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Geometric Spectral Theory

Organized by
Asma Hassannezhad, Bristol
Luc Hillairet, Orléans
Konstantin Pankrashkin, Oldenburg
Iosif Polterovich, Montréal

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Abstract. Spectral geometry is a rapidly developing field with new classes of operators, boundary value problems and geometric objects arising in different applications. At the same time, classical problems continue gaining novel flavors. The main focus of the workshop was on some of the most significant recent developments in geometric spectral theory including geometry of eigenvalues and eigenfunctions, singular spectral problems, and spectral optimization. The talks were complemented by three thematic open problem sessions on the main topics of the meeting.

Mathematics Subject Classification (2020): 58J50, 35J15, 35P20, 49M41, 53C43, 58J32.

Introduction by the Organizers

Spectral theory of partial differential equations is one of the most active areas of modern mathematics. It has numerous applications to other subjects, such as differential geometry, functional analysis, and number theory, as well as to applied mathematics and mathematical physics. A central object of investigation in spectral theory is the eigenvalue problem for the Laplace operator on a Riemannian manifold or a Euclidean domain with suitable boundary conditions. The origins of this problem go back to the celebrated Chladni experiments with vibrating plates and the works of Lord Rayleigh on the theory of sound. Later on it became the focus of spectral geometry, which explores the links between the eigenvalues and eigenfunctions of various operators and geometric properties of the underlying manifolds or domains. There has been a lot recent advances in geometric specral theory. In particular, new topics have emerged, and, moreover, new methods have arisen in the study of long standing open questions, leading to some spectacular breakthroughs.

The workshop was attended by 48 researchers in person and a few online participants. It brought together leading experts as well as promising junior mathematicians working in various domains of analysis and geometry which are related to geometric spectral theory. The scientific program of the workshop consisted of twenty-three 40-minute talks as well as five shorter talks. It allowed many participants, including the younger ones to give an account on their latest reasearch. We also had a few informal evening talks, as well as three thematic open problem sessions. A wide variety of topics has been covered reflecting the vitality of the field, including the latest developments.

The meeting started with two talks featuring different spectral aspects of the magnetic Laplacian. Bernard Helffer spoke about flux and symmetry effects on quantum tunneling. Bruno Colbois shed light on some new isoperimetric inequalities and geometric bounds for the eigenvalues of the magnetic Laplacian on planar domains.

Eigenvalue optimization and connections to questions in geometric analysis have been in the focus of several talks. Mikhail Karpukhin and Antoine Métras presented recent developments on optimization for the Dirac eigenvalues on surfaces and a newly discovered link between this problem and harmonic maps to complex projective spaces. Optimization of spectral quantities have been also discussed in the talk by Rupert Frank in the context of the celebrated Lieb-Thirring inequalities. Romain Petrides spoke about maximization of functionals depending on Steklov eigenvalues and connections to the existence of non-planar free boundary minimal disks in ellipsoids, which was a well-known open problem in the theory of free boundary minimal surfaces. Cristina Trombetti gave an overview of rearrangement techniques and their applications to shape optimization problems.

Richard Laugesen presented some recent results and open problems on Neumann and Robin eigenvalues, including latest progress on optimization of the second Neumann eigenvalue on subdomains of a 2-sphere with an area constraint. Dorin Bucur's talk was concerned with sharp quantitative spectral stability results for Dirichlet eigenvalues. In particular, for domains close to a ball, sharp estimates on the difference between the k-th Dirichlet eigenvalues of the domain and the ball were obtained in terms of the difference between the corresponding first Dirichlet eigenvalues.

Further recent advances on the geometry of eigenvalues have been discussed during the meeting. Kei Funano presented a new universal inequality between consecutive Neumann eigenvalues of a convex Euclidean domain. Alexandre Girouard talked about new techniques of constructing Riemannian manifolds with boundary with a large Steklov spectral gap. Jean Lagacé presented some new techniques on the study of the Steklov spectrum using homogenization theory. Numerical aspects of high-accuracy computations of the Steklov spectra were discussed in the talk by Nilima Nigam. Michael Levitin reported on a recent solution of the Pólya eigenvalue problem for Euclidean balls and and connections to some new estimates on Bessel zeros. The talk of Chris Judge was concerned with new results on the generic simplicity of spectrum. In particular, he showed that the Dirichlet spectrum of an ellipse is generically simple.

Geometric properties of eigenfunctions have been another important theme of the meeting. The talks by Svitlana Mayboroda and Michiel van den Berg were concerned with localization phenomena for eigenfunctions and its connections to the landscape (also known as torsion) function of the domain. Stefan Steinerberger discussed growth of eigenfunctions, in particular, the "spooky action at distance" phenomenon that links a higher than usual growth of an eigenfunction with the existence of pairs of points on the manifold on which the values of the eigenfunction appear to be correlated.

Philippe Charron discussed some major progress on a well-known problem concerning the lower bound on the inradius of nodal domains of Laplace eigenfunctions on higher dimensional Riemannian manifolds. The talks of Sarah Farinelli and David Sher focused on recent developments around Pleijel-type nodal domain theorems, respectively, in the non-smooth setting and in the context of the Robin eigenvalue problem.

Recent progress on spectral asymptotics of different kind has been discussed by several speakers. Luca Rizzi presented new results on the Weyl-type asympotics on singular manifolds, including those having infinite volume and unbounded curvature. Leonid Parnovski discussed a recent proof of the existence of a full asymptotic expansion for the local density of states of a one-dimensional Schrödinger operator with a uniformly smoothly bounded potential. The proof combined classical wave methods with modern gauge transform techniques. Katie Gittins and Marco Vogel talked on different asymptotic questions arising in the study of the heat content and of the Robin eigenvalues on singular domains, respectively.

The talks by Ksenia Fedosova, Daniel Grieser and Vladimir Lotoreichik featured some of the emerging directions in the field, including connections between inhomogeneous Laplace equations and string theory, constructions of quasi-modes of generalized semi-classical operators and existence of embedded eigenvalues in quasi-conical domains.

In addition to the scheduled talks, there were also interesting informal short talks in evening, given by Clara Aldana, Dean Baskin and Sugata Mondal. Last but not least, there were three open problem sessions: on emerging topics (organized by Pavel Exner), on spectral optimization (organized by Pedro Freitas) and on eigenfunctions (organized by Dan Mangoubi). These sessions attracted a lot of attention and motivated a number of fruitful discussions, which occupied the free evenings and the free afternoon on Wednesday in an efficient way. Many speakers also included in their talks some either long-standing, or more recent open problems. A summary of most important open problems that were presented at this conference is included in the present report.

Overall, the workshop was a huge success and a unique opportunity for researchers in different areas of geometric spectral theory to exchange ideas and explore new research venues.

On behalf of all participants, the organizers would like to thank the staff of MFO for the remarkable support in all possible logistical and technical questions. The wonderful atmosphere and the facilities of the MFO were greatly appreciated by everyone.

Workshop: Geometric Spectral Theory

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Abstracts

Geometric aspects of the ground state of magnetic Laplacians on domains of the plane

Bruno Colbois
(joint work with C. Léna, L. Provenzano and A. Savo)

The *magnetic Laplacian* Δ_A is a perturbation of the usual Laplacian Δ by a smooth potential 1-form A. It is defined on a Riemannian manifold M and its formal expression:

$$\Delta_A u = \Delta u + |A|^2 u + 2i\langle \nabla u, A\rangle + iu\,\mathrm{div}\, A$$

acting on smooth complex functions. Here, Δ denotes the usual Laplacian.

We will consider the case of smooth bounded domains in the plane, with two magnetic potentials:

(1) The magnetic potential

$$A = \frac{\beta}{2}(-x_2 dx_1 + x_1 dx_2).$$

with β a positive number, intensity of the field (if $\beta = 0$, Δ_A is the usual Laplacian Δ). We get a family of operators depending on the parameter $\beta \in [0, \infty[$.

As $dA = \beta dx_1 \wedge dx_2$, we say that the magnetic field has constant curvature β.

(2) The *Aharonov-Bohm potential.* Let $p = (a_1, a_2) \in \mathbb{R}^2$. Consider the magnetic potential $A = A_{p,\nu} = \nu A_p$ with

$$A_p = -\frac{(x_2 - a_2)dx_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2} + \frac{(x_1 - a_1)dx_2}{(x_1 - a_1)^2 + (x_2 - a_2)^2}$$

The potential A has a pole at the point p. We have $dA = 0$, so that the curvature of the magnetic field is 0. One get a family of operators depending on the parameter $\nu \in \mathbb{R}$.

Let Ω be a smooth, connected and bounded domain of $\mathbb{R}^2$. We consider the eigenvalue problem for the magnetic Laplacian in Ω with the magnetic Neumann boundary condition:

$$\begin{cases} \Delta_A u = \lambda u\,, & \text{in } \Omega, \\ \langle \nabla u - iAu, N\rangle = 0\,, & \text{on } \partial\Omega. \end{cases} \tag{1}$$

N outer unit normal to $\partial\Omega$. The expression $\nabla^A u = \nabla u - iAu$ is called the magnetic gradient.

As in the case of the Laplacian, a natural question is to study the relation between the first eigenvalue and the geometry of the domain. The first eigenvalue for a domain Ω (the ground state) will be denoted by $\lambda_1^N(\Omega, \beta)$ in case (1) and $\lambda_1^N(\Omega, A_{p,\nu})$ i case (2). In case (1), we have $\lambda_1^N(\Omega, \beta) > 0$ if $\beta > 0$ and in case (2) if $\nu \notin \mathbb{Z}$.

Theorem 1. [2] *Let Ω be a smooth bounded domain in $\mathbb{R}^2$ and $A_{p,\nu}$ be the Aharonov-Bohm potential with pole at p and flux ν. Let Ω^* the disc centered at p such that $|\Omega^*| = |\Omega|$. Then*

$$\lambda_1^N(\Omega, A_{p,\nu}) \leq \lambda_1^N(\Omega^*, A_{p,\nu}).$$

If $\nu \notin \mathbb{Z}$, the equality holds if and only if $\Omega = \Omega^*$

This result leads to natural questions:

Open question: what about the second eigenvalue? For the usual Laplacian, the Szegö-Weinberger inequality concern the second eigenvalue. In this context, it is not clear what is the maximal domain. It may depend on ν.

Regarding the isoperimetric inequality for the ground state $\lambda_1^N(\Omega, \beta)$, the natural question is to compare with the ground state $\lambda_1^N(\Omega^*, \beta)$ of the disk Ω^* of the same area of Ω. We show that if Ω is not simply connected, it is in general not true that

$$\lambda_1^N(\Omega, \beta) \leq \lambda_1^N(\Omega^*, \beta).$$

Open question: is this inequality true for a simply connected domain Ω ?

Lower bounds for the ground state. For the Aharonov-Bohm potential, the case of *doubly convex domains* is well understood. A doubly connected domain in the plane is an annulus $\Omega = F \setminus \bar{G}$ with F and G convex.

Let ν be the flux of the potential and $d(\nu, \mathbf{Z})$ the distance between ν and $\mathbf{Z}$. We have the following lower bound (C-Savo 2018/2021):

Theorem 2. [4]

$$\lambda_1^N(\Omega, A) \geq \frac{\pi^2}{32} \frac{R_{in}^2}{D(F)^4} \frac{\beta(\Omega)}{B(\Omega)} d(\nu, \mathbf{Z})^{\mathbf{2}}$$

where $\beta(\Omega)$ and $B(\Omega)$ are the minimal and maximal width of Ω and $D(F)$ the diameter of F, R_{in} denotes the inradius of F.

The presence of $\beta(\Omega)$ is necessary and optimal.

Open question: for a punctured domain, if $dist(p, \partial\Omega) = \epsilon$, the asymptotic of the ground state is not known.

For the magnetic Laplacian of constant curvature, we say that a domain Ω satisfies the δ-interior ball condition if, for any $x \in \partial\Omega$, there exists a ball of radius δ tangent to $\partial\Omega$ at x and entirely contained in Ω.

Theorem 3. [1] *Let $\mathcal{A}_\delta$ be the family of domains with the δ-interior ball property. If $\Omega \in \mathcal{A}_\delta$, we have:*

If $\beta\delta^2 \leq 1$,

$$\lambda_1^N(\Omega, \beta) \geq C\beta^2\delta^2$$

and if $\beta\delta^2 \geq 1$,

$$\lambda_1^N(\Omega, \beta) \geq C\beta.$$

with C *universal.*

Note that the estimate is good in terms of β: s $\beta \to 0$, the asymptotic estimate of $\lambda_1^N(\beta, \Omega)$ is in β^2, and it is in β as $\beta \to \infty$.

Upper bound for the ground state. For the magnetic Laplacian with constant curvature β, we have the following upper bound

Theorem 4. [1] *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded smooth domain. Then*

$$\lambda_1^N(\Omega, \beta) < \beta.$$

However

Open question: do we have $\lambda_1^N(\Omega, \beta) \leq \Theta_0\beta$ where Θ_0 is the De Gennes constant ($\Theta_0 \sim 0,590106.$) Even for the disc, the question is open.

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Quantum tunneling and flux effects

Bernard Helffer

(joint work with A. Kachmar and M. P. Sundqvist)

The tunneling induced by symmetries is an interesting phenomenon in spectral theory featuring an exponentially small splitting between the ground state and the next excited state energies. The magnetic flux has an effect on the eigenvalue multiplicity which can lead to oscillatory patterns in the spectrum: as the magnetic flux varies, the eigenvalues may cross and split infinitely many times. Hence it is interesting to look at the interaction between symmetry and flux effects.

We explore this question by investigating examples of operators involving the magnetic Laplacian $(-i\,h\nabla - \mathbf{A})^2$ perturbed in various ways by an electric potential, a boundary condition or a magnetic field discontinuity.

We observe interesting flux effects, manifested in endless eigenvalue crossings, when adding symmetry assumptions on the electric potential, the boundary of the domain or the magnetic field discontinuity set.

A braid structure in the distribution of the low lying eigenvalues was predicted heuristically [7, Sec. 15.2.4] and confirmed numerically Bonnaillie-Noël–Dauge–Vial [3] for the magnetic Laplacian on an equilateral triangle with Neumann boundary condition and constant magnetic field.

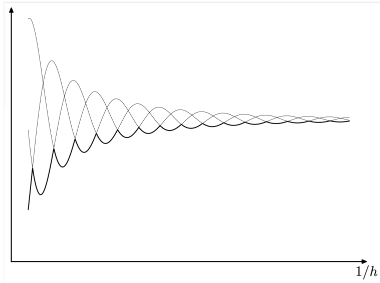

FIGURE 1. A schematic figure of eigenvalues with a braid structure, occuring in the presence of trilateral symmetry is given in the figure. The ground state energy has multiplicity 2 infinitely many times.

For the semi-classical magnetic Laplacian on a simply connected domain with Neumann boundary conditions, the spectrum is related with the spectral properties of an operator which is defined on the boundary. Hence we actually work on another non-simply connected domain (i.e. the boundary) and therefore flux effects are expected to exhibit crossings and splittings of eigenvalues. However, this is not the case when the boundary curvature has for example a unique non-degenerate maximum. In this case the eigenvalues split in the semi-classical limit [6]. It is when non-degenerate minima are exchanged in the presence of symmetries (like in the case of an ellipse or a smoothed equilateral triangle), that eigenvalue crossings are expected to occur along with tunneling effects [4, 7]. We also prove such a behavior for the electro-magnetic Laplacian with 'potential' wells located on the vertices of an equilateral triangle, which to our knowledge is novel.

We consider the electro-magnetic Laplacian on $\mathbb{R}^2$,

$$\mathcal{L}_{h,b} = (-i\,h\nabla - b\mathbf{A})^2 + V,$$

where b, h are positive parameters, $\mathbf{A} = \frac{1}{2}(-x_2, x_1)$ is the vector field generating the unit uniform magnetic field, $\mathrm{curl}\mathbf{A} = 1$, and V is a smooth function.

What we call the wells are the points where V attains its minimum. The pure electric case where $b = 0$ was settled for any number of wells n in [12]-II. We would like to address the case where $b > 0$ and $n \geq 2$. For $n = 2$, this problem was considered in [13] and revisited recently in [5, 10]. The article [13] follows a perturbative approach (i.e. considers the case b relatively small) and assumes the analyticity of the electric potential V, while the results in [5, 10] hold for any $b > 0$ but under the assumption that V is non-positive and defined as a superposition of radially symmetric compactly supported functions. Here we consider the case $n = 3$, when the electric potential V has the form

$$V(x) = v_0(|x - z_1|) + v_0(|x - z_2|) + v_0(|x - z_3|)$$

where v_0 is non positive, radial, with support in $[0, a]$ and a unique non degenerate minimum at 0 and where the wells z_1, z_2, z_3 are located on the vertices of an equilateral triangle with side length L. We prove in [11]

Theorem 1. *Assuming $b > 0$ is fixed and V is given as above, then the three lowest eigenvalues of $\mathcal{L}_{h,b}$ have a braid structure. Moreover,*

$$\limsup_{h \searrow 0} \Big(h \ln \big(\lambda_2(\mathcal{L}_{h,b}) - \lambda_1(\mathcal{L}_{h,b}) \big) \Big) = -\mathfrak{E}_{b,L}(v_0)$$

with $\mathfrak{E}_{b,L}(v_0)$ the same positive quantity as explicitly computed in the double well theorem in [10].

Note that not only the theorem establishes the existence of infinitely many eigenvalue crossings and splittings, but it also establishes an accurate estimate for the magnetic tunneling induced by three symmetric potential wells, thereby extending the recent results of [5, 10] on double wells.

Since our approach combines an abstract approach with the techniques used for treating the interaction between two wells, we can prove the existence of a braid structure in two other cases. We present here results on the pure magnetic Laplacian where the eigenvalue crossings are induced by a combination of the geometry and the flux in the semi-classical limit. We shall describe the results when Ω is a smoothed triangle, i.e. a simply connected domain, invariant under rotation by $2\pi/3$, with three points of maximum curvature that are equidistant with respect to the arc-length distance on the boundary $\Gamma = \partial\Omega$.

(1) **The magnetic Neumann Laplacian under constant magnetic field.** We refer to [7] (and references therein) for an introduction to this case related to Surface Superconductivity. With Ω as above, we consider the magnetic Laplacian on Ω, with uniform magnetic field and the (magnetic) Neumann condition $\nu \cdot (-i\, h\nabla - \mathbf{A})u|_\Gamma = 0$.

The presence of the Neumann boundary condition plays a crucial role. This condition is responsible for the semi-classical localization of the bound states near the boundary of Ω and this is this localization that gives rise to the observed flux effects.

(2) **The Landau Hamiltonian under a magnetic step**. We consider here the magnetic Laplacian on $\mathbb{R}^2$ with the discontinuous magnetic field equal to 1 on Ω and ϑ in $\mathbb{R}^2 \setminus \Omega$ with $\vartheta \in (-1, 0)$. The semi-classical limit for the operator has been studied recently in [1, 2, 9]. The bound states of the system become increasingly concentrated along the discontinuity of B in the semi-classical limit. With Ω as above, we prove. in [11] that the three lowest eigenvalues of the operator have in the semi-classical limit a braid structure.

Recently, a beautiful result has been announced in [8] about the control of the tunneling for a pure magnetic effect. Here the wells correspond to the minima of a variable positive symmetric magnetic field in $\mathbb{R}^2$.

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The Heat Content of polygonal domains

Katie Gittins

Let $D \subset \mathbb{R}^2$ be a bounded set with polygonal boundary ∂D. We impose an initial temperature condition and can also impose boundary conditions on the edges of ∂D. In such a setting, it is natural to ask: how much heat is left inside D at time t? This quantity is the heat content of D. The small-time asymptotic expansions for the heat content of D encode information about the geometry of D and ∂D. Our goal is to explore how these expansions depend upon the geometry and on various combinations of initial and boundary conditions.

We first consider the case where an initial temperature 1 is imposed on D and a Dirichlet boundary condition (cooling) is imposed on ∂D. The solution of the heat equation in this setting, which represents the temperature at $x \in D$ at time $t > 0$, is

$$v_D(x;t) = \int_D dy\, \pi_D(x,y;t),\ x \in D,\ t > 0,$$

where $\pi_D(x, y; t)$ is the Dirichlet heat kernel. The Dirichlet Heat Content is defined as

$$Q_D(t) := \int_D dx \int_D dy\, \pi_D(x, y; t).$$

The small-time asymptotic expansion for $Q_D(t)$ obtained in [4] is

$$Q_D(t) = |D| - \frac{2}{\pi^{1/2}} L(\partial D) t^{1/2} + \sum_{\gamma \in \mathcal{A}} f(\gamma) t + O(e^{-q_D/t}),\ t \downarrow 0,$$

where $L(\partial D)$ denotes the length of the boundary, $\mathcal{A}$ denotes the set of interior angles of ∂D, $f : (0, 2\pi) \to \mathbb{R}$ is given by

$$f(\gamma) := \int_0^\infty d\theta\, \frac{4 \sinh\left((\pi - \gamma)\,\theta\right)}{\sinh\left(\pi\theta\right) \cosh\left(\gamma\theta\right)},$$

and $q_D > 0$ is a constant depending only on D.

We now compare with the small-time asymptotic expansion for another heat flow problem where we do not impose any boundary condition and instead simply allow the heat to flow out of D. More precisely, we consider

$$\begin{cases} \Delta u(x; t) = \frac{\partial u(x;t)}{\partial t}, & x \in D,\ t > 0; \\ \lim_{t \downarrow 0} u(x; t) = \chi_D(x), & x \in \mathbb{R}^2 \setminus \partial D, \end{cases}$$

where $\chi_D(x) = 1$ if $x \in D$ and $\chi_D(x) = 0$ if $x \in \mathbb{R}^2 \setminus \overline{D}$. The solution of the heat equation in this setting is

$$u_D(x; t) = \int_D dy\, p_{\mathbb{R}^2}(x, y; t),\ x \in D,\ t > 0,$$

where $p_{\mathbb{R}^2}(x, y; t) = (4\pi t)^{-1} e^{-|x-y|^2/(4t)}$. We define the Open Heat Content as

$$H_D(t) := \int_D dx \int_D dy\, p_{\mathbb{R}^2}(x, y; t).$$

The small-time asymptotic expansion for $H_D(t)$ obtained in [3] is

$$H_D(t) = |D| - \frac{1}{\pi^{1/2}} L(\partial D) t^{1/2} + \sum_{\gamma \in \mathcal{A}} a(\gamma) t + O(e^{-h_D/t}),\ t \downarrow 0,$$

where $\mathcal{A}$ denotes the set of interior angles, $a : (0, 2\pi) \to \mathbb{R}$ is given by

$$a(\gamma) := \begin{cases} \frac{1}{\pi} + \left(1 - \frac{\gamma}{\pi}\right) \cot \gamma, & \gamma \in (0, \pi) \cup (\pi, 2\pi), \\ 0, & \gamma = \pi, \end{cases} \tag{1}$$

and $h_D > 0$ is a constant depending only on D. If we were to allow D to have polygonal holes, then, in contrast to the Dirichlet case, the expansion for the Open heat content would have additional terms of order t to account for the fact that regions meeting at the same vertex feel each others' presence (see [3]).

It is also possible to obtain the small-time asymptotic expansions for the heat content of D where some edges of ∂D are subject to a Dirichlet boundary condition while the remaining edges of D are Open edges, see [2]. The heat content of D

where each edge of ∂D is subject to either a Dirichlet or a Neumann (insulating) boundary condition can be obtained by reflecting over the Neumann edges.

Finally, we consider the setting where D is subject to a Neumann boundary condition and $\widetilde{D} \subset D$ is a polygonal domain which has some Open edges. More precisely,

$$\begin{cases} \Delta u(x;t) = \frac{\partial u(x;t)}{\partial t}, & x \in D,\, t > 0; \\ \frac{\partial u(x;t)}{\partial n} = 0, & x \in \partial D,\, t > 0; \\ \lim_{t \downarrow 0} u(x;t) = \chi_{\widetilde{D}}(x), & x \in D \setminus \partial \widetilde{D}, \end{cases}$$

where n is the inward-pointing unit normal to ∂D. The solution is

$$u_{D,\widetilde{D}}(t;x) = \int_D dy\ \eta_D(t;x,y)\chi_{\widetilde{D}}(y).$$

where $\eta_D(t;x,y)$ is the Neumann heat kernel for D. We define the heat content of $\tilde{D} \subset D$ as

$$H_{D,\widetilde{D}}(t) := \int_{\widetilde{D}} dx\ u_{D,\widetilde{D}}(t;x).$$

The following result was obtained in joint work with Sam Farrington [6]. There exists a constant $C_{D,\widetilde{D}} > 0$, depending only on D and $\widetilde{D}$, such that, for $t \downarrow 0$,

$$\begin{aligned} H_{D,\widetilde{D}}(t) &= |\widetilde{D}| - \frac{1}{\pi^{1/2}} L(\partial_O \widetilde{D}) t^{1/2} \\ &+ \left(\sum_{\gamma \in \mathcal{A}} a(\gamma) + \sum_{(\gamma,\beta) \in \mathcal{B}} b(\gamma,\beta) + \sum_{(\gamma,\beta,\alpha) \in \mathcal{C}} c(\gamma,\beta,\alpha) \right) t + O\left(e^{-C_{D,\widetilde{D}}/t}\right), \end{aligned}$$

where: $\partial_O \widetilde{D}$ denotes the collection of open edges; $\mathcal{A}, \mathcal{B}, \mathcal{C}$ denote certain subsets of the interior angles; $a : (0, 2\pi) \to \mathbb{R}$ is as in (1); $b(\gamma,\beta)$ is given by

$$b(\gamma,\beta) := \int_0^\infty d\theta\ \frac{\cosh\left(\frac{\pi}{2}\theta\right)\cosh\left(\left(\gamma-\beta\right)\theta\right) - \cosh\left(\left(\frac{\pi}{2}-\gamma-\beta\right)\theta\right)}{2\sinh\left(\left(\gamma+\beta\right)\theta\right)\sinh\left(\frac{\pi}{2}\theta\right)}$$

corresponding to the case where two Neumann and one Open edge meet; and $c(\gamma,\beta,\alpha)$ is given by

$$\begin{aligned} c(\gamma,\beta,\alpha) &:= b(\gamma+\beta,\alpha) + b(\gamma+\alpha,\beta) \\ &\qquad + \int_0^\infty d\theta\ \frac{\cosh\left(\frac{\pi}{2}\theta\right)\left[\cosh\left(\left(\beta+\alpha\right)\theta\right) - \cosh\left(\left(\beta-\alpha\right)\theta\right)\right]}{\sinh\left(\left(\gamma+\beta+\alpha\right)\theta\right)\sinh\left(\frac{\pi}{2}\theta\right)} \end{aligned}$$

corresponding to the case where two Neumann and two Open edges meet.

Using that the angular terms of order t enjoy various symmetry properties, we construct examples of different shapes that have the same first three terms in the small-time asymptotic expansion for $H_{D,\widetilde{D}}(t)$, see [6]. In a similar vein, inspired by the inverse results obtained for the heat trace in [7, 8] and the inverse results obtained for the Dirichlet heat content in [1, 5, 9], we also mention some joint work in progress with Yulun Wu on inverse problems for the small-time asymptotics of the Open heat content on certain polygons.

In addition, we discuss some further questions.

- What are the small-time asymptotic expansions for the heat content of other planar domains? Such asymptotics for some fractal domains in various settings have been considered (see [3] and references therein).
- To what extent can the first three terms in the small-time asymptotic expansions of the heat content distinguish between planar domains?
- What are the small-time asymptotic expansions for the heat content of polyhedra in $\mathbb{R}^3$? What is the precise contribution from the angles?

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Harmonic maps and Dirac eigenvalue optimisation

Mikhail Karpukhin

(joint work with A. Métras and I. Polterovich)

Given a closed surface M one of fundamental problems of spectral geometry is to maximise the eigenvalues of the Laplacian Δ_g in the class of metrics g on M of fixed area. This problem has attracted a significant attention of the geometric analysis community after the observation of Nadirashvili [4] that optimal metrics naturally correspond to harmonic maps to spheres. The goal of the present talk is to report that an analogous phenomenon occurs for eigenvalues of the Dirac operator. To emphasise the similarities we start with a short overview of the theory in the case of the Laplacian.

