Mathematisches Forschungsinstitut Oberwolfach

Report No. 33/2023

DOI: 10.4171/OWR/2023/33

Teichmüller Theory: Classical, Higher, Super and Quantum

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30 July – 4 August 2023

Abstract. Teichmüller spaces play a major role in many areas of mathematics and physical science. The subject of the conference was recent developments of Teichmüller theory with its different ramifications that include the classical, the higher, the super and the quantum aspects of the theory.

Mathematics Subject Classification (2020): Primary: 57K20. Secondary: 20H10, 51M10, 30F35.

Introduction by the Organizers

The workshop *Teichmüller Theory: Classical, Higher, Super and Quantum*, organised by Ken'ichi Ohshika (Gakushin University, Tokyo), Athanase Papadopoulos (IRMA, Strasbourg), Robert C. Penner (IHES Paris) and Anna Wienhard (Heidelberg University) was well attended with 47 participants with broad geographic representation from all continents. The main topic discussed was Teichmüller theory, including its classical aspects and the ramifications, with a focus on recent developments in higher and super Teichmüller theory. The classical part included new results on the metric theory (in particular, the Teichmüller, Weil-Petersson, Thurston and earthquake metrics), identities on the length spectra of hyperbolic surfaces, the de Sitter geometry aspect, the complex analytic aspect with the theory of Kleinian groups, the algebraic aspects of mapping class groups and their relations with 3-manifolds, quantization and physics. The classical curve complex, other graphs on the surface were studied from the point of view of model theory. More generally, several problems in geometry and group actions on manifolds were investigated using ideas of mathematical logic. Higher Teichmüller theory was also well represented. This is the study of connected components of the representations variety $\mathrm{Hom}(\pi_1, G)/G$, where π_1 is the fundamental group of a closed surface and G a Lie group of higher rank. The stress is on the discrete and faithful representations, with the interplay with Higgs bundles, algebraic geometry and the theory of Anosov representations. The workshop program also included the study of Super higher Teichmüller spaces, defined as appropriate subspaces of varieties $\mathrm{Hom}(\pi_1, G)/G$ of flat G-connections on a surface, where π_1 is the fundamental group of the surface and G an appropriate Lie *super*group, as opposed to a suitable ordinary Lie group, thus extending the usual theory.

This workshop was a nice blend of frontline researchers from diverse research backgrounds.

Acknowledgement: The MFO and the workshop organizers would like to thank the National Science Foundation for supporting the participation of junior researchers in the workshop by the grant DMS-2230648, "US Junior Oberwolfach Fellows". Moreover, the MFO and the workshop organizers would like to thank the Simons Foundation for supporting Krishnendu Gongopadhyay and Sumio Yamada in the "Simons Visiting Professors" program of the MFO.

Workshop: Teichmüller Theory: Classical, Higher, Super and Quantum

Table of Contents

Daniele Alessandrini (joint with Sara Maloni, Nicolas Tholozan, Anna Wienhard)
Domains of discontinuity for Anosov representations 1847

Andrea Bourque (joint with Anton Zeitlin)
Flips of $GL(1|1)$ Graph Connections 1850

Ara Basmajian
Identities: Equations on deformation spaces 1851

Valentina Disarlo (joint with Javier de la Nuez Gonzalez and Thomas Koberda)
The curve complex via model theory 1852

James Farre (joint with Or Landesberg and Yair Minsky)
Horocycles, laminations, and Lipschitz maps 1854

Sachiko Hamano
Subharmonicity of a span associated with the moduli disk 1857

Yi Huang (joint with Ken'ichi Ohshika, Huiping Pan, Athanase Papadopoulos)
The earthquake metric 1860

Tsukasa Ishibashi (joint with Hironori Oya, Linhui Shen)
Moduli space of G-local systems and skein algebras 1864

Nariya Kawazumi
A topological proof of Wolpert's formula of the Weil-Petersson symplectic form in terms of the Fenchel-Nielsen coordinates 1866

Sang-hyun Kim (joint with Nhat Minh Doan, Mong Lung Lang, Ser Peow Tan)
An optimal fundamental domain for $\Gamma_0(p)$ and $\Gamma_0(pq)$ 1867

Thomas Koberda (joint with Sang-hyun Kim, J. de la Nuez González)
First order theory of homeomorphism groups of manifolds, and applications 1868

Qiongling Li
Higgs bundles in the Hitchin section over non-compact hyperbolic Riemann surfaces 1870

Hideki Miyachi
Double infinitesimal structures on Teichmüller space 1872

Paul Norbury
Teichmüller Theory: Classical, Higher, Super and Quantum 1876

Andrea Seppi (joint with Graham Smith and Jérémy Toulisse)
Maximal submanifolds of pseudo-hyperbolic space and their applications 1879

Stavros Garoufalidis
Super-representations of 3-manifolds and torsion polynomials 1882

Zhe Sun (joint with Linhui Shen, Daping Weng)
Asymmetric intersection number and Fock–Goncharov duality 1883

Sumio Yamada (joint with Athanase Papadopoulos)
Timelike Convex Geometry 1886

Abstracts

Domains of discontinuity for Anosov representations

Daniele Alessandrini
(joint work with Sara Maloni, Nicolas Tholozan, Anna Wienhard)

Generalized flag manifolds. Let G be a connected semisimple Lie group with finite center. For example, G can be one of the classical matrix groups, such as $\mathrm{SL}(n,\mathbb{R})$, $\mathrm{Sp}(2n,\mathbb{R})$, $\mathrm{SO}_0(p,q)$. We will consider the *generalized flag manifolds* of G: spaces of the form G/Q, where Q is a parabolic subgroup $Q < G$. For example, in the special case when $G = \mathrm{SL}(n,\mathbb{R})$, the spaces G/Q are the projective spaces, the Grassmannians, the partial flag manifolds and the full flag manifolds. These are classical geometric spaces endowed with a rich and interesting geometry.

Anosov representations. Let Γ be a torsion-free Gromov-hyperbolic group. We want to understand the geometric, topological and dynamical properties of an action of Γ on a generalized flag manifold G/Q. These actions correspond to representations

$$\rho : \Gamma \to G\,.$$

We will consider the set of all group homomorphisms of Γ in G, here denoted by

$$\mathrm{Hom}(\Gamma, G) = \{\ \rho : \Gamma \to G\ \}\,.$$

Among these representations, a special place is taken by the Anosov representations. They are the representations with the nicest dynamical properties. They are defined with reference to a parabolic subgroup $P < G$, where P may be different that the parabolic subgroup Q above. We will consider the set of P-Anosov representations, here denoted by

$$\mathrm{Anosov}_P(\Gamma, G) \subset \mathrm{Hom}(\Gamma, G)\,.$$

The P-Anosov representations of Γ are all discrete and faithful, and an important characteristic is that they admit a ρ-equivariant map

$$\xi : \partial_\infty \Gamma \to G/P\,.$$

These representations have the important property of being structurally stable, this means that a small deformation of a P-Anosov representation is still a P-Anosov representation. Equivalently, we can say that the set $\mathrm{Anosov}_P(\Gamma, G)$ is open in $\mathrm{Hom}(\Gamma, G)$. This property is useful for the construction of interesting examples of Anosov representations, see below.

Domains of discontinuity. The dynamics of the action of a P-Anosov representation ρ on a generalized flag manifold G/Q is described by the theory of domains of discontinuity, introduced by Guichard-Wienhard [3] and then generalized and improved by Kapovich-Leeb-Porti [4]. They give conditions for the existence of a cocompact domain of discontinuity for ρ in G/Q. They consider the space of relative positions of a point in G/P and a point in G/Q:

$$R = (G/P \times G/Q)/G\,.$$

This is a finite set with a rich combinatoric structure. Kapovich-Leeb-Porti define the notion of balanced ideal, a set $I \subset R$ with some special properties. This ideal represents the set of "bad" relative positions, that we don't want to have in the domain of discontinuity. Given $s \in G/P$, they define

$$K(s) = \{\ r \in G/Q \ \mid\ [(s,r)] \in I\ \}.$$

Recall that the P-Anosov representation admits the ρ-equivariant curve ξ as above. We want to remove the set

$$K_{\rho,I} = \bigcup_{t \in \partial_\infty \Gamma} K(\xi(t)).$$

The complement of this set is the domain

$$\Omega_{\rho,I} = G/Q \smallsetminus K_{\rho,I}.$$

Kapovich-Leeb-Porti [4] proved that if I is a balanced ideal, the action of ρ on $\Omega_{\rho,I}$ is properly discontinuous, free and cocompact.

The quotient manifold. This construction gives us the closed manifold

$$M_{\rho,I} = \Omega_{\rho,I}/\rho.$$

Since this manifold is a quotient of a domain in G/Q, it carries a $(G, G/Q)$-structure, a geometric structure in the sense of Thurston. This gives us a way to construct manifolds $M_{\rho,I}$ with a large deformation space of $(G, G/Q)$-structures.

One limitation of this theory is that it doesn't say anything about the topology of the manifold $M_{\rho,I}$. Examples of such manifolds were studied by several authors, usually in the case when $\Gamma = \pi_1(S)$ is a surface group, see the references in [1]. From these examples we see that very often the manifold $M_{\rho,I}$ is a bundle over the surface. It would be tempting to conjecture that this is always true, but there are interesting counterexamples, given by Gromov-Lawson-Thurston [2], where $M_{\rho,I}$ cannot fiber over the surface. The examples in [2] are complicated, and suggest that it is impossible to give a general theorem that describes the topology of all the manifolds $M_{\rho,I}$.

Deformations of lattices. We will restrict to a special case, that is still very general and interesting, but is also more tractable. We will fix a connected semisimple Lie group H of real rank 1 with finite center, and we will assume that $\Gamma < H$ is a torsion-free uniform lattice in H. This class of groups Γ includes the surface groups and the fundamental groups of closed oriented (real, complex, quaternionic, octonionic) hyperbolic n-manifolds.

Given a representation $\iota : H \to G$, we can restrict it to Γ and obtain a representation $\rho_0 : \Gamma \to G$. These representations will be called *lattice representations* of Γ, and they are always P-Anosov with reference to some parabolic subgroup of G, see [3].

We can then use the property of structural stability, and deform the lattice representations a little bit. The deformed representation is still P-Anosov, and often it is Zariski-dense in G. We will say that a P-Anosov representation is a *deformation of a lattice representation* if it can be obtained by continuously deforming a lattice representation without leaving the space $\mathrm{Anosov}_{\mathrm{P}}(\Gamma, G)$. In this way we can obtain many interesting examples of P-Anosov representations. Actually, most known Anosov representations are obtained from this construction, just because it is an easy way to construct them.

The topology of the quotient manifold. We will now present the main theorems in [1].

Theorem 1 (A., Maloni, Tholozan, Wienhard)**.** *Let $\Gamma < H$ be a torsion free uniform lattice. Recall that $\Gamma = \pi_1(T)$, where T is a closed oriented (real, complex, quaternionic, octonionic) hyperbolic manifold. Fix a representation $\iota : H \to G$, and restrict it to $\rho_0 : \Gamma \to G$. Fix a $P < G$ such that ρ_0 is P-Anosov. Let $\rho : \Gamma \to G$ be a deformation of the representation ρ_0. Choose a parabolic subgroup Q such that there exists a balanced ideal I of relative positions between G/P and G/Q, and let $M = M_{\rho,I}$ be the quotient $\Omega_{\rho,I}/\rho$.*

Then M is a smooth fiber bundle over the manifold T.

We can describe the structure group of this bundle and characterize the bundle.

Theorem 2 (A., Maloni, Tholozan, Wienhard)**.** *With the same notation as in the previous theorem, let F denote the fiber of the bundle:*

$$F \to M \to T\,.$$

Denote by K the maximal compact subgroup of H, and by $S_H = H/K$ the symmetric space of H, a (real, complex, quaternionic, octonionic) hyperbolic space. Consider the principal K-bundle

$$K \to H/\Gamma \to S_H/\Gamma = T\,.$$

Then F has a K-action, the bundle $M \to T$ has structure group K given by this action, and the bundle $M \to T$ is associated to the principal bundle $H/\Gamma \to T$.

Theorem 1 and 2 fully describe the topology of M, except for the fiber F. We know the topology of the fiber F in many special cases (see [1] and the references therein), but a general description of the fiber F is still not known.

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Flips of $GL(1|1)$ Graph Connections

Andrea Bourque
(joint work with Anton Zeitlin)

Classical Teichmüller space is a component of the character variety $Hom(\pi_1(F), G)/G$, where F is a (punctured) surface, and $G = PSL_2(\mathbb{R})$. In order to probe various generalizations of Teichmüller theory, it is then natural to also consider tweaking parts of the character variety; for instance, what can we say about it for other groups G?

The character variety has a geometric interpretation as the moduli space of flat G-connections. It is possible to reduce the differential geometry of this interpretation to algebra and combinatorics. In particular, the character variety can be identified with the space of G-graph connections on a ribbon graph corresponding to F. In this way, we can get a hands-on description of the character variety.

Another important object of study in Teichmüller theory is the mapping class group. This group acts on classical Teichmüller space. If the surface F admits an ideal triangulation, then there are Whitehead moves, also known as flips, which give new triangulations. Finite sequences of flips generate the mapping class group. The dual to a triangulation is a ribbon graph, and flips also take a ribbon graph to another one. In light of the previous discussion, it is natural to determine how flips interact with graph connections.

In this project we considered the case where G is the supergroup $GL(1|1)$, or more specifically, the component of the identity element. We also restricted to punctured surfaces which admit ideal triangulations. The elements of $GL(1|1)$ can be written in a form which separates the odd and even elements of the superalgebra over which you are working. Using this parametrization, we gave coordinates for the character variety.

Furthermore, the factorization of $GL(1|1)$ was utilized to give a nice description of flips. In particular, we can think of an edge in a graph connection as having one odd element on each end, and one even element in the middle. Using this convention, we were able to give a formula for the action of a flip on a $GL(1|1)$-graph connection, in such a way that the odd ends are fixed. In other words, the pieces of the group elements which can interact with other edges remain the same before and after the flip.

To end, let me give some contexts where the $GL(1|1)$ character variety and our description of it may be useful. One is in a potential super-analogue of the theory of "abelianization" via spectral networks, described by Holland and Neitzke. Another is the quantum $GL(1|1)$ Chern-Simons theory.

Identities: Equations on deformation spaces

Ara Basmajian

Let X be a compact n-dimensional hyperbolic manifold with totally geodesic boundary. A homotopy class (rel the boundary) of a non-trivial arc from the boundary to itself can be realized by an *orthogeodesic*- an oriented immersed geodesic perpendicular to the boundary at its initial and terminal points. This talk will be an introduction to the study of such arcs, their properties, and the identities they satisfy. For example, it is a consequence of the orthospectrum identity ([1]) that the set of all orthogeodesic lengths determine the area ($(n-1)$-dimensional volume) of the boundary of X. In fact, such lengths are also related to the topological entropy of the manifold.

In dimension two, there are special subclasses of orthogeodesics called *prime* orthogeodesics. In work with Hugo Parlier and Ser Peow Tan ([2]) we show that the prime orthogeodesics arise naturally in the study of maximal immersed pairs of pants in X and are intimately connected to regions of X in the complement of the natural collars. These considerations lead to far reaching generalizations of the orthospectrum identity in the form of continuous families of equations (so called *identities*) that remain constant on the deformation space of hyperbolic structures.

