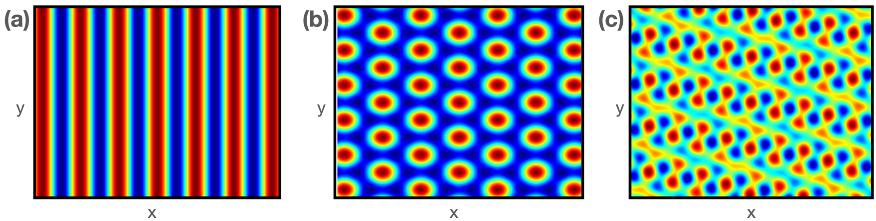

Figure 1: Contour plots of space-filling Turing patterns. Red values are high and blue values are low. (a) Stripes vary in only one spatial dimension, represented by the function $\cos(x)$. (b) Hexagons are the sum of cosine functions $\cos(x) + \cos((x + \sqrt{3}y)/2) + \cos((x - \sqrt{3}y)/2)$. (c) A Turing pattern with $\cos(x)+\cos((x+\sqrt{3}y)/2)+\cos(0.8896x-0.4567y)$, differing from the hexagon pattern by only one cosine function with randomly chosen wave numbers.

We are able to construct fascinating space-filling patterns using different values of k_1, k_2 and adding together different $\cos(k_1x+k_2y)$ functions. In Figure 1 we present three such patterns: (a) stripes that vary in only one dimension, (b) hexagons, and (c) a complicated pattern using only three different cosine functions. As one can see from this figure, these patterns closely resemble the regular patterns that we see in nature, especially on the skin of animals. However, despite the fact that Turing's theory dates back to the 1950s, there is still a lot of work that remains to be done in understanding these space-filling Turing patterns. For example, how do we construct reaction-diffusion models that can accurately reproduce a specific pattern observed in a physical system or in nature? That is, why do certain patterns get selected over others? We can also turn this question around and ask how we can manipulate the system to produce specific patterns: imagine you are given a chemical reaction in a petri dish and you could manipulate the system just so to create stripes, or hexagons, or any other pattern you desire. Finally, all of Turing's results are achieved under the assumption that space is homogeneous, meaning there is nothing special about one point over another. It is natural to ask what happens when this isn't the case. This is something we would like to consider if we were trying to explain the emergence of vegetation patterns where the soil quality is vastly different from one region to another.

3 Localized Patterns

Let us start by imagining we are monitoring a population of some animal within a very vast territory – for example, rabbits. If there are very few rabbits, then without considering any external factors, we would expect the population to go extinct since it would be hard to find a mate to reproduce at a fast enough rate to continue on. On the other hand, if there are a lot of rabbits, we expect the population to continue to grow until the surrounding environment cannot support any more of them. Therefore, without changing anything in this theoretical system, we have two possible outcomes, or *states*, depending entirely on how many rabbits we start with. In many systems that exhibit Turing patterns, we see something similar, that is, homogeneous/featureless states are possible under the same conditions that lead to patterned states - what determines which state is selected is how the system is set up initially. We can think of such a situation as a competition between states.

Although Turing patterns have been studied for decades, it is only recently that the mathematical community has noticed that, in competitive scenarios, Turing patterns can be localized to a small region of a given space [4]. This means that many models describing chemical, biological, physical, or even social processes can support localized patterns that resemble a Turing pattern in a bounded region of space, while remaining relatively homogeneous or featureless outside that region. A well-documented example of this is that of crime hotspots, representing regions in cities with high rates of criminal activity that are highly localized to certain blocks and neighbourhoods. In nature we find such localized patterns in patterns of vegetation (such as the oasis example from the introduction), chemical reactions, and *ferrofluids* which are magnetic liquids that can hold a variety of patterns when exposed to a magnetic field (see, for example, this video).