The (positive) Laplace operator $\Delta_g = \delta_g d$ on a closed connected surface has discrete spectrum

$$0 = \lambda_0(M,g) < \lambda_1(M,g) \leqslant \lambda_2(M,g) \leqslant \ldots \nearrow +\infty,$$

where eigenvalues are written with multiplicities. For a given surface M, let $[g] = \{e^{2\omega}g,\, \omega \in C^\infty(M)\}$ be a conformal class of metrics. We introduce the scale-invariant functionals $\bar\lambda_k(M,g)$ and the corresponding optimal quantity as follows

$$\Lambda_k(M,[g]) = \sup_{h\in[g]} \bar\lambda_k(M,h) := \sup_{h\in[g]} \lambda_k(M,h)\mathrm{Area}(M,h).$$

It is known that $\Lambda_k(M,[g])$ is always finite. The goal is to explicitly determine the values $\Lambda_k(M,[g])$ and to identify metrics for which the supremum is attained.

The first step towards this goal is obtaining the corresponding first-order condition satisfied by the maximising metrics. To explain the result, it is useful to first recall the definition of harmonic maps. A map $\Phi\colon (M,g)\to\mathbb{S}^n\subset\mathbb{R}^{n+1}$ is called harmonic if it satisfies the equation $\Delta_g\Phi = |d\Phi|_g^2\Phi$, where Φ is understood as $\mathbb{R}^{n+1}$-valued vector-function. The Laplacian on surfaces is *conformally covariant*, i.e. it satisfies $\Delta_{e^{2\omega}g} = e^{-2\omega}\Delta_g$, which implies that harmonicity of Φ depends only on the conformal class of g. In particular, for $g_\Phi = \frac{1}{2}|d\Phi|_g^2 g$ one has $|d\Phi|^2_{g_\Phi} = 2$ and $\Delta_{g_\Phi}\Phi = 2\Phi$, i.e. $\lambda_k(M,g_\Phi) = 2$ for some k. Conversely, if $\Phi\colon (M,g)\to\mathbb{S}^n$ is such that $\Delta_g\Phi = 2\Phi$, then

$$0 = \frac{1}{2}\Delta_g(|\Phi|^2) = \Phi\cdot\Delta_g\Phi - |d\Phi|_g^2 = 2 - |d\Phi|_g^2,$$

i.e. $2 = |d\Phi|_g^2$ and Φ is a harmonic map. It turns out that the existence of maps by eigefunctions to spheres is exactly the first-order condition satisfied by any metric critical for a normalised eigenvalue functional.

Theorem 1 (N. Nadirashvili [4], A. El Soufi, S. Ilias [1])**.** *Let $\mathcal{C}$ be a conformal class on M. Suppose that $g\in\mathcal{C}$ is a critical point of $\bar\lambda_k$ in $\mathcal{C}$. Up to scaling one can assume that $\lambda_k(M,g) = 2$. Then there exists a harmonic map $\Phi\colon (M,\mathcal{C})\to\mathbb{S}^n$ such that $g = g_\Phi$ and the components of Φ are $\lambda_k(M,g)$-eigenfunctions of Δ_g. Conversely, for any harmonic $\Phi\colon (M,\mathcal{C})\to\mathbb{S}^n$ the metric g_Φ is $\bar\lambda_k$-critical for k such that $2 = \lambda_k(M,g_\Phi) > \lambda_{k-1}(M,g_\Phi)$.*

We also remark that for a harmonic map $\Phi\colon (M,\mathcal{C})\to\mathbb{S}^n$ the value of $\bar\lambda_k(M,g_\Phi)$ can be naturally expressed in terms of the energy of Φ

$$\bar\lambda_k(M,g_\Phi) = 2\int_M \frac{1}{2}|d\Phi|_g^2\,dv_g = 2E_g(\Phi). \tag{1}$$

The above theorem was originally used by Nadirashvili in [4] to identify a flat metric on equilateral torus as the unique $\bar\lambda_1$-maximising metric on $\mathbb{T}^2$. Since then, many other applications of this correspondence have been found, see [2] and references therein.

To define Dirac operator we need to additionally assume that M is orientable which means that a conformal class $\mathcal{C}$ also induces complex structure. Let $K = (T^{(1,0)}M)^*$ be its canonical line bundle, then a spin structure on M is a holomorphic line bundle S together with a holomorphic isomorphism $S\otimes S\cong K$, which makes S a square root of K. Picking a metric $g\in\mathcal{C}$ allows us to define

$$\bar\partial_g : \Gamma(S) \xrightarrow{\bar\partial} \Gamma(S\otimes\bar K) \to \Gamma(\bar S),$$

where the second map corresponds to the bundle isomorphisms $S\otimes\bar{K}\cong S\otimes\bar{S}\otimes\bar{S}\cong|S|^2\otimes\bar{S}\cong_g\bar{S}$. Similarly, one can define $\partial_g:\Gamma(\bar{S})\to\Gamma(S)$ from the corresponding ∂-operator. The Dirac operator $\mathcal{D}_g$ is

$$\mathcal{D}_g : S\oplus\bar{S}\to S\oplus\bar{S}$$
$$\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}\mapsto 2\begin{pmatrix}0 & \partial_g\\ -\bar{\partial}_g & 0\end{pmatrix}\begin{pmatrix}\psi_+\\ \psi_-\end{pmatrix}.$$

On a compact surface the spectrum of $\mathcal{D}_g$ is discrete and symmetric around zero and we denote it as

$$-\infty\cdots\le-\mu_2\le-\mu_1<0<\mu_1\le\mu_2\le\cdots+\infty.$$

If the kernel of $\mathcal{D}_g$ is non-empty, we do not enumerate the zero eigenvalues. Similarly to Δ_g the Dirac operator $\mathcal{D}_g$ is conformally covariant, i.e $\mathcal{D}_{e^{2\omega}g}=e^{-\omega}\mathcal{D}_g$. This implies that the kernel of $\mathcal{D}_g$ depends only on the conformal class $[g]=\mathcal{C}$.

In the following we fix conformal class $\mathcal{C}$ and the spin structure S on M. Our goal is to study the following quantities

$$\mathrm{M}_k(M,\mathcal{C},S)=\inf_{g\in\mathcal{C}}\bar{\mu}_k(M,g,S):=\inf_{g\in\mathcal{C}}\mu_k(M,g,S)\mathrm{Area}^{\frac{1}{2}}(M,g).$$

In fact, similarly to the Laplacian case, conformal invariance of $\mathcal{D}_g$ guarantees that $\bar{\mu}_k$-critical metrics admit a collection of μ_k-eigenspinors $\psi_1,\dots,\psi_m$ such that $\sum_{j=1}^m|\psi_j|_g^2\equiv1$. However, while for the Laplacian one can then use eigenfunctions to form a map to the sphere, ψ_j are sections of a bundle and the geometric meaning of criticality condition is not immediately clear. Let us write $\psi_j=(\psi_{j+},\psi_{j-})$, where $\psi_{j+}\in\Gamma(S)$, $\psi_{j-}\in\Gamma(\bar{S})$ and define $Z=\{p\in M,\ \psi_j(p)=0,\ j=1,\dots,m\}$. Similar to Kodaira's embedding in complex geometry, one can then form a well-defined map $\Psi\colon M\setminus Z\to\mathbb{CP}^{2m-1}$ given in homogeneous coordinates as

$$\Psi=[\psi_{1+}:\bar{\psi}_{1-}:\dots:\psi_{m+},\bar{\psi}_{m-}].$$

Proposition 2 (KMP [3]). *The set Z is discrete and the map Ψ can be continuously extended across Z. The resulting map $\Psi\colon(M,\mathcal{C})\to\mathbb{CP}^{2m-1}$ is harmonic.*

This proposition shows that $\bar{\mu}_k$-critical metrics correspond to harmonic maps to complex projective spaces. Note, however, that not all harmonic maps to $\mathbb{CP}^{2m-1}$ are maps by eigenspinors, e.g. such maps have restrictions on the pullback of the tautological bundle. To understand these restriction better, we perform the following computation. For two pairs of sections of a line bundle we use the notation $\|$ to express that they differ by multiplication by a section of a (another) line bundle. For any eigenspinor ψ one then has

$$\bar{\partial}\begin{pmatrix}\psi_+\\ \overline{\psi_-}\end{pmatrix}=\begin{pmatrix}\bar{\partial}\psi_+\\ \overline{\partial\psi_-}\end{pmatrix}\;\|\;\begin{pmatrix}-\psi_-\\ \overline{\psi_+}\end{pmatrix}=J\begin{pmatrix}\psi_+\\ \overline{\psi_-}\end{pmatrix},$$

where $J(s_1,s_2)=(-\bar{s}_2,\bar{s}_1)$ can be interpreted as quaternionic multiplication by $\mathbf{j}$. This computation means that for any map by eigenspinors the application of $\bar{\partial}$ can be expressed in terms of the operator J. One can also check that branch points of any harmonic map by eigenspinors have even order. This leads us to the definition of *quaternionic harmonic maps* as harmonic maps to $\mathbb{CP}^{2m-1}$ satisfying a certain interlacing relation between $\bar{\partial}$ and J, and whose branch points have even order, see [3] for details. We show that any quaternionic harmonic map is in fact a map by eigenspinors for a metric and spin structure determined by the map. This establishes a two-sided correspondence between $\bar{\mu}_k$-critical metrics and quaternionic harmonic map, completing the analog of Theorem 1 in the case of Dirac operator. Some concrete applications of this fact to computation of $\Lambda_1(M, \mathcal{C}, S)$ are given in the talk of A. Métras included in the same volume.

We finish the talk by discussing the version of (1) in the case of Dirac eigenvalues. For a map Ψ to $\mathbb{CP}^{2m-1}$ the complexification of the differential induces the maps $\partial\Psi$ between $(1,0)$ vectors and $\bar{\partial}\Psi$ sending $(0,1)$-vectors to $(1,0)$-vectors. One readily sees that $|d\Psi|^2 = 2(|\partial\Psi|^2 + |\bar{\partial}\Psi|^2)$, and hence

$$E(\Psi) = \int_M |\partial\Psi|^2 dv_g + \int_M |\bar{\partial}\Psi|^2 dv_g =: E^{(1,0)}(\Psi) + E^{(0,1)}(\Psi).$$

A straightforward computation shows that if Ψ is a map by $\mathcal{D}_g$-eigenspinors, then $\bar{\mu}_k(M, g, S)^2 = E^{(0,1)}(\Psi)$.

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Optimisation of the first non-zero Dirac eigenvalue

Antoine Métras

(joint work with M. Karpukhin and I. Polterovich)

This talk is a continuation of the talk by Mikhail Karpukhin and focuses on applying the theory developed there to find sharp lower bounds for the first non-zero Dirac eigenvalue on a surface with a fixed conformal class.

Let M be an orientable surface with a conformal class $\mathcal{C}$ and, for a given metric $g \in \mathcal{C}$, we denote by $\mathcal{D}_g$ the Dirac operator acting on spinors $\psi \in \Gamma(S \oplus \bar{S})$ and by μ_k its eigenvalues. We use the notation $\bar{\mu}_k(M, g) = \mu_k(M, g)\mathrm{Area}^{1/2}(M, g)$ for the normalized eigenvalue. We refer the reader to the abstract by Mikhail Karpukhin for the precises definitions of these terms.

As was discussed in the previous talk, the $\bar{\mu}_k$-conformally critical metrics correspond to quaternionic harmonic maps to some projective space $\mathbb{CP}^{2m-1}$. When trying to apply this correspondence to obtain concrete results, one difficulty is that the dimension $2m - 1$ of the projective space is not known beforehand. This is simplified when minimizing the first positive eigenvalue $\bar{\mu}_1$. Indeed, if we have a local minimum for $\bar{\mu}_1$ then we cannot have multiple eigenvalue branches crossing at that point as that would contradict the minimality. Hence local minimum for $\bar{\mu}_1$ correspond to "true" critical points of $\bar{\mu}_1$, i.e. points where the function $\bar{\mu}_1(t) : g(t) \to \bar{\mu}_1(g(t))$ is differentiable for any analytic one-parameter family of metric $g(t) \in \mathcal{C}$, and with derivative 0. As a consequence of this, the quaternionic harmonic map corresponding to such minimum can be chosen to be a map to $\mathbb{CP}^1 \cong \mathbb{S}^2$. We now apply this idea to the cases of the sphere and the torus.

For the sphere, we obtain a new proof of Bär's inequality

Theorem 1 (Bär's inequality [2])**.** *If (M, g) is a sphere then*

$$\bar{\mu}_1(M, g) \geq \bar{\mu}_1(\mathbb{S}^2, g_{round}) = 2\sqrt{\pi}$$

with equality if and only if g is a rescaling of the round metric g_{round}.

The proof is straightforward: given any metric g, we take an eigenspinor $\psi = (\psi_+, \psi_-)$ corresponding to $\mu_1(M, g)$ and construct from it a map $\Psi : M \to \mathbb{CP}^1, \Psi = [\psi_+ : \bar{\psi}_-]$. Then a calculation gives

$$\bar{\mu}_1(M, g)^2 = E^{(0,1)}(\Psi) := \int_M |\bar{\partial}\Psi|^2 dv.$$

So it is enough to show $E^{(0,1)}(\Psi) \geq 4\pi$. But a general formula relating the energy and (anti)-holomorphic energies for maps between surfaces tells us $E^{(0,1)}(\Psi) = -4\pi\deg(\Psi) + E^{(1,0)}(\Psi)$. Since $E^{(1,0)}(\Psi) \geq 0$, the desired result is obtained by showing that $\deg(\Psi) \leq -1$.

For the torus, we first describe the moduli space of conformal classes. Different type of spin structures S exist on the torus, but for the purpose of brevity, we assume the spin structure S is trivial (we have similar results when S is non trivial [4]). By the uniformisation theorem, in each conformal class on the torus, we have a flat metric and we will identify the conformal class with the flat metric $g_{(a,b)}$ on $T_{(a,b)} = \mathbb{R}^2/(\mathbb{Z}\begin{pmatrix}1\\0\end{pmatrix} + \mathbb{Z}\begin{pmatrix}a\\b\end{pmatrix})$ it contains. The moduli space of conformal classes (with the trivial spin structure) is then the region of $\mathbb{R}^2$ delimited by $|a| \leq \frac{1}{2}$ and $a^2 + b^2 \geq 1$, each point (a, b) giving the conformal class $[g_{(a,b)}]$.

We showed that in a large part of the moduli space, the minimal metric for $\bar{\mu}_1$ is a flat metric. More precisely,

Theorem 2 (KMP [4])**.** *For for the trivial spin structure on the torus and all $b > 2\pi$,*

$$\inf_{g \in [g_{(a,b)}]} \bar{\mu}_1(M, g) = \mu_1(g_{(a,b)})\mathrm{Area}^{1/2}(T_{(a,b)}, g_{(a,b)}) = \frac{2\pi}{\sqrt{b}}.$$

Furthermore, in these conformal classes, flat metrics are the only minimizers.

The idea of the proof is that for the flat metric, one can compute explicitly $\bar{\mu}_1(g_{(a,b)}) = \frac{2\pi}{\sqrt{b}}$. So for b large enough, such that $\bar{\mu}_1(g_{(a,b)}) < 2\sqrt{\pi}$, we have the existence of a minimal metric g_0 in the conformal class $[g_{(a,b)}]$ by a result of Ammann [1]. This minimal metric gives rise to a quaternionic harmonic map

$\Psi : T_{(a,b)} \to \mathbb{CP}^1 \cong \mathbb{S}^2$. Then by looking at the different degrees such map can have (either $|\deg(\Psi)| \geq 2$ or $\deg(\Psi) = 0$), we show that $\deg(\Psi) = 0$ as the other case would either imply that the map is holomorphic (if $\deg > 0$) in which case $\mu_1 = 0$ a contradiction, or if $\deg \leq -2$ that $\bar{\mu}_1^2 = E^{(0,1)}(\Psi) > 4\pi$, again a contradiction. Hence $\deg(\Psi) = 0$ and its energy $E(\Psi)$ is equal to $2E^{(0,1)}(\Psi)$. We then use that a harmonic map from the torus to the sphere with energy smaller than 4π must map to a great circle $\mathbb{S}^1 \subset \mathbb{S}^2$ [3]. Hence the map Ψ is a harmonic map to a circle for b large enough and by looking at the explicit formula for these maps, conclude that $|\bar{\partial}\Psi|^2_{g_{(a,b)}}$ is constant. But by our correspondence between the minimal metric g_0 and the quaternionic harmonic map, the metric g_0 is given by $|\bar{\partial}\Psi|^2_{g_{(a,b)}} g_{(a,b)}$.

We conjecture that for the trivial spin structure, the bound $b \geq \pi$ is the best possible for the existence of minimisers, and if $b < \pi$ then we have no minimal metric and the formation of a bubble. Considering Ammann's existence result, our conjecture is

Conjecture 3. *For the trivial spin structure on the torus T, one has*

$$\inf_{g \in [g_{(a,b)}]} \bar{\mu}_1(g) = \begin{cases} \frac{2\pi}{\sqrt{b}} & \text{if } b > \pi; \\ 2\sqrt{\pi} & \text{if } b \leq \pi. \end{cases}$$

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Pleijel nodal domain theorem in non-smooth setting

Sara Farinelli

(joint work with N. de Ponti and I. Y. Violo)

A version of the well known Courant nodal domain theorem [4] states that, given a bounded open set $\Omega \subset \mathbb{R}^n$, any eigenfunctions of the Laplacian with Dirichlet boundary conditions of eigenvalue $\lambda_k(\Omega)$ has at most k nodal domains ($\{\lambda_k(\Omega)\}_k$ denote the eigenvalues in increasing order and counted with multiplicity). This result admits an asymptotic version due to Pleijel [7]. In particular, denoted by N_k the maximal number of nodal domains of an eigenfunction of eigenvalue $\lambda_k(\Omega)$, it holds

$$\limsup_{k \to +\infty} \frac{N_k}{k} = \frac{(2\pi)^n}{\omega_n^2 j^n_{\frac{n-2}{n}}} < 1, \tag{1}$$

where j_α denotes the first positive zero of the Bessel function of order α. An important consequence of (1) is that, except for at most a finite number of k's, eigenfunctions of eigenvalue $\lambda_k(\Omega)$ have strictly less than k nodal domains, or in other words the bound given by the Courant theorem is strict. Pleijel's result was extended to Riemannian manifolds by Bérard and Meyer [3], while the Neumann case was treated by Polterovich in [8] (see also [6]).

The main goal of our work is to extend the Pleijel theorem for Neumann eigenfunctions to the case where the underlying space is non-smooth. In particular we focus on metric measure spaces having a synthetic notion of Ricci curvature bounded below, i.e. RCD spaces. Roughly said, they can be thought as generalised Riemannian manifolds but without any smooth structure and which also admit singularities. Among examples there are Gromov-Hausdorff limits of Riemannian manifolds (having Ricci uniformly bonded below) and Alexandrov spaces. See the survey [1] for an account on this topic. Our main theorem then reads as follows. In the statemet $\mathcal{H}^n$ denotes the n-dimensional Hausdorff measure, while for more on uniform domains see below.

Theorem 1. *Let $(X, \mathsf{d}, \mathcal{H}^n)$ be an $\mathrm{RCD}(K, n)$ space with $K \in \mathbb{R}$ and $n \in \mathbb{N}$, and let $\Omega \subset X$ be a uniform domain. Denoted by N_k the maximal number of nodal domains of a Neumann-Laplacian eigenfunction in Ω of eigenvalue $\lambda_k(\Omega)$, it holds*

$$\limsup_{k \to +\infty} \frac{N_k}{k} \le \frac{(2\pi)^n}{\omega_n^2 j^n_{\frac{n-2}{n}}} < 1.$$

In particular for every $k \in \mathbb{N}$ large enough every Neumann eigenfunction of eigenvalue $\lambda_k(\Omega)$ has less than k nodal domains.

The second part of the statement is particularly interesting because in the RCD setting the Courant nodal domain theorem for eigenfunctions of the Laplacian is not known. This is due to the fact that the main tool needed for its proof, i.e. the unique continuation property for eigenfunctions, is not known (recall that the unique continuation says that an eigenfunction that vanishes on a ball must be identically zero).

The notion of *uniform domain* appearing in the statement is a replacement of the more usual notion of C^k or more in general Lipschitz-boundary domain, which does not make sense in metric spaces. Roughly said, uniform domains are bounded open sets in which each couple of points can be connected by a curve which is both not too-long and stays sufficiently away from the boundary (see e.g. [9] for details). In $\mathbb{R}^n$ examples of uniform domains are Lipschitz domains, but they include also more rough objects like snowflake type domains. Remarkably the validity of the Pleijel theorem in $\mathbb{R}^n$ for Lipschitz domains was open in the Euclidean setting, the best known result being the one of Léna for $C^{1,1}$ domains. Hence from Theorem 1 we can extract the following new version of Pleijel theorem under low boundary-regularity:

Corollary 1. *The Pleijel nodal domain theorem holds for Neumann-eigenfunctions of the Laplacian in any uniform domain in $\mathbb{R}^n$.*

Concerning the proof of Theorem 1, it follows the usual scheme adopted also e.g. in [7, 3, 6], i.e. it combines the Weyl's law with the Faber-Krahn inequality. The main obstacle is of course the presence of singularities. In particular, differently from the Riemannian setting, there might be locations in an RCD space where an almost Euclidean Faber-Krahn inequality as in [3] fails. This required us to obtain a suitable new version of this inequality that holds for sets avoiding some pathological, but small, regions of the ambient space.

For future developments, we note that a limitation of Theorem 1 is that it is only for RCD spaces endowed with the Hausdorff measure, which are usually called *non-collapsed* in the literature. The validity of a version of Theorem 1 for RCD spaces (X, d, μ) with arbitrary reference measure μ remains an open question worth investigating. The main obstacle is the Weyl's law, which plays a key role in the proof of Theorem 1. Indeed for non-collapsed RCD spaces it holds in the usual form, however for general measures μ it can have odd and singular behaviours (see [2] and [5] respectively).

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Asymptotics of Robin eigenvalues for non-isotropic peaks

Marco Vogel

For a given open set $\Omega \subset \mathbb{R}^N$, with a suitable regular boundary, and a parameter $\alpha > 0$, consider the Robin eigenvalue problem

$$\begin{aligned} -\Delta u &= \lambda u \quad \text{in } \Omega \\ \partial_\nu u &= \alpha u \quad \text{on } \partial\Omega \end{aligned}$$

where ∂_ν is the outward normal derivative. We are particularly interested in the strong coupling asymptotics of the eigenvalues λ, i.e. the behavior of the eigenvalues as $\alpha \to \infty$. This was presumably first studied by Lacey, Ockendon and Sabina [5]. For further discussion we need to define operators more rigorous. So let $\Omega \subset \mathbb{R}^N$ be an open set and $\alpha > 0$ such that the quadratic form

$$q_\Omega^\alpha(u,u) = \int_\Omega |\nabla u|^2 \mathrm{d}x - \alpha \int_{\partial\Omega} u^2 \mathrm{d}\sigma, \quad D(q_\Omega^\alpha) = H^1(\Omega),$$

is closed, where $\mathrm{d}\sigma$ denotes the integration with respect to the $(N-1)$-dimensional Hausdorff measure, and denote by Q_Ω^α the self-adjoint operator in $L^2(\Omega)$ associated with q_Ω^α. The asymptotic behavior of the eigenvalues is highly influenced by the regularity/geometry of Ω. We list some results about the strong coupling regime for "nice" domains first and then move on to "bad" domains.

If $\Omega \subset \mathbb{R}^N$, with $N \geq 2$, is a bounded Lipschitz domain, then it is well known that $0 > E_j(Q_\Omega^\alpha) > -K\alpha^2$ for sufficiently large $\alpha > 0$. Levitin and Parnovski [6, Theorem 3.2] showed that the principal eigenvalue for bounded piecewise smooth domains satisfying the uniform interior cone condition behaves as $E_1(Q_\Omega^\alpha) \approx -C_\Omega \alpha^2$. More precisely $-C_\Omega = \inf_{y\in\partial\Omega} \inf \operatorname{spec} Q_{K_y}^1$, where K_y is the tangent cone at $y \in \partial\Omega$. If $\partial\Omega$ is C^1 then Daners and Kennedy [2, Theorem 1.1.] were able to show that $C_\Omega = 1$ for every eigenvalue, i.e. $E_j(Q_\Omega^\alpha) \approx -\alpha^2$. The results mentioned above are all one-term asymptotics, but there are also papers which have proven two-term asymptotics. Exner, Minakov and Parnovski [1, Theorem 1.3] showed for planar domains, which have a closed C^4 Jordan curve as their boundary, that the eigenvalues behave as $E_j(Q_\Omega^\alpha) \approx -\alpha^2 - \gamma^*\alpha$, where γ^* is the maximal curvature of the mentioned Jordan curve. The last result, concerning nice domains, we mention is about planar curvilinear polygons by Khalile, Ourmières-Bonafos and Pankrashkin[3]. They showed that the behavior of the first few eigenvalues are determined by the tangent cones of the vertices. After the "first few" eigenvalues, the leading term of the next eigenvalues is $-\alpha^2$ and the second term in the asymptotic expansion is determined by an operator acting on the edges. The corresponding paper is quite voluminous and technical, in particular certain assumptions on the vertices have to be made. Therefore we refer to [3] for precise statements.

For non Lipschitz domains many different scenarios are possible. If Ω has an outward pointing peak which is "to sharp" the Robin-Laplacian fails to be semibounded from below, see e.g. [7, Lemma 1.2]. However the present paper is motivated by [4], where Kovařík and Pankrashkin looked at isotropic peaks, i.e. there exists $\delta > 0$ such that

$$\Omega \cap (-\delta,\delta)^N = \left\{ (x', x_N) \in \mathbb{R}^{N-1} \times (0,\delta) : \frac{x'}{x_N^q} \in B_1(0) \right\} \subset \mathbb{R}^N,$$

$$\Omega \setminus [-\delta,\delta]^N \text{ is a bounded Lipschitz domain,}$$

with $1 < q < 2$ and $B_1(0)$ being the unit ball centered at the origin in $\mathbb{R}^{N-1}$. They proved that the rate of divergence of the eigenvalues to $-\infty$ is faster than in the pure Lipschitz case. In particular they showed, the eigenvalues behave as $E_j(Q_\Omega^\alpha) \approx \mathcal{E}_j \alpha^{\frac{2}{2-q}}$, with $\mathcal{E}_j < 0$ being the jth eigenvalue of an one dimensional Schrödinger operator. One also observes that the sharper the peak the faster the divergence to $-\infty$. We change the premise of the aforementioned paper in the following way: Consider an open set $\Omega \subset \mathbb{R}^3$, which satisfies

$$\Omega \cap (-\delta,\delta)^3 = \left\{ (x_1,x_2,x_3) \in \mathbb{R}^2 \times (0,\delta) : \left(\frac{x_1}{x_3^p}, \frac{x_2}{x_3^q} \right) \in (-1,1)^2 \right\} \subset \mathbb{R}^3, \tag{1}$$

$$\Omega \setminus [-\delta,\delta]^3 \text{ is a bounded Lipschitz domain,} \tag{2}$$

for some $\delta > 0$ and $1 < p < q < 2$. If a set satisfies condition (1) one says that it has a non-isotropic peak at 0.

Based on the above observation one might expect that the larger power q determines the rate of divergence to $-\infty$, which turns out to be true as described below. For a precise statement we need to define an one dimensional Schrödinger operator. Consider the symmetric differential operator given by

$$C_c^\infty(0,\infty) \ni f \mapsto -f'' + \left(\frac{(p+q)^2 - 2(p+q)}{4s^2} - \frac{1}{s^q} \right) f$$

and denote by $\mathcal{A}_j$ the jth eigenvalue of its Friedrichs extension in $L^2(0,\infty)$. Then the main result reads as follows:

Theorem 1. *Let $j \in \mathbb{N}$ be fixed, then the jth eigenvalue of Q_Ω^α satisfies*

$$E_j(Q_\Omega^\alpha) = \alpha^{\frac{2}{2-q}} \mathcal{A}_j + o\left(\alpha^{\frac{2}{2-q}}\right) \quad \text{as } \alpha \to \infty.$$

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Wave localization

Svitlana Mayboroda

(joint work with G. David and M. Filoche)

Waves of all sorts permeate our world: light (electromagnetic waves), sound (acoustic waves) and mechanical vibrations. Quantum mechanics revealed that, at the atomic level, all matter has a wavelike character, and, very recently, classical gravitational waves have also been detected. Simultaneously, at the cutting edge of today's science it has become possible to map a material atom-by-atom and to manipulate individual atoms, providing us with precise measurements of a world that exhibits myriad irregularities–dimensional, structural, orientational and geometric–simultaneously. For waves, such disorder changes everything. In complex, irregular or random media, waves frequently exhibit the astonishing and mysterious behavior known as "localization"–instead of propagating over extended regions, they remain confined in small portions of the original domain. The Nobel Prize–winning discovery of the Anderson localization in 1958 is only one famous case of this phenomenon. Yet, 60 years later, despite considerable advances in the subject, we still notoriously lack tools to fully understand localization of waves and its consequences. We will discuss modern understanding of the subject, recent results, and the biggest open questions.

