Mathematisches
Forschungsinstitut
Oberwolfach

Member of Leibniz
Leibniz
Association

Oberwolfach Preprints

Number 2023-12

DAMIANO ROSSI

The simplicial complex of Brauer pairs of a finite reductive group

July 2023

Oberwolfach Preprints (OWP)

ISSN 1864-7596

License Information

Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany
https://www.mfo.de

DOI 10.14760/OWP-2023-12

The simplicial complex of Brauer pairs of a finite reductive group

Damiano Rossi

Abstract

In this paper we study the simplicial complex induced by the poset of Brauer pairs ordered by inclusion for the family of finite reductive groups. In the defining characteristic case the homotopy type of this simplicial complex coincides with that of the Tits building thanks to a well-known result of Quillen. On the other hand, in the non-defining characteristic case, we show that the simplicial complex of Brauer pairs is homotopy equivalent to a simplicial complex determined by generalised Harish-Chandra theory. This extends earlier results of the author on the Brown complex and makes use of the theory of connected subpairs and twisted block induction developed by Cabanes and Enguehard.

Introduction

The poset $\mathcal{S}_\ell^\star(G)$ of non-trivial ℓ-subgroups of a finite group G, with respect to a prime ℓ dividing the order of G, gives rise to a simplicial complex $\Delta(\mathcal{S}_\ell^\star(G))$ known as the Brown complex. This simplicial complex was first introduced by Brown in [Bro75] and its homotopy propeties were later described by Quillen in [Qui78]. In particular, for $G = \mathbf{G}^F$ a finite reductive group in characteristic p, Quillen showed that the Brown complex $\Delta(\mathcal{S}_p^\star(\mathbf{G}^F))$ is homotopy equivalent to the Tits building of $\mathbf{G}^F$. More recently, the author considered the remaining non-defining characteristic case, where $p \neq \ell$, and showed in [Ros23a] that the homotopy type of $\Delta(\mathcal{S}_\ell^\star(\mathbf{G}^F))$ can be described in terms of the generic Sylow theory developed by Broué and Malle [BM92]. This result was then used to obtain a connection between the so-called local-global conjectures in group representation theory and certain statements in generalised Harish-Chandra theory introduced in [Ros22] and [Ros23b].

In this paper we extend the above-mentioned homotopy equivalences to a representation theoretic setting by replacing ℓ-subgroups with ℓ-Brauer pairs. More precisely, let B be a Brauer ℓ-block of a finite group G and denote by $\mathcal{S}_\ell^\star(B)$ the poset of non-trivial B-Brauer pairs ordered by inclusion as defined in Section 1. Observe that $\mathcal{S}_\ell^\star(B)$ is non-empty if and only if B has positive defect and that the associated simplicial complex $\Delta(\mathcal{S}_\ell^\star(B))$ yields a natural generalisation of the Brown complex: if $B_0(G)$ is the principal ℓ-block of G, there exists a natural homeomorphism between $\Delta(\mathcal{S}_\ell^\star(B_0(G)))$ and the Brown complex $\Delta(\mathcal{S}_\ell^\star(G))$.

2010 *Mathematical Subject Classification:* 20J05, 20G40, 20C20, 55P91

Key words and phrases: Simplicial complexes, Brauer pairs, finite reductive groups, generalised Harish-Chandra theory.

This work was carried out during a stay of the author at the Mathematisches Forschungsinstitut Oberwolfach funded by a Leibniz Fellowship. The author would like to thank Marc Cabanes for useful comments on an earlier version of this paper.

We now focus on the case where $G = \mathbf{G}^F$ is a finite reductive group in characteristic p as before. When $p = \ell$, in analogy with Quillen's result, it turns out that for each p-block B with positive defect the simplicial complex $\Delta(\mathcal{S}_p^\star(B))$ is homotopy equivalent to the Tits building. This is shown in Theorem 2.2 below. More interesting is the case $p \neq \ell$. In this situation, for a Brauer ℓ-block B, we use the theory of connected subpairs and twisted block induction introduced by Cabanes and Enguehard [CE99] to construct a poset $\mathcal{L}^\star_{e_\ell(q)}(B)$ consisting of pairs $(\mathbf{L}, b_\mathbf{L})$ where $\mathbf{L}$ is an $e_\ell(q)$-split Levi subgroup of $(\mathbf{G}, F)$, with $e_\ell(q)$ the multiplicative order of q modulo ℓ, and $b_\mathbf{L}$ is a Brauer ℓ-block of $\mathbf{L}^F$. We refer the reader to Section 3 for a precise definition. The advantage of the simplicial complex $\Delta(\mathcal{L}^\star_{e_\ell(q)}(B))$ is that it can be determined by generalised Harish-Chandra theory. We can describe the homotopy type of $\Delta(\mathcal{S}_\ell^\star(B))$ as follows.

Theorem A. *Let $\mathbf{G}$ be a connected reductive group defined over an algebraically closed field of characteristic p and consider $F : \mathbf{G} \to \mathbf{G}$ a Frobenius endomorphism endowing the algebraic variety $\mathbf{G}$ with an $\mathbb{F}_q$-rational structure. Suppose that $\ell \in \pi(\mathbf{G}, F)$ as defined in Definition 3.1 and that ℓ does not divide the order of $\mathbf{Z}(\mathbf{G})^F$. Then there exists a $\mathbf{G}^F$-homotopy equivalence*

$$\Delta\left(\mathcal{S}_\ell^\star(B)\right) \simeq \Delta\left(\mathcal{L}^\star_{e_\ell(q)}(B)\right)$$

for every Brauer ℓ-block B of $\mathbf{G}^F$ with non-trivial defect and where $e_\ell(q)$ denotes the multiplicative order of q modulo ℓ.

The paper is structured as follows. In Section 1 we collect some preliminary results on Brauer pairs. In particular, we introduce the notion of almost-centric Brauer pair and show that these pairs control the homotopy type of the simplicial complex $\Delta(\mathcal{S}_\ell^\star(B))$. In Section 2 we consider the case $G = \mathbf{G}^F$ and $p = \ell$, and prove the connection between the Tits building of $\mathbf{G}^F$ and the simplicial complex $\Delta(\mathcal{S}_p^\star(B))$. Finally, in Section 3 we consider the case $p \neq \ell$ and prove Theorem A. We then conclude the paper by showing how [Ros23a, Theorem A] can be recovered from our Theorem A.

