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Incidence Problems in Harmonic Analysis, Geometric Measure Theory, and Ergodic Theory

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Abstract. The workshop *Incidence Problems in Harmonic Analysis, Geometric Measure Theory, and Ergodic Theory* covered interactions between geometric problems involving fractals, dimensions, patterns, projections and incidences, and on the other hand recent developments in Fourier analysis and Ergodic theory which have been inspired by fractal geometric problems, or have been instrumental in solving them.

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Introduction by the Organizers

Many topics in contemporary Harmonic analysis, Geometric measure theory, and Ergodic theory are related to *incidence problems*. A prototypical example is the following: given a collection of narrow tubes in Euclidean space, or in some more exotic geometry, how often can the tubes come together to create "rich" intersections? Questions of such superficially innocent flavour often turn out to be difficult to solve, but are also equally fundamental. Famously, they underpin two central outstanding problems in contemporary Fourier analysis and geometric measure theory: the *Fourier restriction conjecture* and the *Kakeya conjecture*.

In addition to spectacular progress in the Kakeya and restriction conjectures, recent research has established a breadth of further – often unexpected – connections between Harmonic analysis, Geometric measure theory, and Ergodic theory. The common denominator often tends to be an incidence problem. This problem does not always involve straight narrow tubes, as in the case of Kakeya and restriction, but might be stated in terms of circles or parabolae, fractals, sums, products, distances and patterns generated by fractals, or orthogonal or non-linear projections. The applications of such geometric problems range from decoupling theory, Fourier integral operators, and dispersive partial differential equations to problems involving spectral gaps, or effective equidistribution of random walks in groups. Conversely, ideas stemming from Fourier restriction theory, decoupling theory, and additive combinatorics have substantially contributed to progress in old problems in fractal geometry, for example *Falconer's distance set conjecture*, and the broader quest for finding patterns inside fractal sets.

The explosion of recently discovered connections between Harmonic analysis, Geometric measure theory, and Ergodic theory has greatly accelerated the progress in all the fields. The progress has also demonstrated that a dedicated researcher in any one field will benefit from constantly surveying the others. To make the task more manageable, it is indispensable to organise frequent workshops and conferences where experts with somewhat different specialisations may interact and educate each other, and exchange open problems. The Oberwolfach workshop *Incidence Problems in Harmonic Analysis, Geometric Measure Theory, and Ergodic Theory* was designed with this demand in mind.

Here we list a few major topics of the workshop:

(1) Projections and sum-product phenomena. A seminal result in geometric measure theory is *Marstrand's projection theorem* from 1954, which roughly says that orthogonal projections preserve the dimension of planar sets in almost every direction. Making the words "almost every" precise has turned out to be an influential problem. A topic of particularly high recent activity has been to establish "restricted" versions of Marstrand's theorem, where the available space of projections is smaller than in Marstrand's statement, but the lack of "size" is compensated by "curvature". Sharp results have been recently achieved via decoupling theory. Projection problems are also connected with additive combinatorics and sum-product phenomena via the observation that orthogonal projections of a product set $A \times A$ have the form $A + xA$, $x \in \mathbb{R}$.

(2) Advances in sum-product and projection theory have led to powerful applications on the effective equidistribution of random walks in groups. The idea is roughly that, in suitable groups, sum-product theory can be applied to establish growth and Fourier decay for sufficiently high convolution powers. Surprisingly, even the "restricted projections" problem mentioned in (1) has been recently applied in this context.

(3) Patterns in fractal sets. A fundamental theorem of Roth and Szemerédi states that subsets of the integers of positive density contain arbitrarily long arithmetic progressions. This result can be viewed as a manifestation of a research paradigm stating that "thick" fractal sets in $\mathbb{R}^n$ tend to contain "patterns". Thickness can be measured in various different ways, e.g. Lebesgue measure, Newhouse thickness, or Hausdorff and Fourier dimensions. The patterns may be (multi-dimensional) arithmetic progressions, triangles or general simplices, or non-linear patterns such as

$\{x, x+t, x+t^2\}$. Recent progress in pattern-finding problems has been powered by advances in multi-linear singular integrals, and effective convergence theorems for multiple ergodic averages.

(4) A great deal of research in fractal geometry focuses on finding the dimensions of attractors of dynamical systems. The best-understood case is that of self-similar sets and measures, but even in this case fundamental problems remain open, for example characterising the absolute continuity of *Bernoulli convolutions*, and the *exact overlaps conjecture* on the real line. The case of self-affine sets and measures has recently seen many significant advances, combining techniques from ergodic theory and additive combinatorics. A further topic of great current interest is to establish Fourier decay for dynamically generated measures under optimal conditions.

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Workshop: Incidence Problems in Harmonic Analysis, Geometric Measure Theory, and Ergodic Theory

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Abstracts

Dimensions of arithmetic sums of typical self-affine sets

De-Jun Feng
(joint work with Yu-Hao Xie)

In [1], Falconer introduced the concept of affinity dimension and showed that it gives the precise value of the Hausdorff and box-counting dimensions of a typical self-affine sets under a mild assumption (see also [5]). In an ongoing work, we investigate the Hausdorff and box-counting dimensions of arithmetic sums of typical self-affine sets. Let $\{S_i x + a_i\}_{i=1}^{\ell}$ and $\{T_j y + b_j\}_{j=1}^{m}$ be two affine iterated function systems (IFSs) on $\mathbb{R}^d$. Write $\mathbf{a} = (a_1, \ldots, a_\ell)$, $\mathbf{b} = (b_1, \ldots, b_m)$, $\mathbf{S} = (S_1, \ldots, S_\ell)$ and $\mathbf{T} = (T_1, \ldots, T_m)$. Let $E(\mathbf{S}, \mathbf{a})$ and $E(\mathbf{T}, \mathbf{b})$ denote the attractors of these two IFSs.

Definition 1. *We say that* $\mathbf{S}$ *and* $\mathbf{T}$ *are jointly irreducible if for any two two bases* $\{e_1, \ldots, e_d\}$ *and* $\{f_1, \ldots, f_d\}$ *of* $\mathbb{R}^d$*, there always exist a finite word* I *on the alphabet* $\{1, \ldots, \ell\}$ *and another finite word* J *on the alphabet* $\{1, \ldots, m\}$ *such that for every* $k \in \{1, \ldots, d-1\}$,

$$\{S_I e_i : \ 1 \le i \le k\} \cup \{T_J f_j : \ 1 \le j \le d-k\}$$

is a base of $\mathbb{R}^d$*. Here* $S_I = S_{i_1} \cdots S_{i_n}$ *for* $I = i_1 \ldots i_n$.

It is worth pointing out that whenever $d = 2$, the matrix tuples $\mathbf{S}$ and $\mathbf{T}$ are jointly irreducible if and only if not all the matrices S_i, T_j share a common real eigenvector.

Let $\dim_{\mathrm{AFF}} \mathbf{S}$, $\dim_{\mathrm{AFF}} \mathbf{T}$ denote the affinity dimensions of $\mathbf{S}$ and $\mathbf{T}$, respectively (see [1] for the definition of affinity dimension). One of our main results is the following.

Theorem 2. *Assume that* $\|S_i\| < 1/2$, $\|T_j\| < 1/2$ *for all* i, j. *Suppose one of the following conditions holds:*

(i) $\dim_{\mathrm{AFF}} \mathbf{S} + \dim_{\mathrm{AFF}} \mathbf{T} \le 1$*; or*
(ii) $\mathbf{S}$ *and* $\mathbf{T}$ *are jointly irreducible.*

Then for $\mathcal{L}^{\ell d} \times \mathcal{L}^{md}$*-a.e.* $(\mathbf{a}, \mathbf{b}) \in \mathbb{R}^{\ell d} \times \mathbb{R}^{md}$,

$$\begin{aligned} \dim_H E(\mathbf{S}, \mathbf{a}) + E(\mathbf{T}, \mathbf{b}) &= \dim_B E(\mathbf{S}, \mathbf{a}) + E(\mathbf{T}, \mathbf{b}) \\ &= \min\{d, \dim_{\mathrm{AFF}} \mathbf{S} + \dim_{\mathrm{AFF}} \mathbf{T}\}, \end{aligned} \tag{1}$$

where $\dim_H$ *and* $\dim_B$ *stand for the Hausdorff and box-counting dimensions, respectively.*

We remark that the second equality in (1) can break down in certain cases when the assumptions in Theorem 2 do not fulfil. Below we give such an example.

Example 3. *Let $T_i = S_j = T$ for all i, j and assume that $\|T\| < 1/2$. We are able to prove that for $\mathcal{L}^{\ell m} \times \mathcal{L}^{md}$-a.e. $(\mathbf{a}, \mathbf{b})$,*

$$\begin{aligned} \dim_H E(\mathbf{S},\mathbf{a}) + E(\mathbf{T},\mathbf{b}) &= \dim_B E(\mathbf{S},\mathbf{a}) + E(\mathbf{T},\mathbf{b}) \\ &= \max\{s \in [0,d] : \ell m\, \phi^s(T) \geq 1\}, \end{aligned} \tag{2}$$

where $\phi^s(\cdot)$ is the singular value function introduced in [1]*. In particular, take $T = \operatorname{diag}(1/3, 1/5)$, and $\ell = m = 2$. A direct check shows that*

$$\dim_{\mathrm{AFF}} \mathbf{T} + \dim_{\mathrm{AFF}} \mathbf{S} = \frac{2\log 2}{\log 3} > 1 + \frac{\log(4/3)}{\log 5} = \max\{s \in [0,d] : \ell m\, \phi^s(T) \geq 1\}.$$

It arises a natural question how to determine the precise dimensions of $E(\mathbf{S},\mathbf{a})+E(\mathbf{T},\mathbf{b})$ for almost all $(\mathbf{a},\mathbf{b})$ when the assumptions in Theorem 2 do not fulfil. We succeed in proving a computable variational formula for the Hausdorff dimension of $E(\mathbf{S},\mathbf{a}) + E(\mathbf{T},\mathbf{b})$ for almost all $(\mathbf{a},\mathbf{b})$ when all the matrices S_i and T_j are assumed to be diagonal. If, in addition, all the diagonal matrices S_i and T_j are dominated, we are able to provide an easily-computed dimensional formula for the Hausdorff and box-counting dimensions of $E(\mathbf{S},\mathbf{a}) + E(\mathbf{T},\mathbf{b})$ for almost all $(\mathbf{a},\mathbf{b})$. Here a diagonal matrix $\operatorname{diag}(c_1,\ldots,c_d)$ is said to be dominated if $|c_1| \geq \cdots \geq |c_d|$.

The proof of Theorem 2 uses the method of Fourier transform and some transversality techniques developed in [1, 5, 3, 2]. Using a similar approach, we obtain some dimensional properties of typical self-affine sets under orthogonal projections. To be more precise, let $V \subset \mathbb{R}^d$ be a linear space of dimension k and let $P_V : \mathbb{R}^d \to V$ denote the orthogonal projection onto V. Among other things, we show that under the assumptions that $\|S_i| < 1/2$ for all i, and that $\{S_i^{\wedge k}\}_{i=1}^{\ell}$ is irreducible,

$$\dim_H P_V E(\mathbf{S},\mathbf{a}) = \min\{k, \dim_{\mathrm{AFF}}(\mathbf{S})\}$$

for almost all $\mathbf{a}$. Here $A^{\wedge k}$ stands for the k-th exterior product of A.

In a very recent work [4], Pyörälä studied the dimensions of arithmetic sums of concrete planar self-affine sets. Among other things, he proved that if $\{S_i x + a_i\}_{i=1}^{\ell}$ and $\{T_j y + b_j\}_{j=1}^{m}$ are two affine IFSs on $\mathbb{R}^2$ satisfying the strong separation condition, then under the additional assumption that both $\{S_i\}_{i=1}^{\ell}$ and $\{T_j\}_{j=1}^{m}$ are strongly irreducible, the conclusions in (1) (in which $d = 2$) hold except that the following two scenarios occur simultaneously: (a) $\dim_{\mathrm{AFF}} \mathbf{S} > 1 > \dim_{\mathrm{AFF}} \mathbf{T}$ (or $\dim_{\mathrm{AFF}} \mathbf{T} > 1 > \dim_{\mathrm{AFF}} \mathbf{S}$); and (b) the eigenvalues of linear parts T_i, S_j satisfy a certain arithmetic condition. Although Theorem 2 does not apply to any concrete case, it provides a hint that the conclusions in (1) might still hold in the above concrete case even when (a) and (b) occur.

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New bounds for the discretised sum-product problem

William L. O'Regan
(joint work with András Máthé)

Erdős and Volkmann in 1966 [1] showed that for any $\sigma \in [0,1]$ there exists a Borel subgroup of the reals with Hausdorff dimension σ. They conjectured that the same does not hold for Borel subrings, moreover, that there does not exist a Borel subring of the reals with Hausdorff dimension strictly between zero and one. Their conjecture was proved by Edgar and Miller in 2003 [2], using projection theorems of fractal sets. Essentially at the same time, Bourgain independently proved the conjecture [3] via solving the discretised ring conjecture of Katz and Tao from 2001 [4] This is an example of what is referred to as sum-product phenomena, which loosely asserts that a structure is not ring-like.

A classical example of the occurrence of sum-product phenomena is the following theorem from Erdős and Szeremédi from 1983 [5]. They state that there exists an $\epsilon > 0$ and a $C_\epsilon > 0$ such that for every finite subset of integers A at least one of the sumset $A + A$ or the product set $A \cdot A$ is large in the sense that

$$\max(|A + A|,\ |A \cdot A|) \geq C_\epsilon |A|^{1+\epsilon}.$$

Indeed, this asserts that any finite subset of the integers can not even approximately resemble the structure of a ring. They conjectured that a positive constant C_ϵ exists for every $0 < \epsilon < 1$, that is, at least one of $|A + A|$ or $|A \cdot A|$ must be nearly as large as possible.

The discretised sum-product problem (or discretised ring problem) of Katz–Tao is the discretised version of the fractal analogue of the Erdős–Szeremédi problem. Vaguely, it asks/asserts that if $A \subset \mathbb{R}$ behaves like an σ-dimensional set at scale δ in a certain sense, then at least one of $A + A$ and $A \cdot A$ behaves like an $(\sigma + c)$-dimensional set at scale δ (in a different and slightly weaker sense), where the positive constant c should depend only on σ. As previously mentioned, it was first proved in 2003 by Bourgain in, and represented again with weaker non-concentration conditions by Bourgain in 2010 [6]. No explicit bound on the constant was presented. Further examination of Bourgain's papers would suggest that the explicit constant gained following his exact method would be very small. Good explicit constants were gained by Guth, Katz, and Zahl in 2019 [7], by Chen in 2020 [8], and by Fu and Ren [9] in 2022.

The discretised sum-product also has many other applications. For instance it is closely related to the Falconer distance set problem [10] and the dimension of Furstenberg sets. There are also applications to orthogonal projections [11] and fourier decay of measures [12].

The aim of this talk is to provide strong bound for c for the Katz–Tao discretised sum-product problem. We show that c can be taken arbitrarily close to $\sigma/6$ if $\sigma \leq 1/2$ and arbitrarily close to $(1-\sigma)/6$ when $1/2 < \sigma < 1$ [13]. Clearly, c cannot exceed σ nor $1-\sigma$. It is unclear if it is reasonable to conjecture that c can be taken to be (nearly) σ when σ is small (analogously to the Erdős–Szemerédi conjecture).