Denoting the set of orthogeodesics on X by $\mathcal{O}$, the orthospectrum identity reads

$$\ell(\partial X) = \sum_{\mu\in\mathcal{O}} 2\log\coth\frac{\ell(\mu)}{2}$$

where $\ell(\cdot)$ denotes length, and ∂X is the geodesic boundary of X.

To illustrate the more general identities proven in [2], we start with an oriented surface X of genus g and n cusps with $\chi(X) = 2-2g-n \leq -1$ and $(g,n) \neq (0,3)$. Now an orthogeodesic μ determines a closed geodesic on X in the following way: traverse μ in the positive direction until hitting the boundary orthogonally, then turn right along the boundary cusp, loop around it, and traverse μ in the negative direction until hitting the boundary cusp orthogonally, loop around this cusp thereby closing up the curve. This defines a homotopy class of closed curve we denote by γ_μ. If γ_μ avoids the standard cusp neighborhoods of the punctures we say that the orthogeodesic is $\vec{1}$-prime. Denoting the set of $\vec{1}$-prime orthogeodesics by $\mathcal{O}'$ we have

$$\sum_{\mu\in\mathcal{O}'} \frac{2}{e^{\frac{\gamma_\mu}{2}}+1} = n.$$

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The curve complex via model theory

Valentina Disarlo
(joint work with Javier de la Nuez Gonzalez and Thomas Koberda)

Let S be an orientable surface of finite type. In this talk I present the paper [17] in collaboration with Javier de la Nuez Gonzalez (KIAS) and Thomas Koberda (University of Virginia), in which we begin the first study of the curve complex of a surface and other similar graphs as *first-order countable structures* with tools coming from *model theory*, in particular *stability theory*.

The *curve graph* $\mathbb{C}(S)$ is one of the most important tools in Teichmüller theory and geometric group theory. It is a graph where every vertex correspond to a simple closed curve on S and edges correspond to the disjoint relation. It was first introduced by Harvey [9] and employed by Brock-Canary-Minsky in the proof of the Ending Lamination Conjecture. It is now ubiquitous in geometric topology and geometric group theory, following the works of Masur-Minsky [12, 14], Bromberg-Bestvina-Fujiwara [3], Hamenstädt [8], Rafi-Schleimer [18], Masur-Schleimer [13]. In the 1990s Ivanov proved that the automorphism group of the curve complex is the mapping class group. In the following decade, many other graphs have been shown to have the same property (see the survey by McCarthy-Papadopoulos [15]). In response, Ivanov [11] formulated a famous metaconjecture stating that every graph "naturally" associated to a topological surface has the mapping class group as its isomorphism group. Later McCarthy-Papadopoulos [15] gave an example of a natural simplicial complex - the so-called complex of domains - whose automorphism group is larger than the mapping class group and whose coarse geometry is closely related to the curve complex [5]. As the complex serves as a "natural" counterexample to the Ivanov's metaconjecture, McCarthy-Papadopoulos posed the open problem of understanding which of its subgraphs are rigid. Only recently Brendle-Margalit [4] answer the question and described a rich class of its rigid simplicial subcomplexes, providing beautiful applications to the study of the normal subgroups of mapping class group.

In this talk we approach the curve complex and its analogues from the point of view of model theory. Our interest for the model theory of the curve graph is motivated by the fact that the (extended) mapping class group is the *automorphism group* of the curve graph (and many other geometric graphs) also *as models* (see the survey [15]). Many of the rigidity proofs rely on the fact that some "natural" topological properties on curves (i.e. being separating, non-separating, etc) can be expressed with a first-order formula in the language of the curve complex. A detailed reformulation of the Ivanov metaconjecture is presented in our paper [17].

1.1. **Stability in model theory: a few crash notions for non-logicians.** A *structure* $\mathcal{M}$ is given by an underlying set M, some distinguished operations, relations and elements. Given a language $\mathcal{L}$, a $\mathcal{L}$-*structure* is a structure $\mathcal{M}$ where we can interpret all of the symbols of $\mathcal{L}$. Two $\mathcal{L}$-structures $\mathcal{M}$ and $\mathcal{N}$ are *isomorphic* if there is a bijection between the underlying sets $M \to N$ that preserves the interpretation of all function, relation and constant symbols in $\mathcal{L}$. A set $X \subset M^n$ is *definable* in M if $X = \{\bar{a} \in M^n \ : \ M \vDash \phi(\bar{a}, \bar{b})\}$ where ϕ is a $\mathcal{L}$-formula and $\bar{b} \in M^m$. A $\mathcal{L}_0$-structure $\mathcal{N}$ is *definable* in a $\mathcal{L}$-structure $\mathcal{M}$ if and only if we can find a definable subset X of M^n for some n and we can interpret the symbols of $\mathcal{L}_0$ as definable subsets and functions on X so that the new structure is isomorphic to $\mathcal{N}$. One goal of classification theory is to divide all (complete countable) theories into *stable* and *unstable* theories. *Stability theory* is one of the most important branches of model theory. It was initiated in 1965 by Morley [16] and later developed by Shelah [19]. In his proof of the Los conjecture, Morley defined some suitable notion of *rank* of a formula and defined a model ω-*stable* if every definable set has finite *Morley rank.* Most examples of (ω-)stable theories come from number theory, such as the theory of algebraically closed fields.

1.2. **The ω-stability of the curve graph and other geometric graphs.** In 2017 Baudisch-Martin Pizarro-Ziegler proved that the buildings of RAAGs [1] are also ω-stable, providing the first example of a stable theory coming from geometric topology. In paper [17] with Javier de la Nuez Gonzalez and Thomas Koberda, we adapt their strategy to the curve complex and prove the following theorem.

Theorem 1 (de la Nuez Gonzalez - Disarlo - Koberda [17])**.** *Let S_g^b be a surface with genus g and b marked points. Then the first-order theory* $\mathrm{Th}(\mathbb{C}(S))$ *has quantifier elimination with respect to the class of $\forall\exists$-formula. Furthermore,* $\mathrm{Th}(\mathbb{C}(S))$ *is ω-stable with Morley rank bounded above by ω^{3g+b-3}.*

We give sufficient conditions for a geometric graph $X(S)$ of arcs/curves to be interpretable in $\mathbb{C}(S)$. As a consequence we prove that all the “famous” geometric graphs associated to a surface are also ω-stable with finite Morley rank, including the marking graph [14]; the arc graph [6]; the flip graph [7]; the polygonalization graph [2], the arc-and-curve graph [10]. As a further application of the main theorem, we prove that there are “natural” geometric properties that cannot be expressed by a first-order formula in the language of the curve complex:

Corollary 1. *Let S be a surface of positive genus that is not a torus with fewer than two boundary components. The (absolute value of the) integral algebraic intersection number among curves on S is not a definable relation in $\mathbb{C}(S)$.*

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Horocycles, laminations, and Lipschitz maps

James Farre
(joint work with Or Landesberg and Yair Minsky)

There is a rich interplay between the geodesic and horocycle flows on the frame bundle over hyperbolic manifolds. For closed hyperbolic surfaces Σ_0, every horocycle is dense [Hed36] and equidistributed in $T^1\Sigma_0$ [Fur73]. The geometric, topological, and dynamical behaviors of horocycles in geometrically finite hyperbolic surfaces are well understood, e.g., [DS84, Ebe77, Dal00, Bur90, Rob03, Sch05]. We study the behavior of horocyclic and geodesic trajectories on $T^1\Sigma$ where Σ is a $\mathbb{Z}$-cover of a closed hyperbolic surface Σ_0. The main result that we discuss is the construction of a cover $\Sigma \to \Sigma_0$ in which the horocycle orbit closures in $T^1\Sigma$ are completely classified (Theorem 2.1, below). This is the first classification theorem of its kind.

1. Notation and preliminaries

1.1. **Geodesic and horocyclic trajectories.** Let $G = \mathrm{PSL}_2\mathbb{R}$ and $\Gamma_0 \leq G$ be uniform lattice. Let $\varphi : \Gamma_0 \to \mathbb{Z}$ be a surjective homomorphism, and let $\Gamma = \ker\varphi$. We endow G with a right invariant Riemannian metric. Then $G/\Gamma \cong T^1\Sigma$ and $G/\Gamma_0 \cong T^1\Sigma_0$, where $\Sigma \to \Sigma_0$ is a cover deck group $\mathbb{Z}$.

The geodesic flow $\phi_t : T^1\Sigma \to T^1\Sigma$ is given by $x \mapsto a_t x$, where

$$a_t = \begin{pmatrix} e^{t/2} & 0 \\ 0 & e^{-t/2}. \end{pmatrix}$$

Let $A = \{a_t : t \in \mathbb{R}\}$ and let $N = \{n \in G : a_t n a_{-t} \to id,\ t \to \infty\}$ be the contracted horocyclic subgroup corresponding to the stable horocycle flow on $T^1\Sigma$. For $x \in T^1\Sigma$, Nx is the (stable) horocycle through x.

1.2. **Quasi-minimizing points.** Say that $x \in T^1\Sigma$ is *quasi-minimizing* if there is a constant $C \geq 0$ such that

$$d(a_t x, x) \geq t - C, \quad \text{for all } t \geq 0.$$

The following theorem of Eberlein (see also [Dal00]) gives a correspondence between quasi-minimizing points and the "interesting" horocycle orbit closures in $T^1\Sigma$.

Theorem 1.1. [Ebe77] *For $\Gamma \lhd \Gamma_0$ as above, we have $\overline{Nx} \neq T^1\Sigma$ if and only if x is quasi-minimizing.*

Let $\mathcal{Q}$ be the set of quasi-minimizing points in $T^1\Sigma$ and let $\mathcal{Q}^+$ be those quasi-minimizing rays that exit out the +-end of $T^1\Sigma$. With $\mathcal{Q}^-$ defined similarly, we have $\mathcal{Q} = \mathcal{Q}^- \sqcup \mathcal{Q}^+$.

A central object in our investigation is $\mathcal{Q}_\omega \subset T^1\Sigma$, the "$\omega$-limit set of $\mathcal{Q}$ mod $\mathbb{Z}$:"

$$\mathcal{Q}_\omega := \{x \in T^1\Sigma : \exists y \in \mathcal{Q} \text{ s.t. } \{\pi(a_t y)\}_{t>0} \text{ accumulates onto } \pi(x)\},$$

where $\pi : T^1\Sigma \to T^1\Sigma_0$ is the covering projection. The A-orbits of $x \in \mathcal{Q}_\omega$ encode the asymptotic behaviors of the $A_{>0}$-orbits of $y \in \mathcal{Q}$.

1.3. **Tight maps to the circle.** The homomorphism $\varphi : \Gamma_0 \to \mathbb{Z}$ defines uniquely a homotopy class of maps $[f : \Sigma_0 \to S^1]$ to the circle S^1 satisfying $\deg f|_\gamma = \varphi(\gamma)$ for all closed curves $\gamma \subset \Sigma_0$. A Lipschitz map $\tau_0 \in [f : \Sigma_0 \to S^1]$ is called *tight* if

$$\mathrm{Lip}(\tau_0) = \sup_{\gamma \subset \Sigma_0} \frac{|\varphi(\gamma)|}{\ell(\gamma^*)},$$

where $\ell(\gamma^*)$ is the length in Σ_0 of the geodesic γ^* homotopic to γ, and $\mathrm{Lip}(\tau_0)$ is the (minimal) Lipschitz constant of τ_0. The ratio is easily seen to be a lower bound on the Lipschitz constant of any map in $[f : \Sigma_0 \to S^1]$.

Let $\mathrm{stretch}(\tau_0) \subset \Sigma_0$ be the *maximally stretched set* of points whose local Lipschitz constant achieves the global Lipschitz constant of τ_0. The following states that tight maps exist in any homotopy class and that the maximally stretched set contains a non-empty geodesic lamination.

Theorem 1.2 ([GK17]). *There is a tight map τ_0 in the class of $[f : \Sigma_0 \to S^1]$. The set*

$$\lambda_0 := \bigcap_{\tau_0'} \mathrm{stretch}(\tau_0')$$

is a non-empty geodesic lamination on Σ_0 where the intersection runs over tight maps in $[f : \Sigma_0 \to S^1]$.

See also recent work [DU20], which provides an analytic framework for producing circle valued tight maps. By rescaling the metric on the circle S^1, we can assume that every tight map is 1-Lipschitz.

2. Identifying $\mathcal{Q}_\omega$ and horocycle orbit closures

Returning to $\mathbb{Z}$-covers, recall that $\varphi : \Gamma_0 \to \mathbb{Z}$ is a surjective group homomorphism and that $\Gamma = \ker \varphi$. Let λ_0^{cr} be the *chain recurrent* part of the lamination from Theorem 1.2 applied to (the homotopy class of circle maps corresponding to) φ. The following proposition relates the asymptotic trajectories of the quasi-minimizing rays in $T^1\Sigma$ with those $x \in T^1\Sigma_0$ that are tangent to λ_0^{cr}.

Proposition 2.1 ([FLM23])**.** *We have $\pi(\mathcal{Q}_\omega) = T^1\lambda_0^{\mathrm{cr}}$. Furthermore, for all $y \in \mathcal{Q}$, there is an x in $\mathcal{Q}_\omega$ such that $\overline{Nx} = \overline{Ny}$.*

Let $\tau : \Sigma \to \mathbb{R}$ be a lift of a 1-Lipschitz tight map $\tau_0 : \Sigma_0 \to \mathbb{R}/c\mathbb{Z}$ (with $c > 0$ chosen suitably). Abusing notation, we also use τ to denote the map $T^1\Sigma \to \mathbb{R}$ obtained by precomposition with the natural projection $T^1\Sigma \to \Sigma$.

Define a function for $x \in T^1\Sigma$

$$\beta_+(x) = \lim_{t\to\infty} \tau(a_t x) - t \in [-\infty, \infty).$$

Then β_+ is an N-invariant upper semi-continuous function satisfying

$$\beta_+^{-1}(-\infty, \infty) = \mathcal{Q}^+.$$

For $x \in \mathcal{Q}^+$, $\tau(x) - \beta_+(x) \geq 0$ measures "how much time x wastes before exiting the + end of Σ." Since β_+ is N-invariant and $\beta_+^{-1}[\beta_+(x), \infty)$ is closed (by upper semi-continuity), it follows that $\overline{Nx} \subset \beta_+^{-1}[\beta_+(x), \infty)$.

Fix a 1-Lipschitz tight map $\tau_0 : T^1\Sigma_0 \to \mathbb{R}/c\mathbb{Z}$, consider a fiber $k = \tau_0^{-1}(0)$ and let $X = k \cap \lambda_0^{\mathrm{cr}}$. Since τ_0 is locally isometric along λ_0, traveling along the leaves of λ_0^{cr} for time c induces a Poincaré first return $P : X \to X$. The dynamics of P help us understand the behaviors of horocycle orbit closures in $T^1\Sigma$.

Theorem 2.1 ([FLM23])**.** *Suppose $P : X \to X$ is topologically weak mixing. Then for every $x, y \in \mathcal{Q}^+$ we have*

- $\overline{Nx} = \beta_+^{-1}[\beta_+(x), \infty)$*; and*
- *the Hausdorff dimension of $\overline{Nx}$ is 2; and*
- *there is a $t \in \mathbb{R}$ such that $\overline{Nx} = a_t\overline{Ny}$.*

An analogous statement holds for $x, y \in \mathcal{Q}^-$.

In other words, if $P : X \to X$ is weak mixing, then there are exactly two families of non-maximal N-orbit closures in $T^1\Sigma$, each indexed by $\mathbb{R}$. Of course, Theorem 2.1 isn't very useful if we cannot construct examples where $P : X \to X$ is weak mixing.