Today, we will focus on the following result. We establish non-asymptotic estimates from above and below on the integrated density of states of the Schrödinger operator $L = -\Delta + V$, using a counting function for the minima of the localization landscape, a solution to the equation $Lu = 1$.

The density of states of the Schrödinger operator $-\Delta + V$ is one of the main characteristics defining the physical properties of the matter. At this point, most of the known estimates for the integrated density of states pertain to two *asymptotic* regimes, each carrying restrictions on the underlying potentials. The first one stems from the Weyl law and its improved version due to the Fefferman-Phong uncertainty principle. It addresses the energies or eigenvalues $\lambda \to +\infty$ and deteriorates for the potentials oscillating at a wide range of scales. The second one concentrates on the asymptotics as λ tends to 0 for disordered potentials, the so-called Lifschitz tails, and takes advantage of probabilistic arguments and the random nature of the disordered potentials. The goal of the present work is to establish new bounds on the integrated density of states via the counting function of the so-called localization landscape. The main theorem can be viewed as a new version of the uncertainty principle, which, contrary to the above, applies uniformly across the entire spectrum and covers all potentials bounded from below irrespectively of their nature.

Localisation for the torsion function and first Dirichlet eigenfunction

Michiel van den Berg

Let $p \in [1, \infty)$ be fixed, and let (Ω_n) be a sequence of open sets in $\mathbb{R}^m$ with Lebesgue measure $|\Omega_n|$, $0 < |\Omega_n| < \infty, n \in \mathbb{N}$. For $n \in \mathbb{N}$, let $f_n \in L^p(\Omega_n)$, $0 < \|f_n\|_p < \infty$. Define the following collection of sequences

$$\mathfrak{A}((\Omega_n)) = \left\{ (A_n) : (\forall n \in \mathbb{N})(A_n \subset \Omega_n, A_n \text{ measurable}), \lim_{n\to\infty} \frac{|A_n|}{|\Omega_n|} = 0 \right\},$$

and let

$$\kappa = \sup \left\{ \limsup_{n\to\infty} \frac{\|f_n \mathbf{1}_{A_n}\|_p^p}{\|f_n\|_p^p} : (A_n) \in \mathfrak{A}((\Omega_n)) \right\},$$

where $\mathbf{1_A}$ is the indicator function of the set A. Note that $0 \le \kappa \le 1$. We write (f_n) for the sequence of functions $f_n : \Omega_n \to \mathbb{R}, n \in \mathbb{N}$ in the following:

Definition 1. *We say that*

(i) *the sequence* (f_n) *κ-localises in L^p if* $0 < \kappa < 1$,
(ii) *the sequence* (f_n) *localises in L^p if* $\kappa = 1$,
(iii) *the sequence* (f_n) *does not localise in L^p if* $\kappa = 0$.

A very rough quantity of how a measurable function $f : \Omega \mapsto \mathbb{R}$ is distributed is its mean to max ratio

$$\Phi(f) = \frac{\|f\|_1}{|\Omega|\|f\|_\infty}.$$

Lemma 1. *Let* $1 \leq p < \infty$. *If* (f_n) *either localises or κ-localises in L^p, then*

$$\lim_{n\to\infty} \Phi(f_n) = 0.$$

The converse is in general not true. See Lemma 7 in [4] for a proof.

Lemma 2. *For* $n \in \mathbb{N}$, *let* $f_n \in L^2(\Omega_n)$ *with* $\|f_n\|_2 > 0$, *and* $|\Omega_n| < \infty$. *Then* (f_n) *localises in L^2 if and only if*

$$\lim_{n\to\infty} \frac{\|f_n\|_1^2}{|\Omega_n|\|f_n\|_2^2} = 0.$$

Lemma 2 shows that a vanishing L^1-L^2 participation ratio is equivalent to localisation in L^2. For a proof we refer to Lemma 3 in [2]. Also note that

$$\Phi(f_n) \leq \frac{\|f_n\|_1^2}{|\Omega_n|\|f_n\|_2^2}.$$

Definition 2. *The Dirichlet Laplacian $-\Delta$ acting in $L^2(\Omega)$ satisfies the strong Hardy inequality, with constant* $c_\Omega \in (0, \infty)$, *if*

$$\|\nabla w\|_2^2 \geq \frac{1}{c_\Omega} \int_\Omega \frac{w^2}{d_\Omega^2}, \quad \forall\, w \in C_c^\infty(\Omega), \tag{1}$$

where $d_\Omega(x) := \inf\{|x-y| : y \in \mathbb{R}^m \setminus \Omega\}$, $x \in \Omega$ *is the distance to the boundary.*

If Ω is a proper simply connected subset of $\mathbb{R}^2$, then $c_\Omega = 16$. See [1]. Throughout we assume that Ω is connected, and write u_Ω for a first Dirichlet eigenfunction with $u_\Omega \geq 0$ and $\|u_\Omega\|_2 = 1$. The first Dirichlet eigenvalue is denoted by $\lambda(\Omega)$. So $-\Delta u_\Omega = \lambda(\Omega) u_\Omega$, $u_\Omega \in H_0^1(\Omega)$. The torsion function for an open set Ω, $0 < |\Omega| < \infty$ is the unique solution of $-\Delta v = 1$, $v \in H_0^1(\Omega)$, and is denoted by v_Ω.

The following "scenario" for localisation applies to both the first Dirichlet eigenfunction and the torsion function. It asserts that if most of Ω is close to its boundary, then most of the L^2 mass of u_Ω, or of the L^1 mass of v_Ω, is located in that part of Ω where the distance function is large.

Theorem 3. *Let* (Ω_n) *be a sequence of open sets in* $\mathbb{R}^m$ *with* $0 < |\Omega_n| < \infty$, $n \in \mathbb{N}$, *which satisfies* $c := \sup_n c_{\Omega_n} < \infty$ *in* (1).

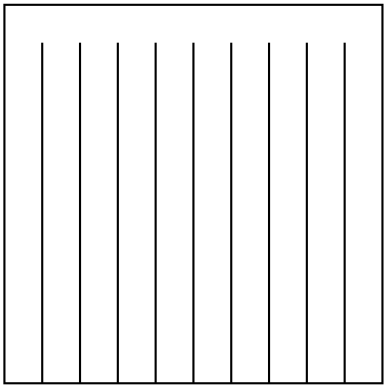

FIGURE 1. The unit square with $n-1$ line segments of length $1-cn^{-\alpha}$ deleted, $0<\alpha<1$, $c>0$ and n such that $cn^{-\alpha}<1$, denoted by $\Omega_{n,\alpha}$.

(i) *If there exists a sequence (A_n) of measurable sets, $A_n \subset \Omega_n$, $n \in \mathbb{N}$, with $\lim_{n\to\infty} \frac{|A_n|}{|\Omega_n|} = 1$, and if $\lim_{n\to\infty} \frac{\sup_{A_n} d_{\Omega_n}}{\sup_{\Omega_n} d_{\Omega_n}} = 0$, then (u_{Ω_n}) localises in L^2.*

(ii) *If there exists a sequence (A_n) of measurable sets, $A_n \subset \Omega_n$, $n \in \mathbb{N}$, with $\lim_{n\to\infty} \frac{|A_n|}{|\Omega_n|} = 1$, and if $\lim_{n\to\infty} \frac{\int_{A_n} d^2_{\Omega_n}}{\int_{\Omega_n} d^2_{\Omega_n}} = 0$, then (v_{Ω_n}) localises in L^1.*

(iii) *If $(B_n) \in \mathfrak{A}((\Omega_n))$ implies $\lim_{n\to\infty} \frac{\int_{B_n} d^2_{\Omega_n}}{\int_{\Omega_n} d^2_{\Omega_n}} = 0$, then (v_{Ω_n}) does not localise in L^1.*

Example 4. *Let $\Omega_{n,\alpha}$ be the simply connected open set in Figure 1.*

(i) *If $0<\alpha<1$, then $(u_{\Omega_{n,\alpha}})$ localises in L^2.*
(ii) *If $0<\alpha<\frac{2}{3}$, then $(v_{\Omega_{n,\alpha}})$ localises in L^1.*
(iii) *If $\frac{2}{3}<\alpha<1$, then $(v_{\Omega_{n,\alpha}})$ does not localises in L^1.*
(iv) *$(v_{\Omega_{n,\frac{2}{3}}})$ κ-localises in L^1 with $\kappa = \frac{c^3}{1+c^3}$.*

The proof of the assertions (i)–(iii) follow immediately from Theorem 3. The proof of (iv) is more delicate. See pp.523–525 in [3].

The second "scenario" for localisation occurs for certain sequences of elongating sets. Localisation in L^2 occurs for the first Dirichlet eigenfunction.

Definition 3. *Points in $\mathbb{R}^m$ are denoted by a Cartesian pair (x_1, x') with $x_1 \in \mathbb{R}$, $x' \in \mathbb{R}^{m-1}$. If Ω is an open set in $\mathbb{R}^m$, then we define its cross-section at x_1 by $\Omega(x_1) = \{x' \in \mathbb{R}^{m-1} : (x_1, x') \in \Omega\}$. A set $\Omega \subset \mathbb{R}^m$ is horn-shaped if it is non-empty, open, and connected, $x_1 > x_2 > 0$ implies $\Omega(x_1) \subset \Omega(x_2)$, and $x_1 < x_2 < 0$ implies $\Omega(x_1) \subset \Omega(x_2)$.*

Let Λ be an open set in $\mathbb{R}^{m-1}$ containing 0. Its first $(m-1)$-dimensional Dirichlet eigenvalue is denoted by $\mu(\Lambda)$, and its $(m-1)$-dimensional Lebesgue measure is denoted by $|\Lambda|_{m-1}$. For $a > 0$ we let $a\Lambda$ be the homothety of Λ by a factor a with respect to 0.

Let $-\infty < c_- \le 0 < c_+ < \infty$. We consider the following class of monotone functions.

$$\mathfrak{F} = \{f : [c_-, c_+] \to [0,1], \text{ non-increasing, continuous on } [0, c_+], \\ \text{non-decreasing, continuous on } [c_-, 0],\ f(0) = 1,\ f(x_1) < 1 \text{ for } x_1 \neq 0\}.$$

Given $f \in \mathfrak{F}$, let

$$f_n : [nc_-, nc_+] \to [0,1],\ f_n(x_1) = f(x_1/n),$$

let $\Omega' \subset \mathbb{R}^{m-1}$ be a non-empty, open, bounded and convex set containing the origin, and let

$$\Omega_n := \Omega_{f_n,\Omega'} = \{(x_1, x') \in \mathbb{R}^m : c_- n < x_1 < c_+ n,\ x' \in f(x_1/n)\Omega'\}.$$

Theorem 5. *If f and Ω' satisfy the hypotheses above, then*
(i) (v_{Ω_n}) *does not localise in L^1.*
(ii) (u_{Ω_n}) *localises in L^2.*

The proof that v_{Ω_n} does not localise follows from the fact that $\Phi(v_{\Omega_n})$ is bounded away from 0 and Lemma 1. For a proof of (ii) we refer to [5]: the main ingredient there is to use Lemma 2 for u_{Ω_n} together with bounds for the heat content of Ω_n for large t.

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Quantum Entanglement and the Growth of Laplacian Eigenfunctions

Stefan Steinerberger

We consider compact Riemannian manifolds (M, g) with or without boundary and Laplacian eigenfunctions

$$-\Delta\phi_k = \lambda_k\phi_k \qquad \text{on } (M, g)$$

with either Dirichlet or Neumann boundary conditions. One of the most basic questions is to understand how much these eigenfunctions can concentrate in a point: to understand the behavior of $\|\phi_k\|_{L^\infty}$. A classic result of Hörmander [6] (see also Avakumovic [2], Grieser [5], Levitan [7], Sogge [8]) is

$$\|\phi_k\|_{L^\infty} \leq c_{(M,g)} \lambda_k^{\frac{d-1}{4}}. \tag{1}$$

This inequality is sharp and attained for the $d-$dimensional sphere $\mathbb{S}^d$. One way of seeing this is via local Weyl laws (see [6]): for any $x \in M$, we have

$$\sum_{k=1}^{n} \phi_k(x)^2 = n + \mathcal{O}(n^{\frac{d-1}{d}}). \tag{2}$$

If we take some 'generic' manifold (say, without any symmetries or an arbitrary domain subjected to a generic diffeomorphism), then numerical experiments indicate that $\|\phi_k\|_{L^\infty}$ tends to grow only very slowly (see e.g. [1]). It is assumed that the growth is perhaps only logarithmic: a guess sometimes mentioned is $\|\phi_k\|_{L^\infty} \leq c\sqrt{\log \lambda_k}$. In contrast, on manifolds on which the eigenvalue problem is explicitly solvable, we frequently encounter eigenfunction growth. On the $d-$dimensional torus $\mathbb{T}^d$ and $d \geq 5$, classical results from number theory imply

$$\|\phi_k\|_{L^\infty} \leq c \lambda_k^{\frac{d-2}{4}}$$

and this bound is best possible (see Bourgain [4]). Toth & Zelditch [9] have established that a uniform bound $\|\phi_k\|_{L^\infty} \leq c$ requires (under some assumptions) the manifold (M,g) to be flat and thus one would perhaps not expect this to be generic.

The main purpose of this work is to introduce and study Ɪ, an object related to the spectral projector Π but rougher. We will argue that it has interesting properties and that these properties can be used to study the growth of eigenfunctions. Given the first n eigenfunctions $\phi_1, \dots, \phi_n$, we define $\text{Ɪ} : M \times M \to \mathbb{R}$ via

$$\text{Ɪ}^{(n)}(x,y) = \sum_{k=1}^{n} \operatorname{sgn}(\phi_k(x)) \phi_k(y), \tag{3}$$

The main result shows that if the eigenfunctions ϕ_k undergo significant concentration (beyond the logarithmic scale), then there exists an interesting type of long-range correlation. The most interesting special case is *spooky action at a distance*: the existence of points $x, y \in M$ such that the sequences $(\phi_k(x))_{k \in \mathbb{N}}$ and $(\phi_k(y))_{k \in \mathbb{N}}$ do *not* behave like independent random variables. This dramatically violates Berry's random wave model [3]. We show that spooky action does indeed occur on $\mathbb{S}^1$ and present numerical examples showing that it seems to happen on many, if not all, general 'structured' manifolds for which eigenfunctions can be computed in closed form.

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Finite rank Lieb–Thirring inequalities

Rupert L. Frank

(joint work with D. Gontier and M. Lewin)

Given $0 \leq U \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$, where $\gamma > \frac{1}{2}$ if $d = 1$ and $\gamma > 0$ if $d \geq 2$, we consider the Schrödinger operator

$$-\Delta - U \qquad \text{in } L^2(\mathbb{R}^d)\,.$$

This is a selfadjoint, lower semibounded operator whose negative spectrum consists of eigenvalues of finite multiplicities with 0 as the only possible accumulation point. We denote these eigenvalues in nondecreasing order and repeated according to multiplicities by $E_n(U)$ and use the convention that $E_n(U) = 0$ if n exceeds the number of negative eigenvalues. Given $N \in \mathbb{N} = \{1, 2, 3, \ldots\}$, we are interested in the smallest possible constant $L^{(N)}_{\gamma,d}$ such that

$$\sum_{n=1}^{N} |E_n(U)|^\gamma \leq L^{(N)}_{\gamma,d} \int_{\mathbb{R}^d} U(x)^{\gamma+\frac{d}{2}}\,dx\,.$$

The constant $L^{(N)}_{\gamma,d}$ may depend on N, γ and d, but not on U. The fact that such a constant exists follows from Sobolev inequalities. Moreover, it is clear that

$$L^{(1)}_{\gamma,d} \leq L^{(2)}_{\gamma,d} \leq L^{(3)}_{\gamma,d} \leq \ldots\,.$$

It is a theorem of Lieb and Thirring (1976) that

$$L_{\gamma,d} := \sup_N L^{(N)}_{\gamma,d} < \infty\,.$$

We are interested in the optimization problem $L^{(N)}_{\gamma,d}$, that is, the problem of maximizing $\sum_{n=1}^{N} |E_n(U)|^\gamma$ over all U with fixed $L^{\gamma+\frac{d}{2}}$-norm. Moreover, we are interested in the behavior of the constants $L^{(N)}_{\gamma,d}$ for large N.

The following result provides a bubble decomposition for optimizing sequences for the optimization problem $L^{(N)}_{\gamma,d}$.

Theorem 1. *Let $\gamma > \frac{1}{2}$ if $d = 1$ and $\gamma > 0$ if $d \geq 2$, and let $N \in \mathbb{N}$. Let $(U_j) \subset L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$ be a sequence of nonnegative functions such that, as $j \to \infty$,*

$$\int_{\mathbb{R}^d} U_j(x)^{\gamma+\frac{d}{2}}\,dx \to 1 \qquad \text{and} \qquad \sum_{n=1}^{N} |E_n(U_j)|^\gamma \to L_{\gamma,d}^{(N)}\,.$$

Then there are $K \in \mathbb{N}$, $0 \leq U^{(1)}, \ldots, U^{(K)} \in L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$, $(a_j^{(1)}), \ldots, (a_j^{(K)}) \subset \mathbb{R}^d$ such that, as $j \to \infty$ along a subsequence,

$$U_j = \sum_{k=1}^{K} U^{(k)}(\cdot - a_j^{(k)}) + o_{L^{\gamma+\frac{d}{2}}}(1) \qquad \text{and} \qquad |a_j^{(k)} - a_j^{(k')}| \to \infty \text{ if } k \neq k'\,.$$

Moreover, there are $N_1, \ldots, N_K \in \mathbb{N}$ such that

$$\sum_{k=1}^{K} N_k = N$$

and, for each $k = 1, \ldots, K$,

$$L_{\gamma,d}^{(N_k)} = L_{\gamma,d}^{(N)} \qquad \text{and} \qquad U^{(k)} \text{ is an optimizer for } L_{\gamma,d}^{(N_k)}\,.$$

Finally, if $K \geq 2$, then

$$E_{N_k+1}(U^{(k)}) = 0 \qquad \text{for all } k = 1, \ldots, K\,.$$

This theorem has the following consequence.

Corollary 2. *Under the assumptions on γ of Theorem 1, for every $N \in \mathbb{N}$ there is an optimizer for $L_{\gamma,d}^{(N)}$.*

Proof. It follows from the theorem that

$$\sum_{k=1}^{K} \int_{\mathbb{R}^d} U^{(k)}(x)^{\gamma+\frac{d}{2}}\,dx = 1 \qquad \text{and} \qquad \sum_{k=1}^{K}\sum_{n=1}^{N} |E_n(U^{(k)})|^\gamma = L_{\gamma,d}^{(N)}\,.$$

As a consequence, each $U^{(k)}$ is an optimizer for $L_{\gamma,d}^{(N)}$. □

Our second main result describes the large-N behavior of $L_{\gamma,d}^{(N)}$.

Theorem 3. *Let $\gamma > \frac{3}{2}$ if $d = 1$, $\gamma > 1$ if $d = 2$, $\gamma > \frac{1}{2}$ if $d = 3$ and $\gamma > 0$ if $d \geq 4$. Then for any $N \in \mathbb{N}$ one has $L_{\gamma,d}^{(2N)} > L_{\gamma,d}^{(N)}$. In particular,*

$$L_{\gamma,d} > L_{\gamma,d}^{(N)} \qquad \text{for all } N \in \mathbb{N}\,.$$

The idea of the proof is to construct a trial potential for the $L_{\gamma,d}^{(2N)}$-problem that consists of two widely separated copies of optimizers of the $L_{\gamma,d}^{(N)}$-problem. It is important that this addition of pieces is not performed at the level of potentials themselves, but rather at the level of their $(\gamma+\frac{d}{2}-1)$-powers. The proof makes use of an exponentially small attraction between these two widely separated bumps, which we can prove under the assumption $\gamma + \frac{d}{2} > 2$.

As a consequence of Theorem 3 one can show that bubbling does not occur.

Corollary 4. *Under the assumptions on γ of Theorem 3, for any $N \in \mathbb{N}$ one has $K = 1$ in Theorem 1. In particular, normalized optimizing sequences for $L^{(N)}_{\gamma,d}$ are relatively compact up to translations in $L^{\gamma+\frac{d}{2}}(\mathbb{R}^d)$.*

Proof. Assume we had $K \geq 2$. Then $N = \sum_k N_k \geq K \min N_k =: KN_*$ and so, by Theorems 1 and 3,

$$L^{(N)}_{\gamma,d} = L^{(N_*)}_{\gamma,d} < L^{(2N_*)}_{\gamma,d} \leq L^{(N)}_{\gamma,d} \,,$$

a contradiction. □

Let us discuss the implications of the above results to the Lieb–Thirring conjecture. Recall that by Weyl asymptotics, as $\alpha \to \infty$,

$$\sum_n |E_n(\alpha U)|^\gamma \sim \alpha^{\gamma+\frac{d}{2}} \iint_{\mathbb{R}^d\times\mathbb{R}^d} (|\xi|^2 - \alpha U(x))_-^\gamma \, \frac{dx\, d\xi}{(2\pi)^d} =: \alpha^{\gamma+\frac{d}{2}} L^{\mathrm{sc}}_{\gamma,d} \int_{\mathbb{R}^d} U(x)^{\gamma+\frac{d}{2}}\, dx.$$

Thus, clearly

$$L_{\gamma,d} \geq \max\{L^{(1)}_{\gamma,d}, L^{\mathrm{sc}}_{\gamma,d}\} \,.$$

Lieb and Thirring had originally conjectured that equality holds in this inequality. This is still believed to be true in dimension $d = 1$ and has been proved for $\gamma \in \{\frac{1}{2}\} \cup [\frac{3}{2}, \infty)$. In higher dimension, however, the situation turns out to be more complicated.

Lemma 5. *If $d \leq 7$ there is a $\gamma_* > 0$ such that*

$$L^{(1)}_{\gamma,d} > L^{\mathrm{sc}}_{\gamma,d} \text{ for all } \gamma < \gamma_* \qquad \text{and} \qquad L^{(1)}_{\gamma,d} < L^{\mathrm{sc}}_{\gamma,d} \text{ for all } \gamma > \gamma_*.$$

If $d \geq 8$, then $L^{(1)}_{\gamma,d} < L^{\mathrm{sc}}_{\gamma,d}$ for all $\gamma > 0$.

In dimension $d = 3$ one has numerically $\gamma_* \approx 0.8627$. It follows from Corollary 4 and Lemma 5 that $L_{\gamma,3} > \max\{L^{(1)}_{\gamma,3}, L^{\mathrm{sc}}_{\gamma,3}\}$ for $\gamma \in (\frac{1}{2}, \gamma_*]$. The same inequality holds for $\gamma \in [\gamma_*, 1)$ by Lemma 5 and a result of Helffer and Robert (1990). In particular, in the regime $\gamma \in (\frac{1}{2}, 1)$ the constant $L_{\gamma,3}$ is given neither by the semiclassical constant $L^{\mathrm{sc}}_{\gamma,3}$ nor by a finite particle constant $L^{(N)}_{\gamma,3}$.

Similarly, in dimension $d = 2$ one has numerically $\gamma_* = 1.1653$. It follows from Corollary 4 and the lemma that $L_{\gamma,2} > \max\{L^{(1)}_{\gamma,2}, L^{\mathrm{sc}}_{\gamma,2}\}$ for $\gamma \in (1, \gamma_*]$. In particular, by continuity we see that there is an $\epsilon > 0$ such that in the regime $\gamma \in (1, \gamma_* + \epsilon)$ the constant $L_{\gamma,2}$ is given neither by the semiclassical constant $L^{\mathrm{sc}}_{\gamma,2}$ nor by a finite particle constant $L^{(N)}_{\gamma,2}$.

It is conceivable that in the regimes where $L_{\gamma,d}$ is neither given by the semiclassical constant nor by a finite particle constant it is instead attained by a periodic U (in the sense that both sides of the Lieb–Thirring inequality are computed on a per volume basis). There is an analytic analogue for this in $d = 1$ at $\gamma = \frac{3}{2}$, as well as numerical evidence in $d = 2$ in a tiny neighborhood around $\gamma = \gamma_*$.

The results presented in this talk are based on the papers [1, 2, 3]. We refer to these papers, as well as the book [4], for further background and references.

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A universal inequality for Neumann eigenvalues of the Laplacian on convex domains in Euclidean space

Kei Funano

Let Ω be a bounded convex domain in $\mathbb{R}^n$. In this setting the Neumann eigenvalues of the Laplacian on Ω is closely related with the diameter of Ω:

$$\mu_1(\Omega) \gtrsim \frac{1}{(\operatorname{Diam}\Omega)^2} \text{ (Payne-Weinberger [16])}$$

and

$$\mu_k(\Omega) \lesssim \frac{n^2k^2}{(\operatorname{Diam}\Omega)^2} \text{ (Kröger [12])} .$$

Here $\mu_k(\Omega)$ is the k-th nontrivial Neumann eigenvalue of the Laplacian on Ω and $a \lesssim b$ stands for $a \leq Cb$ for some absolute (and concrete) constant $C > 0$. Combining these inequalities one can get $\mu_k(\Omega) \lesssim n^2k^2\mu_1(\Omega)$.

On the other hand there is a notion of the *observable diameter* for a bounded domain Ω introduced by Gromov (Refer to [10] for the definition). The observable diameter comes from the study of '*concentration of measure phenomenon*' and it might be interpreted as a substitute of the usual diameter. By definition it depends on a parameter $\kappa \in [0, 1]$. In terms of the observable diameter the first eigenvalue has an upper bound

$$\mu_1(\Omega) \lesssim_\kappa \frac{1}{\operatorname{ObsDiam}_\kappa(\Omega)^2} \text{ (Gromov-V. Milman [11])}$$

for a bounded (but not necessarily convex!) domain Ω and has a lower bound

$$\mu_1(\Omega) \gtrsim_\kappa \frac{1}{\operatorname{ObsDiam}_\kappa(\Omega)^2} \text{ (E. Milman [15])}$$

for a bounded convex domain Ω (See [7, Introduction] to derive these inequalities). Observe that the above two inequalities are dimension-free and one might wonder whether the dimension-free counterpart of the Kröger inequality holds for observable diameter. By the above E. Milman inequality the counterpart is equivalent to the following universal inequality for Neumann eigenvalues:

$$\mu_k(\Omega) \lesssim k^2 \mu_1(\Omega). \tag{1}$$

The inequality (1) was proved by Liu in [14] for any bounded convex domain Ω in $\mathbb{R}^n$, which improved my inequality $\mu_k(\Omega) \lesssim \exp(Ck)\mu_1(\Omega)$ ([8, 7]), where $C > 0$ is an absolute constant. The inequality (1) is sharp with respect to the order of k.

How about the opposite of the inequality (1)? The recent result concerning a quantitative version of Weyl's law by Chitour-Prandi-Rizzi ([4]) and an upper bound of μ_1 in terms of the volume by Kröger ([13]) imply that whenever Ω is a bounded smooth convex domain and $k \geq C(n, \mathrm{inj}_{\partial\Omega})$ we have

$$\mu_k(\Omega) \gtrsim k^{2/n} \mu_1(\Omega), \tag{2}$$

where $\mathrm{inj}_{\partial\Omega}$ is the injectivity radius from $\partial\Omega$. E. Milman obtained a lower bound of μ_1 in terms of a Sobolev-type constant under the convexity assumption ([15]). Using a heat kernel technique Cheng-Li gave a lower bound of μ_k in terms of the Sobolev constant ([1]). Combining these two inequalities shows (2) for any bounded convex domain Ω in $\mathbb{R}^n$ and $k \geq C(n)$.