In the talk we will give the proof. The approach is to start with classical theorems from fractal geometry, or the recent work of Orponen, Shmerkin, and Wang on radial projections [14], that imply that certain arithmetic operations necessarily increase the dimension of any set $A \subset \mathbb{R}$ and then to use information inequalities to extract that simpler arithmetic operations (in this case, addition and multiplication) must already increase the dimension. Bourgain's original proof of the discretised ring conjecture and many improvements since relied on theorems of additive combinatorics (Ruzsa and Plünnecke–Ruzsa inequalities). Our information inequalities make use of both the additive and multiplicative structure of the underlying field. In particular, we also prove a 'ring Plünnecke–Ruzsa inequality'. (We are only interested in the Shannon entropy version of these inequalities.) All these inequalities are immediate corollaries of certain instances of the submodularity inequality, that is, that the conditional mutual information of two random variables given a third is non-negative.

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On the Assouad dimension of the Takagi function

Balázs Bárány
(joint work with Roope Anttila and Antti Käenmäki)

Let $\Phi = \{\varphi_i(x) = A_i x + t_i\}_{i\in\Lambda}$ be a finite collection of invertible, contracting affine maps on $\mathbb{R}^2$ (called iterated function system (IFS)). It is a well-known result of Hutchinson [11] that there exists a unique non-empty compact set X such that $X = \bigcup_{i\in\Lambda}\varphi_i(X)$. The main example we consider in this talk is the graph of the Takagi function.

Let $\lambda \in (0,1)$ and let $b \geq 2$ be an integer. Then the Takagi function is

$$T_{\lambda,b}(x) = \sum_{n=0}^{\infty} \lambda^n \mathrm{dist}(b^n x, \mathbb{Z}).$$

Clearly, $T_{\lambda,b}$ is a 1-periodic map. The original construction is due to Takagi [16] with parameters $b = \lambda^{-1} = 2$ and he showed that the function $T_{1/2,2}$ has no finite derivative at any point. For further properties and discussions, see the survey papers of Allaart and Kawamura [2] and Lagarias [13].

If b is even then the graph of the Takagi function $G_{\lambda,b} = \{(x, T_{\lambda,b}(x)) : x \in [0,1]\}$ is self-affine. For example, for $b = 2$, $G_{\lambda,2} = \varphi_1(G_{\lambda,2}) \cup \varphi_2(G_{\lambda,2})$, where

$$\varphi_1(x,y) = \left(\frac{x}{2}, \lambda y + \frac{x}{2}\right) \text{ and } \varphi_2(x,y) = \left(\frac{x+1}{2}, \lambda y + \frac{1-x}{2}\right). \tag{1}$$

It is easy to see that if $\lambda b > 1$ then $T_{\lambda,b}$ is $\frac{-\log\lambda}{\log b}$-Hölder continuous, furthermore, $\dim_B(G_{\lambda,b}) = 2 + \frac{\log\lambda}{\log b}$, where $\dim_B$ denotes the box-counting dimension. It was shown by B., Rams and Simon [5] (based on B., Hochman and Rapaport [4]) that for every $b \geq 2$ integer and $\lambda \in (b^{-1}, 1)$, $\dim_H(G_{\lambda,b}) = 2 + \frac{\log\lambda}{\log b}$, where $\dim_H$ denotes the Hausdorff dimension.

In this talk, we focus on a third concept of dimension, namely, the Assouad dimension. Let $K \subset \mathbb{R}^2$ be bounded and let $N_r(K)$ be the minimal number of balls of radius $r > 0$ needed to cover K. Then the box-counting dimension is $\dim_B(K) = \lim\limits_{r\to 0+} \frac{\log N_r(K)}{-\log r}$ (if the limit exists, if not then we can define the upper- and lower box-counting dimension by taking limsup and liminf denoted by $\overline{\dim}_B$ and $\underline{\dim}_B$, respectively). Then the Assouad dimension $\dim_A(K)$ is

$$\begin{aligned}\dim_A(K) = \inf\{s > 0 :\ &\text{there exists } C > 0 \text{ such that for every } 0 < r < R \\ &\text{and for every } x \in K,\ N_r(K \cap B(x,R)) \leq C(R/r)^s\}.\end{aligned}$$

Käenmäki, Ojala and Rossi [12] showed that if K is compact then $\dim_A(K) = \max\{\dim_H(T) : T \in \mathrm{Tan}(K)\}$, where $\mathrm{Tan}(K)$ denotes the set of weak-tangent sets of K. Thus, the Assouad dimension of compact sets equals the $\dim^*$-dimension defined by Furstenberg [9]. The Assouad dimension of the attractors of IFSs is recently an active area of research, see [1, 7, 8, 10, 14]. For further properties and discussions, see Fraser [6].

Before we turn to the main result of this talk, we need to define the Furstenberg directions corresponding to the IFS defining the graph $G_{\lambda,2}$. Let

$$X_F = \left\{ \widehat{\begin{pmatrix} 1 \\ y \end{pmatrix}} \in \mathbb{RP}^1 : y \in \left[\frac{-2\lambda}{2\lambda - 1}, \frac{2\lambda}{2\lambda - 1} \right] \right\}.$$

It is easy to see that $A_1^{-1}(X_F) \cup A_2^{-1}(X_F) = X_F$, where A_i are the linear parts of the maps of the IFS defined in (1). For every sequence $\mathbf{i} = (i_1, i_2, \ldots) \in \{1,2\}^{\mathbb{N}}$, there exists a unique $V(\mathbf{i}) \in X_F$ such that $\{V(\mathbf{i})\} = \bigcap_{n=1}^{\infty} A_{i_1}^{-1} \cdots A_{i_n}^{-1}(X_F)$. Moreover, there exists a constant $C > 0$ such that

$$C^{-1} 2^{-n} \leq \|A_{i_n} \cdots A_{i_1} | V(\mathbf{i})\| \leq C 2^{-n} \text{ and } C^{-1} 2^{n} \leq \|A_{i_1}^{-1} \cdots A_{i_n}^{-1} | \theta\| \leq C 2^{n}$$

for every $\theta \in X_F$, every $n \in \mathbb{N}$ and every $\mathbf{i} = (i_1, i_2, \ldots) \in \{1,2\}^{\mathbb{N}}$. These properties allow us to embed the weak-tangent sets of $G_{\lambda,2}$ into $G_{\lambda,2}$ by rank-1 projections and vice versa, to embed the slices of $G_{\lambda,2}$ into some weak-tangent set. Along this line, we obtain the following:

Theorem 1 (Anttila, B., Käenmäki [3]). *For every even $b \in \mathbb{N}$ and $b^{-1} < \lambda < 1$,*

$$\dim_A(G_{\lambda,b}) = 1 + \max_{\substack{\theta \in X_F \\ x \in G_{\lambda,b}}} \dim_H(G_{\lambda,b} \cap \theta(x)) < 2,$$

where $\theta(x)$ denotes the line parallel to $\theta \in \mathbb{RP}^1$ and going through the point $x \in \mathbb{R}^2$.

Computing the dimension of the slices is a very challenging problem. One could conjecture that at least for typical choice of the parameter λ, $\dim_H(G_{\lambda,b}) = \dim_A(G_{\lambda,b})$. However, this is not the case for every pair of λ and b. Yu [17] gave examples for parameters λ, b such that $\dim_H(G_{\lambda,b}) < \dim_A(G_{\lambda,b})$.

In the case of the Takagi function $T_{\lambda,2}$, one can provide equivalent characterisations of the Assouad dimension being equal to the Hausdorff dimension by using a dimension conservation property resembling for the one introduced by Manning and Simon [15]. Let us denote the orthogonal projection from $\mathbb{R}^2$ onto $\theta^{\perp} \in \mathbb{RP}^1$ by proj_θ. Furthermore, let μ be the natural measure on $G_{\lambda,2}$, that is, μ is the push-forward measure of the Lebesgue-measure on the unit interval by the map $x \to (x, T_{\lambda,2}(x))$. For short, let $\mu_\theta := (\mathrm{proj}_\theta)_* \mu$, and let us denote the upper- and lower local dimension of μ_θ at $x \in \theta^{\perp}$ by

$$\underline{d}_{\mu_\theta}(x) = \liminf_{r \to 0} \frac{\log \mu_\theta(B(x,r))}{\log r} \text{ and } \overline{d}_{\mu_\theta}(x) = \limsup_{r \to 0} \frac{\log \mu_\theta(B(x,r))}{\log r}.$$

Theorem 2 (Anttila, B., Käenmäki [3]). *For every $2^{-1} < \lambda < 1$, $\theta \in X_F$ and $x \in G_{\lambda,2}$*

$$\begin{aligned} \underline{d}_{\mu_\theta}(\mathrm{proj}_\theta(x)) + \frac{\log 2}{-\log \lambda} \overline{\dim}_B(G_{\lambda,2} \cap \theta(x)) &= \frac{\log 2}{-\log \lambda}, \text{ and} \\ \overline{d}_{\mu_\theta}(\mathrm{proj}_\theta(x)) + \frac{\log 2}{-\log \lambda} \underline{\dim}_B(G_{\lambda,2} \cap \theta(x)) &= \frac{\log 2}{-\log \lambda}. \end{aligned}$$

This allows us to formulate the following corollary.

Corollary 3. *The following are equivalent:*

- $\dim_A(G_{\lambda,2}) = 2 + \frac{\log \lambda}{\log 2}$,
- *for every* $\theta \in X_F$, $\liminf_{r\to 0} \inf_{x\in G_{\lambda,2}} \frac{\log \mu_\theta(B(\mathrm{proj}_\theta(x),r))}{\log r} = 1$,
- *for every* $\theta \in X_F$ *and every* $q > 1$, *the* L^q*-dimension of* μ_θ *is* 1, *where the* L^q*-dimension is*

$$\liminf_{r\to 0} \frac{\log \int \mu_\theta(B(y,r))^{q-1} d\mu_\theta(y)}{(q-1)\log r}.$$

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Polynomial Fourier decay for smooth images of self-similar measures

Amir Algom

(joint work with Yuanyang Chang, Meng Wu and Yu-Liang Wu)

Let ν be a Borel probability measure on $\mathbb{R}$. For every $q \in \mathbb{R}$ the Fourier transform of ν at q is defined by

$$\mathcal{F}_q(\nu) := \int \exp(2\pi i q x)\, d\nu(x).$$

The motivation behind the work presented in this talk stems from a Theorem of Kaufman [2] from 1984: Recall that for $0 < r < 1$ the corresponding Bernoulli ν_r is defined as the law of the random variable $\sum_{n=0}^{\infty} \pm r^n$, where the $\pm$ are IID Bernoulli-$\frac{1}{2}$ random variables. Kaufman proved that for every $0 < r < \frac{1}{2}$ and every $C^2(\mathbb{R})$ diffeomorphism g such that $g'' > 0$, we have that $g\nu_r$ has polynomial Fourier decay. That is, for some $\alpha > 0$

$$\mathcal{F}_q(g\nu_r) = O\left(\frac{1}{|q|^{\alpha}}\right), \text{ as } |q| \to \infty.$$

This is remarkable, since by a well known Theorem of Erdős (1939), if r^{-1} is a Pisot number then ν_r has no Fourier decay at all. Kaufman's paper was little known, and is tersely written. In 2018, Mosquera and Shmerkin [1] clarified, extended, and quantified this result: Specifically, they extended Kaufman's Theorem to all homogeneous (equi-contractive) self-similar measures on $\mathbb{R}$, and gave explicit bounds on the exponent of the decay (the α).

Since the work of Mosquera and Shmerkin it remained an open problem if one has to assume that the self-similar measure in question is homogeneous. Our main result answers this question negatively; Kaufman's Theorem holds in fact for all self-similar measures:

Theorem 1. *Let μ be a non-atomic self-similar measure, and let $g \in C^2(\mathbb{R})$ be a diffeomorphism such that $g'' > 0$. Then there exists some $\alpha > 0$ such that, as $|q| \to \infty$,*

$$|\mathcal{F}_q(g\mu)| = O\left(\frac{1}{|q|^{\alpha}}\right).$$

The technique used in the proof is flexible enough to yield some interesting corollaries:

(1) All non-atomic self-conformal measures with respect to a $C^{\omega}(\mathbb{R})$ IFS have polynomial Fourier decay, as long as the IFS in question contains at least one non-affine map. This follows by combining our method with a very recent result of Algom, Rodriguez Hertz, and Wang [9].

(2) Let μ be a non-atomic self-similar measure supported on $[1, \infty)$. Then for μ-a.e. x the sequence $\{x^p \mod 1\}_{p \in \mathbb{N}}$ is uniformly distributed. This resolves a recent Conjecture of Baker [8].

Our method relies on a certain large deviations estimate for the Fourier transform proved by Tsuji [3], that extended a previous related estimate of Kaufman [2].

This is morally similar to the arguments of Kaufman and Mosquera–Shmerkin, but the lack of convolution structure presents significant new challenges. Curiously, we make no use of recent methods for Fourier decay such as random walks [5, 6, 7, 11] or additive combinatorics [4, 10].

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Effective methods in geometric measure theory

Marianna Csörnyei

I was trying to explain in my talk some of the recent techniques developed by computer scientists for proving results in geometric measure theory.

First I briefly explained some of the basic definitions.

- For a point $x \in \mathbb{R}^n$, the *Kolgomorov complexity of x with precision r*, denoted by $K_r(x)$, is the length of the shortest program whose output is x with precision r.
- An oracle A is a countable set of information that the program can access 'for free'. The corresponding Kolgomorov complexity is denoted by $K_r^A(x)$.
- The *Hausdorff* and *packing dimension* of x is:

$$\dim_H^A(x) = \liminf_{r\to\infty} \frac{K_r^A(x)}{r}$$

$$\dim_P^A(x) = \limsup_{r\to\infty} \frac{K_r^A(x)}{r}.$$

For every oracle, the Hausdorff and packing dimensions of $x \in \mathbb{R}^n$ are between 0 and n. However, using the oracle corresponding to x, its dimensions become 0. For a general set $E \subset \mathbb{R}^n$:

$$\dim_H^A(E) = \min_A \sup_{x\in A} \dim_H^A(x)$$

and

$$\dim_P^A(E) = \min_A \sup_{x\in A} \dim_P^A(x).$$

This is the 'usual' definition of Hausdorff and packing dimension: I sketched in my talk an argument that the alternative definitions using complexities and oracles above give exactly the same dimensions as the standard ones used in analysis and geometric measure theory.

After that, I explained three recent proofs, by various mathematicians, demonstrating how these notions and techniques can be used for proving results in geometric measure theory. I showed how these techniques were used for proving the following three statements:

(1) For arbitrary sets $E, F \subset \mathbb{R}^n$, and for a.e. $z \in \mathbb{R}^n$, the Hausdorff dimension of $(E+z)\cap F$ is at most $\max(0, \dim(E\times F) - n)$.
(2) For any polyhedral norm, the Hausdorff dimension of the distance set of a set $E \subset \mathbb{R}^n$ is at least $\dim_H E - (n-1)$.
(3) Every planar Besicovitch set has dimension 2.

Thickness and a Gap Lemma in $\mathbb{R}^d$

Alexia Yavicoli

A general problem that comes up repeatedly in geometric measure theory, dynamics and analysis is understanding when two or more "small" compact sets intersect.

My motivation to study intersections comes from studying under which conditions one can guarantee the presence of a pattern (a homothetic copy of a given finite set) in a compact set. This is because there is a homothetic copy of $P = \{p_1, \cdots, p_n\}$ contained in C if and only if there is $a \neq 0$ so that $\bigcap_{1\leq i\leq n}(C - ap_i) \neq \emptyset$.

