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Algebraische Zahlentheorie

Organized by
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Abstract. Algebraic Number Theory is an area of Mathematics that has a legendary history and lies at the interface of Algebra and Number Theory. The last four decades of the last century witnessed rapid developments that led to connections with other areas such as Algebraic Geometry, Representation Theory, Harmonic Analysis, Iwasawa theory, to mention a few. In the last two decades, emergent areas such as p-adic Analysis, p-adic Geometry (p is a prime number) led to additional new facets. More recent developments in Arithmetic Geometry via Perfectoid Spaces and other emerging areas have added newer facets. The lectures in this workshop present current developments in these diverse areas.

Mathematics Subject Classification (2020): 11Dxx, 11Exx, 11Fxx, 11Gxx, 11Mxx, 11Rxx, 11Sxx.

Introduction by the Organizers

The workshop *Algebraische Zahlentheorie*, organised by Guido Kings (Regensburg), Sujatha Ramdorai (Vancouver), Eric Urban (New York) and Otmar Venjakob (Heidelberg) was well attended with over 48 participants with broad geographic representation from all continents. Despite an attempt to include remote participants, only the remote lecture by Xin Wan was actually scheduled and we had zero remote attendance. This workshop was a nice blend of researchers working in different areas of Algebraic Number Theory and the lectures covered various aspects of this subject going from Iwasawa Theory of Selmer groups, Euler systems and arithmetic properties of L-values to the most recent technologies of arithmetic geometry.

The study of motives and their associated Galois representations weaves together various branches of modern algebraic number theory such as Iwasawa theory, special values of L-functions and cohomology theories. As generalisation of Class Field Theory, the Langlands Program with its p-adic versions play a central role and Shimura varieties link the latter with the automorphy of Galois representations. Recently, the work of Clausen and Scholze on 'Condensed Mathematics' has entered the Algebraic Number Theory landscape offering new frameworks for unifying Topology, Algebraic Geometry, Analytic Geometry and Arithmetic Geometry. The lectures in the workshop covered these various topics, and we list them below under broad subtopics.

Galois representations, Euler Systems and GSp_4 The current state of the art in this important topic was the subject of the lectures by David Loeffler, Naomi Sweeting and Fabrizio Andreatta.

- David Loeffler: Euler systems for GSp_4 and applications
- Naomi Sweeting: Tate classes and endoscopy for GSp_4
- Fabrizio Andreatta: Endoscopy for GSp_4 and rational points of elliptic curves

Iwasawa theory and Galois representations

- Xin Wan: A new $+/-$ Iwasawa theory and converse of Gross-Zagier and Kolyvagin theorem
- Giada Grossi: Kolyvagin's conjecture and Iwasawa theory
- Kazim Büyükboduk: On the arithmetic of θ-critical p-adic L-functions
- Samuel Mundy: The nonvanishing of Selmer groups of certain symplectic Galois representations

p-adic Hodge Theory and p-adic Langlands

- Juan Esteban Rodríguez Camargo: The analytic de Rham stack and a Jacquet-Langlands correspondence for locally analytic representations
- Laurent Berger: Super-Hölder functions and vectors
- Rustam Steingart: Analytic cohomology of Lubin-Tate (φ_L, Γ_L)-modules
- Guido Bosco: Rational p-adic Hodge theory for rigid-analytic varieties
- Wieslawa Niziol: Duality for p-adic pro-étale cohomology of analytic varieties

Automorphic forms, Shimura varieties and Cohomology

- Yujie Xu: On the geometry of integral models of Shimura varieties of abelian type
- Matthias Flach: Special values of Zeta functions and Deligne cohomology
- Han-Ung Kufner: Deligne's Conjecture on critical values of L-functions for Hecke characters
- Daniel Kriz: Horizontal p-adic L-functions with applications to L-values
- Romyar Sharifi : Eisenstein cocycles for imaginary quadratic fields

This was the first post-pandemic Algebraische Zahlentheorie Workshop to be held in Oberwolfach and there was a lively atmosphere during the conference. The facilities at Oberwolfach provided for excellent discussions among the participants.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, "US Junior Oberwolfach Fellows".

Workshop: Algebraische Zahlentheorie

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Abstracts

Euler systems for $\mathbf{GSp_4}$ and applications

David Loeffler
(joint work with Sarah Livia Zerbes)

1. Euler systems

If V is a p-adic representation of $\mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q})$, an *Euler system* for V is a collection of co ology classes $z_m \in H^1(\mathbf{Q}(\mu_m), T)$ for varying m, where T is a lattice in V (independent of m), satisfying a norm-compatibility relation involving Euler factors. These are useful because if a non-trivial Euler system exists for V, it implies bounds on Selmer groups attached to V, which play a crucial role in many results on the Bloch–Kato conjecture, the Birch–Swinnerton-Dyer conjecture, and related problems.

2. Euler systems for GSp_4 Galois representations

Let $\mathcal{F}$ be a holomorphic, cuspidal Siegel modular eigenform, of prime-to-p level and ordinary at p. Then there exists a 4-dimensional Galois representation $V_{\mathcal{F}}$ associated to $\mathcal{F}$, whose associated L-function is the spinor L-function $L(\mathcal{F}, s)$. (Here we normalise so that if $\mathcal{F}$ has weight $k_1 \geq k_2 \geq 2$, then the centre of the functional equation is at $s = \frac{k_1+k_2}{2} - 1$.)

Theorem 1 (L.–Skinner–Zerbes)**.** *Suppose $k_2 \geq 3$, and let $j \in \mathbf{Z}$ with $1 \leq j \leq k_2 - 2$. Then $\mathrm{ord}_{s=j} L(\mathcal{F}, s) = 1$, and there exists an Euler system for $V_{\mathcal{F}}^*(1-j)$ which is "morally related" to $L'(\mathcal{F}, j)$.*

Here "morally related" means that the bottom class of this Euler system (denoted $z_{\text{ét}}^{[\mathcal{F},j]}$) is the image in étale cohomology of a special *motivic* cohomology class defined by Francesco Lemma. Lemma has shown that the image of this special class in Deligne–Beilinson cohomology, paired with a de Rham cohomology class $\eta_{\mathcal{F}}$ coming from $\mathcal{F}$, is related to $L'(\mathcal{F}, j)$. Sadly, since the kernel of the map from motivic to étale cohomology is not known to be injective (although this has been conjectured by Bloch and Kato), this does not directly rule out the possibility that the bottom class of the Euler system is zero.

3. Relation to p-adic L-values

The *critical* values of $L(\mathcal{F}, s)$ are $s = j \in \mathbf{Z}$ with $k_2 - 1 \leq j \leq k_1 - 1$ (disjoint from the range where the motivic class is defined).

Theorem 2 (L.–Pilloni–Skinner–Zerbes)**.** *There exists a p-adic L-function $L_p(\mathcal{F})$, which is a bounded rigid-analytic function on $\mathcal{W} = \mathrm{Hom}(\mathbf{Z}_p^\times, \mathbf{C}_p^\times)$, whose values at the characters $x \mapsto x^j \chi(x)$, for j in the critical range and χ a finite-order character, interpolate the L-values $L(\mathcal{F}, \bar{\chi}, j)$.*

Note that the case $k_2 = 2$ is allowed here, unlike the previous theorem. When $k_2 \geq 3$ we have a relation between Euler systems and p-adic L-functions:

Theorem 3 (L.–Zerbes). *If $k_2 \geq 3$, then for every $1 \leq j \leq k_2 - 2$, the value $L_p(\mathcal{F})(j)$ is related to the Euler system for $V^*_{\mathcal{F}}(1-j)$ via a p-adic regulator formula*

$$L_p(\mathcal{F})(j) = \left\langle \eta_{\mathcal{F}}, \log_{\mathrm{BK}}\left(\mathrm{loc}_p\, z_{\text{ét}}^{[\mathcal{F},j]}\right)\right\rangle$$

where $\eta_{\mathcal{F}}$ is a de Rham class associated to $\mathcal{F}$ as above, and $\log_{\mathrm{BK}}$ *is the inverse of the Bloch–Kato exponential map for $V^*_{\mathcal{F}}(1-j)|_{\mathrm{Gal}(\overline{\mathbf{Q}}_p/\mathbf{Q}_p)}$.*

4. Deformation in families

We would like to "analytically continue" the relation between p-adic L-functions and Euler systems given by Theorem 3 from the range $1 \leq j \leq k_2 - 2$ to the critical range $k_2 - 1 \leq j \leq k_1 - 1$ (in which we can relate the p-adic L-value to a complex L-value). Of course, this is meaningless as stated, since one cannot try to analytically continue from a finite set. However, the set

$$\{(k_1, k_2, j) \in \mathbf{Z}^3 : 1 \leq j \leq k_2 - 2 \leq k_1 - 2\}$$

is dense in $\mathcal{W}^3$. So if we can show that all of the objects involved on both sides of Theorem 3 interpolate in p-adic families with (k_1, k_2, j) all varying, we can hope to analytically continue from this set.

For $k_2 \geq 3$, using work of Hida and Tilouine–Urban, one knows that there exists a p-adic family of Siegel modular forms $\mathbb{F}$ over some open $U \ni (k_1, k_2)$ in $\mathcal{W}^2$ whose weight (k_1, k_2) specialisation is $\mathcal{F}$, and an associated family of Galois representations $V_{\mathbb{F}}$. We show the existence of

- a p-adic L-function $L_p(\mathbb{F})$ over $U \times \mathcal{W}$, interpolating the p-adic L-functions of Theorem 2 for all classical specialisations of $\mathbb{F}$;
- a family of Euler system classes $z^{\mathcal{F}}$ interpolating the Euler system classes for all specialisations of $\mathbb{F}$;
- an isomorphism between $\mathbf{D}_{\mathrm{dR}}$ of a subquotient of the Galois representation $V^*_{\mathbb{F}}$, and a space of coherent cohomology classes associated to $\mathbb{F}$, interpolating the p-adic Eichler–Shimura comparison isomorphisms for a dense set of classical specialisations.

The last among these three constructions is the most difficult (despite its apparently tautological appearance); it is an analogue for GSp_4 of Ohta's comparison isomorphism for Hida families for GL_2.

Putting these ingredients together we obtain a version of Theorem 3 which is an identity of functions on $U \times \mathcal{W}^3$. We can now specialise this at (k_1, k_2) to obtain a relation between critical values of $L(\mathcal{F}, j)$ and Euler system classes, hence proving the following theorem:

Theorem 4. *For $\mathcal{F}$ of weight (k_1, k_2) with $k_2 \geq 3$ (and satisfying various other technical conditions), we have the following implication: for any j in the critical range, if $L(\mathcal{F}, j) \neq 0$, then the Bloch–Kato Selmer group of $V_{\mathcal{F}}(j)$ is zero, as predicted by the Bloch–Kato conjecture.*

5. Applications to BSD

Perhaps the most interesting weight is $(2, 2)$, since this is the weight of the eigenforms forms associated to modular abelian surfaces. This is not immediately covered by the previous theorem, since in these non-regular weights, the eigenvariety parametrising p-adic families for GSp_4 may be badly-behaved – it is no longer necessarily étale over weight space. Moreover, there are reasons to believe that the eigenvariety is genuinely not étale at points corresponding to certain abelian surfaces (those induced from elliptic curves over imaginary quadratic fields).

We formulate a criterion we call *deformability* at p, which is a little weaker than étaleness of the eigenvariety, and should also be satisfied in the case of imaginary-quadratic lifts. We can prove the following theorem:

Theorem 5 (L.–Zerbes)**.** *If A is an abelian surface with $\mathrm{End}_{\mathbf{Q}}(A) = \mathbf{Z}$ which is deformable at p (and various other conditions), and $L(A, 1) \neq 0$, then $A(\mathbf{Q})$ and $Ш(A/\mathbf{Q})[p^\infty]$ are finite.*

In ongoing work of my PhD student James Rawson, a computationally practical test for deformability is being developed.

A new $+/-$ Iwasawa theory and converse of Gross-Zagier and Kolyvagin theorem

Xin Wan

We develop a new kind of anticyclotomic local $\pm$-Iwasawa theory at p for Hecke characters of quadratic imaginary fields which is valid for all ramification types of p (split, inert and ramified). As an application we deduce the converse of Gross-Zagier-Kolyvagin them for these CM forms, which states that Selmer rank one implies analytic rank one. To carry out the Iwasawa theory argument we employ a recent construction of a new type of p-adic L-function by Andreatta-Iovita, and a “virtual Heenger family” made via a limiting procedure from a Heegner family along Coleman-Mazur eigencurve constructed by Jetchev-Loeffler-Zerbes.

Suppose ρ is the motive associated to an elliptic curve E over $\mathbb{Q}$, the converse of the Gross-Zagier and Kolyvagin theorem is an important special case predicted by the Bloch-Kato conjecture and states that if the rank of the Selmer group is 1, then the vanishing order of its L-function is exactly 1. In the case when E has no complex multiplication, the result is proved by Skinner under the assumption of finiteness of the p-part of Shafarevich-Tate group, and also by Zhang when E is ordinary at p. These assumptions are removed later by the work of the author and Castella-Wan by using anticyclotomic Iwasawa theory. This converse theorem has important arithmetic implications including average analytic rank of elliptic curves and the result of Bhargava-Skinner-Zhang that at least 66 percent of elliptic curves satisfy the rank part of BSD conjecture.

In the CM case, Rubin proved this converse theorem in the p-ordinary case (i.e. p is split in the quadratic imaginary field $\mathcal{K}$) under the assumption that the p-part of the Shafarevich-Tate is finite. Recently Burungale-Tian removed the assumption of the Shafarevich-Tate group. In the case when E has good supersingular reduction (here p is inert in $\mathcal{K}$), Rubin set up a general local Iwasawa theory at p. More precisely Rubin defined the $\pm$-subspaces H^1_+ (H^1_-) of the rank two (over the anticyclotomic Iwasawa algebra Λ^-) module $H^1_{\mathrm{Iw}}(\mathcal{K}^-_{\infty,p},\psi)$ to be elements specializing to elements in H^1_f (the finite part) at arithmetic points ϕ corresponding to characters of Γ^- of odd (even) powers of p, respectively. Rubin also made a fundamental conjecture stating that

$$H^1_{\mathrm{Iw}}(\mathcal{K}^-_\infty,\psi)=H^1_+\oplus H^1_-.$$

On the other hand Rubin constructed $\pm$-Heegner family $\kappa_\pm$ from cycles over $\mathcal{K}^-_n$ of level prime to p which satisfy the norm relation

$$\mathrm{tr}(\kappa_{p^{n+2}})=-\kappa_{p^n},$$

modified by the $\pm$ p-adic logarithm function.

This conjecture was proved recently by Burungale-Kobayashi-Ota, and from it they also deduced an anticyclotomic Iwasawa main conjecture involving the $\pm$ p-adic L-functions, and the converse of Gross-Zagier and Kolyvagin theorem in this case from Rubin's $\pm$ Heegner point.

We develop a new kind of $\pm$-local Iwasawa theory which applies to all ramification types of p in $\mathcal{K}$ (split, inert or ramified) and allows the CM character to have ramification at p, and apply it to show the converse of Gross-Zagier and Kolyvagin theorem for CM characters of $\mathcal{K}$.

Our main theorem is the following: Let ψ be as above. Suppose $p>3$ and the p-part of the conductor of ψ is p^n for $n\geq 2$. Suppose the rank of the Selmer group for ψ is 1, then the vanishing order of $L(\psi,s)$ at $s=1$ is also 1.

Our method is different from literature: we develop a new kind of $\pm$-local Iwasawa theory, which is valid for all ramification types for p (split, inert and ramified cases). Our theory is analogous to Rubin's $\pm$-theory in format but quite different in nature: we divide the arithmetic points in $\mathrm{Spec}\Lambda^-$ into two parts $\mathcal{X}_1$ and $\mathcal{X}_2$ depending on the Archimedean types instead of the conductors at p. We define the $+$-part ($-$-part) submodules of rank 1 of the local Iwasawa theoretic Galois cohomology at p to be the subspace which specializes in H^1_f (the kernel of the dual exponential map $\exp^*$) at arithmetic points in $\mathcal{X}_1$ ($\mathcal{X}_2$) respectively. The existence and ranks of these submodules are studied by looking at various modules of elliptic units.

Our proof goes along the main line of Castella-Wan.

- For the Rankin-Selberg p-adic L-function, In the case when p is non-split at p, it has been constructed by Andreatta-Iovita as a locally analytic function, which we denote as $\mathcal{L}^{\mathrm{AI}}_{f,\chi}$ (f is the CM form, and χ is some CM character). In our case we need a generalization by Yangyu Fan to Shimura curves, using the techniques developed in his thesis. It is not identically zero by the result of Greenberg again. In this case it splits up into the product of two CM p-adic L-functions.
- To prove the converse theorem it is important to have a family of Heegner points. We first choose a pair (f,χ) where f is a $\mathcal{K}$-CM form of finite

slope at p, and χ is a $\mathcal{K}$-CM character such that $L(f,\chi,s)$ splits up into $L(\psi,s)\cdot L(\psi',s)$ for some ψ'. In other words we move the ramification of ψ to the χ-part. Then we take a Coleman family $\mathcal{F}$ containing f, and use the construction of Jetchev-Loeffler-Zerbes of the 2-variable Heegner family $\kappa_{\mathcal{F},\chi}$ for $\mathcal{F}$.

- One can show that the restriction $\kappa_{f,\chi}$ of $\kappa_{\mathcal{F},\chi}$ (we call this $\kappa_{f,\chi}$ a virtual Heegner family as it is constructed by a p-adic limiting procedure instead of actually Heegner points) has image in $H^1_+(\mathcal{K}_p,\boldsymbol{\psi})\subset H^1(\mathcal{K},\boldsymbol{\psi})$. Moreover there is a basis v_+ of $H^1_+(\mathcal{K}_p,\boldsymbol{\psi})$ such that

(1) $$\kappa_{f,\chi}=\mathcal{L}^{\mathrm{AI}}_{f,\chi}\cdot v_+$$

The main tool proving the above equation is the p-adic Gross-Zagier formula (i.e. analogue of the Bertolini-Darmon-Prasanna formula in this non-split setting) proved by Andreatta-Iovita and generalized by Yangyu Fan. In sum, this virtual Heegner family can play the role of Heegner family in the ordinary case for the Iwasawa theory argument.

- There is also a Rankin-Selberg Iwasawa main conjecture formulated using Andreatta-Iovita's p-adic L-function, which in turn can be deduced from a Rubin type Iwasawa main conjecture (in the generality proved by Johnson-Leung-Kings) for elliptic units. As in the non-CM case, we can use it to prove a Perrin-Riou's main conjecture (note that the ψ' has root number $+1$)

$$\mathrm{char}_{\Lambda^{-,\prime}}\Big(\frac{H^1_+(\mathcal{K},\boldsymbol{\psi})}{\Lambda^{-,\prime}\cdot\kappa_{f,\chi}}\Big)^2=\mathrm{char}_{\Lambda^{-,\prime}}(X^+_{\psi,\mathrm{tor}})\cdot\mathrm{char}_{\Lambda^{-,\prime}}(X^+_{\psi'}).$$

The converse theorem can be deduced from this as before.

Tate classes and endoscopy for GSp_4

Naomi Sweeting

Let g be a classical cuspidal eigenform of weight two for GL_2; then g has a Galois representation, constructed by Deligne as a quotient

$$H^1(X_1(N)_{\overline{\mathbb{Q}}},\overline{\mathbb{Q}}_\ell)\twoheadrightarrow\rho_g$$

for some sufficiently large level N. However, ρ_g also appears in the étale cohomology of many other Shimura varieties; this talk dealt in particular with the Shimura variety $S_K(\mathrm{GSp}_4)$, which is a three-dimensional moduli space of principally polarized abelian surfaces with some level structure determined by an open compact subgroup $K\subset\mathrm{GSp}_4(\mathbb{A}_f)$. Consider the decomposition of the étale cohomology

$$H^3_{\mathrm{et}}(S_K(\mathrm{GSp}_4)_{\overline{\mathbb{Q}}},\overline{\mathbb{Q}}_\ell)=\bigoplus\Pi_f^K\otimes\rho_{\Pi_f},$$

with Π_f the finite part of an automorphic representation of GSp_4 and ρ_{Π_f} a Galois representation. When it is nonzero, ρ_{Π_f} is typically four-dimensional and irreducible [1]. But for certain Π_f corresponding to endoscopic Yoshida lifts, ρ_{Π_f} will be a Tate twist of the two-dimensional representation ρ_g [3]; by Poincaré duality and the Kunneth formula, one deduces the existence of Galois-invariant étale cohomology classes in middle degree four on the product $S_K(\mathrm{GSp}_4) \times X_0(N)$.

The main question of this talk is when these classes arise from algebraic cycles, as predicted by the Tate conjecture. It turns out that a natural special cycle class on $S_K(\mathrm{GSp}_4)$ accounts for *some*, but not all, of the Galois-invariant classes: it only sees the ones corresponding to *globally generic* automorphic representations of GSp_4 [2]. The automorphic representations of GSp_4 are organized by the Langlands program into L-packets, each of which has a unique generic member, so the failure of the special cycle to generate all of the Tate classes of interest is closely related to the existence of nontrivial packet structure on GSp_4. In fact, an analogous result holds over totally real fields, and for Galois-invariant classes in étale cohomology with coefficients in certain automorphic local systems.