In [7, 8] I conjectured that

$$\mu_{k+1}(\Omega) \lesssim \mu_k(\Omega) \tag{3}$$

for any bounded convex domain Ω and any k. Again a quantitative version of Weyl's law by Chitour-Prandi-Rizzi ([4]) confirms (3) for any bounded smooth convex domain Ω and $k \geq C(n, \mathrm{inj}_{\partial\Omega})$. In [6] I proved the universal inequality

$$\mu_{k+1}(\Omega) \lesssim n^4 \mu_k(\Omega) \tag{4}$$

for any bounded convex domain Ω in $\mathbb{R}^n$ and any k. The proof implies the following stronger result: Suppose $\Omega \subseteq \Omega'$ are two bounded convex domains in $\mathbb{R}^n$. Then we have $\mu_{k+1}(\Omega') \lesssim n^4 \mu_k(\Omega)$. In particular it implies a variant of domain monotonicity $\mu_k(\Omega') \lesssim n^4 \mu_k(\Omega)$. In this setting,

$$\mu_k(\Omega') \lesssim n^2 \mu_k(\Omega) \tag{5}$$

is obtained in [5] and this inequality is sharp with respect to the order of n. Hence we cannot hope to confirm the conjecture (3) using the method in [6].

One of key tools used in [6] is the following upper bound for Neumann eigenvalues. Let Ω be a bounded convex domain in $\mathbb{R}^n$ and $\{A_i\}_{i=0}^k$ is a sequence of Borel subsets of Ω. Then we have

$$\mu_k(\Omega) \lesssim \frac{n^2}{(\min_{i \neq j} d(A_i, A_j) \log(k+1))^2} \max_i \left(\log \frac{\mathrm{vol}(\Omega)}{\mathrm{vol}(A_i)} \right)^2. \tag{6}$$

This is a variant of upper bounds for Neumann eigenvalues obtained by Chung-Grigor'yan-Yau [3] (see also [9] for generalization).

I end this abstract by raising some questions:

- Does the inequality (2) hold for any k provided that Ω is bounded and convex?
- The proof of (4) was geometric. Is there analytic proof for (4) or (3)?

- Can we generalize the inequality (4) for a convex domain in a Riemannian manifold of nonnegative Ricci curvature?
- Can we remove the n^2 factor in (6)? If we can it would imply $\mu_{k+1}(\Omega') \lesssim n^2\mu_k(\Omega)$ for any k and any two bounded convex domains $\Omega \subseteq \Omega'$ of $\mathbb{R}^n$.
- Can we prove (5) for convex Ω and non-convex Ω' such that $\Omega \subseteq \Omega'$?
- If we fix Ω' as a unit disk in $\mathbb{R}^n$ can we improve the order of n in (5)?

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Construction of quasimodes for generalized semiclassical operators

Daniel Grieser
(joint work with D. Sobotta)

We consider general parameter-dependent ordinary differential operators:

$$P = (P_h)_h = \sum_{k=0}^{m} C_k(x,h)\partial_x^k\,, \qquad \partial_x := \frac{d}{dx} \tag{1}$$

where the coefficients C_k are real analytic functions of $h \in [0,h_0)$, $x \in I$ for an open interval $I \subset \mathbb{R}$, and $C_m \not\equiv 0$. (In fact, weaker assumptions guaranteeing finite order zeroes of certain associated functions suffice for our results.) The asymptotic behavior of solutions $u = (u_h)_{h>0}$ of $Pu = 0$ (or an inhomogeneous equation $Pu = f$) as $h \to 0$ has been studied intensively for some special classes of operators P. If C_m has no zeroes at $h = 0$ then this is a standard regular perturbation problem. Otherwise this is a singular perturbation problem, and there is a plethora of phenomena that can occur. For instance, if C_m vanishes identically at $h = 0$ then P_0 has lower order than P_h for $h > 0$, so the limit problem $P_0u_0 = 0$ has 'not enough' solutions for standard perturbation theory. A special and much studied case where this happens are so-called *semiclassical* operators, where $C_k(x,h) = h^kD_k(x,h)$ for each k, with D_k smooth in $h \geq 0$. Thus $P = \sum_{k=0}^m D_k(x,h)(h\partial_x)^k$ then. Therefore, we call operators of the form (1) *generalized semiclassical operators*. Our aim is to study the asymptotics problem for general P. Note that this also includes singular operators, where the C_k do not depend on h but C_m may vanish at some points $x \in I$.

The most important example of a semiclassical operator is the Schrödinger operator

$$P = h^2\partial_x^2 + V(x) \tag{2}$$

where our minimal assumptions are that the potential V is smooth with zeroes of finite order. Another case of interest is Bessel's equation $Bu = 0$, where $B = r^2\partial_r^2 + r\partial_r + r^2 - \nu^2$. An important classical problem is to study its solutions as both r and the parameter ν tend to ∞. The substitution $x = \frac{1}{r}$, $h = \frac{1}{\nu}$ transforms B into $x^{-2}h^{-2}P^{\text{Bessel}}$ where

$$P^{\text{Bessel}} = h^2x^4\partial_x^2 + h^2x^3\partial_x + h^2 - x^2 \tag{3}$$

has the form (1). This operator is highly degenerate at $h = 0$, $x = 0$.

Our main result can be roughly stated as follows.

Theorem 1. *For P of the form* (1) *there is a basis of quasimodes which is exponential-polyhomogeneous on a space obtained from $I \times [0,h_0)$ by iterated inhomogeneous blow-ups.*

We give an algorithm for constructing these quasimodes. We need to impose a mild 'separation' condition on P (see below), which we believe to be dispensible, however. We call a function u on the interior of the manifold with boundary $I \times [0, h_0)$ *exponential-polyhomogeneous* if it is smooth and

$$u = e^{\varphi(x)/h^\delta} a(x,h)\,, \quad a(x,h) \overset{h\to 0}{\sim} \sum_{j=0}^{\infty} h^{\gamma_j} a_j(x) \tag{4}$$

with φ and a_j smooth and complex valued, and $\delta \geq 0$, $0 = \gamma_0 < \gamma_1 < \dots \to \infty$. We also generalize this notion to manifolds with corners as they appear after the blow-ups. A function u as in (4) is called a *quasimode* for P if

$$e^{-\varphi/h^\delta} P e^{\varphi/h^\delta} a = O(h^\infty) \text{ as } h \to 0 \quad \text{(i.e. } O(h^N) \text{ for all } N). \tag{5}$$

We conjecture that there is a basis of actual solutions of $Pu = 0$ having the same asymptotic behavior as the quasimodes that we construct. To prove this would require more work, however. The main point of Theorem 1 is its generality and explicitness, and the identification of the essential structures in the problem, some of which we describe below. Also, the algorithm should be easily modifiable to yield the construction of pairs (u_h, E_h) satisfying the approximate eigenvalue equations $(P - E)u = O(h^\infty)$, which yields approximate information about the spectrum, compare [Gri17].

Our construction is an intricate combination of three well-known ideas:

(1) The WKB-construction
(2) Newton polygon analysis
(3) Blow-up

The **WKB construction** is classical for the Schödinger operator (2) with potential $V > 0$. It yields quasimodes (4) with $\delta = 1$ and $\gamma_j = j$. The *phase function* φ and *amplitude* coefficients a_j are obtained by plugging (4) as ansatz into $Pu = 0$ and sorting by powers of h. The least power, h^0, yields the *eikonal equation*

$$E(x, \varphi'(x)) = 0\,, \quad E(x,\zeta) = \zeta^2 + V(x) \tag{6}$$

where the *eikonal polynomial* E is obtained from P by replacing $h\partial_x$ by a formal variable ζ. If φ is chosen to satisfy the eikonal equation then the h^1 power yields the *transport equation*

$$Ta_0 = 0\,, \quad T = 2\varphi'\partial_x + \varphi'' \tag{7}$$

Note that $\varphi' = \pm i\sqrt{V}$ is smooth and has no zeroes by the assumption $V > 0$, so T is a regular (elliptic) operator. Higher powers of h yield a recursive system of equations for the a_j, all involving the same transport operator T, but inhomogeneous: $Ta_j = -a''_{j-1}$. Borel summing the series $\sum_{j=0}^\infty h^j a_j$ yields the quasimode. Note that the same procedure can be used if $V < 0$ everywhere, but now φ is real.

Newton polygon analysis is a classical tool to analyze solutions of polynomial equations in two variables. In our context it appears as follows. By expanding the coefficients in (1) we can write

$$P = \sum_{k,\alpha} c_{k,\alpha}(x) h^\alpha \partial_x^k \tag{8}$$

We write $\lambda = (k, \alpha)$ and denote by $\Lambda_P \subset \{0, \dots, k\} \times \mathbb{N}_0$ the set of those λ for which $c_\lambda \not\equiv 0$. We define the *Newton polygon* of P as

$$\mathcal{P}(\Lambda_P) := \operatorname{conv}\left(\bigcup_{\lambda \in \Lambda} \lambda + \square\right) \cap \mathbb{R}^2_+ \tag{9}$$

where $\mathbb{R}_+ = [0, \infty)$, conv denotes the convex hull and $\square = (-\infty, 0] \times \mathbb{R}_+$ is the second quadrant. The lower boundary of $\mathcal{P}(\Lambda_P)$ is the union of line segments (edges) $\mathcal{L}$ of pairwise different slopes $\delta_{\mathcal{L}} \geq 0$. We denote by $|\mathcal{L}|$ the width of $\mathcal{L}$, i.e. the length of its projection to the k-axis.

The first step in the proof of Theorem 1 is the construction of quasimodes (4), locally near $x_0 \in I$, associated to each edge $\mathcal{L}$, under certain regularity conditions (no blow-ups are needed then). To each edge with $\delta_{\mathcal{L}} > 0$ one can associate an *eikonal polynomial* $E_{\mathcal{L}}(x, \zeta)$, which is a polynomial in ζ of degree $|\mathcal{L}|$ whose coefficients are those coefficients c_λ in (8) with $\lambda \in \mathcal{L}$. Let $x \mapsto \zeta(x)$ be a solution branch of $E_{\mathcal{L}}(x, \zeta(x)) = 0$. Suppose

(Reg) the multiplicity r of $\zeta(x)$ as a zero of $E_{\mathcal{L}}(x, \cdot)$ is constant for x near x_0.

Then there are r independent quasimodes as in (4), where $\delta = \delta_{\mathcal{L}}$ and $\varphi' = \zeta$, with φ and the a_j smooth near x_0.[1] The a_j are obtained via a transport operator associated to $\mathcal{L}$ and the branch ζ. It has order r and leading coefficient $\partial_\zeta^{r-1} E_{\mathcal{L}}(x, \zeta(x))$. If each branch of $E_{\mathcal{L}}$ is regular at x_0 then $a_\mu(x_0) \neq 0$, where μ is the right endpoint of $\mathcal{L}$. If this endpoint condition is satisfied for the (potential) edge with $\delta_{\mathcal{L}} = 0$ and if (Reg) is satisfied for all edges with $\delta_{\mathcal{L}} > 0$ and all solution branches ζ then one obtains $m = \operatorname{ord} P$ independent quasimodes. In the Schrödinger example we have (Reg) $\iff V(x_0) \neq 0$.

The bulk of the work for proving Theorem 1 is in the extension of this result to non-regular operators. In a first extension we show that for branches ζ which are unbounded as $x \to x_0$ (which may exist if $a_\mu(x_0) = 0$) one obtains, under certain conditions, quasimodes in $x > x_0$ and $x < x_0$ of the form (4), but with φ and a only polyhomogeneous, i.e. having generalized Taylor expansions as $x \to x_0 \pm 0$ (uniformly as $h \to 0$). We call operators satsifying these conditions *resolved*. This extension involves the *Newton polyhedron* of P at x_0, which is defined similarly to $\mathcal{P}(\Lambda_P)$ but also involves the vanishing orders of the c_λ at $x = x_0$.

A classical example of a non-resolved operator is $P = h^2\partial_x^2 + x$, where the potential x has a zero ('turning point'). Its solutions have the form $u(x, h) = A(\frac{x}{h^{2/3}})$ with A solving $(\partial_\xi^2 + \xi)A(\xi) = 0$. This could be expanded (for $x > 0$ say) as in (4), but the a_j will behave increasingly singular at $x = 0$ as j increases (like $x^{-3j/2}$). To get a uniform description near $(x, h) = (0, 0)$ we observe that the occurence of the variable $\frac{x}{h^{2/3}}$ indicates that the pull-back of u to the space obtained by **blowing up** $\mathbb{R} \times [0, h_0)$ at $(0, 0)$ (with inhomogeneity $\frac{2}{3}$) should behave

[1] Strictly speaking, this holds under an additional *separation condition*, which states that there should be no points of Λ_P in the interior of $\mathcal{P}(\Lambda_P)$ and 'too close' to $\mathcal{L}$. In the regular case we prove the result also in the non-separated case, but then the quasimodes have additional factors of the form $e^{\psi(x)/h^{\delta'}}$ with $\delta' < \delta$.

well. Indeed, it was shown in [Sob18] that for V having a simple zero there is a basis of quasimodes which are exponential-polyhomogeneous on this blown-up space (and there are corresponding results for true solutions, see e.g. [KS22] for a modern and non-standard treatment).

The general idea now is that if P is not resolved at a point x_0 then the pull-back $\beta^* P$ under the blow-down map β for a blow-up of the point $(x_0, 0) \in I \times [0, h_0)$ will be 'more resolved' than P itself if the inhomogeneity order of the blow-up is chosen appropriately. The core of the proof of Theorem 1 is to show that after finitely many suitable such (iterated) blow-ups the pulled back operator will be resolved everywhere. We refer to the dissertation [Sob23] for details.

In the example of the Bessel equation our algorithm reproduces, on the level of quasimodes, the well-known asymptotics of Bessel functions for large argument and index (see e.g. [Olv97]) which were recently cast in the language of blow-ups by Sher [She23].

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Inner radius of nodal domains in high dimensions

Philippe Charron

(joint work with D. Mangoubi)

Let (M, g) be a closed smooth Riemannian manifold of dimension d. Consider on M an eigenfunction u_λ of the positive Laplace-Beltrami operator $-\Delta_g$ corresponding to an eigenvalue λ. A nodal domain Ω_λ of u_λ is any connected component of the set $\{u_\lambda \neq 0\}$. It is well known that there exists a positive constant $c_{\text{up}} = c_{\text{up}}(M, g)$ independent of λ or u_λ such that every ball of radius bigger than $c_{\text{up}}\lambda^{-1/2}$ contains a zero of u_λ, i.e., the inner radius of Ω_λ is bounded from above:

$$inrad(\Omega_\lambda) \leq c_{\text{up}}\lambda^{-1/2} \ .$$

Given any $C > 0$ and $d \geq 3$, it is possible to construct open sets Ω_C in $\mathbb{R}^d$ such that $\lambda_1(\Omega_C) = 1$ and $inrad(\Omega_C) \leq C$. Indeed, one can take a ball and remove very thin needles. This process does not change λ_1 but it reduces the inner radius dramatically. The interesting question is to find lower bounds for fixed M as λ increases.

In dimension two, it was shown in [7] that the lower bound is of the same order: $inrad(\Omega_\lambda) > C(M)\lambda^{-1/2}$.

In dimensions greater than two, was known previously from [7] and that one has the asymptotic bounds $inrad(\Omega_\lambda) > C(M)\lambda^{-c(d)}$, with $c(d) > 1/2$ and grows with the dimension. Furthermore, if M is analytic then it was proven in [3] that $inrad(\Omega_\lambda) > C\lambda^{-1}$. The bound in [3] is better than the one in [7] when $d > 3$.

Here is the new result that was proven in [1]

Theorem 1. *Let (M, g) be of dimension at least three. Let $x_{\max} \in \Omega_\lambda$ be a point where $|u_\lambda(x_{\max})| = \max_{\Omega_\lambda} |u_\lambda|$. Then*

$$B\left(x_{\max}, c_{\mathrm{lo}}\lambda^{-1/2}(\log\lambda)^{-\frac{(d-2)}{2}}\right) \subset \Omega_\lambda$$

where $c_{\mathrm{lo}} = c_{\mathrm{lo}}(M, g)$ is a positive constant which depends only on (M, g).

The main tools to prove the theorem are the following:

(1) A result from [4] which states that the capacity of the complement of a nodal domain inside a ball of radius $r\lambda^{-1/2}$ centered at a global maximum of the nodal domain is less than Cr^2, with C depending only of M but not on λ.
(2) A Remez inequality for solutions of elliptic equations from [6].
(3) Classical doubling estimates from [2].
(4) Classical gradient estimates from [5].

Here is a summary of the proof:

(1) We start on a cube of radius $r\lambda^{-1/2}$ centered at the maximum of a nodal domain.
(2) We estimate the volume of the complement of the nodal domain by the capacity to volume isoperimetric inequality.
(3) We divide the cube into A^d subcubes such that on each cube, the complement covers at most half of the volume of the subcube. The smaller r is, the larger A can be.
(4) We apply Remez's inequality to the eigenfunction on each layer of subcubes.
(5) If A is too large or the supremum of the eigenfunction on the half-cube is too large, then the doubling of the eigenfunction on the original cube will be larger than the $C\sqrt{\lambda}$ upper bound from [2].
(6) This gives us an upper bound on A, which in turn gives us a lower bound on r. The bound obtained on r is $C(M)(\log(\lambda))^{-\frac{d-2}{2}}$.
(7) Since the function is bounded on the half-cube, we use the gradient estimates for solutions of elliptic equations to obtain that the gradient in the quarter-ball is less than $C(M)/r$. In turn, this gives us that there is a ball of radius Cr around the global maximum such that the eigenfunction does not vanish.

In the talk, I also described how one can refine the calculations to obtain (unpublished) $(\lambda \log\log(\lambda))^{-1/2}$ bounds in dimension 3 by using the capacity-to-volume of thin cylinders.

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Inhomogeneous Laplace Equations and String Theory

Ksenia Fedosova

(joint work with K. Klinger-Logan and D. Radchenko)

Many objects of interest in string theory are functions or, more generally, sections of vector bundles on a locally symmetric space that additionally satisfy certain differential equations. For example, consider a modular surface, $X = \mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$, and an inhomogeneous Laplace equation

$$(\Delta - r(r+1))f(z) = Q(z), \quad z \in X, \tag{1}$$

where Q is a fixed function on X, $r > 0$ and $\Delta = y^2(\partial_x^2 + \partial_y^2)$. The above equation is related to scattering amplitudes in type IIB string theory in 10 space-time dimensions. We are particularly interested in the case, corresponding to 4-graviton scattering, where the inhomogeneous part, Q, is a product of two Eisenstein series with half-integers, or

$$(\Delta - r(r+1))f(z) = E_a(z)E_b(z), \quad z \in X \tag{2}$$

for $a, b \in \mathbb{N}_0 + 1/2$. Note that the condition on r implied that $\Delta - r(r+1)$ is a strictly negative operator. However, we cannot apply the resolvent to the right hand side of (2) to find f, because $E_a(z)E_b(z)$ is not square-integrable; thus, we have resort to other methods of recovering the solution.

In my talk I surveyed two major questions related to solutions of (2):

- Consider a pull-back of f to the hyperbolic upper half-plane $\mathbb{H} = \{z = x + iy \text{ with } y > 0\}$ that we denote by the same letter. The $\mathrm{PSL}_2(\mathbb{Z})$-invariance of f implies its invariance under the transformation of $z \mapsto z+1$, that in turn yields the Fourier expansion of f in the x-variable. Can we recover its Fourier coefficients?

- It turns out that Fourier coefficients of f include sums of the type

$$\sum_{\substack{n_1,n_2\in\mathbb{Z}\setminus\{0\}\\ n_1+n_2=n}} \sigma_0(n_1)\sigma_0(n_2)\Big[\frac{n_2-n_1}{n}\log\left|\frac{n_1}{n_2}\right|+2\Big] \tag{3}$$

for any given $n\in\mathbb{N}$. That leads to the second question: what can we say about values of the sums?

First question. We partially answered the first question in [2]. To solve the equation, we used separation of variables, then, assuming that solutions belong to a certain Picard-Vessiot extension of a differential field, wrote solutions to ordinary differential equations in a form depending on a family of parameters and finally, found the mentioned parameters using a system of computer algebra[1]. This manifested in the following theorem:

Theorem 1. *Let* $(r,a,b)\in\mathbb{N}^3$ *with*

$$|a-b|<r \quad \text{and} \quad a+b+r \text{ is odd} \tag{4}$$

and let $f:\mathbb{H}\to\mathbb{C}$ *be a* 1*-periodic function in the* x*-variable that satisfies* (1)*. Then, for* r, a, b *sufficiently small,* $f(z)=\sum_{n\in\mathbb{Z}}\hat{f}_n(y)e^{2\pi i n x}$ *and there exist* $\alpha_n,\beta_n\in\mathbb{C}$ *such that for* $n\neq 0$,

$$\begin{aligned}\hat{f}_n(y) &= \alpha_n\sqrt{y}K_{r+1/2}(2\pi|n|y)+\beta_n\sqrt{y}I_{r+1/2}(2\pi|n|y)\\ &\quad+\sum_{\substack{n_1,n_2\in\mathbb{Z}\\ n_1+n_2=n}}\sum_{i,j\in\{0,1\}} q^{i,j}(y)K_i(2\pi|n_1|y)K_j(2\pi|n_2|y),\end{aligned} \tag{5}$$

and for $n=0$,

$$\hat{f}_0(y)=\alpha_0 y^{-r}+\beta_0 y^{r+1}+\sum_{\substack{n_1,n_2\in\mathbb{Z}\\ n_1+n_2=0}}\sum_{i,j\in\{0,1\}}\mu^{i,j}(y)K_i(2\pi|n_1|y)K_j(2\pi|n_2|y),$$

where for $\eta\in\mathbb{C}$, I_η *and* K_η *denote the modified Bessel function of the first and second kind of index* η*, respectively, and where* $q^{i,j}=q^{i,j}_{n_1,n_2,\lambda,\alpha,\beta}$ *and* $\mu^{i,j}=\mu^{i,j}_{n_1,n_2,\lambda,\alpha,\beta}$ *are Laurent polynomials in* y.

One of the goals for further research would be to prove that the solution of such form exists for every tuple (r,a,b) satisfying (4) and to find some closed formulas on $q^{i,j}$. Additionally, establishing a connection to the differential Galois theory would be an interesting open problem.

[1] We used a Python program that performed integer arithmetic only.

Second question. The automorphy of f from the theorem above poses certain restrictions on α_n. Very roughly[2], the growth rate of f at the cusp dictates the behavior of Fourier coefficients of f as $y \to 0$. After studying the asymptotic behavior of (5) as $y \to 0$, we obtain that α_n should be equal to a linear combination of

$$(6) \qquad \sum_{\substack{n_1+n_2=n \\ n_1,n_2\in\mathbb{Z}\setminus\{0\}}}^{\infty} \sigma_{r_1}(n_2)\sigma_{r_2}(n_2)|n_1|^{\tau} \text{ and } \sum_{\substack{n_1+n_2=n \\ n_1,n_2\in\mathbb{Z}\setminus\{0\}}}^{\infty} \sigma_{r_1}(n_2)\sigma_{r_2}(n_2)|n_1|^{\tau}\log|n_1|$$

for fixed $n \in \mathbb{Z}$ and various τ. We note that for certain values of the parameters a, b and r, numerical evaluations vaguely hinted that α_n might be equal to zero; in particular, in [3] we stated a conjecture that (3) admits a closed expression and should be evaluated as $(2 - \log\left(4\pi^2|n|\right))\sigma_0(n)$.

For certain other values of a, b and r, however, the "magic" vanishing of α_n does not happen. However, as we show in an on-going work with Danylo Radchenko, there is a correction term involving Fourier coefficients of Hecke eigenfunctions, that manifests in the following theorem:

Theorem 2. *For any $n \in \mathbb{Z}_{>0}$, $d \in \mathbb{Z}_{>0}$ and $r_1, r_2 \in 2\mathbb{Z}_{\geq 0}$,*

$$\sum_{\substack{n_1,n_2\in\mathbb{Z}\setminus\{0\} \\ n_1+n_2=n}} Q_d^{(r_1,r_2)}\Big(\frac{n_2-n_1}{n_1+n_2}\Big)\sigma_{r_1}(n_1)\sigma_{r_2}(n_2)$$
$$= (-1)^d C_d^{(r_1,r_2)}(n)\sigma_{r_1}(n) - C_d^{(r_2,r_1)}(n)\sigma_{r_2}(n) + \frac{a_n}{n^d},$$

where $Q_d^{(r_1,r_2)}$ is a Jacobi function of the second kind,

$$C_d^{(r_1,r_2)}(n) = \begin{cases} \frac{(r_2-1)!(r_1+d)!}{2(r_1+r_2+d)!}\zeta(r_2)n^{r_2} + \binom{d+r_2}{d}\frac{\zeta'(-r_2)}{2} & r_2 \neq 0, \\ \frac{1}{4}\left(H_{d+r_1} + H_d - \log\left|4\pi^2 n\right|\right) & r_2 = 0, \end{cases}$$

where H_d is the d-th harmonic number and $h(\tau) := \sum_{m\geq 1} a_m q^m$ is a cusp form of weight

$$k := 2d + r_1 + r_2 + 2$$

on $SL_2(\mathbb{Z})$, given by $h = \sum_f \lambda_f f$, where f runs over normalized Hecke eigenforms[3] of weight k and level 1, and

$$\lambda_f = \frac{\pi(-1)^{d+r_2/2+1}}{2^k}\binom{k-2}{d}\frac{L^\star(f, d+1)L^\star(f, r_1+d+1)}{\langle f, f\rangle}.$$

Where $\langle f, g\rangle := \int_{\Gamma\setminus\mathbb{H}} f(z)g(z)y^{k-2}dxdy$ is the Petersson inner product, and $L^\star(f, \cdot)$ is the completed L-function of f:

$$L^\star(f, s) = (2\pi)^{-s}\Gamma(s)L(f, s).$$

[2] For a precise statement, see [1, Lemma 2.9].

[3] We say that a Hecke eigenform is normalized if its first non-zero Fourier coefficient is equal to 1.

We note that at the moment, the theorem does not apply to all possible values of a, b, r from Theorem 1 for which we found solutions in the closed form. Moreover, given a highly unexpected[4] form of the correction term, it would be interesting to consider (1) in a different setting, e.g., for other sources, Q, and for other locally symmetric spaces.

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Variation of geometry and spectrum

Chris Judge

(joint work with L. Hillairet)

Karen Uhlenbeck showed showed that the Dirichlet Laplacian acting on a generic bounded domain in $\mathbb{R}^d$ with smooth boundary has one dimensional eigenspaces [5]. Her method also applies to many other contexts in which one has an infinite dimensional space of compactly resolved, self-adjoint, elliptic operators. For example, the Dirichlet Laplacian associated to a generic Riemannian metric on a compact manifold has simple spectrum [5].

However, if the space of operators is only finite dimensional, then Uhlenbeck's method does not apply, and in fact, generic simplicity fails in natural situations. For example, the Laplacian of each flat torus has multiplicities.

Recently, Hezari and Zelditch raised the question of generic simplicity for the Laplacian of the generic ellipse (see Conjecture 6 in [2]). Using a method that we developed in [1], we answer their question for the Dirichlet Laplacian.

Theorem 1. *For all but countably many eccentricities, the Dirichlet Laplacian on an ellipse has simple spectrum.*

I will now sketch a proof of this theorem as it provides an elementary context in which to discuss some of the basic ideas of our general method.

Up to isometry and homothety, each ellipse has the form

$$\Omega_t \ := \ \left\{(x', y) \,:\, (t \cdot x')^2 \ + \ y^2 \ < \ 1\right\}$$

[4] Why would gravitons know values of L-functions?

where $0 < t \leq 1$. To compare the Laplacians on the various Ω_t, we pull back the operators to a common domain, the unit disc $D = \Omega_1$. After making the change of variable $x = tx'$ we find that the Dirichlet energy equals $t^{-1} \cdot q_t$ where

$$q_t(u) \;=\; \int_D \left(t^2 \cdot |u_x|^2 \;+\; |u_y|^2\right)\, dx\, dy$$

and the L^2-norm on Ω_t pulls back to $t^{-1}\|\cdot\|$, where $\|\cdot\|$ is the L^2-norm of the unit disc. Thus, we only need to study the Dirichlet eigenvalues of q_t with respect to the L^2 inner-product $\langle\cdot,\cdot\rangle$ on D.