It is well known that sets of positive Lebesgue measure contain copies of any finite set. On the other hand, Keleti [2] constructed a full dimensional set in the real line that doesn't contain arithmetic progression of length 3. This shows that Hausdorff dimension is not enough to guarantee the presence of patterns such as arithmetic progressions. This poses the problem of finding a different geometrical notion of size that is connected to the presence of arithmetic progressions, as well as other patterns.

In the 1970s, S. Newhouse [3, 4] defined *thickness* on the real line. Thickness is a notion of size of a compact set, and Newhouse gave in his famous *Gap Lemma* a simple checkable robust condition involving thickness that ensures that two compact sets intersect. One issue is that this notion of size is strongly based on the order structure of the reals, which makes it difficult to extend it to higher dimensions. The Gap Lemma is also restricted to the intersection of two sets, so it is still difficult to determine when many compact sets intersect.

In [6] I established a new connection between thickness, winning sets for the Potential Game (a game of Schmidt type) defined in [1]. This allowed me to give a simple thickness criterion to show that many thick compact sets intersect, and I was able to deduce that thick sets contain all finite patterns with size depending on the thickness in a quantitative way.

Joint with Kenneth Falconer [5], we extended of the notion of thickness to "cut-out" sets in higher dimensions. We proved a Gap Lemma and showed the presence of many patterns in thick sets, extending the one-dimensional case. One of the issues with this definition is that we were not able to study totally disconnected sets, which are of great importance in fractal geometry and dynamics. In [7], I was able to give a new definition of thickness in $\mathbb{R}^d$ that can be applied to any compact set. To show that this notion is natural, I proved a Gap Lemma and again showed the presence of patterns in thick sets.

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Orthogonal projections of fractal sets

Shaoming Guo

Let $n \geq 2$. In $\mathbb{R}^n$, let

$$\gamma : [-1, 1] \to \mathbb{R}^n$$

be a smooth non-degenerate curve, that is,

$$\det \left[\gamma'(\theta), \dots, \gamma^{(n)}(\theta)\right] \neq 0$$

for every $\theta \in [-1,1]$. For $1 \leq m \leq n$, define the m-th order tangent space of γ at θ by

$$\mathrm{Tang}_\theta^{(m)} := \mathrm{span}\left[\gamma'(\theta), \dots, \gamma^{(m)}(\theta)\right]$$

Let $\Pi_\theta^{(m)}$ denote the orthogonal projection to $\mathrm{Tang}_\theta^{(m)}$. Let $E \subset [0,1]^n$ be a Borel measurable set. Then it is proven in [3] that

$$\dim\left(\Pi_\theta^{(m)}(E)\right) = \min\{\dim(E), m\}$$

for almost every $\theta \in [-1,1]$. Here dim refers to the Hausdorff dimension.

Low dimensional cases of the above estimate are known before [3]. The case $n = 3, m = 1$ is due to [4] and [5]. Indeed, these two papers also proved very good exceptional set estimates. The case $n = 3, m = 2$ is due to [2]. In [4] and [5], the authors use the approach of incidence bounds; in [2] and [3], the authors use decoupling inequalities that are rooted in the decoupling inequalities of Bourgain, Demeter and Guth [1].

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Sharp Fourier extension from the circle for functions with localized support

Lars Becker

The Tomas-Stein extension inequality for the circle states that there exists a constant $C > 0$ such that for all functions $f \in L^2(\sigma)$

$$\|\widehat{f\sigma}\|_{L^6(\mathbb{R}^2)} \leq C\|f\|_{L^2(\sigma)}\,, \tag{1}$$

where σ is the arc length measure on the unit circle and $\hat{\mu}(\xi) = \int e^{-i\xi x}\,\mathrm{d}\mu(x)$ denotes the Fourier transform of a finite measure μ. We are interested in the conjecture that constant functions are extremizers for this inequality.

This conjecture is motivated by a theorem of Foschi [4], stating that constant functions are extremizers for the Tomas-Stein inequality for the two sphere

$$\|\widehat{f\sigma_2}\|_{L^4(\mathbb{R}^3)} \leq C\|f\|_{L^2(\sigma_2)}\,,$$

where σ_2 denotes the surface measure on the unit sphere in $\mathbb{R}^3$. We outline the proof of Foschi's theorem, and a program by Carneiro, Foschi, Oliveira e Silva and Thiele [3] that adapts parts of it for the unit circle. This program reduces the proof of the conjectured sharp version of (1) to establishing positive semidefiniteness of the quadratic form

$$Q(g) := \int_{(S^1)^6} \delta(\omega_1 + \omega_2 + \omega_3 + \omega_4 + \omega_5 + \omega_6)(|\omega_1 + \omega_2 + \omega_3|^2 - 1)$$

$$(|g(\omega_1, \omega_2, \omega_3)|^2 - g(\omega_1, \omega_2, \omega_3)\overline{g(\omega_4, \omega_5, \omega_6)}) \prod_{j=1}^{6} \mathrm{d}\sigma(\omega_j)$$

on the space of antipodal functions in $L^2(\sigma \times \sigma \times \sigma)$. Here we call a function on $S^1 \times S^1 \times S^1$ antipodal if it is even in each argument.

Numerical computations of Barker, Thiele and Zorin-Kranich [1] verify that Q is positive semidefinite on the space of all antipodal functions with Fourier modes up to degree 120, providing some support for the conjecture. Moreover, their computations show that the eigenfunctions of Q on this space corresponding to small eigenvalues concentrate in space. Hence, understanding the behaviour of Q on functions with localized support seems to be a crucial step for showing positive semidefiniteness of Q.

We present recent work from [2], where we establish positive semidefiniteness of Q on a certain subspace of $L^2(\sigma \times \sigma \times \sigma)$ that contains in particular all functions with localized support. As a corollary, the conjectured sharp version of (1) holds for all functions on the circle that are supported in a $\sqrt{6}/80$-neighbourhood of a pair of antipodal points.

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Quantifying Besicovitch projection theorem

Damian Dąbrowski

Given a compact set $E \subset \mathbb{R}^2$, define its *Favard length* as

$$\mathrm{Fav}(E) = \int_{\mathbb{S}^1} \mathcal{H}^1(\pi_\theta(E))\, d\theta,$$

where $\pi_\theta(x) = \theta \cdot x$ is the orthogonal projection to span(θ). A classical theorem of Besicovitch [1] says the following.

Theorem 1. *If $E \subset \mathbb{R}^2$ satisfies $0 < \mathcal{H}^1(E) < \infty$ and $\mathrm{Fav}(E) > 0$, then there exists a (1-dimensional) Lipschitz graph $\Gamma \subset \mathbb{R}^2$ such that*

$$\mathcal{H}^1(E \cap \Gamma) > 0.$$

This result is called the *Besicovitch projection theorem*, and it is usually stated in the following equivalent way: purely unrectifiable sets with finite length have Lebesgue null orthogonal projection in a.e. direction.

Can this result be made quantitative? The following conjecture is wide open.

Conjecture 2. *If $E \subset [0,1]^2$ is 1-Ahlfors regular and $\mathrm{Fav}(E) \geq C > 0$, then there exist a Lipschitz graph Γ with $\mathrm{Lip}(\Gamma) \lesssim_C 1$ and $\mathcal{H}^1(E \cap \Gamma) \gtrsim_C 1$.*

A weaker version of this conjecture concerning "sets with plenty of big projections" was posed 30 years ago by David and Semmes [4] and recently solved by Orponen [7]. The motivation to study this question comes from the field of quantitative rectifiability: proving this conjecture seems necessary to complete the solution to *Vitushkin's conjecture*, which asks for relationship between Favard length and analytic capacity. Even partial progress on the Conjecture may have applications to this problem, see [2] and [5].

To make progress on the Conjecture, we may start by strengthening the assumption "$\mathrm{Fav}(E) \gtrsim 1$", and assume something better about $\pi_\theta(E)$, or about the projected measure $\pi_\theta \mathcal{H}^1|_E$, than just $\mathcal{H}^1(\pi_\theta(E)) \gtrsim 1$ for many $\theta \in \mathbb{S}^1$. Results of this kind have been shown by Martikainen and Orponen [6], and Orponen [7]. In both papers, one has to assume some nice properties of projections for directions $\theta \in G \subset \mathbb{S}^1$ for a large arc $G \subset \mathbb{S}^1$. Note that the assumption "$\mathrm{Fav}(E) \gtrsim 1$" is equivalent to "there exists a measurable set $G \subset \mathbb{S}^1$ with $\mathcal{H}^1(G) \gtrsim 1$ and such that $\mathcal{H}^1(\pi_\theta(E)) \gtrsim 1$ for all $\theta \in G$." Thus, in order to prove the Conjecture, one needs to be able to deal with measurable sets $G \subset \mathbb{S}^1$, and not just arcs.

In [3] I proved the following result.

Theorem 3. *Let $E \subset [0,1]^2$ be 1-Ahlfors regular with $\mathcal{H}^1(E) \gtrsim 1$. Suppose there exists a measurable set $G \subset \mathbb{S}^1$ with $\mathcal{H}^1(G) \gtrsim 1$ and such that*

$$\|\pi_\theta \mathcal{H}^1|_E\|_{L^\infty} \lesssim 1 \quad \text{for all } \theta \in G. \tag{1}$$

Then, there exist a Lipschitz graph Γ with $\mathrm{Lip}(\Gamma) \lesssim 1$ and $\mathcal{H}^1(E \cap \Gamma) \gtrsim 1$.

The assumption (1) is stronger than the assumptions made on projections in either [6] or [7], but the assumptions made on the set of good directions $G \subset \mathbb{S}^1$ are minimal, while in [6] an [7] the set G had to be an arc. In my talk, I gave an overview of the proofs in [6] and [3], and I explained some difficulties arising from G not being an arc.

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Some remarks on the Mizohata–Takeuchi conjecture

Marina Iliopoulou
(joint work with Anthony Carbery, Bassam Shayya)

Let Σ be a compact patch of a hypersurface in $\mathbb{R}^n$. A major problem in harmonic analysis is the *restriction problem*, set by Stein in the 1960s: understanding the Fourier transform of functions defined on Σ. This is formalised as follows. Define the *extension operator* associated to Σ, denoted by E, as the operator that maps every $g : B^{n-1}(0,1) \to \mathbb{C}$ (where $B^{n-1}(0,1)$ is the unit ball centred at 0 in $\mathbb{R}^{n-1}$) to the function

$$Eg(x) := \int e^{2\pi i \langle x, \Sigma(y)\rangle} g(y)dy, \qquad \text{for all } x \in \mathbb{R}^n.$$

The function Eg is essentially a sum of waves, evolving in different but congruent tubes, whose directions are normal to Σ. The restriction conjecture claims that, when Σ has non-vanishing Gaussian curvature (in which case these normals are all distinct), these waves will interfere very destructively on average, so that

$$\int_{B_R} |Eg|^{\frac{2n}{n-1}} \leq C_\epsilon R^\epsilon \int |g|^{\frac{2n}{n-1}} \qquad \text{for all } R \geq 1, \epsilon > 0,$$

where B_R is the ball centred at 0 with radius R in $\mathbb{R}^n$ (and the contant C_ϵ depends only on Σ and ϵ). The restriction conjecture is only known to hold when $n = 2$.

The Mizohata–Takeuchi conjecture is about L^2-weighted estimates on E, and is open even when $n = 2$. It aims to understand the shape of the level sets of Eg, and specifically how they cluster along lines. It claims that, for every $w : \mathbb{R}^n \to [0, +\infty)$,

$$\int_{B_R} |Eg|^2 w \leq C\|Xw\|_\infty \int |g|^2,$$

where Xw is the X-ray transform of w; that is,

$$Xw(\ell) = \int_\ell w \quad \text{for every line } \ell \text{ in } \mathbb{R}^n$$

(and C is a constant depending only on Σ). The Mizohata–Takeuchi conjecture is a simpler version of another conjecture proposed by Stein, which, together with the Kakeya maximal operator conjecture, would imply the restriction conjecture above.

A possible approach to the Mizohata–Takeuchi conjecture is to try to prove estimates of the form

$$\int_{B_R} |Eg|^{2q} w \leq C_\epsilon R^\epsilon \|Xw\|_\infty \left(\int |g|^2\right)^q,$$

for $q \geq 1$ as close to 1 as possible. We focus on $q = \frac{n}{n-1}$, as it 'mixes' the numerology of the two conjectures above. Bassam Shayya indeed proved the above estimate for $n = 2$ and $q = \frac{n}{n-1} = 2$ in [1]; in other words, he showed that

$$\int_{B_R} |Eg|^4 w \leq C_\epsilon R^\epsilon \|Xw\|_\infty \left(\int |g|^2\right)^2 \tag{1}$$

in two dimensions. His proof is quite complicated, and does not seem to generalise in higher dimensions to imply the desired estimate

$$\int_{B_R} |Eg|^{\frac{2n}{n-1}} w \leq C_\epsilon R^\epsilon \|Xw\|_\infty \left(\int |g|^2\right)^{\frac{n}{n-1}} \tag{2}$$

for all n.

In this talk, we presented two simpler proofs of (1) (in two dimensions), which in our ongoing work we are trying to combine to obtain (2) in all dimensions. The two proofs do not seem to speak to each other: the first is based on the dispersive properties of the operator E, and the second is based on polynomial partitioning, which globally controls the interaction of the tubes carrying the wave packets of E with algebraic varieties (but not the dispersion of E). As the second proof may be viewed as standard, we present here the first one.

Without loss of generality, we may assume that w is the characteristic function of a union of unit balls in $\mathbb{R}^n$, contained in the level set

$$L_\lambda := \{x \in \mathbb{B}_\mathbb{R} : \ \lambda \leq |Eg(x)| < 2\lambda\}$$

of $|Eg|$, for some dyadic $\lambda > 0$. Therefore,

$$\int_{B_R} |Eg|^2 w \geq c\lambda^2 w(B_R),$$

for some absolute constant $c > 0$ (where $w(B_R) := \int_{B_R} w$). On the other hand, denoting by $\mathcal{C}$ the set of centres of the unit balls in supp w, we have that

$$\int_{B_R} |Eg|^{\frac{2n}{n-1}} w \leq C \sum_{x \in \mathcal{C}} |Eg|^2.$$

Now, for every or every $x \in \mathbb{R}^n$,

$$Eg(x) = \int e^{2\pi i \langle x, h(\omega)\rangle} \phi(\omega) g(\omega) d\omega = \langle g, \phi_x \rangle,$$

where ϕ is a bump function on $B^{n-1}(0,1)$, h is a parametrisation of Σ and $\phi_x(\omega) := e^{-2\pi i \langle x, h(\omega)\rangle} \phi(\omega)$. Therefore,

$$|Eg(x)|^2 = Eg(x)\overline{Eg(x)} = Eg(x) \cdot \overline{\langle g, \phi_x \rangle} = Eg(x) \cdot \langle \phi_x, g \rangle = \langle Eg(x)\phi_x, g \rangle,$$

hence

$$\int |Eg|^2 w \leq C \sum_{x \in \mathcal{C}} |Eg(x)|^2 = C \left\langle \sum_{x \in \mathcal{C}} Eg(x)\phi_x, \ g \right\rangle \leq C \left\| \sum_{x \in \mathcal{C}} Eg(x)\phi_x \right\|_2 \|g\|_2.$$