We use recent results of [CF21] to build hyperbolic metrics and circle valued tight maps where the stretch set is an arbitrary oriented, measurable geodesic lamination.

Theorem 2.2 ([FLM23]). *Let S_0 be a closed, oriented surface of negative Euler characteristic. For every cohomology class $\varphi \in H^1(S_0;\mathbb{Z})$ and every oriented geodesic lamination λ_0 that has a measure of full support, if φ and λ_0 satisfy a necessary topological compatibility condition, then there is a hyperbolic metric Σ_0 on S_0 and a 1-Lipschitz tight map $\tau_0 : \Sigma_0 \to \mathbb{R}/c\mathbb{Z}$ in the homotopy class determined by φ such that* $\operatorname{stretch}(\tau_0) = \lambda_0$.

In particular, applying work of [AF07], our construction produces examples satisfying the hypotheses of Theorem 2.1. See [FLM23] for a precise statement.

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Subharmonicity of a span associated with the moduli disk

Sachiko Hamano

In this talk, we give an overview of the main result of [3].

Definition (hydrodynamic differential $\phi^\theta \in \Sigma^\theta(R)$). Let θ be a real number. A holomorphic differential ϕ on an open Riemann surface R is called a *hydrodynamic differential with parameter value θ* (or *Σ^θ-differential* for short) if $\operatorname{Im}[e^{-\pi i\theta/2}\phi]$ is a distinguished differential of Ahlfors [1], namely,

(1) outside a compact set on R, $\phi = d\Phi$ is exact and Dirichlet finite,

(2) on each component of ∂R, $\mathrm{Im}\,[e^{-\pi i\theta/2}\Phi] = const.$ along ∂R.

We denote by $\Sigma^\theta(R)$ the set of Σ^θ-differentials on R.

Definition (closings). For a given open Riemann surface R of genus g $(1 \leq g < \infty)$, we fix a canonical homology basis $\chi^R = \{A_j^R, B_j^R\}_{j=1}^g$ of R modulo dividing cycles. Let S be a closed Riemann surface of the same genus g, $\chi^S = \{A_j^S, B_j^S\}_{j=1}^g$ be a canonical homology basis of S, and ι be a conformal embedding of R into S such that $\iota(A_j^R)$ (resp.$\iota(B_j^R)$) are homologous to A_j^S (resp.B_j^S) on S for all $j = 1, \ldots, g$. Two triplets (S, χ^S, ι) and $(S', \chi^{S'}, \iota')$ are called *equivalent* if there exists a conformal bijection f of S onto S' such that $f \circ \iota = \iota'$. Each equivalence class is referred to as a *closing of* (R, χ^R), which is denoted by $[(R, \chi^R) \overset{\iota}{\hookrightarrow} (S, \chi^S)]$. We denote by $\mathcal{C}(R, \chi^R)$ the set of all closings of (R, χ^R).

Our aim is to study the set $\mathfrak{M}(R, \chi^R)$ of the period matrices of $\mathcal{C}(R, \chi^R)$. The closing $[(R, \chi^R) \overset{\iota}{\hookrightarrow} (S, \chi^S)]$ carries uniquely normal differentials ψ_j $(j = 1, \ldots, g)$ on the closed Riemann surface S. Setting $\tau_{jk} := \int_{B_k^S} \psi_j$ $(j, k = 1, \ldots, g)$, we have the so-called *Riemann period matrix* $T[S, \chi^S, \iota]$:

$$\mathcal{C}(R, \chi^R) \ni [S, \chi^S, \iota] \quad \mapsto \quad T[S, \chi^S, \iota] := \begin{pmatrix} \tau_{11} & \cdots & \tau_{1g} \\ \vdots & \ddots & \vdots \\ \tau_{g1} & \cdots & \tau_{gg} \end{pmatrix} \in \mathfrak{S}_g,$$

where $\mathfrak{S}_g$ is the Siegel upper half space of degree g. To characterize $\mathcal{C}(R, \chi)$, we shall study the set

$$\mathfrak{M}(R, \chi^R) := \{T[S, \chi^S, \iota] \in \mathfrak{S}_g \mid [S, \chi^S, \iota] \in \mathcal{C}(R, \chi^R)\} \quad \subset\subset \mathfrak{S}_g.$$

Oikawa [7] proved that $\mathfrak{M}(R, \chi^R)$ is compact and connected in the Teichmüller space of degree g. In particular, in the case of genus one, Shiba [9] showed that the set $\mathcal{M}(R, \chi^R)$ of moduli is a closed disk in $\mathbb{H}$. In the higher genus cases $(2 \leq g < \infty)$, Schmieder-Shiba [8] proved that the restriction $\mathfrak{M}_j(R, \chi^R)$ of $\mathfrak{M}(R, \chi^R)$ to each diagonal element τ_{jj} of $T[S, \chi^S, \iota]$ is a closed disk in $\mathbb{H}$ for $1 \leq j \leq g$.

Definition (hydrodynamic closing). Let $\theta \in \mathbb{R}/2\mathbb{Z}$ and $\phi^\theta \in \Sigma^\theta(R) \smallsetminus \{0\}$. A closing $\left[(R, \chi^R) \overset{\iota}{\hookrightarrow} (S, \chi^S)\right]$ is called a *hydrodynamic closing of* (R, χ^R) *belonging to* ϕ^θ if $\phi^\theta = \iota^* \psi$ for some holomorphic differential ψ on S. A closing of (R, χ^R) is called *hydrodynamic* if it is a hydrodynamic closing belonging to some hydrodynamic differential on R.

Theorem 1 ([4]; $1 \leq g < \infty$). *Let (R, χ^R) be a marked open Riemann surface of finite genus g. Then, the period matrices T of the closings of (R, χ^R) satisfy a set of inequalities in $\mathfrak{S}_g$:*

$$|\mathbf{c}\ (T - P)\ {}^t\mathbf{c}| \leq \overline{\mathbf{c}}\ Q\ {}^t\mathbf{c}$$

for every $\mathbf{c} \in \mathbb{C}^g \smallsetminus \{\mathbf{0}\}$. Here,

$$\exists 1\ P = \frac{T^\theta + T^{\theta+1}}{2} \in \mathfrak{S}_g \quad \textit{and} \quad \frac{T^\theta - T^{\theta+1}}{2} = e^{\pi i(\theta - \frac{1}{2})} Q \notin \mathfrak{S}_g,$$

where the matrix Q is a positive definite Hermitian matrix, and each T^θ ($-1 < \theta \le 1$) is the $g \times g$ complex matrix which consists of B_k-periods ($k = 1, \ldots, g$) of normal hydrodynamic differentials ϕ_j^θ ($j = 1, \ldots, g$) on (R, χ^R), i.e., $\int_{A_k^R} \phi_j^\theta = \delta_{jk}$, $\tau_{jk}^\theta := \int_{B_k^R} \phi_j^\theta$, $T^\theta := (\tau_{jk}^\theta)$. The equality holds for some $\mathbf{c}$ if and only if T is the period matrix of some hydrodynamic closing of (R, χ^R).

Now, we shall study how $\mathfrak{M}(R(t), \chi^R(t))$ varies when $R(t)$ deforms with complex parameter t from the point of view of several complex variables. Let $\pi : \widetilde{\mathcal{R}} \to \Delta$ be a holomorphic submersion of a 2-dimensional complex manifold $\widetilde{\mathcal{R}}$ to a disk $\Delta = \{t \in \mathbb{C} \mid |t| < r\}$. Assume that $\widetilde{R}(t) := \pi^{-1}(t)$, $t \in \Delta$, is non-compact and irreducible. Let $\pi|_{\mathcal{R}} : \mathcal{R} \to \Delta$ be a sub-holomorphic family of $\widetilde{\mathcal{R}}$ such that each fiber $(\pi|_{\mathcal{R}})^{-1}(t) =: R(t)$ ($\subset\subset \widetilde{R}(t)$) is an open Riemann surface of genus g ($1 \le g < \infty$) with C^ω smooth boundary $\partial R(t) = \sum_{j=1}^{\nu} C_j(t)$ in $\widetilde{R}(t)$. We remark that g and ν do not depend on $t \in \Delta$. We identify $\mathcal{R}$ with the *smooth variation of* $R(t)$ in $\widetilde{\mathcal{R}}$. For $t \in \Delta$, we may take a canonical homology basis $\chi(t) = \{A_k^R(t), B_k^R(t)\}_{k=1}^g$ of $R(t)$ mod dividing cycles $C_j(t)$ ($j = 1, \ldots, \nu$) such that all $A_k^R(t)$ and $B_k^R(t)$ move continuously in $\mathcal{R}$ with $t \in \Delta$.

Theorem 2 ([5]; $g = 1$). *Let $\pi : \mathcal{R} \to \Delta$ be a smooth variation of open tori $(R(t), \chi(t))$, $t \in \Delta$. Assume that $\mathcal{R}$ is pseudoconvex in $\widetilde{\mathcal{R}}$. Then,*

(1) the hyperbolic diameter $\sigma(t)$ of $\mathcal{M}(R(t), \chi^R(t))$ is subharmonic on Δ.
(2) $\sigma(t)$ is harmonic on Δ if and only if $\mathcal{R}$ is a trivial family $\Delta \times R(0)$.

In the proof of Theorem 2, the **unique** closing of $(R(t), \chi^R(t))$ corresponding to each boundary point of $\mathcal{M}(R(t), \chi^R(t))$ was used to establish the variational formulas of ϕ^0 and ϕ^1. However, the method is valid only for Riemann surfaces of **genus one**. In the higher genus cases, even at a boundary point of $\mathfrak{M}_j(R, \chi^R)$, the closing of (R, χ^R) is **not** always uniquely determined.

The purpose of this talk is to extend the theorems in [2] for $\mathbf{a} \in \mathbb{R}^g \setminus \{\mathbf{0}\}$, to the cases of $\phi_{\mathbf{c}}^\theta$ normalized by $\mathbf{c} \in \mathbb{C}^g \setminus \{\mathbf{0}\}$ and every angle $\theta \in \mathbb{R}$ mod 2.

Definition ([3]; the $\mathbf{c}$-span for (R, χ^R)). Let $\mathbf{c} = \mathbb{C}^g \setminus \{\mathbf{0}\}$ and $\theta \in \mathbb{R}$ mod 2. For $\phi_{\mathbf{c}}^\theta \in \Sigma^\theta(R)$, we set $\tau_k^\theta := \int_{B_k} \phi_{\mathbf{c}}^\theta$ ($k = 1, \ldots, g$).

$$\text{The } \mathbf{c}\text{-modulus} \quad \tau_{\mathbf{c}}^\theta := \sum_{k=1}^{g} \left(\int_{A_k} \phi_{\mathbf{c}}^\theta \int_{B_k} \phi_{\mathbf{c}}^\theta \right) = \sum_{k=1}^{g} c_k \, \tau_k^\theta \in \mathbb{C}.$$

$$\text{The } \mathbf{c}\text{-span} \qquad \rho_{\mathbf{c}} := \left| e^{-\pi i \theta} (\tau_{\mathbf{c}}^\theta - \tau_{\mathbf{c}}^{\theta+1}) \right| = \left| \tau_{\mathbf{c}}^1 - \tau_{\mathbf{c}}^0 \right| \ge 0.$$

Let $\mathcal{R}$ be a smooth variation of open Riemann surfaces $(R(t), \chi^R(t))$, $t \in \Delta$, of finite genus g ($1 \le g < \infty$). Fixed $\mathbf{c} = (c_1, \ldots, c_g) \in \mathbb{C}^g \setminus \{\mathbf{0}\}$ which does not depend on t.

Theorem 3 ([3]). *Let $\phi_{\mathbf{c}}^\theta(t, z) = f_{\mathbf{c}}^\theta(t, z) dz$ (for $\mathbf{c} \in \mathbb{C}^g \setminus \{\mathbf{0}\}$ and $\theta \in \mathbb{R}$ mod 2) by use the local parameter z of $\overline{R(t)}$. For $t \in \Delta$,*

$$\frac{\partial^2 \operatorname{Im} [e^{-\pi i \theta} \, \tau_{\mathbf{c}}^\theta(t)]}{\partial t \partial \bar{t}} = -\frac{1}{2} \int_{\partial R(t)} \kappa(t, z) |f_{\mathbf{c}}^\theta(t, z)|^2 |dz| - \left\| \frac{\partial \phi_{\mathbf{c}}^\theta(t, z)}{\partial \bar{t}} \right\|_{R(t)}^2 .$$

Here the Dirichlet norm of a holomorphic differential ϕ *on* R *is denoted by* $\|\phi\|_R^2$*, and* $\kappa(t,z)$ *is the following Levi curvature due to* [6, (1.2)] *for a* C^2*-smooth defining function* $\varphi(t,z)$ *of* $\partial\mathcal{R}$ *in* $\widetilde{\mathcal{R}}$*:*

$$\kappa(t,z) = \left(\frac{\partial^2\varphi}{\partial t\partial\bar{t}} \left|\frac{\partial\varphi}{\partial z}\right|^2 - 2\mathrm{Re}\left\{ \frac{\partial^2\varphi}{\partial\bar{t}\partial z}\frac{\partial\varphi}{\partial t}\frac{\partial\varphi}{\partial\bar{z}} \right\} + \left|\frac{\partial\varphi}{\partial t}\right|^2 \frac{\partial^2\varphi}{\partial z\partial\bar{z}} \right) \left|\frac{\partial\varphi}{\partial z}\right|^{-3} \text{ on } \partial\mathcal{R}.$$

Theorem 4 ([3]). *Let* $\mathcal{R}$ *be a smooth variation of open Riemann surfaces* $(R(t), \chi^R(t))$*,* $t \in \Delta$*, of finite genus* g $(1 \leq g < \infty)$*. Let* $\mathbf{c} \in \mathbb{C}^g \setminus \{\mathbf{0}\}$*. If* $\mathcal{R}$ *is a pseudoconvex domain in* $\widetilde{\mathcal{R}}$*, then*

(1) the $\mathbf{c}$*-span* $\rho_{\mathbf{c}}(t)$ *of* $(R(t),\chi(t))$ *is subharmonic on* Δ*,*

(2) $\rho_{\mathbf{c}}(t)$ *is harmonic on* Δ *if and only if*

(a) each $\tau_{\mathbf{c}}^{\theta}(t)$ $(\theta \in \mathbb{R} \mod 2)$ *is holomorphic on* Δ*,*

(b) $\rho_{\mathbf{c}}(t) \equiv \rho_{\mathbf{c}}(0)$ *for all* $t \in \Delta$*.*

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The earthquake metric

Yi Huang

(joint work with Ken'ichi Ohshika, Huiping Pan, Athanase Papadopoulos)

Earthquakes are natural generalisations of Fenchel–Nielsen twist deformations on Teichmüller space $\mathcal{T}(S)$, where instead of twisting about a simple closed curve, one "twists" along a measured lamination. Remarkably, Thurston's earthquake theorem [8] tells us that any complete hyperbolic metric on a surface S can be deformed to any other via a unique (left) earthquake. This result was famously employed by Kerckhoff in his solution of the Nielsen realisation problem [5], and helped cement the importance of earthquakes in Teichmüller theory.

Our work is the first systematic study of a particular Finsler metric conceived in Thurston's "*Minimal stretch maps between hyperbolic surfaces*" paper [9, pg. 22].

