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Many-Body Quantum Systems

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Abstract. This workshop brought together experts on the analysis of quantum many-body problems and quantum statistical mechanics, with the goal of discussing the state-of-the-art of the field, recent developments as well as challenges for the future. The main topics of discussion concerned the equilibrium and dynamical behavior of (bosonic or fermionic) quantum gases, quantum spin systems, as well as quantum field theory models like the Nelson or Fröhlich model.

Mathematics Subject Classification (2020): 82B10, 82C10, 81Q05, 81Q10, 35Q40, 35Q20, 35Q41, 35Q55, 35Q56.

Introduction by the Organizers

The workshop *Many-Body Quantum Systems*, organized by Christian Hainzl (LMU Munich), Benjamin Schlein (University of Zurich), Robert Seiringer (IST Austria) and Simone Warzel (TU Munich) was attended, at maximal capacity, by 48 participants coming from Europe, North America and Asia. During the workshop we had 21 lectures, covering a large variety of subjects in mathematical physics, with the main focus being on quantum many-body systems, quantum spin systems and models in (non-relativistic) quantum field theory. In addition, we had an after-dinner lecture by Jan Philip Solovej on open problems in the analysis of quantum gases and large atoms.

1. Equilibrium properties of quantum gases

The last few years have seen tremendous progress in the mathematical understanding of the low-energy properties of interacting Bose and Fermi gases. In this line of research, Arnaud Triay presented a rigorous lower bound for the free energy of a dilute Bose gas in the thermodynamic limit at low temperature, matching the celebrated Lee–Huang–Yang formula predicted by physicists more than 65 years ago. An upper bound on the free energy in the Gross–Pitaevskii scaling regime was the topic of the talk by Andreas Deuchert. Giulia Basti's talk concerned an upper bound on the ground state energy at zero temperature that is valid even in the singular case of hard-sphere interactions. Manfred Salmhofer discussed a different approach to the analysis of interacting Bose gases and the emergence of Bose–Einstein condensation, based on a (coherent state) functional integral approach. Fermions at high density were the focus of the talk of Martin Christiansen, presenting a proof of the validity of the Gellmann–Brueckner formula in a suitable (semiclassical mean-field) limit.

2. Time-evolution of large quantum systems

Also the time-evolution of large quantum systems, and the derivation of approximating effective equations, have been closely investigated by the mathematical physics community in recent years. Along this line of research we had four talks at the workshop in Oberwolfach. Cristina Caraci discussed her recent work on quantum fluctuations around the Gross–Pitaevskii equation for the dynamics of dilute Bose gases. Nicolas Rougerie's talk concerned the semiclassical limit of large fermionic systems, and the emergence of a classical description via Vlasov dynamics, in strong magnetic fields. The adiabatic evolution of quantum many-body systems at low temperature was the focus of Marcello Porta's talk. Finally, Marius Lemm reported on bounds on the speed of propagation of bosonic lattice systems, extending the celebrated Lieb–Robinson bounds to systems with infinite-dimensional local Hilbert spaces.

3. Quantum spin systems

Many of the essential properties of many-body quantum systems are being captured by quantum spin systems, which are easier to handle analytically than continuum models. For this reason, they attracted a lot of attention in the last years. Daniel Ueltschi discussed recent progress on the existence of phase transitions in the random interchange model and their possible relation to problems in quantum spin systems. Alessandro Pizzo presented a new method to investigate the spectral gap above the ground state energy in quantum chains, and applied it to the XXZ model in a magnetic field. Chokri Manai explained in his talk the analysis of the low-energy spectrum of quantum spin systems with interactions of mean-field type.

4. Polarons and similar models in quantum field theory

The Fröhlich polaron model is an example of a simple quantum field theory, where a small system (the electron) is coupled to an infinite system (a quantum field). Other models of this kind include the Nelson and Pauli–Fierz models, for instance, and investigations on all these models were discussed during the workshop. Morris Brooks considered the strong coupling limit of the Fröhlich polaron model, deriving a two-term asymptotic expansion of its energy–momentum relation, showing in particular the emergence of the celebrated Landau–Pekar formula for its effective mass. The scattering theory of the massless Nelson model was the topic of Gian Michele Graf's talk. Finally, Marcel Griesemer discussed pointwise bounds on eigenstates of the Nelson and related models.

