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Differentialgeometrie im Grossen

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Abstract. Over the past several decades, classical differential geometry has undergone a remarkable expansion, helped by the integration of tools and insights from neighboring fields like partial differential equations, complex analysis, and geometric topology. In keeping with the spirit of previous gatherings, this meeting aimed to bridge the gaps between researchers working in seemingly disparate subfields of differential geometry, illuminating the connections that unite them.

Amongst other things, this meeting was centered around the theme of scalar curvature, which has recently emerged as a fundamental element across various fields, including differential geometry, metric geometry, topology, and complex geometry. This shared topic presented an ideal opportunity for scholars from these distinct areas to convene, discuss their individual progress, and foster a vibrant exchange of ideas.

Mathematics Subject Classification (2020): 53-XX.

Introduction by the Organizers

The workshop "Differentialgeometrie im Großen" was held from July 2 to July 7, 2023. Bringing together experts in the field of differential geometry and its adjacent disciplines, the event encompassed a diverse range of topics.

There were five 50-minute talks on Monday, Tuesday (three in the morning and two in the afternoon) and three talks in the mornings of Wednesday and Friday. On Thursday, there were three 50-minute talks in the morning, one in the afternoon, as well as three 20-min short talks by young scholars. On Wednesday afternoon, there was a hike. The number of participants on-site was 43 with a few additional participants who tuned into the Zoom stream.

On Monday, the talks delved into geometric implications of scalar curvature and Ricci curvature – with an intriguing highlight on the recent negative resolution of Milnor's Conjecture – Kähler-Ricci flow and Seiberg-Witten theory.

Tuesday morning continued with talks centered around genericity for minimal hypersurfaces (which play a key role in comparison geometry), diameter estimates in Kähler geometry and Einstein 4-manifolds. Tuesday afternoon was devoted to two talks that investigated the subtleties of coordinates at infinity of asymptotically Euclidean manifolds, particularly in connection with the ADM-mass and the Positive Mass Theorem.

The talks on Wednesday focused on minimal submanifolds and mean curvature flow of codimension two, locally conformal Kähler metrics and epsilon regularity in the setting of lower scalar curvature bounds.

Thursday's main talks were on topological obstructions to positive scalar curvature on manifolds with boundary in higher dimensions, macroscopic scalar curvature, Ricci curvature in dimension 3 and holonomy of limits of Einstein 4-manifolds. The short talks by junior researchers were about Morse index of minimal surfaces, Kähler-Einstein metrics and the uniqueness of certain minimal surfaces in the sphere and the ball.

We ended the workshop on Friday with talks on genericity in mean curvature flow, expanding Ricci solitons and the relation of positive scalar curvature with the Urysohn width.

The gathering offered a thorough survey of present-day progress in differential geometry, spotlighting key advancements that have shaped the field. Distinguished researchers from diverse corners of the world enriched the event, encompassing both aspiring graduate students and senior researchers in their respective subfields. The workshop exuded a lively and amiable atmosphere, fostering rich mathematical dialogue and fruitful collaborative efforts, further elevated by the exceptional service of the staff and the picturesque surroundings of Oberwolfach.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, "US Junior Oberwolfach Fellows".

Workshop: Differentialgeometrie im Grossen

Table of Contents

Abstracts

Scalar curvature rigidity of warped product metrics

Bernhard Hanke
(joint work with Christian Bär, Simon Brendle, Yipeng Wang)

In the following, g_{S^n} denotes the standard round Riemannian metric on the n-dimensional sphere S^n.

Theorem 1 (Cecchini-Zeidler [2]). *Let (M,g) be a compact connected Riemannian spin manifold of dimension n where $n \geq 3$ is odd, let $\rho\colon [\theta_-,\theta_+] \to \mathbb{R}$ be a positive, strictly log-concave function (that is, $(\log \rho)'' < 0$), and let*

$$\Phi = (\varphi,\theta)\colon (M,g) \to \big(S^{n-1} \times [\theta_-,\theta_+], g_0 := d\theta \otimes d\theta + \rho(\theta)^2 g_{S^{n-1}}\big)$$

be a smooth 1-Lipschitz map which satisfies $\Phi(\partial M) \subset S^{n-1} \times \{-\theta,+\theta\}$ and is of non-zero degree. Moreover, we assume that

(1) $R_g \geq R_{g_0} \circ \Phi$ on M,
(2) $H_g \geq H_{g_0} \circ \Phi$ on ∂M.

Here R denotes the scalar curvature, and H denotes the mean curvature along the boundary with respect to the exterior normal (i.e., H is positive on the boundary of the round ball).

Then Φ is a Riemannian isometry.

This is a "warped product" version of a classical theorem of Llarull [5] stating that for each closed connected Riemannian spin manifold (M,g) of dimension $n \geq 2$ with $R_g \geq n(n-1)$, each smooth 1-Lipschitz map of non-zero degree $(M,g) \to (S^n, g_{S^n})$ is a Riemannian isometry (for $n \geq 3$ it suffices to assume that the comparison map is Λ^2-contracting).

Similarly to [5], the proof of Theorem 1 uses spectral properties of a twisted Dirac operator

$$D_E\colon C^\infty(M, S \otimes E) \to C^\infty(M, S \otimes E)$$

where $S \to M$ is the spinor bundle, and $E := \varphi^*(\Sigma^+)$ where $\Sigma = \Sigma^+ \oplus \Sigma^- \to S^{n-1}$ is the spinor bundle of the even dimensional Riemannian spin manifold $(S^{n-1}, g_{S^{n-1}})$. In addition to the Llarull setting, one perturbs D_E by a zeroth order operator given by multiplication with the "potential" function $-\frac{in}{2}\psi(\theta)\colon M \to \mathbb{R}$ where $\psi := \frac{\rho'}{\rho}\colon [\theta_-,\theta_+] \to \mathbb{R}$.

Remark 1. *$\psi(\theta)$ is equal to the mean curvature of the level hypersurface*

$$S^{n-1} \times \{\theta\} \subset (S^{n-1} \times [\theta_-,\theta_+], g_0)$$

with respect to the unit normal field $\frac{\partial}{\partial\theta}$.

The index of the Fredholm operator $D_E - \frac{in}{2}\psi(\theta)$ acting on spinor fields $s \in C^\infty(M, S \otimes E)$ subject to the local boundary condition (where ν is the exterior normal and $\bullet$ indicates Clifford multiplication)

$$s = \mp i\nu \bullet s \quad \text{on} \quad \partial M \cap \Phi^{-1}(S^{n-1} \times \{\pm\theta\}) \tag{1}$$

can be shown to be positive, possibly after interchanging Σ^+ and Σ^- and working with the negative of the boundary condition (1). Essentially, this is due to the fact that $\deg(\varphi|_{\partial_- M} \colon \partial_- M \to S^{n-1}) \neq 0$ by assumption, and the Euler characteristic of S^{n-1} is different from zero as $n-1$ is even. We hence obtain a spinor field $s \in C^\infty(M, S \otimes E)$ satisfying (1) on ∂M and $D_E s - \frac{in}{2}\psi(\theta)s = 0$ on M.

Recall the Schrödinger-Lichnerowicz-Weitzenböck (SLW) formula

$$D_E^2 = (\nabla^{S\otimes E})^* \circ \nabla^{S\otimes E} + \tfrac{1}{4}R_g + \mathcal{R}^E \tag{2}$$

where $\nabla^{S\otimes E}$ is the connection on $S \otimes E$ and $\mathcal{R}^E$ is the curvature endomorphim of E. Evaluation of an integrated version of the SLW formula on the spinor field s shows that

$$\nabla_X^{S\otimes E} s + \tfrac{i}{2}\psi(\theta)\, X \bullet s = 0 \tag{3}$$

for all vector fields X on M. In this step it is important that the boundary term in the integrated SLW formula has a favourable sign due to the assumption on H_g in Theorem 1 and because s satisfies (1). It follows from (3) that $s \neq 0$ at each point in M.

Using these facts, an evaluation of the (pointwise) SLW formula (2) on s together with an estimate of $\mathcal{R}^E$ appearing in [5] shows that Φ is a Riemannian isometry, finishing the proof of Theorem 1.

Our first result removes the assumption that n is odd.

Theorem 2 (Bär-Brendle-H.-Wang [1]). *Theorem 1 also holds for even $n \geq 4$.*

The generalisation uses a "suspension trick" which already appears in [5]: Assuming that $n \geq 4$ is even, instead of Φ, one studies the composition

$$M \times S^1_r \xrightarrow{\Phi\times \mathrm{id}} S^{n-1} \times [\theta_-, \theta_+] \times S^1_r \to S^n \times [\theta_-, \theta_+] \tag{4}$$

where S^1_r, $r > 0$, is the 1-sphere equipped with the metric $r^2 \cdot g_{S^1}$ and the second map in this composition involves a 1-Lipschitz map

$$h \colon (S^{n-1} \times S^1, g_{S^{n-1}} + 4g_{S^1}) \to (S^n, g_{S^n})$$

of degree ± 1, where h is independent of r. For the composition (4), the index of the twisted Dirac operator with potential, as considered before, is positive. Theorem 2 is proven by passing to the limit of a sequence of spinor fields in the kernel of this operator as r goes to ∞ and application of the SLW formula to the limit spinor field.

Our second result is a comparison theorem for non-compact and possibly non-complete Riemannian manifolds. It can be considered as a limit case of Theorem 2.

Theorem 3 (Bär-Brendle-H.-Wang [1]). *Let $n \geq 3$, let (Ω, g) be an n-dimensional connected non-compact Riemannian spin manifold without boundary satisfyig $R_g \geq n(n-1)$, and let*

$$\Phi \colon (\Omega, g) \to \big(S^{n-1} \times (0,\pi), d\theta \otimes d\theta + \sin^2(\theta)\, g_{S^{n-1}}\big)$$

be a proper smooth 1-Lipschitz map of non-zero degree.

Then Φ is a Riemannian isometry.

Note that the target of Φ is the n-sphere with north and south poles removed. The idea for the proof of Theorem 3 is to consider

$$M_\delta := \Phi^{-1}(S^{n-1} \times [\delta, \pi - \delta]) \subset \Omega,$$

which is a smooth compact manifold with boundary for generic $\delta \in (0, \frac{\pi}{4})$, and apply Theorem 1, resp. 2, to the restriction $M_\delta \to S^{n-1} \times [\delta, \pi - \delta]$ of Φ.

Of course, the mean curvature condition in Theorem 1 need not be satisfied for this restricted map. This problem is resolved by adjusting the potential function $\psi = \frac{\sin'}{\sin} = \frac{\cos}{\sin} \colon [\delta, \pi - \delta] \to \mathbb{R}$ so that the boundary term in the integrated SLW formula still has a favourable sign.

With some care, for $\delta \to 0$, one obtains a non-zero limit spinor field on Ω satisfying (3) for the original potential function $\psi = \frac{\cos}{\sin}$. From this point, the proof is finished similarly as before.

A proof of Theorem 3 using μ-bubbles is sketched in Gromov's "Four Lectures" [3] for $n \leq 8$. Moreover, a proof for $n = 3$ based on space-time harmonic functions appears in Hirsch-Kazaras-Khuri-Zhang [4], and a spinor geometric proof for $n \geq 3$ was announced by Wang-Xie [6] shortly after [1].

Gromov asked the following question:

Question 1 ([3]). Which subsets $\Sigma \subset S^n$, $n \geq 3$, are "scalar curvature rigid" in the sense that each Riemannian metric g on $S^n \setminus \Sigma$ with $g \geq g_{S^n}$ and $R_g \geq R_{g_{S^n}}$ satisfies $g = g_{S^n}$?

By [5] and Theorem 3, we know that this holds for $\Sigma = \emptyset$ and for Σ consisting of two antipodal points (hence also for Σ consisting of one point). For more general Σ, Question 1 remains open.

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Positive mass theorems and distance estimates for spin initial data sets

Rudolf Zeidler

(joint work with Simone Cecchini, Martin Lesourd)

The positive mass theorem is a landmark result in differential geometry connecting the geometry of scalar curvature to general relativity. In the Riemannian case, the theorem states that on a complete asymptotically flat manifold of nonnegative scalar curvature, the ADM-mass of each asymptotically flat end is nonnegative.

Here an asymptotically flat end is an open subset $\mathcal{E}$ which admits a diffeomorphism to $\mathbb{R}^n \setminus \mathrm{D}_d(0)$ for some $d > 0$, such that the metric satisfies suitable fall-off conditions in this chart, e.g.

$$g_{ij} = \delta_{ij} + \mathcal{O}_2(|x|^{-\tau})$$

for some $\tau > \frac{n-2}{2}$. A Riemannian manifold is called asymptotically flat if outside of a compact subset it consists of finitely many asymptotically flat ends.

More generally, the (spacetime) positive mass theorem considers asymptotically flat *initial data sets* (M, g, k), where (M, g) is an asymptotically flat Riemannian manifold endowed with a symmetric 2-tensor k that has a suitable fall-off, e.g. $k_{ij} = \mathcal{O}_1(|x|^{-\tau-1})$ on the asymptotically flat ends. Conceptually, k is to be understood as the second fundamental form of M being embedded as a spacelike hypersurface in a (hypothetical) spacetime.

In this setting, nonnegativity of the scalar curvature is replaced by the *dominant energy condition* $\mu \geq |J|$, where

$$\mu = \frac{1}{2}(\mathrm{scal}_g + (\mathrm{tr}_g(k))^2 - |k|_g^2), \quad J = \mathrm{div}(k) - \mathrm{d}\,\mathrm{tr}_g(k).$$

The ADM mass then depends on two asymptotic quantities, the *ADM energy* E, a numerical quantity, and the *ADM linear momentum* P, a covector at infinity, both of which are defined for each asymptotically flat end. In this case the positive mass theorem states:

Theorem 1 (Positive mass theorem for initial data sets [10, 11, 9, 4])**.** *Let $(M^{n\geq 3}, g, k)$ be a complete asymptotically flat initial data set that satisfies the dominant energy condition $\mu - |J| \geq 0$. If either $n \leq 7$ or M^n is spin, then* $\mathrm{E} \geq |\mathrm{P}|$ *for each asymptotically flat end.*

Notably, this theorem can be viewed as a separate statement about each asymptotically flat end. Hence one may ask if the conclusion holds for asymptotically flat ends in more general manifolds. In fact, Schoen–Yau conjectured in the Riemannian case that the positive mass theorem holds for an asymptotically flat end inside a general complete manifold of nonnegative scalar curvature (e.g. there may be other ends about which nothing is assumed apart from completeness and the curvature condition). This was in fact settled recently by various authors leading to the following theorem: .