1 Preliminary results on Brauer pairs

Let G be a finite group and fix a prime number ℓ. A *Brauer pair* of G, with respect to the prime ℓ, is a pair (Q, b_Q) where Q is an ℓ-subgroup of G and b_Q is a Brauer ℓ-block of $\mathbf{C}_G(Q)$. Using the Brauer map Br_Q (see [Lin18a, Theorem 5.4.1]) we can define a partial order relation on the set of Brauer pairs. If (P, b_P) and (Q, b_Q) are Brauer pairs of G, then we write $(Q, b_Q) \leq (P, b_P)$ if $Q \leq P$ and there exists a primitive idempotent i such that $\mathrm{Br}_P(i)e_{b_P} \neq 0$ and $\mathrm{Br}_Q(i)e_{b_Q} \neq 0$ (see [Lin18b, Definition 6.3.2]). Here e_{b_P} and e_{b_Q} are the idempotents corresponding to the Brauer ℓ-blocks b_P and b_Q respectively. Given a Brauer pair (Q, b_Q) there exists a unique Brauer ℓ-block B of G such that $(1, B) \leq (Q, b_Q)$ in which case we say that (Q, b_Q) is a *B-Brauer pair*. Moreover, if $(Q, b_Q) \leq (P, b_P)$ then (Q, b_Q) is a B-Brauer pair if and only if (P, b_P) is a B-Brauer pair according to [Lin18b, Proposition 6.3.6]. We denote by $\mathcal{S}_\ell^\star(B)$ the poset of B-Brauer pairs (Q, b_Q) satisfying $Q \neq 1$. Observe that the poset $\mathcal{S}_\ell^\star(B)$ is non-empty if and only if B has non-trivial defect. In fact, maximal B-Brauer pairs are of the form (D, b_D) where D is a defect subgroup of B (see [Lin18b, Theorem 6.3.7]). Moreover, since B is stable under the action of G by conjugation, notice that $\mathcal{S}_\ell^\star(B)$ is actually a G-poset. We denote by $\mathbf{N}_G(Q, b_Q)$ the stabiliser of the Brauer pair (Q, b_Q) under the action of G, that is stabiliser of the block b_Q in the normaliser $\mathbf{N}_G(Q)$. For further details on Brauer pairs we refer the reader to the original paper of Alperin and Brouè [AB79] and to Linckelmann's monograph [Lin18b, Section 6.3].

Next, recall that for every poset $\mathcal{X}$ we can form a simplicial complex $\Delta(\mathcal{X})$ whose simplices are given by totally ordered chains of finite length in $\mathcal{X}$. In particular, if $\mathcal{S}_\ell^\star(G)$ is the set of non-trivial ℓ-subgroups of G, then $\Delta(\mathcal{S}_\ell^\star(G))$ is the *Brown complex* introduced in [Bro75] and further studied in [Qui78]. In the next lemma we show that the Brown complex of G is homeomorphic to the simplicial complex $\Delta(\mathcal{S}_\ell^\star(B_0(G)))$ where $B_0(G)$ is the principal block of G.

Lemma 1.1. *Let ℓ be a prime dividing the order of G and consider the Brown complex $\Delta(\mathcal{S}_\ell^\star(G))$. Then, there is a homeomorphism of $\Delta(\mathcal{S}_\ell^\star(G))$ with $\Delta(\mathcal{S}_\ell^\star(B_0(G)))$ induced by the poset isomorphism*

$$\begin{aligned}\mathcal{S}_\ell^\star(G) &\to \mathcal{S}_\ell^\star(B_0(G))\\ Q &\mapsto (Q, B_0(\mathbf{C}_G(Q))).\end{aligned}$$

Proof. Let Q be an ℓ-subgroup of G and suppose that b_Q is a block of $\mathbf{C}_G(Q)$ such that the Brauer pair (Q, b_Q) belongs to $\mathcal{S}_\ell^\star(B_0(G))$. By Brauer's Third Main Theorem (see [Lin18b, Theorem 6.3.14]) the block b_Q must coincide with the principal block $B_0(\mathbf{C}_G(Q))$ of $\mathbf{C}_G(Q)$ and we conclude that the assignment $Q \mapsto (Q, B_0(\mathbf{C}_G(Q)))$ defines an isomorphism of posets. The map of simplicial complexes obtained by extending this assignment to each simplex of $\Delta(\mathcal{S}_\ell^\star(G))$ is then a homeomorphism. □

In the rest of this section, we reduce the study of the homotopy type of the simplicial complex $\Delta(\mathcal{S}_\ell^\star(B))$ to that of certain subcomplexes whose properties reflects the behaviour of the connected subpairs of Cabanes and Enguehard (see [CE99, Definition-Proposition 2.1]). These results will be used in Section 3 to prove Theorem A. First, we define the set $\mathrm{Ab}_\ell^\star(B)$ consisting of those Brauer pairs (Q, b_Q) belonging to $\mathcal{S}_\ell^\star(B)$ and with Q abelian. The following result extends a well known property of the Brown complex to the simplicial complex of Brauer pairs.

Lemma 1.2. *Let B be an ℓ-block of the finite group G. Then the inclusion of G-posets $\iota : \mathrm{Ab}_\ell^\star(B) \to \mathcal{S}_\ell^\star(B)$ induces a G-homotopy equivalence $\Delta(\iota) : \Delta(\mathrm{Ab}_\ell^\star(B)) \to \Delta(\mathcal{S}_\ell^\star(B))$.*

Proof. Without loss of generality we may assume that B has non-trivial defect, for if otherwise both $\mathcal{S}_\ell^\star(B)$ and $\mathrm{Ab}_\ell^\star(B)$ are empty. Now, fix an element (Q, b_Q) of $\mathcal{S}_\ell^\star(B)$ and denote by $\mathcal{X}$ the set of pairs (P, b_P) of $\mathrm{Ab}_\ell^\star(B)$ satisfying $(P, b_P) \leq (Q, b_Q)$. By Quillen's Theorem A [Qui78, Proposition 1.6] (see also the statement given in [Ros23a, Lemma 1.1]) it is enough to show that the simplicial complex $\Delta(\mathcal{X})$ is $\mathbf{N}_G(Q, b_Q)$-contractible. If c is the unique block of $\mathbf{C}_G(\mathbf{Z}(Q))$ such that $(\mathbf{Z}(Q), c) \leq (Q, b_Q)$ (see [Lin18b, Theorem 6.3.3]), then $(\mathbf{Z}(Q), c)$ is $\mathbf{N}_G(Q, b_Q)$-invariant and we claim that $\Delta(\mathcal{X})$ is $\mathbf{N}_G(Q, b_Q)$-join contractible via $(\mathbf{Z}(Q), c)$ (see [Ros23a, Section 1]). Let $(P, b_p) \in \mathcal{X}$. Since P is contained in Q, we deduce that $\mathbf{Z}(Q)$ centralises P and therefore $P\mathbf{Z}(Q)$ is a well-defined abelian ℓ-subgroup contained in Q. By using [Lin18b, Theorem 6.3.3] once again, we can now find a unique block c_P of $\mathbf{C}_G(P\mathbf{Z}(Q))$ such that $(P\mathbf{Z}(Q), c_P) \leq (Q, b_Q)$. Furthermore, the uniqueness part of [Lin18b, Theorem 6.3.3] implies that $(P\mathbf{Z}(Q), c_P)$ is the join of (P, b_P) and $(\mathbf{Z}(Q), c)$ and therefore $\Delta(\mathcal{X})$ is $\mathbf{N}_G(Q, b_Q)$-contractible according to [Ros23a, Corollary 1.3]. □

The above lemma still holds if we replace $\mathrm{Ab}_\ell^\star(B)$ with the subposet of Brauer pairs (Q, b_Q) such that Q is an elementary abelian ℓ-subgroup. We include this observation in the following remark.

Remark 1.3. By replacing $\mathbf{Z}(Q)$ with the subgroup $\Omega_1(\mathbf{Z}(Q))$ of elements of order ℓ in $\mathbf{Z}(Q)$, the above argument shows that $\Delta(\mathcal{S}_\ell^\star(B))$ is G-homotopy equivalent to the simplicial complex associated to the subposet of Brauer pairs (Q, b_Q) with Q elementary abelian. This fact, together with Lemma 1.1, can be used to recover Quillen's lemma [Qui78, Lemma 2.2] on the Brown complex.