We can now exploit some orthogonality between the exponentials ϕ_x:

$$\begin{aligned}\left\|\sum_{x\in\mathcal{C}} Eg(x)\phi_x\right\|_2^2 &= \sum_{x,y\in C} Eg(x)\overline{Eg(y)}\int_{B^{n-1}(0,1)} \phi_x(\omega)\overline{\phi_y(\omega)}d\omega \\ &= \sum_{x,y\in\mathcal{C}} Eg(x)\overline{Eg(y)}\int_{B^{n-1}(0,1)} e^{2\pi i\langle y-x,h(\omega)\rangle}\phi^2(\omega)d\omega \\ &= \sum_{x,y\in\mathcal{C}} Eg(x)\overline{Eg(y)}E(\phi^2)(\omega) \\ &\leq \sum_{x,y\in\mathcal{C}} |Eg(x)|\cdot|Eg(y)|\cdot\frac{1}{1+|x-y|^{\frac{n-1}{2}}} \\ &\leq C\lambda^2\sum_{x,y\in\mathcal{C}}\frac{1}{1+|x-y|^{\frac{n-1}{2}}},\end{aligned}$$

where we bounded $|E(\phi^2)|$ using non-stationary phase. Plugging this in the earlier estimate, we obtain

$$\lambda w(B_R) \leq C\lambda\left(\sum_{x,y\in\mathcal{C}}\frac{1}{1+|x-y|^{\frac{n-1}{2}}}\right)^{1/2}\|g\|_2,$$

an L^1-weighted estimate on $|Eg|$. Raising both sides to the power 4, rearranging and applying the Cauchy–Schwarz inequality, we obtain

$$\begin{aligned}\lambda^4 w(B_R) &\leq C\frac{1}{w(B_R)^3}\left(\sum_{x,y\in\mathcal{C}}\frac{1}{1+|x-y|^{\frac{n-1}{2}}}\right)^2\left(\int|g|^2\right)^2 \\ &\leq C\sup_{x\in\mathcal{C}}\sum_{y\in\mathcal{C}}\frac{1}{1+|x-y|^{n-1}}\left(\int|g|^2\right)^2.\end{aligned}$$

Fixing the $x\in\mathcal{C}$ that achieves the supremum above, and looking at all y in the roughly $C\log R$ annuli $A(x,r/2,r)$ with centre x, inner radius r and outer radius $2r$ inside B_R, it follows by dyadic pigeonholing that the above sum is dominant in one of these annuli; that is,

$$\lambda^4 w(B_R) \leq C\log R\,\frac{w(B_r)}{r^{n-1}}\left(\int|g|^2\right)^2$$

for some dyadic $1\leq r\leq R$. This trivially implies (1), since

$$\frac{w(B_r)}{r^{n-1}} \leq \|Xw\|_\infty.$$

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Equivalences between different forms of the Kakeya conjecture and duality of Hausdorff and packing dimensions for additive complements

Tamás Keleti
(joint work with András Máthé)

By a *Besicovitch set* $B \subset \mathbb{R}^n$ we mean a set that contains a unit line segment in every direction. By a classical result of Besicovitch [1] for every $n \geq 2$ there exist Besicovitch sets of Lebesgue measure zero.

The *Kakeya conjecture* states that in $\mathbb{R}^n$ every Besicovitch set has Hausdorff dimension n. Davies [2] proved in 1971 that the Kakeya conjecture holds for $n = 2$. It is open for every $n > 2$.

The current best estimates for the Hausdorff dimension of a Besicovitch set $B \subset \mathbb{R}^n$ for $n \geq 3$:

- $n = 3$ Katz-Zahl [6] (2019): $\dim_{\mathrm{H}} B \geq 5/2 + \varepsilon$,
- $n = 4$ Katz-Zahl [7] (2021): $\dim_{\mathrm{H}} B > 3.059$,
- $n = 5$ Hickman-Rogers-Zhang [4] (2022): $\dim_{\mathrm{H}} B > 18/5$,
- $n = 6$ Katz-Tao [5] (2002): $\dim_{\mathrm{H}} B \geq 7 - 2\sqrt{2}$,
- $n > 6$ Katz-Tao [5] (2002), Hickman-Rogers-Zhang [4] (2022): $\dim_{\mathrm{H}} B \geq (2 - \sqrt{2})n + O(1)$.

We do *not* improve any of these estimates.

The Kakeya conjecture is often stated only for compact or Borel B. Often, instead of line segments, we want full lines in every direction. One can also require unit line segments only in a set of directions of positive $n-1$-dimensional Lebesgue measure or $n-1$-dimensional Hausdorff dimension. It was widely believed that all of these variants are equivalent. We prove that this is indeed the case: all of these forms of the Kakeya conjecture are equivalent:

Theorem 1. *For every n there exists a real $d \leq n$ with the following properties:*

(1) *There exists a* compact *Besicovitch set of Hausdorff dimension d.*
(2) *There exists a* closed *set of Hausdorff dimension d that contains a* line *in every direction.*
(3) *If $S \subset \mathbb{R}^n$ contains a line segment in a set of directions of Hausdorff-dimension $n-1$ then* $\dim_{\mathrm{H}} S \geq d$.

Therefore for all possible combinations of variants we considered there exists a set of minimal dimension, and this minimal dimension is always the same for any fixed n. In fact, the sets that have the minimal dimension in (1) and (2) are achieved by typical constructions in the Baire category sense, which extends a result of Körner [10] who proved that typical Besicovitch sets in the Baire category sense have Lebesgue measure zero.

The following conjecture and results show that in general it is not that clear that it does not matter if we take the union of lines or line segments:

Line Segment Extension Conjecture ([8], 2015)**.** *If A is the union of line segments in $\mathbb{R}^n$ and B is the union of the corresponding full lines, then* $\dim_{\mathrm{H}} A = \dim_{\mathrm{H}} B$.

Theorem 2 ([8], 2015). *(i) This conjecture for n would imply that for any Besicovitch set $B \subset \mathbb{R}^n$ we have* $\dim_{\mathrm{H}} B \geq n-1$.

(ii) If this conjecture holds for every $n \geq 2$ then any Besicovitch set has upper Minkowski dimension n.

Theorem 3 ([8], 2015). *The Line Segment Extension Conjecture holds when* $\dim_{\mathrm{H}} A < 2$ *or* $\dim_{\mathrm{H}} B \leq 2$. *In particular the conjecture holds in the plane.*

A lot of important conjectures (e.g. Kakeya problem, Besicovitch (n,k)-set conjecture) and results in geometric measure theory are of this form:

General Principle. *The union of an s Hausdorff-dimensional collection of d Hausdorff-dimensional sets in $\mathbb{R}^n$ has*

- *positive Lebesgue measure if $s+d>n$,*
- *Hausdorff dimension $s+d$ if $s+d \leq n$,*

unless the sets have large intersections.

We prove that *the following form of the General Principle is also equivalent to the Kakeya conjecture*:

General Kakeya Conjecture. *Let E be an arbitrary subset of $\mathbb{R}^n$ and let D be the set of directions in which E contains a line segment. If $D \neq \emptyset$ then*

$$\dim_{\mathrm{H}} E \geq \dim_{\mathrm{H}} D + 1.$$

By a result in [3] this conjecture holds if $\dim_{\mathrm{H}} D \leq 1$. It is not hard to show using a projective transformation that the *General Kakeya Conjecture implies the Line Segment Extension Conjecture*. Thus we have the following implications:

Kakeya Conjecture $\Longleftrightarrow$ General Kakeya Conjecture

$\Downarrow$

Line Segment Extension Conjecture

$\Downarrow$

Every Besicovitch set in $\mathbb{R}^n$ has Hausdorff dimension at least $n-1$

$\Downarrow$

Kakeya Conjecture for upper Minkowski dimension

Let $\dim_{\mathrm{P}}$ denote the (upper) packing dimension and and $A+B = \{a+b : a \in A, b \in B\}$. It is easy to see that

$$A+B=\mathbb{R}^n \text{ (or } \operatorname{int}(A+B) \neq \emptyset) \Longrightarrow \dim_{\mathrm{H}} A + \dim_{\mathrm{P}} B \geq n.$$

We extend this to the following duality result between Hausdorff and packing dimension.

Theorem 4. *For every non-empty Borel set $A \subset \mathbb{R}^n$, we have*

$$\begin{aligned} \dim_H A &= n - \inf\{\dim_P B \,:\, B \subset \mathbb{R}^n \textit{ Borel},\ A+B=\mathbb{R}^n\}, \\ &= n - \inf\{\dim_P B \,:\, B \subset \mathbb{R}^n \textit{ compact } \operatorname{int}(A+B) \neq \emptyset\}. \end{aligned}$$

and

$$\begin{aligned}\dim_P A &= n - \inf\{\dim_{\mathrm{H}} B : B \subset \mathbb{R}^n \text{ } \textit{Borel}, \text{ } A + B = \mathbb{R}^n\} \\ &= n - \inf\{\dim_{\mathrm{H}} B : B \subset \mathbb{R}^n \text{ } \textit{compact}, \text{ } \mathrm{int}(A + B) \neq \emptyset\}.\end{aligned}$$

The link between the above two seemingly unrelated topics is the following lemma, which was essentially also proved by Shmerkin-Suomala [11] in 2018.

Lemma 5. *Let $A \subset \mathbb{R}^n$ be Borel. Then*

(1) *If A has positive Lebesgue measure then there exists a compact set of zero upper Minkowski dimension such that* $\mathrm{int}(A + B) \neq \emptyset$*.*
(2) $\dim_{\mathrm{H}} A > s \Longrightarrow \exists B \subset \mathbb{R}^n$ *compact* $: \overline{\dim}_{\mathrm{M}} B = n - s, \mathrm{int}(A + B) \neq \emptyset,$
(3) $\dim_{\mathrm{H}} A > s \Longrightarrow \exists B \subset \mathbb{R}^n$ *Borel* $: \dim_{\mathrm{P}} B = n - s, \text{ } A + B = \mathbb{R}^n.$

We prove (1) by a random construction. A fairly short argument gives that (2) follows from (1). Clearly, (2) implies (3). Statement (1) is needed for equivalences of the versions of the Kakeya conjecture, (3) implies the first part of Theorem 4.

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Hausdorff-dimension analogue of the Elekes–Ronyái theorem and applications

Orit E. Raz

(joint work with Joshua Zahl)

In a recent work [2], joint with Joshua Zahl, we show how certain ideas from combinatorial geometry can be applied in the context of geometric measure theory. In particular, we prove the following theorem: Let $\gamma \subset [0,1]^2$ be a smooth curve with non-vanishing curvature. Then, for every $E \subset [0,1]^2$ Borel, with $\dim_{\mathrm{Haus}} E = \alpha$, we have

$$\dim_{\mathrm{Haus}}\{q \in \gamma \mid \dim_{\mathrm{Haus}} \Delta_q(E) \leq \alpha/2 + c\} = 0,$$

for $c = c(\alpha) > 0$; here $\Delta_q(E)$ is the set of distances spanned between the point q and the elements of E. The main ingredient in our proof, is showing a *discretized* analogue of the following result from combinatorial geometry: Let $q_1, q_2, q_3 \in \mathbb{R}^2$ be three fixed non-collinear points. Then for every finite point set $P \subset \mathbb{R}^2$, of cardinality $\#P = n$, we have that the set

$$D := \Delta_{q_1}(P) \cup \Delta_{q_2}(P) \cup \Delta_{q_3}(P)$$

is of cardinality $\#D \geq cn^{7/12}$, where $c > 0$ is an absolute constant (this follows by combining the results from Sharir–Solymosi [3] and Solymosi–Zahl [4]).

The "discretized setting" was introduced by Katz and Tao in [1] and can be viewed as an intermediate setting, between the discrete and continuous ones. In particular, it is shown in [1] that certain discretized statements (later verified by Bourgain) imply their continuous analogues (well-known conjectures, at the time open).

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Symmetries of Cantor sets

Michael Hochman

A Cantor set is a non-empty, compact, totally disconnected set with no isolated points. If $X, Y \subseteq \mathbb{R}$ are Cantor sets and $\alpha \geq 0$, we define

$$\mathrm{End}^\alpha(X,Y) \left\{ f : X \to Y \;\middle|\; \begin{array}{c} f(X) = Y \text{ and } f \text{ extends to a } C^\alpha \\ \text{map of an open interval containing} X \end{array} \right\}$$

We abbreviate $\mathrm{End}^\alpha(X) = \mathrm{End}^\alpha(X,X)$ for the semigroup of C^α onto self-maps.

In general, the size of $\mathrm{End}^\omega(X)$ is affected both by α and X. Every Cantor set $X \subseteq \mathbb{R}$ has uncountably many C^0-self maps, but in the regime $0 < \alpha \leq \infty$, the set $\mathrm{End}^\alpha(X)$ may be empty, finite, countably infinite or uncountable, depending on the structure of X.

For example, let X be the central-α Cantor, constructed from $[0,1]$ by removing the open middle segment of relative length α and iterating the procedure. Then every affine map taking X into itself contracts by an integer power of $1-\alpha$, see [1]. Together with a linearization argument, this implies that if $f : X \to X$ is a local diffeomorphism, then the derivative f' takes values that are rational powers of $1-\alpha$, and in particular, if $f \in C^\omega$, it is affine. A similar result holds if $X \subseteq [0,1]$ is a Cantor set invariant and transitive under $\times a$ mod 1 for an integer $a \geq 2$, see [4].

More generally, Funar and Neretin showed that if X is porous ("sparse" in their terminology) then $\mathrm{End}^\alpha(X)$ is countable. But they also showed that there are Cantor sets such that $\mathrm{End}^\infty(X)$ is uncountable [2, Theorem 2,3 and Section 5.1] (note that Theorem 1 of [2] concerns a completely different space than the theorem below and does not imply it).

The main result of this talk is the observation that the real-analytic case $\alpha = \omega$ is much more constrained:

Theorem 1. *If $X, Y \subseteq \mathbb{R}$ is a Cantor set, then $\mathrm{End}^\omega(X, Y)$ is at most countable.*

This result is apparently not known in the fractal geometry community, but after this work was completed we learned that the main step of the argument, which deals with increasing maps of an interval, can be derived from Proposition 3.5 of [5]. Our proof, which is based on different considerations, is more elementary and self-contained (e.g. it does not rely on Szekeres's theorem). We first reduce the , using Bair's theorem and elementary arguments, to bijective maps of an interval preserving a Cantor set. Then we carry out an analysis of the gaps in the Cantor set near fixed points to show that such a map can be fully characterized by certain a finite inter sequence derived from the action on gaps. This proves countability.

Our main theorem suggests many generalizations but we have not succeeded in proving or disproving any of them.

Problem 2. *What can be said about the set of into (rather than onto) self- maps of a Cantor set $X \subseteq \mathbb{R}$?*

The following related question was raised in [3], but remains open:

Problem 3. *If a Cantor set $X \subseteq \mathbb{R}$ has dimension < 1 can there exist uncountably many affine self-maps of X into itself?*

Another direction concerns the algebraic structure of $\mathrm{End}^\omega(X)$:

Problem 4. *Let $\mathrm{Aut}^\omega(X)$ be the invertible elements of $\mathrm{End}^\omega(X)$. What groups arise as [subgroups] if $\mathrm{Aut}^\omega(X)$? When is $\mathrm{Aut}^\omega(X)$ it finitely generated?*

For certain structured sets this question was addressed by Funar and Neretin [2].

It is also natural to consider higher dimensions. Certainly our proof breaks down, as it relies very strongly on the order structure of $\mathbb{R}$. One also must adjust the statement, since if X is contained in a lower-dimensional C^α-manifold M, one may be able to extend a self-map of X in many ways in the transverse direction to the manifold.

Problem 5. *Is the analogue of Theorem 1 true for $X \subseteq \mathbb{R}^d$ and real-analytic maps, provided $M \cap X$ has empty interior in X for every C^ω-submanifold $M \subseteq \mathbb{R}^d$?*

In the complex case, the qualification is unnecessary:

Problem 6. *Is the analogue of Theorem 1 true for Cantor sets $X \subseteq \mathbb{C}$ and analytic maps?*

And finally, going back to the line,

Problem 7. *If μ is a compactly supported, non-atomic Borel probability measure on $\mathbb{R}$, is*

$$\mathrm{End}^{\omega}(\mu) = \{f \in \mathrm{End}^{\omega}(\operatorname{supp}\mu) \mid \mu = \mu \circ f^{-1}\}$$

countable?