In the non-generic case, it is not known whether the Tate classes arise from algebraic cycles, because it is quite difficult to construct (or work with) algebraic cycles that are not special. However, one can at least show that all the Galois-invariant classes arise from Hodge classes under the Betti-étale comparison isomorphism [2]. These Hodge classes are constructing using non-tempered theta lifts on the group GSp_6. The strategy is inspired by the groundbreaking work of Ichino and Prasanna [4], which showed that Tate classes reflecting the Jacquet-Langlands transfer between inner forms of GL_2 also arise from Hodge classes. In the second half of the talk, I gave a schematic overview of the theory of theta lifting, which I then used to sketch a proof of the main results.

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Endoscopy for GSp_4 and rational points of elliptic curves

Fabrizio Andreatta

(joint work with Massimo Bertolini, Marco Seveso, Rodolfo Venerucci)

Let f be a cuspform associated of a semistable elliptic curve A over $\mathbb{Q}$ of conductor N. Let g and h be weight 1 forms associated to Artin representations ρ_g, resp. ρ_h s.t. $\rho := \rho_g \otimes \rho_h$ is self dual. Assume that the the sign of the functional equation of $L(A \otimes \rho, s)$ at the central value $s = 1$ is -1. The BSD conjecture predicts that $\left(A(\overline{\mathbb{Q}} \otimes \rho\right)^{\mathrm{Gal}(\overline{\mathbb{Q}}/\mathrm{Q})})$ is non zero. This implies that the Bloch-Kato Selmer group $\mathrm{Sel}_p(f, g, h)$ is non trivial.

In this work we use p-adic L-functions to provide evidence of the non-trivilaity of $\mathrm{Sel}_p(f, g, h)$. It should be noticed that the relevant p-adic L-function seems to play the role of the derivative of the classical L-function $L(A\otimes\rho, s)$, as envisioned in another context in [BDP].

Assume p is a prime of good ordinary reduction for A. Let $\mathcal{F}$, $\mathcal{G}$ and $\mathcal{H}$ be three Hida families specializing at $(2,1,1)$ as $\mathcal{F}_2 = f$, $\mathcal{G}_1 = g$ and $\mathcal{H}_1 = h$. Greenberg-Seveso [GS] and Hsieh [Hs] have constrcutied a p-adic L-function $L_p(\mathcal{F},\mathcal{G},\mathcal{H})$ interpolating

$$(**)L(\mathcal{F}_k, \mathcal{G}_\ell, \mathcal{H}_m).$$

Here, $L(\mathcal{F}_k, \mathcal{G}_\ell, \mathcal{H}_m)$ is the the central critical value of the Garrett L-function for unbalanced triples $(k,\ell,m)\in\mathbb{N}^3_{\geq 2}$ and $(**)$ is a specific constant for which we refer to loc. cit.

Thm. *If $L_p(\mathcal{F},\mathcal{G},\mathcal{H})\neq 0$, then $\mathrm{Sel}_p(\mathcal{F}_k, \mathcal{G}_\ell, \mathcal{H}_m)\neq 0$ for every $(k,\ell,m)\in\mathbb{N}^3_{\geq 1}$ with $k\geq \ell+m$. In, particular, this holds true for $(2,1,1)$.*

More precisely, we have the following explicit reciprocity law*: for $(k,\ell,m)\in\mathbb{N}^3_{\geq 1}$ with $k\geq\ell+m$, $k\geq 4$ even and ℓ, $m\geq 2$ there exists a class $\kappa(k,\ell,m)\in\mathrm{Sel}_p(\mathcal{F}_k,\mathcal{G}_\ell,\mathcal{H}_m)$ such that*

$$\log_{p,\mathrm{BK}}\big(\mathrm{loc}_p(\kappa(k,\ell,m))\big)(\omega)\sim L_p(\mathcal{F}_k,\mathcal{G}_\ell,\mathcal{H}_m),$$

where $\log_{p,\mathrm{BK}}$ is the Bloch-Kato logarithm and $\omega\in\mathrm{Fil}^0 D_{\mathrm{dR}}(\mathcal{F}_k,\mathcal{G}_\ell,\mathcal{H}_m)$.

The key input is the phenomenon of endoscopy for GSp_4 studied in [We]. Namely, given X the Siegel threefold of full level N and Iwahori level at p and given a positive even integer $k\geq 4$ there exists an automorphic étale sheaf $\mathcal{L}_k$ of parallel weight $\frac{k}{2}-2$ over X and a Siegel modular eigenform F_k of parallel weight $\frac{k}{2}+1$, ordinary for the Siegel U_p-operator, such that the F_k-isotypic component $\mathrm{H}^3_{et}(X_{\overline{\mathbb{Q}}},\mathcal{L}_k)[F_k]$ is isomorphic to the 2-dimensional Galois representation defined by the elliptic cuspoform $\mathcal{F}_k$. Consider integers ℓ, $m\geq 2$ such that $k\geq\ell+m$. The class $\kappa(k,\ell,m)$ is then costructed as follows. Let Y be the modular curve of level $\Gamma(N)\cap\Gamma_0(p)$. Consider the embedding $Y^2\to X$ sending a pair of ellptic curves (E,E'), with their level structures, to the principally polarized abelian surface $E\times E'$, with the product level structures. Let $\iota\colon Y^2\to X\times Y^2$ be the product of this embedding and the identity map on Y^2. Let $\underline{x}:=(k,\ell,m)$ and $\mathcal{L}_{\underline{x}}$ be the p-adic étale sheaf on $X\times Y^2$ given by the product of the pull back of $\mathcal{L}_k$ and the $\ell-2$, resp. the $m-2$ symmetric power of the p-adic Tate module of the universal elliptic curve on Y. One can prove that $\mathrm{H}^0_{\mathrm{et}}\big(Y^2,\iota^*(\mathcal{L}_{\underline{x}})(-3)\big)$ is 1-dimensional with canonical generator $\mathrm{Inv}_{\underline{x}}$. The class $\kappa(k,\ell,m)$ is defined by pushing forward $\mathrm{Inv}_{\underline{x}}$ via

$$\iota_*\colon \mathrm{H}^0_{\mathrm{et}}\big(Y^2,\iota^*(\mathcal{L}_{\underline{x}})\big)\to\mathrm{H}^6_{\mathrm{et}}\big(X\times Y^2,\mathcal{L}_{\underline{x}}\big)$$

and then projecting onto the $\mathcal{F}_k\otimes\mathcal{G}_\ell\otimes\mathcal{H}_m$ isotypic component of $\mathrm{H}^1_{\mathrm{et}}\big(\mathbb{Q},\mathrm{H}^5_{\mathrm{et}}\big((X\times Y^2)_{\overline{\mathbb{Q}}},\mathcal{L}_{\underline{x}}\big)\big)$.

The proof of the explicit reciprocity law uses the existence of models of toroidal compactifications of Y^2 and $X\times Y^2$ having semistable reduction at p and such that ι extends, based on work of de Jong [dJ], Lan [La] and Stroh [St], and a newly developed syntomic formalism with coefficients that allows to reduce the claim to an explicit computation of primitives over the ordinary locus of X.

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On the modularity of elliptic curves over imaginary quadratic fields

ANA CARAIANI

(joint work with James Newton)

In this talk, I reported on recent work [7] establishing the modularity of all elliptic curves over many imaginary quadratic fields, including $\mathbb{Q}(\sqrt{-d})$ with $d = 1, 2, 3, 5$.

Let F be a number field. We say that an elliptic curve E/F is *modular* if either E has complex multiplication or if there exists a cuspidal automorphic representation π of $\mathrm{GL}_2(\mathbb{A}_F)$ of parallel weight 2 whose associated L-function is the same as the L-function of E.

The modularity of all elliptic curves $E/\mathbb{Q}$ was pioneered by Wiles and Taylor–Wiles [10, 11] and completed in work of Breuil–Conrad–Diamond–Taylor [3]. The modularity of elliptic curves over real quadratic fields was established by Freitas–Le Hung–Siksek [8] and there have since been further results over more general totally real fields. The modularity of elliptic curves over imaginary CM fields has historically been more difficult to establish. This is because the systems of Hecke eigenvalues that conjecturally match such elliptic curves contribute to the cohomology of locally symmetric spaces such as Bianchi 3-manifolds, which are not directly related to Shimura varieties.

In [5], Calegari–Geraghty outlined a strategy for proving modularity lifting theorems beyond the setting of Shimura varieties, where the classical Taylor–Wiles method applies. Inspired by Calegari–Geraghty, the potential modularity of elliptic curves over imaginary CM fields was established only recently, in [2] and [4] (independently of each other). Since then, Allen–Khare–Thorne proved many instances of actual modularity in [1]. More precisely, they established the modularity of a positive proportion of elliptic curves over imaginary CM fields together with strong residual modularity results modulo 3 and modulo 5.

In [7], we combine the residual modularity results of [1] with a new modularity lifting theorem and with a careful study of exceptional points on several modular curves of small level to obtain the following result. Let $X_0(15)$ denote the modular curve of level $\Gamma_0(15)$. It is an elliptic curve of rank 0 over $\mathbb{Q}$ (the curve with Cremona label 15A1).

Theorem 1. *Let F be an imaginary quadratic field such that the Mordell–Weil group $X_0(15)(F)$ is finite. Then every elliptic curve E/F is modular.*

A conjecture of Goldfeld on ranks of quadratic twists of elliptic curves predicts that Theorem 1 applies to slightly more than half of imaginary quadratic fields. For more general imaginary CM fields, we are still able to improve on the results of [1] using our new modularity lifting theorem.

Theorem 2. *Let F be an imaginary CM field that is Galois over $\mathbb{Q}$ and such that $\zeta_5 \notin F$. Then 100% of Weierstrass equations defined over F, ordered by their height, define a modular elliptic curve.*

The new modularity lifting theorem used in the proof of Theorem 2 applies in the Barsotti–Tate case. It is proved using a version of the strategy introduced by Kisin in the case of GL_2 over totally real fields, as long as certain key ingredients are in place. The first key ingredient is a *local-global compatibility result* for the Galois representation attached to torsion in the cohomology of the locally symmetric spaces for GL_2/F. We discuss a more general version of this result below. The second key ingredient has to do with the geometry of Barsotti–Tate local deformation rings, whose characteristic 0 points parametrize two-dimensional crystalline Galois representations with parallel Hodge–Tate weights $\{0,1\}$. In general, these deformation rings have two irreducible components, an ordinary and a non-ordinary one. In order to keep track of these two components modulo p, we use the fact that the Barsotti–Tate local deformation rings have generically reduced special fibre. This fact was established in [6] using a version of the Emerton–Gee stack.

We now discuss the local-global compatibility result in its general form. Let $n \geq 2$ be an integer, $K \subset \mathrm{GL}_n(\mathbb{A}_{F,f})$ be a neat compact open subgroup and X_K be the corresponding locally symmetric space for GL_n/F. In [9], Scholze constructed Galois representations $\rho_{\mathfrak{m}}$ attached to systems of Hecke eigenvalues $\mathfrak{m}$ occurring in $H^*(X_K, \mathbb{Z}_p)$. For applications to modularity, it is extremely important to understand the properties of the Galois representations $\rho_{\mathfrak{m}}$. One needs to know whether $\rho_{\mathfrak{m}}$ satisfies some form of local-global compatibility: if v is a prime of F and $G_{F_v} := \mathrm{Gal}(\overline{F}_v/F_v)$, how does the level K_v at which $\mathfrak{m}$ occurs determine the ramification of $\rho_{\mathfrak{m}}|_{G_{F_v}}$? The case when $v \mid p$ is particularly subtle because it is not (a priori) clear how to formulate the correct integral p-adic Hodge theory conditions and because the $\rho_{\mathfrak{m}}$ are constructed via a p-adic interpolation argument that loses track of the level K_v for $v \mid p$ (and also loses track of the weight).

In [2], the first such local-global compatibility results at primes $v \mid p$ were established in two restricted families of cases, described by natural integral conditions: the ordinary case and certain Fontaine–Laffaille cases. Beyond these special cases, Gee–Newton formulated a general conjecture using the potentially semi-stable local deformation rings constructed by Kisin. In [7], we prove their conjecture in the crystalline case, under technical assumptions, but allowing arbitrary n and allowing p to be small and highly ramified in F. We also allow coefficients in general local systems on X_K coming from highest weights for GL_n/F. For applications to Theorems 1 and 2, it is crucial to allow $p = 3$ and $p = 5$, as well as to base change to an extension F'/F that is highly ramified at p. We expect our local-global compatibility result to have many more applications to modularity lifting theorems over CM fields.

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The analytic de Rham stack and a Jacquet-Langlands correspondence for locally analytic representations

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The theory of D-modules is an essential piece in the arsenal of an algebraic geometer, e.g. they are main players in the Hilbert-Riemann correspondence or in Beilinson-Bernstein localization theorem. It turns out that the theory of D-modules on a smooth variety X over $\mathbb{C}$ can be recover as the theory of quasi-coherent sheaves on a suitable stack X_{dR}, introduced by Simpson in [19], that we call nowadays the *de Rham stack*. It is natural to ask whether there are rigid analytic incarnations of these stacks, encoding the theory of $\widehat{\mathcal{D}}$-modules of Ardakov-Wadsley [1]. In the following we explain an affirmative answer of this question and sketch two applications to p-adic automorphic forms on Shimura varieties.

1. The analytic de Rham stack

The recent developments of condensed mathematics and analytic geometry of Clausen and Scholze (cf. [5–7]) have unified classical analytic and algebraic geometry. For instance, both algebraic and analytic spaces belong to the same geometric categories, all spaces appearing in analytic geometry have natural categories of quasi-coherent sheaves, theorems like GAGA, finiteness of coherent cohomology or Serre duality have the same categorical proofs, etc. The theory becomes even more powerful when it is combined with the abstract six-functor formalisms appearing in the work of Mann [9, 10]. We make use of these technologies to define *analytic de Rham stacks* for rigid spaces, provide a six-functor formalism for solid $\widehat{\mathcal{D}}$-modules, generalizing previous work of Bode [3], and show that this theory of D-modules is well behaved in a cohomological sense.

To give an intuition in the stack, we briefly sketch a construction for a separated smooth rigid variety X over a complete extension K of $\mathbb{Q}_p$. Let $\Delta : X \to X \times X$ be the diagonal map, as X is separated $Z = \Delta(X)$ is a Zariski closed subspace of $X \times X$. Let $\Delta X^{\dagger}$ denote the dagger space given by the *overconvergent diagonal*, i.e. the analytic space with underlying topological space $|X|$ but whose structural sheaf are the functions in $X \times X$ that overconverge the diagonal $\Delta(X)$. Then, the analytic de Rham stack of X over K is constructed as the quotient

$$X_{dR} = \mathrm{coeq}(\Delta X^{\dagger} \rightrightarrows X).$$

The intuition behind this presentation is that the analytic de Rham stack identifies points of X that are *overconvergently close*.

The analytic de Rham stack is closely related to the theory of solid locally analytic representations as developed in [12, 13]. For example, let G be a p-adic Lie group and let G^{la} be the analytic space whose underlying space is G and whose sheaf of functions are the locally analytic functions of G. Similarly, let G^{sm} be the analytic space attached to the smooth functions of G, and let $\exp(\mathrm{Lie}G) \subset G^{la}$ be the stalk at the identity. We have a short exact sequence of analytic groups

$$1 \to \exp(\mathrm{Lie}G) \to G^{la} \to G^{sm} \to 1$$

and in fact $G^{sm} = G^{la}_{dR}$. A more interesting application to Shimura varieties will be explained in the next paragraphs.

2. Overconvergent BGG maps

In a recent work [11] Pan has initiated a study of p-adic modular forms via locally analytic representation theory, geometric Sen theory and the geometry of modular curves via the Hodge Tate period map [4, 8, 16]. Part of his work has been extended by the author in [14, 15], we will explain how the theory of analytic stacks allows a more conceptual understanding of these results.

Let $(\mathbf{G}, X)$ be a global Shimura datum, fix K^p a prime to p level and let $K_p \subset \mathbf{G}(\mathbb{Q}_p)$ be a compact open subgroup. Let E be the reflex field of the Shimura datum and fix an embedding $\iota : E \hookrightarrow C := \mathbb{C}_p$. Let Sh_{K_p} be the Shimura variety of level K^pK_p seen as an adic space over C, for simplicity let us assume that the Shimura varities are compact and of Hodge type. Let $\mu : \mathbb{G}_m \to \mathbf{G}_C$ be a fixed Hodge cocharacter, let $\mathbf{P}_\mu \subset \mathbf{G}$ be the parabolic subgroup parametrizing increasing μ-filtrations and let $Fl = \mathbf{G}/\mathbf{P}_\mu$ be the flag variety.

The limit $Sh_\infty = \varprojlim_{K_p} Sh_{K_p}$ is naturally a perfectoid space [16], and there is a $\mathbf{G}(\mathbb{Q}_p)$-equivariant Hodge-Tate period map $\pi_{HT} : Sh_\infty \to Fl$ parametrizing the increasing Hodge-Tate filtration of automorphic local systems over Sh. Let $\widehat{\mathcal{O}}$ be the structural sheaf of Sh_∞. The space Sh_∞ has a natural action of $\mathbf{G}(\mathbb{Q}_p)$, one can then consider the analytic space Sh_∞^{la} with same underlying space as Sh_∞, but whose structural sheaf is the subsheaf $\mathcal{O}^{la} \subset \widehat{\mathcal{O}}$ of locally analytic sections for the action of $\mathbf{G}(\mathbb{Q}_p)$.

Hence, the π_{HT}-map can be restricted to a $\mathbf{G}(\mathbb{Q}_p)$-equivariant map of analytic spaces

$$\pi_{HT}^{la} : Sh_\infty^{la} \to Fl$$

that we can differentiate. This allows a clean passage from $\widehat{\mathcal{D}}$-modules over Fl to p-adic automorphic forms. For example, a consequence of geometric Sen theory is that

$$Sh_{\infty,dR}^{la} = \varprojlim_{K_p} Sh_{K_p,dR}$$

where the limit is taken in the category of analytic stacks, in particular it does not involve a completion process. Moreover, $Sh_{\infty,dR}^{la} \to Sh_{K_p,dR}$ is a K_p^{sm}-torsor, meaning that the data of a K_p-equivariant $\widehat{\mathcal{D}}$-module over Sh_∞^{la} is the same as the data of a $\widehat{\mathcal{D}}$-module over Sh_{K_p}. Therefore, by taking de Rham stacks for π_∞^{la}, and taking quotients by K_p^{sm}, one has a map of analytic stacks

$$\pi_{HT,K_p}^{sm} : Sh_{K_p,dR} \to Fl_{dR}/K_p^{sm}.$$

After some modifications one can also construct an analogue of the previous arrow for twisted $\widehat{\mathcal{D}}$-modules. Moreover, the theory of overconvergent automorphic forms of higher Coleman theory [2] can be encoded as pullbacks of suitable $\widehat{\mathcal{D}}$-modules via π_{HT,K_p}^{sm}. For example, the constructions of the overconvergent Eichler-Shimura or BGG maps for completed cohomology of [15] follow by taking the analogue filtration for the localization of Verma modules over the flag variety and then taking pullbacks along π_{HT,K_p}^{sm}.

3. The local Jacquet-Langlands correspondence for locally analytic representations (joint with Gabriel Dospinescu)

We end with an application for local Shimura varieties and a Jacquet-Langlands correspondence for $\widehat{\mathcal{D}}$-modules. We borrow notation from the previous paragraph; we let $(\mathbf{G}, b, \mu)$ be a local Shimura datum as in [18] *with b basic*, for $K_p \subset \mathbf{G}(\mathbb{Q}_p)$ we let Sh_{K_p} be the finite level local Shimura varities over $C = \mathbb{C}_p$, and let $Sh_\infty = \varprojlim_{K_p} Sh_{K_p}$ be the infinite level Shimura variet. Let $\pi_{HT} : Sh_\infty \to Fl$ be the Hodge-Tate period map. We denote by $\overline{\mathbf{P}}_\mu \subset \mathbf{G}$ the parabolic subgroup parametrizing decreasing μ-filtrations, let $\overline{Fl} = \mathbf{G}/\overline{\mathbf{P}}_\mu$ be the opposite flag variety and let $\pi_{GM} : Sh_{K_p} \to \overline{Fl}$ be the Grothendieck-Messing period map.

Let is write $G = \mathbf{G}(\mathbb{Q}_p)$, and denote by $H = J_b(\mathbf{G})$ the inner form of G associated to b. Let Fl^{adm} and $\overline{Fl}^{adm}$ denote the admissible locus of the flag varieties. The duality of local Shimura varieties implies that $\pi_{HT} : Sh_\infty \to Fl^{adm}$ is a G-equivariant H-torsor while $\pi_{GM} : Sh_\infty \to \overline{Fl}^{adm}$ is a H-equivariant G-torsor. Therefore, one could ask about the comparison between the sheaves $\mathcal{O}^{G-la}$ and $\mathcal{O}^{H-la}$ of locally analytic sections of the structural sheaf $\widehat{\mathcal{O}}$ of Sh_∞. It turns out that both sheaves are the same, and that the equality of analytic vectors follow for more general sheaves of periods attached to affinoid subspaces of the Fargues-Fontaine curve.