Recall that a Brauer pair (Q, b_Q) is called *centric* if b_Q has defect $\mathbf{Z}(Q)$ in $\mathbf{C}_G(Q)$ (see, for instance, [Cab18, Definition 5.6]). In this case, it follows from [Nav98, Theorem 4.8] that $\mathbf{Z}(Q) = \mathbf{Z}(\mathbf{C}_G(Q))_\ell$ and where for any finite abelian group H we define $H_\ell := \mathbf{O}_\ell(H)$. On the other hand, observe that if (Q, b_Q) is a Brauer pair satisfying $\mathbf{Z}(Q) = \mathbf{Z}(\mathbf{C}_G(Q))_\ell$ then the defect groups of b_Q in $\mathbf{C}_G(Q)$ might still be larger than $\mathbf{Z}(Q)$.

Example 1.4. Let G be the direct of product of the cyclic group C_2 with the symmetric group S_3 and consider $\ell = 2$. Denote by $Q = \mathbf{Z}(G)$ the centre of G which is cyclic of order 2, and consider the principal block B_0 of $G = \mathbf{C}_G(Q)$. In this case, $\mathbf{Z}(Q) = Q = \mathbf{Z}(\mathbf{C}_G(Q))_\ell$ while the defect groups of B_0 have order 4. Therefore (Q, B_0) is not centric but satisfies the equality $\mathbf{Z}(Q) = \mathbf{Z}(\mathbf{C}_G(Q))_\ell$.

The above discussion leads to the following definition of almost-centric Brauer pairs.

Definition 1.5. An ℓ-subgroup Q of a finite group G is called *almost-centric* if $\mathbf{Z}(Q) = \mathbf{Z}(\mathbf{C}_G(Q))_\ell$. Furthermore, we say that a Brauer pair (Q, b_Q) is *almost-centric* if the ℓ-subgroup Q is almost-centric. We denote by $\mathcal{S}_\ell^\star(B)^{\mathrm{ac}}$ the subset of $\mathcal{S}_\ell^\star(B)$ consisting of almost-centric Brauer pairs and by $\mathrm{Ab}_\ell^\star(B)^{\mathrm{ac}}$ its intersection with $\mathrm{Ab}_\ell^\star(B)$. Observe that the action of G by conjugation on $\mathcal{S}_\ell^\star(B)$ restricts to the subsets $\mathcal{S}_\ell^\star(B)^{\mathrm{ac}}$ and $\mathrm{Ab}_\ell^\star(B)^{\mathrm{ac}}$.

It follows from the above definition that $\mathcal{S}_\ell^\star(B)^{\mathrm{ac}}$ and $\mathrm{Ab}_\ell^\star(B)^{\mathrm{ac}}$ are G-subposets of $\mathcal{S}_\ell^\star(B)$. We can then refine the statement of Lemma 1.2 and show that the homotopy type of the simplicial complex $\Delta(\mathcal{S}_\ell^\star(B))$ is determined by the abelian almost-centric Brauer pairs in $\mathrm{Ab}_\ell^\star(B)^{\mathrm{ac}}$.

Proposition 1.6. *Let B be an ℓ-block of the finite group G. Then the inclusion of posets $\iota : \mathrm{Ab}_\ell^\star(B)^{\mathrm{ac}} \to \mathcal{S}_\ell^\star(B)$ induces a G-homotopy equivalence $\Delta(\iota) : \Delta(\mathrm{Ab}_\ell^\star(B)^{\mathrm{ac}}) \to \Delta(\mathcal{S}_\ell^\star(B))$.*

Proof. Without loss of generality we may assume that B has non-trivial defect. Moreover, using Lemma 1.2, it suffices to show that the inclusion $\iota : \mathrm{Ab}_\ell^\star(B)^{\mathrm{ac}} \to \mathrm{Ab}_\ell^\star(B)$ induces a G-homotopy equivalence $\Delta(\iota) : \Delta(\mathrm{Ab}_\ell^\star(B)^{\mathrm{ac}}) \to \Delta(\mathrm{Ab}_\ell^\star(B))$. By Quillen's Theorem A [Qui78, Proposition 1.6] (we actually use the stronger form stated in [Ros23a, Lemma 1.1 (ii)]) it is enough to show that, given a Brauer pair $(Q, b_Q) \in \mathrm{Ab}_\ell^\star(B)$, the simplicial complex $\Delta(\mathcal{X})$ is $\mathbf{N}_G(Q, b_Q)$-contractible where $\mathcal{X}$ denotes the poset of pairs $(P, b_P) \in \mathrm{Ab}_\ell^\star(B)^{\mathrm{ac}}$ such that $(Q, b_Q) \leq (P, b_P)$. We claim that the pair $(\mathbf{Z}(\mathbf{C}_G(Q))_\ell, b_Q)$ is an $\mathbf{N}_G(Q, b_Q)$-invariant minimum in the poset $\mathcal{X}$ from which we conclude that $\Delta(\mathcal{X})$ is $\mathbf{N}_G(Q, b_Q)$-contractible thanks to [Ros23a, Corollary 1.3]. First, notice that because Q is abelian it is contained in $\mathbf{Z}(\mathbf{C}_G(Q))_\ell$ and therefore that $\mathbf{C}_G(Q) = \mathbf{C}_\mathbf{G}(\mathbf{Z}(\mathbf{C}_G(Q))_\ell)$ by elementary group theory. In particular, it follows that the Brauer pair $(\mathbf{Z}(\mathbf{C}_G(Q))_\ell, b_Q)$ is well-defined and belongs to $\mathrm{Ab}_\ell^\star(B)^{\mathrm{ac}}$. In addition, noticing that $\mathbf{Z}(\mathbf{C}_\mathbf{G}(Q))_\ell$ is a characteristic subgroup of $\mathbf{C}_G(Q)$ and that $\mathbf{C}_G(Q)$ is normalised by $\mathbf{N}_G(Q)$, we have that $(\mathbf{Z}(\mathbf{C}_G(Q))_\ell, b_Q)$ is invariant under the action of $\mathbf{N}_G(Q, b_Q)$. This shows that $(\mathbf{Z}(\mathbf{C}_G(Q))_\ell, b_Q)$ is an $\mathbf{N}_G(Q, b_Q)$-invariant element of $\mathcal{X}$. Suppose now that (P, b_P) is an almost-centric Brauer pair in $\mathcal{X}$. Since $Q \leq P$ and P is abelian, we deduce that $P \leq \mathbf{C}_G(P) \leq \mathbf{C}_G(Q)$ and hence that $\mathbf{Z}(\mathbf{C}_G(Q)) \leq \mathbf{C}_G(P)$. Then $\mathbf{Z}(\mathbf{C}_G(Q))_\ell \leq \mathbf{Z}(\mathbf{C}_G(P))_\ell = \mathbf{Z}(P) = P$ because P is almost-centric and thus $(\mathbf{Z}(\mathbf{C}_G(Q))_\ell, b_Q) \leq (P, b_P)$ by applying [Lin18b, Theorem 6.3.3] to the inclusions $Q \leq \mathbf{Z}(\mathbf{C}_G(Q))_\ell \leq P$ and recalling that $(Q, b_Q) \leq (P, b_P)$. Therefore $(\mathbf{Z}(\mathbf{C}_G(Q))_\ell, b_Q)$ is a minimum in the poset $\mathcal{X}$ as claimed previously and this completes the proof. □

We conclude this section with a remark on centric Brauer pairs. It follows from the proof of Proposition 1.6 that for every intermediate poset $\mathrm{Ab}^\star_\ell(B)^{\mathrm{ac}} \subseteq \mathcal{P} \subseteq \mathcal{S}^\star_\ell(B)$ the inclusion $\iota : \mathcal{P} \to \mathcal{S}^\star_\ell(B)$ induces an homotopy equivalence of $\Delta(\mathcal{S}^\star_\ell(B))$ with $\Delta(\mathcal{P})$. In particular, $\mathcal{S}^\star_\ell(B)^{\mathrm{ac}}$ and $\mathcal{S}^\star_\ell(B)$ induce homotopy equivalent simplicial complexes. It is interesting to point out that, however, this result fails if we replace almost-centric Brauer pairs with centric Brauer pairs. The following example is due to Gelvin and Møller (see [GM15, Example 6.2]).