Theorem 2 (Positive mass theorem for complete manifolds with arbitrary ends [8, 6, 7, 12, 1, 3])**.** *Let $(M^{n\geq 3}, g)$ be a complete manifold that contains a distinguished asymptotically flat end $\mathcal{E}$ and has nonnegative scalar curvature* $\mathrm{scal} \geq 0$*. Assume either that $n \leq 7$ or that M^n is spin. Then the ADM mass of $\mathcal{E}$ satisfies* $\mathfrak{m}(\mathcal{E}) \geq 0$*, with equality if and only if (M^n, g) is Euclidean space.*

In fact Theorem 2 can be reduced to a version of Theorem 1 with compact boundary [5, 6]. The main trick here is to introduce an artificial tensor k which blows up in a finite distance to the given end in a way to “shield” the selected asymptotically flat end from the rest of the manifold.

In our recent work [2], we extend the methods established in [3] to the spacetime case and prove the initial data set version of Theorem 2 (which also follows from [1]) and a variety of new results. Start with the "shielded version" of the spacetime positive mass theorem.

Definition 1. Let (M^n, g, k) be a Riemannian manifold, not assumed to be complete. We say that (M, g, k) *contains a dominant energy shield* $U_0 \supset U_1 \supset U_2$ if U_0, U_1, and U_2 are open subsets of M such that $U_0 \supset \overline{U}_1$, $U_1 \supset \overline{U}_2$, the closure of U_0 in (M, g) is a complete manifold with compact boundary, and we have the following:

(1) $\mu - |J| \geq 0$ on U_0,
(2) $\mu - |J| \geq \sigma n(n-1)$ on $U_1 \setminus U_2$,
(3) the mean curvature $\mathrm{H}_{\partial \bar{U}_0}$ on $\partial \bar{U}_0$ and the symmetric two tensor k satisfy

$$\mathrm{H}_{\partial \bar{U}_0} - \tfrac{2}{n-1} \left| k(\nu, -)|_{\mathrm{T}\partial \bar{U}_0} \right| > -\Psi(d, l).$$

Here, $\Psi(d, l)$ is the constant defined as

$$\Psi(d, l) := \begin{cases} \frac{2}{n} \frac{\lambda(d)}{1 - l\lambda(d)} & \text{if } d < \frac{\pi}{\sqrt{\sigma} n} \text{ and } l < \frac{1}{\lambda(d)}, \\ \infty & \text{otherwise,} \end{cases}$$

where $d := \mathrm{dist}_g(\partial U_2, \partial U_1)$, $l := \mathrm{dist}_g(\partial U_1, \partial U_0)$, and

$$\lambda(d) := \frac{\sqrt{\sigma} n}{2} \tan\left(\frac{\sqrt{\sigma} n d}{2}\right).$$

More importantly than the precise expression, the crucial aspect of $\Psi(d, l)$ is that it tends to ∞ as d approaches a certain fixed quantity (and similarly for l after fixed $d > 0$). This means that the boundary condition (3) becomes vacuous after one of these thresholds are exceeded. Our first main results then states:

Theorem 3 ([2, Theorem A]). *Let $(M^{n\geq 3}, g, k)$ be an initial data set, not necessarily complete, that contains an asymptotically flat end $\mathcal{E}$ and a dominant energy shield as in Definition 1. Assume that U_0 is spin and that $\overline{\mathcal{E} \setminus U_0}$ is compact. Then $\mathrm{E}_{\mathcal{E}} > |\mathrm{P}_{\mathcal{E}}|$.*

As alluded to earlier, the proof of Theorem 3 in the time-symmetric case (i.e. $k = 0$) can be reduced to a spacetime positive mass theorem by introducing an artificial tensor k that makes use of the strictly dominant energy condition on the shield. However, in the general case this strategy fails because there is already a given k present that cannot be artificially modified without unfavorably modifying the corresponding dominant energy condition. To resolve this issue, we introduce another independent 'time' direction in the spin bundle which allows us to introduce an additional term in the Dirac–Witten operator that is independent of the given tensor k. Currently our main results appear to be only accessible in the spin setting because in the context of minimal hypersurface techniques there seems to be no counterpart of introducing an additional term in an independent timelike direction.

Based on similar techniques, we obtain the following result which shows that embedding an end which violates the positive mass theorem into a complete initial data set is obstructed in a quantitative way:

Theorem 4 ([2, Theorem B])**.** *Let* $(\mathcal{E}, g, k)$ *be an asymptotically flat initial data end of dimension* $n \geq 3$ *such that* $\mathrm{E}_{\mathcal{E}} < |\mathrm{P}_{\mathcal{E}}|$*. Then there exists a constant* $R = R(\mathcal{E}, g, k)$ *such that the following holds: If* (M, g, k) *is an* n*-dimensional initial data set (without boundary) that contains* $(\mathcal{E}, g, k)$ *as an open subset and* $\mathcal{N} = \mathcal{N}_R(\mathcal{E}) \subseteq M$ *denotes the open neighborhood of radius* R *around* $\mathcal{E}$ *in* M*, then at least one of the following conditions must be violated:*

(1) $\overline{\mathcal{N}}$ *(metrically) complete,*
(2) $\mu - |J| \geq 0$ *on* $\mathcal{U}$*,*
(3) $\mathcal{N}$ *is spin.*

Finally, we deduce a new proof of the spacetime positive mass theorem with arbitrary ends and establish a corresponding rigidity result, see [2, Theorems C and D] for details.

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Ricci Curvature, Fundamental Groups, and the Milnor Conjecture

Daniele Semola
(joint work with Elia Bruè, Aaron Naber)

In 1968 John Milnor conjectured that the fundamental group of any smooth, complete Riemannian manifold (M^n, g) with nonnegative Ricci curvature should be finitely generated [8]. In recent joint work [2] with Elia Bruè and Aaron Naber we constructed a family of counterexamples to the Milnor conjecture. More precisely we proved:

Theorem 1. *Let $\Gamma \leq \mathbb{Q}/\mathbb{Z} \subseteq S^1$ be any subgroup. Then there exists a smooth complete 7-dimensional Riemannian manifold (M^7, g) with $\pi_1(M) = \Gamma$ and such that* $\mathrm{Ric} \geq 0$.

In the last sixty years, there have been several partial results toward the resolution of Milnor's conjecture. In particular, the conjecture is known to be true:

- under the stronger assumption that the sectional curvature is nonnegative, by the soul theorem of Cheeger-Gromoll;
- in dimension 3 by the work of G. Liu [7], with previous insights by Schoen-Yau [9] in the case of positive Ricci curvature;
- when the volume growth is maximal, by the work of P. Li [6], subsequently generalized, with a different argument, by Anderson [1];
- when the volume growth is minimal, as proved by Sormani [10].

Moreover, the structure of finitely generated subgroups of the fundamental group is quite well understood. Milnor proved in [8] that they have polynomial growth, hence they must be virtually nilpotent by Gromov's work [3]. More recently, Kapovitch-Wilking have shown in [4] that there is a nilpotent subgroup whose index is bounded by a dimensional constant C_n and that finitely generated subgroups have a dimensional bound on the number of generators.

In our family of counterexamples, the fundamental groups $\Gamma \leq \mathbb{Q}/\mathbb{Z}$ are indeed abelian and they have cyclic finitely generated subgroups. Moreover, the volume growth oscillates between the linear rate $\sim r$ and the $\sim r^4$ rate.

The geometry of the universal covers for the examples in Theorem 1 mimics that of a fractal snowflake. This structure is obtained through an inductive construction, with a twisted gluing mechanism at its hearth. The main technical tool in order to achieve this gluing is an equivariant version of the following:

Lemma 2. *Let g_0 be the standard metric on $S^3 \times S^3$. Then given $\phi \in \mathrm{Diff}(S^3 \times S^3)$ there exists a smooth family g_t of metrics with $\mathrm{Ric}_{g_t} > 0$ such that g_0 is the standard metric and $g_1 = \phi^* g_0$. That is, the orbit $\pi_0 \mathrm{Diff}(S^3 \times S^3) \cdot [g_0]$ of the mapping class group lives in a connected component of $\mathcal{M}_0^+(S^3 \times S^3)$, the space of metrics with strictly positive Ricci curvature.*

The nontriviality of the above lemma comes from the statement, due to Kreck [5], that $\mathrm{Diff}(S^3 \times S^3)$ is not path connected, i.e., there exist self-diffeomorphisms of $S^3 \times S^3$ that are not isotopic to the identity.

Apart from the obvious open question about the validity of Milnor's conjecture in dimensions 4, 5, and 6, the following remain open:

Question 1. Does Milnor's conjecture hold for Ricci flat manifolds?

Question 2. Does Milnor's conjecture hold for Kähler manifolds with $\mathrm{Ric} \geq 0$?

Question 3. Does Milnor's conjecture hold under the additional assumption that the universal cover $(\tilde{M}, \tilde{g})$ has Euclidean volume growth?

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Two-dimensional shrinking gradient Kähler-Ricci solitons

Ronan J. Conlon
(joint work with Richard Bamler, Charles Cifarelli, and Alix Deruelle)

A *shrinking Ricci soliton* is a triple $(M,\, g,\, X)$, where M is a Riemannian manifold endowed with a complete Riemannian metric g and a complete vector field X, such that

$$\mathrm{Ric}(g) + \frac{1}{2}\mathcal{L}_X g = \lambda g \tag{1}$$

for some $\lambda > 0$. The vector field X is called the *soliton vector field.* If $X = \nabla^g f$ for some smooth real-valued function f on M, then we say that $(M,\, g,\, X)$ is *gradient.* In this case, the soliton equation (1) becomes

$$\mathrm{Ric}(g) + \mathrm{Hess}_g(f) - \lambda g = 0,$$

and we call f the *soliton potential.*

Let (M, g, X) be a shrinking Ricci soliton. If g is Kähler and X is real holomorphic, then we say that (M, g, X) is a *shrinking Kähler-Ricci soliton.* Let ω denote the Kähler form of g. If (M, g, X) is in addition gradient, then (1) may be rewritten as

$$\rho_\omega + i\partial\bar{\partial}f = \lambda\omega,$$

where ρ_ω is the Ricci form of ω and f is the soliton potential.

Shrinking Ricci solitons are natural generalisations of Einstein manifolds with positive scalar curvature. As such, their study and classification has become a central topic in both Riemannian and Kähler geometry. Indeed, shrinking Kähler-Ricci solitons are known to exist on certain Fano manifolds that have obstructions to the existence of a Kähler-Einstein metric [WZ04]. Moreover, from the data of a complete shrinking Ricci soliton (M, g, X), one can define an ancient solution $g(t)$, $t < 0$, of the Ricci flow

$$\frac{\partial g(t)}{\partial t} = -2\,\mathrm{Ric}(g(t))$$

with $g(-1) = g$. In this way, non-flat complete shrinking Ricci solitons model finite time Type I singularities of the Ricci flow [Nab10, EMT11], i.e., those singular times of the flow for which there are curvature bounds of the form $|\mathrm{Rm}| \leq C/(T-t)$ near the finite singular time $T > 0$. The study of shrinking Ricci solitons is therefore crucial in potentially implementing higher-dimensional surgery constructions.

While in (real) dimensions 2 and 3, a full classification of shrinking Ricci solitons has been achieved (these are Euclidean or quotients of spheres S^2, S^3 or the cylinder $S^2 \times \mathbb{R}$) [Ham82, Per03], the situation is far from clear in (real) dimension 4. Apart from the obvious examples ($\mathbb{R}^4$ or quotients of $S^4, S^3 \times \mathbb{R}$, and $S^2 \times \mathbb{R}^2$) and the ten del Pezzo surfaces [Tia90, WZ04], the only other example in this dimension was found by Feldman-Ilmanen-Knopf [FIK03] nearly 20 years ago. Its construction was, in part, possible due to its cohomogeneity one $U(2)$-symmetry which allowed the reduction of the soliton equation to a system of ODEs (the solution of which nevertheless still posed a non-trivial problem).

Our main result is the existence of a new complete non-compact shrinking gradient Kähler-Ricci soliton in complex dimension 2.

Theorem 1 ([BCCD22, Theorem A])**.** *Up to automorphism, there exists a unique complete shrinking gradient Kähler-Ricci soliton with bounded scalar curvature on* $\mathrm{Bl}_x(\mathbb{C} \times \mathbb{P}^1)$*, that is, the blowup of* $\mathbb{C} \times \mathbb{P}^1$ *at a fixed point* x *of the standard real torus action on* $\mathbb{C} \times \mathbb{P}^1$*. Moreover, this soliton is invariant under the induced real torus action and appears as a parabolic blowup limit of the Kähler-Ricci flow.*

Being invariant under a real two-dimensional torus action, the new soliton has cohomogeneity 2. Its underlying complex manifold, namely the blowup of $\mathbb{C} \times \mathbb{P}^1$ at one point, was already identified in [CCD22] as the last remaining Kähler surface that was unknown to admit a complete shrinking gradient Kähler-Ricci soliton with bounded scalar curvature. Combined in addition with previous works [Cif20, CDS19], Theorem 1 completes the classification of such solitons on Kähler surfaces.

Theorem 2 ([BCCD22, Theorem B]). *Let (M, g, X) be a complete two-dimensional shrinking gradient Kähler-Ricci soliton with bounded scalar curvature. Then either:*

(i) *M is Fano and g is, up to automorphism, either Kähler-Einstein or the shrinking gradient Kähler-Ricci soliton on M given by* [WZ04]*, or:*
(ii) *(M, g, X) is, up to pullback by an element of $GL(2, \mathbb{C})$, the flat Gaussian shrinking soliton on $\mathbb{C}^2$, or:*
(iii) *(M, g, X) is, up to pullback by an element of $GL(2, \mathbb{C})$, the unique $U(2)$-invariant shrinking gradient Kähler-Ricci soliton of Feldman-Ilmanen-Knopf* [FIK03] *on the total space of $\mathcal{O}(-1)$ over $\mathbb{P}^1$, or:*
(iv) *(M, g, X) is, up to automorphism, the cylinder $\mathbb{C} \times \mathbb{P}^1$, or:*
(v) *(M, g, X) is, up to automorphism, the shrinking gradient Kähler-Ricci soliton of Theorem 1.*

Note that a real four-dimensional shrinking gradient Ricci soliton with bounded scalar curvature automatically has bounded curvature [MW14, Theorem 0.1]. We also remark that the assumption of bounded scalar curvature in the statement of Theorem 2 has recently been removed by Li-Wang [LW23].