5. Effective theories

In the physics literature, complex many-body quantum systems are often described through simplified effective theories. One of the main goals of mathematical statistical mechanics is the rigorous justification of these theories, starting from fundamental, microscopic descriptions of many-body systems. Another important goal for mathematical physicists is the study of the mathematical properties of effective theories. Along this line of research, Mathieu Lewin discussed mathematical properties of the Gross–Pitaevskii equation in infinite space, and conditions on the interaction potential guaranteeing the existence of a phase transition in this model. The Bardeen–Cooper–Schrieffer model of superconductivity was the topic of Barbara Roos's talk, focusing on the effect of system boundaries on the critical temperature, thereby confirming recent predictions in the physics literature based on a numerical simulation of the model.

6. Other

In addition the the talks and topics discussed above, we had a talk by Sven Bachmann explaining how to use the landscape function to count the number of eigenvalues of Schrödinger operators, and a talk by David Gontier on the spectral analysis of Dirac operators. Stefan Teufel reported on properties of subsystems of typical quantum states, and Wojciech de Roeck presented an analysis of the long-time behavior of disordered anharmonic chains.

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Workshop: Many-Body Quantum Systems

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Abstracts

Phase Transition in Gross-Pitaevskii Theory

Mathieu Lewin
(joint work with Phan Thành Nam)

We present here recent results from [4] concerning the Gross-Pitaevskii equation

$$\big(-\Delta + w * |u|^2\big)u = \mu\, u \qquad \text{in } \mathbb{R}^d, \tag{1}$$

where $\mu > 0$ is a fixed parameter and w is a given real-valued even function or measure with $\int_{\mathbb{R}^d} |w| < \infty$ and $\int_{\mathbb{R}^d} w > 0$ (a Dirac delta δ_0 is allowed). Although in [4] we consider any space dimension, here we assume $d \in \{1, 2, 3\}$ for simplicity. The positivity of μ will imply that solutions to (1) must have positive density, in the sense that $\int_{B(\tau,R)} |u|^2 \geq \alpha R^d$ for all $\tau \in \mathbb{R}^d$ and all R large enough, with $\alpha > 0$. In particular, u cannot tend to 0 at infinity. The simplest solution of (1) is the constant function $u(x) = (\mu/\int_{\mathbb{R}^d} w)^{1/2}$ but we will see that there are other more interesting solutions.

We are interested in a particular class of solutions of (1) which we call *"infinite ground states"*. Those are by definition minimizers of the associated free energy

$$\mathcal{F}_\mu(u) := \int_{\mathbb{R}^d} |\nabla u(x)|^2\, dx + \frac{1}{2}\iint_{\mathbb{R}^d\times\mathbb{R}^d} |u(x)|^2|u(y)|^2 w(x-y)\, dx\, dy - \mu\int_{\mathbb{R}^d} |u(x)|^2\, dx. \tag{2}$$

The above integrals are all infinite, so that the concept of minimizers is unclear. We simply ask that the energy goes up when u is perturbed locally, whatever the size of the local perturbation. In other words, we require that "$\mathcal{F}_\mu(v) - \mathcal{F}_\mu(u)$" ≥ 0 for any v that coincides with u outside of a bounded set. The formal difference of the energies can be given a rigorous meaning by writing integrals of differences, for instance $\int_{\mathbb{R}^d}(|\nabla v|^2 - |\nabla u|^2)$ for the first term.

The equation (1) typically describes a *system of infinitely many quantum particles in a mean-field-type approximation* and w is interpreted as the interaction potential between the particles. In particular, it can be used to model the condensate part of an infinite Bose gas. When u is real-valued and $w = \delta$, the same equation can also describe phase separation in binary fluids and living tissues, in which case it is called *Cahn-Hilliard*.

The constant solution $u(x) = (\mu/\int_{\mathbb{R}^d} w)^{1/2}$ describes a *fluid state* of our system, since it is invariant under translations and rotations. The main result which we want to present here states informally that *when the Fourier transform $\widehat{w}$ of the interaction w takes negative values, there exists a unique fluid-solid phase transition at some critical μ_c, that is, the constant solution ceases to be a ground state.* This phenomenon was predicted first in 1957 by Gross in his famous paper [2]. Numerical simulations suggest that u becomes a non trivial periodic function, which is another instance of the crystallization conjecture. For Bose-Einstein condensates,

a periodic infinite ground state u is believed to describe a *supersolid*. Although this state of matter has not yet been observed in solid helium, recent experiments have confirmed its occurrence in ultracold atomic gases, in particular with dipolar interactions.