Example 1.7. Consider $G = C_2 \times S_3$, $\ell = 2$ and B_0 the principal 2-block of G as in Example 1.4. Since $\mathbf{O}_2(G) \neq 1$, it follows from [Qui78, Proposition 2.1 and Proposition 2.4] and Lemma 1.1 that $\Delta(\mathcal{S}^\star_\ell(B_0))$ is contractible. On the other hand the poset of centric B_0-Brauer pairs induces a discrete simplicial complex consisting of three zero dimensional simplexes corresponding to the three Sylow 2-subgroups of G.

2 The defining characteristic case $\ell = p$

Let $\mathbf{G}$ be a connected reductive group defined over an algebraically closed field $\mathbb{F}$ of prime characteristic p and consider a Frobenius endomorphism $F : \mathbf{G} \to \mathbf{G}$ corresponding to an $\mathbb{F}_q$-rational structure on the algebraic variety $\mathbf{G}$ for a power q of p. From now on we restrict our attention to the case where the finite group G coincides with the finite reductive group $\mathbf{G}^F$ consisting of the $\mathbb{F}_q$-rational points in $\mathbf{G}$. In this section we assume that the defining characteristic p of $\mathbf{G}^F$ coincides with the prime ℓ with respect to which Brauer blocks are defined. The non-defining characteristic case, i.e. the case $\ell \neq p$, will be considered in Section 3. We start by recalling the following well-known identity.

Lemma 2.1. *Let $\mathbf{B}$ be an F-stable Borel subgroup of $\mathbf{G}$ with unipotent radical $\mathbf{U}$. Then $\mathbf{C}_{\mathbf{G}^F}(\mathbf{U}^F) = \mathbf{Z}(\mathbf{G}^F)\mathbf{Z}(\mathbf{U}^F)$.*

Proof. By [DM20, Corollary 12.2.4 and Proposition 12.2.14] there exists an F-stable regular unipotent element u of $\mathbf{G}^F$. Recall that u is a p-element of $\mathbf{G}^F$ and that $\mathbf{U}^F$ is a Sylow p-subgroup of $\mathbf{G}^F$ according to [DM20, Proposition 1.1.5 and Proposition 4.4.1]. We can therefore assume that u belongs to $\mathbf{U}^F$. Now, [DM20, Lemma 12.2.3] implies that $\mathbf{C}_{\mathbf{G}^F}(u) = \mathbf{Z}(\mathbf{G}^F)\mathbf{C}_{\mathbf{U}^F}(u)$ and we deduce that

$$\begin{aligned}
\mathbf{C}_{\mathbf{G}^F}(\mathbf{U}^F) &= \mathbf{C}_{\mathbf{G}^F}(u) \cap \mathbf{C}_{\mathbf{G}^F}(\mathbf{U}^F) \\
&= \mathbf{Z}(\mathbf{G}^F)\mathbf{C}_{\mathbf{U}^F}(u) \cap \mathbf{C}_{\mathbf{G}^F}(\mathbf{U}^F) \\
&= \mathbf{Z}(\mathbf{G}^F)\left(\mathbf{C}_{\mathbf{U}^F}(u) \cap \mathbf{C}_{\mathbf{G}^F}(\mathbf{U}^F)\right) \\
&= \mathbf{Z}(\mathbf{G}^F)\mathbf{Z}(\mathbf{U}^F)
\end{aligned}$$

where we used Dedekind's modular law. This completes the proof. □

Let $\mathcal{P}(\mathbf{G}, F)$ be the poset consisting of F-stable parabolic subgroups of $\mathbf{G}$ ordered by inclusion. In this paper we define the *Tits building* $\mathcal{B}(\mathbf{G}, F)$ of the finite reductive group $(\mathbf{G}, F)$ to be the associated simplicial complex $\Delta(\mathcal{P}(\mathbf{G}, F)^{\mathrm{op}})$. Here, for every given poset $\mathcal{X}$, we denote by $\mathcal{X}^{\mathrm{op}}$ the opposite poset given by reverse inclusions. What we just defined is, more precisely, the barycentric subdivision of the Tits building which is homeomorphic to it. We refer the reader to [Ben98, Section 6.8] for further details. The Tits building was shown to be homotopy equivalent to the Brown complex $\Delta(\mathcal{S}_p^\star(\mathbf{G}^F))$ by Quillen in [Qui78, Proposition 2.1 and Theorem 3.1]. The aim of this section is to extend this result to the blockwise set-up considered in this paper and prove that the simplicial complex $\Delta(\mathcal{S}_p^\star(B))$ for a p-block B of $\mathbf{G}^F$ with non-trivial defect is $\mathbf{G}^F$-homotopy equivalent to the Tits building $\mathcal{B}(\mathbf{G},F)$. This is the content of the following theorem. We remark that the homotopy equivalence for the Brown complex $\Delta(\mathcal{S}_p^\star(\mathbf{G}^F))$ can be recovered from the following statement thanks to Lemma 1.1.

Theorem 2.2. *Let B be a p-block of $\mathbf{G}^F$ with non-trivial defect and where p is the defining characteristic of $\mathbf{G}^F$. If $\mathbf{G}$ is simple and $\mathbf{G}^F$ is perfect, then the simplicial complex $\Delta(\mathcal{S}_p^\star(B))$ is $\mathbf{G}^F$-homotopy equivalent to the Tits building $\mathcal{B}(\mathbf{G},F)$.*

Proof. By [Qui78, Theorem 3.1 and Proposition 2.1] we deduce that the Brown complex $\Delta(\mathcal{S}_p^\star(\mathbf{G}^F))$ is homotopy equivalent to the Tits building $\mathcal{B}(\mathbf{G},F)$. To see that this is actually a $\mathbf{G}^F$-homotopy equivalence, notice that the poset of proper parabolic subgroups underlying the Tits building is isomorphic (via a $\mathbf{G}^F$-equivariant map) to the opposite of the poset of proper p-radical subgroups of $\mathbf{G}^F$ while the latter induces a simplicial complex that is $\mathbf{G}^F$-homotopy equivalent to the Brown complex $\Delta(\mathcal{S}_p^\star(\mathbf{G}^F))$ thanks to [TW91, Theorem 2]. Therefore, it suffices to show that $\Delta(\mathcal{S}_p^\star(B))$ is $\mathbf{G}^F$-homotopy equivalent to the Brown complex. Observe that this claim follows already from Lemma 1.1 in the case that B is the principal block. Suppose now that the block B is not principal.