Reflection about the x-axis preserves the unit disc, and it follows that $L^2(D)$ is an orthogonal direct sum of the space, V^+, of functions that are even with respect to the reflection and the space, V^-, of odd functions. We also have $q_t(v_-, v_+) = 0$ if $v_\pm \in V^\pm \cap H_0^1(D)$ and so the spectra of q_t splits into the 'even' spectra and the 'odd' spectra.

Analytic perturbation theory [3] applies to the family $q_t|_{V^\pm}$, and one finds that there exist analytic functions $\lambda_j^\pm : (0,\infty) \to \mathbb{R}$ and $u_j^\pm : (0,\infty) \to V^\pm$ so that $u_j^\pm(t)$ is an eigenfunction of q_t with eigenvalue $\lambda_j^\pm(t)$ and $\{u_j^\pm(t) \,:\, j \in \mathbb{N}\}$ is an orthonormal basis for $V^\pm$ for each t.[1] Each pairwise difference of the various $\lambda_j^\pm$ is analytic, and so such a difference vanishes for at most countably many t or the pair coincides for all t. It follows that Theorem 1 will follow from showing that no two of the various $\lambda_j^\pm$ coincide for all t.

The eigenvalues and eigenfunctions of q_1 are the eigenvalues and eigenfunctions of the unit disc. In particular, an orthonormal basis of $L^2(D)$ is given by products of Bessel functions J_k and sines/cosines. In fact, as a consequence of Siegel's deep work [4] on Bessel functions, one knows that the λ-eigenspace is spanned by

$$\{J_k(\sqrt{\lambda}r)\cos(k\theta), J_k(\sqrt{\lambda}r)\sin(k\theta)\}.$$

The former function is even with respect to the reflection about the x-axis whereas the latter function is odd. It follows that the 'even' spectra of q_1 and the 'odd' spectra of q_1 are both simple.

Therefore, to prove Theorem 1 it suffices to show that no 'even' eigenvalue branch λ_j^+ of q_t coincides (for all t) with an 'odd' eigenvalue branch λ_j^- of q_t. This follows from

Lemma 1. *For each j,*

- $\displaystyle\lim_{t\to 0} \sqrt{\lambda_j^+(t)} \;\in\; \left(\mathbb{Z} + \frac{1}{2}\right) \cdot \pi$
- $\displaystyle\lim_{t\to 0} \sqrt{\lambda_j^-(t)} \;\in\; \mathbb{Z} \cdot \pi.$

The possible limit points in Lemma 1 are 'threshholds' for the 'even' and 'odd' parts of the essential spectra of q_0. For example, an 'even' Weyl sequence associated with the eigenvalue $\pi^2(k + \frac{1}{2})^2/\sqrt{1-\delta^2}$ can be constructed using a Dirac

[1] Note that the indexing $j \mapsto \lambda_j$ has nothing to do with any ordering of the real numbers.

sequence $\phi_n(x)$ associated to the point $\delta \in (-1, 1)$. The sequence of functions $\cos\left(\pi(k+\frac{1}{2})y/\sqrt{1-x^2}\right) \cdot \phi_n(x)$ will be a Weyl sequence for q_0. A similar construction can be made for the odd spectra. Lemma 1 is the statement that even eigenvalue branches converge to 'even' threshholds and odd eigenvalue branches converge to 'odd' thresholds.

The actual proof Lemma of 1 is an application of the method of described in [1]. For example, in the case of 'even' eigenfunction branches, one expands

$$u_t^+(x, y) \;=\; \sum_k u_{k,t}^+(x) \cdot \cos\left(\frac{\pi(k+\frac{1}{2})y}{\sqrt{1-x^2}}\right).$$

One shows that certain sums of $u_{k,t}^+(x)$ defined by a 'spectral window' are small t quasimodes for a related 'separable' quadratic form a_t. This allows one to use techniques from ordinary differential equations to estimate u_t^+. Using such estimates as well as perturbation theory, we show that each eigenvalue branch λ_t^+ must converge to a number of the form $\pi^2\left(k+\frac{1}{2}\right)^2$.

Open questions and future directions:

We do not know yet whether our method extends to show that the Neumann spectra of the generic ellipse is simple. Indeed, in our proof of Lemma 1, the fact that $(1-x^2)^{-\frac{1}{2}}$ is infinite at ± 1 is ameliorated by the fact that Dirichlet eigenfunctions vanish on the boundary. In the Neumann case, more care needs to be taken.

The method should extend to 3-dimensional ellipsoids. The even-odd decomposition of eigenspaces of the disk should be replaced by the decomposition of eigenspaces of the ball induced by rotations about the x-axis and reflections in the $x-y$ and $x-z$ planes.

The proof of Theorem 1 that I have sketched here depends crucially on Siegel's work on Bessel functions. The 'super-separation' of eigenvalues proven in [1] for triangles might lead one to a 'Siegel-free' proof. However, unlike triangles, the boundary of the ellipse is 'not pointy at its top'. At the moment, this represents an obstacle to proving 'super-separation'.

One can try to apply the method to other situations. For example, does the generic Bunimovitch stadium have simple eigenvalues?

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High Accuracy Computation for Steklov eigenproblems

Nilima Nigam
(joint work with K. Imeri and K. Patil)

In recent years the use of techniques from numerical analysis to study questions in spectral geometry has lead to a fruitful collaboration between the two fields, [4]. High-accuracy discretizations can lead us to formulate novel conjectures about spectral properties [3], and conversely, analytical results on the behaviour of eigenvalues and eigenfunctions can lead to novel discretrization approaches, [1].

Steklov eigevalue problems for planar elliptic problems present fascinating challenges for numerical approximation: the eigenfunctions may concentrate near the boundary or decay rapidly away from it. Additionally, the spectra are sensitive to the boundary regularity: the spectra of smooth domains converge exponentially to those of equal-perimeter disks. These facts highlight the need for high-accuracy computational strategies.

In this talk I began by providing a high-level description of discretization approaches which are widely used in the numerical analysis of Steklov eigenvalue problems. These can broadly be classified as *volumetric* and *boundary-based* approaches. Within these, one may use collocation-based strategies, or variational ones; one may seek approximation by polynomials, trigonometric polynomials or other bases (including the use of 'particular solutions' in MPS). The particular question of interest should dictate the method being used. For instance, for high-accuracy computation of eigenvalues, boundary-based approaches are very successful. If one seeks to use eigenvalues as part of a proof, the validated numerics approaches (via finite elements) are effective.

I next showed one boundary-based approach using single-layer potentials. In this approach, we recast the Steklov problem as a generalized eigenvalue problem in terms of single and (the adjoint of) the double layer operator. This has the advantage that the unknown eigenfunction (in this case, the unknown density of the layer potential) needs to be determined only on the boundary. For planar domains with Lipschitz boundaries, this eigenvalue problem can be discretized using well-known quadrature schemes which yield very high accuracy eigenpairs. I showed how the error in approximation behaved for domains with smooth boundary, as well as for polygonal domains. The method is effective in capturing the impact of boundary curvature on the spectra of Steklov problems, and is now being tried to understand the impact of boundary curvature on the asymptotic behaviour of eigenvalues for curvilinear polygons [6]. The computed Steklov eigenfunctions can be used to design spectrally-accurate approximation of Robin boundary value problems following the ideas of Auchmuty et al. [2, 5].

As a final demonstration of these ideas, I presented some recent computional work on the Steklov problem for the Helmholtz operator. In this case, the layer potentials used have to be modified to be appropriate for the Helmholtz operator. This problem presents many challenges- for starters, the eigenvalues may be negative. If the wave number is an interior Dirichlet eigenvalue, an entirely different approach for computation must be implemented. These and other questions concerning Steklov eigenpairs will likely keep us busy for the forseeable future.

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Neumann and Robin eigenvalues on curved surfaces—open problems

Richard S. Laugesen

Consider the Neumann eigenvalues of the Laplace–Beltrami operator on a subdomain of the sphere. The first Neumann eigenvalue is zero.

Is the second Neumann eigenvalue maximal for a spherical cap, among subdomains of the 2-sphere with specified area? Yes for simply connected subdomains with area up to 94% of the the sphere (it is an open problem to try to get up to 100%), and also yes for arbitrary subdomains with area at most half that of the sphere provided the domain lies outside a complementary cap (it is an open problem to drop the complementary cap restriction).

More generally, is the second Robin eigenvalue maximal for a spherical cap, among subdomains of the 2-sphere or hyperbolic space with specified area? Yes for simply connected subdomains with suitable areas and Robin parameters, in a result that extends the 94% spherical Neumann result and also extends a known Euclidean Robin result. Viewing this picture in the parameter plane is worth 1000 words, for terra incognita stares us right in the face: the eigenvalue maximization problem is completely open for hyperbolic domains with positive Robin parameter.

This report builds in particular on work by Bandle [3, 4], Langford and Laugesen [18], Ashbaugh and Benguria [2], Bucur, Martinet and Nahon [8], and Freitas and Laugesen [16].

Euclidean motivations

The Neumann eigenfunctions of the Laplacian on a planar domain $\Omega \subset \mathbb{R}^2$ satisfy

$$\begin{cases} -\Delta u = \mu u & \text{in } \Omega \\ \dfrac{\partial u}{\partial n} = 0 & \text{on } \partial\Omega \end{cases}$$

with eigenvalues

$$0 = \mu_1(\Omega) < \mu_2(\Omega) \leq \mu_3(\Omega) \leq \cdots \to \infty.$$

We fix the area of Ω and ask:

1. For which free membrane is the ground tone highest?
2. For which insulated shape does heat equilibrate fastest?

Both questions are in fact asking: what shape maximizes $\mu_2(\Omega)$? The answer is the domain with maximal symmetry, that is, a disk, by the Szegő–Weinberger theorem [21, 22]. But does **curvature** (positive, or negative) change the answer? And does the answer change for domains with **holes** (non-simply connected)?

Simply connected domains

To pursue such curvature effects, we consider now the Neumann eigenvalues on a spherical domain $\Omega \subset \mathbb{S}^2$. They arise from the same equation as above except now Δ represents the spherical Laplacian. What shape maximizes $\mu_2(\Omega)$?

Theorem 1 (Bandle [3, 4], building on Szegő's conformal mapping method)**.**

$$\begin{matrix} \textit{simply connected } \Omega \\ \textit{area}(\Omega) \leq 0.50\, \textit{area}(\mathbb{S}^2) \end{matrix} \implies \begin{matrix} \mu_2(\Omega) \leq \mu_2(\textit{spherical cap} \\ \qquad \textit{with same area as } \Omega) \end{matrix}$$

The intuition is that the spherical cap has no "long" directions, and hence has higher frequency μ_2 than any competitor domain.

Langford and I recently managed to improve Bandle's 50% result to 94%.

Theorem 2 (Langford–Laugesen [18])**.**

$$\begin{matrix} \textit{simply connected } \Omega \\ \textit{area}(\Omega) \leq 0.94\, \textit{area}(\mathbb{S}^2) \end{matrix} \implies \begin{matrix} \mu_2(\Omega) \leq \mu_2(\textit{spherical cap} \\ \qquad \textit{with same area as } \Omega) \end{matrix}$$

The method cannot at present do better than $\simeq 94\%$. But surely the result should hold up to 100%, meaning simply connected domains of any size?

Conjecture 1. *The previous theorem can be improved from 0.94 to 1.00, so that the spherical cap maximizes μ_2 among simply connected domains of any given area.*

One approach would be to find a way to use certain qualitative information that is currently discarded in the proof.

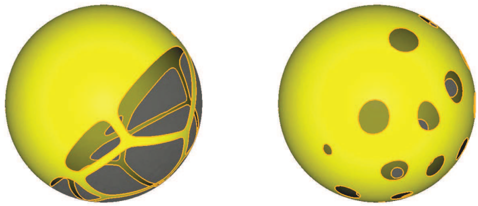

FIGURE 1. Counterexamples found numerically by Martinet [20]. Left: $0.64\,\mathrm{area}(\mathbb{S}^2)$. Right: $0.86\,\mathrm{area}(\mathbb{S}^2)$.

MULTIPLY CONNECTED DOMAINS WITH EXCLUSION CONSTRAINTS

For domains with holes, a well known result uses Weinberger's mass transplantation method to prove:

Theorem 3 (Chavel [9, 10] and Ashbaugh–Benguria [2])**.**

$$\begin{array}{r}\Omega \subset \textit{hemisphere in } \mathbb{S}^2\\ \mathcal{C} = \textit{cap of same area}\end{array} \quad \Longrightarrow \quad \mu_2(\Omega) \le \mu_2(\mathcal{C})$$

Notice the area of Ω is at most the area of the hemisphere, which is $0.5\,\mathrm{area}(\mathbb{S}^2)$. A surprising improvement appeared last year.

Theorem 4 (Bucur–Martinet–Nahon [8])**.**

$$\begin{array}{r}\Omega \subset \mathbb{S}^2 \setminus \mathcal{C}\\ \mathcal{C} = \textit{cap of same area}\end{array} \quad \Longrightarrow \quad \mu_2(\Omega) \le \mu_2(\mathcal{C})$$

Again $\mathrm{area}(\Omega) \le 0.5\,\mathrm{area}(\mathbb{S}^2)$, but the improvement is that Ω is permitted to "spread out more".

Conjecture 2. *Can the exclusion constraint $\Omega \subset \mathbb{S}^2 \setminus \mathcal{C}$ be dropped from the last result? That is, does the cap $\mathcal{C}$ maximize μ_2 among all domains of given area $\le 0.5\,\mathrm{area}(\mathbb{S}^2)$?*

Some restriction on the area is necessary, as Martinet [20] has constructed numerical counterexamples to the conjecture when area $\gtrsim 0.64\,\mathrm{area}(\mathbb{S}^2)$; see Figure 1. Rigorous counterexamples with only four holes are in progress by Bucur, Laugesen, Martinet and Nahon, when area $\gtrsim 0.80\,\mathrm{area}(\mathbb{S}^2)$, with the intuition behind the examples being to cut holes in the domain at hot spots of the cap.

ROBIN — SIMPLY CONNECTED DOMAINS

Now that we have described open problems for the second Neumann eigenvalue, let us generalize to the Robin problem on a subdomain of the sphere:

$$\begin{cases} -\Delta u = \lambda u & \text{in } \Omega \subset \mathbb{S}^2 \\ -\dfrac{\partial u}{\partial n} = \alpha u & \text{on } \partial\Omega \end{cases}$$

Horizontal coord. $t = \pm$ area
sphere: $+$
hyperbolic space: $-$

$\lambda_2(\mathcal{C}, \beta/L(\mathcal{C})) = 0$ when $\beta = -2\pi$

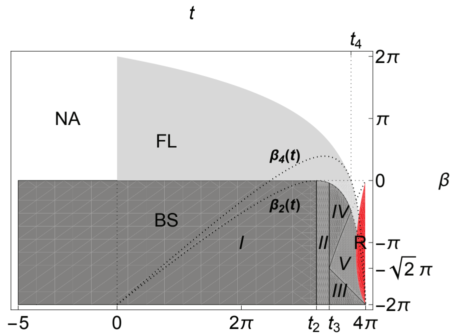

FIGURE 2. The spherical Neumann theorem by Langford and Laugesen covers $t \le t_4 \simeq (0.94)4\pi$ with $\beta = 0$. The Euclidean Robin theorem by Freitas and Laugesen [14] covers $t = 0$ with $|\beta| \le 2\pi$. The hyperbolic Robin case $t < 0$ holds for $-2\pi \le \beta \le 0$. Open problem: the method is Not Applicable for hyperbolic domains with positive Robin parameter (2nd quadrant) or where the 2nd eigenfunction of the cap is radial (red region).

with eigenvalues

$$\lambda_1(\Omega,\alpha) < \lambda_2(\Omega,\alpha) \le \lambda_3(\Omega,\alpha) \le \cdots \to \infty.$$

Choose a number $\beta \in \mathbb{R}$ and fix the area(Ω). We take the Robin parameter to be $\alpha = \beta/L$ where $L =$perimeter(Ω). Scaling the Robin parameter in this way by a length is physically natural because the Robin boundary condition implies that α has dimensions of 1/length.

Building on the Neumann case, we ask: is $\lambda_2(\Omega,\beta/L)$ maximal when Ω is a spherical cap $\mathcal{C}$? The next theorem says yes, provided the area and Robin parameter lie in certain regimes.

Theorem 5 (Langford–Laugesen [18]).

$$\begin{array}{r} \textit{simply connected } \Omega \\ (\textit{area}(\Omega),\beta) \in \\ \textit{shaded region in Figure 2} \end{array} \implies \begin{array}{c} \lambda_2(\Omega,\beta/L(\Omega)) \\ \le \lambda_2(\mathcal{C},\beta/L(\mathcal{C})) \end{array}$$

As the caption on Figure 2 makes clear, different trial functions would be needed when the 2nd eigenfunction is purely radial (the red region in the figure). Our method cannot handle that case, and cannot handle domains in hyperbolic space with positive Robin parameter. Is the cap (geodesic disk) still maximal in those situations?

Further Robin results for multiply connected domains can be found in [15, 19].

Extremizing the first eigenvalue

This report focuses on the second Robin eigenvalue, but of course much is known about the first Robin eigenvalue, while much remains unknown.

Write $A(\Omega)$ for the area of the domain and $L(\Omega)$ for its perimeter.

Positive Robin parameter — known results. If $\beta > 0$ and $A(\Omega)$ is given then $\lambda_1(\Omega, \beta/\sqrt{A(\Omega)})$ is minimal for the:

- disk in 2 dimensions (Bossel [5]),
- ball in higher dimensions (Daners [12]),
- geodesic ball in the sphere and hyperbolic space (Chen, Cheng, and Li [11]).

Thus the Rayleigh–Faber–Krahn type result for the first Robin eigenvalue with positive Robin parameter holds in all three standard constant curvature spaces.

Negative Robin parameter — open problems on Euclidean domains. Suppose $\beta < 0$ and $A(\Omega)$ is given. Is $\lambda_1(\Omega, \beta/\sqrt{A(\Omega)})$ maximal for the disk (Bareket conjecture)?

This conjecture was proved for small $\beta < 0$ and disproved for all large $\beta < 0$ (using annular counterexamples) by Freitas and Krejčiřík [13]. The conjecture remains plausible but unproven for simply connected planar domains and for convex domains in all dimensions [1, Section 5.3].

A variant of the conjecture says that if the perimeter $L(\Omega)$ is given then $\lambda_1(\Omega, \beta/\sqrt{L(\Omega)})$ is maximal for the disk, as has been proved by Antunes, Freitas and Krejčiřík [1] for arbitrary planar domains and by Bucur, Ferone, Nitsch and Trombetti [6] for convex domains in all dimensions. The conjecture is true also for domains close to a ball. For more, see the survey chapter by Bucur, Freitas and Kennedy [7], and a recent paper by Khalile and Lotoreichik [17].

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Weyl's law for singular Riemannian manifolds

Luca Rizzi

(joint work with Y. Chitour and D. Prandi)

In this work [4], we focus on Weyl's-type asymptotics for the Laplace-Beltrami operator of a class of singular Riemannian structures. We first discuss a simple but representative model. Consider $\mathbb{S}^2 \subset \mathbb{R}^3$. Let X and Y be the generators of rotations around the x and y axis, respectively. We define a Riemannian structure by declaring X and Y to be orthonormal. These vector fields are collinear on the equator $S = \{(x,y,z) \in \mathbb{S}^2 \mid z = 0\}$, and hence the metric tensor we defined is singular on S. This is an almost-Riemannian structure in the sense of [1, 2]. In coordinates $(\theta, z) \in (0, 2\pi) \times (-1, 1)$, the Laplace-Beltrami operator Δ is

$$-\Delta = \frac{\partial^2}{\partial z^2} + z^2 \frac{\partial^2}{\partial \theta^2} + \left(\frac{1}{z} - z\right)\frac{\partial}{\partial z}. \tag{1}$$

It turns out that Δ is essentially self-adjoint on $L^2(\mathbb{S}^2 \setminus S, d\mu_g)$, with compact resolvent [2]. The spectrum can be computed, cf. [3], and it satisfies the following non-classical Weyl's asymptotics:

$$N(\lambda) \sim \frac{1}{4}\lambda \log \lambda, \qquad \lambda \to \infty. \tag{2}$$

Despite the problem taking place on a relatively compact space, the total Riemannian volume is infinite and the curvature is unbounded. Hence, on-diagonal small-time heat kernel estimates blow up at the singular region. It is not clear how to deduce the asymptotics of $N(\lambda)$ using classical Tauberian techniques.

The class of singular structures that we study in this paper is inspired by the Grushin sphere, and it is determined by the control on the blow-up of intrinsic quantities such as curvature, injectivity radius, as formalized by the following.

Assumption A. *Let $(\mathbb{M}, g)$ be a non-complete Riemannian manifold. Let δ be the distance from the metric boundary of $\mathbb{M}$. We assume that there exists a neighbourhood $U = \{\delta < \varepsilon_0\}$ on which the following hold:*

(a) *regularity: δ is smooth;*
(b) *convexity: the level sets of δ are convex, i.e.,* $\mathrm{Hess}(\delta) \leq 0$*;*
(c) *curvature control: there exists $C > 0$ such that* $|\mathrm{Sec}| \leq C\delta^{-2}$*;*
(d) *injectivity radius control: there exists $C > 0$ such that* $\mathrm{inj} \geq C\delta$*.*

Let Δ be the Friedrichs extension of the Laplace-Beltrami operator on $(\mathbb{M}, g)$. To quantify the rate of growth of the volume at the singularity, let $\mathbb{M}^\infty_\varepsilon$ be the set at distance greater than $\varepsilon > 0$ from the metric boundary, and define

$$\upsilon(\lambda) := \mathrm{vol}\left(\mathbb{M}^\infty_{1/\sqrt{\lambda}}\right). \tag{3}$$

Our main result is a precise Weyl's law under an additional assumption on the volume growth, ruling out rapid oscillations and growth.

Theorem 1. *Let $\mathbb{M}$ be an n-dimensional Riemannian manifold with compact metric completion, satisfying Assumption* **A**. *Then, if υ is slowly varying, we have*

$$\lim_{\lambda\to\infty} \frac{N(\lambda)}{\lambda^{n/2}\upsilon(\lambda)} = \frac{\omega_n}{(2\pi)^n}. \tag{4}$$

Recall that υ is slowly varying if $\upsilon(a\lambda) \sim \upsilon(\lambda)$ as $\lambda \to \infty$ for all positive a. Examples of slowly varying functions are logarithms and their iterations

$$\log\lambda, \qquad \log_k \lambda = \log_{k-1}\log\lambda, \qquad k = 2, 3, \dots, \tag{5}$$

and any rational function with positive coefficients formed with the above. This class also contains functions with non-logarithmic growth such as

$$\exp\left((\log\lambda)^{\alpha_1}\dots(\log_k\lambda)^{\alpha_k}\right), \qquad 0 < \alpha_i < 1. \tag{6}$$

The assumptions of Theorem 1 are verified for the Grushin sphere, and more generally for generic 2-dimensional ARS without tangency points [2]. In these cases, $\upsilon(\lambda) = \sigma\log\lambda$ for some $\sigma > 0$ depending on the structure.

We now turn to the inverse problem of building structures with prescribed large eigenvalues asymptotic. Our next main result can be seen as a counterpart at infinity of a celebrated result of Colin de Verdière [5] stating that, for any finite sequence of numbers $0 < \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_m$, one can find a compact Riemannian manifold such that these numbers are the first m eigenvalues

Theorem 2. *Let M be an n-dimensional compact manifold, $S \subset M$ be a closed submanifold, and $\upsilon : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing slowly varying function. Then, there exists a Riemannian structure on M, singular at S, such that Weyl's law* (4) *holds for the non-complete Riemannian manifold $\mathbb{M} = M \setminus S$.*

A technical result of independent interest, key in the proof of Theorem 1, is the following universal formula[1] for the remainder of the eigenvalue counting function on a compact Riemannian manifold with convex boundary.

Theorem 3. *Let (M, g) be a compact n-dimensional Riemannian manifold with convex boundary ∂M. Let $K \geq 0$ such that $|\operatorname{Sec}(M)| \leq K$. Then, there exists a constant $C > 0$, depending only on n, such that the following estimate holds for the counting function for Dirichlet or Neumann eigenvalues:*

$$\left| \frac{N(\lambda)}{\frac{\omega_n}{(2\pi)^n}\operatorname{vol}(M)\lambda^{n/2}} - 1 \right| \leq \frac{C}{\log(1 + \sqrt{\lambda/\lambda_0})}, \qquad \forall \lambda > 0, \tag{7}$$

with $\sqrt{\lambda_0} = \min\left\{\operatorname{inj}(M), \frac{\operatorname{inj}_\partial(M)}{4}, \frac{\pi}{\sqrt{K}}\right\}^{-1}$.

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[1] By "universal", we mean that it depends on the structure only through a handful of geometrical invariants, and fixed dimensional constants.

Nodal counts for Steklov and Robin eigenfunctions

David Sher

(joint work with A. Hassannezhad)

Let Ω be a $d+1$-dimensional manifold with smooth boundary $\partial\Omega$. Let $q \in L^\infty(\Omega)$ and $h \in L^\infty(\partial\Omega)$. Consider the following generalized eigenvalue problem, with Δ the non-negative definite Laplacian and ∂_n the outward normal:

$$\begin{cases} (\Delta + q - \mu)u = 0 & \text{in } \Omega; \\ \partial_n u + hu = 0 & \text{on } \partial\Omega. \end{cases} \tag{1}$$

If we fix h and q, this is a Robin problem with eigenvalue parameter μ. If on the other hand we fix q and μ and let $h = -\sigma$ be a constant, this is a Steklov (or Steklov-Helmholtz) problem with eigenvalue parameter σ. This observation, colloquially called *Steklov-Robin duality*, dates back to Friedlander [4].

Our goal is to take advantage of this duality to say things about nodal counts for various eigenvalue problems. Recall the Courant and Pleijel nodal domain theorems. If we let $\Omega \subseteq \mathbb{R}^{d+1}$ be a domain, then the Courant nodal domain theorem says that the number N_k of nodal domains of any kth Dirichlet eigenfunction ϕ_k is at most k. The Pleijel theorem is the stronger asymptotic statement that

$$\limsup_{k\to\infty} \frac{N_k}{k} \leq \gamma(d+1),$$

where $\gamma(d+1)$ is an explicit constant, strictly less than 1 (see [8] or [6] for the exact expression).

The first result I discuss here is the main theorem of [5]. We consider the Steklov problem with potential and generalize the Courant theorem. That is, in (1), we set $\mu = 0$ and let h be the eigenvalue parameter; the Steklov eigenfunctions u_k and eigenvalues σ_k satisfy

$$\begin{cases} (\Delta + q)u_k = 0 & \text{in } \Omega; \\ \partial_n u_k = \sigma_k u_k & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Our main theorem in this setting is the following:

Theorem 1: ([5]) Let Ω be a connected Lipschitz domain in a manifold M. Let $q \in L^\infty(\Omega)$. Let d_q be the number of non-positive Dirichlet eigenvalues of $\Delta + q$. Then for all k, the number of nodal domains N_k of u_k satisfies

$$N_k \leq k + d_q.$$

The idea of the proof is straightforward. In [5], we show that the conditions on Ω and q are sufficient for Steklov-Robin duality to hold. Then by that duality, the kth Steklov eigenfunction is the $(k+d_q)$th Robin eigenfunction.

The second result is one of the key theorems in [6]. We now consider (1) as a Robin problem, and generalize the Pleijel nodal domain theorem:

Theorem 2: ([6]) Let Ω be a $(d+1)$-dimensional manifold with $C^{1,1}$ boundary. Let $h \in L^\infty(\partial\Omega)$. Then there exists a positive number $\epsilon(d)$ depending only on d for which, if N_k is the number of nodal domains of a kth Robin eigenfunction,

$$\limsup_{k\to\infty} \frac{N_k}{k} \leq \gamma(d+1) - \epsilon(d+1).$$

This theorem builds on an extensive set of prior results. First, the Pleijel theorem was proven for manifolds, with Dirichlet boundary conditions, by Bérard and Meyer in 1982 [1]. The first example of the theorem in the Neumann case was for analytic domains by Polterovich [9]. The Robin case was considered by Léna in [7] under the additional hypothesis that $h \geq 0$, but without the ϵ improvement. That ϵ improvement was demonstrated in the Dirichlet case by Bourgain, Donnelly, and Steinerberger [2, 3, 10]. Our proof is an adaptation of the method of Léna [7], with some new Green's theorem estimates to handle the possibility that $h \leq 0$ might be negative, and incorporating the argument of Steinerberger [10] to get the ϵ improvement.