We are able to rigorously prove the above claim under very general assumptions on w. We assume that w is *superstable* [6], that is, can be written in the form

$$w = \varepsilon\, \chi_{B(0,\varepsilon)} + w_2, \qquad \varepsilon > 0, \tag{3}$$

where w_2 is stable in the sense that $\iint_{\mathbb{R}^{2d}} \rho(x)\rho(y)w_2(x-y)dx\,dy \geq 0$ for all non-negative $\rho \in L^2(\mathbb{R}^d)$. We also assume that w is decays fast enough at infinity:

$$\begin{cases} w_2(x) \geq -\frac{\kappa}{1+|x|^s} & \forall x \in \mathbb{R}^d, \\ w(x) = w_2(x) \leq \frac{\kappa}{|x|^s} & \forall |x| \geq \kappa, \end{cases} \tag{4}$$

for some $s > d$ and $\kappa \geq \varepsilon$. Close to the origin we just assume that w_+ is integrable and can even allow a positive Dirac delta at 0.

Theorem 1 (Uniform bounds)**.** *Let $\Omega \subset \mathbb{R}^d$ be any open set (bounded or unbounded) and $\mu > 0$. Any solution $u \in H^1_{\rm unif}(\Omega)$ of the GP equation* (1) *with the Dirichlet condition $u_{|\partial\Omega} = 0$ satisfies the uniform bound*

$$\|u\|_{L^\infty(\Omega)} \leq C\sqrt{\mu}\left(1 + \mu^{\frac{d}{4}}\right) \tag{5}$$

where C depends on w and d, but not on u, Ω and μ. If Ω_n is a sequence of bounded domains such that $B(0,n) \subset \Omega_n$, then any sequence $u_n \geq 0$ of minimizers for the energy (2) *in $H^1_0(\Omega_n)$ is bounded in L^∞ and converges locally uniformly to a non-negative infinite ground state u, after extraction of a subsequence.*

The previous result provides the existence of non-negative infinite ground states in $L^\infty(\mathbb{R}^d)$ for all $\mu > 0$. The proof of the bound (5) is delicate under our general assumptions on w. We had to use a combination of elliptic PDE techniques with a localization method introduced by Ruelle in statistical mechanics in [6].

When the Fourier transform $\widehat{w}$ is non-negative, the energy $\mathcal{F}_\mu$ is a strictly convex function of $|u|^2$ and Dirichlet minimizers in a bounded set Ω are unique, up to multiplication by a constant phase factor. This is not true anymore in the whole space, due to the possibility of varying the boundary condition "at infinity".

Theorem 2 ((Non)uniqueness for positive-definite interaction)**.** *Assume $\widehat{w} \geq 0$.*

(i) The constant $u(x) = (\mu/\int_{\mathbb{R}^d} w)^{1/2}$ is an infinite ground state for all $\mu > 0$.

(ii) If $d = 1$ and $s > 2$, then it is the ***unique*** *infinite ground state, up to multiplication by a constant phase factor.*

(iii) If $d = 2$, $s > 3$ and $\widehat{w}$ is radial decreasing, then for μ small enough there exists an infinite ground state with a non-trivial phase. More precisely, we have $|u(x)| > 0$ for all $|x| \geq R_0$ and u takes the form

$$u(x) = |u(x)|\frac{x_1 + ix_2}{\sqrt{x_1^2 + x_2^2}}e^{i\psi(x)}, \qquad \forall |x| \geq R_0,$$

with a smooth ψ. In other words, u has topological degree one at infinity.

Recall that s is the polynomial decay of w at infinity. The last solution in 2D describes an infinite quantum fluid rotating about a point (a vortex). It can be seen to have an energy infinitely higher than the constant, but it is nevertheless an infinite ground state, protected by its topological degree at infinity. Existence of such vortices has a long history in the case of the Ginzburg-Landau equation, which corresponds to taking $w = \delta_0$. In this setting it was even proved that the constant and the vortex are the only two infinite ground states, up to translations, complex conjugation and multiplication by a phase [5, 7]. It would be interesting to extend the uniqueness to a general w and the existence to any positive μ. Our proof uses that the equation converges to Ginzburg-Landau after an appropriate rescaling in the limit $\mu \to 0$.