To prove Theorem 1, we employ a novel approach in which we construct the soliton indirectly as a blowup limit of a specific Kähler-Ricci flow on a compact Kähler manifold, using recent estimates obtained by Bamler [Bam20a, Bam20b, Bam20c] together with Kähler-Ricci flow techniques to control the singularity formation of this flow. More precisely, we consider the blowup $N := \mathrm{Bl}_x(\mathbb{P}^1 \times \mathbb{P}^1)$ of $\mathbb{P}^1 \times \mathbb{P}^1$ at one point x and show that there is a toric Kähler-Ricci flow that contracts the exceptional divisor and exactly one other (-1)-curve at the singular time $T > 0$. The volume of N close to the singular time is $\sim (T - t)$. Using the estimates from [Bam20c], we analyse possible blowup models of this flow. Thanks to the toricity of the flow and the topology of the underlying manifold, we are able to exclude orbifold singularities from appearing in the limit and show that the singularity is close to a *smooth* shrinking Kähler-Ricci soliton at most scales. By [CCD22], regions that are close enough to such Kähler-Ricci solitons must contain a complex curve of self-intersection 0 or -1. The areas of these curves, which are determined by the Kähler class of the evolving metric, shrink by at most a linear rate. This observation allows us to bound from below the scales at which the flow exhibits closeness to a shrinking Kähler-Ricci soliton. Consequently, we see that the flow must be of Type I. As a result, the singularity formation near every point can be described by a shrinking Kähler-Ricci soliton. Among these, we are able to exclude the soliton of Feldman-Ilmanen-Knopf [FIK03] because the volume of N under the flow converges to zero [CCD22]. This only leaves $\mathbb{C} \times \mathbb{P}^1$ and its blowup [CCD22] as possible blowup limits. Complex geometric reasons allow us to further rule out a $\mathbb{C} \times \mathbb{P}^1$ forming near the exceptional divisor, demonstrating that a new soliton, namely that given by Theorem 1, must exist.

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Seiberg-Witten theory and moduli spaces of 4-manifolds and metrics

Hokuto Konno

(joint work with Jianfeng Lin)

The Seiberg–Witten equations are partial differential equations defined on a smooth 4-manifold. Seiberg–Witten theory has been effectively used in 4-dimensional topology and geometry. It is a natural idea to consider *families* of Seiberg–Witten equations associated to a family of 4-manifolds, and use it to study the moduli spaces of 4-manifolds and metrics. Here, by the moduli space of 4-manifolds, we meant the classifying space $\mathrm{BDiff}(X)$ of the diffeomorphism group $\mathrm{Diff}(X)$ of a smooth 4-manifold X. The moduli space of metrics (with a certain condition, such as of metric of positive scalar curvature) usually means the quotient $\mathrm{Met}(X)/\mathrm{Diff}(X)$ of the space of metrics $\mathrm{Met}(X)$ by the diffeomorohism group

$\mathrm{Diff}(X)$. However, here we mainly consider the *homotopy quotient* which we denote by $\mathrm{Met}(X)/\!\!/\mathrm{Diff}(X)$. The homotopy quotient is defined as the quotient

$$\mathrm{Met}(X)/\!\!/\mathrm{Diff}(X) = (\mathrm{EDiff}(X) \times \mathrm{Met}(X))/\mathrm{Diff}(X),$$

where $\mathrm{EDiff}(X)$ is a contractible space acted by $\mathrm{Diff}(X)$ freely. The homotopy quotient $\mathrm{Met}(X)/\!\!/\mathrm{Diff}(X)$ is the moduli space of *pairs* of (M, g) where M is a manifold diffeomorphic to X and g is a metric on M. In other words, $\mathrm{Met}(X)/\!\!/\mathrm{Diff}(X)$ classifies fiber bundles with fiber X with fiberwise metrics. Similarly, if we denote by $\mathrm{PSC}(X)$ the space of positive scalar curvature metrics on X, the homotopy quotient $\mathrm{PSC}(X)/\!\!/\mathrm{Diff}(X)$ classifies fiber bundles with fiber X with fiberwise positive scalar curvature metrics.

The first striking result by families gauge theory in the study of the diffeomorphism groups of 4-manifolds is work by Ruberman [8]. He proved that there are self-diffeomorphisms of some 4-manifolds which are topologically isotopic to the identity but smoothly not. Ruberman used 1-parameter family version of Donaldson invariant. A similar argument works also using 1-parameter families Seiberg–Witten theory (see such as [9, 1]). Also, 1-parameter families Seiberg–Witten invariant has been used by Ruberman [9] to prove the disconnectivity of $\mathrm{PSC}(X)$ for some 4-manifolds X. (However, there is no known example of orientable 4-manifolds for which the moduli spaces of the positive scalar curvature metrics $\mathrm{PSC}(X)/\mathrm{Diff}(X)$ has non-trivial topology. Ruberman [9] gave such an example of unorientable 4-manifolds.)

Going to the moduli space $\mathrm{BDiff}(X)$ of a 4-manifold, a basic question is whether we can compute (co)homology of the moduli space $\mathrm{BDiff}(X)$. This is important since cohomologies in $H^*(\mathrm{BDiff}(X))$ one-to-one correspond to characteristic classes of smooth fiber bundles with fiber X. Even for a 2-dimensional manifold X, it is known to be extremely hard to determine the cohomology group $H^*(\mathrm{BDiff}(X))$. Nevertheless, a result by Harer [4] in dimension 2 and its higher dimensional analog by Galatius and Randal-Williams [2] say that $H^*(\mathrm{BDiff}(X))$ can be computed *stably* in the following sense.

Let W be a manifold with non-empty boundary, and let $\mathrm{Diff}_\partial(W)$ denote the group of diffeomorphisms that are the identity near boundary. Suppose that W is even dimensional, say of dimension $2n$. By extending identity, one can define a natural map

$$s : \mathrm{Diff}_\partial(W) \to \mathrm{Diff}_\partial(W \# S^n \times S^n),$$

which we call the *stabilization map.* The results by Harer [4] when $2n = 2$ and by Galatius and Randal-Williams [2] when $2n \geq 6$ show the following: let W be a simply-connected compact manifold W, and suppose that $\dim W$ is even but not 4. Then, for any $k \geq 0$, there exists $N > 0$ such that the induced map

$$s_* : H_k(\mathrm{BDiff}_\partial(W \# N S^n \times S^n); \mathbb{Z}) \to H_k(\mathrm{BDiff}_\partial(W \# (N+1) S^n \times S^n); \mathbb{Z})$$

are isomorphisms for all $N \gg k$. This result is called *homological stability.* This result together with the computation of the *stable homology*

$$\lim_{N \to +\infty} H_k(\mathrm{BDiff}_\partial(W \# N S^n \times S^n))$$

due to Madsen–Weiss [7] in dimension 2 and in other even dimensions due to Galatius and Randal-Williams [3] compute the (co)homology $H_k(\mathrm{BDiff}_\partial(W\#NS^n \times S^n))$ for large N, at least over $\mathbb{Q}$-coefficient.

In contrast, the following holds in dimension 4, which we call *homological instability*:

Theorem 1 ([6]). *Let X be a simply-connected smooth closed 4-manifold and let $\mathring{X}$ denote the punctured X: $\mathring{X} = X \setminus \mathrm{int}(D^4)$. Then, for any $k > 0$, there exists a sequence $0 < N_1 < N_2 < \cdots \to +\infty$ such that*

$$s_* : H_k(\mathrm{BDiff}_\partial(\mathring{X}\#N_iS^2 \times S^2); \mathbb{Z}) \to H_k(\mathrm{BDiff}_\partial(\mathring{X}\#(N_i+1)S^2 \times S^2); \mathbb{Z})$$

are not isomorphisms for all $i \geq 1$.

More strongly, we can prove that both of the injectivity and surjectivity fail infinitely many times for s_*.

On the moduli space of 4-manifolds with positive scalar curvature metrics, we can prove the following. Let $\mathrm{Diff}^+(X)$ denote the orientation-preserving diffeomorphism group of an oriented manifold X. Let

$$\iota : \mathrm{PSC}(X)/\!\!/\mathrm{Diff}^+(X) \to \mathrm{Met}(X)/\!\!/\mathrm{Diff}^+(X)$$

denote the map induced from the inclusion $\mathrm{PSC}(X) \hookrightarrow \mathrm{Met}(X)$.

Theorem 2 ([6]). *Let X be a simply-connected smooth closed oriented 4-manifold. Then, for any $k > 0$, there exists a sequence $0 < N_1 < N_2 < \cdots \to +\infty$ such that*

$$\iota_* : H_k(\mathrm{PSC}(X\#N_iS^2 \times S^2)/\!\!/\mathrm{Diff}^+(X); \mathbb{Z}) \to H_k(\mathrm{Met}(X\#N_iS^2 \times S^2)/\!\!/\mathrm{Diff}^+(X); \mathbb{Z})$$

are not isomorphisms for all $i \geq 1$.

For this theorem, we can prove the surjectivity fails infinitely many times, but we do not know anything about the injectivity.

The proofs of the above two theorems use a characteristic class

$$\mathbb{SW}^k_{\text{half-tot}}(X) \in H^k(\mathrm{BDiff}^+(X); \mathbb{Z}/2)$$

for an oriented closed smooth 4-manifold, defined when $b^+(X) \geq k+2$. This is defined, loosely speaking, by counting the moduli spaces of solutions to the Seiberg–Witten equations over each k-cell of $\mathrm{BDiff}^+(X)$ (after taking a CW approximation) for all spin^c structures on X with formal dimension $-k$, taking into account the charge conjugation symmetry on spin^c structures. The construction is inspired by a numerical 1-parameter families Seiberg–Witten invariant $\mathbb{SW}^k(X, f)$ of a diffeomorphism $f : X \to X$ (which may not preserve a given spin^c structure) by Ruberman [9], together with a construction of a characteristic class $\mathbb{SW}^k(X, \mathfrak{s}) \in H^k(\mathrm{BDiff}(X, \mathfrak{s}))$ depending on a choice of spin^c structure by the author [5].

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Improved generic regularity of minimizing hypersurfaces

Christos Mantoulidis

(joint work with Otis Chodosh, Felix Schulze)

Let Γ be a smooth, closed, oriented, $(n-1)$-dimensional submanifold of $\mathbf{R}^{n+1}$. Among all smooth, compact, oriented hypersurfaces $M \subset \mathbf{R}^{n+1}$ with $\partial M = \Gamma$, does there exist one with *least area*?

Foundational results in geometric measure theory can be used to produce an integral n-current T with least mass ("minimizing") among all those with boundary equal to the multiplicity-one current represented by Γ. When $n+1 \leq 7$, it is known that T is supported on a smooth, compact, oriented hypersurface that solves the original differential geometric problem (see [1, 2, 3, 4, 5]). When $n+1 \geq 8$, smooth minimizers can fail to exist (see [6]) but it is nevertheless known that away from a compact set $\operatorname{sing} T \subset \mathbf{R}^{n+1} \setminus \Gamma$ of Hausdorff dimension $\leq n-7$, the support of T will be a smooth precompact hypersurface with boundary Γ (see [7, 5]).

A fundamental result of Hardt–Simon [8] shows that the singularities of 7-dimensional minimizing currents in $\mathbf{R}^8$, which are necessarily isolated points, can be eliminated by a perturbation of the prescribed boundary Γ, thus yielding solutions to the original geometric problem in $\mathbf{R}^8$ for the perturbed boundary.

In recent work motivated from our past results on mean curvature flow (see, e.g., [9, 10]) we obtained a generic regularity result for minimizers in higher ambient dimensions:

Theorem ([11], [12]). *Let $\Gamma^{n-1} \subset \mathbf{R}^{n+1}$ be a smooth, closed, oriented, submanifold. There exist arbitrarily small perturbations Γ' of Γ such that every minimizing integral n-current with boundary $[\![\Gamma']\!]$ is of the form $[\![M']\!]$ for a smooth, precompact, oriented hypersurface M' with $\partial M' = \Gamma'$ and $\operatorname{sing} M' = \bar{M}' \setminus M'$ satisfies*

$$\operatorname{sing} M' = \emptyset \text{ if } n+1 \leq 10, \text{ otherwise } \dim_H \operatorname{sing} M' \leq n - 9 - \varepsilon_n$$

where $\varepsilon_n \in (0,1]$ is an explicit dimensional constant.

Let us discuss what goes into the proof of this theorem. Let us denote

$$\mathcal{M}(\Gamma) = \{\text{minimizing integral } n\text{-currents in } \mathbf{R}^{n+1} \text{ with boundary } [\![\Gamma]\!]\}.$$

We agree to the following simplifying assumptions (see [12] for the general case):

- Γ is connected.
- $\mathcal{M}(\Gamma)$ is a singleton.

The above and the standard regularity theory guarantee that $\mathcal{M}(\Gamma) = \{[\![M]\!]\}$ for a smooth, precompact, oriented hypersurface $M \subset \mathbf{R}^{n+1}$ with $\partial M = \Gamma$, $\operatorname{sing} M = \bar{M} \setminus M \subset\subset \mathbf{R}^{n+1} \setminus \Gamma$, and $\dim_H \operatorname{sing} M \leq n - 7$.

Now set $\Gamma_0 := \Gamma$ and perturb Γ smoothly to $(\Gamma_s)_{s \in (-\delta,\delta)}$ by s times the unit normal to M along Γ (recall that $\operatorname{sing} M \cap \Gamma = \emptyset$) for some small $\delta > 0$. Accordingly, for each $s \in (-\delta, \delta)$, let $\mathcal{M}(\Gamma_s)$ be the set of all minimizers with boundary data Γ_s; each such is still of the form $[\![M_s]\!]$, with M_s enjoying similar a priori regularity as M. A cut-and-paste argument implies that

$$(\ddagger) \qquad [\![M_s]\!] \in \mathcal{M}(\Gamma_s),\ [\![M_{s'}]\!] \in \mathcal{M}(\Gamma_{s'}),\ s \neq s' \implies \bar{M}_s \cap \bar{M}_{s'} = \emptyset.$$

Define

$$\mathcal{L} = \cup_{s \in (-\delta,\delta)} \cup_{[\![M_s]\!] \in \mathcal{M}(\Gamma_s)} \bar{M}_s,$$
$$\mathcal{S} = \cup_{s \in (-\delta,\delta)} \cup_{[\![M_s]\!] \in \mathcal{M}(\Gamma_s)} \operatorname{sing} M_s.$$

In view of ($\ddagger$), the following "timestamp" function is well-defined:

$$\mathbf{t} : \mathcal{L} \to (-\delta, \delta),$$
$$\mathbf{t}(x) = s \text{ for all } x \in \bar{M}_s,\ [\![M_s]\!] \in \mathcal{M}(\Gamma_s),\ s \in (-\delta, \delta).$$

We are now ready to state the two main tools required for our main theorem.

Tool A ([12]). *It holds that $\dim_H \mathcal{S} \leq n - 7$.*

Tool B ([12]). *The timestamp function $\mathbf{t} : \mathcal{L} \to (-\delta, \delta)$ above is α-Hölder on $\mathcal{S}$ for every $\alpha \in (0, 2 + \varepsilon_n)$, where $\varepsilon_n \in (0,1]$ is an explicit dimensional constant.*

To obtain the **Theorem** from **Tools A, B** one can invoke a Sard-type covering argument of Figalli–Ros-Oton–Serra, who successfully proved a generic regularity result for free boundary singularities in the obstacle problem using tools similar to **A, B**.