We have a number of open questions on which progress or thoughts would be more than welcome!

(1) For any boundary condition, we can define the *Pleijel constant* to be the true value of $\limsup \frac{N_k}{k}$. In [6] we prove that the Pleijel constant for a ball is "generically" independent of the Robin boundary condition, in the sense that it is the same for almost all values of σ. This leads to the natural question: **is it always true that the Pleijel constant for a Robin problem is equal to the Pleijel constant for the Dirichlet problem on the same domain?**

(2) **Can we say anything about nodal counts for the Dirichlet-to-Neumann eigenfunctions (that is, the restrictions to the boundary of the Steklov eigenfunctions)?** The difficulty is that the Dirichlet-to-Neumann operator is nonlocal. Its square has the same principal symbol as the boundary Laplacian, which is local, but it is not clear if one can take advantage of this.

(3) **Are there any other consequences of Steklov-Robin duality?** This is in part considered in ongoing work with K. Gittins, A. Hassannezhad, and C. Léna.

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Manifolds with arbitrarily large Steklov eigenvalues

Alexandre Girouard
(joint work with P. Polymerakis)

Let (Ω, g) be a compact, connected Riemannian manifold of dimension $n \geq 2$, with boundary $\partial\Omega$. The Steklov eigenvalues of (Ω, g) form an unbounded sequence $0 = \sigma_0 < \sigma_1 \leq \sigma_2 \leq \cdots \nearrow +\infty$, where each eigenvalue is repeated according to its multiplicity. For background and open problems on the spectral geometric perspective, see [8, 5]. The litterature is pregnant with interesting upper bounds for the spectral gap σ_1. They are based on the following characterisation:

$$\sigma_1 = \min\Big\{ \frac{\int_\Omega |\nabla_g f|^2 \, dV_g}{\int_{\partial\Omega} f^2 \, dA_g} \; : \; f \in W^{1,2}(\Omega), \int_{\partial\Omega} f \, dA_g = 0 \Big\}. \tag{1}$$

Let Ω be a compact surface ($n = 2$) of genus γ. Let b be the number of connected components of the boundary $\partial\Omega$, and let $L = |\partial\Omega|$ be its length. In 2011, Fraser and Schoen [7] proved that

$$\sigma_1 L \leq 2\pi(\gamma + b). \tag{2}$$

A natural question when presented with such an upper bound is to decide if the geometric quantities that are involved are truly relevant. Rather than trying to explain what I mean by that, I will give examples. Let's start by the obvious: one could not simply remove L from inequality (2) since the LHS would not be scale-invariant anymore, while the RHS is. In regard to (2), Kokarev [9] proved in 2014 that

$$\sigma_1 L \leq 8\pi(\gamma + 1). \tag{3}$$

This inequality is better than (2) provided that $b > 4 + 3\gamma$, and if one is willing to compromise on the constant 2π, then the number of boundary components b is not truly relevant. What about the genus γ of the surface Ω? Could we remove it from (3)? If true, this would provide a universal upper bound for $\sigma_1 L$. Alas, this is impossible. Indeed, in [6], we constructed a sequence of compact surfaces Ω_m of genus $\gamma = 1 + m$ with connected boundary ($b = 1$) such that $\sigma_1(\Omega_m)|\partial\Omega_m| \geq Cm$ for some contant $C > 0$. Not only does this prevent the possibility of removing γ from (3), but it also shows that the exponent on γ is optimal. This situation is typical: on the one hand, we are seaking and proving upper bounds that involve various geometric quantities, and on the other we want to construct examples where σ_1 becomes arbitrarily large while some of the geometric quantities remain bounded.

Let me give a second example. In [4], we obtained upper bounds for Steklov eigenvalues of hyper-surfaces $\Omega \subset \mathbb{R}^{n+1}$ that have a prescribed boundary $\partial\Omega = \Sigma$. It follows from our work that there is a constant $C_\Sigma > 0$ depending only on the submanifold $\Sigma \subset \mathbb{R}^n$, such that

$$\sigma_1(\Omega) \leq C_\Sigma \mathrm{Vol}(\Omega). \tag{4}$$

This bound is interesting, but similarly to the previous discussion, it raises the question to know if there is a universal bound, independant of the volume of Ω, or if the presence of $\mathrm{Vol}(\Omega)$ in the RHS of inequality (4) is required. In other words, given a closed submanifold $\Sigma \subset \mathbb{R}^n$ of dimension $n-2$, the question is to determine if the following quantity is finite or not:

$$\sup\{\sigma_1(\Omega) : \Omega \subset \mathbb{R}^n \text{ is a submanifold with } \partial\Omega = \Sigma\}.$$

For instance, it raises the question to know how large σ_1 can be among all surfaces $\Omega \subset \mathbb{R}^3$ that have the same boundary $\partial\Omega = S^1 \times \{0\}$. In an ongoing project with Panagiotis Polymerakis, we use a relative version of the the Nash–Kuiper C^1 isoperimetric embedding theorem to transfer known intrinsic constructions to embedded ones. An interesting point is that the characterization (1) of σ_1 does not involve any higher derivative of the metric, hence C^1 regularity is sufficient to work with it. In particular, we use the results of [6] described above to show that the existence of a sequence of surfaces $\Omega_n \subset \mathbb{R}^3$ with $S^1 \times \{0\}$ such that $\sigma_1(\Omega, g_\epsilon) \xrightarrow{\epsilon\to 0} +\infty$. Notice that we do not know what these surfaces Ω_n look like, since the machinery of Nash and Kuiper is rather cumbersome and does not readily provide a construtive argument. This is akin to the situation with existence of isoperimetric embeddings of the flat torus in $\mathbb{R}^3$, which were known to exists since the work of Nash and Kuiper, but where constructed explicitly only recently by Vincent Borrelli and his collaborators of the *Hévéa project*[1]. A similar argument was already used by Colbois, Dryden and El Soufi [2] when studying upper bounds for the eigenvalues λ_k of the Laplace operator of submanifolds.

In yet another direction, Fraser and Schoen [7] considered metrics constrainted to a conformal class $C = [g_0]$ and proved that any metric $g \in C$ satisfies

$$\sigma_1(\Omega, g)\mathrm{Vol}_g(\partial\Omega) \leq nV_{rc}(\Omega, C)\mathrm{Vol}_g(\Omega)^{\frac{n-2}{n}},$$

where V_{rc} is a conformal invariant known as the *relative conformal volume.* Could a similar inquality hold without involving $\mathrm{Vol}_g(\Omega)$? This could tentatively mean that

$$\sigma_1(\Omega, g)\mathrm{Vol}_g(\partial\Omega)^{1/(n-1)} \leq K,$$

for some constant $K = K(\Omega, C)$ depending only on the conformal class C, and where the exponent on $\mathrm{Vol}_g(\partial\Omega)_g$ is adjusted for scale-invariance. Again, this is impossible. In [3] we proved that for any manifold Ω of dimension $n \geq 3$,

[1] http://hevea-project.fr/

there exists a family of conformal metrics $g_\epsilon = \rho_\epsilon^2 g$ such that $\rho_\epsilon \equiv 1$ on $\partial\Omega$ and $\sigma_1(\Omega, g_\epsilon) \xrightarrow{\epsilon\to 0} +\infty$. In this construction, the conformal factor ρ_ϵ are very large away from the boundary, and they act as a kind of barrier for the Dirichlet energy

$$\int_\Omega |\nabla_g f|^2 \, dV_g = \int_\Omega |\nabla_{g_0} f|_{g_0}^2 \rho_\epsilon^{n-2} \, dV_{g_0}$$

that appears in the characterisation (1) of σ_1.

This construction raised the question to know if it is possible for a compact manifold of unit volume with prescribed boundary to have arbitrarily large σ_1, when removing the conformal constraint. In [1], we obtained a family of such examples. On any compact Riemannian manifold (Ω, g) of dimension $n \geq 4$ such that $\partial\Omega$ admits a unit Killing vector field ξ with dual 1-form η whose exterior derivative is nowhere zero, we proved the existence of a family g_ϵ of Riemannian metrics that coincide with g on $\partial\Omega$ such that $\mathrm{Vol}_{g_\epsilon}(\Omega) = 1$ and $\sigma_1(\Omega, g_\epsilon) \to \infty$ as $\epsilon \searrow 0$. Euclidean balls of even dimension ≥ 4 are amongst the manifolds covered by this theorem. However, the dimensional constraint and technical asumption are unsettling.

In a second ongoing project with Panagiotis Polymerakis, we are proposing a new family of examples, that are both simpler and more flexible. Let (M^m, g_M) be a compact, connected Riemannian manifold with boundary. Let (F^k, g_F) be a closed Riemannian manifold. We consider the prodct $\Omega = M \times F$ equiped with the direct sum $g = g_M \oplus g_F$. If $m > k \geq 1$, then we prove the existence of a family g_ϵ of Riemannian metrics that coincide with g on $\partial\Omega$ such that $\mathrm{Vol}_{g_\epsilon}(\Omega) = 1$ and $\sigma_1(\Omega, g_\epsilon) \to \infty$ as $\epsilon \searrow 0$. A particularly satisfying example is that of the solid torus $\mathbb{D} \times S^1$. The metrics g_ϵ are obtained as warped product with large warping factor. The proof is based on comparison inequalities involving an auxiliary Steklov problem for the Bakry-Emery Laplacian.

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Quasi-conical domains with embedded eigenvalues

Vladimir Lotoreichik
(joint work with D. Krejčiřík)

An open set $\Omega \subset \mathbb{R}^d$, $d \geq 2$, is called *quasi-conical* if it contains a ball of an arbitrarily large radius. We consider the self-adjoint *Dirichlet Laplacian* $-\Delta_{\rm D}^{\Omega}$ in the Hilbert space $L^2(\Omega)$ defined as

$$-\Delta_{\rm D}^{\Omega} := -\Delta u, \qquad \mathrm{dom}\,(-\Delta_{\rm D}^{\Omega}) := \big\{u \in H_0^1(\Omega)\colon \Delta u \in L^2(\Omega)\big\}.$$

For any quasi-conical domain Ω, the spectrum of the Dirichlet Laplacian $-\Delta_{\rm D}^{\Omega}$ coincides with the set $[0,\infty)$. Existence or absence of the positive (embedded) eigenvalues for such an operator is a subtle question. In the classical works, embedded eigenvalues were excluded for certain classes of quasi-conical domains. In particular, Rellich proved in [4] absence of embedded eigenvalues for the Dirichlet Laplacian on the connected exterior of a compact set. Later, Jones excluded in [2] embedded eigenvalues for domains conical at infinity. By that he understands a connected domain, which coincides with an unbounded and not necessarily round conical domain outside a compact set.

An example of a connected quasi-conical domain with an embedded eigenvalue has not been constructed in the literature before. The aim of our work [3] was to fill in this gap. Our main result reads as follows.

Theorem 1.
- (i) *For any $\lambda > 0$, one can construct a connected quasi-conical domain $\Omega \subset \mathbb{R}^d$, $d \geq 2$, such that $\lambda \in \sigma_{\rm p}(-\Delta_{\rm D}^{\Omega})$.*
- (ii) *This construction can be performed in such a way that simultaneously $\sigma_{\rm ac}(-\Delta_{\rm D}^{\Omega}) = \varnothing$.*

The proof relies on an explicit construction. We build a tower of cubes, whose sizes are growing and tend to infinity. The cubes are placed on the top of each other as shown in Figure 1. In the centers of the common boundaries of the cubes

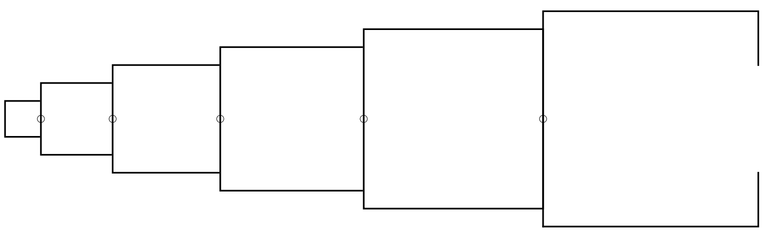

Figure 1. The disconnected quasi-conical open set with places for the small holes to be dug indicated by circles.

we dig small holes in the places indicated by circles. We show that the sizes of the holes can be chosen so small that the Dirichlet Laplacian on such a domain has an embedded eigenvalue and no absolutely continuous spectrum. Our result is not quantitative in the sense that we do not obtain estimates on the sizes of the holes, which are sufficient for the desired spectral properties to hold.

Our construction is dual to the celebrated "rooms-and-passages" domain, which was used in [1] to construct an example of a bounded connected domain, on which the Neumann Laplacian has a non-empty essential spectrum. In our construction the holes can also be replaced by thin passages connecting the cubes. Moreover, the cubic shape of the rooms is not essential.

The argument we use to show the existence of an embedded eigenvalue relies on the perturbation theory of linear operators and proceeds in infinitely many steps. We start with the disjoint union of cubes and in each step we dig a new hole. Throughout the construction we keep track on a certain simple eigenvalue and the corresponding eigenfunction of the Dirichlet Laplacian on the union of first n cubes with $(n-1)$ holes. When choosing the size of the n-th hole we require that the change of this selected eigenvalue and the respective eigenfunction satisfies certain smallness condition. A subsequence of the eigenfunctions constructed in these steps converges in an appropriate sense to a function, which we show is an eigenfunction of the Dirichlet Laplacian on a tower of cubes with all holes dug.

The absence of the absolutely continuous is shown by means of verifying that, for a sufficiently large power, the resolvent power difference of the Dirichlet Laplacians on the tower of cubes with all holes dug and on the disjoint union of cubes is a trace-class operator provided that the sizes of the holes are chosen sufficiently small.

It remains an open problem whether the sizes of the holes can be chosen so that both the absolutely continuous and the singular continuous spectra of the Dirichlet Laplacian on such a domain are empty. In this case, we would get the Dirichlet Laplacian on a connected quasi-conical domain with dense point spectrum.

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Sharp stability of the Dirichlet spectrum near the ball

Dorin Bucur

(joint work with J. Lamboley, M. Nahon and R. Prunier)

Let us denote by $\mathcal{A}$ the class of all open sets $\Omega \subset \mathbb{R}^N$ of measure equal to 1. For such a set Ω, one denotes by

$$\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq ... \to +\infty$$

the eigenvalues of the Dirichlet Laplacian on Ω and by $T(\Omega)$ the torsional rigidity of Ω, namely $T(\Omega) = \int_\Omega w_\Omega$, where $-\Delta w_\Omega = 1$ in Ω and $w_\Omega \in H_0^1(\Omega)$. This talk is intended to answer the following question: if $\Omega \in \mathcal{A}$ is such that $\lambda_1(\Omega)$ is close to $\lambda_1(B)$, where B is the ball of the same measure as Ω, is it true that for every $k \in \mathbb{N}$, the eigenvalue $\lambda_k(\Omega)$ is close to $\lambda_k(B)$? Is this quantifiable?

The results presented in the talk are being published in the preprint [3]. In a short way, they can be described as.

Result 1. For every $k \in \mathbb{N}$, there exists a constant $C_{N,k}$ such that for every $\Omega \in \mathcal{A}$ satisfying $\lambda_1(\Omega) \le 2\lambda_1(B)$ one has

$$|\lambda_k(\Omega) - \lambda_k(B)| \le C_{N,k}(\lambda_1(\Omega) - \lambda_1(B))^{\frac{1}{2}}.$$

In fact, the result holds in a stronger form which can be written as

$$|\lambda_k(\Omega) - \lambda_k(B)| \le C_N k^{2+\frac{4}{N}} |\Omega|^{\frac{1}{2}} (T^{-1}(\Omega) - T^{-1}(B))^{\frac{1}{2}}.$$

The power $\frac{1}{2}$ is in general sharp. We refer the reader to some previous inequalities of this type obtained by Bertrand and Colbois [2] and by Mazzoleni and Pratelli [6].

However, if $\lambda_k(B)$ is simple, the higher power 1 can be expected on the right hand side. Indeed, we have obtained the following.

Result 2. Assume $\lambda_k(B)$ is simple. Then the following inequality holds true

$$|\lambda_k(\Omega) - \lambda_k(B)| \le C_{N,k}(\lambda_1(\Omega) - \lambda_1(B)).$$

Moreover, this inequality can be reinforced with the variation of the torsional rigidities on the right hand side.

In case $\lambda_k(\Omega)$ is not simple, e.g. $\lambda_{k-1}(\Omega) < \lambda_k(\Omega) = \cdots = \lambda_{k+j}(\Omega) < \lambda_{k+j+1}(\Omega)$ our result reads:

Result 3.

$$|\sum_{i=0}^{j}(\lambda_{k+i}(\Omega) - \lambda_k(B))| \le C_{N,k}(\lambda_1(\Omega) - \lambda_1(B)).$$

As well, this inequality can be reinforced with the variation of the torsional rigidities on the right hand side.

There are several consequence of these inequalities. For instance, if $\lambda_k(\Omega)$ is simple, then the minimizer of the shape functional

$$\min_{\Omega \in \mathcal{A}} \lambda_1(\Omega) + \delta\lambda_k(\Omega)$$

is a ball, provided δ is sufficiently close to 0 (being positive or negative).

Moreover, a second consequence is the full reverse Kohler-Jobin inequality, which proves a conjecture raised in [1]. This reads: there exists $p_N > 0$ such that for every $p \ge p_N$ the solution of the shape optimization problem

$$\max_{\Omega \in \mathcal{A}} \lambda_1(\Omega) T^p(\Omega)$$

is the ball.

While the proof of the power $\frac{1}{2}$-result is based on direct comparisons and use of test functions, the power 1-results are much more technical and require the analysis of vectorial free boundary problems of degenerate type (see [4], [5]).

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Existence of non planar free boundary minimal disks into ellipsoids

Romain Petrides

A surface D with boundary (here the disk) is a free boundary minimal surface into a surface S of $\mathbb{R}^3$ if it is a minimal surface and if it meets S orthogonally along the boundary. Of course, equatorial disks, which are planar satisfy this property on ellipsoids. We prove existence of embedded non planar free boundary minimal disks into ellipsoids [Pet23b]. This existence question was raised by Dierkes, Hildebrandt, Küster and Wohlrab [DHKW92]. Notice that if the target ellipsoid is a sphere all branched immersed free boundary minimal disks have to be planar, by Nitsche [Nit85].

Our result is comparable to the recent answer of a question by Yau [Yau87] by Haslhofer and Ketover [HK19]: there are non planar embedded minimal spheres into sufficiently elongated ellipsoids of $\mathbb{R}^4$. Their result was proved by a mixing of a min-max variational method and a mean-curvature flow method. It was next extended in different ways in [BP22] by bifurcation methods and in [Pet23a] by shape optimization of combinations of Laplace eigenvalues on spheres.

The latest method is performed with Steklov eigenvalues on the disk $\mathbb{D}$ instead of Laplace eigenvalues in order to prove our result. Indeed, branched free boundary minimal immersions into ellipsoids can be seen as critical objects of functionals depending on Steklov eigenvalues with respect to a Riemannian metric on the surface (see [Pet21]). Non planar disks are then expected as maximizers of well chosen linear combinations of the first and second Steklov eigenvalues renormalized by the length, $\bar{\sigma}_1(g), \bar{\sigma}_2(g)$ among metrics g on $\mathbb{D}$. We also work with the linear combinations of their inverse which are more related to Weinstock's inequality [Wei54]: $\bar{\sigma}_1^{-1} + \bar{\sigma}_2^{-1} \geq \pi^{-1}$ (with equality only realized by flat disks). We give four steps of proof:

Step 1 The infimum of $h_t^1 = (\bar{\sigma}_1 + t\bar{\sigma}_2)^{-1}$ or $h_t^2 : \bar{\sigma}_1^{-1} + t\bar{\sigma}_2^{-1}$ is realized by a metric g_c for any parameter $t \geq 0$. Since g_c is critical, we obtain by [Pet21] a possibly branched free boundary minimal immersion $\Phi : \mathbb{D} \to \mathbb{R}^{1+n}$ into an ellipsoid

$$E_n = \{(x_0, x') \in \mathbb{R}^{1+n}; \sigma_1(g_c)x_0^2 + \sigma_2(g_c)\left|x'\right|^2 = 1\}$$

such that the first coordinate ϕ_0 is a first Steklov eigenfunction with respect to g_c and the others are independent second eigenfunctions,

Step 2 $n = 1$ or $n = 2$, which means that the minimal surface $\Phi(\mathbb{D})$ is (possibly branched) immersed into $\mathbb{R}^3$ (maybe planar),

Step 3 $\Phi(\mathbb{D})$ cannot be planar for good choices of t,

Step 4 Φ is an embedding.

Step 1 and Step 4 are the most technical steps but all the steps need a particular attention since they are not automatic in the general context of shape optimization of eigenvalues. For instance, by Weinstock's inequality, h_t^2 is only realized by the flat planar disk for $0 \leq t \leq 1$: a careful choice of t is needed in Step 3. Keep also in mind that for Step 1, $\bar{\sigma}_2 < 4\pi$ ([HPS75] [GP10]), where 4π is sharp (see [GP10] or (2)) and corresponds to the disjoint union of two flat disks of same boundary length $\mathbb{D} \sqcup \mathbb{D}$. Here, we expect that the sequence of optimal metrics converge to $\mathbb{D} \sqcup \mathbb{D}$ as $t \to +\infty$, and that the associated minimal immersions for t large have to be non planar to allow this convergence.

In order to prove Step 1, we check the sufficient condition of the paper [Pet21]

$$\inf_{g \text{ on } \mathbb{D}} h(\bar{\sigma}_1(g), \bar{\sigma}_2(g)) < \inf_{g \text{ on } \mathbb{D}\sqcup\mathbb{D}} h(\bar{\sigma}_1(g), \bar{\sigma}_2(g)) = h(0, 4\pi) \tag{1}$$

for the combination h we choose. For h_t^2, it is automatic since $h_t^2(0, 4\pi) = +\infty$. For h_t^1, we use new explicit asymptotic computations for a natural one parameter family of test metrics g_ϵ degenerating to a disjoint union of two flat disks of same boundary length as $\epsilon \to 0$ (see [Pet23b]):

$$\bar{\sigma}_1(g_\epsilon) \sim \frac{2\pi}{\ln \frac{1}{\epsilon}} \text{ and } \bar{\sigma}_2(g_\epsilon) - 4\pi \sim -16\pi\epsilon. \tag{2}$$

Step 2 is an immediate consequence of the sharp bound 2 on the multiplicity of first and second Steklov eigenvalues on disks [Jam16].

In Step 3, we prove by contradiction that for t large enough, the minimizer cannot be planar. If $\Phi : \mathbb{D} \to co(E_1)$ is planar, it has to be a difformorphism as a consequence of [Kne26]: Φ is harmonic into a convex set, $\Phi : \mathbb{S}^1 \to E_1$ is embedded because it is immersed (consequence of the Hopf lemma) and one coordinate vanishes only twice on the boundary because it is a first eigenfunction. We then explicitly compute first and second eigenvalues of the critical planar ellipses (or disks) and prove that they cannot be critical or minimizers for the chosen functionals.

In Step 4, we prove in [Pet23b] that if a cricital metric with respect to combinations of first and second eigenvalues $g_c = e^{2v_c}(dx^2 + dy^2)$, is even with respect to the coordinates (x, y), then Φ is an embedding. It relies on a careful study of nodal sets of the three symmetric coordinates of Φ and a deep use of a generalization of Kneser's theorem [AN21].

As a conclusion, since we are not able to prove that our minimizers have theses symmetries, we finally look for minimizers among metrics with these symmetries. The previous steps still work in this context (in particular (1) as a consequence of (2)) and the existence result is proved.

Open questions:

- Are there *non-embedded* free boundary minimal disks into ellipsoids by first and second eigenfunctions ? Of course, if they exist, they are non planar and cannot have the previous symmetries. We only proved in [Pet23b] that they cannot have any branched point
- Are the non planar minimizers of $\{h_t^1\}_{t\geq 0}$ and $\{h_t^2\}_{t\geq 0}$ the same ? More generally, is there a choice of combinations of first and second Steklov eigenvalues having a non planar minimizer which is different to the previous ones ?
- Given the parameter q, is the non-planar free boundary minimal disk with the previous symmetries into the ellipsoid $\{x_0^2 + q(x_1^2 + x_2^2) = 1\}$ unique ?
- Is there a non-planar free boundary minimal disk into $\{x_0^2+px_1^2+qx_2^2 = 1\}$ for $1 < p < q$? (for instance, $1, p, q$ are the 3 first Steklov eigenvalues ?)

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On the rearrangement of a function and its applications

Cristina Trombetti

(joint work with A. Alvino,V. Amato, A. Gentile, C.Nitsch)

Let Ω be an open, bounded set of $\mathbb{R}^N$, $N \geq 2$ and let $h : x \in \Omega \to [0, +\infty[$ be a measurable function, then the decreasing rearrangement h^* of h is defined as follows:

$$h^*(s) = \inf\{t \geq 0 : |\{x \in \Omega : |h(x)| > t\}| < s\} \quad s \in [0, \Omega].$$

The Schwarz rearrangement of h is defined as follows

$$h^\sharp(x) = h^*(\omega_N |x|^N) \quad x \in \Omega^\sharp,$$

where ω_N denotes the measure of the unit ball in $\mathbb{R}^N$ and $\Omega^\sharp$ is the ball centered at the origin with the same Lebesgue measure as Ω. It is easily checked that h, h^* and $h^\sharp$ are equi-distributed, i.e.

$$|\{x \in \Omega : |h(x)| > t\}| = |\{s \in (0, |\Omega| : h^*(s) > t\}| = |\{x \in \Omega^\sharp : h^\sharp(x) > t\}| \quad t \geq 0$$

and then if $h \in L^p(\Omega)$, $1 \leq p \leq \infty$, then $h^* \in L^p(0, |\Omega|)$, $h^\sharp \in L^p(\Omega^\sharp)$, and

$$||h||_{L^p(\Omega)} = ||h^*||_{L^p(0,|\Omega|)} = ||h^\sharp||_{L^p(\Omega^\sharp)}.$$

Comparison results à la Talenti have been widely studied in the last decades, after in his seminal paper [11] Talenti proved that, if $f \in L^2(\Omega)$, u is the solution to

$$\begin{cases} -\Delta u = f & \text{in } \Omega \\ \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

and v is the solution to

$$\begin{cases} -\Delta v = f^\sharp & \text{in } \Omega^\sharp \\ \\ v = 0 & \text{on } \partial\Omega^\sharp, \end{cases}$$

then $u^\sharp(x) \leq v(x)$ for all x in $\Omega^\sharp$. It is impossible to make a comprehensive list of all the results developed in the wake of this fundamental achievement. Generalization to semilinear and nonlinear elliptic equations are, for instance, in [2, 12], anisotropic elliptic operators are considered for instance in [1], while parabolic equation are handled for instance in [2]. Higher order operators have been investigated for instance in [5, 13] and two textbooks which provide survey on Talenti's technique and collect as well many other references are [9, 10]. However, to our knowledge, in literature there are no comparison results related to Talenti techniques, concerning Robin boundary conditions.

In this talk we explore the possibility to generalize Talenti's result to the case of Robin boundary conditions [3]. We mention however that, when $f = 1$, a similar result has been proved in [6] with a completely different argument. We recall the Theorem of Giarrusso and Nunziante ([7, 8]).

Theorem 1. *Let $\Omega \subset \mathbb{R}^n$ be a bounded open set, let $\Omega^\sharp$ be the centered ball, let $p \geq 1$, let $f \colon \Omega \to \mathbb{R}$ be a measurable function, let $H \colon \mathbb{R}^n \to \mathbb{R}$ be measurable non-negative functions and let $K \colon [0, +\infty) \to [0, +\infty)$ be a strictly increasing real-valued function such that*

$$0 \leq K(|y|) \leq H(y) \qquad \forall y \in \mathbb{R}^n \qquad \textit{and } K^{-1}(f) \in L^p(\Omega).$$

Let $v \in W_0^{1,p}(\Omega)$ be a function that satisfies

$$\begin{cases} H(\nabla v) = f(x) & \textit{a.e. in } \Omega \\ v = 0 & \textit{on } \partial\Omega \end{cases},$$

denoting by $z \in W_0^{1,p}(\Omega^\sharp)$ the unique spherically decreasing symmetric solution to

$$\begin{cases} K(|\nabla z|) = f_\sharp(x) & \textit{a.e. in } \Omega^\sharp \\ z = 0 & \textit{on } \partial\Omega^\sharp \end{cases},$$

then,

$$\|v\|_{L^1(\Omega)} \leq \|z\|_{L^1(\Omega^\sharp)}. \tag{1}$$

In [4] we generalize the above result to the BV-settings.