Since B has non-trivial defect by assumption, [Hum71] implies that B has defect group $U = \mathbf{U}^F$ a Sylow p-subgroup of $\mathbf{G}^F$. Moreover, following the proof of [CE04, Theorem 6.18] we can find an irreducible character $1 \neq \zeta$ of $\mathbf{Z}(\mathbf{G}^F)$ that parametrises the p-block B. More precisely, by Lemma 2.1 we have $\mathbf{C}_{\mathbf{G}^F}(U) = \mathbf{Z}(\mathbf{G}^F)\mathbf{Z}(U)$ and therefore $\epsilon_\zeta := |\mathbf{Z}(\mathbf{G}^F)|^{-1}\sum_{z\in\mathbf{Z}(\mathbf{G}^F)}\zeta(z)z^{-1}$ determines a primitive idempotent of the group algebra $\mathbb{F}\mathbf{C}_{\mathbf{G}^F}(U)$ where $\mathbb{F}$ is the algebraically closed field of characteristic p over which $\mathbf{G}$ is defined. Moreover, if we denote by B_U the corresponding p-block of $\mathbf{C}_{\mathbf{G}^F}(U)$, then we have the inclusion of Brauer pairs $(1,B)\leq(U,B_U)$ and therefore (U,B_U) belongs to $\mathcal{S}_p^\star(B)$. Now, [Lin18b, Theorem 6.3.3] shows that for every non-trivial p-subgroup Q of $\mathbf{G}^F$ contained in U there exists a unique p-block B_Q of $\mathbf{C}_{\mathbf{G}^F}(Q)$ such that $(Q,B_Q)\leq(U,B_U)$ and, by applying [Lin18b, Proposition 6.3.6], it follows that (Q,B_Q) belongs to $\mathcal{S}_p^\star(B)$. We can therefore define a map of posets from $\mathcal{S}_p^\star(\mathbf{G}^F)$ to $\mathcal{S}_p^\star(B)$ by sending the p-subgroup Q to the B-Brauer pair (Q,B_Q). Observe that each Brauer pair (Q,B_Q) is uniquely determined by Q and B and thus the map defined above is an isomorphism of posets. Furthermore, since $\mathbf{G}^F$ acts trivially on B and ζ, we conclude that this map is $\mathbf{G}^F$-equivariant. We can now conclude that the induced map of simplicial complexes is a $\mathbf{G}^F$-homotopy equivalence between $\Delta(\mathcal{S}_p^\star(\mathbf{G}^F))$ and $\Delta(\mathcal{S}_p^\star(B))$ as claimed above. This concludes the proof according to the previous paragraph. □

3 The non-defining characteristic case $\ell \neq p$

We keep $\mathbf{G}$, F, p and q as in Section 2 and assume now that $p\neq\ell$. We define $e_\ell(q)$ to be the multiplicative order of q modulo ℓ. The aim of this section is to prove Theorem A and obtain a description of the homotopy type of the simplicial complex $\Delta(\mathcal{S}_\ell^\star(B))$ for each ℓ-block B of $\mathbf{G}^F$

in terms of $e_\ell(q)$-Harish-Chandra theory. This result also extends [Ros23a, Theorem A] according to Lemma 1.1.

To start, we recall the definition of the set $\pi(\mathbf{G},F)$ from [Ros23a, Definition 2.1]. Let $\mathbf{G}_{\mathrm{sc}} := ([\mathbf{G},\mathbf{G}])_{\mathrm{sc}}$ be the group introduced in [GM20, Example 1.5.3 (b)] and consider a pair $(\mathbf{G}^*,F^*)$ in duality with $(\mathbf{G},F)$ as defined in [GM20, Definition 1.5.17]. We refer the reader to [GM20, 2.7.14] for the definition of good primes.

Definition 3.1. Let $\pi'(\mathbf{G},F)$ be the set of primes ℓ that are good for $\mathbf{G}$, do not divide 2, q or $|\mathbf{Z}(\mathbf{G}_{\mathrm{sc}})^F|$, and satisfy $\ell \neq 3$ whenever $(\mathbf{G},F)$ has a rational component of type ${}^3\mathbf{D}_4$. Then, we define $\pi(\mathbf{G},F)$ to be the set of primes $\ell \in \pi'(\mathbf{G},F)$ not diving $|\mathbf{Z}(\mathbf{G})^F : \mathbf{Z}^\circ(\mathbf{G})^F|$ nor $|\mathbf{Z}(\mathbf{G}^*)^F : \mathbf{Z}^\circ(\mathbf{G}^*)^F|$.

We will make use of the theory of Φ_e-tori and e-split Levi subgroups as introduced in [BM92] where e is a positive integer. In particular, for an F-stable Levi subgroup $\mathbf{T}$, we denote by $\mathbf{T}_{\Phi_e}$ its Sylow Φ_e-torus which is well-defined thanks to [BM92, Theorem 3.4]. Then, we say that an F-stable Levi subgroup $\mathbf{L}$ of $\mathbf{G}$ is *e-split* if it satisfies $\mathbf{L} = \mathbf{C}_\mathbf{G}(\mathbf{Z}^\circ(\mathbf{L})_{\Phi_e})$. In the next lemma we collect some well-known results on centralisers of abelian ℓ-subgroups and their connection to e-split Levi subgroups.

Lemma 3.2. *Let Q be an abelian ℓ-subgroup of $\mathbf{G}^F$ and assume that ℓ is good for $\mathbf{G}$. Then:*

(i) $\mathbf{H} := \mathbf{C}^\circ_\mathbf{G}(Q)$ *is an F-stable Levi subgroup of $(\mathbf{G},F)$;*

(ii) $\mathbf{L} := \mathbf{C}_\mathbf{G}(\mathbf{Z}^\circ(\mathbf{H})_{\Phi_e})$ *is an e-split Levi subgroup of $(\mathbf{G},F)$ and $\mathbf{H} \leq \mathbf{L}$;*

(iii) if ℓ does not divide $|\mathbf{Z}(\mathbf{G})^F : \mathbf{Z}^\circ(\mathbf{G})^F|$, then $\mathbf{L} = \mathbf{C}^\circ_\mathbf{G}(\mathbf{Z}(\mathbf{L})^F_\ell)$;

(iv) if ℓ does not divide $|\mathbf{Z}(\mathbf{G}^*)^F : \mathbf{Z}^\circ(\mathbf{G}^*)^F|$, then $\mathbf{C}^\circ_\mathbf{G}(Q)^F = \mathbf{C}_{\mathbf{G}^F}(Q)$;

(v) if $\ell \in \pi(\mathbf{G},F)$ and $e = e_\ell(q)$, then $\mathbf{L} = \mathbf{G}$ if and only if $Q \leq \mathbf{Z}(\mathbf{G})^F_\ell$;

Proof. The statement in (i) follows from [CE04, Proposition 13.16 (ii)] while (ii) is an immediate consequence of the definition of e-split Levi subgroup. For (iii) and (iv) see [CE04, Proposition 13.19] and [CE04, Proposition 13.16 (i)] respectively. Finally (v) follows from [Ros23a, Lemma 3.10]. □