Definition 1 (earthquake norm and earthquake metric)**.** Given an arbitrary point $x \in \mathcal{T}(S)$, every tangent vector in $\mathrm{T}_x\mathcal{T}(S)$ can be uniquely expressed as the derivative $\mathbf{e}_\lambda(x) := \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} E_{t\lambda}(x)$ of a path $E_{t\lambda}(x)$ of earthquakes of x with respect to a measured lamination λ (see [5, Proposition 2.6]). Thurston shows [9] that the function

$$\|\cdot\|_e : \mathrm{T}_x\mathcal{T}(S) \to \mathbb{R}, \quad \|\mathbf{e}_\lambda(x)\|_e = \ell_\lambda(x)$$

defines an asymmetric norm on every tangent space, and hence defines a Finsler metric. He refers to $\|\cdot\|_e$ as the *earthquake norm*, and we dub the induced metric $d_e(\cdot,\cdot)$ the *earthquake metric.*

We establish basic properties of the earthquake metric, and discover surprising connections to both the Thurston (Lipschitz) metric and the Weil–Petersson metric.

Main results

In [2, Section 3.4.5], Barbot and Fillastre clarified a duality first asserted by Thurston on the last lines of pages 20 and 21 of [9]:

Theorem 1 (infinitesimal duality)**.** For every $x \in \mathcal{T}(S)$, there is a linear isomorphism between the normed spaces $(\mathrm{T}_x\mathcal{T}(S), \|\cdot\|_e)$ and $(\mathrm{T}^*_x\mathcal{T}(S), \|\cdot\|^*_{\mathrm{Th}})$, where $\|\cdot\|^*_{\mathrm{Th}}$ denotes the Finsler conorm for the Thurston metric.

As an immediate consequence of the above result and the infinitesimal rigidity of the Thurston metric [4, 7], we obtain:

Corollary 2. The extended mapping class group of S is precisely the isometry group of $(\mathcal{T}(S), d_e)$.

Mirzakhani [6] showed that earthquake flows on Teichmüller space is measurably isomorphic to the horocyclic flow. Since horocycles are highly inefficient ways of navigating hyperbolic space, this heuristically suggests that earthquakes are *not* candidates for geodesics on the earthquake metric. Thurston asserted this in [9], and we verify the following:

Theorem 3. Sufficiently long earthquake paths in $\mathcal{T}(S)$ cannot be geodesics with respect to the earthquake metric.

Despite appearances, Theorem 3 does not divorce earthquake paths and the earthquake metric: we show that the earthquake metric measures the infimal amount of some form of work needed to deform between two hyperbolic metrics using a sequence of earthquakes.

Definition 2. (magnitude)**.** Consider a finite sequence of earthquake paths joining $x_1, \ldots, x_{m+1}$ such that $x_{i+1} = E_{\lambda_i}(x_i)$. We refer to the quantity $\sum_{i=1}^m \ell_{\lambda_i}(x_i)$ as the *magnitude* of this piecewise earthquake path.

Theorem 4. (magnitude minimization)**.** For arbitrary $x, y \in \mathcal{T}(S)$, the earthquake distance $d_e(x, y)$ is equal to the infimal magnitude over the collection of piecewise earthquake paths from x to y.

Although we presently know comparatively little about the geodesy of the earthquake metric, we are beginning to unravel its metrical properties. To begin with, we obtain the following comparisons between the earthquake norm and the following norms of Finsler/Riemannian metrics on $\mathcal{T}(S)$:

- the Teichmüller norm $\|\cdot\|_{\mathrm{T}}$,
- the Thurston norm $\|\cdot\|_{\mathrm{Th}}$,
- and the Weil–Petersson norm $\|\cdot\|_{\mathrm{wp}}$.

Theorem 5. There are positive constants C_0, C_1, C_2, C_3, C_4 depending only on the topology of S, so that for any $x \in \mathcal{T}(S)$ and any $v \in \mathrm{T}_x\mathcal{T}(S)$,

$$C_0 \ell_{\mathrm{sys}}(x) \mathrm{Log}\left(\tfrac{1}{\ell_{\mathrm{sys}}(x)}\right) \|v\|_{\mathrm{Th}} \leq \|v\|_e \leq C_1 \|v\|_{\mathrm{wp}} \leq C_2 \|v\|_{\mathrm{Th}} \leq C_3 \|v\|_{\mathrm{T}} \leq \tfrac{C_4 \|v\|_{\mathrm{wp}}}{\ell_{\mathrm{sys}}(x)},$$

where $\ell_{\mathrm{sys}}(x)$ denotes the length of the systole on x, and $\mathrm{Log}(x) := \max\{1, \log(x)\}$.

Using the above comparisons, we show:

Theorem 6. There are constants C_1, C_2 depending only on the topology of S such that for any two points x, y in $\mathcal{T}(S)$, we have

$$d_e(x, y) \leq C_1 d_{\mathrm{wp}}(x, y) \leq C_2 d_{\mathrm{Th}}(x, y).$$

The fact that the earthquake metric is bounded above by a multiple of the Weil–Petersson metric, which is incomplete, suggests that the earthquake metric is incomplete. This is a nuanced statement: one needs to clarify what incompleteness means for asymmetric metrics. We do so and further define a general notion of metric completion for asymmetric metrics.

The first inequality in Theorem 6 also tells us that the completion of the Teichmüller space, endowed with the earthquake metric, should be a quotient of the Weil–Petersson metric completion. This is known to be the augmented Teichmüller space [1], obtained by adding boundary strata of stable curves to $\mathcal{T}(S)$. In fact:

Theorem 7. The completion of the earthquake metric is precisely the augmented Teichmüller space.

Corollary 8. The completion of the moduli space $\mathcal{M}(S)$ with respect to the earthquake metric is the Deligne–Mumford compactification.

This is a novel instance of the real analytic hyperbolic viewpoint of moduli space being able to access the complex analytic/algebraic geometric structure of the moduli space. This is reminiscent of Wolpert's result [10] that the Weil–Petersson metric, which has complex analytical origins, is equal to a Riemannian metric, also defined in terms of the behaviour of the hyperbolic lengths of closed geodesics, (also) introduced by Thurston.

Given this growing list of similarities between the earthquake metric and the Weil–Petersson, one naturally wonders if they might be metrically equivalent in some sense. The following result on the behaviour of these two metrics near the thin part of moduli space can be used to tell us that they are not bi-Lipschitz.

Theorem 9. Let $\overline{\mathcal{M}(S)}$ denote the completion of the moduli space with respect to the earthquake metric. There exists $C_S > 1$, depending only on the topology of S, such that for any $x \in \mathcal{M}(S)$,

$$2\ell_{\text{sys}}(x)\text{Log}\frac{1}{\ell_{\text{sys}}(x)} \leq d_e(x, \partial\overline{\mathcal{M}(S)}) \leq 2C_S\ell_{\text{sys}}(x)\text{Log}(\frac{1}{\ell_{\text{sys}}(x)}),$$

where $\ell_{\text{sys}}(x)$ denotes the length of the systole on x, and $\text{Log}(x) := \max\{1, \log(x)\}$. Furthermore, as $\ell_{\text{sys}}(x) \to 0$

$$\frac{d_e(x, \partial\overline{\mathcal{M}(S)})}{2\ell_{\text{sys}}(x)\text{Log}(\frac{1}{\ell_{\text{sys}}(x)})} \to 1.$$

However, the two metrics are coarsely equivalent:

Theorem 10. $(\mathcal{T}(S), d_e)$ is quasi-isometric to $(\mathcal{T}(S), d_{\text{wp}})$.

This means that the $(\mathcal{T}(S), d_e)$ is also quasi-isometric to the pants graph [3], which suggests that to efficiently navigate the Teichmüller space with respect to the earthquake metric (especially when travelling over long distances), it might make sense to use the pants graph as a guide and to "zig-zag" between thin parts of Teichmüller space where the systolic pants decomposition is short.

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Moduli space of G-local systems and skein algebras

Tsukasa Ishibashi
(joint work with Hironori Oya, Linhui Shen)

Setting. Let G be a simply-connected semisimple algebraic group over $\mathbb{C}$ with a fixed maximal unipotent subgroup $U^+ \subset G$. Let $(\Sigma, \mathbb{M})$ be a marked surface, which consists of a compact oriented surface Σ and a fixed non-empty finite set $\mathbb{M} \subset \partial\Sigma$ of marked points. Here we assume that each boundary component has at least one marked point and $-2\chi(\Sigma) + |\mathbb{M}| > 0$. Let us simply denote a marked surface by Σ, with a choice of $\mathbb{M}$ understood.

Background. Let $\mathcal{A}_{G,\Sigma}$ denote the moduli space of decorated twisted G-local system [FG06]. By choosing a basepoint $\xi \in T'\Sigma$ of the punctured tangent bundle $T'\Sigma = T\Sigma \smallsetminus$ (0-section) and a suitable collection of curves, we get a description as a quotient stack $\mathcal{A}_{G,\Sigma} = [A_{G,\Sigma}/G]$, where

$$A_{G,\Sigma} \simeq \mathrm{Hom}^{\mathrm{tw}}(\pi_1(T'\Sigma, \xi), G) \times (G/U^+)^{\mathbb{M}}$$

is an affine G-variety. In particular, we are interested in the $\mathbb{C}$-algebra $\mathcal{O}(\mathcal{A}_{G,\Sigma}) = \mathcal{O}(A_{G,\Sigma})^G$ of global (algebraic) functions.

Since its introduction by Fock–Goncharov [FG06], the cluster K_2-structure on $\mathcal{A}_{G,\Sigma}$ has been discovered after sequential works, including [FG06, Le19, GS19]. Here, a *cluster K_2-structure* means a particular kind of birational atlas parametrized by combinatorial data $\boldsymbol{i} = (\{A_i\}_{i\in I}, \varepsilon)$ called *seeds*, each giving rise to a birational map

$$\psi_{\boldsymbol{i}} : \mathcal{A}_{G,\Sigma} \dashrightarrow (\mathbb{C}^*)^I.$$

Their transition maps $\psi_{\boldsymbol{i}'} \circ \psi_{\boldsymbol{i}}^{-1} : (\mathbb{C}^*)^I \dashrightarrow (\mathbb{C}^*)^I$ are positive rational maps of particular form, controlled by combinatorial operations called *seed mutations* $\mu_k : \boldsymbol{i} \to \boldsymbol{i}'$ [FG09]. Put in another way, we can start with an initial seed $\boldsymbol{i}_\triangle$ associated with a "decorated" triangulation $\triangle$ of Σ, and the other seeds are recursively created via seed mutations. The above mentioned works establish that the entire collection of seeds does not depend on the initial choice.

In particular, the cluster K_2-structure defines a *cluster algebra*

$$\mathscr{A}_{\mathfrak{g},\Sigma} := \langle \mathrm{CV} \rangle_{\mathbb{C}} \subset \mathcal{K}(\mathcal{A}_{G,\Sigma})$$

in the field of rational functions on $\mathcal{A}_{G,\Sigma}$. Here CV denotes the set of all cluster coordinates A_i appearing in some seed in this collection, together with the inverses of "frozen" coordinates assigned on the boundary of Σ.

Geometric model of the cluster algebra. Our goal is to understand $\mathscr{A}_{\mathfrak{g},\Sigma}$ in terms of the geometry of the moduli space $\mathcal{A}_{G,\Sigma}$. There is a related algebra $\mathscr{U}_{\mathfrak{g},\Sigma} \subset \mathcal{K}(\mathcal{A}_{G,\Sigma})$ called the *upper cluster algebra*, which is the function algebra of the cluster K_2-variety. In the general context of cluster algebra, showing $\mathscr{A} = \mathscr{U}$ is an important problem. One inclusion $\mathscr{A} \subseteq \mathscr{U}$ is known as the Laurent phenomenon of cluster variables.

Theorem 1 ([IOS22, Theorem 1]). *For a finite-dimensional simple Lie algebra* $\mathfrak{g}$ *admitting a non-trivial minuscule representation (namely, not of type* E_8, F_4, G_2*) and a connected marked surface* Σ *with* $|\mathbb{M}| \geq 2$*, we have*

$$\mathscr{A}_{\mathfrak{g},\Sigma} = \mathscr{U}_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}).$$

Here $\mathcal{A}^{\times}_{G,\Sigma} \subset \mathcal{A}_{G,\Sigma}$ *denotes a certain Zariski open subspace.*

In particular, our theorem tells us that all of the cluster coordinates are global functions on $\mathcal{A}^{\times}_{G,\Sigma}$, which are *a priori* rational functions.

Skein model of the cluster algebra. The (upper) cluster algebras admit quantizations $\mathscr{A}^q_{\mathfrak{g},\Sigma}$ and $\mathscr{U}^q_{\mathfrak{g},\Sigma}$ by means of Berenstein–Zelevinsky [BZ05]. Here our choice of compatibility matrices follows that of Goncharov–Shen [GS19].

In [Mul16], Muller obtained the equality

$$\mathscr{A}^q_{\mathfrak{sl}_2,\Sigma} = \mathscr{S}^q_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}] = \mathscr{U}^q_{\mathfrak{sl}_2,\Sigma}$$

if $|\mathbb{M}| \geq 2$, where $\mathscr{S}^q_{\mathfrak{sl}_2,\Sigma}[\partial^{-1}]$ denotes the boundary localization of the Kauffman bracket skein algebra with an appropriate skein relations at the marked points. In my joint works [IY23, IY22] with Wataru Yuasa, we obtained an inclusion $\mathscr{S}^q_{\mathfrak{g},\Sigma}[\partial^{-1}] \subset \mathscr{A}^q_{\mathfrak{g},\Sigma}$ for the cases $\mathfrak{g} = \mathfrak{sl}_3, \mathfrak{sp}_4$, where the former denotes certain higher rank generalizations of the Muller skein algebra. Combining with the geometric technique in [IOS22], we obtain:

Theorem 2 ([IOS22, Theorem 2]). *For* $\mathfrak{g} = \mathfrak{sl}_2, \mathfrak{sl}_3, \mathfrak{sp}_4$ *and a connected marked surface* Σ *with* $|\mathbb{M}| \geq 2$*, we have*

$$\mathscr{A}_{\mathfrak{g},\Sigma} = \mathscr{U}_{\mathfrak{g},\Sigma} = \mathcal{O}(\mathcal{A}^{\times}_{G,\Sigma}) = \mathscr{S}^1_{\mathfrak{g},\Sigma}[\partial^{-1}].$$

Here the last one denotes the classical specialization $q = 1 \in \mathbb{C}$ *of the Muller type skein algebra mentioned above.*

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A topological proof of Wolpert's formula of the Weil-Petersson symplectic form in terms of the Fenchel-Nielsen coordinates

Nariya Kawazumi

Let Σ_g be a closed connected oriented surface of genus $g \geq 2$. Given any pants decomposition $\{C_i\}_{i=1}^{3g-3}$ of the surface, one can define the Fenchel-Nielsen coordinates

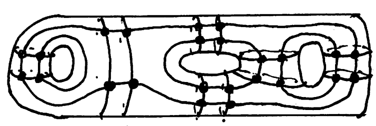
$$(\ell_i, \tau_i)_{i=1}^{3g-3} : \mathcal{T}_g \overset{\cong}{\to} (\mathbb{R}_+ \times \mathbb{R})^{3g-3}$$

of the Teichmüller space $\mathcal{T}_g$ of genus g. Wolpert [8] described the Weil-Petersson symplectic form ω_{WP} on the space $\mathcal{T}_g$ as

$$\omega_{\mathrm{WP}} = \sum_{i=1}^{3g-3} d\tau_i \wedge d\ell_i.$$

The formula can be deduced from his duality theorem [7], whose topological proof of the formula was already given by Goldman [1, 4.11 Theorem]. In this report, we give an outline of an alternative topological proof of the formula using a cell decomposition of the surface with an explicit groupoid cocycle representing each point of the space $\mathcal{T}_g$.