Proposition ([13, Proposition 7.7]). *Let $S \subset \mathbf{R}^n$, $0 < d \leq n$, and $0 < \beta < \alpha$. Assume that $\mathcal{H}^d(S) < \infty$ and that $f : S \to (-1,1)$ is α-Hölder continuous.*

(1) If $d \leq \beta$, then $\mathcal{H}^{d/\beta}(f(S)) = 0$.
(2) If $d > \beta$, then for a.e. $t \in (-1,1)$ we have $\mathcal{H}^{d-\beta}(f^{-1}(t)) = 0$.

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Diameter estimates in Kähler geometry

Bin Guo

(joint work with Duong H. Phong, Jian Song, and Jacob Sturm)

The diameter is an important geometric invariant associated to a Riemannian metric. Bounds for the diameter are essential for the study of geometric convergence of a family of Riemannian manifolds. Previously known results require the conditions on the curvature. For example, the Myers' theorem states that the diameter is bounded if the Ricci curvature is bounded below by a *positive* constant. Recently, in [1] we develop a general theory of diameter estimates for Kähler metrics, which in particular does not require any assumptions on the Ricci curvature. The nonlinear analysis of complex Monge-Ampère equations allows us to derive uniform estimates of the Green's functions [2], from which the diameter estimates follow.

Let (X, ω_X) be an n-dimensional compact Kähler manifold equipped with a Kähler metric ω_X. Let γ be a non-negative continuous function and $A, B, K > 0$, $p > n$ be given parameters. We define a subset of $\mathcal{W} := \mathcal{W}(X, \omega_X, n, A, p, K, \gamma)$ of the space of Kähler metrics on X by

$$\mathcal{W} = \left\{ \omega : \ (V_\omega)^{-1} \frac{\omega^n}{\omega_X^n} \geq \gamma, \ [\omega] \cdot [\omega_X]^{n-1} \leq A, \ \mathcal{N}_{X,\omega_X,p}(\omega) \leq K \right\},$$

where $V_\omega = \int_X \omega^n = [\omega]^n$ is the volume of ω, and

$$\mathcal{N}_{X,\omega_X,p}(\omega) = \frac{1}{V_\omega}\int_X \left|\log\left((V_\omega)^{-1}\frac{\omega^n}{\omega_X^n}\right)\right|^p \omega^n,$$

is the p-Nash entropy of ω relative to ω_X. Our main result is:

Theorem. [1] *Let X be an n-dimensional connected Kähler manifold, and let γ be a nonnegative continuous function on X satisfying*

$$\dim_{\mathcal{H}}\{\gamma = 0\} < 2n-1,\ \gamma \geq 0,$$

where $\dim_{\mathcal{H}}$ *is the Hausdorff dimension. Then for any $A, K > 0$ and $p > n$, there exist $C > 0$, $c > 0$ that depend on n, A, p, K, γ, and $\alpha = \alpha(n,p) > 0$ such that for any $\omega \in \mathcal{W}$, the following hold:*

(a) The Green's function G associated to ω:

$$\int_X |G(x,\cdot)|\omega^n + \int_X |\nabla G(x,\cdot)|_\omega \omega^n + \left(-\inf_{y\in X} G(x,y)\right) V_\omega \leq C$$

for any $x \in X$;

(b) The diameter: $\mathrm{diam}(X,\omega) \leq C$;

(c) The volume element: for any $x \in X$ and any $R \in (0,1]$,

$$\frac{\mathrm{Vol}_\omega(B_\omega(x,R))}{\mathrm{Vol}_\omega(X)} \geq cR^\alpha.$$

An immediate consequence of the theorem is the Gromov-Hausdorff (pre)compactness of the Kähler metrics in $\mathcal{W}$. The theorem also applies to the cases of long-time (normalized) Kähler-Ricci flow

$$\frac{\partial\omega}{\partial t} = -\mathrm{Ric}(\omega) - \omega, \quad t \in [0,\infty);$$

and the finite-time KRF

$$\frac{\partial\omega}{\partial t} = -\mathrm{Ric}(\omega), \quad t \in [0,T),\ T < \infty$$

when the volumes do not collapse to zero. The idea of proof is to verify that the metrics along these flows lie in the set $\mathcal{W}$ for some bounded parameters p, A, K. In particular, along these flows, the diameters are uniformly bounded and these metrics are pre-compact in the Gromov-Hausdorff topology.

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Einstein Manifolds, Weyl Curvature, & Conformally Kähler Geometry

Claude LeBrun

In what follows, an Einstein metric will be understood to mean a Riemannian metric g of constant Ricci curvature on some smooth n-manfold M; equivalently, it is a metric g whose Ricci tensor r satisfies [3]

$$r = \lambda g$$

for a real constant λ that is called the Einstein constant of g. Any metric of constant sectional curvature is Einstein, but the converse is only generally true if the manifold M has dimension $n = 2$ or 3. By contrast, this talk concerns the transitional case of $n = 4$, where Einstein metrics on closed manifolds generally do not have constant sectional curvature, but nonetheless exhibit strong rigidity phenomena that are reminiscent of the constant-sectional-curvature paradigms.

As an illustration of this last point, there are a number of smooth compact 4-manifolds M where we completely understand the moduli space

$$\mathscr{E}(M) = \{\text{Einstein metrics on } M\}/(\mathit{Diff}(M) \times \mathbb{R}^+)$$

of Einstein metrics modulo isometries and and constant rescalings. For example, if M admits a real or complex-hyperbolic metric, the moduli space consists of exactly one point [4, 8], and the given locally-symmetric metric represents the *unique* Einstein geometry on M. By contrast, the Einstein moduli spaces for the 4-torus and $K3$ are more complicated, with dimension 9 and 57, respectively, but these examples still enjoy an important feature of the previous ones, because the corresponding moduli spaces once again turn out to be connected [1, 2, 6, 7, 13, 15].

Now, for some of the 4-manifolds M we've just discussed, the key to understanding the moduli space $\mathscr{E}(M)$ lies in showing that every Einstein metric on M must be Kähler. This strategy can however only stand a chance of working when the 4-manifold admits a symplectic structure. Are there other 4-manifolds where a symplectic structure helps reveal something about Einstein metrics? The following result [5, 9] shows that this question can sometimes be surprisingly fruitful:

Theorem 1. *Suppose that M is a smooth compact oriented 4-manifold which carries a symplectic form ω. Then M admits an (a priori unrelated) Einstein metric g with $\lambda > 0$ if and only if*

$$M \approx_{\mathit{diff}} \begin{cases} \mathbb{CP}_2 \# k\overline{\mathbb{CP}}_2, & 0 \leq k \leq 8, \text{ or} \\ S^2 \times S^2 \end{cases}$$

These ten diffeotypes exactly catalog the smooth compact 4-manifolds that arise as Del Pezzo surfaces, which are by definition the compact complex 2-manfolds with ample anti-canonical line bundle. The proof shows that each of these 4-manifolds admits a $\lambda > 0$ Einstein metric that is conformally equivalent to a Kähler metric with vanishing Bach tensor. Conversely, Seiberg-Witten theory, the theory of pseudo-holomorphic curves, and the Hitchin-Thorpe inequality are together used to show that no other smooth compact 4-manifold can admit both a $\lambda > 0$ Einstein metric and an orientation-compatible symplectic structure.

But while Theorem 1 asserts that the Einstein moduli space is non-empty for each of the ten Del Pezzo 4-manifolds, it otherwise tells us nothing at all regarding the structure of the relevant moduli spaces. For example, are these Einstein moduli spaces connected? We currently just don't know. However, we do at least now have two different Riemannian characterizations of one component of each moduli space. The first such characterization [10] involves the use of a global self-dual harmonic 2-form:

Theorem 2. *Let (M, g) be a compact oriented connected Einstein 4-manifold, and suppose that there is a self-dual harmonic 2-form ω on (M, g) with the property that $W_+(\omega, \omega) > 0$ at every point of M, where W_+ denotes the self-dual Weyl curvature. Then M is orientedly diffeomorphic to a Del Pezzo surface, and g is conformally Kähler. Conversely, each Del Pezzo 4-manifold M carries Einstein metrics with this property, and these metrics precisely sweep out a single connected component of the Einstein moduli space $\mathscr{E}(M)$.*

This may seem reasonably satisfying, but the use of a global harmonic form unfortunately makes this criterion rather non-local. Fortunately, a purely local characterization was later proposed by Peng Wu, and two entirely different proofs of this characterization were then given by Wu [14] and the present author [11]:

Theorem 3. *Let (M, g) be a simply-connected compact oriented Einstein 4-manifold whose self-dual Weyl curvature $W_+ : \Lambda^+ \to \Lambda^+$ satisfies $\det(W_+) > 0$ at every point of M. Then M is orientedly diffeomorphic to a Del Pezzo surface, and g is conformally Kähler. Conversely, each Del Pezzo 4-manifold M carries Einstein metrics with this property, and these metrics precisely sweep out a single connected component of the Einstein moduli space $\mathscr{E}(M)$.*

In the author's current joint work with Tristan Ozuch, these criteria have also proved useful in classifying the Kähler-Einstein orbifolds that are Gromov-Hausdorff limits of $\lambda > 0$ Einstein 4-manifolds. These results, which extend the previous work of Odaka-Spotti-Sun [12], should be publicly available very soon.

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Coordinates are messy in General (Relativity)

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(joint work with Melanie Graf, Jan Metzger)

It is customary to use the existence of an asymptotic coordinate chart with certain properties as a venue to defining asymptotic flatness of the underlying manifold. This has proven extremely successful for the study of (total) mass which turns out to be independent of the choice of such an asymptotic coordinate chart and plays a central role in many important results. The situation turns out to be drastically different for the (total) center of mass which is not generally well-defined; this is usually addressed by assuming seemingly suitable additional properties of the asymptotic coordinate charts; we show that these additional properties are not a good solution of the underlying issue(s), see Section 3. See also [3] for a review.

1. Setup and context

A *general relativistic initial data set (IDS)* is a (smooth) 3-dimensional Riemannian manifold (M, g) carrying a symmetric $(0,2)$-tensor field K, a function μ and a 1-form J satisfying the *Einstein constraint equations*

$$\begin{aligned} \mathrm{R} - |K|^2 + (\operatorname{tr} K)^2 &= 2\mu \\ \operatorname{div}(K - (\operatorname{tr} K) g) &= J \end{aligned} \tag{1}$$

on M. Here, R, $|\cdot|$, tr, and div denote the scalar curvature of and the tensor norm, trace, and divergence w.r.t. the metric g, respectively. IDSs arise as spacelike hypersurfaces (or "time-slices") in 4-dimensional relativistic spacetimes (i.e., time-orientable Lorentzian manifolds) $(\mathfrak{L}, \mathfrak{g})$ satisfying the Einstein equation $\mathfrak{Ric} - \frac{1}{2}\mathfrak{R}\,\mathfrak{g} = \mathfrak{T}$, where $\mathfrak{Ric}$, $\mathfrak{R}$ denote the Ricci and scalar curvature of the Lorentzian metric $\mathfrak{g}$, respectively, and $\mathfrak{T}$ is the stress-energy-momentum tensor encoding the physical properties of the system. In this framework, the metric g of the IDS arises as the metric induced on the spacelike hypersurface $M \subset \mathfrak{L}$, K plays the role of its second fundamental form, and μ and J are suitable components of $\mathfrak{T}$ restricted to M called the *energy density* and *momentum density*, respectively.

An IDS (M, g, K, μ, J) models an isolated system if it is *asymptotically Euclidean (AE)*, i.e., if there is a $k \geq 2$, an $\varepsilon > 0$, a compact set $C \subset M$, a ball $B \subset \mathbb{R}^3$, and a diffeomorphism $\Phi \colon M \setminus C \to \mathbb{R}^3 \setminus \overline{B}$ such that the decay conditions

$$(2) \qquad (\Phi_* g)_{ij} = \delta_{ij} + \mathcal{O}_k(r^{-\frac{1}{2}-\varepsilon}), \quad (\Phi_* K)_{ij} = \mathcal{O}_{k-1}(r^{-\frac{3}{2}-\varepsilon})$$

hold as $r = |\vec{x}| \to \infty$ w.r.t. the coordinates (x^i) induced on M by Φ and the integrability conditions

$$(3) \qquad \Phi_* \mu, (\Phi_* J)_i \in L^1(\mathbb{R}^3 \setminus \overline{B}) \text{ or } \Phi_* \mu, (\Phi_* J)_i = \mathcal{O}(r^{-3-2\varepsilon})$$

hold as $r = |\vec{x}| \to \infty$, see also Section 2. The index k in the decay conditions (2), (3) indicates that derivatives up to order k must decay as corresponding derivatives of the argument of $\mathcal{O}_k$. The decay in (2), (3) can be modeled in weighted C^k- or Sobolev spaces.

2. Mass, energy, and linear momentum

In the 1960's, Arnowitt, Deser, and Misner [1] gave by now well-known definitions of *(total) energy* $E_{ADM} \in \mathbb{R}$, *(total) linear momentum* $\vec{P}_{ADM} \in \mathbb{R}^3$, and *(total) mass* $m_{ADM} \in \mathbb{R}$ of an AE IDS in terms of flux integrals "at infinity", e.g.

$$(4) \qquad E_{\text{ADM}} = \frac{1}{16\pi} \lim_{r\to\infty} \int_{\mathbb{S}^2_r(0)} \sum_{i,j=1}^{3} (g_{ij,i} - g_{ii,j}) \frac{x^j}{r} dA_\delta,$$

where $\mathbb{S}^2_r(0)$ denotes a coordinate sphere of radius r around the coordinate origin in the chosen asymptotic coordinates (x^i) and dA_δ denotes the area measure induced on it by the Euclidean metric. In the 1980's, Bartnik [2] and Chruściel [7] proved that E_{ADM} and m_{ADM} are independent of the choice of asymptotic coordinates (and that $\vec{P}_{\text{ADM}}$ transforms appropriately under change of asymptotic coordinates). One can modify an example by Denissov and Soloviev [8] to see why $\varepsilon > 0$ is necessary in (2): Take $g = \delta$ on $M = \mathbb{R}^3 \setminus \overline{B_1(0)} \ni \vec{x}$ and choose new asymptotic coordinates $\vec{y} = \vec{x} + \frac{a\vec{x}}{\sqrt{|\vec{x}|}}$ for some $a \neq 0$. Then $(M, \delta, K = 0, \mu = 0, J = 0)$ is an IDS which satisfies (2), (3) for $\varepsilon = 0$ w.r.t. the asymptotic coordinates (y^i). Yet, one computes that $E_{\text{ADM}} \propto a$, in particular $E_{\text{ADM}} \neq 0$ even though we are just looking at Euclidean space which should not contain any mass/energy. On the other hand, the conditions in (3) are motivated by the analogous integrability condition(s) on the matter density of a Newtonian gravitating system in a Lebesgue or indefinite Riemann sense, respectively.