Of course, if the support is a Cantor set, this is answered by the results of the present paper (because any map preserving μ must preserve its support), but if the support an interval the answer is far from clear. The problem may be viewed as a relative of Furstenberg's celebrated $\times 2, \times 3$ problem, but the two are not directly related since here we consider more general measures and maps but ask for a weaker conclusion. We remark, that for $\times m$ invariant measures of positive dimension, we resolved Problem 7 in [4]. It is possible that also the problem above becomes easier under the assumption that $\dim \mu > 0$ even this remains open.

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Entropy Rates in Fractal Geometry

Lauritz Streck

We start by describing the problem setting. We let ν be a measure with finite support on the integers and we let λ be a real number with $|\lambda| < 1$. Let $\xi_0, \xi_1, \dots$ be independent and identically distributed with respect to ν. We define the self similar measure ν_λ on $\mathbb{R}$ by

$$\nu_\lambda := \operatorname{law}\left(\sum_{j=0}^{\infty} \xi_j \lambda^j\right).$$

For $\nu = \operatorname{Unif}\{0,1\}$ the Bernoulli measure, this measure is commonly known as a *Bernoulli convolution*. Questions about the Hausdorff dimension, absolute continuity (with respect to the Lebesgue measure) and Fourier decay of self-similar measures have a long and varied history; we refer the readers to [6].

For the study of the dimension of ν_λ, a crucial quantity is the *Garsia Entropy.* It is defined as

$$h_\lambda(\nu) := \lim_{n\to\infty} \frac{H(X_n)}{n}$$

for algebraic λ, where $X_n := \sum_{j=0}^{n-1} \xi_j \lambda^j$ and $H(\cdot)$ is the Shannon entropy. To highlight its importance, we mention that entropy rates played an important role in Hochman's breakthrough [5]. As a corollary of this, one can deduce that for algebraic λ,

$$\dim(\nu_\lambda) = \min\left(1, \frac{h_\lambda(\nu)}{\log \lambda^{-1}}\right).$$

Using Hochman's result, Varjú showed that $\dim(\nu_\lambda^{\mathrm{Ber}}) = 1$ for transcendental $\lambda \in (0.5, 1)$ [8]. In his proof, a crucial step is estimating the Garsia entropy.

The Garsia entropy is always bounded in terms of the algebraic complexity of λ, irrespective of ν. To make this precise, we define the *Mahler measure* as

$$M_\lambda := |a_d| \prod_{\sigma:\mathbb{Q}(\lambda)\to\mathbb{C}, |\sigma(\lambda)|>1} |\sigma(\lambda)|,$$

where a_d is the leading coefficient of the minimal polynomial of λ over $\mathbb{Z}[X]$ and $\sigma(\lambda)$ ranges over the Galois conjugates of λ. It is well-known that then

$$h_\lambda(\nu) \le \log M_\lambda \tag{1}$$

for any ν (see for example [7], Lemma 3.3). We say that a pair λ, ν has *maximal entropy* if $h_\lambda(\nu) = \log M_\lambda$. It was observed that roughly speaking, good things happen if maximal entropy is attained. For example, it was shown that for ν the Bernoulli measure and specific algebraic λ, namely those for which λ^{-1} is a so-called Garsia number, ν_λ is absolutely continuous [4] with bounded density and has *power Fourier decay* [1]. Here, power Fourier decay means that there are δ and C such that

$$|\widehat{\nu_\lambda}(x)| \le C|x|^{-\delta}$$

for all $x \in \mathbb{R}$. For those combinations of λ and ν, maximal entropy is attained (and such λ are the only ones with maximal entropy if ν is the Bernoulli measure). Although this is a relatively special family and both absolute continuity and power Fourier decay are known to hold generically, they are the only explicit examples for which both are known.

As negative examples, it is known that ν_λ is singular [2] and has dimension drop [3] if λ^{-1} is a so-called Pisot number and ν is the Bernoulli measure. In the proof, Erdos shows that ν_λ does not have Fourier decay and deduces singularity from this. Garsia introduced the Garsia entropy to show that if ν_λ had Hausdorff dimension one, it would also be absolutely continuous. Garsia's proof is related to the phenomenon of maximal entropy because he essentially shows (if not in this language) that in the case of Pisot numbers, Hausdorff dimension one implies maximal entropy which in turn implies absolute continuity.

This raises the question of how special the phenomenon of maximal entropy is and whether there is an underlying reason for its relation to absolute continuity of ν_λ. We denote the Galois conjugates of an algebraic λ by λ_i.

Theorem 1 (Theorem 1.1 in [7]). *Let $|\lambda| < 1$ be algebraic such that $|\lambda_i| \neq 1$ for all Galois conjugates. Let ν be a finitely supported probability measure on $\mathbb{Z}$. If there is an i with $|\lambda_i| > 1$, then $h_\lambda(\nu) < \log M_\lambda$.*

In the case that $|\lambda_i| < 1$ for all i, $M := M_\lambda$ is an integer. In this case, the measures ν with maximal entropy can be described explicitly. In [7], the notion of complete vanishing at level m for ν. Stating this condition would make this report too technical but we just say that it amounts to prescribing a finite list of finite sets $E_i \subset \mathbb{R}/\mathbb{Z}$ such that complete vanishing at level m is equivalent to $\exists i : \forall \theta \in E_i : \widehat{\nu}(\theta) = 0$. The sets E_i can be made completely explicit by considering certain finite groups associated with λ. The condition of complete vanishing thus just specifies a finite union of lower-dimensional subspaces of $l^1(\mathbb{Z})$ of the form $\{\nu : \widehat{\nu}(E_i) = 0\}$ and a measure lies in one of the subspaces if and only if there is complete vanishing. The theorem below says that a measure has maximal entropy if and only if it lies in one of the subspaces of this form.

Theorem 2 (Theorem 1.2/1.3 in [7]). *Let λ be an algebraic number such that $|\lambda_i| < 1$ for all conjugates and let ν be a finitely supported probability measure on $\mathbb{Z}$. The pair ν, λ attains maximal entropy if and only if there is an m with complete vanishing at level m. In this case, the measure ν_λ is absolutely continuous on $\mathbb{R}$ with bounded density and has power Fourier decay.*

The theorem gives a list of subspaces the measures with maximal entropy lie in. Following the proof, they could in principle be made completely explicit, as described in Example 1.5 in [7]. In general, the size of this list will depend on the length of the interval ν is supported on and on the biggest denominator in the finite set $\{\theta \in \mathbb{Q} : \widehat{\nu}(\theta) = 0\}$. However, in some cases, this bound can be improved: If a_1 and M are coprime, where $a_d x^d + \cdots + a_1 X + M \in \mathbb{Z}[X]$ is the minimal polynomial of $\beta := \lambda^{-1}$, the bound on m in the theorem depends only on M and on the length of the interval the measure ν is supported on (see Theorem 1.4 in [7]). In this case, the bounds on m are sharp.

To end this document, we briefly talk about the proof of Theorems 1 and 2. One considers the completions K_ν of the field $K := \mathbb{Q}(\lambda)$ with respect to some absolute value ν. These completions could be either $\mathbb{R}, \mathbb{C}$ or finite field extensions of some $\mathbb{Q}_p$. The product

$$A_\lambda := \prod_{\nu : |\lambda|_\nu < 1} K_\nu,$$

where ν runs over all absolute of K is then a natural setting to consider questions about the Garsia entropy. One has a natural diagonal embedding of K into A_λ and a contracting action of λ on A_λ. The Mahler measure M_λ is exactly the factor by which the action of λ shrinks sets in A_λ (when measured in terms of the Haar measure on A_λ).

On this space, one obtains a self-affine measure μ_λ as the limit measure of the random variables $X_n = \sum_{0 \le j < n} \xi_j \lambda^j$ embedded into A_λ. This measure projects onto ν_λ and turns out to be nicer than ν_λ in many respects - for example, the points in its support are exponentially separated.

Both Theorem 1 and Theorem 2 are proven by showing the corresponding, more general, result for μ_λ. The key point of the proof is to adapt the argument of Erdos and Garsia to the setting of A_λ. From this argument, one can deduce that maximal entropy must entail the vanishing of certain Fourier coefficients. This reduces the problem to one more combinatorial in nature that can be solved explicitly.

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Semisimple linear random walks on the torus I

Nicolas de Saxcé
(joint work with Weikun He)

Given a sequence $(g_n)_{n\geq 1}$ of identically distributed independent random variables with law μ on $\mathrm{SL}_d(\mathbb{R})$, we study the random walk $(s_n)_{n\geq 1}$ defined by

$$s_n = g_n g_{n-1} \dots g_1.$$

More precisely, we shall be interested in the law μ^{*n} of s_n, as n goes to infinity. Recall Kesten's law of large numbers for products of matrices.

Theorem 1 (Law of large numbers). *If $\int \log\|g\|\mu(\mathrm{d}g) < +\infty$, then there exists $\lambda_1 \geq 0$ such that almost surely and in L^1*

$$\lim \frac{1}{n} \log\|g_1 \dots g_n\| \to \lambda_1.$$

In other words, $\|g_1 \dots g_n\| = e^{n(\lambda_1 + o(1))}$. We therefore want to study the rescaled measure

$$\mu_n = (e^{-n\lambda_1})_*(\mu^{*n}),$$

seen as a measure on $M_d(\mathbb{R})$.

We make two important assumptions on the subgroup Γ generated by the support of μ:

(1) Γ is unbounded, and spans $E = M_d(\mathbb{R})$ as a vector space.

(2) Γ is included in $\mathrm{SL}_d(\mathbb{Z})$.

We study the properties of μ_n at scale

$$\delta = e^{-n\lambda_1}.$$

Let us make some elementary observations:

(1) *(Controlled support)* By the law of large numbers, μ_n is mostly supported on the ball $B_E(0,\delta^{-\varepsilon})$, for $\varepsilon > 0$ arbitrarily small.

(2) *(Exponential growth)* The assumption that Γ spans E implies that Γ is not virtually solvable. By the Tits alternative, it must contains a free group, and therefore, the support of μ_n contains at least e^{cn} points, for some $c > 0$.

(3) *(Separation)* Since $\Gamma \subset \mathrm{SL}_d(\mathbb{Z})$, the support of μ_n is δ-separated.

Writing $N(X,\delta)$ for the cardinality of a maximal δ-separated subset of X, the above implies that there exists $\sigma > 0$ such that for all large enough n, for some subset $S \subset B_E(0,\delta^{-\varepsilon}) \cap \mathrm{Supp}\mu_n$,

$$N(S,\delta) \geq \delta^{-\sigma}.$$

This means that μ_n has positive box-dimension at scale δ. Our goal will be to exploit the multiplicative structure of μ_n to show that μ_n in fact has positive *Fourier dimension*, i.e. that one can get a polynomial upper bound on the Fourier transform of μ_n.

Recall that if E^* denotes the space of linear forms on E, the Fourier transform of a measure ν on E is the function on E^* given by

$$\forall \xi \in E^*, \quad \widehat{\nu}(\xi) = \int_E e^{-i\langle \xi, x\rangle}\nu(\mathrm{d}x).$$

We show the following theorem.

Theorem 2 (Fourier decay for random walks)**.** *Under the above assumptions on* Γ, *and assuming also that for some* $\varepsilon > 0$, $\int_{\mathrm{SL}_d(\mathbb{R})} \|g\|^{\varepsilon}\mu(\mathrm{d}g) < +\infty$, *there exists* $\alpha_0 > 0$ *such that for all* $\xi \in E^*$ *such that* $\|\xi\| \leq \delta^{-\alpha_0}$,

$$|\widehat{\mu_n}(\xi)| \leq \|\xi\|^{-\tau}.$$

The proof of this result can be split into two parts:

(A) Fourier decay of multiplicative convolutions in E.

(B) Non-concentration of semi-simple random walks.

The first part is independent of the random walk setting; its goal is to show that if a probability measure ν on E has positive dimension and satisfies some non-concentration condition at scale δ, then some multiplicative convolution power of ν will exhibit some Fourier decay at scale δ. We show this using a strategy developed by Bourgain in the case $E = \mathbb{R}$, based on the discretized sum-product estimate and a flattening lemma.

The goal of the second part is to check that the rescaled measure μ_n associated to a random walk on $\mathrm{SL}_d(\mathbb{Z})$ satisfies the non-concentration condition needed in order to apply the Fourier decay result from the first part. The proof is based on the fact that if the Zariski closure of Γ is semisimple, then the convolution operator associated to μ has a uniform spectral gap on all Cayley graphs $\pi_p(\Gamma)$, where $\pi_p\colon \mathrm{SL}_d(\mathbb{Z}) \to \mathrm{SL}_d(\mathbb{Z}/p\mathbb{Z})$ is the reduction modulo p, a property first discovered by Bourgain and Gamburd in the setting of $\mathrm{SL}_2(\mathbb{Z})$, and then generalized to all semisimple algebraic groups by Salehi Golsefidy and Varjú.

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Semisimple linear random walks on the torus, II

Weikun He

(joint work with Nicolas de Saxcé)

This is the sequel to Nicolas de Saxcé's talk. The goal is to explain how to use sum-product theorems to derive quantitative equidistribution for random walks on the torus by toral automorphisms.

Namely, we consider the action of the group of automorphisms $\mathrm{GL}_d(\mathbb{Z})$ on the d-dimensional torus $\mathbb{R}^d/\mathbb{Z}^d$, with $d \geq 2$. Given a probability measure μ on the group and a starting point x on the torus, a random walk is defined. We are interested in questions regarding the quantitative equidistribution in law of such random walks. In concrete terms, for positive integer n, let ν_n denote the image measure of $\mu \times \cdots \times \mu$ by the map $(g_1, \ldots, g_n) \in \mathrm{GL}_d(\mathbb{Z})^n \mapsto g_n \cdots g_1 x \in \mathbb{R}^d/\mathbb{Z}^d$. We ask whether ν_n converges in the weak-$*$ topology to the Haar measure on $\mathbb{R}^d/\mathbb{Z}^d$ and how fast or how slow is that convergence.

We explain a recent result giving a satisfactory answer in a semisimple setting, that is, when the support of the measure μ generates a subgroup whose Zariski closure is semisimple. Put into informal words, it is the following.

Theorem 1 (Equidistribution)**.** *A semisimple linear random walk on a torus equidistributes unless the orbit of the starting point is not dense.*

Theorem 2 (Quantitative equidistribution)**.** *The equidistribution is fast unless the starting point is close to a small invariant closed subset.*

The precise and rigourous statement of the quantitative equidistribution is sharp up to a constant. In particular, this is a generalisation of the work of Bourgain–Furman–Lindenstrauss–Mozes [1]. Moreover, this gives rise to new proofs of the classification of orbit closures due to Guivarc'h–Starkov [3] and Muchnik [6] and the classification of stationary measures due to Benoist–Quint [2].

The proof follows the strategy of Bourgain-Furman-Lindenstrauss-Mozes, where Fourier analysis is combined with incidence geometry (or sum-product estimates). In the original proof of Bourgain–Furman–Lindenstrauss–Mozes, the sum-product side appears in the form of projection theorems. Instead of that, we make use of another appearance of the sum-product phenomenon, Fourier decay. To be more precise, we use the Fourier decay of random walks on the space of matrices, which is explained in Nicolas de Saxcé's talk. This is the main new input in our proof.