We finally state our main theorem concerning the fluid-solid phase transition.

Theorem 3 (Fluid-solid phase transition)**.** *Assume that $\widehat{w}$ takes negative values. There exists a $0 < \mu_c < \infty$ such that the following holds.*

*(i) For $\mu \leq \mu_c$, the constant solution is an infinite ground state. It is the **unique real-valued** one if μ is small enough.*

*(ii) For $\mu > \mu_c$, the constant solution is **not** an infinite ground state. Any infinite ground state u satisfies*

$$\max_{B(\tau,R)} |u| \geq c + \min_{B(\tau,R)} |u|, \qquad \forall \tau \in \mathbb{R}^d,$$

for some $c, R > 0$.

In [4] we provide explicit upper and lower bounds on μ_c. The upper bound is obtained by finding the value of μ for which the constant solution becomes linearly unstable. The lower bound is more complicated; the first result in this direction appeared in [1] in the special case $w \geq 0$. It would be very interesting to prove that u is periodic for $\mu > \mu_c$, as is suggested by numerical simulations [3]. The inequality (ii) means, at least, that an infinite ground state must oscillate everywhere in space.

References

[1] A. Aftalion, X. Blanc, and R. L. Jerrard, *Mathematical issues in the modelling of supersolids*, Nonlinearity, 22 (2009), p. 1589.

[2] E. P. Gross, *Unified Theory of Interacting Bosons*, Physical Review, 106 (1957), pp. 161–162.

[3] M. Lewin, *Mean-field limit of Bose systems: rigorous results*, in Proceedings of the International Congress of Mathematical Physics, Santiago de Chile, 2015. ArXiv e-prints.

[4] M. Lewin and P. T. Nam, *Positive-density ground states of the Gross-Pitaevskii equation*, in preparation, (2023).

[5] P. Mironescu, *Les minimiseurs locaux pour l'équation de Ginzburg-Landau sont à symétrie radiale*, C. R. Acad. Sci. Paris Sér. I Math., 323 (1996), pp. 593–598.

[6] D. Ruelle, *Superstable interactions in classical statistical mechanics*, Comm. Math. Phys., 18 (1970), pp. 127–159.

[7] E. Sandier, *Locally minimising solutions of $-\Delta u = u(1 - |u|^2)$ in $\mathbf{R}^2$*, Proc. Roy. Soc. Edinburgh Sect. A, 128 (1998), pp. 349–358.

The free energy of dilute Bose gases at low temperatures

Arnaud Triay

(joint work with Florian Haberberger, Christian Hainzl, Phan Thành Nam and Robert Seiringer)

In 1957, Lee, Huang and Yang published a celebrated work [9] on the expansion in the dilute limit of the eigenvalues of a system of Bosons at low temperature. The rigorous justification of their approximation has been a mathematical challenge since then. Following their work, we consider N interacting Bosons in a box of side length $L > 0$. The Hamiltonian of the system is

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + \sum_{1\leq i<j\leq N} V(x_i - x_j)$$

which acts on symmetric space $L^2_s([-L/2, L/2]^3)^N$ and where V is a non-negative, radial, compactly supported and integrable function.

The most famous formula of Lee, Huang and Yang concerns the lowest eigenvalue of H_N,

$$\lambda_0(H_N) \simeq 4\pi\mathfrak{a}\rho N\left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho\mathfrak{a}^3}\right) \tag{1}$$

where $\rho = N/L$ is the density of the system and $\mathfrak{a}$ the scattering length of the potential V.

The mathematical derivation of this formula has been achieved over the past 60 years in a series of seminal works. Dyson proved the leading order upper bound already in 1957 [5], but it was only until 1998 that Lieb and Yngvason gave the proof of the matching lower bound [12]. The second order, which is known as the Lee-Huang-Yang correction, was proven as an upper bound ten years later by Yau and Yin [15] (see also [2] for an alternate proof) and it was only recently that the lower bound of the LHY correction was established by Fournais and Solovej [6, 7].