A first consequence of this relation is the compatibility of the functor of Jacquet-Langlands of Scholze [17] with the passage to locally analytic vectors: let F be a finite extension of $\mathbb{Q}_p$, $\mathbf{G} = \mathrm{GL}_n(F)$ and D is a division algebra of invariant $1/n$. Then $Fl^{adm} = \Omega$ is the Drinfeld space and $\overline{Fl}^{adm} = \overline{Fl}$ is the whole flag variety. Given a representation π of G one constructs a proétale sheaf $\mathcal{F}_\pi$ by descent along $\pi_{GM} : Sh_\infty \to Fl$, and one defines the Jacquet-Langlands functor

$$JL(\pi) := R\Gamma_{proet}(Fl, \mathcal{F}_\pi).$$

The compatibility with the passage to locally analytic vectors implies that if π is a Banach representation of G, one has that

$$JL(\pi)^{H-la} = JL(\pi^{G-la}).$$

Finally, a last consequence of geometric Sen theory and the equality $\mathcal{O}^{G-la} = \mathcal{O}^{H-la}$ is an equivalence of $\widehat{\mathcal{D}}$-modules, indeed, we have an isomorphism of equivariant de Rham stacks

$$\overline{Fl}^{adm}_{dR}/H^{sm} = Fl^{adm}_{dR}/G^{sm}.$$

As immediate implication one has an isomorphism of smooth $G \times H$-representations given by the de Rham cohomology of local Shimura varieties

$$\varinjlim_{G_n \subset G} DR(Sh_{G_n}) = \varinjlim_{H_n \subset H} DR(Sh_{H_n}).$$

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Horizontal p-adic L-functions with applications to L-values

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(joint work with Asbjørn Nordentoft)

Fix a modular form f of weight $k \geq 2$. The study of non-vanishing of twisted L-values $L(f, \chi, k/2)$ for χ varying in various families of Dirichlet characters (i.e. finite-order) has long been the interest of both algebraic and analytic number theorists. One version of this question is the following.

Question. *Fix $d \in \mathbb{Z}_{\geq 1}$ and let χ vary through all Dirichlet characters of order d, ordered by conductor. Let*

$$\mathcal{K}_d(X) = \{\chi \text{ primitive of conductor } D, \text{ order } d \ : D \leq X\}.$$

What is the asymptotic behavior of

$$\#\{\chi \in \mathcal{K}_d(X) : L(f, \chi, k/2) \neq 0\}$$

as $X \to \infty$?

When restricting to quadratic χ (i.e. of order $d = 2$) and f the weight $k = 2$ newform associated to an elliptic curve E over $\mathbb{Q}$ (i.e. $L(f, s) = L(E, s)$), this question is addressed by Goldfeld's conjecture [3]: when ordering quadratic χ by conductor, the order of vanishing $\mathrm{ord}_{s=1} L(f, \chi, s)$ is 0 for 50% of such χ and 1 for 50% of such χ. Goldfeld's conjecture has seen considerable progress in recent years, see for example [5], [7], [8], [1] and [6]. In [6], one of the authors verified Goldfeld's conjecture for the congruent number family (as well as other families of complex multiplication elliptic curves) by establishing a p-converse theorem and invoking known results on distributions of 2-Selmer ranks in such families.

For $d > 2$ there are no root number obstructions that force vanishing on $L(f, \chi, k/2)$ and so it is conjectured that $L(f, \chi, k/2) \neq 0$ for 100% of χ of order d. In fact, if d is a prime then the random matrix heuristics of [2] support this conjecture and even make the stronger claim that $L(f, \chi, k/2) \neq 0$ for all but finitely many χ when $d > 5$. Unconditional results toward these conjectures have proven difficult to establish and, unlike in the $d = 2$ case, currently seem out of reach of analytic techniques. The main results of this talk are to give the first unconditional results toward these heuristics for general d. The proofs of these results rely on a new construction of *horizontal p-adic L-functions*, which are power series $F \in R[[T]]$ over p-adically complete rings R, associated to elements of group rings $\nu_n \in R[\Pi_n]$, where $\Pi_n = G_1 \times \ldots \times G_n$ for finite abelian groups G_n, compatible with respect to the natural projections $R[\Pi_n] \to R[\Pi_{n-1}]$. These latter compatibilities are often called *horizontal norm relations* in the context of Euler systems or Mazur-Tate θ-elements, whence the name horizontal p-adic L-function. The construction of horizontal p-adic L-functions relies on a new kind of patching for the group rings $R[\Pi_n]$ when the latter satisfies Taylor-Wiles conditions; these p-adic L-functions satisfy a new kind of interpolation property, wherein special values are related to *twisted moments* of $L(f, k/s)$, i.e. linear combinations of the $L(f, \chi, k/2)$ for χ varying over all characters of Π_n, with coefficients given by certain cyclotomic units. For other examples of similar twisted moments, see [9] and [10]. Moreover, in certain cases the derivative $\frac{d}{dT}F$ are related to the Kolyvagin derivatives of $\nu = \varprojlim_n \nu_n \in \varprojlim_n R[\Pi_n]$.

Given a Hecke newform f, we say that an integer $d \geq 1$ is *f-good* if for all prime divisors $p|d$, the residual $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$-representation $\bar{\rho}_f$ at a prime of the Hecke field λ above p is irreducible. When f is attached to an elliptic curve $E/\mathbb{Q}$ (i.e. $L(f, s) = L(E, s)$), we say d is E-good if it is f-good. We prove the following general non-vanishing result:

Theorem. *Let f be a Hecke newform of weight k and let $d \in \mathbb{Z}_{\geq 1}$ be f-good. Then for any $A > 0$ we have*

$$\#\{\chi \in \mathcal{K}_{2d}(X) : L(f, \chi, k/2) \neq 0\} \gg_{A,f,d} \frac{(\log\log X)^A X}{\log X}.$$

Furthermore, if $L(f, k/2) \neq 0$, then for all f-good $d \geq 1$ we have

$$\#\{\chi \in \mathcal{K}_{d}(X) : L(f, \chi, k/2) \neq 0\} \gg_{A,f,d} \frac{(\log\log X)^A X}{\log X}.$$

Furthermore, our methods allow us to obtain non-vanishing results modulo powers of p and at the same time control the number of prime factors of the conductor:

Theorem. *Let $E/\mathbb{Q}$ be an elliptic curve with $L(E, 1) \neq 0$. Let p be an be a E-good prime and let $n = \mathrm{ord}_p(\mathrm{Num}(L(E, 1)/\Omega_E^+))$ be the p-adic valuation of the numerator of the algebraic part of the central L-value (where Ω_E^+ is the real Néron* *period). Then there exist infinitely many order p Dirichlet characters χ with conductors divisible by at most $n+1$ primes such that $L(E,\chi,1)/\Omega_E^+ \not\equiv 0 \pmod{p^{2n}\overline{\mathbb{Z}}}$. Moreover, the number of such χ-twists with $L(E,\chi,1)/\Omega_E^+ \not\equiv 0 \pmod{p^{2n}\overline{\mathbb{Z}}}$ and conductor $\leq X$ is $\gg X/\log X$.*

One can refine the previous theorem when $E/\mathbb{Q}$ satisfies Kurihara's conjecture, which posits the existence of $r \in \mathbb{Z}_{>0}$ (where r is essentially the p^∞-Selmer rank of E) and r *Kolyvagin primes* $q_1, \ldots, q_r$ (i.e. primes satisfying certain conditions with respect to (E,p)) such that a certain linear combination of modular symbols of level dividing $q_1 \cdots q_r$ does not vanish modulo p. This can be seen as a topological incarnation of Kolyvagin's conjecture for $E/\mathbb{Q}$, which posits the existence of certain non-vanishing mod p classes in the Galois cohomology of $E[p]$.

Theorem. *Assume that $E/\mathbb{Q}$ satisfies Kurihara's conjecture at p. Then for all E-good integers d divisible by p, we have that for all $A > 0$,*

$$\#\{\chi \in \mathcal{K}_d(X) : L(E,\chi,1) \neq 0\} \gg_{A,E,d} \frac{(\log\log X)^A X}{\log X}.$$

If moreover $r = 1$ in the context of Kurihara's conjecture then there exists a Kolyvagin prime q_1 such that for a positive proportion of Dirichlet characters χ of order p and conductor dividing $q_1\ell$ where ℓ is a prime we have $L(E,\chi,1) \neq 0$.

Kurihara's conjecture is known to be true in many cases; by work of Kim [4] and forthcoming work of Burungale-Castella-Grossi-Skinner, it is known to be true in certain cases assuming the cyclotomic Iwasawa main conjecture for $(E/\mathbb{Q}, p)$, the latter having been established under mild assumptions by Skinner-Urban [11]. The proofs of all these results use the *propagation principle*:

Propagation Principle. *Let χ_0 be a Dirichlet character such that $L(E,\chi_0,1) \neq 0$. Let p^m be E-good. Then there exist infinitely many Dirichlet characters χ of order p^m such that $L(E,\chi\chi_0,1) \neq 0$.*

This principle is proven using the construction of a horizontal p-adic L-function $F_{\chi_0} \in \mathbb{Z}_p[[T]] \cong \mathbb{Z}_p[[\mathbb{Z}_p]]$ attached to (E,χ_0) and a system of Mazur-Tate elements indexed by Taylor-Wiles primes. We will briefly describe the construction of the horizontal p-adic L-function below; for now, let us admit its existence and explain the idea of the proof of the propagation principle when $m = 1$. By assumption $F_{\chi_0}(0) = L(E,\chi_0,1)/\Omega_E^+ \cdot C \neq 0$ for some $C \in \overline{\mathbb{Q}}^\times$. Thus $F_{\chi_0} \neq 0$; this implies that for all but finitely many characters $\psi : \mathbb{Z}_p \to \overline{\mathbb{Q}}_p^\times$ we have $F_{\chi_0}(\psi) \neq 0$. Using the fact that $F_{\chi_0}(\psi)$ is an explicit linear combination of $L(E,\chi\chi_0,1)$ where χ is a Dirichlet character of order p on Π_n (defined above), i.e. the conductor of χ divides the product of the first n Taylor-Wiles primes, and whose last component is given by the order p character $\psi^{p^{n-1}}$ (here p^n is the conductor of ψ), we conclude the existence of at least one such χ with $L(E,\chi\chi_0,1) \neq 0$. Allowing ψ to vary, this gives the desired conclusion.

We briefly describe the construction of the horiztonal p-adic L-function in the above context. From modular symbols attached to $G_i = \mathrm{Gal}(\mathbb{Q}(\mu_{\ell_i})/\mathbb{Q})$, ℓ_i Taylor-Wiles primes for (E,χ_0) so that we have a surjective map $G_i \to \mathbb{Z}/p$, we get an element $\nu \in \varprojlim_n \mathbb{Z}_p[\Pi_n]$ which projects under $\varprojlim_n \mathbb{Z}_p[\Pi_n] \to \varprojlim_n \mathbb{Z}_p[(\mathbb{Z}/p)^n] = \mathbb{Z}_p[[(\mathbb{Z}/p)^\infty]]$ to an element $\tilde{\nu}$, here giving $(\mathbb{Z}/p)^\infty = \varprojlim_n (\mathbb{Z}/p)^n$ the natural inverse limit topology (with the discrete topologies on each $\mathbb{Z}/p$). Then using the "digit bijection" of *sets* $(\mathbb{Z}/p)^\infty \xrightarrow{\sim} \mathbb{Z}_p, (a_n)_{n\geq 0} \mapsto \sum_{n=0}^\infty \tilde{a}_n p^n$, $0 \leq \tilde{a}_n \leq p-1, \tilde{a}_n \equiv a_n \pmod{p}$, we get an induced $\mathbb{Z}_p$-*module* isomorphism $\mathbb{Z}_p[[(\mathbb{Z}/p)^\infty]] \xrightarrow{\sim} \mathbb{Z}_p[[\mathbb{Z}_p]] = \mathbb{Z}_p[[T]]$. The image of $\tilde{\nu}$ under this $\mathbb{Z}_p$-module homomorphism is F_{χ_0}.

Similar results to the above are also established for E/K where K is an imaginary quadratic field, for the twists $L(E/K, \chi, 1)$ (resp. $L'(E/K, \chi, 1)$) by order d ring class characters χ over K.

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Eisenstein cocycles for imaginary quadratic fields

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(joint work with Emmanuel Lecouturier, Sheng-Chi Shih, Jun Wang)

We aim to establish a particular connection between the geometry of GL_2 and the arithmetic of GL_1 over an imaginary quadratic field, with an Eisenstein ideal serving as the intermediary between them.

First, let us explain what is known over $\mathbb{Q}$. Consider the modular curve $X_1(N)$ of level N over $\mathbb{C}$ and its cusps $C_1^\infty(N)$ that do not lie over $\infty \in X_0(N)$. Let us use $+$ to denote fixed part under complex conjugation, or projection to such a part. We set $\mathbb{Z}' = \mathbb{Z}[\frac{1}{2}]$ for now. Busuioc and I constructed [1, 5] a map

$$\tilde{\Pi}\colon H_1(X_1(N), C_1^\infty(N), \mathbb{Z}')^+ \to (K_2(\mathbb{Z}[\mu_N, \tfrac{1}{N}]) \otimes \mathbb{Z}')^+,$$
$$[c:d]^+ \mapsto \{1-\zeta_N^c, 1-\zeta_N^d\}^+$$

for $c, d \in \mathbb{Z}$ with $(c, d, N) = 1$ and $c, d \not\equiv 0 \bmod N$, where $[c:d]$ is the Manin symbol that is the class of the geodesic from $\frac{a}{c}$ to $\frac{b}{d}$ for $a, b \in \mathbb{Z}$ with $ad - bc = 1$, and where $\{1-\zeta_N^c, 1-\zeta_N^d\}$ is the Steinberg symbol of cyclotomic N-units, for $\zeta_N = e^{2\pi i/N}$. This map $\tilde{\Pi}$ restricts to a map

$$\Pi\colon H_1(X_1(N), \mathbb{Z}')^+ \to (K_2(\mathbb{Z}[\mu_N]) \otimes \mathbb{Z}')^+.$$

Let I be the the Eisenstein ideal in the modular Hecke algebra of weight 2 and level $\Gamma_1(N)$, generated by $T_\ell - \ell - \langle \ell \rangle$ for primes ℓ not dividing Np and $U_\ell^* - 1$ for primes ℓ dividing Np. Some years ago, I made the following conjecture:

Conjecture 1. *The map $\tilde{\Pi}$ is Eisenstein, i.e., $\tilde{\Pi} \circ T = 0$ for all $T \in I$.*

Moreover, I expect that $\Pi \bmod I$ is an isomorphism. In a 2012 preprint [2], Fukaya and Kato proved an earlier form of the conjecture (see [5]), as well a result towards the latter statement.

Theorem 1 (Fukaya-Kato)**.** *The conjecture holds on p on p-parts for $p \mid N$, i.e., $\tilde{\Pi} \otimes \mathbb{Z}_p$ is Eisenstein.*

The proof involves viewing Π as the composition with specialization at the cusp ∞ of a zeta map carrying Manin symbols to cup products of Siegel units on $Y_1(N)$ over $\mathbb{Z}[\frac{1}{N}]$. The zeta map is Hecke-equivariant, as seen by a p-adic regulator computation, and the specialization-at-∞ map is (basically) Eisenstein.

In [6], Venkatesh and I proved the following:

Theorem 2 (S.-Venkatesh)**.** *The map Π is Eisenstein away from the level: i.e., $\Pi \circ (T_\ell - \ell - \langle \ell \rangle) = 0$ for primes ℓ not dividing Np.*

In this result, Π is constructed as the restriction of a cocycle $\Theta_N\colon \Gamma_0(N) \to (K_2(\mathbb{Z}[\mu_N]) \otimes \mathbb{Z}')^+$. This cocycle is in turn the pullback by $(1, \zeta_N) \in \mathbb{G}_m(\mathbb{Q}(\mu_N))$ of the restriction to $\Gamma_0(N)$ of a cocycle $\Theta\colon \mathrm{GL}_2(\mathbb{Z}) \to K_2(\mathbb{Q}(\mathbb{G}_m^2)) \otimes \mathbb{Z}'$, which lands in a small enough subgroup to make the pullback well-defined.

The cocycle Θ is constructed out of Kato's analogue of the Gersten complex, using the pullback action of the canonical right action of $\mathrm{GL}_2(\mathbb{Z})$ on $\mathbb{G}_m^2$. The key point is that the Eisenstein property of Θ can be easily verified (as can parabolicity and an explicit formula) by examining the action of Hecke operators on the $\mathrm{GL}_2(\mathbb{Z})$-fixed class determined by $1 \in \mathbb{G}_m(\mathbb{Q})$.

There is an evident potential analogue in the setting of Bianchi spaces and elliptic units, asked for in a paper of Fukaya, Kato, and I [3]; some evidence for its existence can be found in a much earlier work of Goncharov [4]. That is, let F be an imaginary quadratic field with integer ring $\mathcal{O}$, and let N be an ideal of $\mathcal{O}$.

Consider the Bianchi space $\tilde{Y}_1(N)$ and its minimal compactification $\tilde{X}_1(N)$ for the analogue of $\Gamma_1(N)$ in $\mathrm{GL}_3(\mathcal{O})$. (This space is a disjoint union of $h = |\mathrm{Cl}(F)|$

quotients of complex upper half-space, with the class of the identity corresponding to $\Gamma_1(N)$ more directly.) Its first homology relative to non-infinity cusps contains Manin-type symbols $[c:d]$ of Cremona for $c, d \in \mathcal{O} - N$ with $(c, d, N) = 1$.

Let $F(N)$ denote the ray class field of F of conductor N, denote its integer ring by $\mathcal{O}(N)$, and now set $\mathbb{Z}' = \mathbb{Z}[\frac{1}{6}]$. Let E be an elliptic curve with CM by $\mathcal{O}$ and θ_E a theta-function attached to it. Fix a point P of order N on E. Let $\Phi\colon \mathrm{Cl}_F(N) \to \mathrm{Gal}(F(N)/F)$ be the Artin map.

The following is then the obvious question:

Question. Does there exist an Eisenstein map

$$\Pi\colon H_1(\tilde{X}_1(N), \mathbb{Z}') \to K_2(\mathcal{O}(N)) \otimes \mathbb{Z}'$$

that is the restriction of a map $\tilde{\Pi}$ on homology relative to non-infinty cusps that takes the Cremona symbol $[u:v]$ to the Steinberg symbol $\{\theta_E(cP), \theta_E(dP)\}$?

Some remarks: if F is non-Euclidean, then the Cremona symbols do not generate relative homology. Also, the theta-function θ_E does not exist absent some choice of divisor of degree zero, whereas we might wish θ_E to correspond to the divisor of degree one that is the class of $0 \in E$ to rid ourselves of this choice. In work in progress, we nevertheless show that the answer to our question is basically "yes". The following is a rough statement of our result.

Theorem 3. *Let $c \in \mathcal{O}$ be a non-unit prime to N. There exists a map*

$${}_c\Pi\colon H_1(\tilde{Y}_1(N), \mathbb{Z}') \to K_2(\mathcal{O}(N)) \otimes \mathbb{Z}'$$

such that ${}_c\Pi \otimes \mathbb{Z}[\frac{1}{5}]$ is Eisenstein away from the level and which is compatible with the existence of $\tilde{\Pi}$.

In fact, on p-parts for $p \geq 7$ prime to N, we show the finer result that ${}_c\Pi$ factors through the homology of $\tilde{X}_1(N)$ and that there exists a map Π with $(Nc^2 - \Phi(c))\Pi = {}_c\Pi$ that is Eisenstein away from the level and has an explicit description compatible with being the restriction of a map $\tilde{\Pi}$ as in the above question.

Our approach extends that of S.-Venkatesh; it contains numerous subtleties not present over $\mathbb{Q}$. We define cocycles for all h^2-products of representatives of the h isomorphism classes $\{E_i \mid 1 \leq i \leq h\}$ of elliptic curves with CM by $\mathcal{O}$, where $E_1(\mathbb{C}) \cong \mathbb{C}/\mathcal{O}$. The representatives are then taken to be Galois conjugate curves defined over $L = F(\mathfrak{f})$, where $\mathfrak{f}$ is a suitable auxiliary ideal of $\mathcal{O}$ prime to N. We consider the Gersten complexes $\mathsf{K}_{i,j}$ of Kato for each of these products to obtain a cocycle for a group $\Gamma_{i,j}$ related to $\mathrm{GL}_2(\mathcal{O})$ for each of them. The complex $\mathsf{K}_{i,j}$ enables us to construct a first cohomology class $\Theta_{i,j}\colon \Gamma_{i,j} \to K_2(L(E_i \times E_j)) \otimes \mathbb{Z}'$ out of the $\Gamma_{i,j}$-fixed degree 0 formal sum $c^2(0) - (E_i \times E_j)[c]$ of points.

There are no non-principal Hecke operators on cohomology groups of terms of $\mathsf{K}_{i,j}$. We define a notion of a Δ-module system to consider all of the classes $\Theta = (\Theta_{i,j})_{i,j}$ at once. We find that $T_\mathfrak{n}\Theta = (N\mathfrak{n} + S_\mathfrak{n})\Theta$ for all primes of $\mathcal{O}$ not dividing N, where $T_\mathfrak{n}$ and $S_\mathfrak{n}$ are certain Hecke operators.