Next, we recall the notion of connected subpairs and of twisted block induction introduced in [CE99, Section 2]. We say that $(U,b_U)^\circ$ is a *connected subpair* of $(\mathbf{G},F)$ if U is an abelian ℓ-subgroup of $\mathbf{G}^F$ such that $U \leq \mathbf{C}^\circ_\mathbf{G}(U)$ and b_U is a Brauer ℓ-block of $\mathbf{C}^\circ_\mathbf{G}(U)^F$. If V is another abelian ℓ-subgroup of $\mathbf{G}^F$ with $V \leq U$, then there exists a unique Brauer ℓ-block b_V of $\mathbf{C}^\circ_\mathbf{G}(V)^F$ such that $\mathrm{Br}_U(b_V)b_U \neq 0$ in which case we write $(V,b_V)^\circ \triangleleft (U,b_U)^\circ$. Observe that, because the index of $\mathbf{C}^\circ_\mathbf{G}(U)^F$ in $\mathbf{C}_{\mathbf{G}^F}(U)$ is a power of ℓ, there exists a unique ℓ-block $\widehat{b}_U$ of $\mathbf{C}_{\mathbf{G}^F}(U)$ that covers b_U. The relation between the Brauer pair $(U,\widehat{b}_U)$ and the connected subpair $(U,b_U)^\circ$ is described in [CE99, Proposition 2.2] and will be used in what follows without further reference. Next, assume that ℓ is good for $\mathbf{G}$ and consider an $e_\ell(q)$-split Levi subgroup $\mathbf{L}$ of $(\mathbf{G},F)$. For any ℓ-block $b_\mathbf{L}$ of $\mathbf{L}^F$ it was shown in [CE99, Theorem 2.5], using Deligne–Lusztig induction, that $b_\mathbf{L}$ corresponds to a unique ℓ-block $b_\mathbf{G}$ of $\mathbf{G}^F$. This uniquely defined ℓ-block of $\mathbf{G}^F$ is denoted by $b_\mathbf{G} = \mathbf{R}^\mathbf{G}_\mathbf{L}(b_\mathbf{L})$ (see [CE99, Notation 2.6]). Furthermore, whenever $\mathbf{L} = \mathbf{C}^\circ_\mathbf{G}(\mathbf{Z}(\mathbf{L})^F_\ell)$, we have the inclusion of connected subpairs $(1,\mathbf{R}^\mathbf{G}_\mathbf{L}(b_\mathbf{L}))^\circ \triangleleft (\mathbf{Z}(\mathbf{L})^F_\ell, b_\mathbf{L})^\circ$.

Using the notion of twisted block induction we can now introduce an analogue of the simplicial complex $\Delta(\mathcal{S}_\ell^\star(B))$ adapted to finite reductive groups. These two complexes will then be shown to be homotopy equivalent.

Definition 3.3. Assume that ℓ is good for $\mathbf{G}$ and let $e = e_\ell(q)$. For every ℓ-block B of $\mathbf{G}^F$ we define the set $\mathcal{L}_e^\star(B)$ of pairs $(\mathbf{L}, b_\mathbf{L})$ where $\mathbf{L}$ is an e-split Levi subgroup of $(\mathbf{G}, F)$ such that $\mathbf{L} < \mathbf{G}$ and $b_\mathbf{L}$ is an ℓ-block of $\mathbf{L}^F$ satisfying $\mathbf{R}_\mathbf{L}^\mathbf{G}(b_\mathbf{L}) = B$.

Observe that $\mathbf{G}^F$-acts by conjugation on the set $\mathcal{L}_e^\star(B)$ and consider the order relation on $\mathcal{L}_e^\star(B)$ given by $(\mathbf{L}, b_\mathbf{L}) \leq (\mathbf{K}, b_\mathbf{K})$ if and only if $\mathbf{R}_\mathbf{L}^\mathbf{K}(b_\mathbf{L}) = b_\mathbf{K}$, for every $(\mathbf{L}, b_\mathbf{L})$ and $(\mathbf{K}, b_\mathbf{K})$ belonging to $\mathcal{L}_e^\star(B)$. Then, we can construct a $\mathbf{G}^F$-simplicial complex $\Delta(\mathcal{L}_e^\star(B))$ as explained in Section 1. We point out that the order relation $\leq$ defined above is closely related to the one introduced in [CE99, Notation 1.11] (see also [Ros22, Section 3 and Section 4]).

Our aim is now to show that the simplicial complex of Brauer pairs $\Delta(\mathcal{S}_\ell^\star(B))$ is homotopy equivalent to $\Delta(\mathcal{L}_e^\star(B))$. First we construct a suitable underlying map of posets. By Proposition 1.6 we can restrict our attention to the poset of almost-centric abelian Brauer pairs.

Proposition 3.4. *Suppose that $\ell \in \pi(\mathbf{G}, F)$ does not divide the order of $\mathbf{Z}(\mathbf{G})^F$. For every ℓ-block B of $\mathbf{G}^F$ with non-trivial defect there exists a map of $\mathbf{G}^F$-posets*

$$\phi : \mathrm{Ab}_\ell^\star(B)^{\mathrm{ac}} \to \mathcal{L}_e^\star(B)^{\mathrm{op}}$$

given by sending a Brauer pair (Q, b_Q) to the pair $(\mathbf{L}, b_\mathbf{L})$ where $\mathbf{L} = \mathbf{C}_\mathbf{G}(\mathbf{Z}^\circ(\mathbf{C}_\mathbf{G}^\circ(Q))_{\Phi_e})$ and the block $b_\mathbf{L}$ is determined by the inclusion $(\mathbf{Z}(\mathbf{L})_\ell^F, b_\mathbf{L}) \leq (Q, b_Q)$.

Proof. Fix $(Q, b_Q) \in \mathrm{Ab}_\ell^\star(B)^{\mathrm{ac}}$ and define $\mathbf{H} := \mathbf{C}_\mathbf{G}^\circ(Q)$ and $\mathbf{L} := \mathbf{C}_\mathbf{G}(\mathbf{Z}^\circ(\mathbf{H})_{\Phi_e})$. By Lemma 3.2 (i-ii) we know that $\mathbf{H}$ is an F-stable Levi subgroup and $\mathbf{L}$ an e-split Levi subgroup of $(\mathbf{G}, F)$ with $\mathbf{H} \leq \mathbf{L}$. In particular, it follows from the definition of Levi subgroup that $\mathbf{Z}(\mathbf{L})_\ell^F \leq \mathbf{Z}(\mathbf{H})_\ell^F$. On the other hand, $\mathbf{Z}(\mathbf{H})_\ell^F = \mathbf{Z}(\mathbf{H}^F)_\ell$ while $\mathbf{H}^F = \mathbf{C}_{\mathbf{G}^F}(Q)$ by Lemma 3.2 (iv). Now, using the fact that Q is almost-centric, we have $\mathbf{Z}(\mathbf{H})_\ell^F = \mathbf{Z}(\mathbf{C}_{\mathbf{G}^F}(Q))_\ell = Q$ and therefore we conclude that $\mathbf{Z}(\mathbf{L})_\ell^F \leq Q$. Then, according to [Lin18b, Theorem 6.3.3] there exists a unique block $b_\mathbf{L}$ of $\mathbf{C}_{\mathbf{G}^F}(\mathbf{Z}(\mathbf{L})_\ell^F)$ such that $(\mathbf{Z}(\mathbf{L})_\ell^F, b_\mathbf{L}) \leq (Q, b_Q)$. Observe that $\mathbf{C}_{\mathbf{G}^F}(\mathbf{Z}(\mathbf{L})_\ell^F) = \mathbf{L}^F$ by Lemma 3.2 (iii-iv) and so $b_\mathbf{L}$ is a block of $\mathbf{L}^F$. We now define

$$\phi(Q, b_Q) := (\mathbf{L}, b_\mathbf{L})$$

and claim that ϕ is a well-defined map of $\mathbf{G}^F$-posets. To start, we check that the pair $(\mathbf{L}, b_\mathbf{L})$ actually belongs to $\mathcal{L}_e^\star(B)$. Since Q is non-trivial and ℓ does not divide the order of $\mathbf{Z}(\mathbf{G})^F$, it follows from Lemma 3.2 (v) that $\mathbf{L} < \mathbf{G}$. Next, once again using the fact that $\mathbf{L} = \mathbf{C}_\mathbf{G}^\circ(\mathbf{Z}(\mathbf{L})_\ell^F)$, we can apply [CE99, Theorem 2.5] to obtain the inclusion of connected subpairs $(1, \mathbf{R}_\mathbf{L}^\mathbf{G}(b_\mathbf{L}))^\circ \trianglelefteq (\mathbf{Z}(\mathbf{L})_\ell^F, b_\mathbf{L})^\circ$. The latter is equivalent to the inclusion of Brauer pairs $(1, \mathbf{R}_\mathbf{L}^\mathbf{G}(b_\mathbf{L})) \leq (\mathbf{Z}(\mathbf{L})_\ell^F, b_\mathbf{L})$ according to [CE99, Proposition 2.2 (v)] because ℓ does not divide $|\mathbf{Z}(\mathbf{G}^*)^F : \mathbf{Z}^\circ(\mathbf{G}^*)^F|$. Then, we get