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Large copies of large configurations in large sets

Vjekoslav Kovač

(joint work with Polona Durcik, Kenneth J. Falconer, Luka Rimanić, Mario Stipčić and Alexia Yavicoli)

There exist patterns in large but otherwise arbitrary structures.

This is the main maxim of the Ramsey theory, but it is also widespread in other areas of mathematics. Results that are addressed here can be collectively called the *Euclidean density theorems* and they belong to the intersection of the arithmetic combinatorics and the geometric measure theory. These results study *large sets*, as this is what the word "density" stands for. In the present context, a measurable subset A of the unit cube $[0,1]^d$ is considered *large* if its Lebesgue measure is positive. On the other hand, a measurable subset A of the whole space $\mathbb{R}^d$ is considered *large* if it occupies a positive portion of the space, i.e., its (appropriately defined) upper (Banach) density $\overline{\delta}(A)$ is positive. The Euclidean density theorems search inside A for congruent (i.e., isometric) copies of given *configurations* (*patterns*) from a prescribed family $\mathcal{P} = \{P_\lambda : \lambda \in (0,\infty)\}$, indexed by a certain "size" parameter λ. Typically, P_λ is the dilate by λ of a fixed point configuration P, i.e., $P_\lambda = \lambda P$.

Using the Lebesgue density theorem one can easily find all kinds of finite configurations inside a positive measure set A. Moreover, generalizing the Steinhaus theorem one can even find inside A all sufficiently small dilates of a given finite point configuration P. Therefore, we have to ask for more in order to obtain a meaningful result in this setting. There are two types of results that we are generally aiming for. The first one is the *"all large scales" formulation.*

ALS: *For every measurable set $A \subseteq \mathbb{R}^d$ satisfying $\overline{\delta}(A) > 0$ there exists a number $\lambda_0 = \lambda_0(\mathcal{P}, A) > 0$ such that for every $\lambda \geq \lambda_0$ the set A contains a congruent copy of P_λ.*

This is a rather strong but only qualitative claim, as the number λ_0 depends on more than just the density $\overline{\delta}(A)$. The second one is *"an interval of scales" formulation*, sometimes also known as a *compact formulation* (after [1]).

IOS: *Take a number $0 < \delta \ll 1$ and a measurable set $A \subseteq [0,1]^d$ with measure at least δ. Then the set of scales $\lambda \in (0, \infty)$ such that A contains a congruent copy of P_λ contains an interval of length at least $\varepsilon = F_{\mathcal{P},d}(\delta)$.*

This is a weaker but quantitative claim and it enables a competition to find better dependencies of ε on δ.

This whole topic was initiated by a question of Székely [17] on whether a positive upper density set $A \subseteq \mathbb{R}^2$ realizes all sufficiently large distances (i.e., in our terminology, whether an ALS result holds for $P = \{0, 1\}$), which has been subsequently popularized by Erdős [8]. It was answered affirmatively by Furstenberg, Katznelson, and Weiss [11], and independently also by Falconer and Marstrand [10] and Bourgain [1]. Since then, a lot of work has been done in the aforementioned natural special case, when a fixed pattern P is scaled by the usual Euclidean dilations. The most general known positive result, in both ALS and IOS formulations, is due to Lyall and Magyar [16] and it holds when $P = \Delta_1 \times \cdots \times \Delta_m$ is a Cartesian product of vertex-sets Δ_j of nondegenerate simplices. The most general negative result is still due to Graham [12], who showed that ALS (and similarly IOS) results fail for configurations that cannot be inscribed in a sphere.

The purpose of this note is to inform the reader on where to look for the most recent developments on the topic, which have become possible primarily due to recent breakthroughs in the field of the multilinear harmonic analysis. We can "change the rules" slightly in one of the following ways, in order to open new interesting research directions.

Quantitative bounds. We might want to improve bounds in the IOS formulations. Already when P is a set of vertices of an n-dimensional rectagular box and A is a measurable subset of $[0,1]^{2n}$, the approach of Lyall and Magyar [16] gives an interval of a very small length; namely ε^{-1} is a tower of exponentials of height n of the number $\delta^{-3\cdot 2^n}$. Durcik and the author [4] have increased this to a "more reasonable" bound, $\varepsilon = (\exp(\delta^{-C(n,P)}))^{-1}$. A bound of the same type was later shown, more generally, for products of simplices by Durcik and Stipčić [7].

Anisotropic dilations. One can start with a configuration P and generate the collection $\mathcal{P}$ by applying to it anisotropic power-type scalings, namely $(x_1, \ldots, x_n) \mapsto (\lambda^{a_1} b_1 x_1, \ldots, \lambda^{a_n} b_n x_n)$, where a_j, b_j are fixed positive parameters. It was shown in [13] that analogues of many classical results from [1, 15, 16] remain valid in this modified context.

Sizes in ℓ^p. Already Bourgain [1] noted that ALS results fail for a triple of collinear points P. Cook, Magyar, and Pramanik [2] came up with an idea to study three-term arithmetic progressions $x, x+t, x+2t \in \mathbb{R}^d$, but evaluate sizes of their gaps t in other ℓ^p norms. Thew showed the ALS formulation whenever $p \neq 1, 2, \infty$ and d is sufficiently large. This was generalized to *corners* (x, y), $(x+t, y)$, $(x, y+t) \in (\mathbb{R}^d)^2$ by Durcik, Rimanić, and the author [5], but longer arithmetic progressions are still an open problem at the time of writing. As opposed to that, the IOS formulation (with a still "reasonable" length ε) was shown by Durcik and the author [4]. It turns out that for n-term arithmetic progressions one needs to avoid precisely the values $1, 2, \ldots, n-1, \infty$ for p. Finally, certain mixtures of three-term progressions or corners and product-type configurations were explored by the same authors in [3].

Very dense sets. Falconer, Yavicoli, and the author [9] considered measurable sets with density $\overline{\delta}(A)$ sufficiently close to 1 that A must contain all large dilates of all n-point configurations. Nontrivial upper and lower bounds for the critical density were shown in that paper, but its sharp asymptotics as $n \to \infty$ is currently still unknown.

Nonlinear configurations. Kuca, Orponen, and Sahlsten [14] showed that every compact set $K \subseteq \mathbb{R}^2$ of Hausdorff dimension sufficiently close to 2 contains a pair of distinct points of the form (x, y), $(x, y) + (u, u^2)$. This can be thought of as a continuous variant of the Furstenberg–Sárközy theorem (on $\mathbb{R}^2$ instead of $\mathbb{Z}$). One naturally wonders what stronger property of this type holds for sets of positive Lebesgue measure. Durcik, Stipčić, and the author [6] showed, among other things, that a positive measure set $A \subseteq [0,1]^2$ contains a point $(x_0, y_0) \in A$ such that A nontrivially intersects parabolae $y - y_0 = a(x - x_0)^2$ for a whole interval $I \subseteq (0, \infty)$ of parameters $a \in I$. Larger nonlinear configurations could be an interesting topic to study.

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Counting Rational Points near Flat Hypersurfaces

Rajula Srivastava
(joint work with Niclas Technau)

Let $\mathcal{M} \subseteq \mathbb{R}^n$ be a compact C^1-hypersurface, and $\|\cdot\|_2$ be the Euclidean norm on $\mathbb{R}^n$. The distance of a point $\mathbf{y} \in \mathbb{R}^n$ to $\mathcal{M}$ is defined by

$$\operatorname{dist}(\mathbf{y}, \mathcal{M}) := \inf_{\mathbf{x} \in \mathcal{M}} \|\mathbf{x} - \mathbf{y}\|_2.$$

Given $Q \geq 1$ and $\delta \geq 0$, we are interested in counting the number of rational points with denominator $q \in [Q, 2Q)$, located in a δ/q-neighbourhood of $\mathcal{M}$. This is captured by

$$\mathrm{N}_{\mathcal{M}}(\delta, Q) := \# \left\{ (q, \mathbf{p}) \in \mathbb{Z} \times \mathbb{Z}^n : \, q \in [Q, 2Q), \, \operatorname{dist}\Big(\frac{\mathbf{p}}{q}, \mathcal{M}\Big) \leq \frac{\delta}{q} \right\}. \tag{1}$$

Using compactness, we can cover $\mathcal{M}$ by a finite collection of open charts. Thus, without loss of generality, we may assume (using the implicit function theorem) that $\mathcal{M}$ can be represented in the Monge form as the graph

$$\{(\mathbf{x}, f(\mathbf{x})) : \mathbf{x} \in \mathcal{B}(\mathbf{0}, 1)\}$$

over the unit open ball in $\mathbb{R}^{n-1}$ of a C^1 scalar valued function f. We call this the *normalised Monge form* of $\mathcal{M}$. We now control the counting function (1) in terms of

$$\mathrm{N}_f(\delta, Q) = \sum_{(q, \mathbf{a}) \in \mathbb{Z} \times \mathbb{Z}^{n-1}} \mathbf{1}_{[1,2)}\big(\frac{q}{Q}\big) \mathbf{1}_{\mathcal{B}(\mathbf{0},1)} \left(\frac{\mathbf{a}}{q}\right) \mathbf{1}_{[0,\delta]} \left(\|q f(\frac{\mathbf{a}}{q})\|_{\mathbb{R}/\mathbb{Z}} \right),$$

where $\|\cdot\|_{\mathbb{R}/\mathbb{Z}}$ denotes the distance to the nearest integer. How large should we expect $\mathrm{N}_f(\delta, Q)$ to be? Suppose we replace the term $\|qf(\mathbf{a}/q)\|_{\mathbb{R}/\mathbb{Z}}$ by a uniformly distributed random variable $R : \mathcal{B}(\mathbf{0}, 1) \to [0, 1/2]$. There are about Q^{n-1} relevant choices for $\mathbf{a}$, and Q choices for q, while the constraint $\|R\| < \delta$ holds with probability δ. Thus it is reasonable to expect that

$$\mathrm{N}_f(\delta, Q) \asymp \delta Q^n. \tag{2}$$

However, this random model cannot predict the size of $\mathrm{N}_f(\delta, Q)$ correctly when δ is rather small in terms of Q, because $\mathrm{N}_f(\delta, Q)$ counts, in particular, the points *lying on* the manifold — there might be relatively many such points, or none at all!

For instance, the cylinder

$$\mathcal{Z} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1^2 + x_2^2 = 3\}$$

is readily seen to contain no point of $\mathbb{Q}^n$ at all. A short computation in combination with the fact that any non-zero integer is at least one in absolute value shows that any rational point contributing to $\mathrm{N}_{\mathcal{Z}}(\delta, Q)$ when $\delta = o(Q^{-1})$ must in fact be a point on $\mathcal{Z}$. Hence $\mathrm{N}_{\mathcal{Z}}(\delta, Q) = 0$ in the range $\delta \in (0, (Q \log Q)^{-1})$, once Q is sufficiently large– violating (2) dramatically.

However, if there is no "local obstruction", then the probabilistic guess (2) should be accurate up to a Q^{ε} of room in the expected range for δ, meaning that

$$\mathrm{N}_f(\delta, Q) \asymp \delta Q^n \text{ uniformly for any } Q \geq 1, \text{ and } \delta \in (Q^{\varepsilon-1}, 1/2) \tag{3}$$

where $\varepsilon > 0$ is fixed. The implied constants are allowed to depend on f and ε.

The term "local obstruction" is deliberately vague. While there is some understanding of geometric properties which have to be excluded for (3) to be true, it is currently not known what a precise characterisation all such local obstructions should be. In a recent break-through, J.-J. Huang showed the following:

Theorem 1 ([1])**.** *Let $n \geq 3$ and $\mathcal{M}$ be immersed by $f : [-1, 1]^{n-1} \to \mathbb{R}$. Suppose that f is $\max(\lfloor \frac{n-1}{2} \rfloor + 5, n + 1)$-many times continuously differentiable. Let*

$$H_f(\mathbf{x}) := \Big(\frac{\partial^2}{\partial x_i \partial x_j} f(\mathbf{x})\Big)_{i,j \leq n-1}$$

be the Hessian of f at $\mathbf{x} \in \mathbb{R}^{n-1}$. If $\mathcal{M}$ is not locally flat, i.e. the Gaussian curvature $\det H_f(\mathbf{x})$ *does not vanish for any $\mathbf{x} \in [-1, 1]^{n-1}$, then* (3) *holds.*

In other words, uniform curvature information rules out any local obstruction for hypersurfaces. Thus (3) is true for a wide class of hypersurfaces in all dimensions $n \geq 3$. The main innovation in [1] was an elegant bootstrap procedure relying on projective duality and the method of stationary phase from harmonic analysis.

On the other hand, our main result (Theorem 3 below) establishes a *new heuristic* for a rich class of flat or rough hypersurfaces. Indeed we consider hypersurfaces whose Hessian determinant (along with several other higher order derivatives) either vanishes or blows up at an isolated point, say the origin. To handle such surfaces, we have to introduce an additional new term to account for this local flatness/roughness. This term arises quite naturally from underlying harmonic analysis but also from purely geometric considerations (volumes of corresponding Knapp caps).

Main Results. In order to describe the class of hypersurfaces we consider, some more notation is needed. Let $\mathcal{H}_d(\mathbb{R}^{n-1})$ denote the collection of functions $f : \mathbb{R}^{n-1} \to \mathbb{R}$ which are homogeneous of degree d, that is $f(\lambda \mathbf{x}) = \lambda^d f(\mathbf{x})$ for all $\mathbf{x} \in \mathbb{R}^{n-1}$ and all $\lambda \geq 0$.

Definition 2. *For a real number $d \neq 0$, denote by $\mathcal{H}_d^{\mathbf{0}}(\mathbb{R}^{n-1})$ the set of all smooth functions $f \in \mathcal{H}_d(\mathbb{R}^{n-1})$ whose Hessian $H_f(\mathbf{x})$ remains invertible away from the origin; that is*

$$\mathcal{H}_d^{\mathbf{0}}(\mathbb{R}^{n-1}) := \{f \in \mathcal{H}_d(\mathbb{R}^{n-1}) : f \text{ is smooth on } \mathbb{R}^{n-1} \setminus \{\mathbf{0}\}, \\ \text{and } \det H_f(\mathbf{x}) \neq 0 \text{ if } \mathbf{x} \neq \mathbf{0}\}.$$

The simplest example of a function in $\mathcal{H}_d^{\mathbf{0}}(\mathbb{R}^{n-1})$ to keep in mind is

$$f(\mathbf{x}) := \|\mathbf{x}\|_2^d.$$

Theorem 3 (Main Theorem)**.** *Let $n \in \mathbb{Z}_{\geq 3}$. Let $d > \frac{2(n-1)}{2n-3}$ be a real number. Fix $\varepsilon > 0$, and $f \in \mathcal{H}_d^{\mathbf{0}}(\mathbb{R}^{n-1})$. Then*

$$\mathrm{N}_f(\delta, Q) \asymp \delta Q^n + \left(\frac{\delta}{Q}\right)^{\frac{n-1}{d}} Q^n$$

for any $Q \geq 1$ and $\delta \in (Q^{\varepsilon-1}, 1/2)$, provided $\frac{2(n-1)}{2n-3} < d < n-1$.

If $d \geq n-1$ then

$$\delta Q^n + \left(\frac{\delta}{Q}\right)^{\frac{n-1}{d}} Q^n \ll \mathrm{N}_f(\delta, Q) \ll \delta Q^n + \left(\frac{\delta}{Q}\right)^{\frac{n-1}{d}} Q^{n+k\varepsilon}$$

for any $Q \geq 1$ and $\delta \in (Q^{\varepsilon-1}, 1/2)$, where $k = k(n) > 0$ depends only on n.