Less known is the Lee-Huang-Yang formula for the low-lying eigenvalues

$$\lambda_j(H_N) \simeq \lambda_0(H_N) + \sum_{k\neq 0} n_k\sqrt{k^4 + 16\pi\mathfrak{a}k^2},$$

where $n_k \in \mathbb{N}_0$ is the occupation number of the mode $k \in 2\pi\mathbb{Z}^3/L$. Deriving this expansion for each eigenvalue is probably out of reach, therefore one might focus on macroscopic quantities instead, like the free energy of the system. Assuming the validity of the above formula leads to

$$-T\log \operatorname{Tr} e^{-\frac{H_N}{T}} \simeq \lambda_0(H_N) + \frac{T}{(2\pi)^3}\sum_p \ln\left(1 - e^{-\sqrt{p^4 + 16\pi\frac{\rho\mathfrak{a}}{T}p^2}}\right) \tag{2}$$

where $T > 0$ is the temperature, the trace is taken over the symmetric space and the sum is over $p \in 2\pi\mathbb{Z}^3/(LT^{1/2})$.

We report a rigorous justification of the lower bound of (2) in the dilute limit $\rho\mathfrak{a}^3 \to 0$ and for temperatures $T \lesssim \rho\mathfrak{a}$. As one can easily check, in the regime

$T \simeq \rho\mathfrak{a}$ the free energy of the thermal excitations in (2) is of the same order as the famous Lee-Huang-Yang correction in (1). The first order of was already derived by Seiringer [13] (lower bound) and Yin [16] (upper bound), for a range of temperatures allowing to go up to the order of the critical temperature $T_c(\rho) \simeq \rho^{2/3}$ where there might not be condensation.

More precisely, in [8], we prove the following theorem.

Theorem 1. *Assume V is non-increasing, then there is some $\nu > 0$, such that in the dilute limit $\rho\mathfrak{a}^3 \to 0$, for any $0 \leq T \leq \rho\mathfrak{a}(\rho\mathfrak{a}^3)^{-\nu}$,*

$$\begin{aligned} -\lim_{\substack{N\to\infty \\ N/L^3\to\rho}} \frac{T}{L^3} \log \operatorname{Tr} e^{-\frac{H_N}{T}} \geq 4\pi\mathfrak{a}\rho^2 \left(1 + \frac{128}{15\sqrt{\pi}}\sqrt{\rho\mathfrak{a}^3}\right) \\ + \frac{T^{5/2}}{(2\pi)^3} \int_{\mathbb{R}^3} \log\left(1 - e^{-\sqrt{p^4 + \frac{16\pi\rho\mathfrak{a}}{T}p^2}}\right) \mathrm{d}p \\ - \mathcal{O}((\rho\mathfrak{a})^{5/2}(\rho\mathfrak{a}^3)^{\nu}). \end{aligned}$$

Our proof relies on the subadditivity of the free energy, which allows us to reduce the problem to studying smaller subsystems. At shorter length scales, the gap of the kinetic operator is much bigger, allowing to prove Bose-Einstein condensation [11] and use Bogoliubov's approximation [1, 14, 10, 3]. However, to resolve the energy at LHY precision, one needs to consider length scales slightly larger than the Gross–Pitaevskii one $\ell \gg (\rho\mathfrak{a})^{-1/2}$ in order to avoid pollution coming from the boundary effects.

This strategy of decomposing into smaller subsystems was already used in [12] for the proof of the leading order lower bound in (1). The downside of this method is that the smaller subsystems arise with Neumann boundary conditions, which we cannot afford to remove by localizing without loosing the LHY precision. Working with Neumann boundary conditions is more involved than on the torus with periodic boundary conditions. The main reason is that the Neumann momentum is not conserved by the interaction potential as in the periodic case where $\langle e_p \otimes e_q, V(x-y)e_r \otimes e_s\rangle_{\mathbb{T}^3} = \delta_{p+q,r+s}\hat{V}(p-r)$, with $e_p(x) = e^{ipx}$. The analysis in configuration space is also intricate and was already investigated in [4] where almost LHY order was reached. One of the main novelties of our work is to resolve this problem by combining the momentum and the configuration space approaches. We construct a symmetrized version of the full space scattering function by a reflexion method, defining in fact the kernel of a diagonal operator in the Neumann basis.

References

[1] N. N. Bogoliubov, *About the theory of superfluidity*, Izv. Akad. Nauk SSSR, 11 (1947), p. 77.

[2] G. Basti, S. Cenatiempo, and B. Schlein, *A new second order upper bound for the ground state energy of dilute Bose gases*, Forum Math. Sigma, 9 (2021), E74.