$$\left(1, \mathbf{R}_\mathbf{L}^\mathbf{G}(b_\mathbf{L})\right) \leq \left(\mathbf{Z}(\mathbf{L})_\ell^F, b_\mathbf{L}\right) \leq (Q, b_Q)$$

and therefore that $\mathbf{R}_\mathbf{L}^\mathbf{G}(b_\mathbf{L}) = B$ because (Q, b_Q) is a B-Brauer pair and by the uniqueness part of [Lin18b, Theorem 6.3.3]. This shows that $(\mathbf{Z}(\mathbf{L})_\ell^F, b_\mathbf{L})$ belongs to the set $\mathcal{L}_e^\star(B)$. Furthermore, it follows from the construction of $(\mathbf{L}, b_\mathbf{L}) = \phi(Q, b_Q)$ that for every $g \in \mathbf{G}^F$ we have $\mathbf{L}^g = \mathbf{C}_\mathbf{G}(\mathbf{Z}^\circ(\mathbf{C}_\mathbf{G}^\circ(Q^g))_{\Phi_e})$ and $(\mathbf{L}^g, b_\mathbf{L}^g) \leq (Q^g, b_Q^g)$. In other words, we have $\phi((Q, b_Q)^g) = \phi(Q, b_Q)^g$.

To conclude, let $(P, b_P) \in \mathrm{Ab}_\ell^\star(B)^{\mathrm{ac}}$ satisfying $(Q, b_Q) \le (P, b_P)$ and set $(\mathbf{K}, b_{\mathbf{K}}) := \phi(P, b_P)$. We want to show that $(\mathbf{K}, b_{\mathbf{K}}) \le (\mathbf{L}, b_{\mathbf{L}})$. By the discussion above, we already know that $\mathbf{Z}(\mathbf{L})_\ell^F \le Q$ and similarly that $\mathbf{Z}(\mathbf{K})_\ell^F \le P$. Moreover, using the fact that $Q \le P$, we deduce that $\mathbf{C}^\circ_{\mathbf{G}}(P) \le \mathbf{C}^\circ_{\mathbf{G}}(Q)$ and therefore that $\mathbf{K} \le \mathbf{L}$ and $\mathbf{Z}(\mathbf{L})_\ell^F \le \mathbf{Z}(\mathbf{K})_\ell^F$.

$$\begin{tikzcd}
 & & P & b_P \\
b_Q & Q \arrow[ur, dash] & & \\
 & & \mathbf{Z}(\mathbf{K})_\ell^F \arrow[u, dash] & b_{\mathbf{K}} \\
b_{\mathbf{L}}, c_{\mathbf{L}} & \mathbf{Z}(\mathbf{L})_\ell^F \arrow[uu, dash] \arrow[ur, dash] & &
\end{tikzcd}$$

By the definition of ϕ we know that $(\mathbf{Z}(\mathbf{L})_\ell^F, b_{\mathbf{L}}) \le (Q, b_Q)$ and $(\mathbf{Z}(\mathbf{K})_\ell^F, b_{\mathbf{K}}) \le (P, b_P)$. Let now $c_{\mathbf{L}}$ be the unique block satisfying $(\mathbf{Z}(\mathbf{L})_\ell^F, c_{\mathbf{L}}) \le (\mathbf{Z}(\mathbf{K})_\ell^F, b_{\mathbf{K}})$ and observe that it satisfies $(\mathbf{Z}(\mathbf{L})_\ell^F, c_{\mathbf{L}}) \le (P, b_P)$. On the other hand, since $(Q, b_Q) \le (P, b_P)$ we get $(\mathbf{Z}(\mathbf{L})_\ell^F, b_{\mathbf{L}}) \le (P, b_P)$ and the uniqueness of [Lin18b, Theorem 6.3.3] implies that $c_{\mathbf{L}} = b_{\mathbf{L}}$. This shows that $(\mathbf{Z}(\mathbf{L})_\ell^F, b_{\mathbf{L}}) \le (\mathbf{Z}(\mathbf{K})_\ell^F, b_{\mathbf{K}})$ holds inside $\mathbf{G}$ which is true if and only if $(1, b_{\mathbf{L}}) \le (\mathbf{Z}(\mathbf{K})_\ell^F, b_{\mathbf{K}})$ holds inside $\mathbf{L}$ according to [CE99, Proposition 2.2 (i)] and because ℓ does not divide $|\mathbf{Z}(\mathbf{G}^*)^F : \mathbf{Z}^\circ(\mathbf{G}^*)^F|$. Then, by applying [CE99, Theorem 2.5] to the block $b_{\mathbf{K}}$ of the e-split Levi subgroup $\mathbf{K}$ of $(\mathbf{L}, F)$, we conclude that $b_{\mathbf{L}} = \mathbf{R}_{\mathbf{L}}^{\mathbf{K}}(b_{\mathbf{K}})$. This finally shows that $(\mathbf{K}, b_{\mathbf{K}}) \le (\mathbf{L}, b_{\mathbf{L}})$ as wanted and the proof is now complete. □

Next, we show that the fibres of the map ϕ are contractible. Recall that, given a poset $\mathcal{X}$ and an element $x \in \mathcal{X}$, we denote by $\mathcal{X}_{\ge x}$ the subposet consisting of those $y \in \mathcal{X}$ such that $x \le y$.

Lemma 3.5. *Consider the setting of Proposition 3.4. For every pair $(\mathbf{L}, b_{\mathbf{L}})$ of $\mathcal{L}_e^\star(B)$, the simplicial complex $\Delta(\mathcal{X})$ induced by the fibre $\mathcal{X} := \phi^{-1}(\mathcal{L}_e^\star(B)^{\mathrm{op}}_{\ge(\mathbf{L}, b_{\mathbf{L}})})$ is $\mathbf{N}_{\mathbf{G}^F}(\mathbf{L}, b_{\mathbf{L}})$-contractible.*