One can consider a natural cell decomposition of the surface Σ_g based on the decomposition $\{C_i\}_{i=1}^{3g-3}$: there are 2 faces, which we call 'squares', associated with each simple closed curve C_i, and 2 faces, which we call 'hexagons', included in each pair of pants. (For example, see the below figure.)



For each point of the Teichmüller space of the surface, there exists a unique groupoind cocycle on the cell complex representing the point with some normalization condition. By a classical argument essentially due to Keen [3, 4], we can compute the cocycle explicitly. (It would be very interesting if one could get similar explicit cocycles also for higher cases.) As was shown by Goldman [1], the Weil-Petersson symplectic form equals the cup product on the first cohomology of the surface with values in $\mathfrak{sl}_2(\mathbb{R})$ with adjoint action. We compute the cup product of two first variations of the explicit cocycle on the cell decomposition to prove the formula in the title. In our computation, the symplectic form is localized in the squares. In higher cases, the results of Goldman [2], Wienhard-Zhang [6] and Sun-Wienhard-Zhang [5] tell us that the hexagons also have nontrivial contributions to the symplectic form.

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An optimal fundamental domain for $\Gamma_0(p)$ and $\Gamma_0(pq)$

Sang-hyun Kim

(joint work with Nhat Minh Doan, Mong Lung Lang, Ser Peow Tan)

For each positive integer N, let us consider the congruence subgroup

$$\Gamma_0(N) := \left\{ A \in \mathrm{PSL}(2,\mathbb{Z}) \;\middle|\; A \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}.$$

While it is immediate from the Kurosh subgroup theorem that $\Gamma_0(N)$ can be written as a free product of copies of finite cyclic groups $\mathbb{Z}_2$ and $\mathbb{Z}_3$ along with copies of $\mathbb{Z}$, precisely describing matrix representatives of such free factors is far from being a trivial task. Using the classical Reidemeister–Schreier process, Rademacher described such matrix representatives for the case when N is a prime [6]; see [1] for a general case.

A geometrically intuitive method of finding such representatives was discovered by Kulkarni [3]. His method used planar hyperbolic geometry, and turned out to be computationally efficient, and applicable to all the finite index subgroups of $\mathrm{PSL}(2,\mathbb{Z})$; many modern computer algebra software including MAGMA uses his algorithm. His method is directly related to the task of finding a fundamental domain of such a finite index subgroup. In the special case of the subgroup $\Gamma_0(N)$, he made concrete computation for all the primes not exceeding 100 and conjectured that there exists a freely independent generating set of $\Gamma_0(N)$ for a prime N such that the $(2,1)$ components of all the generators are either 0 or N.

In this talk, we settle Kulkarni's conjecture and also extend it to the case when N is the square of a prime; we also describe a similar result for the case when $N = pq$

with p and q being sufficiently close primes. Our proof comes from an investigation of an optimal fundamental domain of $\Gamma_0(N)$. More concretely, let us denote by $m(\Gamma_0(N))$ the smallest possible value of the maximum of the denominators of the cusps of a fundamental domain for $\Gamma_0(N)$. A fundamental domain of $\Gamma_0(N)$ is *optimal* if it realizes $m(\Gamma_0(N))$. Our main result is to establish that

$$\lfloor \sqrt{N} \rfloor \leq m(\Gamma_0(N)) \leq \lfloor \sqrt{4N/3} \rfloor,$$

in the case when N is a prime or its square. Kulkarni's conjecture follows from this main result. We remark that our near-optimal choice of a fundamental domain can also be utilized to obtain equidistribution results of closed geodesics on the quotient orbifolds $\mathbb{H}^2/\Gamma_0(N)$ [2, 4, 5].

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First order theory of homeomorphism groups of manifolds, and applications

Thomas Koberda

(joint work with Sang-hyun Kim, J. de la Nuez González)

In this talk, we describe an approach to investigating problems in geometric topology and group actions on manifolds using ideas informed by mathematical logic. Basic motivating questions are as follows.

Question 1. *Let M be a compact, smooth manifold, and let $r \in [1, \infty)$. Does there exist a countable, or perhaps finitely generated group $\Gamma_{r,M} \leq \mathrm{Diff}^r(M)$ such that for all $s > r$, all actions of $\Gamma_{r,M}$ on M by C^s diffeomorphisms are trivial (or have a large kernel at least)?*

This question was answered by Kim and the speaker for manifolds in dimension one in [1], cf. [2]. In dimension one, it is possible to exploit features specific to dimension one, such as linear orderability and circular orderability of groups of homeomorphisms, in order to explicitly produce examples of the desired groups. In higher dimensions, the dynamics of group actions are much wilder, and it is not clear where to begin looking for candidate groups.

Another basic motivating question connects this talk more directly with the main theme of this workshop. Recall that the mapping class group of a compact, orientable surface is the group of isotopy classes of homeomorphisms, and this group naturally acts by isometries on the Teichmüller space of the surface.

Question 2. *Let Σ be a compact, orientable surface of genus at least three. Is the mapping class group of Σ a linear group? Can this question be understood in a way that is more intrinsic to Σ?*

We approach the motivating questions by studying the homeomorphism groups of manifolds, viewed as abstract groups. It was proved by Whittaker [8] that if M and N are compact manifolds then an abstract isomorphism between $\mathrm{Homeo}_0(M)$ and $\mathrm{Homeo}_0(N)$ is induced by a homeomorphism between M and N, where here the subscript indicates the identity components of the underlying groups.

The first step in approaching the motivating questions is to weaken the hypotheses of Whittaker's Theorem. To do this, we view homeomorphism groups as abstract structures in the language of groups, and investigate their first order theory; see [5, 7] for model theoretic background.

In 1989, Rubin [6] conjectured that two homeomorphism groups of manifolds are elementarily equivalent to each other if and only if the underlying manifolds are homeomorphic, which would greatly strengthen Whittaker's Theorem. In [3], we prove an even stronger version of Rubin's Conjecture, at least for compact manifolds.

Theorem 1. *Let M be a compact, connected manifold. There is a sentence ϕ_M in the language of group theory such that for each compact manifold N, we have ϕ_M holds in $\mathrm{Homeo}_0(N)$ if and only if M and N are homeomorphic.*

For a homeomorphism group or diffeomorphism group of a compact smooth manifold, we may now produce many candidate groups for resolving Question 1. Specifically, one considers a countable subgroup which is elementarily equivalent to the ambient homeomorphism or diffeomorphism group, which exists by the classical Löwenheim–Skolem Theorem.

Conjecture. *Let M be a compact, connected, smooth manifold, and let $\Gamma_{r,M} \leq \mathrm{Diff}^r(M)$ be an elementary subgroup. Then for $s > r$, every action of $\Gamma_{r,M}$ on M by C^s diffeomorphisms is trivial.*

As for Question 2, we investigate the expressive power of the language of groups in the context of homeomorphism groups. In [4], we prove that the first order theory of homeomorphism groups of manifolds can interpret large collections of subsets of $\mathrm{Homeo}(M)$, such as arbitrary finite tuples and arbitrary sequences.

Theorem 2. *The group $\mathrm{Homeo}_0(M)$ interprets weak second order logic.*

As a corollary, we get an answer to Question 2 above:

Corollary. *There is a sentence that, for a compact surface Σ, is true in the homeomorphism group of Σ if and only if the mapping class group of Σ is linear.*

Many further avenues relating model theory, dynamics of group actions, and geometric topology remain to be explored.

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Higgs bundles in the Hitchin section over non-compact hyperbolic Riemann surfaces

Qiongling Li

Let X be a Riemann surface and $(E, \bar{\partial}_E, \theta)$ be a Higgs bundle on X. Let h be a Hermitian metric of E. We obtain the Chern connection $\nabla_h = \bar{\partial}_E + \partial_{E,h}$ and the adjoint θ^{*h} of θ. The metric h is called a harmonic metric of the Higgs bundle $(E, \bar{\partial}_E, \theta)$ if $\nabla_h + \theta + \theta^{*h}$ is flat, i.e., $\nabla_h \circ \nabla_h + [\theta, \theta^{*h}] = 0$. It was introduced by Hitchin [4], and it has been one of the most important and interesting mathematical objects. A starting point is the study of the existence and the classification of harmonic metrics. If X is compact, the results of Hitchin [4] and Simpson [8] show that a Higgs bundle is polystable of degree 0 if and only if it admits a harmonic metric. Together with the work of Corlette [2] and Donaldson [3], one obtains the non-Abelian Hodge correspondence which says the moduli space of polystable $SL(n,\mathbb{C})$-Higgs bundles is isomorphic to the representation variety of the surface group $\pi_1(S)$ into $SL(n,\mathbb{C})$. The study of harmonic metrics for Higgs bundles in the non-compact case was pioneered by Simpson [8, 9], and pursued by Biquard-Boalch [1] and Mochizuki [7].

Let $\vec{q} = (q_2, \cdots, q_n)$, where q_j is a holomorphic j-differential on X. One can naturally construct a Higgs bundle $(K_{X,n}, \theta(\vec{q}))$ as follows. Let K_X be the canonical line bundle of X. The multiplication of q_j induces the following morphisms:

$$K_X^{(n-2i+1)/2} \to K_X^{(n-2i+2(j-1)+1)/2} \otimes K_X \quad (j \le i \le n).$$

We also have the identity map for $i = 1, \dots, n-1$:

$$K_X^{(n-2i+1)/2} \to K_X^{(n-2(i+1)+1)/2} \otimes K_X.$$

They define a Higgs field $\theta(\vec{q})$ of $K_{X,n} = \oplus_{i=1}^{n} K_X^{(n+1-2i)/2}$. The natural pairings $K_X^{(n-2i+1)/2} \otimes K_X^{-(n-2i+1)/2} \to \mathcal{O}_X$ induce a non-degenerate symmetric bilinear form $C_{K,X,n}$ of $K_{X,n}$. There exists a basis of $SL(n,\mathbb{C})$-invariant homogeneous polynomials p_i of deg $i(i = 2,\cdots,n)$ on $sl(n,\mathbb{C})$ such that $p_i(\theta(\vec{q})) = q_i$. The Hitchin fibration is from the moduli space of polystable $SL(n,\mathbb{C})$-Higgs bundles to the vector space $\oplus_{i=2}^{n} H^0(X, K_X^i)$ given by

$$[(E,\theta)] \longmapsto (p_2(\theta),\cdots,p_n(\theta)).$$

Such Higgs bundles $(K_{X,n}, \theta(\vec{q}))$ were introduced by Hitchin in [5] for compact hyperbolic Riemann surfaces. They form a section of the Hitchin fibration. For this reason, for arbitrary (not necessarily compact) Riemann surfaces, we call $(K_{X,n}, \theta(\vec{q}))$ Higgs bundles in the Hitchin section.

For the compact hyperbolic surface case, Hitchin in [5] showed that $(K_{X,n}, \theta(\vec{q}))$ are always stable and the Hitchin section corresponds to Hitchin component, a connected component in the representation variety of $\pi_1(X)$ into $SL(n,\mathbb{R})$ which contains embedded Fuchsian representations. In particular, when $n = 2$, the Hitchin section parametrize the Teichmüller space. Hitchin component has been the central object in the field of higher Teichmüller theory.

We want to study Higgs bundles in the Hitchin section in general case: tuples of holomorphic differentials on an arbitrary non-compact Riemann surfaces, e.g., unit disk, of infinite topology, etc. We focus on the following natural question.

Given a tuple of holomorphic differentials $\vec{q} = (q_2,\cdots,q_n)$ on a non-compact Riemann surface X,
(1) does there exist a harmonic metric on $(K_{X,n}, \theta(\vec{q}))$ compatible with $C_{K,X,n}$?
(2) If so, can one find a notion of "best" harmonic metric such that it uniquely exists?

Main Results

Suppose X is a non-compact hyperbolic Riemann surface, equivalently, it is not $\mathbb{C}$ nor $\mathbb{C}^*$. Let g_X be the unique complete hyperbolic Kähler metric on X. Let $h_X = \oplus_{k=1}^{n} a_k \cdot g_X^{-\frac{n+1-2k}{2}}$, where a_k are some fixed constants. Such a_k's are chosen so that h_X is a harmonic metric for the Higgs bundle $(K_{X,n}, \theta(\mathbf{0}))$.

Let $F_k = \oplus_{l\le k} K_X^{\frac{n+1-2l}{2}}$. Then $\{0 \subset F_1 \subset F_2 \subset \cdots \subset F_n\}$ forms an increasing filtration of $K_{X,n}$. We call a Hermitian metric h on $K_{X,n}$ **weakly dominates** h_X if

$$\det(h|_{F_k}) \le \det(h_X|_{F_k}), \quad 1 \le k \le n-1.$$

Our main result in this paper is the following two theorems, as an answer to the above question.

Theorem 1. *([6]) On a non-compact hyperbolic surface X, there exists a harmonic metric h on $(K_{X,n}, \theta(\vec{q}))$ satisfying (i) h weakly dominates h_X; (ii) h is compatible with $C_{K,X,n}$.*

As a result, the associated harmonic map $f : (\widetilde{X}, \widetilde{g_X}) \to SL(n,\mathbb{R})/SO(n)$ satisfies the energy density $e(f) \ge \frac{n^2(n^2-1)}{6}$. The equality holds if $\vec{q} = 0$.

Theorem 2. *([6]) On a non-compact hyperbolic surface X, suppose $q_i (i = 2, \cdots, n)$ are bounded with respect to g_X. Then there uniquely exists a harmonic metric h on $(K_{X,n}, \theta(\vec{q}))$ satisfying (i) h weakly dominates h_X; (ii) h is compatible with $C_{K,X,n}$.*

Moreover, h is mutually bounded with h_X.

As an application of Theorem 2, we reprove the existence and uniqueness of a harmonic metric on $(K_{X,n}, \theta(\vec{q}))$ over a compact hyperbolic Riemann surface. Note that our proof here does not invoke the Hitchin-Kobayashi correspondence by using the stability of Higgs bundle.

Theorem 3. *Given a tuple of holomorphic differentials $\vec{q} = (q_2, \cdots, q_n)$ on a compact hyperbolic surface X, there uniquely exists a harmonic metric h on $(K_{X,n}, \theta(\vec{q}))$ satisfying h is compatible with $C_{K,X,n}$.*

Moreover, h weakly dominates h_X.

Acknowledgement. The author would like to thank the organisers of Oberwolfach's workshop Teichmüller Theory: Classical, Higher, Super and Quantum, 30 July - 4 August 2023, for the organization and invitation.

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Double infinitesimal structures on Teichmüller space

Hideki Miyachi

First of all, I would like to thank the organizers, Professor Ken'ichi Ohshika, Professor Athanase Papadopoulos, Professor Robert C. Penner, and Professor Anna Wienhard for giving me a great opportunity. I thank MFO for giving an atmosphere to concentrate on Mathematics, and their warm hospitalities. I also thank Professor Krishnendu Gongopadhyay for editing this report.

In my talk at 2023 conference at MFO, I announced a description of a model space of the holomorphic double tangent space for the Teichmüller space $\mathcal{T}_g$ of

Riemann surfaces of genus $g \geq 2$. In this note, we give a survey of (a part of) results on this research. The details will be appeared elsewhere.