In Figure 2 we illustrate localized patterns that can be found as solutions to mathematical equations. When the system is only able to vary in one spatial dimension, the localized patterns have a rolling patch resembling a cosine function in their centre, which eventually reduces to a featureless background state as one moves away in either direction. In two or more spatial dimensions localized patterns can be much more intricate, coming from the fact that Turing patterns are significantly more complicated in higher dimensions. There are many open problems related to localized patterns. First, it is still unknown which patterns are possible in two and higher spatial dimensions, making this an area of significant mathematical research. Second, a lot of research has been dedicated to understanding the exact conditions that lead to this competition in our mathematical models, so that we can recognize it in a variety of pattern-forming systems. Third, localized patterns can only exist in systems where there is competition between states, and so it is important to understand the response to the change of the external settings of a system to see how these patterns react. In some cases, changing parameters destroys the competition in the system. The result is often that either the localized central patch expands outwards forever, consuming the featureless pattern, or the localized patch collapses on itself, eventually resulting in a completely featureless state. Knowing which state wins out can have significant repercussions on the ability of the model to form patterns in the real-world, since one can then design physical systems that exhibit or inhibit patterns.

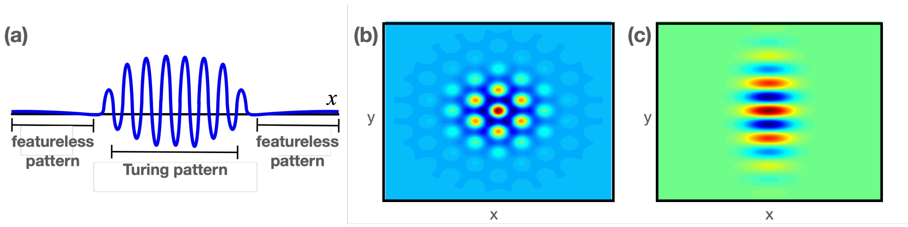

Figure 2: Localized patterns take the form of Turing patterns in a confined region of space, while outside of this region they are featureless. (a) In one spatial dimension localized patterns look like a sinusoidal function in the middle and are constant away from the centre. In two spatial dimensions localized patterns can take more intricate forms, such as (b) hexagons or (c) elongated stripes.

4 Spreading Waves

Let us now move on to processes that change with time in a regular way and let us consider the example of an infectious disease spreading across the US state of Virginia. In the north-east corner of Virginia is the Washington DC metropolitan area, which is one of the largest population centres in North America and an international hub, due to its political significance. One would then expect that an infectious disease would start where the population density is high, such as in the northeast, and slowly spread over the entire state from there, as illustrated in Figure 3.

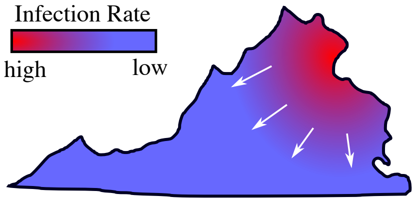

Figure 3: A theoretical infectious disease spreading from the Washington DC metropolitan area over the whole of Virgina.

From a mathematical perspective, there are a number of questions that arise from this situation. First, how can we model such disease spread? Let us start with just the spatial domain. We could simply consider ourselves on a two-dimensional domain that represents the whole of Virginia, or we could simplify our model by dividing the state into distinct regions. This is known as "coarse-graining" the domain. For example we could consider the congressional district of Virginia to be points, or nodes, connected by edges only if they share a border, as in Figure 4. After choosing how to model our domain, we could then ask how fast the disease spreads in it. Numerous studies have shown that there are small, but measurable, differences between the modeled spread of the disease when we use a continuous spatial domain versus when we coarse-grain the domain [5]. Knowing which model offers better predictions is case-dependent, and typically requires comparisons with available historical data. Hence, we see that there are both modelling and mathematical complexities that can arise when attempting to understand naturally occurring patterns and waves. There is no simple answer as to how to simulate such phenomena, and so mathematicians research each possible model independently to compare the outcomes and note their discrepancies.