The other implied constants in this theorem depend on ε, f, n, and d but not on δ, or Q.

An interesting consequence of the above theorem is that the original heuristic (3) is indeed true for a large subclass of the locally flat hypersurfaces we consider! Thus having non-vanishing Gaussian curvature is a sufficient but not a necessary condition for (3) to be valid. On the other hand, the 'flat term' that we introduce is indeed required to count rational points near manifolds corresponding to functions of large enough homogeneity, as a straightforward reformulation of our theorem shows.

Corollary 4. *Keep the notation and assumptions as in Theorem 3. Let $\varepsilon > 0$ be small in terms of d and n.*
(i) If $\frac{2(n-1)}{2n-3} < d \leq 2(n-1)$, then

$$\mathrm{N}_f(\delta, Q) \asymp \delta Q^n \text{ for } Q \geq 1 \text{ and } \delta \in (Q^{\varepsilon-1}, 1/2).$$

(ii) If $d > 2(n-1)$, then

$$\left(\frac{\delta}{Q}\right)^{\frac{n-1}{d}} Q^n \ll \mathrm{N}_f(\delta, Q) \ll \left(\frac{\delta}{Q}\right)^{\frac{n-1}{d}} Q^n Q^{k\varepsilon} \text{ for } Q \geq 1 \text{ and } \delta \in (Q^{\varepsilon-1}, Q^{-\frac{n-1-k\varepsilon d}{d-(n-1)}})$$

while

$$\mathrm{N}_f(\delta, Q) \asymp \delta Q^n \text{ for } Q \geq 1 \text{ and } \delta \in [Q^{-\frac{n-1-k\varepsilon d}{d-(n-1)}}, 1/2).$$

All the implied constants may depend on ε, f, n, d but not on δ, or Q.

Indeed, values of $d > 2$ lead to hypersurfaces whose Gaussian curvature vanishes at the origin. On the other hand, $d < 2$ leads to unbounded Gaussian curvature near the origin (and indeed the hypersurface is not even C^2), unless $d = 0$.

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Intersections and projections in the first Heisenberg group

Terry Harris

This talk discusses the problem of generalising the Marstrand–Mattila projection theorem to the first Heisenberg group. In Euclidean space, the Marstrand–Mattila projection theorem says that the Hausdorff dimension of sets of dimension $\leq m$ generically does not decrease under projection onto m-planes, and that projections of sets of dimension $> m$ generically have positive m-dimensional Lebesgue measure. The conjectured Heisenberg analogues are similar, with the Hausdorff dimension defined through a natural metric related to the group structure, and with projections defined through a natural semidirect decomposition of the Heisenberg group into homogeneous subgroups.

Although the Heisenberg group $\mathbb{H}$ has the same topology as $\mathbb{R}^3$, it has Hausdorff dimension 4 with respect to the Heisenberg metric, and the vertical 2-planes in $\mathbb{H} = \mathbb{R}^3$ that we project onto have dimension 3. The main conjecture (due to Balogh, Durand–Cartagena, Fässler, Mattila and Tyson [1]) says that if $\dim A = s$, then for $s \leq 3$ the Hausdorff dimension of A does not decrease generically under vertical projection, and if $s > 3$ then vertical projections of A generically have positive area. In the range $s \leq 1$, this conjecture has been known since 2011 (it was solved in this range in the same paper it was introduced). This conjecture was recently solved in the range $s \in [0, 2] \cup \{3\}$ by Fässler and Orponen [5], with partial results between 2 and 3. This talk presents a result of the author [7] which solves the conjecture in the range $s > 3$, building on the ideas of Fässler and Orponen. The problem is still open for $s \in (2, 3)$.

The proof uses the endpoint trilinear Kakeya inequality (in particular, the *endpoint* result without $\delta^{-\epsilon}$ loss is crucial, though the exponent p is not). I do not know if this is the first application which requires the endpoint version. The endpoint trilinear Kakeya was first proved by Guth [6]. The affine-invariant version (used in the proof) was first proved by Bourgain and Guth [3], and a simpler proof was later found by Carbery and Valdimarsson [4]. Guth's original version of trilinear Kakeya is probably also sufficient for the application. (One only needs some dependence $C(\rho)$ on the angle ρ of transversality; the precise dependence is not important. By a scaling argument, the version with coefficients a_T in front of the functions χ_T is implied by the version without coefficients (without any $\delta^{-\epsilon}$ loss), since repetitions of tubes in the inequality are allowed.)

The method also gives a solution to the problem of extending the Marstrand–Mattila intersection theorem to the first Heisenberg group. Previously this was known only for generic intersections with vertical planes ([2]), but it is now solved for generic intersections with horizontal lines too.

Notes/corrections. In my talk/preprint I implied that the affine-invariant version of trilinear Kakeya was originally due to Carbery–Valdimarsson rather than Bourgain–Guth. This will be corrected in the proofs of the preprint. Thanks to R. Zhang for this correction.

In my talk I mentioned that the SL_2 Kakeya conjecture was solved by Katz-Wu-Zahl, but forgot to mention that it was also solved by Fässler-Orponen.

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L^p-estimates of projections, analytic interpolation, and applications

Bochen Liu

There are two equivalent ways to define the Hausdorff dimension of a Borel set E in the Euclidean space by measures. The first one is

$$\dim_{\mathcal{H}} E = \sup\{s : \exists\, \mu \in \mathcal{M}(E), C_s(\mu) := \sup_x \frac{\mu(B(x,r))}{r^s} < \infty\}.$$

Here $\mathcal{M}(E)$ denotes the class of finite Borel measures supported on E. We call a measure $\mu \in \mathcal{M}(E)$ satisfying $C_s(\mu) < \infty$ for some s a Frostman measure on E.

The second equivalent definition is

$$\dim_{\mathcal{H}} E = \sup\{s : \exists\, \mu \in \mathcal{M}(E), I_s(\mu) < \infty\},$$

where

$$I_s(\mu) := \iint |x-y|^{-s}\, d\mu(x)\, d\mu(y) = c \int |\widehat{\mu}(\xi)|^2\, |\xi|^{-d+s}\, d\xi.$$

We call $I_s(\mu)$ the s-dimensional energy of μ.

Let $\pi_e(x) = x \cdot e$, $x \in \mathbb{R}^2$, $e \in S^1$, denote the orthogonal projection. One of the most classical results in geometric measure theory is the Marstrand projection theorem, which states that, for every Borel set $E \subset \mathbb{R}^2$, $\dim_{\mathcal{H}} E > 1$, the set $\pi_e(E) \subset \mathbb{R}$ has positive Lebesgue measure for almost all $e \in S^1$. Marstrand's original proof is very complicated. In 1968, Kaufman gave a one-line proof using Fourier analysis. Moreover, he proved that, if $\dim_{\mathcal{H}} E > 1$ then there exists a Frostman measure μ on E such that for almost all $e \in S^1$ the induced measure $\pi_e\mu$ on $\pi_e(E)$ has L^2 density with respect to the 1-dimensional Lebesgue measure. Similar results hold in higher dimensions, with orthogonal projections denoted by $\pi_V : \mathbb{R}^d \to V \approx \mathbb{R}^n$, where $V \in G(d,n)$ is a n-dimensional subspace of $\mathbb{R}^d$ and $G(d,n)$ denotes the Grassmannian. We refer to [5] for details of these classical results.

Of course $\pi_V\mu$ has L^2 density is not a necessary condition for $|\pi_V(E)| > 0$. But for technical reasons the L^2-method is the most popular approach to this type of problems. There is also some discussion on L^p estimates of orthogonal projections [6], but for a while it does not draw much attention due to the lack of geometric applications.

In 2019, Orponen [4] proved the following L^p estimate on radial projections that solves the visibility problem: for $x \neq y$ in $\mathbb{R}^d$, denote by

$$\pi^y(x) := \frac{x-y}{|x-y|} \in S^{d-1}$$

the radial projection. Then for every $s > d-1$, $s+t > 2(d-1)$, there exists $p > 1$ such that

$$\int \|\pi^y\mu\|^p_{L^p(S^{d-1})}\, d\nu(y) \leq C \cdot I_s(\mu)^{p/2} \cdot I_t(\nu)^{1/2}$$

for all finite Borel measures μ, ν on $\mathbb{R}^d$ of disjoint supports. As a corollary, if $E, F \subset \mathbb{R}^d$ are Borel sets satisfying $\dim_{\mathcal{H}} E > d-1$, $\dim_{\mathcal{H}} E + \dim_{\mathcal{H}} F > 2(d-1)$, then there exists $y \in F$ such that $\pi^y(E) \subset S^{d-1}$ has positive surface measure.

Later Orponen's L^p estimate plays an important role in the breakthrough on Falconer distance conjecture. See, for example, [3][2].

We would like to point out that for all applications above, the existence of some $p > 1$ is enough. So a natural question is, does the range of p matter? Recently, Dąbrowski, Orponen, Villa [1] proved that, if μ is a Frostman measure on $\mathbb{R}^d$ of exponent $s > n$, then

$$\int \|\pi_V\mu\|^p_{L^p(\mathcal{H}^n)}\, d\gamma_{d,n}(V) < \infty,\ \forall\, 2 \leq p < 2 + \frac{s-n}{d-s}.$$

This estimate has applications in incidence estimates, Furstenburg-type problems, and sum-product problems. Moreover, their range of p does help improve the exponents.

As a summary, we have L^p estimates on orthogonal and radial projections, and the range of p on orthogonal projections has geometric applications. So it is very natural to ask if one can improve L^p estimates of radial projections and find geometric applications that the range of p helps. We would like to remind the reader that Orponen's range in [4] is $1 \leq p < \min\left\{\frac{t}{2(d-1)-s},\ 2-\frac{t}{d-1}\right\}$ that is always less than 2.

In [7] we prove the following result on mixed-norm estimates of projections.

Theorem 1. *Suppose μ, ν are Frostman measures of exponents s, t respectively, of disjoint supports, and $s+t > 2(d-1)$, then*

$$\int \|\pi^y \mu\|^q_{L^p(S^{d-1})} d\nu(y) < \infty$$

given

$$q = 2 + \frac{s+t-2n}{d-s},\ 1 \leq p < \frac{2(d-1)}{d-1+t} \cdot \left(1 + \frac{s+t-2(d-1)}{2(d-s)}\right).$$

In fact we prove a more general result on

$$\pi^y \mu(V) := \int_{y+V^\perp} \mu \, d\mathcal{H}^{d-n} = \pi_V \mu(\pi_V y),\ V \in G(d,n)$$

assuming μ has continuous density. Notice that the map $\pi^y\mu(V)$ is not a projection when $n \geq 2$, because two points cannot determine an n-plane. So in general we prefer calling our result a mixed-norm estimate of orthogonal projections as $\pi^y \mu(V) = \pi_V \mu(\pi_V y)$.

Then, by solving for $p > m$, we obtain the following geometric application.

Theorem 2. *Suppose $E, F \subset \mathbb{R}^d$ satisfying $\dim_{\mathcal{H}} F > t(d, m, \dim_{\mathcal{H}} E)$, then there exists $y \in F$ such that*

$$\gamma_{d,m}(\{W \in G(d,m) : W = Span\{x_1 - y, \dots, x_m - y\} : x_i \in E\}) > 0.$$

Here the threshold $t(d, m, \dim_{\mathcal{H}} E)$ can be computed explicitly. A remark is, when $m = 1$ our result coincides with Orponen's result on radial projection (the visibility problem), and when $m \geq 2$ previous results do not help as they could not go beyond $p = 2$ as pointed out above.

The technique in our proof has its own interest. To work on radial projections, the argument of Dąbrowski, Orponen, Villa seem not to help. So we need new understandings of L^p estimates of projections. Our idea is to run analytic interpolations on the Riesz potential

$$\mu_z := \frac{\pi^{\frac{z}{2}}}{\Gamma(\frac{z}{2})} |\cdot|^{-d+z} * \mu(x).$$

To make the interpolation work, we introduce a new notion called the s-dimensional amplitude, defined by

$$A_s(\mu) := \sup_{x \in \mathbb{R}^d} \int |x-y|^{-s} \, d\mu(y).$$

This definition is very natural and it can be easily checked that it gives another equivalent definition of Hausdorff dimension of Borel sets, namely

$$\dim_{\mathcal{H}} E = \sup\{s : \exists\, \mu \in \mathcal{M}(E),\ A_s(\mu) < \infty\}.$$

Unlike $C_s(\mu)$ introduced at the very beginning, $A_s(\mu)$ is well defined for μ_z, more precisely, if $0 < s - Re(z) < d$, then

$$\begin{aligned} A_s(\mu_z) = \left\| | \cdot |^{-s} * \mu_z \right\|_{L^\infty} &= C \left\| | \cdot |^{-s} * | \cdot |^{-d+z} * \mu \right\|_{L^\infty} \\ &\leq C \left\| | \cdot |^{-s+Re(z)} * \mu \right\|_{L^\infty}. \end{aligned}$$

Then we propose a general mechanism to study $\|T\mu\|_p$, where T is a linear operator and μ is a Frostman measure, by interpolating between $\|T\mu_z\|_k$, $k \in \mathbb{Z}_+$. Moreover, if one applies this mechanism to orthogonal projections, it provides an alternative proof of the result of Dąbrowski, Orponen, Villa. For our mixed-norm estimate, it follows in the same routine with more complicated computation.

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Sharp isoperimetric inequalities on the hypercube

David Beltran

(joint work with José Madrid and Paata Ivanisvili)

Let $n \geq 1$ be an integer and $\{0,1\}^n$ be the hypercube of dimension n. One can regard $\{0,1\}^n$ as a graph where two vertices $x, y \in \{0,1\}^n$ are joined by an edge if the vectors $x = (x_1, \ldots, x_n)$ and $y = (y_1, \ldots, y_n)$ differ in exactly one coordinate. We denote such an edge by (x, y). Let $A \subset \{0,1\}^n$ and $A^c := \{0,1\}^n \backslash A$. Define the edge boundary of A by $\nabla A := \{(x,y) : x \in A, y \in A^c\}$. Associated to ∇A, we define the function $h_A : \{0,1\}^n \to \mathbb{R}$ by $h_A(x) = 0$ if $x \in A^c$, and if $x \in A$ we let $h_A(x)$ be the number of edges joining x with a vertex in A^c. Let μ be the uniform probability measure on $\{0,1\}^n$ and let $\mathbb{E}$ denote the expectation operator.

The classical edge isoperimetric inequality in $\{0,1\}^n$ says that

$$\frac{|\nabla A|}{2^n} = \mathbb{E} h_A \geq \mu(A)^* \log_2 \Big(\frac{1}{\mu(A)^*} \Big), \tag{1}$$

where $t^* = \min\{t, 1-t\}$ for all $t \in [0,1]$. Note this becomes an equality on subcubes of co-dimension k for any $0 \leq k \leq n$. It has been of interest to obtain lower bounds for $\mathbb{E} h_A^\beta$ for other values of $1/2 \leq \beta < 1$. The case $\beta = 1/2$ was first considered by Talagrand in [3]. By an inductive argument, he proved that there is a universal constant $K > 1$ such that

$$\mathbb{E}\sqrt{h_A} \geq K\mu(A)^* \sqrt{\log_2(1/\mu(A)^*)} \tag{2}$$

for all $A \subseteq \{0,1\}^n$. Furthermore, he also proved that

$$\mathbb{E}\sqrt{h_A} \geq \sqrt{2}\mu(A)(1-\mu(A)) \tag{3}$$

holds for all $A \subset \{0,1\}^n$. Bobkov and Götze [1] came up with an improved and elegant inductive argument and, in particular, one of their results implies that $\sqrt{2}$ can be replaced by $\sqrt{3}$ in (3). Recently, Kahn and Park [2] showed that

$$\mathbb{E}h_A^{\log_2(3/2)} \geq 2\mu(A)(1-\mu(A)), \tag{4}$$

and the constant 2 in the right-hand side of (4) is sharp: the inequality becomes an equality for subcubes of co-dimension 1 and 2. Our main result is an improved version of (4), which becomes tight on all subcubes (and not only co-dimension 1 and 2).