Proof. We claim that the pair $(\mathbf{Z}(\mathbf{L})_\ell^F, b_{\mathbf{L}})$ is the minimum of the poset $\mathcal{X}$. First, observe that $\mathbf{L}^F = \mathbf{C}_{\mathbf{G}^F}(\mathbf{Z}(\mathbf{L})_\ell^F)$ by Lemma 3.2 (iii-iv) and therefore that $\mathbf{Z}(\mathbf{L})_\ell^F$ is an almost-centric abelian ℓ-subgroup as defined in Section 1. Then, since $(\mathbf{L}, b_{\mathbf{L}})$ coincides with $\phi(\mathbf{Z}(\mathbf{L})_\ell^F, b_{\mathbf{L}})$, it follows that $(\mathbf{Z}(\mathbf{L})_\ell^F, b_{\mathbf{L}})$ belongs to the fibre $\mathcal{X}$. Consider now a Brauer pair $(P, b_P) \in \mathcal{X}$ and set $(\mathbf{K}, b_{\mathbf{K}}) := \phi(P, b_P)$. By the definition of $\mathcal{X}$ we have $(\mathbf{K}, b_{\mathbf{K}}) \le (\mathbf{L}, b_{\mathbf{L}})$ and so $b_{\mathbf{L}} = \mathbf{R}_{\mathbf{K}}^{\mathbf{L}}(b_{\mathbf{K}})$. Then, [CE99, Theorem 2.5] shows that $(1, b_{\mathbf{L}})^\circ \le (\mathbf{Z}(\mathbf{K})_\ell^F, b_{\mathbf{K}})^\circ$ holds in $(\mathbf{L}, F)$. By [CE99, Proposition 2.2 (i) and (v)] the latter is equivalent to the inclusion of Brauer pairs $(\mathbf{Z}(\mathbf{L})_\ell^F, b_{\mathbf{L}}) \le (\mathbf{Z}(\mathbf{K})_\ell^F, b_{\mathbf{K}})$. However, by the definition of ϕ, we know that $(\mathbf{Z}(\mathbf{K})_\ell^F, b_{\mathbf{K}}) \le (P, b_P)$ and therefore we conclude that $(\mathbf{Z}(\mathbf{L})_\ell^F, b_{\mathbf{L}}) \le (P, b_P)$. This shows that $(\mathbf{Z}(\mathbf{L})_\ell^F, b_{\mathbf{L}})$ is the minimum of $\mathcal{X}$ and, since $(\mathbf{Z}(\mathbf{L})_\ell^F, b_{\mathbf{L}})$ is also $\mathbf{N}_{\mathbf{G}^F}(\mathbf{L}, b_{\mathbf{L}})$-invariant, the result follows by applying [Ros23a, Corollary 1.3]. □

We can finally prove Theorem A. This follows by combining Proposition 1.6, Proposition 3.4, and Lemma 3.5.

Proof of Theorem A. Set $e = e_\ell(q)$. By applying Proposition 1.6 and recalling that opposite posets induce homeomorphic simplicial complexes, it is enough to show that $\Delta(\mathrm{Ab}_\ell^\star(B)^{\mathrm{ac}})$ is $\mathbf{G}^F$-homotopy equivalent to $\Delta(\mathcal{L}_e^\star(B)^{\mathrm{op}})$. For this purpose, consider the map ϕ of $\mathbf{G}^F$-posets constructed in Proposition 3.4. Then, the induced map $\Delta(\phi)$ of simplicial complexes is a $\mathbf{G}^F$-homotopy equivalence according to Quillen's Theorem A (see the formulation given in [Ros23a, Lemma 1.1]) whose hypotheses are satisfied thanks to Lemma 3.5. □

We conclude the paper by observing that [Ros23a, Theorem A] is a direct consequence of our Theorem A. In fact, by Lemma 1.1 we know that the Brown complex $\Delta(\mathcal{S}_\ell^\star(\mathbf{G}^F))$ is $\mathbf{G}^F$-homotopy equivalent to the simplicial complex $\Delta(\mathcal{S}_\ell^\star(B_0(\mathbf{G}^F)))$ where $B_0(\mathbf{G}^F)$ denotes the principal ℓ-block of $\mathbf{G}^F$. On the other hand, if we denote by $\mathcal{L}_{e_\ell(q)}^\star(\mathbf{G},F)$ the poset of proper e-split Levi subgroups of $(\mathbf{G},F)$ (see [Ros23a, Section 1.4]), then the assignment $\mathbf{L}\mapsto(\mathbf{L},B_0(\mathbf{L}^F))$ defines an isomorphism of between the $\mathbf{G}^F$-posets $\mathcal{L}_{e_\ell(q)}^\star(\mathbf{G},F)$ and $\mathcal{L}_{e_\ell(q)}^\star(B_0(\mathbf{G}^F))$ thanks to Brauer's Third Main Theorem (see [Lin18b, Theorem 6.3.14]). Finally, Theorem A implies that $\Delta(\mathcal{S}_\ell^\star(B_0(\mathbf{G}^F)))$ is $\mathbf{G}^F$-homotopy equivalent to $\Delta(\mathcal{L}_{e_\ell(q)}^\star(B_0(\mathbf{G}^F)))$ and we conclude that $\Delta(\mathcal{S}_\ell^\star(\mathbf{G}^F))$ is $\mathbf{G}^F$-homotopy equivalent to $\Delta(\mathcal{L}_{e_\ell(q)}^\star(\mathbf{G},F))$ as in [Ros23a, Theorem A].

References

[AB79] J. Alperin and M. Broué. Local methods in block theory. *Ann. of Math. (2)*, 110(1):143–157, 1979.

[Ben98] D. J. Benson. *Representations and cohomology. II*, volume 31 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 1998. Cohomology of groups and modules.

[BM92] M. Broué and G. Malle. Théorèmes de Sylow génériques pour les groupes réductifs sur les corps finis. *Math. Ann.*, 292(2):241–262, 1992.

[Bro75] K. S. Brown. Euler characteristics of groups: the p-fractional part. *Invent. Math.*, 29(1):1–5, 1975.

[Cab18] M. Cabanes. Local methods for blocks of finite simple groups. In *Local representation theory and simple groups*, EMS Ser. Lect. Math., pages 179–265. Eur. Math. Soc., Zürich, 2018.

[CE99] M. Cabanes and M. Enguehard. On blocks of finite reductive groups and twisted induction. *Adv. Math.*, 145(2):189–229, 1999.

[CE04] M. Cabanes and M. Enguehard. *Representation theory of finite reductive groups*, volume 1 of *New Mathematical Monographs*. Cambridge University Press, Cambridge, 2004.

[DM20] F. Digne and J. Michel. *Representations of finite groups of Lie type*, volume 95 of *London Mathematical Society Student Texts*. Cambridge University Press, second edition, 2020.

[GM20] M. Geck and G. Malle. *The character theory of finite groups of Lie type: A guided tour*, volume 187 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, 2020.

[GM15] M. Gelvin and J. M. Møller. Homotopy equivalences between p-subgroup categories. *J. Pure Appl. Algebra*, 219(7):3030–3052, 2015.

[Hum71] J. E. Humphreys. Defect groups for finite groups of Lie type. *Math. Z.*, 119:149–152, 1971.

[Lin18a] M. Linckelmann. *The block theory of finite group algebras. Vol. I*, volume 91 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 2018.

[Lin18b] M. Linckelmann. *The block theory of finite group algebras. Vol. II*, volume 92 of *London Mathematical Society Student Texts.* Cambridge University Press, Cambridge, 2018.

[Nav98] G. Navarro. *Characters and blocks of finite groups*, volume 250 of *London Mathematical Society Lecture Note Series.* Cambridge University Press, Cambridge, 1998.

[Qui78] D. Quillen. Homotopy properties of the poset of nontrivial p-subgroups of a group. *Adv. in Math.*, 28(2):101–128, 1978.

[Ros22] D. Rossi. Counting conjectures and e-local structures in finite reductive groups. *arXiv:2204.00428*, 2022.

[Ros23a] D. Rossi. The Brown complex in non-defining characteristic and applications. *arXiv:2303.13973*, 2023.

[Ros23b] D. Rossi. A local-global principle for unipotent characters. *arXiv:2301.05151*, 2023.

[TW91] J. Thévenaz and P. J. Webb. Homotopy equivalence of posets with a group action. *J. Combin. Theory Ser. A*, 56(2):173–181, 1991.

Mathematisches Forschungsinstitut Oberwolfach
Schwarzwaldstr. 9-11,
77709 Oberwolfach-Walke,
Germany

Email address: damiano.rossi.math@gmail.com