We pull back the restrictions of representatives of the $\Theta_{i,j}$ to the analogues of $\Gamma_0(N)$ via Galois-conjugate points $(0, P_j)$ of order N on $E_i \times E_j$ to obtain cocycles valued in $K_2(F(N\mathfrak{f})) \otimes \mathbb{Z}'$. These cocycles come from cocycles valued in $K_2(F(N)) \otimes \mathbb{Z}'$ independent of a choice of $\mathfrak{f}$. Away from the level, the Eisenstein property transfers to these pulled back, or specialized, classes. The specialized classes come in Galois orbits, reducing the number of them we need to consider to h. Their restrictions to the analogue of $\Gamma_1(N)$ together give the map ${}_c\Pi$. Its partial explicit description can be gleaned by equating residues of values of $\Theta_{1,1}$ with sums of tame symbols of Steinberg symbols of theta-functions.

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Kolyvagin's conjecture and Iwasawa theory

Giada Grossi

(joint work with Ashay Burungale, Francesc Castella, and Christopher Skinner)

Let $E/\mathbb{Q}$ be an elliptic curve and p be an odd prime of good ordinary reduction for E. In 1991 Kolyvagin conjectured that the system of cohomology classes derived from Heegner points on the p-adic Tate module of E over an imaginary quadratic field K is non-trivial (see [10]). We report on a joint work with A. Burungale, F. Castella and C. Skinner, in which we prove Kolyvagin's conjecture in the cases where an anticyclotomic Iwasawa Main Conjecture for E/K is known. In particular, we provide the first known cases when p is an Eisenstein prime.

Let K be a quadratic imaginary field of odd conductor $D_K \neq -3$ and such that p and all primes dividing the conductor N of E split in K. Assume also that $E(K)[p] = \{0\}$. For any integer $n \geq 1$, using the modular parametrisation of the elliptic curve, one can construct the Heegner points

$$P_n \in E(K[n])$$

defined over $K[n]$, the ring class field of K of conductor n. The Kummer map yields a class in $H^1(K[n], T/p^M)$, where $T = T_p(E)$ is the p-adic Tate module of E and $M \geq 0$. Applying the *Kolyvagin derivative* and using the assumption $E(K)[p] = \{0\}$ (which yields the surjectivity of the restriction map), one can build a collection of classes

$$\{\kappa_n \in H^1(K, T/p^{M(n)})\}_{n \in \mathcal{N}},$$

where $\mathcal{N}$ is the set of square-free products of inert primes ℓ coprime to p and N such that $M(\ell) = \min\{\mathrm{ord}_p(\ell+1), \mathrm{ord}_p(a_\ell)\} > 0$, $M(n) = \min\{M(\ell) : \ell \mid n\}$, and by convention $1 \in \mathcal{N}$ with $M(1) = \infty$.

Such collection of classes forms a *Kolyvagin system*; in particular if $\ell n, n \in \mathcal{N}$, one can show that:

- the restriction to the cohomology group $H^1(K_\ell, T/p^{M(n)})$ of the completion K_ℓ of K at the unique prime of K above ℓ of the class κ_n, denoted by $\mathrm{loc}_\ell(\kappa_n)$, lies in the unramified subspace $H^1_f(K_\ell, T/p^{M(n)}) \subset H^1(K_\ell, T/p^{M(n)})$;
- the image $\mathrm{loc}^s_\ell(\kappa_{n\ell})$ of $\mathrm{loc}_\ell(\kappa_{n\ell})$ in the singular quotient $H^1_s(K_\ell, T/p^{M(n\ell)})$ of $H^1(K_\ell, T/p^{M(n\ell)})$ is non trivial (unless $\mathrm{loc}_\ell(\kappa_{n\ell}) = 0$);
- the elements $\mathrm{loc}^s_\ell(\kappa_{n\ell})$ and $\mathrm{loc}_\ell(\kappa_n)$ have the same order (in the corresponding local cohomology groups with coefficients T/p^k for $k \leq M(n)$, $M(n\ell)$.

These properties follow from the definition of the classes and the so called *norm relations* between the point $P_n, P_{n\ell}$. One also has an explicit description of the bottom class κ_1 which is simply obtained as the image via the Kummer map in $H^1(K, T)$ of the Heegner point $P_K := \mathrm{Tr}_{K[1]/K} P_1 \in E(K)$. The celebrated work of Gross–Zagier [6] gives the following:

$$\text{(GZ)} \qquad \kappa_1 \neq 0 \quad \Leftrightarrow \quad L'(E/K, 1) \neq 0.$$

In [10] Kolyvagin conjectured that even when the analytic rank of E/K is not one, the *system* $\{\kappa_n\}_n$ is non-trivial, namely:

Conjecture 1 (Kolyvagin)**.** *There exists $n \in \mathcal{N}$ such that $\kappa_n \neq 0$.*

The first major progress towards this conjecture is due to W. Zhang [16], who proved Kolyvagin's conjecture using level raising techniques when $p \neq 2, 3$ is a prime of good ordinary reduction for E under the assumption that

$$\text{(sur)} \qquad \bar{\rho}_E : G_{\mathbb{Q}} = \mathrm{Gal}(\bar{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{Aut}_{\mathbb{F}_p}(E[p]) \text{ is surjective}$$

and $\bar{\rho}_E$ satisfies certain ramification hypotheses. More recently, some of the hypotheses have been relaxed by N. Sweeting [13] using an ultrapatching method for bipartite Euler systems.

The main result of [1] shows that Iwasawa theory can be used to prove new results about Conjecture 1:

Theorem 1. *Let $E/\mathbb{Q}$ be an elliptic curve with good ordinary reduction at $p > 2$ and K a quadratic imaginary field as above. Conjecture 1 holds true if the rational anticyclotomic main conjecture for E/K holds.*

In particular, Conjecture 1 holds true if

(i) *$E[p]^{ss} \simeq \mathbb{F}_p(\phi) \oplus \mathbb{F}_p(\phi^{-1}\omega)$ as $G_{\mathbb{Q}}$-module, where $\phi : G_{\mathbb{Q}} \to \mathbb{F}_p^\times$ is a character such that $\phi|_{G_p} \neq 1, \omega$, with G_p a decomposition group at p and ω the Teichmüller character;*

(ii) *$p > 3$ is such that $E[p]$ is an irreducible $G_{\mathbb{Q}}$-module.*

In case (i), we rely on the work [4] where we prove the (integral) anticyclotomic main conjecture at Eisenstein primes, strengthening our previous work [3] which applied only in the co-rank one case (when $\kappa_1 \neq 0$). Case (ii) follows from the main conjecture proved in [2]. Previous results interlacing Iwasawa theory and Kolyvagin's conjecture (in the case where (sur) holds) appeared in [9].

Strategy of the proof. We briefly mention which are the main ingredients in the proof. Firstly, using Heegner points of p-power conductor, one can construct a Λ-adic Kolyvagin system, where $\Lambda = \mathbb{Z}_p[[\Gamma]]$ and Γ is the Galois group of the anticyclotomic $\mathbb{Z}_p$-extension K_∞ of K. It is a collection of classes

$$\{\kappa_{\Lambda,n} \in H^1(K, T\hat{\otimes}\Lambda/p^{M(n)})\}_{n\in\mathcal{N}}.$$

For any character $\alpha : \Gamma \to \mathbb{Z}_p^\times$, we can consider the *specialization* of the Λ-adic system at α, giving classes $\kappa_n(\alpha) \in H^1(K, T\otimes\mathbb{Z}_p(\alpha)/p^{M(n)})$. The main ingredients of the proof will then be:

(a) The existence of a character $\alpha_m : \Gamma \to \mathbb{Z}_p^\times$ with $\alpha_m \equiv 1 \bmod p^m$ for $m \gg 0$ such that

$$\kappa_1(\alpha_m) \neq 0.$$

This follows from Mazur's conjecture, proved by Cornut and Vatsal.

(b) A suitable control theorem/explicit reciprocity law characterizing the objects appearing in the Iwasawa main conjecture *specialized* at α_m. Combining such results with the (integral) anticyclotomic main conjecture, we will get:

$$\mathrm{length}_{\mathbb{Z}_p} Ш(E\otimes\alpha_m/K)[p^\infty] + \mathrm{ord}_p\,\mathrm{Tam}(E/K) = 2\cdot\mathrm{ind}(\kappa_1(\alpha_m)),$$

where $Ш(E\otimes\alpha_m/K)[p^\infty]$ denotes the Shafarevic-Tate group of $E[p^\infty]\otimes\alpha_m$, $\mathrm{Tam}(E/K)$ denotes the product of the Tamagawa factors of E/K and $\mathrm{ind}(\kappa_1(\alpha_m))$ denotes the maximal power of p dividing the class $\kappa_1(\alpha_m)$. If one only has the rational main conjecture, an extra term (independent on α_m) appears in the equality, causing no harm for the proof of the result.

(c) A Kolyvagin system bound (with an *error term* $\mathcal{E}$, which is non-zero only when (sur) does not hold, but crucially not depending on α_m) for the Selmer groups of $E[p^\infty]\otimes\alpha_m$ given any weak Kolyvagin system $\{\kappa'_n\}$ with $\kappa'_1 \neq 0$:

$$\mathrm{length}_{\mathbb{Z}_p} Ш(E\otimes\alpha_m/K)[p^\infty] \leq 2\cdot\mathrm{ind}(\kappa'_1) + \mathcal{E}.$$

The idea of the proof is then to bound the *index of divisibility* of the Kolyvagin system showing that if p^t divides κ_n for every n, then, choosing a suitable α_m as in (a), p^t also divides $\kappa_n(\alpha_m)$ for every n and we can consider the weak Kolyvagin system $\{\kappa'_n := p^{-t}\kappa_n(\alpha_m)\}$. Applying the bound in (c) and the equality in (b), we get a bound on t.

Further results. The above strategy also yields a bound on the index of divisibility μ_∞ of the Kolyvagin system in terms of the Tamagawa factors of E. Let $m_r := \min\{\mathrm{ind}(\kappa_n) : n \text{ is divisible by exactly } r \text{ primes}\}$. One shows that $m_r \geq m_{r+1}$ for every $r \geq 0$. Let

$$\mu_\infty = \lim_{r\to\infty} m_r.$$

Note that Conjecture 1 is equivalent to $\mu_\infty < \infty$. In the case where (sur) holds, we are working on upgrading the bound $\mu_\infty \leq \mathrm{ord}_p \operatorname{Tam}(E/\mathbb{Q})$ obtained by the above argument to an exact equality, replacing the Kolyvagin system bound in (c) (which was proved in [4]) with the exact structure theorem

$$\mathrm{length}_{\mathbb{Z}_p} Ш(E \otimes \alpha_m/K)[p^\infty] = 2 \cdot (\mathrm{ind}(\kappa_1(\alpha_m)) - \mu_\infty).$$

That leads to the *refined* form of Conjecture 1, as formulated in [17, Conjecture 4.5]:

$$\mu_\infty = \mathrm{ord}_p \operatorname{Tam}(E/\mathbb{Q}).$$

One could also ask about the non-vanishing of other Kolyvagin systems. For example we can consider the Kolyvagin system κ_n^{Kato}, for n in some set $\mathcal{N}_{\mathrm{Kato}}$, obtained from the Euler system constructed by Kato [7] over cyclotomic extensions of $\mathbb{Q}$. A similar strategy applies: the non-vanishing results in (a) above are provided by Rohrlich's work [11] and Kato's explicit reciprocity law in [7] and the Kolyvagin system bound in (c) proved by Mazur–Rubin is generalized in our work to the case where (sur) does not hold. This allows us to prove the following:

Theorem 2. *Let $E/\mathbb{Q}$ be an elliptic curve without CM, and let p be an odd prime of good ordinary reduction for E such that $E(\mathbb{Q})[p] = 0$. Assume that the rational cyclotomic Iwasawa main conjecture holds, then*

$$\text{there exists } n \in \mathcal{N}_{\mathrm{Kato}} \text{ such that } \kappa_n^{\mathrm{Kato}} \neq 0.$$

In particular, $\{\kappa_n^{\mathrm{Kato}}\} \neq 0$ in the cases (i) and (ii) in Theorem 1.

For Eisenstein primes, we apply the new results on Mazur's cyclotomic main conjecture proved in [4]. Case (ii) follows from the main conjecture proved in [7, 12, 15]. Previous results on the non-vanishing of Kato's Kolyvagin system (in the case where (sur) holds) were proved in [8], where also an analogue of the refined Kolyvagin conjecture is formulated. We also prove such conjecture using Mazur–Rubin's structure theorem and the strategy outlined above in the Heegner point case.

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On the arithmetic of θ-critical p-adic L-functions

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(joint work with Denis Benois)

The construction and the basic arithmetic properties of p-adic L-functions for elliptic modular forms are well-established, save the mysterious case of θ-critical forms. Bellaïche in [5] gave a construction of a p-adic L-function in the θ-critical scenario, but its arithmetic properties (e.g. link to basic arithmetic invariants associated to the underlying motive) remained mysterious. The purpose of our joint work with Denis Benois [1, 2] that we summarize here is to unveil these properties. A starting pointing is to reconstruct this p-adic L-function by interpolation in étale cohomology (and in the style of Perrin-Riou), as this approach is naturally better connected with arithmetic. Much of this note will focus on this new construction of θ-critical p-adic L-functions.

1. The set-up and background

Fix forever a prime number $p > 2$ and a positive integer N coprime to p. Let $g \in S_k(\Gamma_1(Np))$ be a cuspidal eigenform, which is normalized and new away from p, of weight $k \geq 2$. Let us fix a finite extension E of $\mathbb{Q}_p$ that contains an isomorphic copy of the Hecke field of g.

The eigenform g corresponds to a point x_0 in the cuspidal eigencurve $\mathcal{C}_N^{\leq k-1}(E)$ of tame level N and slope $\leq k-1$. The eigencurve is equipped with the weight map into the weight space: $\mathcal{C}_N^{\leq k-1} \xrightarrow{w} \mathcal{WS} = \mathrm{Hom}_{\mathrm{cts}}(\mathbb{Z}_p^\times, \mathbb{G}_\mathrm{m})$. We normalize the weight map so that $w(x_0) = k$.

The eigenform g is θ-critical if the weight map w is ramified at x_0. From here on, let us assume that g is θ-critical and let e denote the ramification index of w at x_0. Let us fix a sufficiently small affinoid neighborhood $\mathcal{W} = \mathrm{Spm}(\mathcal{O}_\mathcal{W}) \subset \mathcal{WS}$ in the weight space about k, such that $\mathcal{O}_\mathcal{W} = E\langle Y/p^{er}\rangle$ is a Tate algebra over E with parameter Y/p^{er}, for some $r \gg 0$. On shrinking $\mathcal{W}$ as necessary, x_0 belongs to a unique connected component $\mathcal{X} = \mathrm{Spm}(\mathcal{O}_\mathcal{X})$ of $w^{-1}(\mathcal{W})$, where

$\mathcal{O}_{\mathcal{X}} = \mathcal{O}_{\mathcal{W}}[X]/(X^e - Y) \simeq E\langle X/p^r\rangle$. The reader may benefit from thinking about X as the Hecke operator $U_p - \alpha \cdot \mathrm{Id}$, where $\alpha \in E$ is the eigenvalue with which U_p acts on g.

It turns out that (cf. [5, Prop. 2.11]) g is necessarily of critical slope (i.e. $v_p(\alpha) = k-1$) and $g = f^\alpha$ is the p-stabilization of a unique newform $f \in S_k(\Gamma_1(N), \varepsilon_f)$. Let $\beta := p^{k-1}\varepsilon_f(p)/\alpha$ be the other (slope-zero) root of the Hecke polynomial of f at p. Let us denote by V_f Deligne's Galois representation attached to f, which is normalized so that its determinant is given by $\chi^{1-k}\varepsilon_f^{-1}$, where χ denotes the cyclotomic character. Then the requirement that $g = f^\alpha$ be θ-critical is equivalent to any one of the following conditions.

- $\mathrm{Fil}^{k-1}\mathbf{D}_{\mathrm{cris}}(V_f) = \mathbf{D}_{\mathrm{cris}}(V_f)^{\varphi=\alpha}$, cf. [3].
- $V_f|_{G_{\mathbb{Q}_p}} = V_f^{(\alpha)} \oplus V_f^{(\beta)}$ splits.

A conjecture of Greenberg (which we do not assume) predicts that such f necessarily has CM. If this is the case, a conjecture of Jannsen implies that $e = 2$, cf. [4]. In what follows, we will assume that $e = 2$.

2. Critical p-adic L-functions

One defines p-adic L-functions by interpolation of L-values. These often characterize the p-adic L-function. However, with its expected growth properties, one may not characterize the p-adic L-function $L_p(f^\alpha)$ attached to the critical-slope eigenform f^α by interpolation properties. The problem is that the interpolation range is not sufficiently large. An idea (that goes back to Stevens) is to construct a 2-variable p-adic L-function $L_p(\mathcal{X}) \in \mathcal{O}_{\mathcal{X}}\widehat{\otimes}\mathcal{H}(\Gamma)$ to rigidify this problem, and define $L_p(f^\alpha) := L_p(\mathcal{X})_{|_{x_0}}$. Here, $\mathcal{H}(\Gamma) \supset \Lambda(\Gamma)$ is Perrin-Riou's extended Iwasawa algebra, parametrizing cyclotomic variation. The point is that if $\mathcal{W}$ is sufficiently small, all classical points $x_0 \neq y \in \mathcal{X}(E)$ are of non-critical slope, and the 2-variable p-adic L-function $L_p(\mathcal{X})$ should be characterized by the requirement that $L_p(\mathcal{X})_{|_y}$ coincides with the Manin–Vishik p-adic L-function. It still remains to understand the interpolation properties of $L_p(f^\alpha) := L_p(\mathcal{X})_{|_{x_0}}$.

In our work with Denis Benois, we execute this general idea in the context of overconvergent étale cohomology of Andreatta–Iovita–Stevens. In what follows, we describe our results in this vein.

2.1. We extend the results of Andreatta–Iovita–Stevens, Bellaïche and Chenevier to prove that there exists Galois representations $V_{\mathcal{X}} \simeq \mathcal{O}_{\mathcal{X}}^{\oplus 2}$ and $V'_{\mathcal{X}} \simeq \mathcal{O}_{\mathcal{X}}^{\oplus 2}$ which interpolate Deligne's representations over $\mathcal{X}$, and which come equipped with a Galois equivariant $\mathcal{O}_{\mathcal{W}}$-linear pairing $V'_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{W}}} V_{\mathcal{X}} \xrightarrow{\langle\,,\rangle_{\mathcal{W}}} \mathcal{O}_{\mathcal{W}}$. One may then extend the scalars to obtain an $\mathcal{O}_{\mathcal{X}}$-valued pairing $V'_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{W}}} (\mathcal{O}_{\mathcal{X}} \otimes_{\mathcal{O}_{\mathcal{W}}} V_{\mathcal{X}}) \xrightarrow{\langle\,,\rangle_{\mathcal{X}}} \mathcal{O}_{\mathcal{X}}$.

2.2. In [1], we constructed a big Beilinson–Kato class $\mathrm{BK}_{\mathrm{Iw}}(\mathcal{X}) \in H^1_{\mathrm{Iw}}(\mathbb{Q}, V'_{\mathcal{X}}(1)) := H^1_{\mathrm{Iw}}(\mathbb{Q}, V'_{\mathcal{X}}(1)\widehat{\otimes}\Lambda(\Gamma))$ interpolating Kato's classes for eigenforms f_y corresponding to classical points $y \in \mathcal{X}^{\mathrm{cl}}(E) \subset \mathcal{X}(E)$.

2.3. By the work of Kedlaya–Pottharst–Xiao, the (φ,Γ)-module $\mathbb{D}^\dagger_{\mathrm{rig}}(V_{\mathcal{X}})$ of rank-2 admits a triangulation $\mathbb{D}\subset\mathbb{D}^\dagger_{\mathrm{rig}}(V_{\mathcal{X}})$ (which is a *non-saturated* (φ,Γ)-submodule of rank one; the non-saturation is another incarnation of θ-criticality). The $\mathcal{O}_{\mathcal{X}}$-module $\mathbb{D}_{\mathrm{cris}}(\mathbb{D}) := \mathbb{D}^{\Gamma=1}$ is free of rank one, and Nakamura has constructed a big exponential map EXP on it. We fix a generator η, and define our 2-variable "Kato" p-adic L-function

$$L_p^{\mathrm{K}}(X) := \langle \mathrm{res}_p(\mathrm{BK}_{\mathrm{Iw}}(\mathcal{X})), 1\otimes \mathrm{EXP}(X\eta) + X\otimes \mathrm{EXP}(\eta)\rangle_{\mathcal{X}} \in \mathcal{O}_{\mathcal{X}}\widehat{\otimes}\mathcal{H}(\Gamma)\,.$$

2.4. The 2-variable p-adic L-function $L_p^{\mathrm{K}}(\mathcal{X})$ has the following interpolation properties.

- K0) For all $x_0\neq y\in\mathcal{X}^{\mathrm{cl}}(E)$, $L_p^{\mathrm{K}}(X)_{|_y}$ coincides (up to a non-zero factor) with the Manin–Vishik p-adic L-function attached to the corresponding eigenform f_y.
- K1) $L_p^{\mathrm{K}}(X,\rho\chi^j)_{|_{x_0}}=0$ for integers $1\leq j\leq k-1$ and finite-order characters ρ of Γ.
- K2) If Property (GP) holds (see §2.5 below), $\dfrac{\partial}{\partial X}L_p^{\mathrm{K}}(X,\rho\chi^j)_{|_{x_0}}\doteq\dfrac{L(f,\rho^{-1},j)}{\Omega_f^{\pm}}$.