[3] C. Boccato, C. Brennecke, S. Cenatiempo, B. Schlein. *Bogoliubov Theory in the Gross-Pitaevskii limit*, Acta Math. **222** (2019), no. 2, 219–335.

[4] C. Boccato and R. Seiringer. The Bose gas in a box with Neumann boundary conditions. *Annales Henri Poincaré*, 24, No. 5, (2023) pp. 1505–1560.
[5] F. J. Dyson, *Ground state energy of a hard-sphere gas*, Phys. Rev., 106 (1957), pp. 20–26.
[6] S. Fournais and J. P. Solovej, *The energy of dilute Bose gases*, Ann. of Math., 192 (2020), pp. 893–976.
[7] ———, *The energy of dilute Bose gases II: The general case*, Inventiones mathematicae, 232, no 2, (2023) p. 863–994.
[8] F. Haberberger, C. Hainzl, P. T. Nam, R. Seiringer, and A. Triay, *The free energy of dilute Bose gases at low temperatures.* arXiv:2304.02405.
[9] T. D. Lee, K. Huang, and C. N. Yang, *Eigenvalues and eigenfunctions of a Bose system of hard spheres and its low-temperature properties*, Phys. Rev., 106 (1957), p. 1135.
[10] M. Lewin, P. T. Nam, S. Serfaty, and J. P. Solovej. *Bogoliubov spectrum of interacting Bose gases*, Comm. Pure Appl. Math. 68 (2015), pp. 413–471.
[11] E. H. Lieb and R. Seiringer, *Proof of Bose–Einstein condensation for dilute trapped gases*, Phys. Rev. Lett., 88 (2002), p. 170409.
[12] E. H. Lieb and J. Yngvason, *Ground state energy of the low density Bose gas*, Phys. Rev. Lett., 80 (1998), pp. 2504–2507.
[13] R. Seiringer. Free energy of a dilute Bose gas: lower bound. *Commun. Math. Phys.* 279 (2008), pp. 595–636.
[14] R. Seiringer. The excitation spectrum for weakly interacting bosons. *Commun. Math. Phys.* 306 (2011), pp. 565–578.
[15] H.-T. Yau and J. Yin, *The second order upper bound for the ground energy of a Bose gas*, J. Stat. Phys., 136 (2009), pp. 453–503.
[16] J. Yin. Free energies of dilute Bose gases: upper bound. *J. Stat. Phys.* 141 (2010), pp. 683–726.

Quantum Fluctuations of Many-Body Dynamics around the Gross-Pitaevskii Equation

Cristina Caraci

(joint work with Jakob Oldenburg and Benjamin Schlein)

We consider a system of N interacting bosons confined by an external field in a volume of order one. The Hamilton operator describing the system, acting on $L^2_s(\mathbb{R}^{3N})$, is given by

$$H_N^{\mathrm{trap}} = \sum_{j=1}^N \left[-\Delta_{x_j} + V_{\mathrm{ext}}(x_j)\right] + \sum_{i<j}^N N^2 V(N(x_i - x_j)) \tag{1}$$

with $V_{\mathrm{ext}}(x) \to \infty$ as $|x| \to \infty$ and $V \geq 0$ compactly supported. It is well known that every sequence of approximate ground states of (1), exhibit complete Bose-Einstein condensation in the minimizer of the Gross-Pitaevskii functional.

We are interested in studying the dynamics of an equilibrium state of (1), after switching off the trap. That is to say, at zero temperature, we look at the solution of the time-dependent Schrödinger equation

$$i\partial_t \psi_{N,t} = H_N \psi_{N,t} \tag{2}$$

where $H_N = H_N^{\mathrm{trap}} - \sum_{j=1}^N V_{\mathrm{ext}}(x_j)$, for initial data $\psi_{N,0}$ approximating the ground state of (1). In [9, 10], it was first proven that the time-evolution $\psi_{N,t}$ of a $\psi_{N,0}$ exhibiting Bose-Einstein condensate in a one-particle state $\varphi \in L^2(\mathbb{R}^3)$ still

exhibits Bose-Einstein condensation, in a new one-particle state φ_t, given by the solution of the nonlinear time-dependent Gross-Pitaevskii equation

$$i\partial_t\varphi_t = -\Delta\varphi_t + 8\pi\mathfrak{a}|\varphi_t|^2\varphi_t$$