Spreading waves can be observed in many places other than disease propagation. We recall that in the previous section we said that when we perturb

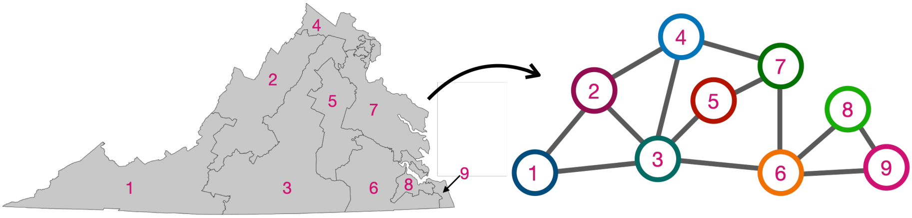

Figure 4: We can use the congressional districts of Virginia to coarse-grain the state of Virginia. The result is a set of nodes, representing the different districts, and edges connecting them. Edges are only present between nodes if they share a border.

models so as to destroy the competition between Turing and featureless states, one state typically overtakes the other as time goes on. Current research is investigating this on a theoretical level by initiating models with half of the domain taken to be a Turing pattern and the other half taken to be a featureless state. One possibility is illustrated in Figure 5 where the Turing pattern advances from left to right as time goes on, eventually replacing the featureless state everywhere in space. Questions abound in relation to this kind of wave propagation. Does one state advance into the other with constant speed? If the speed is constant, what is it? What conditions are necessary in the model for one state to overtake the other? How can we manipulate this invasion process to slow down or even stall the advance? Recent work is trying to answer many of these questions using phenomenological pattern-forming mathematical models [1], but moving to more complicated and realistic models remains a challenge due to the complexity of the methods involved. Finally, we can ask ourselves how to mathematically predict how spreading waves interact with each other when they collide. A real-world example of this can be observed in this video of the Belousov–Zhabotinsky chemical reaction in a petri dish, where we can see the spreading rings bounce off each other and attach to create more complex patterns.

5 Rotating Waves

In the previous section we discussed waves that move outward linearly. Another type of wave continually rotates around a fixed point in the domain. We refer to these waves as rotating waves and they typically come in the form of spirals, as illustrated in Figure 6. From a mathematical perspective, far less is understood about spiral rotating waves than the other patterns and waves

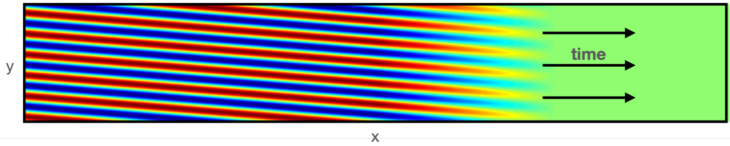

Figure 5: A striped Turing pattern propagating into and overtaking a featureless state. As time goes on, the stripes advance to fill the domain from left to right.

discussed here. Mathematicians have been able to show some ways in which they arise as solutions to mathematical equations, however these studies are primarily focussed on phenomenological models that are significantly simpler than those that describe important chemical or biological processes. One such biological process where these waves can be observed is on the surface of fertilized starfish eggs, where they are produced by billions of activated proteins that tell the egg to start dividing and begin the formation of the organism. See the press article [2] for a non-technical overview of the work being done in this direction.

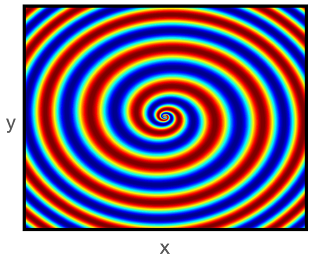

Figure 6: A spiral wave rotates with constant speed about its centre.

Many mathematical studies of spiral waves have been limited to a single spiral which is rotating and fills the entire spatial domain. However, studies employing phenomenological models have shown that if the domain has isolated blemishes, such as dead or defective cells on the surface of biological tissue, then the spiral will anchor itself on these blemishes [3]. What is far less understood is how multiple spiral waves interact with each other. For example, the Belousov–Zhabontinsky chemical reaction mentioned earlier is also known to support spiral waves that rotate and sometimes wander over the domain. This chemical reaction gives experimentalists a tabletop experiment that generates spiral waves to study and manipulate to better inform the scientific community of their behaviour in more complex processes, such as on the surface of fertilized eggs. One such experiment can be observed in this video where we see that the spirals push ripples outwards in every direction, leading to collisions with ripples generated by other spirals and eventually self-organizing into a complex pattern that is being continually generated by the spirals near the centre. Spiral waves have proven themselves to be significantly more difficult to study using traditional mathematical tools, and so the community's understanding of them lags far behind that of spreading waves. One example of where we are still catching up is to simply identify the systems spiral waves can be found in and determining what exactly this means for the application of the associated models.

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