We would like to think of property (K1) as an extreme exceptional zero phenomenon. Our main motivation was to understand the arithmetic aspects of this phenomenon, which lead us to define an (Iwasawa theoretic) $\mathcal{L}$-invariant, extended Selmer groups, whose behaviour indeed justify this stand point. I have discussed some of this in my talk, but space does not permit to record them as part of these notes, and the interested reader may refer to [2].

2.5. A p-adic L-function $L_p^{\mathrm{S}}(X)$ with analogous interpolation properties has been constructed by Bellaïche, using modular symbols (i.e. interpolating Betti cohomology). We can indeed compare his p-adic L-function with ours: We have $L_p^{\mathrm{K}}(X)=u\cdot L_p^{\mathrm{S}}(X)$ for some $u\in\mathcal{O}_{\mathcal{X}}$. Note that $u\in\mathcal{O}_{\mathcal{X}}^\times$ if and only if (on shrinking $\mathcal{X}$ as necessary) $u(x_0)\neq 0$. We remark that $u(x_0)\in E$ is a canonically defined constant, and its non-vanishing is equivalent to the following property:

(GP) $\big(V_k := V_{\mathcal{X}}/(X^2)\big)_{|_{G_{\mathbb{Q}_p}}}\not\supset V^{(\alpha)}\oplus V^{(\alpha)}$.

2.6. Let us make a note of various interesting properties of $V_k\simeq(E[X]/(X^2))^{\oplus 2}$, which is the infinitesimal deformation of V_f along the eigencurve. It turns out that V_k is not de Rham (since $H^1_{\mathrm{g}}(\mathbb{Q},\mathrm{ad}^0V_f)=0$), and it turns out that $\mathbf{D}_{\mathrm{cris}}(V_k)$ is 3-dimensional, with Hodge–Tate weights $(0,1-k,1-k)$, and φ-eigenvalues β and α with multiplicity 2. The property (GP) is then equivalent to require that $\mathbb{D}_{\mathrm{cris}}\big(\mathbb{D}_k:=\mathbb{D}/(X^2)\big)\not\subset\mathrm{Fil}^{k-1}\mathbf{D}_{\mathrm{cris}}(V_k)$, i.e. "triangulation is in general position relative to Hodge filtration" on $\mathbf{D}_{\mathrm{dR}}(V_k)$. Our discussion in §2.5 shows that if $L_p^{\mathrm{K}}(X)_{|_{x_0}}\in\mathcal{H}(\Gamma)$ is not identically zero, then the above properties of V_k hold true. In other words, we may study the properties of the deformation of V_f along the eigencurve with the aid of our p-adic L-function.

2.7. We conclude our note with an instance where we can check the required non-vanishing statements. Suppose that $f = f_E$ is the newform attached to an elliptic curve $E/\mathbb{Q}$ of analytic rank one. Since f admits a θ-critical p-stabilization, it follows from a work of Serre (combined with the discussion in §1) that f has CM. Combining the work of Lei–Loeffler–Zerbes [6] (which establishes a comparison of $L_p^{\mathrm{K}}(X)_{|x_0}$ with a branch of Katz p-adic L-function with empty interpolation range) with that of Rubin's [7] (expressing the logarithms of Heegner points with the value of a Katz p-adic L-function outside its range of interpolation), we prove that $L_p^{\mathrm{K}}(X)_{|x_0}$ is not identically zero. This suggests that the proof of (GP) or the non-vanishing of the Iwasawa theoretic $\mathcal{L}$-invariant $\mathcal{L}_{\mathrm{Iw}}$ will require a highly non-trivial transcendental input (akin to the non-vanishing of a non-critical value of Katz p-adic L-function outside its range of interpolation).

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Super-Hölder functions and vectors

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(joint work with Sandra Rozensztajn)

1. Motivation

Let $K_\infty = \mathbf{Q}_p(\mu_{p^\infty})$ be the cyclotomic extension of $\mathbf{Q}_p$. The Galois group $\Gamma = \mathrm{Gal}(K_\infty/\mathbf{Q}_p)$ is isomorphic to $\mathbf{Z}_p^\times$ via the cyclotomic character. The action of Γ on K_∞ extends to a continuous action of Γ on $\widehat{K}_\infty$. How can we recover K_∞ from the p-adic Banach representation $\widehat{K}_\infty$ of Γ? The space K_∞ is the space of smooth vectors $\widehat{K}_\infty^{\mathrm{sm}} = \{x \in \widehat{K}_\infty$ such that $\mathrm{Stab}(x)$ is open in $\Gamma\}$. The space K_∞ is also (see [BC16]) the space of locally analytic vectors $\widehat{K}_\infty^{\mathrm{la}} = \{x \in \widehat{K}_\infty$ such that the orbit map $\gamma \mapsto \gamma(x)$ is a locally analytic function on $\Gamma\}$.

Let $\mathbf{E} = \mathbf{F}_p(\!(X)\!)$ and $\mathbf{E}_n = \mathbf{F}_p(\!(X^{1/p^n})\!)$ for $n \geq 0$ and $\mathbf{E}_\infty = \cup_{n\geq 0}\mathbf{E}_n$ and let $\widetilde{\mathbf{E}}$ be the X-adic completion of $\mathbf{E}_\infty$. The group $\Gamma = \mathbf{Z}_p^\times$ acts on $\mathbf{E}$ by $a \cdot f(X) = f((1+X)^a - 1)$, and this action extends to $\widetilde{\mathbf{E}}$. The motivation for our work was the following analogue of the above question: how can we recover $\mathbf{E}_\infty$ from the valued

$\mathbf{F}_p$-representation $\widetilde{\mathbf{E}}$ of Γ? One can prove that $\widetilde{\mathbf{E}}^{\mathrm{sm}} = \mathbf{F}_p$, so smooth vectors are not enough. In order to answer the question, we define super-Hölder functions, that seem to be a characteristic p analogue of locally analytic functions.

2. Super-Hölder functions

Let G be a uniform pro-p-group of rank d and let $G_i = G^{p^i}$ for $i \geq 0$ (for example, one could take $G = \mathbf{Z}_p^d$, so that $G_i = p^i\mathbf{Z}_p^d$). Let M be an $\mathbf{F}_p$-vector space, equipped with a valuation val_M for which it is separated and complete. We say that a function $f : G \to M$ is super-Hölder if there exist constants $\lambda, \mu \in \mathbf{R}$ and $e > 0$ such that $\mathrm{val}_M(f(g) - f(h)) \geq p^\lambda \cdot p^{ei} + \mu$ whenever $gh^{-1} \in G_i$, for all $g, h \in G$ and $i \geq 0$. We let $\mathcal{H}_e^{\lambda,\mu}(G, M)$ denote the corresponding space of functions. For example, the map $\mathbf{Z}_p \to \mathbf{F}_p[\![X]\!]$ given by $a \mapsto (1 + X)^a$ belongs to $\mathcal{H}_1^{0,0}(\mathbf{Z}_p, \mathbf{F}_p[\![X]\!])$.

These super-Hölder functions seem to be the analogue in characteristic p of locally analytic functions. As further evidence, take $G = \mathbf{Z}_p$ and let M be as above. If $\{m_n\}_{n\geq 0}$ is a sequence of M with $m_n \to 0$, the map $z \mapsto \sum_{n\geq 0} \binom{z}{n} m_n$ defines a continuous function $\mathbf{Z}_p \to M$. Conversely, every continuous function $\mathbf{Z}_p \to M$ can be written in this way in one and only one way. Such a function is then in $\mathcal{H}_e^{\lambda,\mu}(\mathbf{Z}_p, M)$ if and only if $\mathrm{val}_M(m_n) \geq p^\lambda \cdot p^{ei} + \mu$ whenever $n \geq p^i$, for all $i \geq 0$. This criteria (see §1.3 of [BR22]) is the analogue of a criteria of Amice characterizing locally analytic functions in terms of their Mahler expansion.

3. Super-Hölder vectors

We now assume that M is endowed with an $\mathbf{F}_p$-linear action of G by isometries. We say that $m \in M$ is a super-Hölder vector if the orbit map $g \mapsto g(m)$ is a super-Hölder function $G \to M$. We denote by $M^{G\text{-}e\text{-sh},\lambda,\mu}$ the elements for which the orbit map is in $\mathcal{H}_e^{\lambda,\mu}(G, M)$. Let $M^{G\text{-}e\text{-sh},\lambda} = \cup_\mu M^{G\text{-}e\text{-sh},\lambda,\mu}$ and $M^{G\text{-}e\text{-sh}} = \cup_\lambda M^{G\text{-}e\text{-sh},\lambda}$. If H is an open uniform subgroup of G, note that $M^{G\text{-}e\text{-sh}} = M^{H\text{-}e\text{-sh}}$.

We can now answer the above question. Let $M = \widetilde{\mathbf{E}}$, with $\mathrm{val}_M = \mathrm{val}_X$, and let $G = 1 + p^k\mathbf{Z}_p$ with $k \geq 1$ (or $k \geq 2$ if $p = 2$). Theorem 2.9 of [BR22] now says that $\widetilde{\mathbf{E}}^{1+p^k\mathbf{Z}_p\text{-1-sh}} = \mathbf{E}_\infty$. More precisely, $\widetilde{\mathbf{E}}^{1+p^k\mathbf{Z}_p\text{-1-sh},k-n} = \mathbf{E}_n$ for $n \geq 0$. The proof of this result in [BR22] uses Colmez' analogue in $\widetilde{\mathbf{E}}$ of Tate's normalized trace maps. In [BR23], we prove a more general result that implies the above one: see §5 of this report.

4. (φ, Γ)-modules

Let $\Gamma = \mathbf{Z}_p^\times$. In this report, a (φ, Γ)-module is a finite dimensional $\mathbf{F}_p(\!(X)\!)$-vector space $\mathbf{D}$, endowed with a semilinear injective Frobenius map $\varphi : \mathbf{D} \to \mathbf{D}$ (acting by $f(X) \mapsto f(X^p)$ on $\mathbf{F}_p(\!(X)\!)$), and a compatible action of Γ. These objects correspond, via Fontaine's equivalence (see [Fon90]), to $\mathbf{F}_p$-linear representations of $\mathrm{Gal}(\mathbf{Q}_p^{\mathrm{alg}}/\mathbf{Q}_p)$. Such an object has a Γ-stable lattice, which allows us to define an X-adic valuation on $\mathbf{D}$. Proposition 3.9 of [BR22] says that $\mathbf{D} = \mathbf{D}^{1+p^k\mathbf{Z}_p\text{-1-sh},k}$.

Let ψ be the usual map on $\mathbf{D}$, defined by $\psi(y) = y_0$ if one writes $y \in \mathbf{D}$ as $y = \sum_{i=0}^{p-1}(1+X)^i \varphi(y_i)$ with $y_i \in \mathbf{D}$. Following Colmez (see [Col10]), let $\mathbf{D}^+$ be the set of $x \in \mathbf{D}$ such that $\{\varphi^i(x)\}_{i \geq 0}$ is bounded, and let $\mathbf{D}^\sharp$ be the largest sub $\mathbf{F}_p[\![X]\!]$-module of finite rank of $\mathbf{D}$ that is stable under ψ and on which ψ is surjective. For example, if $\mathbf{D} = \mathbf{F}_p(\!(X)\!)$, then $\mathbf{F}_p(\!(X)\!)^+ = \mathbf{F}_p[\![X]\!]$ and $\mathbf{F}_p(\!(X)\!)^\sharp = X^{-1} \cdot \mathbf{F}_p[\![X]\!]$. Let $M = \varprojlim_\psi \mathbf{D}^\sharp = \{(y_0, y_1, \ldots)$ where $y_i \in \mathbf{D}^\sharp$ and $\psi(y_{i+1}) = y_i$ for all $i \geq 0\}$.

The space M is an $\mathbf{F}_p[\![X]\!]$-module; we can define an X-adic valuation on it. The group Γ acts on M by isometries (note: the X-adic topology on M is not the natural topology of M, and the action of Γ on M is not continuous for the X-adic topology). There is a map $i : \mathbf{D}^+ \to M$ given by $y \mapsto (y, \varphi(y), \varphi^2(y), \ldots)$. We then have $M^{1+p^k\mathbf{Z}_p\text{-1-sh},k} = i(\mathbf{D}^+)$. When $\mathbf{D} = \mathbf{F}_p(\!(X)\!)$, this result is proved in §3.4 of [BR22]. The $\mathbf{D} \mapsto \varprojlim_\psi \mathbf{D}^\sharp$ construction is an important part of the construction of the p-adic local Langlands correspondence for $\mathrm{GL}_2(\mathbf{Q}_p)$, and the previous result shows that we can "invert" this construction using super-Hölder vectors.

5. The field of norms

We now explain how super-Hölder vectors allow us to recover the field of norms of certain extensions by decompleting their tilt. This material is in [BR23]. Let K be a finite extension of $\mathbf{Q}_p$, and let K_∞ be an almost totally ramified Galois extension of K, whose Galois group Γ is a p-adic Lie group of dimension ≥ 1. Such an extension is then deeply ramified (equivalently, $\widehat{K}_\infty$ is perfectoid) and also strictly arithmetically profinite (see [Win83]). One can then attach two objects to K_∞/K. The first object is the field $\widetilde{\mathbf{E}}_{K_\infty}$, the fraction field of $\varprojlim_{x \mapsto x^p} \mathcal{O}_{K_\infty}/p$ (now called the tilt of $\widehat{K}_\infty$). This is a perfect valued field of characteristic p, on which Γ acts by isometries.

The second object is the field of norms. Let $\mathcal{E} = \{E/K$ such that E/K is finite and $E \subset K_\infty\}$. Let $X_K(K_\infty) = \varprojlim_{\mathrm{N}_{F/E}} E = \{(x_E)_{E \in \mathcal{E}}$ with $x_E \in E$ and such that $\mathrm{N}_{F/E}(x_F) = x_E$ whenever $E \subset F\}$. The set $X_K(K_\infty)$ can be given (see [Win83]) a natural structure of a valued field of characteristic p, on which Γ acts by isometries. It is then isomorphic to $k_{K_\infty}(\!(\pi)\!)$ where k_{K_∞} is the residue field of K_∞ and π is a norm compatible sequence of uniformizers. Furthermore (see ibid), there is a natural map $X_K(K_\infty) \to \widetilde{\mathbf{E}}_{K_\infty}$, and $\widetilde{\mathbf{E}}_{K_\infty}$ is the completion of the perfection $\cup_{n \geq 0} X_K(K_\infty)^{1/p^n}$ of $X_K(K_\infty)$.

Theorem A of [BR23] says that $\cup_{n \geq 0} X_K(K_\infty)^{1/p^n} = \widetilde{\mathbf{E}}_{K_\infty}^{\Gamma\text{-}d\text{-sh}}$. In the "cyclotomic" case, with $K_\infty = \mathbf{Q}_p(\mu_{p^\infty})$, we have $d = 1$ and $X_K(K_\infty) = \mathbf{F}_p(\!(X)\!)$ and $\widetilde{\mathbf{E}}_{K_\infty} = \widetilde{\mathbf{E}}$ and the action of Γ on $\widetilde{\mathbf{E}}$ is the one coming from $a \cdot f(X) = f((1+X)^a - 1)$. Hence the result above implies the answer to the question formulated at the beginning.

6. Examples

Here are two examples of super-Hölder functions with interesting properties.

6.1. A locally analytic function that has a nonisolated zero is locally constant at this point. Here is a function $f : \mathbf{Z}_p \to \mathbf{F}_p[\![X]\!]$ that is super-Hölder and has a nonisolated zero but is nowhere locally constant.

Set $f(0) = 0$ and if $a \in \mathbf{Z}_p^\times$ and $i \geq 0$, let $f(p^i a) = ((1+X)^a - (1+X))^{p^i}$.

6.2. If $\alpha \in \mathbf{Z}_{\geq 1}$, then $\sum_{n \geq 0} X^{p^{n\alpha}+p^{-n}} \in \mathbf{F}_p[\![X]\!]$ is a super-Hölder vector for the action of $1 + 2p\mathbf{Z}_p$ on $\mathbf{F}_p[\![X]\!]$ with $e = \alpha/(1+\alpha)$, but not for $e > \alpha/(1+\alpha)$.

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Analytic cohomology of Lubin-Tate (φ_L, Γ_L)-modules

Rustam Steingart

The goal of my talk was to explain the some finiteness results for analytic cohomology of Lubin-Tate (φ_L, Γ_L)-modules over relative Robba rings and a variant of Shapiro's Lemma for Iwasawa cohomology from [Ste22a],[Ste22b]. These results are applied in work in progress (joint with Milan Malčić, Otmar Venjakob and Max Witzelsperger) on a variant of the local ε-conjecture for analytic (φ_L, Γ_L)-modules.

1. Analytic Lubin-Tate (φ_L, Γ_L)-modules

Let $L/\mathbb{Q}_p$ be a finite extension and fix an embedding $L \to \mathbb{C}_p$. Let $\varphi_L(T) \in o_L[\![T]\!]$ be a Frobenius power series for some uniformiser π_L of L. We denote by L_∞ the Lubin-Tate extension attached to φ_L and set $\Gamma_L := \mathrm{Gal}(L_\infty/L)$. Fontaine showed that the category of étale $(\varphi_{\mathbb{Q}_p}, \Gamma_{\mathbb{Q}_p})$-modules is equivalent to the category of p-adic representation of $G_{\mathbb{Q}_p}$. If one wants to reconstruct the invariants attached to a representation by p-adic Hodge theory, one has to work over the Robba ring $\mathcal{R}_L$ consisting of Laurent series with coefficients in L, which converge on the half-open annulus $[r, 1)$ for some $r \in [0, 1)$. The equivalence of categories still works over the Robba ring in the classical case (cf. [CC98]). In the Lubin-Tate case the category of étale (φ_L, Γ_L)-modules over $\mathcal{R}_L$ is equivalent to the category of overconvergent representations and if $L \neq \mathbb{Q}_p$ then there exist Galois representations which are not overconvergent. A sufficient condition for overconvergence of a representation is analyticity. An L-linear representation V of G_L is called L-analytic if the $\mathbb{C}_p$-semilinear representation $\mathbb{C}_p \otimes_{L,\sigma} V$ is trivial for every $\sigma \neq \mathrm{id}$. Berger showed that the category of L-analytic representations is equivalent to the category of analytic étale (φ_L, Γ_L)-modules over $\mathcal{R}_L$ (cf. [Ber16]). A (φ_L, Γ_L)-module over $\mathcal{R}_L$ is called L-analytic if the derived action of Γ_L is L-bilinear.

2. Analytic cohomology and Iwasawa cohomology

Interestingly, the categories of overconvergent (resp. analytic) (φ_L, Γ_L)-modules are not stable under extensions. By the usual recipe one can produce from an extension $0 \to M \to E \to \mathcal{R}_L \to 0$ a continuous cocycle of the monoid $\varphi_L^{\mathbb{N}_0} \times \Gamma_L$ with values in M, which is L-analytic (with repsect to the discrete topology on $\varphi_L^{\mathbb{N}_0}$) if and only if E is analytic. More generally, we can define the analytic cohomology as the cohomology of the complex

$$C^\bullet_{an}(\varphi_L^{\mathbb{N}_0} \times \Gamma_L, M) \cong \mathrm{Tot}[C^\bullet_{an}(\Gamma_L, M) \xrightarrow{\varphi_L - 1} C^\bullet_{an}(\Gamma_L, M)]$$

of locally L-analytic cochains with values in M. There is a natural map to the complex of continuous cochains which induces an isomorphism (resp. an injection) when taking H^0 (resp. H^1). The analytic nature of the complex makes it difficult to obtain results for higher cohomology groups. The key inputs to make the theory more algebraic are the following: Let us fix an open subgroup $U \subset \Gamma_L$ isomorphic to o_L. By the results of [ST01] the rigid analytic variety $\mathfrak{X}_U$ parametrising locally L-analytic characters of o_L is isomorphic to an open unit disc after base change to a sufficiently large field K (e.g. $K = \mathbb{C}_p$). The global sections of $\mathfrak{X}_U$ are isomorphic to the algebra $D(U, L)$ of locally L-analytic distributions. If we henceforth assume that K is large enough, we can assume that $D(U, K)$ is isomorphic to a ring of convergent power series in some variable Z. By adapting the results from [Koh11] one can show that the analytic cohomology $H^i_{an}(U, M)$ is isomorphic to $\mathrm{Ext}^i_{D(U,K)}(K, M)$ which leads to a purely algebraic description of analytic cohomology. The latter groups can be computed by the two-term complex $[M \xrightarrow{Z} M]$, which in turn means that $C^\bullet_{an}(\varphi_L^{\mathbb{N}_0} \times U, M)$ can be computed by a three term complex $C_{\varphi_L, Z}(M) : [M \xrightarrow{(\varphi_L - 1, Z)} M^2 \xrightarrow{Z \oplus (1 - \varphi_L)} M]$ similar to the classical Herr complex (in the classical case (assuming $p \neq 2$) one takes $Z = \gamma - 1$ for a topological generator γ of $\Gamma_{\mathbb{Q}_p}$). Using the explicit description of the complex and general results on bounded complexes of Banach modules one can deduce the perfectness of analytic cohomology which leads to similar results as in [KPX14]. Using the left-inverse Ψ of φ_L one can define $C_\Psi(M) : [M \xrightarrow{\Psi - 1} M]$. In the classical case this complex computes Iwasawa cohomology and can be related to the Galois cohomology of the cyclotomic deformation of a (φ, Γ)-module by a variant of Shapiro's Lemma. There is an analogue of the latter comparison in our case. One can define a family $\mathbf{Dfm}(M)$ of (φ_L, Γ_L)-modules over $\mathfrak{X}_U$ which is roughly speaking given as the base change $D(U, K)\hat{\otimes}_K M$. In [Ste22b] it is shown that the natural map $C_\Psi(M) \to C_{\Psi, Z}(\mathbf{Dfm}(M))$ is a quasi-isomorphism provided that the cohomology groups of $C_\Psi(M)$ are coherent sheaves on $\mathfrak{X}_U$. The latter condition is necessary due to the finiteness properties of the right-hand side. We do not know whether this condition holds in general. We show it in some cases in [Ste22b]. This question is closely related to understanding the $D(U,K)$-structure on the heart $\mathcal{C}(M) := (1-\varphi_L)(M^{\Psi=1})$. We expect that the heart is projective of rank $[\Gamma_L : U]\mathrm{rk}(M)$. This result would imply the validity of the Euler-Poincaré formula for analytic cohomology.