3. Center of mass and RT conditions

By analogy to Newtonian gravity, we should expect that defining the (total) center of mass requires to assume better integrability, namely $\mu x^i \in L^1(\mathbb{R}^3 \setminus \overline{B})$ or $\mu = \mathcal{O}(r^{-4-2\varepsilon})$ as $r = |\vec{x}| \to \infty$ (see [5]). From special relativity, we should expect that the center of mass of an AE IDS should depend on the relative velocity/momentum measured by the observer locating the center of mass, i.e., should depend on the second fundamental form K of the IDS. Indeed, with Sakovich [6], we gave a definition of (total) center of mass via asymptotic foliations by *spacetime constant mean curvature surfaces* with leaves $\Sigma_{\mathcal{H}}$ satisfying $\mathcal{H}^2 = H^2 - (\mathrm{tr}_{\Sigma_{\mathcal{H}}} K)^2 = \text{const.}$, where H denotes the mean curvature of $\Sigma_{\mathcal{H}}$ in (M, g), $\mathrm{tr}_{\Sigma_{\mathcal{H}}} K$ denotes the (partial) trace of K over $\Sigma_{\mathcal{H}}$, and $\mathcal{H}$ equals the Lorentzian length of the codimension 2 or "spacetime" mean curvature vector. Our definition has many desirable properties [6], yet the (coordinate) center of mass of the foliation

$$\lim_{\mathcal{H}\searrow 0} \frac{1}{|\Sigma_{\mathcal{H}}|} \int_{\Sigma_{\mathcal{H}}} \vec{x}\, dA_\delta \tag{5}$$

does not always converge [5, 6, 3, 4], even when assuming additional decay of μ.

This problem is well-known to occur for other definitions of center of mass [9, 3, 4] and has a counterpart in Newtonian gravity: Indeed, if one has a – relativistic or Newtonian – isolated system for which the center of mass converges w.r.t. asymptotic coordinates (x^i), changing the (asymptotic) coordinates by the implicit transformation $\vec{y} + \vec{a} \sin \ln(|\vec{y}|) = \vec{x}$ for a fixed vector $\vec{a} \in \mathbb{R}^3$, $\vec{a} \neq \vec{0}$, necessarily leads to a divergence of the center of mass w.r.t. (y^i) (see [3, 4] for more details).

The divergence issue sketched above is usually "remedied" by assuming additional asymptotic parity assumptions known as *weak (resp. strong) Regge–Teitelboim (RT) conditions* [10], requiring that for $\eta = \frac{1}{2}$ (resp. $\eta = 1$), one has

$$g_{ij}^{\mathrm{odd}} = \mathcal{O}_2(r^{-\frac{1}{2}-\eta-\varepsilon}), \quad K_{ij}^{\mathrm{even}} = \mathcal{O}_1(r^{-\frac{3}{2}-\eta-\varepsilon}), \quad \mu^{\mathrm{odd}}, J_i^{\mathrm{odd}} = \mathcal{O}(r^{-3-\eta-\varepsilon}) \tag{6}$$

as $r = |\vec{x}| \to \infty$. Here, the even and odd parts are taken w.r.t. the asymptotic coordinates (x^i). However, with Graf and Metzger [4], we prove that

(1) not all AE IDSs possess asymptotic coordinate charts satisfying the RT conditions,
(2) the RT conditions do not transform equivariantly under asymptotic (coordinate) translations.

Here, asymptotic (coordinate) translations represent a subgroup of the asymptotic Poincaré group of the underlying asymptotically flat spacetime $(\mathfrak{L}, \mathfrak{g})$. This suggests that the Regge–Teitelboim conditions are not the final answer to addressing the well-definition of the center of mass. Instead, we are looking for geometric, Poincaré equivariant (PDE) conditions selecting asymptotic coordinates (x^i) on an arbitrary AE IDS with $\mu x^i, J_j x^i \in L^1(\mathbb{R}^3 \setminus \overline{B})$ with respect to which the coordinate center of mass (5) of the STCMC-foliation will always converge.

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ADM mass for C^0 metrics and distortion under Ricci-DeTurck flow

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In recent years considerable attention has been devoted to the study of Riemannian metrics with lower scalar curvature bounds in various nonsmooth settings. In the C^0 setting, Gromov [1] showed that pointwise lower scalar curvature bounds are preserved under uniform convergence. Somewhat more recently, the Ricci and Ricci-DeTurck flows have emerged as useful tools in this setting, since they provide a smoothing of the metric under which the scalar curvature has a well-behaved evolution equation (a Ricci-DeTurck flow is a parabolic flow that is related to a Ricci flow via pullback by a family of diffeomorphisms). For instance, in [2], Bamler provided a Ricci-DeTurck flow proof of Gromov's [1] result.

In light of this context, it is natural to ask whether other metric quantities associated with the scalar curvature may be formulated using only C^0 data of the metric, and whether anything can be learned about these quantities by letting the metric evolve by Ricci or Ricci-DeTurck flow. One such quantity is the ADM mass. Recall that if (M^n, g) is a smooth Riemannian manifold and $\Phi : M \setminus K \to \mathbb{R}^n \setminus \overline{B(0,1)}$ is a smooth coordinate chart for M, where K is a compact subset of M, then the ADM mass is given by:

$$m_{ADM}(g) := \lim_{r\to\infty} \frac{1}{4\pi(n-1)\omega_{n-1}} \int_{\mathbb{S}(r)} \sum_{i=1}^{n} (\partial_i g_{ij} - \partial_j g_{ii})\nu^j \, dS, \tag{1}$$

where the coordinate expression in the integrand corresponds to the coordinates Φ,

$$\mathbb{S}(r) = \{x \in \mathbb{R}^n : (x^1)^2 + \cdots + (x^n)^2 = r^2\},$$

ν denotes the outward unit normal to $\mathbb{S}(r)$ with respect to the Euclidean metric, ω_{n-1} denotes the Euclidean volume of the $(n-1)$-dimensional unit sphere, and dS denotes the Euclidean surface measure on $\mathbb{S}(r)$. Henceforth, if we wish to emphasize the coordinate chart Φ, then we write $m_{ADM}(g, \Phi)$.

A priori it is not clear whether the limit (1) should always exist, or whether the limit depends on the choice of Φ, but Bartnik [3, Theorems 4.2 and 4.3] (see also [4] for the asymptotically Minkowski case) showed that under certain conditions the ADM mass does indeed exist, is finite, and is independent of choice of coordinate chart:

Theorem 1 (cf. [3]). *Let (M^n, g) be a smooth Riemannian manifold. Suppose that for some compact set $K \subset M$ there exists a coordinate chart $\Phi : M \setminus K \to \mathbb{R}^n \setminus \overline{B(0,1)}$ for M such that, for some $\tau > \frac{1}{2}(n-2)$, we have*

$$|(\Phi_* g)_{ij} - \delta_{ij}|\big|_x = O(|x|_\delta^{-\tau}), \tag{2}$$

$$|\partial_k (\Phi_* g)_{ij}|\big|_x = O(|x|_\delta^{-\tau-|k|}) \text{ for } |k| = 1, 2, \tag{3}$$

where δ denotes the Euclidean metric and k is a multiindex, and

$$\int_M |R(g)| < \infty, \tag{4}$$

where $R(g)$ denotes the scalar curvature of g. Then the limit from (1) with respect to Φ exists, is finite, and is independent of choice of Φ satisfying (2) and (3).

Henceforth we will say that a continuous Riemannian metric is C^0-*asymptotically flat* if, for some smooth coordinate chart, it satisfies (2) but not necessarily (3). In view of the well-known Riemannian Positive Mass Theorem, we henceforth impose the condition that the C^0 Riemannian metric have nonnegative scalar curvature "in the sense of Ricci flow". Broadly speaking, this condition is a pointwise condition for C^0 metrics, which, when it holds globally, implies that a Ricci-DeTurck flow starting from the C^0 metric will have nonnegative scalar curvature (in the classical sense) for all positive times; see [5] and [6] for the precise statement.

Towards the C^0 setting, observe that $m_{ADM}(g, \Phi)$ is computed by integrating over a single coordinate sphere, but when the limit $m_{ADM}(g, \Phi)$ exists, one may alternatively compute $m_{ADM}(g, \Phi)$ by integrating over a family of spheres weighted by some test function, since, if $\varphi : \mathbb{R} \to \mathbb{R}$ is any smooth function with $\int_{.9}^{1.1} \varphi(\ell) d\ell \neq 0$, then

$$\begin{aligned} &\frac{\int_{.9r}^{1.1r} \varphi(\frac{\ell}{r}) \int_{\S(\ell)} \sum_{i=1}^n (\partial_i g_{ij} - \partial_j g_{ii}) \nu^j \, dS d\ell}{4\pi(n-1)\omega_{n-1} r \int_{.9}^{1.1} \varphi(\ell) d\ell} \\ &= \frac{\int_{.9}^{1.1} \varphi(\ell) \int_{\S(\ell r)} \sum_{i=1}^n (\partial_i g_{ij} - \partial_j g_{ii}) \nu^j \, dS d\ell}{4\pi(n-1)\omega_{n-1} \int_{.9}^{1.1} \varphi(\ell) d\ell} \xrightarrow[r\to\infty]{} m_{ADM}(g, \Phi). \end{aligned} \tag{5}$$

The significance of the left-hand side of (5) is that it may be expressed solely in terms of the C^0 data of g after integrating by parts; we denote the result of this calculation by $M_{C^0}(g, \Phi, \varphi, r)$; see [6, Definition 2.1] for the details. In order to get a well-defined limit at infinity we replace φ with a family of functions φ^r that vary with r. Specifically, the φ^r are time-zero slices of an evolving family of smooth functions satisfying a particular backwards evolution that is chosen in order to bound the distortion of the "C^0 local mass" under Ricci-DeTurck flow; see [6, Lemma 2.6]. We summarize this fact in the following theorem (see [6, Theorem 2.9] for the precise statement):

Theorem 2. *Let M^n be a smooth manifold, and g a continuous Riemannian metric on M. Suppose there is a smooth coordinate chart $\Phi : M \setminus K \to \mathbb{R}^n \setminus \overline{B(0,1)}$ for M, where K is some compact set. For any smooth cutoff function $\varphi : \mathbb{R} \to \mathbb{R}^{\geq 0}$ with* $\mathrm{Supp}(\varphi) \subset\subset (.9, 1.1)$ *and for any $r > 0$, there exists a smooth family of functions $(\varphi^r)_{r>0} : \mathbb{R} \to \mathbb{R}$ such that $\varphi^r \xrightarrow[r\to\infty]{C^\infty} \varphi$, and there exists a quantity $M_{C^0}(g, \Phi, \varphi^r, r)$, depending on only the C^0 data of $\Phi_* g$, for which the following is true:*

(1) If g is C^2 and $m_{ADM}(g, \Phi)$ exists, then $m_{ADM}(g, \Phi) = \lim_{r\to\infty} M_{C^0}(g, \Phi, \varphi^r, r)$.
(2) If g has nonnegative scalar curvature in the sense of Ricci flow on $M \setminus K$ and the decay condition (2) holds for Φ for some $\tau > (n-2)/2$, then $\lim_{r\to\infty} M_{C^0}(g, \Phi, \varphi^r, r)$ exists, is either finite or $+\infty$, and is independent of choice of such Φ and φ. Moreover, this limit is finite if and only if a particular condition involving the scalar curvature of time slices of Ricci-DeTurck flows associated to g is satisfied.

Furthermore, in the case that M has multiple ends, the result holds if $M \setminus K$ is replaced by a neighborhood of an end of M that is diffeomorphic to $\mathbb{R}^n \setminus \overline{B(0,1)}$.

Theorem 2 follows from a monotonicity result for the C^0 local mass; see [6, Theorem 2.11]:

Theorem 3. *Let M be a smooth manifold and g a C^0 metric on M. Suppose that U_1 and U_2 are open subsets of M for which, for $m = 1, 2$, there exist coordinate charts $\Phi^m : U_m \to \mathbb{R}^n \setminus \overline{B(0,1)}$ that determine the same end of M and such that, for some $\tau_m > \frac{1}{2}(n-2)$, (2) holds for Φ^m and τ_m. If g has nonnegative scalar curvature in the sense of Ricci flow and $\varphi_m \geq 0$ are smooth cutoff functions with* $\mathrm{Supp}(\varphi_m) \subset\subset (.9, 1.1)$*, then, for all sufficiently large r,*

$$M_{C^0}(g, \Phi^1, \varphi_1^{200r}, 200r) \geq M_{C^0}(g, \Phi^2, \varphi_2^r, r) - cr^{-\omega}$$

for some $\omega > 0$, where c and ω do not depend on r.

Question 1. How does $\lim_{r\to\infty} M_{C^0}(g, \Phi, \varphi^r, r)$ relate to Huisken's isoperimetric mass [7, Definition 11] and Jauregui's isocapacitary mass [8, Definition 4]?

Question 2. If g satisfies the hypotheses of the second item in Theorem 2 and moreover g has nonnegative scalar curvature in the sense of Ricci flow everywhere on M, do we have $\lim_{r\to\infty} M_{C^0}(g, \Phi, \varphi^r, r) \geq 0$?

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Minimal submanifolds and mean curvature flows of codimension 2 via adiabatic limits

Daniel Stern
(joint work with Davide Parise and Alessandro Pigati)

Since the 1970s, the correspondence between Allen–Cahn-type energies $E_\epsilon(u) := \int_M \frac{\epsilon}{2}|du|^2 + \frac{W(u)}{\epsilon}$ for scalar-valued functions $u : M \to \mathbb{R}$ and the $(n-1)$-area functional for hypersurfaces has inspired many developments in geometry and PDE. While early results emphasized convergence of minimizers u_ϵ to area-minimizing hypersurfaces as $\epsilon \to 0$ in a BV sense, in the '90s, a series of papers confirmed a long-suspected link between the gradient flow of E_ϵ and the mean curvature flow of hypersurfaces, culminating in Ilmanen's work [10] proving convergence of solutions to the parabolic Allen–Cahn equations $\frac{\partial u_\epsilon}{\partial t} = \Delta u_\epsilon - \frac{W'(u_\epsilon)}{\epsilon^2}$ to $(n-1)$-Brakke flows in a measure-theoretic sense. In the decades since, Ilmanen's results have been refined in various directions, e.g. in Tonegawa's work showing that the limiting hypersurfaces appearing in those weak mean curvature flows come with integer multiplicity–i.e., they are Brakke flows of *integral* varifolds [20]. Recently, the stationary case of these results–giving convergence of critical points of E_ϵ to minimal hypersurfaces–has been exploited to great effect by the geometry community to study the space of minimal hypersurfaces in Riemannian manifolds. In particular, work of Guaraco [8] and Gaspar–Guaraco [7] uses the variational theory for the functionals E_ϵ as an appealing regularization of the GMT min-max framework of Almgren–Pitts for producing minimal hypersurfaces, while work of Chodosh–Mantoulidis provides some dramatic geometric applications, including the first proof of the multiplicity one conjecture for min-max minimal surfaces in 3-manifolds [4] and surprising regularity results for min-max geodesic networks on surfaces [5].