1. Motivation

The purpose of this research is to develop the complex geometry of the Teichmüller space. Especially, I am now focusing to develop the complex Finsler geometry with the Teichmüller metric. In a technique to study a complex Finsler bundle $E \to M$ with a Finsler fiber metric F, we consider a projective bundle $\mathbb{P}(E) \to M$ and the pull-back bundle

$$\begin{array}{ccc} \tilde{E} & \longrightarrow & E \\ \downarrow & & \downarrow \\ \mathbb{P}(E) & \longrightarrow & M. \end{array}$$

We define an *Hermitian metric* $g_F(Z,Z) = \sum_{1\leq j,k\leq n} \frac{\partial^2 F}{\partial \zeta_i \partial \overline{\zeta_j}}(z,\zeta) Z_i \overline{Z_j}$ on $\tilde{E}$, where (z,ζ) and Z are coordinates of E and of the fiber of $\tilde{E}$ respectively. Then, the geometry of $E \to M$ with a Finsler metric F is studied from the bundle $\tilde{E} \to \mathbb{P}(E)$ with the Hermitian metric g_F (e.g. [3]). To develop the complex Finsler geometry of the Teichmüller metric under this technique, we need to *formulate* the double tangent space, which is the tangent space to the tangent bundle over the Teichmüller space.

2. Notation

2.1. **Double tangent space.** Let M be a complex manifold and TM the holomorphic tangent bundle. The *(holomorphic) double tangent space* T_vTM at $v \in TM$ is the holomorphic tangent space to TM at $v \in TM$.

Naively speaking, double tangent vectors appear as second derivatives of holomorphic maps from a 2-dimensional polydisk. Namely, consider $M = \mathbb{C}^n$ for simplicity. Let v_t ($|t| < \varepsilon$) be a holomorphic family of tangent vectors to $\mathbb{C}^n$. Then, we have a holomorphic map $F\colon \{|t| < \varepsilon\} \times \{|s| < \varepsilon\} \to \mathbb{C}^n$ such that $\frac{\partial F}{\partial s}(t,0) = v_t$ for $|t| < \varepsilon$. The family $\{v_t\}_{|t|<\varepsilon}$ is thought of a path in $T\mathbb{C}^n = \mathbb{C}^n \times \mathbb{C}^n$. Hence, the double tangent vector of the family $\{v_t\}_{|t|<\varepsilon}$ is given by the pair $(u, \dot{v})$ where $u = \frac{\partial F}{\partial t}(0,0)$ and $\dot{v} = \frac{\partial^2 F}{\partial t \partial s}(0,0)$.

2.2. **Kodaira-Spencer theory.** Let M be a closed Riemann surface of genus g. Let Θ_M be the sheaf of germs of holomorphic vector fields on M. Let $\mathcal{A}^{p,q}(\Theta_M)$ be the sheaf of germs of (p,q)-C^∞ vector fields on M. Let $\mathcal{U} = \{U_i\}_{i\in I}$ be a locally finite covering of M. We assume that $H^1(U_i, \Theta_M) = 0$ for $i \in I$. Then, the (holomorphic) tangent space to $\mathcal{T}_g$ at a marked Riemann surface with conformal structure M is identified with $H^1(M, \Theta_M) \cong H^1(\mathcal{U}, \Theta_M)$ (cf. [2, §7.2.4] and [4, Lemma 5.2]). Indeed. the exact sequence

$$(2.1) \qquad 0 \to C^*(\mathcal{U}, \Theta_M) \to C^*(\mathcal{U}, \mathcal{A}^{0,0}(\Theta_M)) \xrightarrow{-\overline{\partial}} C^*(\mathcal{U}, \mathcal{A}^{0,1}(\Theta_M)) \to 0$$

leads the Dolbeaut isomorphism

$$H^0(M, \mathcal{A}^{0,1}(\Theta_M))/\overline{\partial} H^0(M, \mathcal{A}^{0,0}(\Theta_M)) \cong H^1(M, \Theta_M)$$

where the left-hand side is thought of as the space of (infinitesimal) smooth Beltrami differentials and the first cohomology with coefficient Θ_M.

3. Model spaces

In the following, for a sheaf $\mathcal{F}$ on M and a k-the cochain $C \in C^k(\mathcal{U}, \mathcal{F})$, $C_{i_0 i_1 \cdots i_k} \in \Gamma(U_{i_0} \cap \cdots \cap U_{i_k}, \mathcal{F})$ the $(i_0, \cdots, i_k)$-th component of C. For $\alpha \in C^0(\mathcal{U}, \Theta_M)$, $X, Y \in C^1(\mathcal{U}, \Theta_M)$, we consider the following cochains $K(\alpha, X)$, $K'(\alpha, X) \in C^1(\mathcal{U}, \Theta_M)$, $\zeta(X, Y) \in Z^2(\mathcal{U}, \Theta_M)$ by

$$\begin{aligned} K(\alpha, X)_{ij} &= [\alpha_i, X_{ij}] \\ K'(\alpha, X)_{ij} &= [\alpha_j, X_{ij}] \\ \zeta(X, Y)_{ijk} &= \frac{1}{2}([X_{ij}, Y_{jk}] + [Y_{ij}, X_{jk}]) \end{aligned}$$

for $i, j, k \in I$. The cocycle $\zeta(X, Y)$ is called the *primary obstruction* (cf. [4, §5.1]). For $Y \in Z^1(\mathcal{U}, \Theta_M)$, we define two linear maps

$$\begin{aligned} D_0^Y &: C^0(\mathcal{U}, \Theta_M)^{\oplus 2} \to Z^1(\mathcal{U}, \Theta_M) \oplus C^1(\mathcal{U}, \Theta_M) \\ D_1^Y &: Z^1(\mathcal{U}, \Theta_M) \oplus C^1(\mathcal{U}, \Theta_M) \to C^1(\mathcal{U}, \Theta_M) \oplus C^2(\mathcal{U}, \Theta_M) \end{aligned}$$

by

$$\begin{aligned} D_0^Y(\alpha, \beta) &= (\delta\alpha, \delta\beta + K(\alpha, Y)) \\ D_1^Y(X, \dot{Y}) &= \left(\dot{Y} + \dot{Y}^* + [X, Y], \delta\left(\dot{Y} + \frac{1}{2}[X, Y]\right) - \zeta(X, Y)\right) \end{aligned}$$

We can check that $D_1^Y \circ D_0^Y = 0$ and define the *model space of the double tangent space for cocycle* Y by

$$\mathbb{T}_Y[\mathcal{U}] = \mathrm{Ker}(D_1^Y)/\mathrm{Im}(D_0^Y).$$

The equivalence classes of $(X, \dot{Y}) \in \mathrm{Ker}(D_1^Y)$ is denoted by $[\![X, \dot{Y}]\!]_Y$. For $\beta \in C^0(\mathcal{U}, \Theta_M)$, two model spaces $\mathbb{T}_Y[\mathcal{U}]$ and $\mathbb{T}_{Y+\delta\beta}[\mathcal{U}]$ are isomorphic by

$$[\![X, \dot{Y}]\!]_Y \mapsto [\![X, \dot{Y} + K'(\beta, X)]\!]_{Y+\delta}.$$

This define the model space $\mathbb{T}_{[Y]}[\mathcal{U}]$ for $[Y] \in H^1(\mathcal{U}, \Theta_M)$. By taking the direct limit via the refinements of locally finite coverings, we define the model space $\mathbb{T}_{[Y]}$ for $[Y] \in H^1(M, \Theta_M)$.

The model space $\mathbb{T}_{[Y]}[\mathcal{U}]$ is canonically identified with the double tangent space $T_{[Y]}T\mathcal{T}_g$ at $[Y] \in H^1(\mathcal{U}, \Theta_M) \cong H^1(M, \Theta_M)$ as follows. Notice that any holomorphic map from a polydisk D to the Teichmüller space is thought of as a holomorphic family of closed Riemann surfaces of genus g over D, since the Teichmüller space $\mathcal{T}_g$ is the universal space of such holomorphic families (cf. [5], [6]).

Let $\mathcal{M} \to D = \{(t, s) \mid |t| < \varepsilon, |s| < \varepsilon\}$ be a holomorphic family of closed Riemann surfaces of genus g over D with the fiber M at $(0, 0) \in D$. Since the fiber space

$\mathcal{M} \to D$ is trivial as a differentiable family, a family $\{U_i \times D\}_{i\in I}$ can be thought of as a locally finite covering of $\mathcal{M} = M \times D$ for coordinates. Let $U_i \times D \ni (p,t,s) \to (z_i^{t,s}(p),t,s)$ is the chart and set $z_{ij}^{t,s} = z_i^{t,s} \circ (z_j^{t,s})^{-1}$ for i, $j \in I$ with $U_i \cap U_j \neq \varnothing$. We define

$$X_{ij}^t = \frac{\partial z_{ij}^{t,0}}{\partial t} \circ z_{ji}^{t,0} \frac{\partial}{\partial z_i^{t,0}},\ Y_{ij}^t = \left.\frac{\partial z_{ij}^{t,s}}{\partial s}\right|_{s=0} \circ z_{ji}^{t,0} \frac{\partial}{\partial z_i^{t,0}},\ \dot{Y}_{ij} = \left.\frac{\partial Y_{ij}^t}{\partial t}\right|_{t=0} \frac{\partial}{\partial z_i^{0,0}}.$$

We set $\dot{Y}_{ii} = 0$, $X_{ij} = X_{ij}^0$ and $Y_{ij} = Y_{ij}^0$. Then, it can be shown that $(X, \dot{Y}) \in \mathrm{Ker}(D_1^Y)$.

The short exact sequence (2.1) leads the *Dolbeaut type presentation* $\mathbb{T}_Y[\mathcal{U}] = \mathcal{L}_1/((\overline{\partial} \oplus \overline{\partial})\mathcal{L}_0)$ where

$$\mathcal{L}_1 = \{(\mu, \dot{\nu}) \in \Gamma(M, \mathcal{A}^{0,1}(\Theta_M)) \oplus C^0(\mathcal{U}, \mathcal{A}^{0,1}(\Theta_M) \mid \delta\dot{\nu} + [\mu, Y] = 0\}$$
$$\mathcal{L}_0 = \{(\xi, \eta) \in \Gamma(M, \mathcal{A}^{0,0}(\Theta_M)) \oplus C^0(\mathcal{U}, \mathcal{A}^{0,1}(\Theta_M) \mid \delta\eta + [\xi, Y] = 0\}.$$

4. Differential formulas

The holomorphic cotangent bundle $T^*\mathcal{T}_g$ of the Teichmüller space is presented by the holomorphic bundle $\mathcal{Q}_g \to \mathcal{T}_g$ of holomorphic quadratic differentials via the pairing

$$\langle \mu, q \rangle = \iint_M \mu(z)q(z)dxdy$$

for $\mu \in H^0(\mathcal{U}, \mathcal{A}^{0,1}(\Theta_M))$ and $q \in H^0(M, \Omega_M^{\otimes 2})$ where Ω_M is the sheaf of germs of holomorphic 1-forms on M. The pairing is a holomorphic function on the Whitney product $T\mathcal{T}_g \oplus T^*\mathcal{T}_g = T\mathcal{T}_g \oplus \mathcal{Q}_g$. In [1], Hubbard and Masur formulated the tangent space at $q_0 \in \mathcal{Q}_g$ as the hypercohomology group $H^1(\mathbb{L}_{q_0})$ of the sequence $\mathbb{L}_{q_0}\colon \Theta_M \overset{L.(q_0)}{\to} \Omega_M^{\otimes 2}$, where $L_X(q_0)$ is the Lie derivative of q_0 along X.

With our formulation of the double tangent space, we obtain the (intrinsic) differential formulas for the pairing on $T\mathcal{T}_g \oplus \mathcal{Q}_g$, the L^1-norm $\|\cdot\|_1$ on $\mathcal{Q}_g$, and the Teichmüller metric κ_T on $T\mathcal{T}_g$ as follows:

$$D\langle\cdot,\cdot\rangle|_{(v_0,q_0)}[V,W] = -\frac{1}{2i}\iint_M (dA_i - L_{\xi_i}(d(\eta_i q_i^0)))d\overline{z} \wedge dz$$
$$D\|\cdot\|_1|_{q_0}[W] = \frac{1}{4i}\iint_M \frac{\overline{q_i^0}}{|q_i^0|}\left(\varphi_i - L_{\xi_i}(q_i^0)\right) d\overline{z} \wedge dz$$
$$D\kappa_T|_{v_0}[V] = \frac{1}{4i}\iint_M (\dot{\nu}_i Q_i + \kappa_T(v_0)L_{\xi_i}(|Q_i|))\, d\overline{z} \wedge dz,$$

where

- $v_0 = [Y] \in H^1(M, \Theta_M)$ is a tangent vector, $Q = \{Q_i\}_{i\in I} \in H^0(M, \Omega_M^{\otimes 2})$ defines the corresponding Teichmüller Beltrami differential to v_0, and $q_0 = \{q_i^0\}_{i\in I} \in H^0(M, \Omega_M^{\otimes 2})$;
- $V = [\![X, \dot{Y}]\!]_Y \in \mathbb{T}_Y[\mathcal{U}] \cong T_{v_0}T\mathcal{T}_g$ and $W = [X, \varphi] \in H^1(\mathbb{L}_{q_0}) \cong T_{q_0}T^*\mathcal{T}_g$;
- $\dot{\nu} = \{\dot{\nu}_i\}_{i\in I}$ is the second coordinate of the Dolbeaut presentation of $V = [\![X, \dot{Y}]\!]_Y$;

- $\xi \in C^0(\mathcal{U}, \mathcal{A}^{0,0}(\Theta_M))$ and $\eta \in C^0(\mathcal{U}, \mathcal{A}^{0,0}(\Theta_M))$ with $\delta\xi = X$ and $\delta\eta = Y$;
- $\eta_i q_i^0 = \eta_i(z) q_i^0(z) dz$; and
- $A_i = A_i(z)dz$ satisfies

$$A_i(z_{ij}^0(z))\frac{dz_{ij}^0}{dz}(z) - A_j(z) = \dot{Y}_{ji}(z)q_j^0(z) + Y_{ji}(z)\varphi_j(z) + L_{X_{ji}}(\eta_i q_i^0)(z)$$

for $z \in z_j^0(U_i \cap U_j)$ and $i,\, j \in I$, where $\varphi = \{\varphi_i\}_{i\in I}$.

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Teichmüller Theory: Classical, Higher, Super and Quantum

Paul Norbury

1. A new finite measure on the moduli space of curves

The main aim of this talk is to define a new finite measure on the moduli space of oriented hyperbolic surfaces. The measure is defined using spin structures via a procedure analogous to the definition of the Weil-Petersson metric. In fact, it arises naturally out of the super Weil-Petersson metric defined over the moduli space of super curves. We summarise the construction of the measure here:

- construct a natural Euler form $e(E)$ of a vector bundle E defined over the moduli space of spin curves;
- together with the Weil-Petersson form, $e(E)\exp\omega^{WP}$ defines a finite measure which pushes forward to a measure on the moduli space of curves.

The total measure can be identified with the volume of the moduli space of super hyperbolic surfaces. It can be calculated in many examples, and conjecturally satisfies a recursion analogous to Mirzakhani's recursion for Weil-Petersson volumes of moduli spaces of hyperbolic surfaces. This conjecture has been verified in many cases, including the Neveu-Schwarz case where it coincides with the recursion of Stanford and Witten [3]. The Ramond case produces deformations of the Neveu-Schwarz volume polynomials, again satisfying Mirzakhani-like recursion relations.