3. Applications

The results above are applied in joint work in progress with Milan Malčić, Otmar Venjakob and Max Witzelsperger. In [Nak14] Nakamura generalises Kato's ε-conjecture to cyclotomic (φ,Γ)-modules over the Robba ring. Roughly speaking, the conjecture says that to a family M of (φ,Γ)-modules over an affinoid, one can attach a line bundle $\Delta(M)$ (essentially the determinant of the Galois cohomology of M) and a unique trivialisation of $\Delta(M)$ which interpolates the ε-constants of Deligne at de Rham points. Inspired by his construction, we explore a variant for analytic (φ_L,Γ_L)-modules over $\mathcal{R}_K$. The preceding finiteness results allow us to define the fundamental line $\Delta(M)$ as the determinant of analytic cohomology. For an L-analytic de Rham representation, we have a decomposition $D_{dR}(V) = \bigoplus_{\sigma\colon L\to\overline{L}} D_{dR,\sigma}(V)$ (similarly for $D_{pst}(V)$). In contrast to Nakamura's variant we restrict to the ε-constants attached to the identity component. We construct the desired ε-isomorphisms in the rank one case as global sections on $\mathfrak{X}_{\Gamma_L}$, which satisfy the interpolation property at de Rham points.

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Deligne's Conjecture on critical values of L-functions for Hecke characters

Han-Ung Kufner

In this talk we discussed a proof of Deligne's rationality conjecture for Hecke L-functions. We first recall the statement of the conjecture in this case and afterwards give a sketch of the proof.

1. The Statement

1.1. **Notations.** All number fields will be equipped with a fixed embedding into $\mathbb{C}$. Let L and T be number fields. We write J_L for the set of embeddings $\mathrm{Hom}(L, \mathbb{C})$ and let I_L denote the free abelian group on J_L. For $\alpha \in I_L$ and $\ell \in L^\times$ we set $\ell^\alpha = \prod_{\sigma \in J_L} \sigma(\ell)^{\alpha(\sigma)}$. We also denote $d(\alpha) = \sum_{\sigma \in J_L} \alpha(\sigma)$. Let χ be an algebraic Hecke character of L with values in T of conductor $\mathfrak{f} \subset \mathcal{O}_L$. We also regard χ as a continuous homomorphism $\chi : \mathbb{A}_L^\times \to T^\times$.

1.2. **Hecke L-functions.** For each embedding $\iota \in J_T$, we consider the L-series given by

$$L(\iota \circ \chi, s) = \sum_{\mathfrak{a} \subset \mathcal{O}_L, (\mathfrak{a}, \mathfrak{f}) = 1} \frac{(\iota \circ \chi)(\mathfrak{a})}{N\mathfrak{a}^s},$$

which converges absolutely for $s \in \mathbb{C}$ with sufficiently large real part. Using the usual identification $T \otimes \mathbb{C} \cong \mathbb{C}^{J_T}$, we assemble these into a $T \otimes \mathbb{C}$-valued L-function

$$L(\chi, s) = \left(L(\iota \circ \chi, s)\right)_{\iota \in J_T}.$$

The function $L(\chi, s)$ admits a meromorphic continuation and satisfies a functional equation. We say that χ is *critical* if the Γ-factors on both sides of the functional equation have no pole at $s = 0$. If χ is critical, it follows that L is either totally real or that L contains a CM-field.

1.3. **The period $c^+(\chi)$.** Attached to χ is a motive (for absolute Hodge cycles) $M(\chi)$ defined over L and with coefficients in T such that its motivic L-function coincides with the function $L(\chi, s)$ defined above. Let $RM(\chi)$ denote the restriction of scalars of $M(\chi)$ from L to $\mathbb{Q}$, $RM(\chi)_B^+$ the subspace in the Betti realization fixed by the involution induced by complex conjugation and let $F^\bullet$ denote the Hodge filtration. If χ is critical, the composite

$$I_+ : RM(\chi)_B^+ \otimes \mathbb{C} \to RM(\chi)_B \otimes \mathbb{C} \xrightarrow{I_\infty^{-1}} RM(\chi)_{\mathrm{dR}} \otimes \mathbb{C} \to RM(\chi)_{\mathrm{dR}}/F^0 \otimes \mathbb{C},$$

where I_∞ denotes the comparison isomorphism between de Rham and Betti realizations, is an isomorphism. This allows to define a period

$$c^+(\chi) = \det(I_+) \in (T \otimes \mathbb{C})^\times,$$

where the determinant is calculated with respect to T-bases of $RM(\chi)_B^+$ and $RM(\chi)_{\mathrm{dR}}/F^0$. Note that $c^+(\chi)$ is really only well-defined up to element in $T^\times$.

1.4. **Main result.** In our talk we discussed the following result which is an instance of a more general conjecture of Deligne ([2]) for the motive $M(\chi)$:

Theorem 1. *Let χ be a critical algebraic Hecke character of L with values in T. Then the value $L(\chi, 0)$ agrees with $c^+(\chi)$ up to a factor in T.*

Previously, the theorem was known if L is a totally real field (due to work of Euler, Siegel and Klingen), if L is imaginary quadratic (Goldstein-Schappacher) and more generally when L is a CM-field (Blasius [1]). For arbitrary critical χ, the result above was announced by Harder-Schappacher ([4]) but details were never published. For arbitrary number fields L containing a CM-field, Kings-Sprang ([5]) were able to relate the L-value $L(\chi, 0)$ to periods of abelian varieties with complex multiplication by L up to an algebraic integer. This allowed them to establish Deligne's conjecture up to a factor in $T \otimes \overline{\mathbb{Q}}$. Central to their approach is a novel construction of equivariant coherent cohomology classes attached to CM abelian varieties.

2. Sketch of proof

2.1. **Outline.** It is a key insight of Blasius that one can recover $c^+(\chi)$ (up to a factor in $T^\times$) in terms of a period construction on a different motive $RM(\Xi)$. The advantage of this motive is that it admits an alternative description in terms of abelian varieties with CM by L. This makes it possible to define an element in $RM(\Xi)_{\mathrm{dR}}$ using the aforementioned classes of Kings-Sprang. By reformulating and extending their results in this context, one shows that the associated period coincides with the L-value $L(\chi, 0)$ and one concludes by using Blasius' result.

2.2. **Blasius' period relation.** Let us fix a number field L containing a CM-field and a critical algebraic Hecke character $\chi : \mathbb{A}_L^\times \to T^\times$. Then the embeddings σ of L on which the infinity-type χ_a of χ takes on negative values form a CM-type Φ of L (lifted from its maximal CM-subfield) and one can split up χ_a in the form

$$\chi_a = \beta - \alpha,$$

where α (resp. β) is supported on Φ (resp. $\overline{\Phi}$) such that $\alpha(\sigma) \geq 1$ and $\beta(\overline{\sigma}) \geq 0$ for all $\sigma \in \Phi$. Let E denote the reflex field of (L, Φ) and $\Phi^* \in I_E$ denote the reflex CM-type of Φ. Consider the algebraic Hecke character

$$\Xi = (\chi \circ \Phi^*)^{-1} \cdot N_{E/\mathbb{Q}}^{d(\beta)} \cdot \varepsilon_\Phi : \mathbb{A}_E^\times \to T^\times,$$

where ε_Φ is a certain sign character attached to Φ.

Theorem 2 (Blasius [1])**.** *Let $n = d(\alpha) + d(\beta)$. Then $\dim_T F^n RM(\Xi)_{\mathrm{dR}} = 1$ and there exists a "special" 1-dimensional T-subspace $\Delta \subset RM(\Xi)_B \otimes \overline{\mathbb{Q}}$ such that, for any non-zero elements $\omega \in F^n RM(\Xi)_{\mathrm{dR}}$ and $\delta \in \Delta$, one has*

$$I_\infty(\omega) = (2\pi i)^{d(\beta)} c^+(\chi) \cdot \delta \mod T^\times.$$

2.3. **Relation to CM abelian varieties.** Let F/E be a finite extension and A/F an abelian variety of CM-type (L, Φ) with CM-character $\psi : \mathbb{A}_F^\times \to L^\times$. The dual abelian variety $A^\vee$ canonically acquires a CM-structure from A and has CM-character $\overline{\psi}$. Comparing infinity-types, we see that for sufficiently large F/E, one has

$$\Xi \circ N_{F/E} = \psi^\alpha \overline{\psi}^\beta.$$

Here it is necessary to also enlarge the field of values T suitably. This is no problem since Deligne's conjecture is invariant under extension of coefficients. Extending a technique of Goldstein-Schappacher ([3]) to motives, the above identity allows to recover $M(\Xi)$ as a direct summand in

$$R_{F/E}(h^1(A)^\alpha \otimes_T h^1(A^\vee)^\beta)$$

Here $h^1(A)^\alpha$ is constructed from $h^1(A)$ and a motive for the Hecke character ψ^α. By construction, one can attach to every $\gamma \in h^1(A)_B$ an element $\gamma^\alpha \in h^1(A)_B^\alpha$.

2.4. **Relation to L-values.** For every $\sigma \in J_F$, let $\gamma_\sigma \in h^1(A^\sigma)_B$ be an L-basis and let $\gamma_\sigma^\vee \in h^1(A^{\sigma,\vee})_B$ denote dual basis with respect to the canonical pairing $h^1(A^\sigma) \otimes h^1(A^{\sigma,\vee}) \to 2\pi i L$. The following theorem uses results of Kings-Sprang as the main input and reformulates them in our context:

Theorem 3. *There is a class $\mathcal{EK} \in F^n(h^1(A)^\alpha \otimes h^1(A^\vee)^\beta)_{\mathrm{dR}}$ such that*

$$I_\infty(\mathcal{EK}^\sigma) = t_\sigma \cdot (2\pi i)^{d(\beta)} (L(\iota \circ \chi, 0)_{\iota|_E = \sigma|_E} \cdot (\gamma_\sigma^\alpha \otimes (\gamma_\sigma^\vee)^\beta),$$

where $t_\sigma \in T^\times$ is a certain factor (depending on the choice of γ_σ's).

Consider the image of $\mathcal{EK}$ in $RM(\Xi)_{\mathrm{dR}}$ and simply denote it by $\mathcal{EK}$ again. The above theorem then implies

$$I_{\infty, RM(\Xi)}(\mathcal{EK}) = \sum_{\sigma \in J_F} I_\infty(\mathcal{EK}^\sigma) = (2\pi i)^{d(\beta)} \cdot L(\chi, 0) \cdot \delta$$

for an element $\delta \in RM(\Xi)_B \otimes \overline{\mathbb{Q}}$. Finally, one checks that δ lies in the special subspace Δ from Blasius' Theorem.

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On the geometry of integral models of Shimura varieties of abelian type

Yujie Xu

The construction of smooth (resp. normal) integral models of Shimura varieties plays an important part in number theory. In this report, we discuss some recent advancements on the geometry of integral models of Shimura varieties of abelian type. Such results as Theorems 4 and 5 have been useful in various aspects of number theory, e.g. in the construction of p-adic L-functions using Euler systems (see for example [24]), in the arithmetic intersection theory of special cycles on Shimura varieties and their integral models as in the Kudla-Rapoport program and Gross-Zagier program etc. (see for example [18]).

Let (G, X) be a Shimura datum of Hodge type[1], i.e. it is equipped with an embedding $(G, X) \hookrightarrow (\mathrm{GSp}(V, \psi), S^{\pm})$, where V is a $\mathbb{Q}$-vector space equipped with a symplectic pairing ψ. The embedding of Shimura data induces an embedding of Shimura varieties $\mathrm{Sh}_K(G, X) \hookrightarrow \mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$ for suitable choices of compact opens $K \subset G(\mathbb{A}_f)$ and $K' \subset \mathrm{GSp}(\mathbb{A}_f)$. For K' sufficiently small, the moduli interpretation of the Siegel modular variety $\mathrm{Sh}_{K'}(\mathrm{GSp}, S^{\pm})$ naturally gives rise to an integral model $\mathcal{S}_{K'}(\mathrm{GSp}, S^{\pm})$. We consider the integral model $\mathcal{S}_K(G, X)$ of $\mathrm{Sh}_K(G, X)$ with hyperspecial (resp. parahoric) level structure K_p at p, as constructed in [1] (resp. [2]), which is initially defined as the normalization of the closure of $\mathrm{Sh}_K(G, X)$ inside $\mathcal{S}_{K'}(\mathrm{GSp}, S^{\pm})$. One of the results discussed in this report concerns the author's recent work [15], which shows that this construction can be simplified, in that the normalization step is redundant, and that $\mathcal{S}_K(G, X)$ is simply the closure of $\mathrm{Sh}_K(G, X)$ inside $\mathcal{S}_{K'}(\mathrm{GSp}, S^{\pm})$. For the precise statements, see Theorem 4 and Theorem 5.

A key input in the above-mentioned result lies in the author's recent joint work with Gleason and Lim [20], which resolves the following long-standing conjecture (see [22]) on the geometry of affine Deligne–Lusztig varieties, which parametrize mod p isogeny classes on (global) integral models of Shimura varieties. More precisely, let (G, b, μ) be a p-adic Shtuka datum, and $X_\mu^{K_p}(b)$ its associated affine Deligne–Lusztig variety at level K_p. Let I be the Galois group of $\breve{\mathbb{Q}}_p$. In [21], Kottwitz defined a map $\kappa_G : G(\breve{\mathbb{Q}}_p) \to \pi_1(G)_I$, which induces the map ω_G in Conjecture 1. We refer the reader to [20] for the precise notations.

Conjecture 1. *If (b, μ) is Hodge-Newton irreducible, the following map is bijective*

$$\omega_G : \pi_0(X_\mu^{K_p}(b)) \to c_{b,\mu}\pi_1(G)_I^{\varphi}$$

Theorem 1 (Gleason-Lim-Xu). [20] *For all p-adic shtuka datum (G, b, μ) and all parahoric subgroups $K_p \subseteq G(\mathbb{Q}_p)$, Conjecture 1 holds.*

[1]Since abelian type integral models are built out of Hodge type integral models as building blocks, most of our results such as Theorems 4 and 5 hold for general abelian type integral models.

As a corollary, we have the following "CM lifting" theorem on the integral model $\mathcal{S}_K(G,X)$, which is an analogue of the classical CM lifting theorem: every abelian variety over a finite field lifts–up to a finite field extension–to a CM abelian variety in characteristic zero.

Theorem 2 (Gleason-Lim-Xu)**.** [20] *Every mod p isogeny class on $\mathcal{S}_K(G,X)$ contains a CM-liftable point.*

The following corollaries concern the geometry of integral models $\mathcal{S}_K(G,X)$.

Corollary 1 (Gleason-Lim-Xu)**.** [20] *(a) The "almost product structure" of the Newton strata in $\mathcal{S}_{K_p,\bar{\mathbb{F}}_p}(G,X)$ holds.*
(b) Every EKOR stratum in $\mathcal{S}_{K_p}(G,X)_{\overline{\mathbb{F}}_p}$ is quasi-affine.

A further consequence of our Theorem 1 is the following result on p-adic uniformization. Let $\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ be the local Shimura varieties considered in [23].

Theorem 3 (Gleason-Lim-Xu)**.** [20] *$\mathcal{M}^{\mathrm{int}}_{\mathcal{G},b,\mu}$ is representable by a formal scheme $\mathcal{M}_{\mathcal{G},b,\mu}$, and we obtain a p-adic uniformization isomorphism of $\mathcal{O}_{\breve{E}}$-formal schemes*

$$I_x(\mathbb{Q})\backslash(\mathcal{M}_{\mathcal{G},b,\mu}\times \mathbf{G}(\mathbb{A}_f^p)/\mathbf{K}^p)\to(\mathcal{S}_{\mathbf{K}}\widehat{\otimes_{\mathcal{O}_E}\mathcal{O}_{\breve{E}}})_{/\mathcal{I}(x)}.$$

The following result uses Theorem 2 as a key ingredient.

Theorem 4 (Xu)**.** [15,16] *For $K\subset G(\mathbb{A}_f)$ small enough, there exists some $K'\subset$ $\mathrm{GSp}(\mathbb{A}_f)$, such that we have a closed embedding ("the Hodge embedding")*

$$\mathcal{S}_K(G,X)\hookrightarrow\mathcal{S}_{K'}(\mathrm{GSp},S^{\pm})$$

More precisely, the normalization step $\mathcal{S}_K(G,X)\xrightarrow{\nu}\mathcal{S}_K^-(G,X)$ is redundant as the closure $\mathcal{S}_K^-(G,X)$ is already smooth (resp. normal), and the integral model $\mathcal{S}_K(G,X)$ has a moduli interpretation inherited from that of $\mathcal{S}_{K'}(\mathrm{GSp},S^{\pm})$.

In particular, the Hodge morphism is a closed embedding in the PEL case[2], where we consider integral models constructed in [3] (resp. [4]). For the result in that case, see [16] for details. Note that, in this case, finiteness of the Hodge morphism follows from finiteness of certain fppf-cohomology, see [17] for details.

In the general Hodge type case, the mod p points of the integral model $\mathcal{S}_K(G,X)$ can be interpreted as abelian varieties equipped with certain "mod p Hodge cycles", which come from reduction mod p of Hodge cycles in characteristic zero. These mod p Hodge cycles are indeed motivated cycles in characteristic p in the sense of [7–9]. We denote the mod p Hodge cycle at a mod p point $x\in\mathcal{S}_K(G,X)$ by a tuple $(s_{\alpha,\ell,x},s_{\alpha,\mathrm{cris},x})$, which is determined by either its ℓ-adic étale component $s_{\alpha,\ell,x}$ or its cristalline component $s_{\alpha,\mathrm{cris},x}$ (see Proposition 1). This is analogous to the case of Hodge cycles in characteristic 0, which are determined by either their étale components or their de Rham components [10].

More specifically, let $\mathcal{S}_K^-(G,X)$ be the closure of $\mathrm{Sh}_K(G,X)$ in $\mathcal{S}_{K'}(\mathrm{GSp},S^{\pm})$. By a criterion in [5] (resp. [6]), two mod p points $x,x'\in\mathcal{S}_K(G,X)(k)$ that have the

[2] This terminology can cause ambiguity. The reader should note that we mean Kottwitz' (hyperspecial) or Rapoport-Zink's (parahoric) PEL moduli problem.

same image in $\mathcal{S}_K^-(G,X)(k)$ are equal if and only if $s_{\alpha,\mathrm{cris},x} = s_{\alpha,\mathrm{cris},x'}$. Therefore, to show that the normalization morphism is an isomorphism, it reduces to proving the following statement on cohomological tensors:

Proposition 1 (Xu)**.** [15] $s_{\alpha,\ell,x} = s_{\alpha,\ell,x'} \Longrightarrow s_{\alpha,\mathrm{cris},x} = s_{\alpha,\mathrm{cris},x'}$.

By the CM lifting result, i.e. Theorem 2, these cohomological tensors lift, up to G-isogenies, to Hodge cycles on CM abelian varieties. A key observation is that when two mod p points $x, x' \in \mathcal{S}_K(G,X)(k)$ map to the same image in $\mathcal{S}_K^-(G,X)(k)$, they can be CM-lifted using the same torus, whose cocharacter induces the filtration on the Dieudonné modules $\mathbb{D}(\mathcal{A}_x) = \mathbb{D}(\mathcal{A}_{x'})$ which then identifies the filtrations on the Dieudonné modules associated to CM-liftable mod p points, giving rise to an isogeny *in characteristic zero* between the two CM lifts. This observation allows us to match up the mod p cristalline tensors using the input from ℓ-adic étale tensors, precisely due to the rationality of Hodge cycles in characteristic zero and the existence of an isogeny lift in characteristic zero.