It is natural to ask whether any meaningful analogs of this correspondence can be found in higher codimension. In the late 1990s and early 2000s, a series of papers (e.g., [11, 12, 1, 2]) suggested a possible analog in codimension two via the formally similar Ginzburg–Landau energies $E_\epsilon^{GL}(u) := \int_M |du|^2 + \frac{(1-|u|^2)^2}{\epsilon^2}$ for *complex–valued maps* $u : M \to \mathbb{C}$. The results of [11, 12, 1] show that critical points u_ϵ subject to the natural energy bound $E_\epsilon(u_\epsilon) = O(|\log \epsilon|)$ give rise as $\epsilon \to 0$ to stationary rectifiable $(n-2)$-varifolds, with [2] proving a parabolic analog. However, in spite of the formal similarities with the scalar Allen–Cahn energies, the behavior of solutions to the elliptic or parabolic Ginzburg–Landau equations differs fundamentally from that of their Allen–Cahn counterparts: most significantly, while the energy of Allen–Cahn solutions u_ϵ is highly concentrated in an $O(\epsilon)$-neighborhood of the zero set $u_\epsilon^{-1}\{0\}$, the energy of Ginzburg–Landau solutions v_ϵ spreads over a large annular region around $v_\epsilon^{-1}\{0\}$, giving the energy a more non-local flavor, with nontrivial interactions between distant components of $v_\epsilon^{-1}\{0\}$. One concrete consequence of these qualitative differences can be seen in the problem of *integrality*: while the results of [9, 20] show that the limit hypersurfaces obtained from solutions of the elliptic and parabolic Allen–Cahn equations have the structure of *integral* varifolds, the results of [1, 2] only show rectifiability in the Ginzburg–Landau setting, leaving the question of their integrality open. Last spring, with Pigati, we showed that in fact integrality fails in general even in the stationary case, exhibiting families of critical points of E_ϵ^{GL} for which the density of the limit varifold may take prescribed values anywhere in the range $\{\pi\} \cup [2\pi, \infty)$ [17].

On the other hand, in a series of papers with Pigati and Parise–Pigati [16, 14, 15], we identified a compelling analog of the Allen-Cahn–minimal hypersurface correspondence in codimension two, replacing the complex Ginzburg–Landau functionals with the *self-dual $U(1)$-Higgs functionals* from gauge theory, defined for sections $u \in \Gamma(L)$ and hermitian connections ∇ on Hermitian line bundles $L \to M$ by

$$E_\epsilon(u, \nabla) := \int_M |\nabla u|^2 + \epsilon^2 |F_\nabla|^2 + \frac{(1-|u|^2)^2}{4\epsilon^2}.$$

Finite-energy solutions over $\mathbb{R}^2$ were classified by Taubes [18, 19], and solve a special first-order system known as the *vortex equations*; solutions of this first-order system were later studied on higher-dimensional Kähler manifolds as well, where a correspondence was exhibited between the moduli space of solutions and the space of complex hypersurfaces [3, 6]. In [16], Pigati and I showed that bounded-energy critical points of E_ϵ on arbitrary manifolds converge in a measure-theoretic sense to stationary, integral varifolds of codimension two, with applications to the existence theory for minimal varieties of codimension two. In many ways the analysis is strikingly similar to that of [10, 9] in the Allen–Cahn setting, with a delicate balancing between the Yang–Mills term $\epsilon^2 |F_\nabla|^2$ and the nonlinear potential $\frac{(1-|u|^2)^2}{4\epsilon^2}$ playing a role analogous to the balancing between Dirichlet and potential terms in [10, 9]. Later, with Parise [14], we established a Γ-convergence theory analogous to that of [13] in this setting, showing convergence of the functionals E_ϵ to the area functional on integral $(n-2)$-cycles at the variational level.

In forthcoming work with Parise and Pigati [15], we demonstrate convergence of gradient flows $(u_t^\epsilon, \nabla_t^\epsilon)$ for the energy E_ϵ to codimension-two integral Brakke flows in a measure-theoretic sense–analogous in spirit to the work of Ilmanen [10] and Tonegawa [20] in the Allen–Cahn setting. Together with the Γ-convergence theory developed in [14], this can be used to produce nontrivial weak mean curvature flows starting from any initial integral cycle of codimension two, similar to those produced by Ilmanen via elliptic regularization. With an eye to geometric applications, it would be interesting to understand whether this approximation can be used to obtain any kind of gauge-theoretic regularization for the *Lagrangian* mean curvature flow of surfaces in Kähler-Einstein manifolds of complex dimension two.

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Towards canonical locally conformally Kähler metrics

Cristiano Spotti
(joint work with Daniele Angella, Simone Calamai, Francesco Pediconi)

Finding interesting notions of best metrics for non-Kähler complex manifolds turned out to be a quite subtle problem, in particular due to the many possible different generalizations of familiar notions in Kähler geometry. Partially guided by the idea that canonical metrics should be of some help in the study of moduli spaces of complex manifolds, in [1] we investigated an extension of the infinite dimensional Donaldson-Fujiki moment map picture in the so-called locally conformally Kähler case, introducing a possible notion of canonical metrics in this setting. In particular, it's worth noting that the co-differential of the Lee form (as well as the Chern scalar curvature) appears in the proposed equation for the metric. Moreover, as a by-product of this setup, we also obtained a Futaki type obstruction, which may lead to notion of stability.

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Epsilon regularity for spaces with scalar curvature lower bounds

Robin Neumayer

A central theme in geometric analysis is the study of the structure and a priori regularity of a Riemannian manifold when certain constraints are imposed on the curvature tensor. Following a remarkably successful program in the past three decades studying (lower and two-sided) bounds on the Ricci curvature, there has been interest in recent years in the structure and regularity of manifolds that only have bounds on the scalar curvature.

A first step toward develop such a theory is an epsilon regularity theorem, which in the context of lower bounds on scalar curvature says: if a Riemannian manifold (M, g) has $R_g \geq -\epsilon$ and is "ϵ-flat," then locally it is geometrically close to Euclidean space, where the notions of ϵ-flat and geometric closeness and defined appropriately.

For Riemannian manifolds with almost-Euclidean lower bounds on Ricci curvature, ϵ-flatness is measured by the volumes of balls via a non-collapsing assumption $\mathrm{vol}_g(B_g(x, 1)) \geq (1-\epsilon)\omega_n$. Here ω_n is the volume of a Euclidean unit ball. By Bishop-Gromov volume comparison, if such a non-collapsing condition holds and $\mathrm{Ric}_g \geq -\epsilon g$ in $B_g(x, 1)$ for ϵ small enough, then in fact

$$(1) \qquad (1 - c_n\epsilon)\omega_n r^n \leq \mathrm{vol}_g(B(x, r)) \leq (1 - c_n\epsilon)\omega_n r^n \quad \text{for all } r \in (0, 1]\,.$$

For spaces with (weaker) almost-Euclidean lower bounds on scalar curvature, it is reasonable to expect that a stronger form of noncollapsing must be imposed to obtain an epsilon regularity theorem. One notion of non-collapsing is formulated in terms of the Perelman entropy, a one-parameter family of optimal constants $\mu(g,\tau)$ for a log-Sobolev inequality on M at varying scales $\tau^{1/2} > 0$, introduced by Perelman [4] as a monotone quantity for the Ricci flow. For any complete Riemannian manifold (M,g), $\inf_{\tau>0}\mu(g,\tau) \leq 0$ with equality if and only if (M,g) is isometric to Euclidean space. The assumption $\inf_{\tau\in(0,2]}\mu(g,\tau) \geq -\epsilon$ is another notion of non-collapsing or ϵ-flatness. It relates to volume non-collapsing in the following way: for any $\epsilon > 0$, there exists $\delta > 0$ such that if (M,g) is complete with bounded curvature and has $R_g \geq -\delta$ and $\inf_{\tau\in(0,2]}\mu(g,\tau) \geq -\delta$, then the volume lower bound in (1) holds. Here R_g is the scalar curvature.

With Lee and Naber in [1] (see also [2]), we investigate complete Riemannian manifolds with almost nonnegative scalar curvature and almost-Euclidean Perelman entropy. We construct examples in dimensions $n \geq 4$ showing that distance functions are not uniformly controlled under these assumptions and thus "geometric closeness" in an epsilon regularity theorem cannot hold with respect to a metric space distance such as the Gromov-Hausdorff distance or intrinsic flat distance in this context. Instead, we prove an ϵ-regularity theorem that measures geometric closeness with respect to a new notion called the d_p distance that is based on the distance between $W^{1,p}$ Sobolev spaces.

These examples show d_p closeness is in a sense optimal, However, this is an atypical notion of regularity and it is important to understand under what additional assumptions the geodesic distance is controlled and thus more conventional regularity estimates hold. A shared feature of the examples we construct is that the degeneration of distance functions leads to blowup of the volumes of geodesic balls. This suggests the possibility of imposing a notion of ϵ-flatness that implies both upper and lower bounds on the volumes of balls along the lines of (1).

Indeed, in [3], we prove a different type of ϵ-regularity result. Here, for a complete Riemannian manifold (M,g), in addition to almost-Euclidean lower bounds on the scalar curvature, $R_g \geq -\epsilon$, and Perelman entropy, $\inf_{\tau\in(0,2]}\mu(g,\tau) \geq -\epsilon$, we assume an almost-Euclidean upper bound on volumes of balls: $\mathrm{vol}_g(B_g(x,r)) \leq (1+\epsilon)\omega_n r^n$ for all $x \in M$ and $r \in (0,2]$. These assumptions imply that geodesic balls of radius $r \in (0,1]$ are Gromov-Hausdorff close and bi-$W^{1,p}$ homeomorphic to Euclidean balls. The Gromov-Hausdorff closeness can alternatively be deduced from work of B. Wang [5], while the $W^{1,p}$ estimates are based on a decomposition theorem shown in [1]. We also prove a compactness and limit space structure theorem under the same assumptions.

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Positive scalar curvature on manifolds with boundary and their doubles

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(joint work with Shmuel Weinberger)

In high dimensions, there is now a good obstruction theory for psc (positive scalar curvature) Riemannian metrics. For example, it is known that on a closed connected spin manifold M^n with $n \geq 5$, the question of whether or not M admits a psc metric only depends on the image in $ko_n(B\pi_1(M))$ of the ko-fundamental class $[M] \in ko_n(M)$ under the classifying map for the universal cover $c\colon M \to B\pi_1(M)$. And on a closed oriented totally non-spin manifold (totally non-spin means that there are no spin covers), the same holds with ko_n replaced by ordinary integral homology H_n. In some nice cases, one knows the GLR (Gromov-Lawson-Rosenberg) conjecture that in the spin case, the vanishing of $c_*([M])$ is both necessary and sufficient for M to admit a psc metric.

So we wanted to see if there is a corresponding theory for compact manifolds X with non-empty boundary. In this case, imposing boundary conditions is essential, since Gromov observed a long time ago that if $\partial X \neq \emptyset$, then X always admits a Riemannian metric on X with positive sectional curvature, albeit with no good properties near the boundary. Thus we consider two different boundary conditions:

(1) the metric on X is required to be a product metric $dt^2 + g_{\partial X}$ in a neighborhood of ∂X; or
(2) the boundary ∂X is required to have positive mean curvature H (with the sign convention that X is mean convex, or that pushing ∂X away from the interior makes the $(n-1)$-dimensional area increase).

With regard to boundary condition (1), there is a very satisfactory obstruction theory.

Theorem 1 ([4]). *The question of whether or not a compact connected spin manifold X of dimension $n \geq 6$ with non-empty boundary admits a psc metric which is a product metric in a neighborhood of the boundary only depends on the image of the ko-fundamental class $[X, \partial X]$ in $ko_n(B\Gamma, B\Lambda)$ under the classifying map c. Here $\Gamma = \pi(X)$ and $\Lambda = \pi(\partial X)$ are the fundamental groupoids. (Note that we need groupoids and not just groups since ∂X can be disconnected.) If X is oriented and totally non-spin, the same holds with ko_n replaced by H_n.*

In many cases (when everything is spin), one can deduce from the index theorem of Chang, Weinberger, and Yu, or of Schick and Seyedhosseini [5], that vanishing of the image of $[X, \partial X]$ in $ko_n(B\Gamma, B\Lambda)$ is both *necessary and sufficient* for a existence of a psc metric on X with product structure near the boundary.

Clearly, if ∂X does not admit a psc metric, then boundary condition (1) can never be satisfied. However, boundary condition (2) is weaker, in that according to Bär and Hanke [1, Corollary 40], if ∂X admits a psc metric g_0, then existence of solutions to condition (1) restricting to g_0 on the boundary is equivalent to existence of solutions to condition (2) restricting to g_0 on the boundary. But when ∂X does not admit a psc metric, it may still be possible to satisfy condition (2).

In [2, Theorem 5.7], it was shown that existence of a solution to boundary condition (2) implies that $M = \mathrm{Dbl}(X, \partial X)$, the double of X along its boundary, admits psc.

Conjecture 1 (Doubling Conjecture)**.** The converse to the Gromov-Lawson Theorem holds, in that existence of a psc metric on $M = \mathrm{Dbl}(X, \partial X)$ implies existence of a psc metric on X with $H > 0$ on the boundary.

Our other main results are proofs of this in many cases, using surgery techniques coming from [3].

1. Open Problems

(1) Is the Doubling Conjecture always true? In particular, does it fail when $X = Y \times [0, 1]$, Y a smooth simply connected non-spin 4-manifold with a non-zero Seiberg-Witten invariant (so that Y does not admit a psc metric)? Note that in this case, $M = \mathrm{Dbl}(X, \partial X) = Y \times S^1$ *does* admit a psc metric, so our Theorem 1 about obstructions to product-type boundary conditions definitely fails in this case.
(2) What can one say about the homotopy type of the space of psc metrics on X satisfying boundary conditions (1) or (2)? A bit is known at present, but there is a lot more work to be done.

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Volume and macroscopic scalar curvature

Roman Sauer

(joint work with Sabine Braun)

We report on the following result in [2] which is the non-sharp macroscopic cousin of the well-known conjecture that rationally essential manifolds do not admit a metric of positive scalar curvature (the stateminent for $R = 1$ readily implies the one for all $R > 0$ by scaling the metric). Let $V_{(\widetilde{M},\widetilde{g})}(R)$ denote the maximal volume of an R-ball on the universal cover of a Riemannian manifold (M, g).