2. The moduli space of spin hyperbolic surfaces

Define the moduli space of oriented, spin, hyperbolic surfaces with fixed length geodesic boundary components by

$$\mathcal{M}^{\text{spin}}_{g,n}(L_1,...,L_n)=\Big\{(\Sigma,\beta_1,...,\beta_n)\mid\Sigma \text{ genus g, oriented, spin, hyperbolic surface,}$$
$$\text{geodesic boundary } \partial\Sigma=\sqcup\beta_i, L_i=\ell(\beta_i)\Big\}/\sim$$

where the quotient is by isometries preserving each β_i. An oriented, spin, hyperbolic surface is defined by a hyperbolic representation

$$\rho:\pi_1\Sigma\xrightarrow{\text{hyp.}}SL(2,\mathbb{R})$$

which determines a quotient $\Sigma\simeq\mathbb{H}/\rho(\pi_1\Sigma)$ and an associated flat real rank 2 vector bundle $T^{\frac{1}{2}}_{\Sigma}\to\Sigma$ which we identify with the associated locally constant sheaf. A boundary component $\beta\subset\partial\Sigma$ is defined to be

- Neveu-Schwarz if $\operatorname{tr}(\rho(\beta))<0$
- Ramond if $\operatorname{tr}(\rho(\beta))>0$.

Homological considerations imply that the number of Ramond boundary components is even, which decomposes the moduli space into 2^{n-1} components

$$\mathcal{M}^{\text{spin}}_{g,n}=\bigsqcup_{\sigma\in\{0,1\}^n}\mathcal{M}^{\text{spin}}_{g,\sigma}$$

for $\sigma=(\sigma_1,...,\sigma_n)$, where $\sigma_i=1$ corresponds to a Neveu-Schwarz boundary component β_i and $\sigma_i=0$ corresponds to a Ramond boundary component. Each component $\mathcal{M}^{\text{spin}}_{g,\sigma}$ is connected except for the case of $\sigma=(1^n)$ which decomposes further into two connected components labeled odd and even.

3. A vector bundle equipped with a natural Euler form

Define a bundle with fibre $E_{g,n}|_{\Sigma}=H^1_{dR}(\Sigma,T^{\frac{1}{2}}_{\Sigma})$, the sheaf cohomology of the locally constant sheaf of sections of $T^{\frac{1}{2}}_{\Sigma}$, which can be calculated via a twisted de Rham cohomology.

$$\begin{array}{c}E_{g,n}\hookleftarrow H^1_{dR}(\Sigma,T^{\frac{1}{2}}_{\Sigma})\\ \downarrow\\ \mathcal{M}^{\text{spin}}_{g,\sigma}\end{array}$$

The moduli space of spin, hyperbolic surfaces is the reduced space of the moduli space of super hyperbolic surfaces, and $E_{g,n}$ arises as the normal bundle of the reduced space inside the super space

$$\mathcal{M}^{\text{spin}}_{g,n}(L_1,...,L_n)\to\widehat{\mathcal{M}}_{g,n}(L_1,...,L_n).$$

The Euler form is defined via a natural holomorphic structure and hermitian metric on $E_{g,n}$ which gives rise to a Chern connection. The bundle $E_{g,n}$ has a holomorphic structure due to a natural isomorphism

$$H^1_{dR}(\Sigma,T^{\frac{1}{2}}_{\Sigma})\cong H^1(\mathcal{C},\theta^\vee)^\vee.$$

where $\mathcal{C}$ is a compact curve related to Σ by $\Sigma = \mathcal{C} - D$ for $D = \{p_1, ..., p_n\} \in \mathcal{C}$. The sheaf $\theta^{\otimes 2} \cong \omega_{\mathcal{C}}^{\log}$ is a spin structure corresponding to the spin structure on Σ.

The Hermitian metric on $E_{g,n}$ arises similarly to the Weil-Petersson metric so we first recall that definition. For $[C] \in \mathcal{M}_{g,n}$,

$$T_{[C]}\mathcal{M}_{g,n} \cong H^1(C, T_C(-D)) \cong H^0(C, K_C^{\otimes 2}(D))^\vee$$

and the Weil-Petersson metric is defined on the cotangent bundle by

$$\langle \eta_1, \eta_2 \rangle := \int_\Sigma \frac{\overline{\eta_1}\eta_2}{h}, \qquad \eta_1, \eta_2 \in H^0(C, K_C^2(D)) \tag{3.1}$$

where h is the hyperbolic metric. Analogously, for $E_{g,n}$,

$$H^1_{dR}(\Sigma, T_\Sigma^{\frac{1}{2}}) \cong H^1(C, T_C^{\frac{1}{2}}(-D))^\vee \cong H^0(\overline{\Sigma}, K_C^{3/2}(D))$$

where $T_C^{\frac{1}{2}}(-D)$ is essentially $\theta^\vee$. (More precisely $T_C^{\frac{1}{2}}(-D)$ is the push-forward of $\theta^\vee$ from a twisted curve to its underlying coarse curve—see [2].) The 3/2 differentials give the analogue of holomorphic quadratic differentials used to define the Weil-Petersson metric. Define a Hermitian metric on $E_{g,n}$ by

$$\langle \eta, \xi \rangle := \int_\Sigma \frac{\overline{\eta}\xi}{\sqrt{h}}, \qquad \eta, \xi \in H^0(\overline{\Sigma}, K_{\overline{\Sigma}}^{3/2}(D)) \tag{3.2}$$

where $\sqrt{h}$ is the hyperbolic metric on the spin bundle $T_\Sigma^{\frac{1}{2}}$. This hermitian metric and the holomorphic structure on the rank N vector bundle $E_{g,n}$ defines a unique Chern connection A. With this we define an Euler form and measure

$$\mu = e(E_{g,n}) \exp \omega^{WP}, \qquad e(E_{g,n}) := \left(\frac{1}{4\pi}\right)^N \mathrm{pf}(F_A). \tag{3.3}$$

The push-forward under the forgetful map $\mathcal{M}_{g,\sigma}^{\mathrm{spin}} \to \mathcal{M}_{g,n}$ defines a measure μ_σ on $\mathcal{M}_{g,n}$ for each $\sigma \in \{0,1\}^n$.

The proof that (3.2) is well-defined requires convergence of the integral if Σ is non-compact. The proof that this defines a *finite* measure uses an extension of the Hermitian metric to the compactification by stable spin curves, which requires convergence of the same integral in (3.2), proven as follows. Consider a local coordinate z with $z = 0$ corresponding to a point of D and a cusp of the metric. Locally, the hyperbolic metric is given by $h = \frac{|dz|^2}{|z|^2(\log|z|)^2}$ and the 3/2 differentials are given by $\eta = \frac{f(z)dz^{3/2}}{z}$ and $\xi = \frac{g(z)dz^{3/2}}{z}$ where $f(z)$ and $g(z)$ are holomorphic at $z = 0$. The local contribution to the metric $\int_{|z|<\varepsilon} \frac{\overline{f}g \log|z||dz|^2}{|z|}$ exists since

$$\int_{|z|<\varepsilon} \frac{|\log|z||}{|z|}|dz|^2 = \int_0^\varepsilon |\log r| dr d\theta = 2\pi|\varepsilon \log \varepsilon - \varepsilon| < 2\pi \quad \Leftarrow \quad \varepsilon < 1. \tag{3.4}$$

Note that similar issues arise in the definition of the Weil-Petersson metric, however in this case the convergence of (3.1) holds whereas the extension to the Deligne-Mumford compactification does not.

4. Volume polynomials

A powerful method pioneered by Mirzakhani [1] to study Weil-Petersson volumes is to consider a family of symplectic deformations $\omega(L_1, ..., L_n)$ of the Weil-Petersson form for $(L_1, ..., L_n) \in \mathbb{R}^n_{\geq 0}$ where $\omega^{WP} = \omega(0, ..., 0)$, and corresponding volumes $V_{g,n}(L_1, ..., L_n) := \int_{\mathcal{M}_{g,n}} \exp \omega(L_1, ..., L_n)$. We use the same deformation of the Weil-Petersson form to study the measure defined in (3.3). Define

$$\widehat{V}_{g,n}(s, L_1, ..., L_n) := \sum_{m=0}^{\infty} \frac{s^m}{m!} \int_{\mathcal{M}^{\text{spin}}_{g,(1^n,0^m)}} e(E^\vee_{g,(1^n,0^m)}) \exp \omega(L_1, ..., L_n, 0^m)$$

which deforms the symplectic form only at the Neveu-Schwarz points, and allows an arbitrary number, or gas, of Ramond punctures.

Conjecture 1.

$$L_1 \widehat{V}_{g,n}(s, \vec{L}) = \frac{1}{2} \int_0^\infty \int_0^\infty xy \widehat{D}(L_1, x, y) P(x, y) dx dy + \sum_{i=2}^{n} \int_0^\infty x \widehat{R}(L_1, L_i, x) Q(x) dx$$

$$P(x,y) = \widehat{V}_{g-1,n+1}(s,x,y,L_2,..,L_n) + \sum_{\substack{g_1+g_2=g \\ I \sqcup J = \{2,..,n\}}} \widehat{V}_{g_1,|I|+1}(s,x,L_I) \widehat{V}_{g_2,|J|+1}(s,y,L_J)$$

$$Q(x) = \widehat{V}_{g,n-1}(s,x,L_2,..,\hat{L}_i,..L_n)$$

The conjecture is proven up to $O(s^4)$, and in particular the $s = 0$, or Neveu-Schwarz, case is derived via supergeometry by Stanford and Witten [3] and proven rigorously in [2]. The $(0,1)$ disk case, which needs an extra term, is also proven:

$$\widehat{V}_{0,1}(s, L) = \frac{s^2}{2!} + \frac{1}{2L} \int_0^\infty \int_0^\infty xy D(L, x, y) \widehat{V}_{0,1}(s, x) \widehat{V}_{0,1}(s, y) dx dy.$$

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Maximal submanifolds of pseudo-hyperbolic space and their applications

Andrea Seppi

(joint work with Graham Smith and Jérémy Toulisse)

Motivated by the asymptotic Plateau problem in the hyperbolic three-space, which is the question of existence of a minimal surface having a prescribed Jordan curve as asymptotic boundary, in this talk we discussed a similar problem in the pseudo-hyperbolic space $\mathbb{H}^{p,q}$ of any signature (p, q). This is the space of unoriented negative lines in the pseudo-Euclidean space $\mathbb{R}^{p,q+1}$, and its boundary at infinity $\partial_\infty \mathbb{H}^{p,q}$ is thus the space of unoriented isotropic lines. The main result of [11] is the following:

Theorem 1. *Given any non-negative $(p-1)$-sphere Λ in $\partial_\infty \mathbb{H}^{p,q}$, there exists a unique complete maximal submanifold Σ in $\mathbb{H}^{p,q}$ such that $\partial_\infty \Sigma = \Lambda$.*

To explain the statement, a subset Λ of $\partial_\infty \mathbb{H}^{p,q}$ is non-negative (resp. positive) if, for any triple of pairwise distinct points x, y, z of Λ, their span in $\mathbb{R}^{p,q+1}$ does not contain any negative definite 2-plane (resp. has signature $(2,1)$). A submanifold of $\mathbb{H}^{p,q}$ is complete if its first fundamental form is a complete Riemannian metric, and it is maximal if its second fundamental form has vanishing trace. Theorem 1 is a generalization of the results obtained in [2] and [5] for $q = 1$, and in [6] and [10] for $p = 2$.

A key feature of this result is that we consider submanifolds of dimension p, namely having the maximal possible dimension among spacelike submanifolds. For spacelike submanifolds of dimension lower than p — although no general results of existence were available at the time of the talk — uniqueness certainly does not hold, as one easily sees by considering Jordan curves in $\partial_\infty \mathbb{H}^3$ spanning several minimal surfaces (from [1] or [9]), and embedding $\mathbb{H}^3$ as a totally geodesic submanifold of $\mathbb{H}^{p,q}$ for $p \geq 3$.

We have briefly discussed the strategy of the proof of the existence part of Theorem 1, which consists in a continuity method, via a deformation from Λ to the boundary of a totally geodesic copy of $\mathbb{H}^p$. While the most difficult part of the proof consists in achieving the openness of the set of solutions, the closedness follows from the following "dichotomy" that completely describes the non-compactness of the space of complete maximal submanifolds.

Proposition 1. *Let $\{\Sigma_n\}_{n\in\mathbb{N}}$ be a sequence of complete maximal p-dimensional submanifolds of $\mathbb{H}^{p,q}$. Up to subsequences, either Σ_n converges smoothly on compact sets to a complete maximal submanifold, or Σ_n converges in the Hausdorff topology to a Lipschitz submanifold which is foliated by lightlike geodesics.*

In the second part of the talk, we have discussed applications to the study of Anosov representations. In the following, let Γ be a word hyperbolic group whose Gromov boundary is homeomorphic to S^{p-1}. Following [7], a representation ρ of Γ in $\mathrm{PO}(p, q+1)$, which is the isometry group of $\mathbb{H}^{p,q}$, is positive P_1-Anosov (i.e., roughly speaking, it admits a proximal limit set Λ_ρ which is a positive sphere) if and only if it has finite kernel and $\rho(\Gamma)$ acts convex cocompactly on $\mathbb{H}^{p,q}$ (i.e. there exists a convex region, which can be taken to be the convex hull of Λ_ρ, on which $\rho(\Gamma)$ acts properly discontinuosly and cocompactly). An immediate consequence of Theorem 1 is:

Corollary 1. *If $\rho : \Gamma \to \mathrm{PO}(p, q+1)$ is a positive P_1-Anosov representation, then $\rho(\Gamma)$ preserves a unique complete maximal submanifold Σ in $\mathbb{H}^{p,q}$, and the action of $\rho(\Gamma)$ on Σ is properly discontinuous and cocompact.*

Corollary 1 has been an important ingredient in [4], which exhibited new examples of "higher higher Teichmüller spaces", that is, connected components of the space of representations of Γ into a Lie group G which consist entirely of discrete and faithful representations. Indeed, [4] proves that the space of positive P_1-Anosov representations is a union of connected components in the space of representations of Γ in $\mathrm{PO}(p, q+1)$. One key idea to achieve this result is to consider a sequence $\{\rho_n\}_{n\in\mathbb{N}}$ of positive P_1-Anosov representations converging to some representation ρ_∞. By Corollary 1, each ρ_n preserves a complete maximal submanifold Σ_n, and by the dichotomy of Proposition 1, ρ_∞ preserves the limit Σ_∞, which however might a priori be degenerate in the sense of Proposition 1. Then [4] proves a converse of Corollary 1, even in case the invariant submanifold is degenerate in the sense of Proposition 1, thus showing that ρ_∞ is positive P_1-Anosov. A posteriori, combining [4] and [11], Corollary 1 is actually an "if and only if".

Corollary 1 permits to make some partial progress on the question of which hyperbolic groups Γ as above do admit Anosov representations.

Corollary 2. *If Γ is torsion-free and admits a positive P_1-Anosov representation $\rho : \Gamma \to \mathrm{PO}(p, q+1)$, then Γ is isomorphic to the fundamental group of a closed smooth p-dimensional manifold M whose universal cover is diffeomorphic to $\mathbb{R}^p$.*

Observe that, from [3], when $p \geq 6$ any torsion-free hyperbolic group Γ with Gromov boundary homeomorphic to S^{p-1} is isomorphic to the fundamental group of a closed topological p-dimensional manifold M whose universal cover is homeomorphic to $\mathbb{R}^p$. However, for $p = 4k$, $k \geq 2$, there are examples where M cannot be made a smooth manifold as in Corollary 2. The following result follows.