It is worth pointing out that, in the case where the aforementioned cohomological tensors are algebraic–for example, at points where the Hodge conjecture is true–the family of Hodge cycles (tensors) s_α that naturally lives over the Hodge type integral model $\mathcal{S}_K(G,X)$ becomes a flat family of algebraic cycles over $\mathcal{S}_K(G,X)$. In this case, $s_{\alpha,\ell,x} = s_{\alpha,\ell,x'}$ implies that the two algebraic cycles corresponding to the two ℓ-adic cycles are ℓ-adic cohomologically equivalent, hence numerically equivalent, and we only need to show that they are also cristalline-cohomologically equivalent. Recall that the Grothendieck Standard Conjecture D says that numerical equivalence and cohomological equivalence agree for algebraic cycles. The proof of 1 thus follows from a cristalline realisation of this Standard Conjecture D, for points on the integral model of Hodge type and their associated cristalline tensors, which are mod p Hodge cycles. Our result essentially establishes, unconditionally, rationality for mod p Hodge cycles that live on mod p points of Hodge type Shimura varieties. In general, without the algebraicity of s_α, Proposition 1 is essentially an instance of a conjecture due to Yves André on the rationality of motivated cycles in characteristic $p > 0$ [7, 9], which has implications for the Tannakian category constructed by Langlands-Rapoport [11].

Finally, we state the following analogue of Theorem 4 for toroidal compactifications of integral models of Hodge type constructed in [12] (the PEL cases were constructed earlier in [13]). Combining Theorem 4 with an analysis as in [14] on the boundary components of toroidal compactifications, one obtains the following result, a special case of which has been used in [24] to construct Euler systems.

Theorem 5 (Xu)**.** [15] *Let (G,X) be a Shimura datum of Hodge type. For each $K \subset G(\mathbb{A}_f)$ sufficiently small, there exist collections Σ and Σ' of cone decompositions, and $K' \subset \mathrm{GSp}(\mathbb{A}_f)$, such that we have a closed embedding of toroidal compactifications of integral models $\mathcal{S}_K^\Sigma(G,X) \hookrightarrow \mathcal{S}_{K'}^{\Sigma'}(\mathrm{GSp}, S^\pm)$ extending the Hodge embedding of integral models.*
In particular, the normalization step is redundant, and $\mathcal{S}_K^\Sigma(G,X)$ can be constructed by simply taking the closure of $\mathrm{Sh}_K(G,X)$ inside $\mathcal{S}_{K'}^{\Sigma'}(\mathrm{GSp}, S^\pm)$.

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Special values of Zeta functions and Deligne cohomology

Matthias Flach

The main topic of the talk was a construction in homological algebra: Gluing the solid and liquid categories of Clausen and Scholze [4]. The motivation for such a construction comes from Deligne cohomology and its role in describing values of Zeta functions.

1. Motivation: Deligne cohomology and Zeta values

Let $\mathcal{X}$ be a regular scheme, proper and flat over $\mathrm{Spec}(\mathbb{Z})$. One has the Beilinson regulator

$$B_n : R\Gamma(\mathcal{X}, \mathbb{Z}(n)) \to R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}}, \mathbb{Z}(n))$$

from motivic cohomology to Deligne cohomology. This map is important in describing Zeta-values. For example:

Theorem 1. *(Borel) Let $\mathcal{X} = \mathrm{Spec}(\mathcal{O}_F)$ for a number field F and $n \geq 2$. Then $\mathrm{coker}(B_n)$ has compact cohomology groups and $\zeta_F(n) \sim_{\mathbb{Q}^\times} \mathrm{covol}(B_n)$.*

In particular one should view $R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{R}}, \mathbb{Z}(n))$ as an object of the derived category of locally compact abelian groups $D^b(\mathrm{LCA})$ [3]. This category was also used in [1], [2] to formulate a conjecture describing $\mathrm{ord}_{s=n}\,\zeta(\mathcal{X}, n)$ and $\zeta^*(\mathcal{X}, n) \in \mathbb{R}^\times/\{\pm 1\}$ for any $\mathcal{X}$ and any $n \in \mathbb{Z}$ (under some assumptions). To make progress on these conjectures one wants to view Deligne cohomology as a variant of motivic cohomology which takes into account the topology of the base field $\mathbb{R}$ or $\mathbb{C}$. The definition of $R\Gamma(\mathcal{X}, \mathbb{Z}(n))$ in terms of algebraic cycles (higher Chow complexes) seems unsuitable for this goal but the K-theoretic definition

$$R\Gamma(\mathcal{X}, \mathbb{Z}(n)) \simeq \mathrm{gr}^n_{Mot} K(\mathcal{X})[-2n]$$

has an analogue for Deligne cohomology. One has

$$R\Gamma_{\mathcal{D}}(\mathcal{X}_{/\mathbb{C}}, \mathbb{Z}(n)) \simeq \mathrm{gr}^n_{Mot} K_{\mathcal{D}}(\mathcal{X}_{/\mathbb{C}})[-2n]$$

where

$$K_{\mathcal{D}}(\mathcal{X}_{/\mathbb{C}}) := K^{top}(\mathcal{X}(\mathbb{C})) \times_{HP(\mathcal{X}_{\mathbb{C}})} HC^-(\mathcal{X}_{\mathbb{C}}).$$

In particular, there is an exact triangle

$$HC(\mathcal{X}_{\mathbb{C}})[1] \to K_{\mathcal{D}}(\mathcal{X}_{/\mathbb{C}}) \to K^{top}(\mathcal{X}(\mathbb{C})). \tag{1}$$

The theory of condensed sets of Clausen and Scholze [4], [5] allows to view $K(\mathcal{X}_{\mathbb{C}})$ as a condensed spectrum whose underlying spectrum is however still the usual algebraic K-theory spectrum. The hope is that a suitably modified K-theory functor will have underlying spectrum $K_{\mathcal{D}}(\mathcal{X}_{/\mathbb{C}})$. This modified functor should take values in the category CCC introduced in the next section and the exact triangle (1) should coincide with (2) in the next section.

2. Complete complexes of condensed abelian groups

Recall the two full reflective stable subcategories of $D\,\mathrm{Cond}(\mathbf{Ab})$ [5]

$$D\mathrm{Liquid} \subset D\,\mathrm{Cond}(\mathbf{Ab}); \quad D\mathrm{Solid} \subset D\,\mathrm{Cond}(\mathbf{Ab})$$

where Liquid is formed with respect to a fixed parameter $0 < p \leq 1$. For the left adjoints $L_\wr$, $L_\blacksquare$ of these two inclusions we shall also use the notations

$$M \hat{\otimes} \mathbb{R} := L_\wr(M); \quad M^\blacksquare := L_\blacksquare(M)$$

respectively. We would like to define a smallest full reflective stable subcategory of $D\,\mathrm{Cond}(\mathbf{Ab})$ containing both DLiquid and DSolid. This category will then also naturally contain $D^b(\mathrm{LCA})$ (note that for example $\mathbb{R}/\mathbb{Z}$ is neither solid nor liquid). Following the localization theory of presentable categories we make the following definition.

Definition 1. *Let*

$$i : \mathrm{CCC} \subseteq D\,\mathrm{Cond}(\mathbf{Ab})$$

be the full subcategory of local objects for the (strongly saturated) class of morphisms $S_\wr \cap S_\blacksquare$ where $S_\wr$, resp. $S_\blacksquare$, is the (strongly saturated) class of morphisms mapped to equivalences under $L_\wr$, resp. $L_\blacksquare$. We call CCC *the ∞-category of* complete complexes of condensed abelian groups*.*

Remark 1. CCC depends on the same parameter $0 < p \leq 1$ as does Liquid which we leave implicit.

Proposition 1.

a) The ∞-category CCC *is stable and has all limits and colimits. The inclusion i has a left adjoint L and preserves limits and colimits.*

b) The stable ∞-category CCC *has a semiorthogonal decomposition*

$$(D\mathrm{Solid}, D\mathrm{Liquid})$$

with associated functorial exact triangle

$$\underline{R\,\mathrm{Hom}}(\mathbb{R}, M) \to M \to M^\blacksquare \to \tag{2}$$

for all objects M of CCC*. Here $\underline{R\,\mathrm{Hom}}$ is the internal Hom in $D\,\mathrm{Cond}(\mathbf{Ab})$. The stable ∞-category* CCC *has another semiorthogonal decomposition*

$$(D\mathrm{Liquid}, D\mathrm{Solid}^c)$$

with associated functorial exact triangle

$$M^c \to M \to M \hat{\otimes} \mathbb{R} \to \tag{3}$$

where $D\mathrm{Solid}^c \subset \mathrm{CCC}$ is a full stable subcategory equivalent but not equal to DSolid.

c) The ∞-category CCC *has a symmetric monoidal structure $\hat{\otimes}$ preserving colimits in both variables, and so that the localization functor*

$$L : D\,\mathrm{Cond}(\mathbf{Ab}) \to \mathrm{CCC}$$

is symmetric monoidal. The inclusion

$$D\mathrm{Liquid} \subset \mathrm{CCC}$$

is symmetric monoidal.

d) The stable ∞-category CCC *has a t-structure so that L is right t-exact, the inclusions*

$$D\mathrm{Liquid} \subset \mathrm{CCC}, \quad D\mathrm{Solid} \subset \mathrm{CCC}$$

are t-exact and $\mathrm{CCC}^{\heartsuit}$ *is generated under colimits and extensions by objects $L\mathbb{Z}[S]$ for S profinite. If $M \in \mathrm{CCC}^{\heartsuit}$ then $\pi_i(M) = 0$ for $i \neq 0, -1$. If $M \in \mathrm{CCC}^{\heartsuit}$ and $\pi_0(M) = 0$ then $M = 0$.*

e) Define

$$\mathrm{CCond}(\mathbf{Ab}) := \mathrm{CCC} \cap \mathrm{Cond}(\mathbf{Ab}).$$

Then $\mathrm{CCond}(\mathbf{Ab})$ *is an additive subcategory of* $\mathrm{CCC}^{\heartsuit}$ *containing* Solid *and* Liquid. *Denoting by π_i^{cc} the homotopy object for the t-structure in d) there is a natural transformation*

$$\eta_i : \pi_i \to \pi_i^{cc}$$

for any $i \in \mathbb{Z}$. For $M \in \mathrm{CCC}$ and $i \in \mathbb{Z}$ the following are equivalent

i) $\pi_i(M) \to \pi_i^{cc}(M)$ *is an equivalence*
ii) $\pi_{i+1}^{cc}(M), \pi_i^{cc}(M) \in \mathrm{CCond}(\mathbf{Ab})$
iii) $\pi_i(M) \in \mathrm{CCond}(\mathbf{Ab})$

Denote by

$$\mathrm{D\,CCond}(\mathbf{Ab}) \subseteq \mathrm{CCC}$$

the full subcategory of objects satisfying i)-iii) for all $i \in \mathbb{Z}$. Then $\mathrm{D\,CCond}(\mathbf{Ab})$ *is an additive subcategory of* CCC *containing* DSolid *and* DLiquid.

f) There are fully faithful embeddings

$$\mathrm{LCA} \subset \mathrm{CCond}(\mathbf{Ab}), \qquad D^b(\mathrm{LCA}) \subset \mathrm{D\,CCond}(\mathbf{Ab})$$

where LCA *denotes the category of locally compact abelian groups and* $D^b(\mathrm{LCA})$ *was defined in* [3]*. If $A^\bullet$ and $B^\bullet$ are bounded complexes of locally compact abelian groups of finite ranks in the sense of* [3]*[Def. 2.6] there is an isomorphism*

$$A^\bullet \hat{\otimes} B^\bullet \xrightarrow{\sim} A^\bullet \otimes^{\mathbb{L}}_{HS} B^\bullet$$

where $\otimes^{\mathbb{L}}_{HS}$ was defined in [3]*[Rem. 4.3]. If $A, B \in \mathrm{LCA}$ have finite ranks and*

$$\pi_0(A \hat{\otimes} B) \in \mathrm{LCA}$$

then $\pi_0(A\hat{\otimes}B) \simeq A \otimes_{HS} B$ where $A \otimes_{HS} B$ denotes the underived tensor product defined in [3]*[Def. 3.13].*

Example 1. *If M is a commutative Lie group with tangent space $T_\infty M$ then (2) canonically identifies with the exponential triangle*

$$T_\infty M \xrightarrow{\exp} M \to M^{\blacksquare} \to$$

associated to M. This follows from the fact that continuous homomorphisms between Lie groups are automatically analytic together with

$$\mathrm{Hom}_{an}(\mathbb{R}, M) \simeq T_\infty M.$$

Moreover, $M^{\blacksquare} \simeq K(\pi_0 M, 0) \times K(\pi_1 M, 1)$ is a perfect complex of abelian groups representing the homotopy type of M under the Dold-Kan equivalence.

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The nonvanishing of Selmer groups of certain symplectic Galois representations

Samuel Mundy

Fix a prime p. For an integer $n \geq 1$, let G_n denote the split orthogonal group $SO(n, n+1)$. Let π be a cuspidal automorphic representation of G_n which is discrete series at infinity. Assuming the validity of the main results of Arthur's book [1], one can attach to π a continuous, semisimple Galois representation $\rho_\pi : G_{\mathbb{Q}} \to GL_{2n}(\overline{\mathbb{Q}}_p)$ such that $\rho_\pi \cong \rho_\pi^\vee(1)$, and such that ρ_π factors through $GSp_{2n}(\overline{\mathbb{Q}}_p)$. Normalized this way, the center of the functional equation for the L-function $L(s, \rho_\pi)$ is $s = 0$.

Write ρ for half the sum of positive roots for G_n (with respect to a fixed pinning), and χ_{π_∞} for (the dominant weight which represents) the infinitesimal character of the archimedean component π_∞ of π. Also write $\lambda_0 = \chi_{\pi_\infty} - \rho$. The following is our main theorem, which depends on [1] in several ways.

Theorem. *Assume that the weight λ_0 is regular, that the order of vanishing $\mathrm{ord}_{s=0} L(s, \rho_\pi)$ is odd, and that π is unramified at p and 2. Then the Bloch–Kato Selmer group $H^1_f(\mathbb{Q}, \rho_\pi)$ is nonzero.*

We remark that this is in accordance with the Bloch–Kato conjectures, which in this case predict that

$$\mathrm{ord}_{s=0} L(s, \rho_\pi) = \dim_{\overline{\mathbb{Q}}_p} H^1_f(\mathbb{Q}, \rho_\pi).$$

We also remark that when $n = 1$, this theorem is a consequence of the work of Skinner–Urban [5], whose general method underlies the proof of our main theorem.

We now discuss some aspects of the proof of this theorem. The main point is to construct congruences between Eisenstein series built from π and cusp forms. To this end, let $M = GL_1 \times G_n$, which is a Levi subgroup of the parabolic subgroup

$$P = \left\{ \begin{pmatrix} t & * & * \\ & g & * \\ & & t^{-1} \end{pmatrix} \middle| \; t \in GL_1,\ g \in G_n \right\}.$$

of G_{n+1}. Let $e_1 : M \to GL_1$ be the projection. Then we let Π be the unique irreducible quotient of the (unitary) parabolic induction

$$\mathrm{Ind}_{P(\mathbb{A})}^{G_{n+1}(\mathbb{A})}(|e_1|^{1/2} \otimes \pi).$$

We would like to p-adically deform Π into a generically cuspidal family of cohomological automorphic representations of G_{n+1}. The cohomological properties of Π itself are described in the theorem below.

For a dominant weight λ, let V_λ be the representation of $G_{n+1}(\mathbb{C})$ of highest weight λ. This determines local systems on the locally symmetric spaces attached to G_{n+1}, and we consider their cohomology now. Let K_∞ denote a maximal compact subgroup of $G_{n+1}(\mathbb{R})$, and let

$$H^i(\widetilde{X}, \lambda) = \varinjlim_{\substack{K_f \subset G_{n+1}(\mathbb{A}_f) \\ \text{open compact}}} H^i(G_{n+1}(\mathbb{Q}) \backslash G_{n+1}(\mathbb{A}) / K_f K_\infty),$$

which is naturally a $G_{n+1}(\mathbb{A}_f)$-module.

Theorem. *If $\mathrm{ord}_{s=0} L(s, \rho_\pi)$ is odd, then the finite part Π_f of Π occurs in $H^i(\widetilde{X}, \lambda_0)$ N times, where $N = \binom{n+1}{n/2}$ if n is even, and $N = \binom{n+1}{(n+1)/2}$ if n is odd.*

We now describe the p-adic deformation. Fix an embedding $\overline{\mathbb{Q}}_p \subset \mathbb{C}$. Let $K_f^p \subset G_{n+1}(\mathbb{A}_f^p)$ be an open compact subgroup such that Π_f has fixed vectors under $K_f^p G_{n+1}(\mathbb{Z}_p)$. We define $\mathcal{H}^p = C_c^\infty(K_f^p \backslash G_{n+1}(\mathbb{A}_f)/K_f^p, \mathbb{Q}_p)$. Let $\mathcal{U}_p$ be the commutative algebra of U_p-operators for $G_{n+1}(\mathbb{Q}_p)$, and define $\mathcal{H} = \mathcal{H}^p \otimes_{\mathbb{Z}_p} \mathcal{U}_p$. We single out a particular critical p-stabilization Π_f^{crit} of Π_f. Then using the machinery of Urban's eigenvariety [6] we prove the following.

Theorem. *There is an affinoid rigid space $\mathfrak{X}$ over $\mathbb{Q}_p$ which is finite over a neighborhood of λ_0 in the $(n+1)$-dimensional weight space of G_{n+1}, and a $\mathbb{Q}_p$-linear map $J_{\mathfrak{X}} : \mathcal{H} \to \mathcal{O}(\mathfrak{X})$ satisfying the following two properties: First, for general points $x \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$ lying above a regular classical weight λ, we have that $x \circ J_{\mathfrak{X}}$ is an irreducible summand of the trace of $\mathcal{H}$ on cuspidal cohomology, $\mathrm{tr}(\cdot | H^*_{cusp}(\widetilde{X}, \lambda))$; second, there is a point $x_0 \in \mathfrak{X}(\overline{\mathbb{Q}}_p)$ above λ_0 such that $x_0 \circ J_{\mathfrak{X}}$ contains $\mathrm{tr}(\cdot | \Pi_f^{crit})$ as a summand.*

Using this we construct a Galois representation $\rho_{\mathfrak{X}} : G_{\mathbb{Q}} \to GL_{2n+2}(\mathrm{Frac}(\mathcal{O}(\mathfrak{X})))$ interpolating those attached to the representations whose traces are given by $x \circ J_{\mathfrak{X}}$

for x over general regular classical weights λ. One constructs a lattice $\mathcal{L}$ in $\rho_{\mathfrak{X}}$ such that $x_0 \circ \mathcal{L}$ has unique irreducible quotient ρ_π. One has either

$$\mathcal{L} \sim \begin{pmatrix} 1 & * & * \\ & \chi_{\mathrm{cyc}} & *_1 \\ & & \rho_\pi \end{pmatrix}, \qquad \text{or} \qquad \mathcal{L} \sim \begin{pmatrix} \chi_{\mathrm{cyc}} & *_2 & * \\ & 1 & * \\ & & \rho_\pi \end{pmatrix},$$

with $*_1$ nontrivial. We rule out the second case, in which one can show that $*_2$ is nontrivial; checking Selmer conditions (more on this below) shows that $0 \neq *_2 \in H^1_f(\mathbb{Q}, \overline{\mathbb{Q}}_p(1)) = 0$, which is a contradiction. Thus we are in the first case, and $*_1$ gives a nontrivial class in $H^1_f(\mathbb{Q}, \rho_\pi^\vee(1)) = H^1_f(\mathbb{Q}, \rho_\pi)$.

For ℓ a bad prime for π, let us write (r_0, N_0) for the Weil–Deligne representation attached to $\rho_\pi|_{G_{\mathbb{Q}_\ell}}$, and for x over general regular classical weights λ, write (r, N) for the Weil–Deligne representation attached to $x \circ \mathcal{L}$. Then $0 \oplus 0 \oplus N_0 \prec N$ for x near x_0. (The relation $N_1 \prec N_2$ means that the Zariski closure of the adjoint orbit of N_2 contains N_1). To check the Selmer conditions of $*_1$ or $*_2$ at ℓ, it suffices to show that $0 \oplus 0 \oplus N_0 \sim N$ (that is, they are conjugate by a matrix in GL_{2n+2}). Assuming otherwise for sake of contradiction, then using work of Atobe [2], building on that of Moeglin, which gives explicit descriptions in terms of Langlands parameters of the Jacquet modules of the ℓ-component $\Pi_{x,\ell}$ of the representation whose trace is $x \circ J_{\mathfrak{X}}$, we show that $\mathrm{tr}(\cdot|\,\mathrm{Jac}_{L(\mathbb{Q}_\ell)}(\Pi_\ell))$ contains a constituent which $\lim_{x\to x_0} \mathrm{tr}(\cdot|Jac_{L(\mathbb{Q}_\ell)}(\Pi_{x,\ell}))$ does not; here, L is the Levi of G_{n+1} which supports a supercuspidal, call it σ, which supports both Π_ℓ and $\Pi_{x,\ell}$, and the traces are taken for Hecke operators in the Hecke algebra $\mathcal{H}(\sigma)$ associated with a type for σ. Such types exists (if $\ell \neq 2$) by Miyauchi–Stevens [4].

In fact, Miyauchi–Stevens construct types for $G_{n+1}(\mathbb{Q}_\ell)$ which are covers of those of $L(\mathbb{Q}_\ell)$. Let $\widetilde{\mathcal{H}}(\sigma)$ be the Hecke algebra associated with a type for G_{n+1} which covers σ. By Bushnell–Kutzko [3], there is a map $t : \mathcal{H}(\sigma) \to \widetilde{\mathcal{H}}(\sigma)$ such that

$$\mathrm{tr}(f|\,\mathrm{Jac}_{L(\mathbb{Q}_\ell)}(\Pi_{x,\ell})) = \mathrm{tr}(t(f)|\Pi_{x,\ell})$$

for any $f \in \mathcal{H}(\sigma)$, and similarly for Π_ℓ. But the above theorem implies that the trace of such operators $t(f)$ on $\Pi_{x,\ell}$ are analytic in x, which gives the desired contradiction.