Theorem. *There is a dimensional constant $\epsilon(d) > 0$ with the following property. For every rationally essential Riemannian manifold (M, g) of dimension d and every $R > 0$ we have*

$$V_{(\widetilde{M},\widetilde{g})}(R) > \epsilon(d) \cdot R^d.$$

If $\epsilon(d)$ could be chosen to be the volume of a Euclidean d-ball, then the above conjecture would follow. A closed oriented manifold is *rationally essential* if its classifying map sends the fundamental class to a non-zero class in rational homology. Guth proves the above theorem in [3] for closed aspherical manifolds. The extension to rationally essential manifolds needs a number of new tools from topological dynamics of actions on Cantor spaces and equivariant topology.

We also report on recent work of H. Alpert who made the constant $\epsilon(d)$ explicit: $\epsilon(d) = 1/d!$. To this end, she combined the above ideas from topological dynamics with recent work of Papasoglu [4].

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On the Hamilton-Lott conjecture

Alix Deruelle

(joint work with Felix Schulze and Miles Simon)

In this report, we consider smooth solutions $(M^n, g(t))_{t\in(0,T)}$ to Ricci flow defined on smooth, connected manifolds satisfying for $t \in (0, T)$,

$$|\mathrm{Rm}(g(t))| \leq \frac{c_0}{t} \quad \text{and} \quad \mathrm{Ric}(g(t)) \geq -g(t), \tag{1}$$

where c_0 is a positive time-independent constant. This setting has been shown to occur in many situations, one prominent one being that of expanding solitons with non-negative curvature operator coming out of cones with non-negative curvature operator: see for example [11], [6], [12], [1].

Using [12, Lemma 3.1], we see that the above setting guarantees that the distances $d_t := d(g(t))$ on M converge locally in a strong sense. More explicitly, assuming (1), we have:

$$\text{for all } x_0 \in M, \text{ there exists a connected open set } U_{x_0} \subset M \text{ and } S(x_0) > 0 \quad (2)$$
$$\text{s.t.} \quad e^{t-s} d_s \geq d_t \geq d_s - c_0\sqrt{t-s} \text{ for } 0 < s \leq t < S(x_0), \text{ on } U_{x_0},$$

and hence there exists a unique limit $d_0 := \lim_{t\to 0} d_t$ on U_{x_0}, which is attained uniformly, such that (U_{x_0}, d_0) is a metric space. Note that if for different points $x_0, y_0 \in M$ we obtain $d_0 = \lim_{t\to 0} d_t$ on U_{x_0} and $\tilde{d_0} = \lim_{t\to 0} d_t$ on U_{y_0} then we have $d_0 = \tilde{d_0}$ on $U_{x_0} \cap U_{y_0}$, since they both are obtained as the uniform limit of $d(g(t))$ as $t \to 0$. For this reason we do not include a quantifier depending on x_0 in the definition of $d_0 := \lim_{t\to 0} d_t$ on U_{x_0}. We are interested in the following problem:

Problem 1. *Let $(M_i^n, g_i(t))_{t\in(0,T)}, i = 1, 2$ be two smooth, connected (possibly incomplete) Ricci flows satisfying* (1) *and* (2) *and converging locally to the same metric space, up to an isometry as t tends to* 0*, that is:* $\lim_{t\to 0} d(g_1(t)) = d_{0,1}$ *on* $U_{p,1}$ $\lim_{t\to 0} d(g_2(t)) = d_{0,2}$*, on* $U_{p,2}$ *and* $\psi_0(U_{p,1}) = U_{p,2}$*, where* $\psi_0 : U_{p,1} \to U_{p,2} = \psi_0(U_{p,1})$ *is a homeomorphism with* $\psi_0^*(d_{0,2}) = d_{0,1}$*. Then we are concerned with the following problems:*

What further assumptions on the regularity of d_0 *ensure that*

- *there is a suitable gauge in which we can compare the solutions* $(g_1(t))_{t\in(0,T)}$ *and* $(g_2(t))_{t\in(0,T)}$ *effectively?*
- *the solutions* $(g_1(t))_{t\in(0,T)}$ *and* $(g_2(t))_{t\in(0,T)}$ *remain close to one another for t close to zero in this gauge?*

Our fundamental regularity assumption on d_0 is the following *Reifenberg* property:

(3)
$$\text{for all } p \in M, \text{ for all } x \in U_p, \text{ every tangent cone at } x \text{ of } (U_p, d_0) \text{ is } (\mathbb{R}^n, d(\delta)),$$

where $d(\delta)$ stands for Euclidean distance. In fact, if (M, d_0) is the local limit of a non-collapsing sequence of complete, smooth Riemannian manifolds with bounded curvature and Ricci curvature uniformly bounded from below, this condition can be turned into a uniform Reifenberg condition,

(4)
for all $p \in M$ and for all $\epsilon > 0$, there exist $r > 0$ and a neighborhood $U_p \subset\subset M$ such that $d_{GH}(B_{s^{-1}d_0}(x, 1), \mathbb{B}(0, 1)) < \epsilon$, for all $s < r$, and for all $x \in U_p$ such that $B_{d_0}(x, s) \subset\subset U_p$.

Assumption (3) is infinitesimal and means that there are no *singular* points in (U_p, d_0) for any $p \in M$. However, it does not necessarily mean that the distance is induced by a Hölder continuous Riemannian metric, let alone a Lipschitz Riemannian metric: see [5, Theorem 1.2] for such a counterexample.

We assume as a further regularity assumption on d_0 that there is a bi-Lipschitz chart around each point in (U_p, d_0) given by distance coordinates, and that the Lipschitz constant is ϵ_0 close to 1:

(5)

For any $x_0 \in U_p$, there is a radius $R = R(x_0) > 0$ such that $B_{d_0}(x_0, 4R) \subset\subset U_p$ and points $a_1, \ldots, a_n \in B_{d_0}(x_0, 3R)$ such that the map

$$D_0 : \begin{cases} B_{d_0}(x_0, 4R) & \to \quad \mathbb{R}^n \\ \qquad\quad x & \to \quad (d_0(a_1, x) - d_0(a_1, x_0), \ldots, d_0(a_n, x) - d_0(a_n, x_0)), \end{cases}$$

is a $(1 + \epsilon_0)$ bi-Lipschitz homeomorphism on $B_{d_0}(x_0, 2R)$.

Assumption (5) is local and always holds true in the case that (U_p, d_0) is an Alexandrov space with curvature bounded from below satisfying (3): see [2, Theorem 10.8.4]. For example, this would be the case, if we assume (3) and replace the lower bound on the Ricci curvature in (1), by $\sec_{g(t)} \geq -1$, $t \in (0, T)$, where sec refers to sectional curvature.

Our main result is an initial stability estimate which addresses the question posed in Problem 1: see [8, Theorem 1.2].

As an application of this result, we show that the approach of Lott [10] leads to a resolution of a conjecture posed by Hamilton [4, Conjecture 3.39] and Lott [10, Conjecture 1.1] provided the metric has bounded curvature.

Recall that a Riemannian manifold (M^n, g) is uniformly Ricci pinched if $\mathrm{Ric}(g) \geq 0$ and there exists a constant $c > 0$ such that on M,

$$\mathrm{Ric}(g) \;\geq\; c\,\mathrm{R}_g g,$$

in the sense of quadratic forms where R_g denotes the scalar curvature of the metric g. Notice that such a condition is invariant under rescalings.

Theorem 1. *Let (M^3, g) be a smooth complete Riemannian manifold with bounded and uniformly pinched Ricci curvature. Then (M^3, g) is either smoothly isotopic to a spherical space form or flat. In particular, if M is non-compact then (M^3, g) is flat.*

Hamilton introduced the Ricci flow in [9], and in the case that (M^3, g) is compact with non-negative uniformly pinched Ricci curvature and the scalar curvature is positive at least at one point, the paper shows that the volume preserving Ricci flow of (M^3, g) exists for all time and converges smoothly to a spherical space form. In the case that (M^3, g) is compact and has non-negative uniformly pinched Ricci curvature and the scalar curvature is zero everywhere, then M^3 is Ricci-flat and hence flat. That is, the results of Hamilton imply Theorem 1 immediately in the case that M^3 is compact.

In case (M^3, g) is non-compact with bounded curvature, Lott [10] has proven Theorem 1 under the assumption that the sectional curvature of g has a lower bound decaying at least quadratically in the distance from a fixed point, improving a result of Chen-Zhu [3] where it is assumed that the metric g has non-negative sectional curvature. Finally, Theorem 1 can be interpreted as an extension of Myers' theorem in dimension 3.

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Holonomy of limits of Einstein 4-manifolds

Tristan Ozuch

(joint work with Claude LeBrun)

Recent developments [Biq13, Biq16, Ozu22a, Ozu22b] have led to a complete reconstruction of the moduli space of Einstein 4-metrics in a Gromov-Hausdorff (GH) neighborhood of any noncollapsed singular metric.

This new description gave more structure to the moduli space of Einstein 4-manifolds close to its boundary, [Ozu20].

This arbitrarily precise reconstruction has been use to show that not every singular (or synthetic in Naber's sense) Einstein 4-metric could be a limit of smooth Einstein 4-manifolds, [Ozu21b]. Spherical and hyperbolic orbifolds with the simplest singularities cannot be limits of Einstein 4-manifolds in the Gromov-Hausdorff sense. A natural question becomes: what are the possible limits of smooth Einstein 4-manifolds? Do these limits have a specific structure?

In [Ozu21a] indications that being Kähler may be a necessary condition for an Einstein singular space to be a limit of smooth Einstein 4-manifolds with expected topology. The natural question is then: Can any Kähler-Einstein singular metric be desingularized by Einstein 4-manifolds? It is known [Kol07, OSS16] that this is not the case for some Kähler-Einstein limits. Can this still be achieved by Einstein metrics that are not necessarily Kähler, [And10]?

With Claude LeBrun, we however show that the Kähler condition is not sufficient when the scalar curvature is positive. Specifically, our work demonstrates that if a real Einstein 4-metric is close to a conformally Kähler, Einstein orbifold with positive scalar curvature, it must be conformally Kähler, which rules out most desingularizations.

To prove this, we combine the above reconstruction and a flexible criterion developed by Wu [Wu21] and improved by LeBrun [LeB21] to detect Kähler-Einstein metrics among Einstein metrics: an Einstein metric with positive scalar curvature is conformally Kähler if and only if $\det W^+ > 0$ at every point, where W^+ is the selfdual part of the Weyl curvature seen as an endomorphism on 2-forms.

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Morse index of minimal surfaces in $\mathbb{R}^3$ and the half-space

Shuli Chen

In this talk I report about my work [2] on the study of the Morse index of minimal surfaces in $\mathbb{R}^3$ and minimal surfaces with free boundary in a half-space of $\mathbb{R}^3$. We observe that for a minimal surface with free boundary in a half-space, the Neumann index can be bounded from below by the sum of the Dirichlet index and the Dirichlet nullity. We use this to answer a question of Ambrozio, Buzano, Carlotto, and Sharp [1] concerning the non-existence of index two embedded minimal surfaces with free boundary in a half-space. Using the reflection symmetry, we also give a simplified proof of a result of Chodosh and Maximo [3] showing the index of the Costa deformation family $M_{2,x}, x \geq 1$ is bounded below by 4. Here $M_{2,1}$ denotes the Costa surface. More generally, for each $k \geq 2$, the surfaces $M_{k,x}, x \geq 1$ denote the 1-parameter family of embedded minimal surfaces of genus $k-1$ with three ends constructed by Hoffman–Meeks. Lastly, we discuss about work in progress about computing the index of $M_{k,x}$ when the parameter x is sufficiently large.

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Kähler–Einstein metrics on families of Fano varieties

Chung-Ming Pan

(joint work with Antonio Trusiani)

Fano varieties and their families are central objects in complex geometry. They often have rich geometry as they can have many interesting birational models, and they are also terminal objects in the classification theory of projective varieties. Moreover, singular Fano varieties arise naturally as degenerations of Fano manifolds from a moduli space point of view. The construction of well-behaved moduli spaces of Fano varieties has recently been advanced by studying Fano Kähler–Einstein metrics.

The resolution of the Yau–Tian–Donaldson (YTD) correspondence by Chen–Donaldson–Sun [CDS15] established a deep connection between the existence of Kähler–Einstein metrics on Fano manifolds and an algebro-geometric notion called "K-stability". On mildly singular varieties, it is still possible to define "singular" Kähler–Einstein metrics. These metrics are genuine Kähler–Einstein metrics on the smooth part of varieties and have "bounded potentials" near the singular locus. Specifically, on a $\mathbb{Q}$-Fano variety X, considering a fixed Kähler metric $\omega \in c_1(-K_X)$, singular Kähler–Einstein metrics are obtained by solving the following complex Monge–Ampère equation:

$$(\omega + \mathrm{dd^c}\varphi)^n = e^{-\varphi}\mu \quad \text{and} \quad \varphi \in \mathrm{PSH}(X,\omega) \cap L^\infty(X),$$

where μ is an adapted probability measure associated to ω (i.e. $\mathrm{Ric}(\mu) = \omega$). The Kähler–Einstein problem remains meaningful if X has Kawamata log terminal (klt) singularities, wherein the measure μ possesses finite mass near the singular set. In the singular context, pluripotential theory has led to significant breakthroughs in understanding the singular Kähler–Einstein problem, as the contributions in [BBEGZ19, BBJ21, LTW22, Li22].

As mentioned earlier, the degenerate families of Fano varieties are important in the (relative) Minimal Model Program and moduli theory. Therefore, it becomes highly desirable to comprehend the behavior of singular Kähler–Einstein metrics in families of Fano varieties.

In the sequel, we focus on one-parameter families of $\mathbb{Q}$-Fano varieties. More precisely, we assume that the family $\pi : \mathcal{X} \to \mathbb{D}$ fulfills the following setting:

Setting. *Let $\mathcal{X}$ be an $(n+1)$-dimensional $\mathbb{Q}$-Gorenstein variety and let $\pi : \mathcal{X} \to \mathbb{D}$ be a proper holomorphic surjective map with connected fibres. In addition, suppose that $-K_{\mathcal{X}/\mathbb{D}}$ is relatively ample, the central fibre X_0 is klt, and $\mathrm{Aut}(X_0)$ is finite.*

In [PT23], we first provide an analytic proof of the (Euclidean) openness of the existence of unique Kähler–Einstein metric:

Theorem 1 ([PT23, Thm. A]). *Under Setting, if the central fibre X_0 admits a Kähler–Einstein metric, then so do the nearby fibres.*

In the case where the family $\pi : \mathcal{X} \to \mathbb{D}$ is smooth, Koiso [Koi83] showed the openness using an implicit function theorem argument. For families that are smoothings of the central fibre, Spotti, Sun and Yao [SSY16, Thm. 1.1] established the openness through a sophisticated cone metrics technique originally introduced by Chen–Donaldson–Sun. However, these two approaches strongly use the smoothness of generic fibres and they are hard to be extended to the case of a general family as considered in our situation. Therefore, we resort to the variational approach and pluripotential theory, which have successfully addressed the YTD conjecture in the singular setting to proceed with our proof.