Corollary 3. *For every $k \geq 2$, there exists a torsion-free hyperbolic group Γ with Gromov boundary homeomorphic to S^{4k-1} which does not admit any positive P_1-Anosov representation into $\mathrm{PO}(4k, q+1)$.*

Based on the questions during and after the talk, and the feedbacks from the participants, it appears that the existence of new examples of higher higher Teichmüller spaces has been accepted as a very interesting phenomenon, opening up many questions in higher Teichmüller theory. There has been also a rich discussion, and interest, around Corollaries 2 and 3 and related questions. Finally, we have briefly mentioned another application to the topology of the quotient of Guichard-Wienhard's domain of discontinuity Ω_ρ, introduced in [8], in the space of maximally isotopic subspaces of $\mathbb{R}^{p,q+1}$. Indeed, it can be proved that $\Omega_\rho/\rho(\Gamma)$ has a fiber bundle structure over the manifold M constructed in Corollary 2.

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Super-representations of 3-manifolds and torsion polynomials

Stavros Garoufalidis

Torsion polynomials connect the genus of a hyperbolic knot (a topological invariant) with the discrete faithful representation (a geometric invariant). Using a new combinatorial structure of an ideal triangulation of a 3-manifold that involves edges as well as faces, we associate a polynomial to a cusped hyperbolic manifold that conjecturally agrees with the $\mathbb{C}^2$-torsion polynomial, which conjecturally detects the genus of the knot. The new combinatorics is motivated by super-geometry in dimension 3, and more precisely by super-Ptolemy assignments of ideally triangulated 3-manifolds and their $\mathrm{OSp}_{2|1}(\mathbb{C})$-representations. Joint work with Seokbeom Yoon which has appeared in four parts:

- *Twisted Neumann–Zagier matrices*, (with S. Yoon),
  preprint 2021, arXiv:2109.00379
- *1-loop equals torsion equals for fibered 3-manifolds*, (with N. Dunfield and S. Yoon),
  preprint 2023, 2304.00469.
- *Super-representations of 3-manifolds and torsion polynomials*, (with S. Yoon),
  preprint 2022, 2301.11018.
- *Asymptotically multiplicative quantum invariants*, (with S. Yoon),
  preprint 2022, 2211.00270.

Extensive computer code to test the 1-loop equals torsion conjecture for the about 70000 cusped hyperbolic manifolds in the `SnapPy` census was written by Nathan Dunfield, and this included testing the conjecture (which is proven for fibered cusped hyperbolic manifolds) for the census manifold `t07035` made out of 8 tetrahedra, whose invariant trace field is degree 90, which is fibered with a genus 122 surface and whose $\mathbb{C}^2$ and $\mathbb{C}^3$-torsion polynomials have degrees 486 and 729 and coefficients in the trace field.

Asymmetric intersection number and Fock–Goncharov duality

Zhe Sun

(joint work with Linhui Shen, Daping Weng)

I will talk about the Fock–Goncharov duality conjecture and webs on surfaces.

In [FG06], Fock and Goncharov introduce a pair of mutually dual moduli spaces $(\mathcal{X}_{G,\hat{S}}, \mathcal{A}_{G^L,\hat{S}})$, which generalizes the enhanced Teichmüller space and the decorated Teichmüller space [Pen87]. In *loc. cit.*, Fock and Goncharov conjecture that the ring of regular functions on one space has a canonical linear basis parameterized by the tropical points of the dual space, and the parametrization is equivariant under the action of the cluster modular group. They prove the conjecture for $G = \mathrm{SL}_2$ by relating Thurston's transversely integer measured laminations [Thu79] to the tropical $\mathcal{A}$ coordinates using the topological intersection numbers with the ideal edges of the ideal triangulation, and to the trace functions of the laminations. As pointed out by Goncharov and Shen [GS15], when the surface $\hat{S}$ has marked points on its boundary, the space $\mathcal{X}_{G,\hat{S}}$ should be replaced by $\mathcal{P}_{G,\hat{S}}$, an enhancement of $\mathcal{X}_{G,\hat{S}}$. It is further conjectured in *loc.cit.* that when the tropical points of $\mathcal{A}_{G^L,\hat{S}}$ are cut out by the condition that the tropical potential function being non-negative, the dual space should be changed into the character variety $\mathcal{L}_{G,\hat{S}}$, and the modified duality is proven for $G = \mathrm{SL}_2$ in *loc. cit.*. By a different approach using scattering diagrams and broken lines to construct theta bases, Gross, Hacking, Keel and Kontsevich [GHKK18] proved the duality conjecture under certain conditions where the conditions are checked for most of our cases [GS18, GS19]. But their theta bases are hard to compute explicitly. Recently, Mandel and Qin [MQ23] showed that for PGL_2, the theta bases and the Fock–Goncharov's bracelet bases are the same. We are eager to see the generalization of Fock–Goncharov's orignial approach to the duality for general Lie groups.

The SL_3-*web* is the collection of $\mathbb{Z}_{>0}$-weighted oriented arcs, $\mathbb{Z}_{>0}$-weighted oriented loops and oriented 3-valent graphs (1-valent on the boundary) with only sinks (all pointing inward) and sources (all pointing outward). By the condition that we have only sinks and sources, the interior faces of the web have only $2k$-gons. The SL_3-web is *reduced* if it is a disjoint collection without any self-intersection, and there is no 0-faces, 2-faces, 4-faces in the interior, and at least d-gons for $d \geq 5$ along the boundary. Kuperberg [K96] introduces these reduced webs to construct natural bases of the tensor invariant spaces of irreducible representations of SL_3. Motivated by Fock–Goncharov's original approach, the duality was established for $G = \mathrm{SL}_3$ by using web bases as follows. Let $\mathcal{W}^{\mathcal{A}}_{3,\hat{S}}$ be the collection of reduced $(\mathrm{SL}_3, \mathcal{A})$-webs which ends at boundary intervals. Following the confluence theory used by Kuperburg [K96] for the polygon cases, Sikora and Westbury [SW07] showed the trace functions of the reduced $(\mathrm{SL}_3, \mathcal{A})$-webs provide a linear basis of the ring $\mathcal{O}(\mathcal{L}_{\mathrm{SL}_3,\hat{S}})$ of regular functions of SL_3 character variety. But for the rank ≥ 3 Lie group cases, the confluence property fails, which means that we have more than two webs for a given tropical point.

Let $\mathcal{A}^+_{\mathrm{PGL}_3,\hat{S}}(\mathbb{Z}^t)$ be the cone of nonnegative tropical points introduced by Goncharov and Shen. Douglas and Sun [DS20a, DS20b] constructed a mapping class group equivariant bijection[1]

$$\Phi: \mathcal{W}^{\mathcal{A}}_{3,\hat{S}} \xrightarrow{\sim} \mathcal{A}^+_{\mathrm{PGL}_3,\hat{S}}(\mathbb{Z}^t).$$

Therefore the tropical set $\mathcal{A}^+_{\mathrm{PGL}_3,\hat{S}}(\mathbb{Z}^t)$ parameterizes a linear basis of $\mathcal{O}(\mathcal{L}_{\mathrm{SL}_3,\hat{S}})$. This parametrization can be extended to a canonical bijection between a basis of $\mathcal{O}(\mathcal{P}_{\mathrm{PGL}_3,\hat{S}})$ and $\mathcal{A}_{\mathrm{PGL}_3,\hat{S}}(\mathbb{Z}^t)$ by adding $\mathbb{Z}$-weighted loops around the punctures and $\mathbb{Z}$-weighted arcs around the boundary corners. When $\hat{S}$ is a punctured surface without marked points, Kim [Kim20] showed that the highest degree of the above trace functions are the tropical coordinates of $\mathcal{A}_{\mathrm{PGL}_3,\hat{S}}(\mathbb{Z}^t)$. However, till here, the generalization of the intersection number in Fock–Goncharov's original approach for SL_2 is still missing.

Joint with Linhui Shen and Daping Weng, we introduce the asymmetric geometric intersection numbers between ordered pairs of the SL_3-webs. Using that, we prove the mutation equivariant property of the bijection Φ between the reduced SL_3-webs and $\mathcal{A}^+_{\mathrm{PGL}_3,\hat{S}}(\mathbb{Z}^t)$.

Definition 1 (Intersection number)**.** *For any* SL_3*-webs* W, V *on* S *intersecting transversely, suppose* p *lies on the oriented edge* $e_p(W)$ *of* W *and* $e_p(V)$ *of* V*. We define the* intersection number of the ordered pair (W,V) at p

$$\varepsilon_p(W,V) := \begin{cases} 1 & \text{if } e_p(W) \text{ crossing } e_p(V) \text{ towards the right side of } e_p(V); \\ 2 & \text{if } e_p(W) \text{ crossing } e_p(V) \text{ towards the left side of } e_p(V). \end{cases}$$

Then the intersection number of (W,V)

$$i(W,V) := \sum_p \varepsilon_p(W,V).$$

The intersection number of $([W],[V])$

$$i([W],[V]) := \inf_{w\in[W],v\in[V]} \{i(w,v)\}$$

among all the w*,* v *in the homotopy classes of* W*,* V *respectively such that they intersect each other transversely.*

For any seed $\mathbf{s}$ of the moduli space $\mathcal{A}_{\mathrm{SL}_3,\hat{S}}$, there is a family of $(\mathrm{SL}_3,\mathcal{X})$-webs ending at marked points and punctures, corresponding to cluster variables in the seed $\mathbf{s}$. We define $i_{\mathbf{s}}$ for any reduced $(\mathrm{SL}_3,\mathcal{A})$-web W to be the intersection numbers between W and this particular family of $(\mathrm{SL}_3,\mathcal{X})$-webs. Then we prove that

Theorem 1. [SSW] *For the seed* $\mathbf{s}$ *associated with an arbitrary ideal triangulation* $\mathcal{T}$ *of* $\hat{S}$*, we have* $i_{\mathbf{s}} = \Phi_{\mathcal{T}}$*. For two seeds* $\mathbf{s}$ *and* $\mathbf{s}'$ *related by any mutation* μ *in the flip from the ideal triangulation* $\mathcal{T}$ *to* $\mathcal{T}'$ *of the square* $\square$*, we have*

$$\mu \circ i_{\mathbf{s}} = i_{\mathbf{s}'}.$$

[1] Frohman and Sikora [FS22] provided a different topological integer coordinate system for the reduced SL_3-webs depending on the ideal triangulation chosen.

The fact that each mutation changes one tropical $\mathcal{A}$ coordinate into another is understood as replacing the intersection number with one $(\mathrm{SL}_3, \mathcal{X})$-web V by the intersection number with another $(\mathrm{SL}_3, \mathcal{X})$-web V'. The proof of mutation equivariant is to interprete the tropical mutation formula as the tropicalized SL_3-skein relation for V intersecting V'.

Further research. For the Sp_4 case, joint with Tsukasa Ishibashi and Wataru Yuasa (in progress) [ISY], we construct the intersection number coordinates for the crossroad webs [IY22].

In [FP16], Fomin and Pylyavskyy conjectured that all the cluster variables could be represented by the webs. If their conjecture is true, we believe that our mutation equivariance property should be true for any mutation between any seeds.

It is interesting to see the positivity of the structure constants for the reduced SL_3-web basis. Further more, if the structure constants are positive, it is interesting to see their log-concave property conjectured by Okounkov [Ok03] in a more general setting.

We also would like to investigate the relation among webs, k-differentials, the possible higher geodesic measured laminations where the webs are the integer points inside and the compactification of the higher Teichmüller spaces in the future.

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Timelike Convex Geometry

Sumio Yamada

(joint work with Athanase Papadopoulos)

This topic concerns a Finsler generalization of the de Sitter space. Thurston has called the de Sitter space as "exterior hyperbolic space" and our idea is to exploit further the "exterior" aspect of the Lorentian geometry, and to introduce a new convex geometry in the context of projective space. In particular, we introduce a new class of metric spaces (X, d) which satisfies the timelike inequality, which is the reverse triangle inequality

$$d(p, r) \geq d(p, q) + d(q, r),$$

a notion first axiomatized by H. Busemann in 1967 in [3]. The logarithm of cross ratios defined in the exterior of a pair of convex sets on the standard sphere S^n has been shown in [1] to satisfy the timelike inequality, due to its variational formulation.

Among the examples of the timelike spaces equipped with Hilbert metrics constructed using the cross ratio, an important subclass is when the two convex sets (the past set Ω_1 and the future set Ω_2) are antipodal, i,e. $\Omega_2 = -\Omega_1 := \tilde{\Omega}_1$. In that case, a great circle tangential to one of the convex sets is automatically tangential to the other, making the cross ratio to become one, consequently making the distance between any pair of points on the great circle becomes zero. Infinitesimally each tangent vector to the circle is lightlike, a situation reminiscent of the the de Sitter space. Indeed the de Sitter space can be realized when the past and future convex sets $\Omega, \tilde{\Omega}$ are antipodal discs of the sphere S^n. As the projective transformations of the ambient space $\mathbb{R}^{n+1}$ preserves cross ratios, the projective transformations leaving the convex sets are isometries of the timelike geometry. Hence we call this construction of timelike metric space $(S^n \setminus (\Omega \cup \tilde{\Omega}))$ "generalized de Sitter spaces" in analogy with the situation where the Hilbert geometry in the interior of a convex set can be regarded as " generalized hyperbolic space." Recall that historically the first model of a hyperbolic plane was the Klein-Beltrami model, which is nothing but the Hilbert geometry of the unit disc in $\mathbb{R}^2$.

We have constructed in [2] a timelike space based on the 2-dimensional spherical simplex

$$\Delta_2 = \{(x_1, x_2, x_3) \,|\, x_1^2 + x_2^2 + x_3^2 = 1, \text{ and } x_i > 0 \ (i = 1, 2, 3)\}.$$

and its antipodal set $\tilde{\Delta}_2$ in S^2. The space $(S^2 \setminus (\Delta_2 \cup \tilde{\Delta}_2))$ is isometric to a union of six normed spaces, each corresponding to one of the 6 spherical orthants, with an abelian subgroup of the isometry group, induced by the affine map

$$(x, y) \mapsto (\lambda_1 x, \lambda_2 y) \qquad (\lambda_i > 0)$$

of $\mathbb{R}^2$, which corresponds to the hyperplane $\{x_i = i\}$. It also have a discrete isometric action of $\mathbb{Z}_2 \times \mathbb{Z}_3$, where $\mathbb{Z}_2$ is the antipodal map $p \mapsto -p$ in $\mathbb{R}^3$ and $\mathbb{Z}_3$ is generated by the rotation around the central axis $(x_1, x_2, x_3) \mapsto (x_2, x_3, x_1)$.

This construction in [2] is not special to the dimension 2. Hence the n-dimensional spherical disc and n-dimensional spherical simplex are the two extreme cases of the generalized de Sitter spaces, and there are many in between them interpolating the two spaces. In the classical Hilbert geometry, namely the Hilbert metric defined by the cross ration *inside* a convex set, this interpolation was important in describing the moduli space of projective structures on Riemann surfaces as in the work of W. Goldman [4]. The subject is now called "divisible convex sets." We believe that our construction of the generalized de Sitter spaces merits an analogous attention in the coming years.

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