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Pólya's conjecture for Euclidean balls and related questions

Michael Levitin
(joint work with N. Filonov, I. Polterovich, and D. A. Sher)

1. Introduction

Let $\Omega \subset \mathbb{R}^d$, $d \geq 2$, be a bounded open set with Lipschitz boundary, and let

$$\lambda_1^{\aleph}(\Omega) \leq \lambda_2^{\aleph}(\Omega) \leq \dots \qquad \text{and} \qquad \mathcal{N}_\Omega^{\aleph}(\lambda) := \#\{k \in \mathbb{N} : \lambda_k^{\aleph}(\Omega) \leq \lambda^2\},$$

with $\aleph \in \{\mathrm{D}, \mathrm{N}\}$, be the eigenvalues of the Dirichlet and Neumann Laplacian on Ω and their counting functions, respectively. The celebrated Pólya's conjecture (1954) states that for any such domain Ω one has

$$\text{(1)} \qquad \mathcal{N}_\Omega^{\mathrm{D}}(\lambda) \leq W_\Omega(\lambda) := C_d |\Omega|_d \lambda^d \leq \mathcal{N}_\Omega^{\mathrm{N}}(\lambda), \qquad \text{with } C_d = \frac{(4\pi)^{-d/2}}{\Gamma\left(\frac{d}{2}+1\right)},$$

for *all* $\lambda \geq 0$. Here $W_\Omega(\lambda)$ is the leading term in the Weyl asymptotics of the eigenvalue counting functions.

Until very recently, the only case when (1) was proved in full generality (in [5] in the Dirichlet case and in [3] in the Neumann case) has been that of *tiling domains* Ω such that $\mathbb{R}^d$ can be covered, up to a set of measure zero, by a union of disjoint congruent copies of Ω. Additionally, Laptev showed in the Dirichlet case that if Pólya's conjecture holds for $\Omega \subset \mathbb{R}^d$, it also holds for $\Omega \times I \subset \mathbb{R}^{d+1}$, where I is any finite interval.

In a recent paper [1], we have proved the following

Theorem 1. *Pólya's conjecture* (1) *holds*

(a) *in the Dirichlet case if* $\Omega \subset \mathbb{R}^d$, $d \geq 2$, *is a ball;*
(b) *in the Neumann case if* $\Omega \subset \mathbb{R}^2$ *is a disk;*
(c) *in both the Dirichlet and the Neumann cases if* $\Omega \subset \mathbb{R}^2$ *is a finite circular sector of an arbitrary aperture.*

We also showed, in either the Dirichlet or the Neumann case, that if Pólya's conjecture (1) holds for $\Omega \subset \mathbb{R}^d$ and Ω' tiles Ω, then it also holds for Ω'.

The proof of Theorem 1 is quite involved and requires some important novel ingredients which we discuss below.

2. Uniform estimates of zeros of Bessel functions and their derivatives

It is well known that the eigenvalues of the Dirichlet Laplacian in a disk can be expressed in terms of zeros of Bessel functions. Can one find *elementary* (that is, without the use of special functions) and *uniform* (that is, one expression in all the cases) enclosures for these zeros? This question has a very long history, and we mention only Watson, Hethcote, Elbert–Laforgia and Qu–Wong among many works on the subject.

Concentrating only on the kth positive zero $j_{\nu,k}$ of the Bessel function J_ν, we state the following new bounds.

Theorem 2 ([6, 1, 2])**.** *Let $\nu \geq 0$, and let the functions $\underline{\theta}_\nu, \overline{\theta}_\nu : (\nu, +\infty) \to \mathbb{R}$ be defined by*

$$\overline{\theta}_\nu(x) := \sqrt{x^2 - \nu^2} - \nu \arccos\frac{\nu}{x} - \frac{\pi}{4}, \qquad \underline{\theta}_\nu(x) := \overline{\theta}_\nu(x) - \frac{3x^2 + 2\nu^2}{24(x^2 - \nu^2)^{3/2}}.$$

Then these functions are monotone increasing (and therefore invertible) and

$$\overline{\theta}_\nu^{-1}\left(\pi k - \frac{\pi}{2}\right) < j_{\nu,k} < \underline{\theta}_\nu^{-1}\left(\pi k - \frac{\pi}{2}\right) \qquad \text{for all } k \in \mathbb{N}. \tag{2}$$

Similar two-sided bounds are obtained for zeros $y_{\nu,k}$ of the Bessel functions Y_ν and for zeros $j'_{\nu,k}$ and $y'_{\nu,k}$ of the derivatives J'_ν and Y'_ν.

Leaving aside the question of whether bounds (2) are elementary enough, we remark that in practice they provide very tight enclosures for Bessel zeros, for example $171.71092 < j_{10,50} < 171.71167$ with a relative error less than $5 \cdot 10^{-6}$. The lower bound in (2) is needed for the proof of Theorem 1 and was first established in [6] via the analysis of the so-called Bessel *phase function*. The full proof of Theorem 2 and its extensions, using the Sturm comparison theorem, will be given in [2].

We note that the bounds in (2) arise from the asymptotics of the Bessel phase function for large ν. It would be interesting to know if similar techniques may be used to obtain uniform bounds for zeros of other special functions or, more generally, of solutions of some class of second order ODEs.

3. Counting lattice points

Since the eigenvalue counting function of the Dirichlet Laplacian in the unit disk $\mathbb{D} \subset \mathbb{R}^2$ is given by

$$\mathcal{N}_{\mathbb{D}}^{\mathrm{D}}(\lambda) = \#\{k : j_{0,k} \leq \lambda\} + 2\sum_{m=1}^{\infty} \#\{k : j_{m,k} \leq \lambda\},$$

and because $j_{\nu,1} > \nu$ for all $\nu \geq 0$, applying the lower bound (2) yields

$$\mathcal{N}_{\mathbb{D}}^{\mathrm{D}}(\lambda) < \sum_{m=0}^{\lfloor\lambda\rfloor} \kappa_m \left\lfloor G_\lambda(m) + \frac{1}{4}\right\rfloor =: \mathcal{P}^{\mathrm{D}}(\lambda), \tag{3}$$

where $\kappa_m = \begin{cases} 1, & m = 0, \\ 2, & m > 0, \end{cases}$ and $G_\lambda(z) := \frac{1}{\pi}\left(\sqrt{\lambda^2 - z^2} - z \arccos \frac{z}{\lambda}\right)$. Thus, Pólya's conjecture for the Dirichlet Laplacian in the unit disk will follow immediately if we can show that $\mathcal{P}^{\mathrm{D}}(\lambda) < \frac{\lambda^2}{4} = W_{\mathbb{D}}(\lambda)$ for all $\lambda > 0$.

We remark that the sum $\mathcal{P}^{\mathrm{D}}(\lambda)$ appearing in the right-hand side of (3) is the weighted count of shifted lattice points of the form $\left(m, k - \frac{1}{4}\right)$ under the graph of $G_\lambda(z)$ in the first quadrant, where $(m, k) \in (\mathbb{N} \cup \{0\}) \times \mathbb{N}$, and points with $m > 0$ are counted twice. Various bounds on such lattice point counts can be traced from the fundamental works of van der Corput to important recent results of Laugesen and Liu, see [4] and references therein. We need to improve these bounds by taking into account the properties of the derivative $G'_\lambda(z)$ via the following general result.

Theorem 3. *Let $b > 0$, and let $g : [0, b] \to \mathbb{R}$ be a monotone decreasing convex function with $g(b) = 0$ and such that $|g(z) - g(w)| \leq \frac{1}{2}|z - w|$ for all $z, w \in [0, b]$. Then*

$$\sum_{m=0}^{\lfloor b \rfloor} \kappa_m \left\lfloor g(m) + \frac{1}{4} \right\rfloor \leq 2 \int_0^b g(z)\,\mathrm{d}z,$$

with equality attained only when g is identically zero.

Since it is easy to check that $g = G_\lambda$ with $b = \lambda$ satisfies the conditions of Theorem 3, and that $\int_0^\lambda G_\lambda(z)\,\mathrm{d}z = \frac{\lambda^2}{8}$, we immediately obtain the proof of Theorem 1(a) for $d = 2$. Other cases in Theorem 1 are more complicated and require some extra work: in particular, the proof for the Neumann problem in the disk leaves a gap for $\lambda \in [3, 14]$ which we close with the help of a rigorous computer-assisted argument (using only integer arithmetic).

Once more, it would be interesting to extend Theorem 3 (and its counterpart for the shift $\frac{3}{4}$ arising in the Neumann problem) to wider classes of functions, weights and shifts.

Further extensions of Theorem 1 will be addressed in [2].

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Homogenisation as control: mimicking eigenvalue problems

Jean Lagacé

(joint work with A. Girouard, A. Henrot and M. Karpukhin)

Homogenisation theory has long been established as an important tool in shape optimisation [1]. In particular, it provides a method for relaxation of the space of allowable domains. However the method is traditionally limited to euclidean spaces, or at least to spaces with a natural group action. In [5], we have presented a construction of a homogenisation limit on manifolds in the study of shape optimisation for the Steklov problem. Similar ideas were also present in [2, 3, 4, 6, 8] where homogenisation was used as a control mechanism to explore the behaviour of the Steklov problem.

1. The weighted Steklov and Laplace problems

Let (M, g) be a smooth, connected, closed Riemannian manifold of dimension $d \geq 2$ and $\Omega \subset M$ be a connected open set with Lipschitz boundary $\partial\Omega$. Consider the weighted Laplace and Steklov eigenvalue problems

$$\begin{cases} -\Delta f = \lambda \beta f & \text{in } M \end{cases} \qquad \begin{cases} \Delta u = 0 & \text{on } \Omega \\ \partial \nu u = \sigma \rho u & \text{on } \partial\Omega, \end{cases}$$

where $0 < \beta \in \mathrm{C}^\infty(M)$ and $0 < \rho \in \mathrm{C}^\infty(\partial\Omega)$. In either case, the eigenvalues form discrete sequences $0 = \lambda_0(M, g, \beta) < \lambda_1(M, g, \beta) \leq \lambda_2(M, g, \beta) \leq \ldots \nearrow \infty$ and $0 = \sigma_0(\Omega, g, \rho) < \sigma_1(\Omega, g, \beta) \leq \sigma_2(\Omega, g, \beta) \leq \ldots \nearrow \infty$ accumulating only at infinity. They satisfy the variational characterisations

$$\lambda_k(M, g, \beta) = \min_{E_k \subset \mathrm{W}^{1,2}(M)} \max_{f \in E_k \setminus \{0\}} \frac{\int_M |\nabla f|^2 \, \mathrm{d}v_g}{\int_M f^2 \beta \, \mathrm{d}v_g}$$

and

$$\sigma_k(\Omega, g, \rho) = \min_{E_k \subset \mathrm{W}^{1,2}(\Omega)} \max_{u \in E_k \setminus \{0\}} \frac{\int_\Omega |\nabla u|^2 \, \mathrm{d}v_g}{\int_{\partial\Omega} f^2, \rho \, \mathrm{d}A_g}$$

where E_k is a subspace of dimension $k + 1$.

Theorem 1. *Let (M, g) be a smooth connected closed Riemannian manifold and $0 < \beta \in \mathrm{C}^\infty(M)$. There is a family of domains $\Omega^\varepsilon \subset M$ (described in Section 3) such that for every $k \in \mathbb{N}$, $\sigma_k(\Omega^\varepsilon, g, 1) \to \lambda_k(\Omega, g, \beta)$ as*

$\varepsilon \to 0$. Furthermore, the associated eigenfunctions on Ω^ε can be extended to M in such a way that they converge weakly in $\mathrm{W}^{1,2}(M)$ to the relevant eigenfunction, up to maybe taking a subsequence.

Theorem 2. *Let (M, g) be a smooth connected closed Riemannian manifold, $\Omega \subset M$ an open set with Lipschitz boundary and $0 < \beta \in \mathrm{C}^\infty(\partial\Omega)$. There is a family of domains $\Omega^\varepsilon \subset \Omega$ (described in Section 4) such that for every $k \in \mathbb{N}$, $\sigma_k(\Omega^\varepsilon, g, 1) \to \sigma_k(\Omega, g, \rho)$ as $\varepsilon \to 0$. Furthermore, the associated eigenfunctions on Ω^ε can be extended to Ω in such a way that they converge weakly in $\mathrm{W}^{1,2}(\Omega)$ to the relevant eigenfunction, up to maybe taking a subsequence.*

Both of these theorems will be proven using a similar homogenisation limit.

2. A tesselation of a manifold

The homogenisation construction at hand is reliant on a tesselation of an arbitrary manifold. For $\varepsilon > 0$, let $\mathbf{Z}^\varepsilon \subset M$ be a maximal ε-separated set; that is a set where every point is at distance at least ε from each other, yet if we added any point to $\mathbf{Z}^\varepsilon$ it would no longer be ε-separated. Let $\mathbf{V}^\varepsilon := \{V_z : z \in \mathbf{Z}^\varepsilon\}$ be the Voronoĭ tesselation associated with $\mathbf{Z}^\varepsilon$, where for every $z \in \mathbf{Z}^\varepsilon$

$$V_z := \{x \in M : \operatorname{dist}(x, z) \le \operatorname{dist}(x, \zeta) \text{ for all } \zeta \in \mathbf{Z}^\varepsilon\}.$$

The tiles satisfy a few convenient properties.

- For every $z \in \mathbf{Z}^\varepsilon$, $B(z, \varepsilon) \subset V_z \subset B(z, 3\varepsilon)$.
- For ε sufficiently small and $z \in \mathbf{Z}^\varepsilon$ V_z is geodesically convex.
- The total number of tiles is of the order of ε^{-d}, and each has volume on the order of ε^d.

3. From Steklov to weighted Laplace

In order to prove Theorem 1, we will use the tesselation from the previous section as a starting point. For every $z \in \mathbf{Z}^\varepsilon$, remove a ball of radius $r_\varepsilon = a_d^{-1}\beta(z)\varepsilon^{\frac{d}{d-1}}$ around z, where a_d is the area of the unit sphere in $\mathbb{R}^d$. Then, Ω^ε consists of M with those balls removed. The following facts are not hard to prove:

- as $\varepsilon \to 0$, $\operatorname{vol}(\Omega^\varepsilon) \to \operatorname{vol}(M)$;
- as $\varepsilon \to 0$, the boundary measure of $\partial\Omega^\varepsilon$ converges in the weak-$*$ sense of measures to $\beta \mathrm{d}v_g$. In particular, $\operatorname{per}(\partial\Omega^\varepsilon) \to \int_M \beta \, \mathrm{d}v_g$.

Together, this is not enough to prove convergence of the eigenvalues. The convenient spectral convergence [4, Proposition 4.11] gives us two additional conditions which may be verified. First, that there is an equibounded family of extension operator $J^\varepsilon : \mathrm{W}^{1,2}(\Omega^\varepsilon) \to \mathrm{W}^{1,2}(M)$. This is classical in the euclidean case [10], but the proof extends without too much hassle to the Riemannian setting. The second condition is that the boundary measure of $\partial\Omega^\varepsilon$ converges, in fact, in the dual norm of $\mathrm{W}^{1,p}(M)^*$ for some $1 < p < \frac{d}{d-1}$. Together, this proves the claimed convergence of the eigenvalues and eigenfunctions.

From this, we get the following corollary

Corollary 1. *Let $\mathcal{G}$ be the class of metrics on a closed surface M. Then,*

$$\sup_{\substack{g \in \mathcal{G} \\ \beta \in \mathrm{C}^\infty(M)}} \lambda_k(M, g, 1) \int_M \beta \, \mathrm{d}v_g \le \sup_{\substack{g \in \mathcal{G} \\ \Omega \subset M}} \sigma_k(\Omega, g, 1) \operatorname{per}(\partial\Omega).$$

This is proven by starting from a metric and density pair (g, β) close to maximal on M for λ_k, then using the homogenisation process in order to approximate those eigenvalues with Steklov eigenvalues. Note that this can also be done within a conformal class, and that the reverse inequality to Corollary 1 has been proven in [9], so that these suprema are in fact equal.

4. From Steklov to weighted Steklov

In this situation, the tesselation from Section 2 is applied considering $\partial\Omega$ itself as a closed manifold of dimension $d-1$. This time, rather than remove a hole around every $z \in \mathbf{Z}^\varepsilon$, we push the boundary of Ω inwards as a cone with base V_z and the right height so that the boundary measure of this cone is $\operatorname{per}(V_z)\rho(z)$. We note that this is possible only if $\rho \geq 1$ everywhere, however this assumption can be later lifted in [2] and [6]. Furthermore, such a cone can be analytically described but would require a bit too much extra notation, see [6, p. 7], importantly its height is of order ε. We put Ω^ε as the domain obtained by the perturbation described above, and in the Hausdorff metric $\Omega^\varepsilon \to \Omega$. Here again we use [4, Proposition 4.11], and have convergence of the boundary measure $\mathrm{d}a_{\partial\Omega^\varepsilon}$ to $\rho \mathrm{d}a_{\partial\Omega}$ in the dual norm on $\mathrm{W}^{1,p}(\Omega)^*$, for $1 < p < \frac{d}{d-1}$. We get the following corollary about flexibility of the Steklov spectrum, which appears in [2] for simply connected domains and [6] for general surfaces.

Corollary 2. *Let Ω_1, Ω_2 be two compact surfaces with boundary in the same conformal class. Then, there is a sequence $\Omega^\varepsilon \subset \Omega_2$, converging to Ω_2 in the Hausdorff metric, such that $\sigma_k(\Omega^\varepsilon, g, 1)\operatorname{per}(\Omega^\varepsilon) \to \sigma_k(\Omega_1, g, 1)\operatorname{per}(\Omega_1)$ as $\varepsilon \to 0$.*

The proof of Corollary 2 hinges on three facts. Fix a conformal diffeomorphism $\phi : \Omega_2 \to \Omega_1$. Then, following [7, Theorem 1.7], the Steklov problem on Ω_1 is isospectral to the weighted problem

$$\begin{cases} \Delta u = 0 & \text{in } \Omega_2 \\ \partial_\nu u = \sigma|\phi'|u & \text{on } \partial\Omega_2. \end{cases}$$

Applying the homogenisation Theorem 2 we obtain convergence of the eigenvalues. Finally, $\operatorname{per}(\partial\Omega^\varepsilon) \to \int_{\partial\Omega_2} \rho \, \mathrm{d}a_g = \operatorname{per}(\partial\Omega_1)$.

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Classical wave methods and modern gauge transforms: Spectral asymptotics in the one dimensional case

Leonid Parnovski
(joint work with J. Galkowski and R. Shterenberg)

Let (M, g) be a smooth, compact, connected Riemannian manifold of dimension d, V be a smooth positive potential and $-\Delta_g$ be the Laplace–Beltrami operator on M. Then the operator $-\Delta_g + V$ has discrete spectrum, $0 = \lambda_0^2 < \lambda_1^2 \le \lambda_2^2 \le \dots$, with $0 \le \lambda_j \to \infty$. The following conjecture has been made by physicists:

Conjecture 1 (Sommerfeld–Lorentz, 1910)**.** *Let*

$$N(\lambda) := \#\{\, j \,:\, \lambda_j \le \lambda\}.$$

Then,

$$N(\lambda) = \frac{\mathrm{vol}_g(M)\,\mathrm{vol}_{\mathbb{R}^d}(B_1)}{(2\pi)^d}\lambda^d + o(\lambda^d). \tag{1}$$

Despite seemingly being very difficult (D.Hilbert did not think it would be proved in his lifetime), this conjecture was proved by H. Weyl in 1911. One of the ways of proving it is using the Laplace transform of $N(\lambda)$. Consider $u(t) := \mathrm{tr}(e^{t\Delta_g}) = \sum_j e^{-t\lambda_j^2}$, $t > 0$.

Theorem 1 (Minakshisundaram–Pleijel - 1949)**.** *Let (M, g) be a smooth, compact Riemannian manifold of dimension d. Then, there are $\{a_j\}_{j=1}^\infty$ such that for all N we have, as $t \to 0^+$:*

$$u(t) = \frac{\mathrm{vol}(M)}{(4\pi t)^{\frac{d}{2}}} + \sum_{j=1}^{N-1} a_j t^{-\frac{d}{2}+j} + O(t^{-\frac{d}{2}+N}). \tag{2}$$

This theorem implies (1); one can even make a ‘naive conjecture’ that if one takes the inverse Laplace transform of the RHS of (2), one would get a complete asymptotic expansion of $N(\lambda)$:

Naive Conjecture 1. *Let $N(\lambda) := \#\{j \,:\, \lambda_j \le \lambda\}$. Then, there are $\{b_j\}_{j=1}^\infty$ such that for all N we have:*

$$N(\lambda) = \frac{\mathrm{vol}_g(M)\,\mathrm{vol}_{\mathbb{R}^d}(B_1)}{(2\pi)^d}\lambda^d + \sum_{j=1}^{N-1} b_j\lambda^{d-j} + O(\lambda^{d-N}). \tag{3}$$

This conjecture is clearly false, and the obstructions to (3) being valid are closed geodesic trajectories. Similarly, if one considers local density of states

$$e(-\Delta_g + V, \lambda)(x) := 1_{(-\infty,\lambda^2]}(-\Delta_g + V)(x, x),$$

then the main obstructions to it having a complete power asymptotic expansion as $\lambda \to \infty$ are closed loops (geodesics that start and finish at x). This naturally leads to the question of whether the complete asymptotics exist once there are no loops. The absence of loops is clearly impossible if M is compact, so let us consider the simplest non-compact setting of $M = \mathbb{R}^d$.

We say $V \in C_b^\infty(\mathbb{R}^d)$ if $V \in C^\infty$ and for all $\alpha \in \mathbb{N}^d$, there are $C_\alpha > 0$ such that

$$\|\partial_x^\alpha V\|_{L^\infty} \leq C_\alpha.$$

The following conjecture was formulated in 2016:

Conjecture 2 (LP–Shterenberg 2016)**.** *Suppose $V \in C_b^\infty(\mathbb{R}^d)$. Then, there are $\{a_j(x)\}_{j=0}^\infty$ such that for any $N > 0$,*

$$e(-\Delta_{\mathbb{R}^d} + V, \lambda)(x) = \sum_{j=0}^{N-1} a_j(x)\lambda^{d-j} + O(\lambda^{d-N}).$$

This conjecture was earlier proved in the situations when V is periodic, almost-periodic (with several extra assumptions), or compactly supported. This talk presents the proof of this conjecture on the one-dimensional case, without any restrictions on $V \in C_b^\infty(\mathbb{R}^d)$:

Theorem 2 (Galkowski – LP – Shterenberg 2022)**.** *Let $V \in C_b^\infty(\mathbb{R};\mathbb{R})$. Then there are $\{a_j(x)\}_{j=0}^\infty$ such that for all $N > 0$, there is $C_N > 0$ satisfying*

$$\left| e(-\Delta_{\mathbb{R}} + V, \lambda)(x) - \sum_{j=0}^{N-1} a_j(x)\lambda^{1-2j} \right| \leq C_N \lambda^{1-2N}.$$

Moreover, $a_j(x)$ can be determined from a finite (j-dependent) number of derivatives of V at x.

Open Problems: Emerging Topics

Coordinated by Pavel Exner

The following is a summary of the open problems presented on Monday 21st of August 2023.

Jussi Behrndt: The fate of Landau levels under δ-perturbations. The singularly perturbed self-adjoint Landau Hamiltonian

$$H_\alpha = (i\nabla + A)^2 + \alpha\delta_\Sigma$$

in $L^2(\mathbb{R}^2)$ with a δ-potential supported on a finite $C^{1,1}$-smooth curve Σ was first studied in [1]. Here $A = \frac{1}{2}B(-x_2, x_1)^\top$ is the vector potential, $B > 0$ is the strength of the homogeneous magnetic field, and $\alpha \in L^\infty(\Sigma)$ is a position-dependent real coefficient modeling the strength of the singular interaction on the curve Σ. It turns out that the perturbation $\alpha\delta_\Sigma$ is compact in resolvent sense and hence the essential spectrum of H_α coincides with the (essential) spectrum of the unperturbed Landau Hamiltonian $H_0 = (i\nabla + A)^2$, that is,

$$\sigma_{\text{ess}}(H_\alpha) = \sigma_{\text{ess}}(H_0) = \sigma(H_0) = \big\{\Lambda_q = B(2q+1) : q \in \mathbb{N}_0\big\}.$$

It is shown in [1] that the singular perturbation $\alpha\delta_\Sigma$ smears the Landau levels Λ_q into eigenvalue clusters of H_α, and the accumulation rate of the eigenvalues within these clusters is determined in terms of the capacity of the support of α. The fate of the Landau levels Λ_q was studied in more detail in [2], where it is shown with the help of a Berezin-Toeplitz type operator that for positive perturbations the lowest Landau level Λ_0 is not an eigenvalue of H_α, but the higher Landau levels Λ_q, $q \in \mathbb{N}$, may remain eigenvalues of H_α. This effect is illustrated in [2] for the special case that Σ is a circle. The fact that Landau levels may remain eigenvalues under singular pertubations is in stark contrast to the results for classical sign definite potentials in [8]. A more detailed investigation on the multiplicity of the perturbed Landau levels for general finite $C^{1,1}$-smooth curves Σ remains open.

Anne-Sophie Bonnet-Ben Dhia: A sign-changing eigenvalue problem on the sphere. Consider a partition of the sphere $\mathbb{S}^2$ into two regular subdomains ϖ_+ and ϖ_- such that ϖ_- is contained in an hemisphere and such that their common boundary $\partial\varpi_+ = \partial\varpi_-$ is smooth (at least $\mathcal{C}^2$) and connected. Suppose that ε is a real valued function defined on $\mathbb{S}^2$, piecewise constant, which takes a strictly positive value ε_+ on ϖ_+ and a strictly negative value ε_- on ϖ_-. We consider the following eigenvalue problem: find $\lambda \in \mathbb{C}$ and $\varphi \in \mathrm{H}^1(\mathbb{S}^2)$, $\varphi \neq 0$ such that

$$\int_{\mathbb{S}^2} \varepsilon \nabla_S \varphi \cdot \nabla_S \overline{\psi}\, d\omega = \lambda(\lambda+1) \int_{\mathbb{S}^2} \varepsilon\, \varphi \overline{\psi}\, d\omega, \quad \forall \psi \in \mathrm{H}^1(\mathbb{S}^2)$$

where ∇_S denotes the surfacic gradient on the sphere. This problem arises when studying time-harmonic electromagnetic waves in presence of a conical tip of material with negative electromagnetic constants.

It has been proved in [10] that the spectrum is discrete as soon as the contrast $\varepsilon_-/\varepsilon_+$ is not equal to -1. The problem is non-selfadjoint and eigenvalues are complex. People are interested in eigenvalues of the form $\lambda = -1/2 + i\eta$ with $\eta \in \mathbb{R}$, $\eta \neq 0$, because there are linked to very strong singularities of the electromagnetic field. When $\partial\varpi_+ = \partial\varpi_-$ is a circle, one can prove that such eigenvalues do not exist when the contrast is below -1. The *open question* is the following: under which condition on the interface $\partial\varpi_+ = \partial\varpi_-$ does this result remain true?

Pedro Freitas: Pólya–type inequalities. Given a domain Ω in $\mathbb{R}^n$ with measure $|\Omega|$, Pólya's conjecture states that its Dirichlet Laplacian eigenvalues $\lambda_1 \leq \lambda_2 \leq ...$, satisfy $\lambda_k(\Omega) \geq C_n \left(\frac{k}{|\Omega|}\right)^{2/n}$, where $C_n = \frac{4\pi^2}{\omega_n^{2/n}}$ is the constant in the leading term in the corresponding Weyl asymptotics. The conjecture is known not to hold for manifolds in general, but recent results for the highly symmetric case of (round) $n-$hemispheres H_+^n show that it is possible to obtain inequalities of a similar type by adding a correction term [6]. In the case of H_+^n we have

$$\lambda_k\left(H_+^n\right) \geq C_n \left(\frac{k}{\left|H_+^n\right|}\right)^{2/n} - \frac{(n-1)(n-2)}{6}.$$

Open problem: Are there similar inequalities valid for families of manifolds?