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Rational p-adic Hodge theory for rigid-analytic varieties

Guido Bosco

Let p be a prime number. In this talk, we explained how one can use the condensed and solid formalisms, recently developed by Clausen–Scholze, to study the rational p-adic Hodge theory of general rigid-analytic varieties, without properness or smoothness assumptions. The study of this subject for varieties that are not necessarily proper is in part motivated by the desire of finding a geometric incarnation of the p-adic Langlands correspondence in the p-adic cohomology of local Shimura varieties.

Let us begin by observing that the p-adic (pro-)étale, de Rham, etc. cohomology groups of non-proper rigid-analytic varieties are usually huge. Therefore, it becomes important to exploit the topological structure that such cohomology groups carry in order to study them. But, in doing so, one quickly runs into several topological issues, mainly due to the fact that the category of topological abelian groups is *not* abelian. On the other hand, the category of *condensed abelian groups* is a nice abelian category, containing most topological abelian groups of interest as well as new objects that were invisible in the topological world; moreover, Clausen–Scholze defined a full abelian subcategory of condensed abelian groups, called the category of *solid abelian groups*, which play the role of "complete modules", [4].

Thus, to state our first main result, given a commutative solid ring A, we denote by $\mathrm{Mod}_A^{\mathrm{solid}}$ the symmetric monoidal category of A-modules in solid abelian groups, endowed with the solid tensor product $\otimes^{\blacksquare}_A$, and we write $D(\mathrm{Mod}_A^{\mathrm{solid}})$ for the associated derived ∞-category.

We denote by K a complete discretely valued non-archimedean extension of $\mathbb{Q}_p$ with perfect residue field. We fix an algebraic closure $\overline{K}$ of K and we let $\mathcal{G}_K := \mathrm{Gal}(\overline{K}/K)$ denote the absolute Galois group of K. We denote by $C := \widehat{\overline{K}}$ the completion of $\overline{K}$, by $\mathcal{O}_C$ its ring of integers, and by k_C its residue field. We let $A_{\mathrm{inf}} := W(\mathcal{O}_C^\flat)$, we denote by

$$Y_{\mathrm{FF}} := \mathrm{Spa}(A_{\mathrm{inf}}, A_{\mathrm{inf}}) \setminus V(p[p^\flat])$$

the mixed characteristic punctured open unit disk, and we write B for the condensed ring of analytic functions on Y_{FF}.

In the following theorem, which relies on results in p-adic Hodge theory pioneered by Bhatt–Morrow–Scholze [1] (in particular, on results of Le Bras, [11], and Česnavičius–Koshikawa, [3]), we state the existence of a cohomology theory for rigid-analytic varieties over C, which interpolates between other rational p-adic cohomology theories for such varieties — namely, the rational p-adic pro-étale cohomology, the Hyodo–Kato cohomology defined by Colmez-Nizioł, [6, §4] (a refinement of the de Rham cohomology endowed with a (φ, N)-module structure), and the infinitesimal cohomology over B^+_{dR}, [1, §13], [9] (a deformation of the de Rham cohomology along the pro-infinitesimal thickening $B^+_{\mathrm{dR}} \to C$).

Theorem 1 ([2], [11]). *There exists a cohomology theory for rigid-analytic varieties X over C*

$$R\Gamma_B(X)$$

called B-cohomology, *taking values in* $D(\mathrm{Mod}_B^{\mathrm{solid}})$, *endowed with a filtration*

$$\mathrm{Fil}^\star R\Gamma_B(X)$$

and a φ_B*-semilinear automorphism*

$$\varphi : R\Gamma_B(X) \to R\Gamma_B(X)$$

preserving $\mathrm{Fil}^\star$, *and satisfying the following comparison results.*

(1) (Pro-étale comparison) *Let* $i \geq 0$. *Define the* syntomic Fargues–Fontaine cohomology of X with coefficients in $\mathbb{Q}_p(i)$ *as the complex of* $D(\mathrm{Mod}_{\mathbb{Q}_p}^{\mathrm{solid}})$

$$R\Gamma_{\mathrm{syn,FF}}(X, \mathbb{Q}_p(i)) := \mathrm{Fil}^i R\Gamma_B(X)^{\varphi = p^i}.$$

We have a natural isomorphism in $D(\mathrm{Mod}_{\mathbb{Q}_p}^{\mathrm{solid}})$

$$\tau^{\leq i} R\Gamma_{\mathrm{syn,FF}}(X, \mathbb{Q}_p(i)) \xrightarrow{\sim} \tau^{\leq i} R\Gamma_{\mathrm{pro\acute{e}t}}(X, \mathbb{Q}_p(i)).$$

(2) (Hyodo–Kato comparison) *Assume X connected and paracompact. We have a natural isomorphism in* $D(\mathrm{Mod}_B^{\mathrm{solid}})$

$$R\Gamma_B(X) \simeq (R\Gamma_{\mathrm{HK}}(X) \otimes^{\mathrm{L}\blacksquare}_{\breve{C}} B_{\log})^{N=0}$$

compatible with the action of φ, and the action of $\mathcal{G}_K$ in the case when X is the base change to C of a rigid-analytic variety over K. Here, we denote $\breve{C} := W(k_C)_{\mathbb{Q}_p}$, we write $B_{\log}$ for the log-crystalline condensed period ring (endowed with a $(\varphi, N, \mathcal{G}_K)$-module structure satisfying $B_{\log}^{N=0} = B$), and we denote by

$$R\Gamma_{\mathrm{HK}}(X) \in D_{(\varphi,N)}(\mathrm{Mod}_{\breve{C}}^{\mathrm{solid}})$$

the Hyodo–Kato cohomology of X, satisfying éh*-descent and*

$$R\Gamma_{\mathrm{HK}}(X) = R\Gamma_{\mathrm{cris}}(\mathfrak{X}_{\mathcal{O}_C/p}/W(k_C)^0)_{\mathbb{Q}_p}$$

in the case X has a semistable formal model $\mathfrak{X}$ over $\mathcal{O}_C$.

Moreover, we have a natural isomorphism in $D(\mathrm{Mod}_{B_{\mathrm{dR}}^+}^{\mathrm{solid}})$

$$R\Gamma_{\mathrm{HK}}(X) \otimes^{\mathrm{L}\blacksquare}_{\breve{C}} B_{\mathrm{dR}}^+ \simeq R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+)$$

where the right hand side denotes the infinitesimal cohomology over B_{dR}^+.

(3) (De Rham comparison) *Assume X as in (2). We have a natural isomorphism in* $D(\mathrm{Mod}_{B_{\mathrm{dR}}^+}^{\mathrm{solid}})$

$$R\Gamma_B(X) \otimes^{\mathrm{L}\blacksquare}_{B} B_{\mathrm{dR}}^+ \simeq R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+)$$

compatible with the Hyodo-Kato comparison.

If $X = X_0 \widehat{\otimes}_K C$ with X_0/K, we have a natural isomorphism in $D(\mathrm{Mod}_{B_{\mathrm{dR}}^+}^{\mathrm{solid}})$

$$R\Gamma_{\mathrm{inf}}(X/B_{\mathrm{dR}}^+) \simeq R\Gamma_{\mathrm{dR}}(X_0) \otimes^{\mathrm{L}\blacksquare}_{K} B_{\mathrm{dR}}^+$$

compatible with the action of $\mathcal{G}_K$*, where* $R\Gamma_{\mathrm{dR}}(X_0)$ *denotes the* éh*-de Rham cohomology of* X_0*,* [8] *(agreeing with the usual de Rham cohomology in the smooth case).*

We emphasize that the definition of the B-cohomology theory, for which we refer the reader to [2], is purely in terms of X and it is global in nature, contrary to the definition of the Hyodo–Kato cohomology which is first defined in the semistable reduction case and then globalized (using that locally for the éh-topology X has a semistable formal model, thanks to the alterations of Hartl and Temkin).

Moreover, we remark that Theorem 1 can be reinterpreted in terms of the Fargues–Fontaine curve $\mathrm{FF} = Y_{\mathrm{FF}}/\varphi^{\mathbb{Z}}$, thus giving a cohomology theory for rigid-analytic varieties over C, taking values in filtered solid quasi-coherent complexes over the Fargues–Fontaine curve, which compares to the rational p-adic pro-étale cohomology, the Hyodo–Kato cohomology, and the infinitesimal cohomology over B_{dR}^+.

As a corollary of Theorem 1, we obtain the following result, expressing the rational p-adic pro-étale cohomology in terms of de Rham data. For smooth rigid-analytic varieties, a similar result was obtained by Colmez–Nizioł, [6, Theorem 1.1], by a different method, namely via the syntomic cohomology of Fontaine–Messing combined with alterations.

Theorem 2. *Let* X *be a connected, paracompact, rigid-analytic variety defined over* K*. For any* $i \geq 0$*, we have a* $\mathcal{G}_K$*-equivariant isomorphism in* $D(\mathrm{Mod}^{\mathrm{solid}}_{\mathbb{Q}_p})$

$$\begin{aligned}&\tau^{\leq i} R\Gamma_{\mathrm{pro\acute{e}t}}(X_C, \mathbb{Q}_p(i))\\ &\simeq \tau^{\leq i}\,\mathrm{fib}((R\Gamma_{\mathrm{HK}}(X_C) \otimes^{\mathrm{L}\blacksquare}_{\breve{C}} B_{\log})^{N=0,\varphi=p^i} \to (R\Gamma_{\mathrm{dR}}(X) \otimes^{\mathrm{L}\blacksquare}_{K} B^+_{\mathrm{dR}})/\operatorname{Fil}^i).\end{aligned}$$

We remark that Theorem 2 can be extended to the case X is defined over C, using the infinitesimal cohomology over B_{dR}^+.

From such general derived comparison results, one can deduce in some special cases a refined description of the single rational p-adic (pro-)étale cohomology groups in terms of de Rham data. For X a proper (possibly singular) rigid-analytic variety over C, we prove in [2] a version of the semistable conjecture for X; in the case when X is the base change to C of a rigid-analytic variety X_0 defined over K, this result relies on the degeneration at the first page of the Hodge-de Rham spectral sequence associated to X_0 ([12, Corollary 1.8], [8, Proposition 8.0.8]). Another case in which the Hodge-de Rham spectral sequence simplifies is for smooth Stein spaces, thanks to Kiehl's acyclicity theorem. In this case, we show the following theorem which reproves results of Colmez–Dospinescu–Nizioł [5] (in the semistable reduction case) and Colmez–Nizioł [7].

Theorem 3. *Let* X *be a smooth Stein space over* C*. For any* $i \geq 0$*, we have a short exact sequence in* $\mathrm{Mod}^{\mathrm{solid}}_{\mathbb{Q}_p}$

$$0 \to \Omega^{i-1}(X)/\ker d \to H^i_{\mathrm{pro\acute{e}t}}(X, \mathbb{Q}_p(i)) \to (H^i_{\mathrm{HK}}(X) \otimes^{\blacksquare}_{\breve{C}} B_{\log})^{N=0,\varphi=p^i} \to 0.$$

A recent conjecture of Hansen, [10, Conjecture 1.10], suggests that any local Shimura variety is a Stein space, therefore Theorem 3 potentially applies to any such variety.

As a final curiosity, we note that for smooth affinoid rigid spaces, the Hodge-de Rham spectral sequence simplifies similarly to smooth Stein spaces, thanks to Tate's acyclicity theorem. Using this, we prove in [2] a version of Theorem 3 for smooth affinoid rigid spaces over C of dimension 1; we observe that, in this case, the de Rham and Hyodo–Kato cohomology groups in the condensed world are examples of new objects that were invisible in the topological world.

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Duality for p-adic pro-étale cohomology of analytic varieties

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(joint work with Pierre Colmez, Sally Gilles)

Let p be a prime. Let K be a finite extension of $\mathbf{Q}_p$. Let $\overline{K}$ be an algebraic closure of K and let $C = \widehat{\overline{K}}$ be its p-adic completion; let $\mathcal{G}_K = \mathrm{Gal}(\overline{K}/K)$. Or analytic varieties are separated.

1. Arithmetic duality

We have just finished writing a proof of the following result.

Theorem 1. (Poincaré duality for curves) *Let X be a smooth, geometrically irreducible, dagger variety of dimension 1 over K. Then:*

(1) *There exists a natural trace map of solid $\mathbf{Q}_p$-vector spaces*

$$\mathrm{Tr}_X : H^4_{\text{proét},c}(X, \mathbf{Q}_p(2)) \xrightarrow{\sim} \mathbf{Q}_p.$$

(2) *Let $i, j \in \mathbb{Z}$. The pairing*

$$H^i_{\text{proét}}(X, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} H^{4-i}_{\text{proét},c}(X, \mathbf{Q}_p(2-j)) \xrightarrow{\cup} H^4_{\text{proét},c}(X, \mathbf{Q}_p(2)) \xrightarrow{\mathrm{Tr}_X} \mathbf{Q}_p[-4]$$

is perfect, i.e., it induces isomorphisms

$$\gamma_{X,i} : \quad H^i_{\text{proét}}(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} H^{4-i}_{\text{proét},c}(X, \mathbf{Q}_p(2-j))^*,$$
$$\gamma^c_{X,i} : \quad H^i_{\text{proét},c}(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} H^{4-i}_{\text{proét}}(X, \mathbf{Q}_p(2-j))^*,$$

where $(-)^* := \underline{\mathrm{Hom}}_{\mathbf{Q}_p}(-, \mathbf{Q}_p)$.

Remark 1. (i) If X is a Stein variety over K with an exhaustive covering by affinoids $\{U_n\}_{n\in\mathbf{N}}$, $U_n \Subset U_{n+1}$, then the compactly supported pro-étale cohomology is defined as

$$\mathrm{R}\Gamma_{\text{proét},c}(X, \mathbf{Q}_p(j)) := \mathrm{fib}(\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(j)) \to \mathrm{R}\Gamma_{\text{proét}}(\partial X, \mathbf{Q}_p(j))),$$
$$\mathrm{R}\Gamma_{\text{proét}}(\partial X, \mathbf{Q}_p(j)) := \mathrm{colim}_n \mathrm{R}\Gamma_{\text{proét}}(X \setminus U_n, \mathbf{Q}_p(j)).$$

If X is of dimension 1 then $\mathrm{R}\Gamma_{\text{proét},c}(X, \mathbf{Q}_p)$ is the same as Huber's étale cohomology with compact support. We think that this is true in any dimension.

(ii) Let X be a smooth dagger variety over K, of dimension 1. Then if X is proper the pro-étale cohomology groups are finite; if X is Stein, $H^i_{\text{proét}}(X, \mathbf{Q}_p(j))$ is nuclear Fréchet and $H^i_{\text{proét},c}(X, \mathbf{Q}_p(j))$ is of compact type; if X is a dagger affinoid then it is the opposite.

(iii) If X is Stein, we actually prove a derived duality in $\mathcal{D}(\mathbb{Q}_{p,\square})$, i.e., the cup product pairing gives a natural quasi-isomorphism

$$\gamma_X : \mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} \mathbb{D}(\mathrm{R}\Gamma_{\text{proét},c}(X, \mathbf{Q}_p(2-j))[4]),$$

where $\mathbb{D}(-, \mathbf{Q}_p) := \mathrm{R}\underline{\mathrm{Hom}}_{\mathbf{Q}_p}(-, \mathbf{Q}_p)$.

Having that, we get the quasi-isomorphism $\gamma_{X,i}$ above by taking cohomology and using the fact that $\underline{\mathrm{Ext}}^s(H^i_{\text{proét},c}(X, \mathbf{Q}_p(j)), \mathbf{Q}_p) = 0$, $s \geq 1$, because $H^i_{\text{proét},c}(X, \mathbf{Q}_p(j))$ is of compact type (hence a colimit of Smith spaces, which are projective solid objects). The quasi-isomorphism $\gamma^c_{X,i}$ follows because $H^i_{\text{proét},c}(X, \mathbf{Q}_p(j))$ is reflexive (in the classical world, in fact).

(iv) In higher dimensions we venture the following conjecture:

Conjecture 1. *Let X be a smooth, geometrically irreducible, Stein variety of dimension d over K. Then:*

(1) *The cohomology groups $H^i_{\text{proét}}(X, \mathbf{Q}_p(j))$ and $H^i_{\text{proét},c}(X, \mathbf{Q}_p(j))$ are nuclear Fréchet and of compact type, respectively.*

(2) *There exists a natural trace map of solid $\mathbf{Q}_p$-vector spaces*

$$\mathrm{Tr}_X : H^{2d+2}_{\text{proét},c}(X, \mathbf{Q}_p(d+1)) \xrightarrow{\sim} \mathbf{Q}_p.$$

(3) *The pairing*

$$\begin{aligned} &\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(j)) \otimes^{\square}_{\mathbf{Q}_p} \mathrm{R}\Gamma_{\text{proét},c}(X, \mathbf{Q}_p(d+1-j)) \\ &\xrightarrow{\cup} \mathrm{R}\Gamma_{\text{proét},c}(X, \mathbf{Q}_p(d+1)) \xrightarrow{\mathrm{Tr}_X} \mathbf{Q}_p[-2d-2] \end{aligned}$$

is perfect, i.e., it induces (quasi-)isomorphisms

$$\begin{aligned} &\gamma_X : \mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} \mathbb{D}(\mathrm{R}\Gamma_{\text{proét},c}(X, \mathbf{Q}_p(d+1-j))[2d+2]), \\ &\gamma_{X,i} : \quad H^i_{\text{proét}}(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} H^{2d+2-i}_{\text{proét},c}(X, \mathbf{Q}_p(d+1-j))^*, \\ &\gamma^c_{X,i} : \quad H^i_{\text{proét},c}(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} H^{2d+2-i}_{\text{proét}}(X, \mathbf{Q}_p(d+1-j))^*. \end{aligned} \tag{1}$$

We note that the duality (quasi-)isomorphisms (1) hold for X proper, smooth, and algebraic by Galois descent from the geometric Poincaré duality (proved by Mann and Zavyalov).

(v) The starting point of our work on arithmetic dualities was the following computation:

Example 1. *Let $X = D$ be the open unit disc. The we compute (noncanonically)*

$$\begin{aligned} H^1_{\text{proét}}(X, \mathbf{Q}_p(1)) &\simeq (\mathcal{O}(D)/K) \oplus H^1(\mathcal{G}_K, \mathbf{Q}_p(1)), \\ H^3_{\text{proét},c}(X, \mathbf{Q}_p(1)) &\simeq \mathcal{O}(\partial D)/\mathcal{O}(D) \oplus H^1(\mathcal{G}_K, \mathbf{Q}_p). \end{aligned}$$

These groups are dual via the Galois and coherent duality:

$$\begin{aligned} H^i(\mathcal{G}_K, \mathbf{Q}_p) &\simeq H^{2-i}(\mathcal{G}_K, \mathbf{Q}_p(1))^*, \\ H^0(D, \Omega^1_D) &\simeq H^1_c(D, \mathcal{O}_D)^*. \end{aligned}$$

We used here that $\mathcal{O}(D)/K \xrightarrow{\sim} H^0(D, \Omega^1_D)$, $\mathcal{O}(\partial D)/\mathcal{O}_D \xrightarrow{\sim} H^1_c(D, \mathcal{O}_D)$. We note that the coherent duality is a K-duality, which can be transformed into $\mathbf{Q}_p$-duality because $[K : \mathbf{Q}_p] < \infty$.

(vi) **Solid versus classical functional analysis**. Most of our work could be done in the set-up of classical functional analysis. We had to pass to the solid formalism because (a) we needed a well-behaved derived dual (b) we needed topological Hochschild-Serre spectral sequences.

2. Geometric duality

Our work on geometric dualities is still in progress. We are writing down a proof of the following result (which works in any dimension):

Theorem 2. (Verdier Duality) *Let X be a smooth, Stein rigid analytic variety over C, connected, dimension d. Then there is a natural quasi-isomorphism*

$$\mathrm{R}\Gamma_{\text{proét}}(X, \mathbf{Q}_p(j)) \xrightarrow{\sim} \mathrm{R}\operatorname{Hom}_{VS}(\mathrm{R}\Gamma_{\text{proét},c}(X, \mathbf{Q}_p(d+1-j))[2d], \mathbf{Q}_p(1)),$$

where VS is the category of solid Vector Spaces, i.e., v-sheaves of solid $\mathbf{Q}_p$-vector spaces on Perf_C.

The strategy is to pass to syntomic cohomology (via a geometric version of a comparison theorem), represent syntomic cohomology via a complex of solid quasi-coherent sheaves on the Fargues-Fontaine curve, prove a Poincaré duality for this complex, and then project it down to the VS category. The Poincaré duality on the curve reduces to Hyodo-Kato duality on the whole curve and $\mathbf{B}_{\mathrm{dR}}^{+}$-duality at infinity (both of which are known). The functional analytic problems can be solved because all the infinite data "come from the base" and can be "taken out" via a projection formula.

Remark 2. It is likely that Conjecture 1 will follow from the above theorem via Galois descent (as is the classical algebraic case)

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