Theorem 1. *Let $\beta_0 := \log_2(3/2)$. Then the inequality*

$$\mathbb{E}h_A^{\beta_0} \geq \mu(A)^*(\log_2(1/\mu(A)^*))^{\beta_0} \tag{5}$$

holds for all $A \subset \{0,1\}^n$. In particular, if $\mu(A) \leq 1/2$ we have

$$\mathbb{E}h_A^{\beta} \geq \mu(A)(\log_2(1/\mu(A)))^{\beta} \qquad \text{for all} \quad \beta \geq \log_2(3/2). \tag{6}$$

The equality holds in both (5) and (6) for any subcube $A \subset \{0,1\}^n$.

Our second main result is an improvement of the constant $\sqrt{3}$ in Bobkov's inequality $\mathbb{E}\sqrt{h_A} \geq \sqrt{3}\mu(A)(1-\mu(A))$.

Theorem 2. *For any $\beta \in [1/2, \log_2(3/2)]$ we have*

$$\mathbb{E}h_A^{\beta} \geq C_\beta\, \mu(A)(1-\mu(A)) \tag{7}$$

for all $A \subset \{0,1\}^n$, where $C_\beta = 2\sqrt{2^{\beta+1}-2}$.

The theorem applied to the case $\beta = 1/2$ gives the improved constant $C_{1/2} = 1.82... > \sqrt{3} = 1.73....$ Testing the inequality $\mathbb{E}h_A^\beta \geq C_\beta\mu(A)(1-\mu(A))$ on subcubes of co-dimension 2, we obtain the upper bound $C_\beta \leq 2^{\beta+2}/3$ for $1/2 \leq \beta \leq \log_2(3/2)$. For $\beta = \log_2(3/2)$ the inequality (7) coincides with (4).

We also obtain a further improvement of (4) in the sense that we are able to lower the exponent $\log_2(3/2) = 0.5849...$ to 0.53 when restricted to sets of measure $\mu(A) \geq 1/2$.

Theorem 3. *If $\mu(A) \geq 1/2$ then*

$$\mathbb{E}h_A^{0.53} \geq 2\mu(A)(1-\mu(A)).$$

This result can be immediately applied to make progress on a conjecture of [2] regarding separating the hypercube. Furthermore, all of our results have applications to two-sided boundary isoperimetric bounds.

The proofs of all theorems are based on induction in the dimension $n \geq 1$. In particular, we follow the induction scheme of [2], but in the proof of Theorem 1 we enhance the induction using the inequality (4) for sets A of measure $\mu(A) \in [1/4, 1/2]$. This allows to focus our attention to $\mu(A) \in [0, 1/4]$. In order to close the induction, we need to verify the non-negativity of certain functions of two-variables. We refer to the article for further details.

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Quantitative norm convergence for multiple ergodic averages of several commuting transformations

Christoph Thiele

(joint work with Polona Durcik, Lenka Slavíková)

Given a probability space X, a bijective and measure preserving map $T : X \to X$, and a bounded measurable function f on X, the classical ergodic averages are defined as

$$A_N f(x) = \frac{1}{N} \sum_{n=1}^{N} f(T^n x).$$

One may be interested in convergence of these averages as $N \to \infty$ in $L^2(X)$, this is guaranteed by von Neumann's theorem, or convergence in an almost everywhere sense, this is Birkhoff's theorem. One may also be interested in quantitative convergence results in norm or almost everywhere. One form of quantitative convergence is expressed by variation norm estimates. The r-variation in norm of the sequence of functions $A_N f$ is defined as

$$\|A_N f\|_{V^r(N)} := \sup_J \sup_{0<N_0<N_1<\cdots<N_J} \left(\sum_{j=1}^{J} \left\| A_{N_j} f - A_{N_{j-1}} f \right\|_2^r \right)^{1/r}.$$

It was observed by Jones, Ostrovskii, and Rosenblatt that for $f \in L^2(X)$ and $r \geq 2$ we have

$$\|A_N f\|_{V^r(N)} \leq C \|f\|_2.$$

This is a strong form of convergence of $A_N f$ in $L^2(X)$ norm.

Given commuting bijective measure preserving transformations $T_1, T_2, \ldots, T_d$ and functions $f_1, f_2, \ldots, f_d$, one can form the multiple ergodic averages

$$A_N(f_1, f_2, \ldots, f_N)(x) = \frac{1}{N} \sum_{n=1}^{N} \prod_{k=1}^{d} f_k(T_k^n x).$$

For functions $f_k \in L^\infty(X)$, norm convergence in $L^2(X)$ is known due to the work of Tao, with earlier work in $d = 2$ by Conze and Lesigne on the one hand and Furstenberg and Weiss on the other hand. Pointwise almost everywhere convergence of these multiple averages is not known for any $d \geq 2$ and remains a very interesting open question. We are interested in bounds for the variation in norm. This has been shown for two commuting transformations in [2] for $r \geq 2$. In this talk, we present work in progress with P. Durcik and L. Slavíková for three commuting transformations and $r > 4$. It is not known, whether the threshold $r = 4$ is sharp, in principle $r \geq 2$ as for one or two commuting transformations may be true as well. Present techniques however fail unless $r > 4$.

Our approach to these multiple ergodic averages is to use the Calderon transfer principle to pass to estimates in harmonic analysis. We define for 3 functions in $\mathbb{R}^3$

$$A_N(f_1, f_2, f_3)(x) = \frac{1}{N} \int_0^N f(x_1 + t, x_2, x_3) f(x_1, x_2 + t, x_3) f(x_1, x_2, x_3 + t)\, dt.$$

Here $\mathbb{R}^d$ is to be compared with $\mathbb{Z}^3$, which in turn occurs as it parameterizes the orbit of a point x under the action of T_1, T_2, T_3. We the prove bounds for the variation norm of $A_N(f_1, f_2, f_3)$ as announced above in the ergodic setting. We use the theory of singular Brascamp Lieb forms, which has seen rapid progress in recent years, see for example [1], [3].

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Sharp sparse domination results for Bochner–Riesz means at the critical index

Andreas Seeger

(joint work with David Beltran and Joris Roos)

New results from [2] were presented on the Bochner–Riesz means R_t^λ of index λ, defined by $\widehat{R_t^\lambda f}(\xi) = (1 - t^{-2}|\xi|^2)_+^\lambda \widehat{f}(\xi)$. Given $1 \leq p < \frac{2d}{d+1}$, $d \geq 2$, the value $\lambda(p) := d\left(\frac{1}{p} - \frac{1}{2}\right) - \frac{1}{2}$ is referred to as the *critical index*. It is conjectured that

R_t^λ are bounded operators on L^p for $\lambda > \lambda(p)$. This conjecture was proved by Carleson–Sjölin [6] for $d = 2$, $1 < p < 4/3$; and in higher dimensions many partial results are known (the currently best results are in [12]). For the critical $\lambda(p)$ one should have a weak type (p,p) inequality; this was first proved for $p = 1$ and $\lambda(1) = \frac{d-1}{2}$ by Christ [7] and extended to weighted weak type estimate involving A_1 weights by Vargas [21]. For other p see [8], [18], [19] and finally the black-box result by Tao [20].

We aim to prove an extension of Vargas' $L^1(w) \to L^{1,\infty}(w)$ result, with $w \in A_1$:

Theorem 1. *For every $w \in A_1$ there exists an exponent $p_1(w) > 1$ such that for $1 \le p < p_1(w)$, the operators $R_t^{\lambda(p)}$, $t > 0$, are uniformly bounded as operators from $L^p(w)$ to $L^{p,\infty}(w)$. Moreover, $\lim_{t\to\infty} \|R_t^{\lambda(p)} f - f\|_{L^{p,\infty}(w)} = 0$ for all $f \in L^p(w)$.*

Essential for the proof of this result are new sparse domination results for Bochner–Riesz means at the critical index.

To formulate the sparse domination results, we need to review some definitions (for basic ideas on sparse bounds [15, 16, 17, 5, 10, 1]). Let $\mathfrak{D}$ denote a *dyadic lattice* in the sense of the monograph by Lerner and Nazarov [17, §2]. For a locally integrable function f, a cube $Q \in \mathfrak{D}$ and $1 \le p < \infty$, let $\langle f\rangle_{Q,p} = (|Q|^{-1}\int_Q |f(y)|^p dy)^{1/p}$. The collection $\mathfrak{S} \in \mathfrak{D}$ is called *γ-sparse* if for every $Q \in \mathfrak{S}$ there is a measurable subset $E_Q \subset Q$ so that $|E_Q| \ge |Q|/2$ and $\{E_Q : Q \in \mathfrak{S}\}$ is a collection of pairwise disjoint sets. Let $1 \le p, q < \infty$. For a sparse family $\mathfrak{S}$ of cubes we define a sparse form $\Lambda^{\mathfrak{S}}_{p,q}$ by $\Lambda^{\mathfrak{S}}_{p,q}(f_1, f_2) = \sum_{Q\in\mathfrak{S}} |Q|\langle f_1\rangle_{Q,p}\langle f_2\rangle_{Q,q}$; moreover a corresponding maximal form $\Lambda^*_{p,q}(f_1, f_2) = \sup_{\mathfrak{S}\text{ sparse}} \Lambda^{\mathfrak{S}}_{p,q}(f_1, f_2)$, where the sup is taken over all sparse families (which are allowed to be subcollections of different dyadic lattices). These definitions are of interest in the range $p \le q < p'$. A linear operator $T : C_c^\infty(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)$ satisfies a (p,q) sparse bound if for all f_1, $f_2 \in C_c^\infty$ the inequality $|\langle Tf_1, f_2\rangle| \le C\Lambda^*_{p,q}(f_1, f_2)$ holds with some constant C independent of f_1, f_2. In this case, we say that T belongs to the space $\mathrm{Sp}(p, q; \mathbb{R}^d)$ and we denote by $\|T\|_{\mathrm{Sp}(p,q;\mathbb{R}^d)}$ the best constant. A $\mathrm{Sp}(p, q)$ bound for T with $q < p'$ implies that T is of weak type (p,p) and restricted strong type (q', q'). The above theorem can de deduced by combining $\mathrm{Sp}(p, q)$-bounds with the fact that every A_1 weight belongs to a reversed Hölder class RH_σ for some $\sigma > 1$, and a result of Frey and Nieraeth [11] on weighted weak type inequalities for weights belonging to $A_1 \cap \mathrm{RH}_{(q'/p)'}$.

Given $0 < \lambda \le \frac{d-1}{2}$, let $\mathcal{T}_d(\lambda)$ denote the closed trapezoid with corners

$$P_1 = \left(\tfrac{2\lambda+d+1}{2d}, \tfrac{d-2\lambda-1}{2d}\right), \qquad P_2 = \left(\tfrac{2\lambda+d+1}{2d}, \tfrac{d-1}{2d} + \tfrac{\lambda(d+1)}{d(d-1)}\right),$$
$$P_3 = \left(\tfrac{d-1}{2d} + \tfrac{\lambda(d+1)}{d(d-1)}, \tfrac{2\lambda+d+1}{2d}\right), \quad P_4 = \left(\tfrac{d-2\lambda-1}{2d}, \tfrac{2\lambda+d+1}{2d}\right).$$

Standard necessary conditions ([4], see also [1]) suggest the conjecture that sparse bounds for R_t^λ and $\lambda > 0$ hold for all $(\frac1p, \frac1q) \in \mathcal{T}_d(\lambda)$. It was observed in [4, 14] that for $(\frac1p, \frac1q)$ in the interior of the trapezoid, $\mathrm{Sp}(p, q)$ bounds for R_t^λ can be obtained via a single-scale analysis, with affirmative results depending on the partial knowledge on the Bochner–Riesz conjecture. In our work we focus on the endpoint cases in which $(\frac{1}{p}, \frac{1}{q})$ belongs to the boundary of $\mathcal{T}_d(\lambda)$. Note that the critical case $\lambda = \lambda(p)$, and $p < \frac{2d}{d+1}$ corresponds to the vertical line segment P_1P_2.

Almost sharp results at the critical line P_1P_2 had been obtained in the case $\lambda = \frac{d-1}{2}$ (that is, $p = 1$) by Conde-Alonso–Culiuc–Di Plinio–Ou [9]; namely they proved a $(1, q)$ sparse bound for all $q > 1$. Partial results on that line for $p > 1$ were obtained in two dimensions by Kesler and Lacey [13] with the restrictive assumption that $q > 4$. We fully state our results in two dimensions where they are most complete. For $0 < \lambda < 1/2$, let $p_\lambda = \frac{4}{3+2\lambda}$ (so that $\lambda(p_\lambda) = \lambda$).

Theorem 2. *For $0 < \lambda < 1/2$, we have $R_t^\lambda \in \mathrm{Sp}(p_\lambda, q; \mathbb{R}^2)$, for $q > \frac{4}{1+6\lambda}$.*

This means that in two dimension we obtain the $\mathrm{Sp}(p, q)$ bound for R_t^λ for all $(\frac{1}{p}, \frac{1}{q})$ in $\mathcal{T}_2(\lambda) \setminus \overline{P_2P_3}$ and it remains open what happens on the line segment $\overline{P_2P_3}$. The case $\lambda = 1/6$ (where $P_2 = (\frac{5}{6}, \frac{1}{2})$, the Stein-Tomas endpoint) is special and allows a complete result:

Theorem 3. *For $\lambda_* = 1/6$ we have $R_t^{\lambda_*} \in \mathrm{Sp}(p, q; \mathbb{R}^2)$ for every $(\frac{1}{p}, \frac{1}{q}) \in \mathcal{T}_2(\lambda_*)$.*

In higher dimensions, we obtain similar optimal results but only for a partial range of λ. This is natural in view of the currently incomplete knowledge on $L^p \to L^r$ bounds for Bochner–Riesz type operators. Our main result here ([2, 3]) is a conditional result, which can be seen as a sparse analogue of Tao's black box result on weak-type estimates at the critical index [20]. Fix $\frac{2(d+1)}{d+3} \le p_\circ < \frac{2d}{d+1}$. Assume that for all $r_\circ \in [p_\circ, \frac{d-1}{d+1}p_\circ')$, (a) the Fourier restriction operator maps $L^{p_\circ}(\mathbb{R}^d) \to L^{r_\circ}(S^{d-1})$ and (b) the Bochner–Riesz operator $(1 - \rho(D))_+^\lambda$ maps $L^{p_\circ}(\mathbb{R}^d) \to L^{r_\circ}(\mathbb{R}^d)$ for all $\lambda > \lambda(r_\circ)$. The conclusion than is that $R_t^{\lambda(p)} \in \mathrm{Sp}(p, q)$ for $1 \le p < p_\circ$ and $q > q_{opt} := \frac{(d-1)p}{d+1-2p}$. This can be used to obtain $\mathrm{Sp}(p, q)$ bounds for $R_t^{\lambda(p)}$ (up to the q-endpoint) in a range including $1 \le p < \frac{2(d+2}{d+4}$.

Finally, for the special value $\lambda_*(d) = \frac{d-1}{2(d+1)}$ we can do better and obtain an optimal result for the entire trapezoid $\mathcal{T}_d(\lambda_*)$, analogous to the two-dimensional result in Theorem 3.

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