Another instance where Pólya's conjecture fails is for the Robin Laplacian with positive boundary parameter α. Based on the results obtained in [5] for rectangles and unions of rectangles, we conjecture that similar results will hold for convex and general domains in $\mathbb{R}^n$:

Conjecture: Given positive numbers α and M, there exist positive constants a and b depending on α, M and n only, such that $\rho_k(\Omega) \geq ak^{2/(2n-1)}$ (convex Ω) and $\rho_k(\Omega) \geq bk^{1/n}$, (general Ω) for all $k \in \mathbb{N}$, where $|\Omega| = M$.

Daniel Grieser: Behavior of the spectrum under degeneration. This contribution is about a class of problems, rather than a single problem. There are many isolated results dealing with the behavior of spectral quantities associated to families $(P_\varepsilon)_{0<\varepsilon<\varepsilon_0}$ of elliptic operators (assumed to be self-adjoint with discrete spectrum here) which degenerate as $\varepsilon \to 0$. One class of examples is $P_\varepsilon = \Delta$ on domains $\Omega_\varepsilon \subset \mathbb{R}^n$ (bounded, with some choice of boundary conditions; one may replace $\mathbb{R}^n$ by a Riemannian manifold), e.g.: 'small enclosures', i.e. $\Omega_\varepsilon = \Omega \setminus \overline{B_\varepsilon(\{p\})}$ where Ω is a fixed domain, $p \in \Omega$ and we denote $B_\varepsilon(S) := \{x \in \mathbb{R}^n : \operatorname{dist}(x, S) < \varepsilon\}$ for a set S; 'dumbbells', i.e. $\Omega_\varepsilon = \Omega \cup \Omega' \cup B_\varepsilon(L)$ where Ω, Ω' are domains with disjoint closure and L is a line segment (or curve) connecting them; 'fat graphs', i.e. $\Omega_\varepsilon = B_\varepsilon(G)$, where G is a metric graph embedded in $\mathbb{R}^n$; 'smoothings', e.g. $\Omega_\varepsilon =$ a polygon in $\mathbb{R}^2$ whose corners are smoothed at a scale of ε. Another class of examples is semi-classical operators, e.g. $P_\varepsilon = \varepsilon^2\Delta + V$ on $\Omega \subset \mathbb{R}^n$ for a potential function V on Ω. Some of these results (always concerning the asymptotic behavior as $\varepsilon \to 0$) are about single eigenvalues, i.e. $\lambda_k(\varepsilon)$ for fixed k, others are about spectral invariants involving all eigenvalues, e.g. $\det \Delta$, or about the resolvent or heat kernel.

There are some well-known general ideas how to deal with such problems, e.g. the idea of model problems (rescaled limits), often in combination with the method of matched asymptotic expansions, and they are often applied in a somewhat ad hoc way. I want to propose that research effort should be directed towards developing more general systematic approaches for dealing with such problems. The language of manifolds with corners and blow-ups, as originally developed by Richard Melrose (see e.g. [9]), is a suitable framework for this. An attempt at a somewhat general setup for applying these ideas in the context of degenerating families of domains (as in the examples above) was given in [7]. See also the extended abstract of my talk on generalized semiclassical operators in this volume.

Konstantin Pankrashkin: Approximating spheres by tori. Let k_1 and k_2 be the principal curvatures on a connected compact smooth surface $\Sigma \subset \mathbb{R}^3$ and define $H_\Sigma := \|k_1 - k_2\|_\infty$. As known from the classical differential geometry, the minimal value H_Σ is zero, which is only attained by the spheres. If Σ has genus $g \geq 1$, then by Marques-Neves-Willmore inequality and Gauss-Bonnet theorem one has

$$H_\Sigma^2\,|\Sigma| \geq \int_\Sigma (k_1 - k_2)^2 = \int_\Sigma (k_1 + k_2)^2 - 4\int_\Sigma k_1 k_2 \geq 8\pi^2 + 16\pi(g-1),$$

while one of the inequalities is strict, which results in the strict lower bound

$$H_\Sigma > \tfrac{2\pi}{\sqrt{|\Sigma|}}\sqrt{2+\tfrac{4(g-1)}{\pi}}.$$

I propose to look at the following problem: minimize H_Σ under all Σ having a fixed surface area and a fixed genus g. Of interest are lower bounds for the infimum and any description of minimizing surfaces Σ, if they exist. Probably the case of the tori ($g=1$) is the most interesting one. We remark that the quantity H_Σ appears as the 'size' of the essential spectrum of Dirac operators with some special transmission conditions along Σ, see [3].

Pavel Exner: Multiplicity of the waveguide discrete spectrum. Consider a *soft quantum waveguide*, for simplicity in two dimensions, described by the Schrödinger operator $H_{\Gamma,V} = -\Delta + V(\mathrm{dist}(x,\Gamma))$, where Γ is an infinite, appropriately regular (say, C^3) curve parametrized by its arc length, and V is a potential supported in $[-a,a]$. We assume (i) Γ is *not straight* but it is straight outside a compact, or at least *asymptotically straight*, (ii) $|\Gamma(s)-\Gamma(s')| \to \infty$ as $|s-s'|\to\infty$, (iii) the *tubular neighborhood* of Γ with halfwidth a does not intersect itself, and (iv) $\sigma_{\mathrm{disc}}\big(-\frac{\mathrm{d}^2}{\mathrm{d}t^2}+V(t)\big) \neq \emptyset$. In this situation the *discrete spectrum of* $H_{\Gamma,V}$ *is nonempty*, see e.g. [4]. Clearly, $\epsilon_1 := \inf \sigma\big(H_{\Gamma,V}\big)$ is a simple eigenvalue.

Question: Are the other eigenvalues – provided they exist – also simple?

Let us add a couple of remarks: (a) the question is not void; for suitably chosen Γ and V, one can get any finite number of eigenvalues, (b) the question is open also for 'hard-wall' waveguides in which the potential confinement is replaced by Dirichlet condition, and (c) the quasi-onedimensional character is vital; if the generating curve is replaced by a graph, the spectrum may not be simple. An example is a symmetric star-shaped potential channel with an extended enough connecting region which can have degenerate eigenvalues as observed by Thomas Hoffmann-Ostenhof.

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Open problems: Spectral Optimisation

Coordinated by Pedro Freitas

The following is a summary of the open problems presented on Wednesday 23rd of August 2023.

Pavel Exner: Optimization of potential well families. Consider a family of N radial potentials supported in non-overlapping balls $B_\rho(y_i)$ with the centers in $Y = \{y_i\}$ given by the real-valued function $L^2(0,\rho) \ni V \geq 0$ giving rise to the Schrödinger operator $H_{V,Y} = -\Delta - \sum_i V(x - y_i)$ in $L^2(\mathbb{R}^\nu)$ with $\nu = 2, 3$. If the points of Y are distributed over a planar circle, the principal eigenvalue of $H_{V,Y}$ is *maximixed*, uniquely up to rotations, by the configurations in which all the neighboring points have the same angular distance $\frac{2\pi}{N}$ [7]; one *conjectures* that the symmetric arrangement is also a maximizer if the points of Y are placed on a loop-shaped curve having the same arc-length distances.

The problem becomes more difficult is Y is a subset of a *sphere* Σ; it reminds the well-known and notoriously difficult *Thomson problem*.

Conjecture: If the balls $B_\rho(y_i)$ centered at Σ do not overlap, $\inf \sigma(H_{V,Y})$ is *maximized*, uniquely up to Euclidean transformations, by three *simplices*, with $N = 2$ (a pair antipodal points), $N = 3$ (equilateral triangle), and $N = 4$ (tetrahedron), and furthermore by *octahedron* with $N = 6$ and *icosahedron* with $N = 12$. These extrema are independent of V.

Note that point-interactions counterpart of these conjectures are valid [8]. The results for the other N are rather difficult to find and expected to depend on V.

Antoine Henrot: Ratio and partial monotonicity of Neumann eigenvalues. Let Ω be a convex domain in the plane and $0 = \mu_0(\Omega) < \mu_1(\Omega \leq \mu_2(\Omega) \leq \ldots$ its Neumann eigenvalues. We are interested in the two following problems.

Ratio μ_k/μ_1. According to the Payne-Weinberger inequality [14], together with the Kröger inequality [12] (rediscovered by Henrot-Michetti [11]), we have the following bound for the ratio μ_k/μ_1:

$$\frac{\mu_k(\Omega)}{\mu_1(\Omega)} < \left(\frac{2j_{0,1}}{\pi} + (k-1)\right)^2.$$

Numerical simulations, see for example Antunes-Henrot [1], suggest the sharp inequality $\frac{\mu_2(\Omega)}{\mu_1(\Omega)} \leq 4$, equality being attained by rectangles $(0;L) \times (0,\ell)$ with $L \geq 2\ell$. Our first problem is: can we prove this inequality or more generally the inequality

$$\frac{\mu_k(\Omega)}{\mu_1(\Omega)} \leq k^2?$$

Partial monotonicity. It is well-known that Neumann eigenvalues do not satisfy domain monotonicity (by contrast with Dirichlet eigenvalues). With L. Cavallina, K. Funano, A. Lemenant, I. Lucardesi and S. Sakaguchi, this led us to study the problems: for a given convex D, solve

$$\min\{\mu_k(\Omega), \Omega \text{ convex}, \ \Omega \subset D\}. \tag{1}$$

For $k = 1$, this problem has no solution, but, in general, it has a solution for $k \geq 2$.

In particular, it seems that if we take D to be the unit square, D is itself the solution of (1) for $k = 2$. This leads to the following open problem: Let D be the unit square, prove that for any convex subdomain $\Omega \subset D$, we have $\mu_2(\Omega) \geq \mu_2(D)$.

Mikhail Karpukhin: Universal bounds for magnetic and Bakry-Emery Laplacians. Let (M, g) be a closed Riemannian surface. For a real-valued 1-form A on M one has a differential d_A given by $d_A = d - iA$. The magnetic Laplacian is then defined as $\Delta_A = d_A^* d_A$, where d_A^* is the formal adjoint of d_A. Let $\bar\lambda_i(M, g, A) := \lambda_i(M, g, A)\mathrm{Area}(M, g)$ denote its normalised eigenvalues. Then one can show that critical pairs (g, A) of $\lambda_i(M, g, A)$ correspond to harmonic maps to $\mathbb{CP}^n$. This raises a question of whether there is a natural way to obtain critical pairs (g, A). However, it is easy to see that $\inf_{(g,A)} \bar\lambda_i(M, g, A) = 0$ and it was proven in [2] that $\sup_A \bar\lambda(\mathbb{S}^2, g_{\mathbb{S}^2}, A) = +\infty$. The next natural choice is $\sup_g \inf_A \bar\lambda_i(M, g, A)$. Thus, the open question is to show that for a fixed (M, g) and $i > 1$ one has

$$\inf_A \bar\lambda_i(M, g, A) \geqslant C(M, g) > 0.$$

There is a similar question for Bakry-Emery (or Witten) Laplacian. For $\sigma = e^{-f}$, $f \in C^\infty(M)$ the operator L_σ can be defined via

$$\int_M \langle \nabla_g \phi, \nabla_g \psi \rangle \, \sigma dv_g = \int_M \phi(L_\sigma \psi) \, \sigma dv_g.$$

Denoting the eigenvalues of the operator as $\lambda_i(M, g, e^{-f})$ consider the (somewhat unusual) normalisation $\bar\lambda_i(M, g, e^{-f}) = \lambda_i(M, g, e^{-f}) \int_M f e^{-f} \, dv_g$. The reason for this choice is that one can again show that critical pairs (g, f) of $\bar\lambda_i(M, g, e^{-f})$ correspond to natural geometric objects, mean curvature shrinkers. In this case, the open question is: does there exist an upper bound for $\bar\lambda_i(M, g, e^{-f})$ independent of g and f? This question was studied in [6] for a different normalisation.

Jean Lagacé: Asymptotic spectral optimisation. On domains $\Omega \subset \mathbb{R}^2$, consider the Laplace eigenvalue problem $\Delta f + \lambda f = 0$, with either the Dirichlet or Neumann boundary condition. Denote by $\left\{\lambda_k^D(\Omega), k \in \mathbb{N}\right\}$ and $\left\{\lambda_k^N(\Omega), k \in \mathbb{N}_0\right\}$ the sequences of Dirichlet and Neumann eigenvalues of a domain, respectively.

Consider the following shape optimisation question: amongst a class $\mathcal{R}$ of domains, which domain Ω_k^* minimises (respectively maximises) the functional $\lambda_k^D(\Omega)\,\mathrm{Area}(\Omega)$ (respectively $\lambda_k^N(\Omega)\,\mathrm{Area}(\Omega)$). We are interested in the asymptotic behaviour of Ω_k^* as $k \to \infty$. Note that if $\mathcal{R}$ is the class of all bounded open sets, the celebrated conjecture of Pólya stating that for all $k \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^2$ bounded and open

$$\lambda_k^N(\Omega)\,\mathrm{Area}(\Omega) \leq 4\pi k \leq \lambda_k^D(\Omega)\,\mathrm{Area}(\Omega)$$

is equivalent to Ω_k^* having a bounded open set as a limit [5]. The two-term Weyl law heuristically tells us that Ω_k^* should converge to a disk (as it minimises the isoperimetric ratio) when $k \to \infty$. This is however out of reach for the moment.

I propose the following open problem: let $\mathcal{R}_n$ be the class of all polygons with n sides. Prove that within that class the optimisers for λ_k^N or λ_k^D converge to the regular n-gon as $k \to \infty$. While the general problem is the most interesting, adding a convexity assumption would still be very good. In principle the following programme should work towards this.

(1) Prove that there exists an optimiser within $\mathcal{R}_n$, using that the space of polygons with at most n sides is a finite dimensional orbifold.
(2) Prove that polygons that are 'too degenerate' cannot be optimisers for k large. If one restricsts themselves to convex polygons, this is simply a diameter bound. One may also have to exclude polygons with 'bad dynamics', that is a lot of closed orbits to the billiard flow.
(3) Prove that a Weyl law with sharp remainder holds uniformly within the class of polygons whose geometry remains far from degenerate. Then use the general heuristic around isoperimetric minimiser to conclude. This will probably require a good understanding of the wave propagator kernel on polygons.

Richard Laugesen: Neumann and Robin eigenvalues. The Rayleigh quotient for the Robin eigenvalue problem on a domain Ω with Robin parameter α is

$$\frac{\int_\Omega |\nabla u|^2\,dx + \alpha \int_{\partial\Omega} u^2\,dS}{\int_\Omega u^2\,dx}.$$

Robin gap monotonicity. For a convex bounded domain $\Omega \subset \mathbb{R}^n$, is the Robin spectral gap given by the difference between the first two Robin eigenvalues, namely, $(\rho_2 - \rho_1)(\alpha)$, strictly increasing as a function of the Robin parameter $\alpha > 0$?

If so, then the limiting cases $\alpha = 0$ and $\alpha = +\infty$ would imply that the Neumann gap is a lower bound for the Dirichlet gap:

$$\mu_2 < \lambda_2 - \lambda_1$$

where $0 = \mu_1 < \mu_2$ are the first and second Neumann eigenvalues of the Laplacian and $0 < \lambda_1 < \lambda_2$ are the first and second Dirichlet eigenvalues.

The conjecture was raised by Smits [15, Section IV], and it is known to hold for disks and rectangular boxes. See the discussion by Laugesen [13, Section II].

Robin second eigenvalue concavity. For a convex bounded domain $\Omega \subset \mathbb{R}^n$, is the second Robin eigenvalue $\rho_2(\alpha)$ a concave function of $\alpha > 0$?

The conjecture was raised and discussed by Laugesen [13, Section II]. Note the first Robin eigenvalue $\rho_1(\alpha)$ is certainly concave, by the Rayleigh characterization of $\rho_1(\alpha)$ as a minimum of an affine function of α.

Maximization of the third Neumann eigenvalue with upper curvature bound. Is the third Neumann eigenvalue maximal for a disjoint union of two disks with constant curvature K, among simply connected surfaces with curvature bounded above by K?

The conjecture is known for $K = 0$ by Girouard and Polterovich [10], but is open when $K < 0$, and when $K > 0$. Note: in the constant curvature setting the conjecture is known by Bucur and Henrot [3] for Euclidean ($K = 0$), Freitas and Laugesen [9] for hyperbolic ($K < 0$), and Bucur, Martinet and Nahon [4] for spherical domains ($K > 0$).

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Open Problems: Geometry of Eigenfunctions

Coordinated by Dan Mangoubi

The following is a summary of the open problems presented on Thursday, 24th of August, 2023.

Sugata Mondal: A question about second Neumann eigenfunctions of triangular domains. A second Neumann eigenfunction u of the Laplacian approximates the temperature distribution of an insulated domain for large times. The 'hot spots' conjecture, due to Rauch, is the assertion that u does not assume its maximum value in the interior of the domain. The conjecture is known to be false for non-simply connected domains in the plane [5]. Among simply connected domains, obtuse triangles [3], convex domains with bi-axial symmetry [9] and lip domains [1] are known to satisfy the conjecture. The case of a general acute triangular domain in the plane was open until recently. In 2020 we announced [10] a proof of the following: For any second Neumann eigenfunction u of an acute triangle T, there is at most one non-vertex critical point of u. We later discovered a gap in the proof of this statement. Although we were able to give an alternative proof of the hot spots conjecture for all acute triangles in 2021 [11], the above statement still remains an open question.

Stefan Steinerberger: Oscillatory * Oscillatory = Smooth. Let $G = (V, E)$ be a finite combinatorial graph and assume that G has a bipartite subgraph H such that H has almost as many edges as G. Alternatively, we would think of G as a graph obtained from taking a bipartite graph and adding a few edges. It can then be observed that eigenfunctions of $L = D - A$ corresponding to large eigenvalues have the property that their pointwise product is 'smooth' (which, in the discrete setting, means that the Rayleigh Ritz quotient is relatively small or, equivalently, that the product can be written as a linear combination of eigenfunctions with relatively small frequency). This is a kind of aliasing phenomenon on graphs: high-frequency eigenvectors are themselves rigorously structured in a way that is uncovered by pointwise multiplication. A theory of this phenomenon is largely missing.

Bernard Helffer: Can we extend Courant's nodal theorem to the sub-Laplacian? Our aim is to present some of the results by Eswarathasan–Letrouit of [7] and attached open problems. We consider in an open set $\Omega \subset \mathbb{R}^n$ the Dirichlet realization of a sublaplacian

$$\sum X_j^* X_j + c(x)\,,$$

where the X_j are C^∞ real vector fields satisfying the so called Hörmander condition [8]. These operators are known to be hypoelliptic [8] and under an additional condition at the boundary (at each point there exists a vector field X_i which is transverse to the boundary), we have regularity up to the boundary. We can consider for its discrete spectrum all the questions that have been solved along the years concerning the Dirichlet realization of the Laplacian: Local structure of the nodal sets, density of the nodal sets, Courant's theorem, Pleijel's theorem. In the case of Courant's theorem the question of Unique Continuation property (or substitute) plays an important role [2, 4].

Svitlana Mayboroda: Harmonic measure and localization for Robin's problem. Together with G. David, S. Decio, M. Filoche and M. Michetti we have recently proved that the Robin harmonic measure on a fractal domain has the dimension of the underlying fractal, a surprising result contradicting the physicists' intuition. I discussed possible further directions to clarify its nature and more generally understanding of diffusion and of waves on fractal domains with Robin boundary conditions.

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Reporter: Antoine Métras

Participants

Dr. Clara Lucia Aldana
Universidad del Norte
Departamento de Matematicas y
Estadistica. Edificio J. Ofic. 66.
Km. 5 Vía Puerto Colombia
081007 Barranquilla
COLOMBIA

Dr. Dean Baskin
Department of Mathematics
Texas A & M University
Mailstop 3368
College Station, TX 77843-3368
UNITED STATES

Prof. Dr. Jussi Behrndt
Institut für Mathematik
Technische Universität Graz
Steyrergasse 30
8010 Graz
AUSTRIA

Dr. Stine Marie Berge
Institut für Analysis
Leibniz Universität Hannover
Humboldtstr. 24
30167 Hannover
GERMANY

Prof. Dr. Dmitriy Bilyk
School of Mathematics
University of Minnesota
Vincent Hall 328
206 Church Street S.E.
Minneapolis, MN 55455
UNITED STATES

Dr. Anne-Sophie Bonnet-Ben Dhia
ENSTA Paris
UMA-POEMS
828, Boulevard des Maréchaux
91762 Palaiseau Cedex
FRANCE

Prof. Dr. Dorin Bucur
Laboratoire de Mathématiques
Université Savoie Mont Blanc
73376 Le Bourget-du-Lac
FRANCE

Tirumala Chakradhar
Department of Mathematics
University of Bristol
Fri Building, Woodland Road
Bristol BS8 1UG
UNITED KINGDOM

Dr. Philippe Charron
Département de Mathématiques
Université de Geneve
Case Postale 64
2-4 rue du Lievre
1211 Genève 4
SWITZERLAND

Prof. Dr. Bruno Colbois
University of Neuchâtel
2000 Neuchâtel
SWITZERLAND

Prof. Dr. Pavel Exner
Department of Theoretical Physics
Nuclear Physics Institute
Czech Academy of Sciences
Hlavní 130
250 68 Řež
CZECH REPUBLIC

Dr. Sara Farinelli
Lagrange Mathematics and Computing
Research Center
103 Rue De Grenelle
75007 Paris
FRANCE

Dr. Ksenia Fedosova
Mathematisches Institut
Universität Freiburg
Ernst-Zermelo-Str. 1
79104 Freiburg i. Br.
GERMANY

Prof. Dr. Rupert L. Frank
Institut für Mathematik
Ludwig-Maximilians-Universität
München
Theresienstraße 39
80333 München
GERMANY

Prof. Dr. Pedro Freitas
Departamento de Matematica
Instituto Superior Tecnico
Avenida Rovisco Pais, 1
Lisboa 1049-001
PORTUGAL

Dr. Kei Funano
Graduate School of Information Science
Tohoku University
Aramaki, Aoba-ku
Sendai 980-8579
JAPAN

Dr. Alexandre Girouard
Dept. de Mathématiques et de
Statistique
Pavillon Alexandre-Vachon
Université Laval
1045, av. de la Médecine
Québec G1V0A6
CANADA

Dr. Katie Gittins
Department of Mathematical Sciences
Mathematical Sciences & Computer
Science Building
Durham University
Upper Mountjoy Campus, Stockton
Road
Durham DH1 3LE
UNITED KINGDOM

Prof. Dr. Carolyn S. Gordon
Department of Mathematics
Dartmouth College
27 N. Main Street
Hanover, NH 03755-3551
UNITED STATES

Prof. Dr. Daniel Grieser
Institut fuer Mathematik
Carl v. Ossietzky-Universität Oldenburg
Fakultät V: Mathematik &
Naturwissensch.
26111 Oldenburg
GERMANY

Dr. Asma Hassannezhad
School of Mathematics
University of Bristol
Woodland Road
Bristol BS8 1UG
UNITED KINGDOM

Prof. Dr. Bernard Helffer
Laboratoire de Mathématiques
Jean Leray UMR 6629
Université de Nantes, B.P. 92208
2 Rue de la Houssiniere
44322 Nantes Cedex 03
FRANCE

Prof. Dr. Antoine Henrot
Institut Elie Cartan
-Mathématiques-
Université de Lorraine Nancy
Boite Postale 239
54506 Vandoeuvre-lès-Nancy Cedex
FRANCE

Prof. Dr. Luc Hillairet
Institut Denis Poisson
Université d'Orleans
rue de Chartres, B. P. 6759
45067 Orléans Cedex 2
FRANCE

Prof. Dr. Christopher Judge
Department of Mathematics
Indiana University
Bloomington IN 47405-4301
UNITED STATES

Prof. Dr. Mikhail Karpukhin
Department of Mathematics
University College London
Gower Street
London WC1E 6BT
UNITED KINGDOM

Dr. Jean Lagacé
Department of Mathematics
Kings College
Strand
London WC2R 2LS
UNITED KINGDOM

Prof. Richard S. Laugesen
Department of Mathematics
University of Illinois Urbana-Champaign
Urbana, IL 61801
UNITED STATES

Prof. Dr. Michael Levitin
Department of Mathematics and Statistics
University of Reading
Pepper Lane
Reading RG6 6AX
UNITED KINGDOM

Dr. Aleksandr Logunov
Section de mathématiques
Université de Genève
rue du Conseil-Général 7-9
1205 Genève 4
SWITZERLAND

Dr. Vladimir Lotoreichik
Department of Theoretical Physics
Nuclear Physics Institute
Czech Academy of Sciences
Hlavní 130
250 68 Řež
CZECH REPUBLIC

Prof. Dr. Eugenia Malinnikova
Dept. of Mathematics
Stanford University
Building 380
Stanford, CA 94305
UNITED STATES

Prof. Dr. Dan Mangoubi
Einstein Institute of Mathematics
Edmond J. Safra Campus
The Hebrew University of Jerusalem
Jerusalem 9190401
ISRAEL

Prof. Dr. Svitlana Mayboroda
Department of Mathematics
University of Minnesota
127 Vincent Hall
206 Church Street S. E.
Minneapolis, MN 55455
UNITED STATES

Dr. Antoine Métras
School of Mathematics
University of Bristol
Woodland Road
Bristol BS8 1UG
UNITED KINGDOM

Dr. Sugata Mondal
Department of Mathematics,
University of Reading
Reading RG6 6AX
UNITED KINGDOM

Dr. Nilima Nigam
Dept. of Mathematics
Simon Fraser University
8888 University Drive
Burnaby BC V5A 1S6
CANADA

Prof. Dr. Carlo Nitsch
Dipartimento di Matematica e Appl.
Universita di Napoli
Complesso Monte S. Angelo
Via Cintia
80126 Napoli
ITALY

Dr. Thomas Ourmières-Bonafos
Centre de Mathématiques &
d'Informatique
Aix-Marseille Université
39, Rue Joliot-Curie
13453 Marseille Cedex 13
FRANCE

Prof. Dr. Konstantin Pankrashkin
Carl von Ossietzky University of
Oldenburg
Institute of Mathematics
Carl von Ossietzky Str. 9-11
26129 Oldenburg
GERMANY

Dr. Leonid Parnovski
Department of Mathematics
University College London
Gower Street
London WC1E 6BT
UNITED KINGDOM

Dr. Romain Petrides
Institut de Mathématiques de Jussieu -
Paris Rive Gauche (IMJ-PRG)
UP7D - Campus des Grands Moulins
P.O. Box 7012
75205 Paris Cedex 13
FRANCE

Prof. Dr. Iosif Polterovich
Dept. of Mathematics and Statistics
University of Montreal
CP 6128, succ. Centre Ville
Montréal QC H3C 3J7
CANADA

Prof. Dr. Olaf Post
Fachbereich IV - Mathematik
Universität Trier
54286 Trier
GERMANY

Prof. Dr. Luca Rizzi
SISSA
International School for Advanced
Studies
via Bonomea 265
34136 Trieste
ITALY

Dr. Tommaso Rossi
Institut für Angewandte Mathematik
Universität Bonn
Endenicher Allee 60
53115 Bonn
GERMANY

Dr. David Sher
Department of Mathematical Sciences
DePaul University
2320 N Kenmore Ave.
Chicago, IL 60614
UNITED STATES

Prof. Dr. Anna Siffert
WWU Münster
48161 Münster
GERMANY

Stefan Steinerberger
Department of Mathematics
University of Washington, Seattle
Seattle, WA 98195
UNITED STATES

Prof. Dr. Cristina Trombetti
Dipartimento di Matematica e Appl.
Universita di Napoli
Complesso Monte S. Angelo
Via Cintia
80126 Napoli
ITALY

Prof. Dr. Michiel van den Berg
School of Mathematics
University of Bristol
Woodland Road
Bristol BS8 1UG
UNITED KINGDOM

Marco Vogel
Institut für Mathematik
Carl von Ossietzky Universität
Oldenburg
26129 Oldenburg
GERMANY

Prof. Dr. Tobias Weich
Institut für Mathematik
Universität Paderborn
Warburger Straße 100
33098 Paderborn
GERMANY