Furthermore, the finiteness of the automorphism group on the central fibre is an important hypothesis as there are arbitrary small deformations of the Mukai–Umemura threefold which do not admit Kähler–Einstein metrics (see [Tia97, Don07]).

From the algebraic perspective, recently, the (Zariski) openness of K-stability has been established by Blum and Liu [BL22]. As a consequence, the openness of a unique Kähler–Einstein metric can be deduced by combining the openness of K-stability with the singular YTD correspondence [LTW22]. Notably, our analytic approach to achieve Euclidean openness does not depend on the aforementioned two deep results.

Our results go beyond by establishing uniform a priori estimates on the Kähler–Einstein potentials, and the continuous variation of the Kähler–Einstein currents:

Theorem 2 ([PT23, Thm. B]). *Under Setting, letting $\omega \in c_1(-K_{\mathcal{X}/\mathbb{D}})$ be a Kähler metric, there exists a uniform constant $C > 0$ such that for all t sufficiently close to 0,*

$$\operatorname*{osc}_{X_t} \varphi_t < C.$$

where $\varphi_t \in \mathrm{PSH}(X_t, \omega_t) \cap L^\infty(X_t)$ is the Kähler–Einstein potential with respect to the Kähler metric $\omega_t := \omega_{|X_t}$.

In addition, $(\varphi_t)_t$ varies smoothly away from the singular locus, and for all $k \in \{0, 1, \cdots, n\}$, the following map is continuous near $0 \in \mathbb{D}$

$$t \longmapsto \omega_{\mathrm{KE},t}^k \wedge [X_t] \in (\mathcal{D}_{(n-k,n-k)}(\mathcal{X}))',$$

where $(\mathcal{D}_{(n-k,n-k)}(\mathcal{X}))'$ is the space of currents of bidimension $(n-k, n-k)$.

To accomplish Theorem 1 and 2, we introduce and explore a notion of convergence for sequences of quasi-plurisubharmonic functions on a sequence of fibres. Roughly speaking, the convergence is defined by patching functions along local isomorphisms on the smooth part while overlooking the singular locus. Subsequently, we analyze the upper semi-continuity of the energy functional and the continuity of weighted integration of the adapted measures with respect to this convergence. These two essential properties form the lower semi-continuity of the Ding functional, which plays a crucial role in our proofs.

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Uniqueness and Morse index of minimal surfaces in the sphere and ball

David Wiygul

(joint work with Alessandro Carlotto, Mario B. Schulz, Nikolaos Kapouleas)

I will present some recent work with Alessandro Carlotto and Mario Schulz that partially complements earlier work with Nicos Kapouleas, which I will also review. Specifically, I will review the result [5] with Kapouleas that each of Lawson's embedded minimal surfaces in the round 3-sphere (constructed in [6]) is uniquely determined by its genus and symmetry group, and I will review the calculation [4], also with Kapouleas, of the Morse index of an infinite subfamily (namely $\{\xi_{g,1}\}_{g=0}^{\infty}$ in the notation of [6]) of these same surfaces. Then I will describe the construction [1] with Carlotto and Schulz of a family of embedded free boundary minimal surfaces in the Euclidean 3-ball having the same topological type and symmetry group as the members of a previously identified family (constructed in [3]), and I will finally describe some index estimates [2] for our new surfaces, including the result that each has equivariant index exactly two.

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Mean curvature flow with generic initial data

Felix Schulze

(joint work with Otis Chodosh, Kyeongsu Choi and Christos Mantoulidis)

Mean curvature flow is the natural heat equation for submanifolds. A family of surfaces $M(t) \subset \mathbb{R}^3$ flows by mean curvature flow if

$$\left(\tfrac{\partial}{\partial t}\mathbf{x}\right)^{\perp} = \mathbb{H}_{M(t)}(\mathbf{x}), \tag{1}$$

where $\mathbb{H}_{M(t)}(\mathbf{x})$ denotes the mean curvature vector of $M(t)$ at $\mathbf{x}$. When the initial surface $M(0)$ is compact, mean curvature flow is guaranteed to become singular in finite time. The simplest way to analyze such a singularity is to parabolically dilate around a singular point in space-time. Huisken's monotonicity formula guarantees that a subsequential limit of such dilations will weakly limit to a *tangent flow* which will be a weak solution to (1), evolving only by homothety. Ilmanen has showed that such a tangent flow will be supported on a smooth surface $\Sigma \subset \mathbb{R}^3$ so that $(-\infty, 0) \mapsto \sqrt{-t}\Sigma$ is a solution to (1). Such a surface is called a *self-shrinker*.

A deeper understanding of such tangent flows is necessary in order to continue the flow past the onset of singularities (either by constructing a flow with surgery, or by showing that weak solutions to the flow have good partial regularity and well-posedness properties). The two main obstructions to doing so are the potential presence of *multiplicity* and of tangent flows supported on self-shrinkers whose scalar mean curvature *changes sign.* By a result of Huisken [Hui90] (cf. [CM12a]) a shrinker whose mean curvature does not change sign is either a plane or a round sphere or cylinder. Such singularities (when they occur with multiplicity one) are known to be well-behaved, thanks to regularity results of Colding–Minicozzi [CM16] and the resolution of the mean convex neighborhood conjecture by Choi–Haslhofer–Hershkovits [CHH22].

In this work, we show that the second obstruction does not occur for *generic* initial data $M(0) \subset \mathbb{R}^3$. Our main result is roughly as follows:

Theorem (Mean curvature flow of generic initial data in $\mathbb{R}^3$)**.** *For $M(0) \subset \mathbb{R}^3$ closed embedded surface, there is $M'(0) \subset \mathbb{R}^3$ an arbitrarily small C^∞ normal graph over $M(0)$ so that the mean curvature flow $t \mapsto M'(t)$ satisfies one of the following conditions:*

- *all tangent flows at singular points are multiplicity one shrinking spheres or cylinders, or*
- *at the first time this fails, some tangent flow has multiplicity ≥ 2.*

This confirms—up to the potential occurrence of multiplicity—a long-standing conjecture of Huisken [Ilm03, # 8]. The well-known *multiplicity-one conjecture* posits that multiplicity never occurs, but this is widely open even for generic initial data.

There is strong evidence that the *generic* hypothesis in the above result is necessary. Indeed, Ilmanen–White have indicated [Whi02] a construction of a closed surface $M(0) \subset \mathbb{R}^3$ whose mean curvature flow is not well-posed after the onset of singularities. (On the other hand, if $M(0) \subset \mathbb{R}^3$ has genus zero, an important work of Brendle [Bre16] shows that the statement above holds without "generic.")

Remark. *By combining our main result with a surgery construction by Daniels-Holgate* [DH22]*, one can construct a mean curvature flow with surgery for a generic initial $M(0) \subset \mathbb{R}^3$ until the first time multiplicity occurs for the weak flow above.*

The no-cylinder conjecture. At the first time singularities appear (or more generally, the first time that some singularity other than multiplicity-one spherical/cylindrical singularities occur) we have seen that any tangent flow is supported on a smooth self-shrinker Σ (possibly with multiplicity). A fundamental result of Wang [Wan16a] shows that if $\Sigma \subset \mathbb{R}^3$ is a non-compact self-shrinker then any end of Σ is smoothly asymptotic—with multiplicity one—to a smooth cone or a cylinder.

Ilmanen has asked if it is possible that a non-cylindrical shrinker has a cylindrical end (the *no-cylinder conjecture* [Ilm03, # 12]). See [Wan16b] for some partial progress towards non-existence. Moreover, we note that are by now many constructions of self-shrinkers (both numerical and rigorous) but no example with a cylindrical end has been found.

The main new ingredient this work can be stated as follows:

Even if a non-cylindrical shrinker with a cylindrical end exists, it does not arise generically (with multiplicity one at the first singular time).

This builds on our previous work with Mantoulidis [CCMS20] where we proved that asymptotically conical and compact shrinkers do not arise generically, but left open the possibility of a shrinker with a cylindrical end.

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Expanding Ricci solitons asymptotic to cones with non-negative scalar curvature

Eric Chen

(joint work with Richard Bamler)

The Ricci flow starting from a compact Riemannian manifold (M^n, g_0) consists of a family of metrics $g(t)$ on M^n which satisfies $\frac{\partial}{\partial t} g = -2\mathrm{Ric}_g$ and $g(0) = g_0$, for $t \in [0, T)$ and some maximal time $T \in (0, \infty]$. When $n = 2$, after normalizing to fix the volume, the flow always converges to a metric of constant Gauss curvature, providing another proof of the uniformization theorem [4]. When $n = 3$, a complete understanding of singularity formation as $t \to T$ makes possible Ricci flow with surgery and its subsequent applications to three-manifold topology [7, 8, 9].

To develop a theory of Ricci flow with surgery in higher dimensions for potential new topological applications, especially in dimension $n = 4$, an understanding of singularity formation would again be essential [1]. One family of singularities arising in higher dimensions are the conical singularities; in fact, any asymptotically conical gradient shrinking soliton arises as a singularity model for some compact Ricci flow [10]. Recent work of Bamler further indicates the importance of this class—any four-dimensional compact Ricci flow with a finite-time singularity has a blowup model which is either a smooth compact gradient shrinking soliton, $S^2\times\mathbb{R}^2$, $S^3/\Gamma\times\mathbb{R}$, or a cone $(\mathbb{R}_+\times N^3, dr^2+r^2h)$ with non-negative scalar curvature [2].

The appearance of such conical singularities along the flow might be resolved by cutting and gluing in an expanding Ricci soliton, if one asymptotic to the same cone exists. When restricting to cones with positive curvature operator, the existence of such expanders is known [5], and using these it is indeed possible to resolve such isolated conical singularities [6]. Relaxing to cones of non-negative scalar curvature, with Richard Bamler we obtain the existence of expanding Ricci solitons asymptotic to such cones over S^3/Γ [3].

Theorem 1. *For any cone metric dr^2+r^2h on $\mathbb{R}_+\times S^3/\Gamma$ with non-negative scalar curvature, there is a gradient expanding soliton metric g on $\mathbb{R}^4/\Gamma$ with non-negative scalar curvature that is asymptotic to it.*

This result arises from a degree theory we establish for the natural projection map from the space of asymptotically conical expanding solitons on a fixed smooth orbifold with boundary X^4 to the space of cone metrics with non-negative scalar curvature. Let $\mathcal{M}_{\mathrm{grad},R\geq 0}(X)$ denote the space of isometry classes $[(g,\nabla f,\gamma)]$ of expanding gradient solitons with non-negative scalar curvature on the interior of X with metric g and potential field ∇f, asymptotic to the conical metric γ on a tubular neighborhood of ∂X via a fixed set of coordinates at infinity. Also let $\mathsf{Cone}_{R\geq 0}(\partial X)$ denote the space of conical metrics γ on $\mathbb{R}_+\times\partial X$. Then when X has isolated singularities, a regular boundary, and satisfies certain additional topological assumptions, we have (roughly):

Theorem 2. *The projection map $\Pi:\mathcal{M}_{\mathrm{grad},R\geq 0}(X)\to\mathsf{Cone}_{R\geq 0}(\partial X)$ is proper in a suitable topology and has a well-defined, integer valued degree, $\deg_{\exp}(X)\in\mathbb{Z}$. This degree is an invariant of the smooth structure of X.*

If the expander degree $\deg_{\exp}(X)\in\mathbb{Z}$ is nonzero, then Π is surjective, and therefore any member of $\mathsf{Cone}_{R\geq 0}(\partial X)$ is indeed the asymptotic cone of some expanding soliton on X. When $X\approx D^4$ or D^4/Γ, an argument showing that the Gaussian expander is the unique expander asymptotic to the flat cone over S^3 implies that $\deg_{\exp}(X)=1$, from which the existence result stated earlier follows.

One issue that arises during our construction is that $\mathcal{M}_{\mathrm{grad},R\geq 0}(X)$ may not have a local Banach manifold structure, due to analytical properties of the linear operator $L_f:=\triangle-\nabla_{\nabla f}+2\mathrm{Rm}$ associated with the expanding soliton equation. We must therefore work with the space $\mathsf{GenCone}(\partial X)$ of generalized cone metrics $\gamma=dr^2+r(dr\otimes\beta+\beta\otimes dr)+r^2h$ on $\mathbb{R}_+\times\partial X$ together with the larger space of expanding solitons $\mathcal{M}(X)$ asymptotic to these. This adds additional complications, but also yields deformations to infinitely many non-gradient expanding solitons near any asymptotically conical gradient expander with non-negative scalar curvature; all gradient expanders must be asymptotic to elements of $\mathsf{Cone}(\partial X)$.

For possible applications to Ricci flow with surgery on four-manifolds, it would be desirable to extend our existence result to cover all cones with non-negative scalar curvature. The link of such a cone must be diffeomorphic to a connected sum $(S^3/\Gamma_1)\#\cdots\#(S^3/\Gamma_k)\#(\#^\ell S^2\times S^1)$, and therefore in light of our degree theory it is natural to ask:

Question. Is there any relation between the degrees $\deg_{\exp}(X_1)$, $\deg_{\exp}(X_2)$, and $\deg_{\exp}(X_1\#_\partial X_2)$? What is $\deg_{\exp}(D^3\times S^1)$?

Above, the connected sum involves cutting along half-balls intersecting the boundaries ∂X_i to study expanders asymptotic to cones over $\partial X_1\#\partial X_2$. If these degrees can be shown to be nonzero, this would yield natural candidates for resolving the conical singularities encountered along the Ricci flow in four dimensions.

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Urysohn width and scalar curvature

Yevgeny Liokumovich

(joint work with Davi Maximo; Zhichao Wang; Boris Lishak, Alexander Nabutovsky, Regina Rotman; Otis Chodosh, Chao Li)

I discuss the interplay between results about PSC and macroscopic PSC. These include:

- (with D. Maximo [3]) There is a Morse foliation of PSC closed 3-manifolds by surfaces of controlled area, diameter and genus.
- (with Z. Wang [4]) There is a Morse foliation of PSC complete non-compact 3-manifolds by surfaces of controlled area and diameter.

- (with B. Lishak, A. Nabutovsky, and R. Rotman [2]) A metric space with positive macroscopic PSC has controlled Urysohn width.
- (with O. Chodosh and C. Li [1]) PSC 4-manifolds with $\pi_2(M) = 0$ admits finite cover homeomorphic to S^4 or $\sharp S^3 \times S^1$.

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