with the initial data $\varphi_{t=0} = \varphi$. Here $\mathfrak{a} > 0$ denotes the scattering length of the interaction potential V. By means of the one-particle reduced density $\gamma_{N,t}$ associated with $\psi_{N,t} \in L^2_s(\mathbb{R}^{3N})$ of (2), it turns out that

$$\lim_{N\to\infty} \langle\varphi_t, \gamma_{N,t}\varphi_t\rangle = 1 \tag{3}$$

for any fixed $t \in \mathbb{R}$, if (3) holds true at time $t = 0$. Other related results have been obtained in [13, 2, 5]. The convergence (3) implies that $\gamma_{N,t} \to |\varphi_t\rangle\langle\varphi_t|$ in the trace-class topology. However, (3) does not provide an approximation for the many-body wave function $\psi_{N,t}$ in the strong $L^2(\mathbb{R}^{3N})$ topology. To obtain a norm-approximation, it is not enough to approximate the evolution of the condensate. It is instead crucial to take into account the evolution of its orthogonal excitations. Aim of this talk is to present the results in [8] that provide a norm approximation for $\psi_{N,t}$ for a suitable class of initial data.

The precise statement of our theorem can be found in [8, Theorem 2.2], and it requires the introduction of several technical details. Here we report the main ideas behind the approximation of $\psi_{N,t}$.

To describe the evolution of the excitations we follow the original idea of [16]. Working in Fock space $\mathcal{F} = \bigoplus_{n\geq 0} L^2_s(\mathbb{R}^3)^{\otimes n}$, we construct a fluctuation dynamics $\mathcal{U}_N(t;s)$. This map not only factors out the condensate from the evolution $\psi_{N,t}$ (using a unitary map from [11]), but takes also into account the correlation structure generated by the two-body problem, through the action of an appropriate Bogoliubov transformation, as in [5]. As a result, we can approximate the dynamics $\mathcal{U}_N(t;s)$ by a simpler fluctuation dynamics $\mathcal{U}_{2,N}(t;s)$ generated by a quadratic-in creation and annihilation operators - self-adjoint operator $\mathcal{J}_{2,N}(t)$, that satisfies the equation

$$i\partial_t\mathcal{U}_{2,N}(t;s) = \mathcal{J}_{2,N}(t)\,\mathcal{U}_{2,N}(t;s) \tag{4}$$

with $\mathcal{U}_{2,N}(s;s) = 1$ for all $s \in \mathbb{R}$.

The main difference with respect to [3, 4, 5], where a similar dynamics were constructed, is the implementation of a cubic transformation, similarly as in [6, 12]. However, this transformation happen to be only auxiliary to have the correct energy of the norm of the difference between the evolution vector and the approximate one. Hence, the final approximation turns out to be quasi-free.

Using (4), we approximate the evolution of the macroscopic correlations by the Bogoliubov dynamics $\mathcal{U}_{2,N}(t)$, which still depends on N. It is thus natural to ask whether $\mathcal{U}_{2,N}(t)$ approaches a limiting, N-independent, quadratic evolution $\mathcal{U}_{2,\infty}(t)$, as N tends to infinity. The answer is affirmative, and is the content of Theorem 2.3 in [8].

Moreover, since $\mathcal{U}_{2,\infty}$ is generated by a quadratic operator, the evolution $\mathcal{U}_{2,\infty}(t)$ acts as time-dependent Bogoliubov transformations, and its action on annihilation and creation operators can be calculated explicitly. As a consequence, using the

approximation in terms of the Bogoliubov dynamics $\mathcal{U}_{2,\infty}(t)$, we can establish a central limit theorem for the evolution of initial data approximating ground states ψ_N of the Hamiltonian (1). For details we refer to Theorem 2.5 in [8]. The proof is based on previous results [1, 7, 15, 14].

References

[1] G. Ben Arous, K. Kirkpatrick, B. Schlein. *A central limit theorem in many-body quantum dynamics.* Comm. Math. Phys. **321**, (2013) no. 2: 371–417.

[2] N. Benedikter, G. de Oliveira and B. Schlein. *Quantitative derivation of the Gross-Pitaevskii equation.* Comm. Pure Appl. Math. **68** (8), (2014) 1399–1482.

[3] C. Boccato, S. Cenatiempo, B. Schlein. *Quantum many-body fluctuations around nonlinear Schrödinger dynamics.* Ann. Henri Poincaré, **18**, (1) (2017) 113–191.

[4] C. Brennecke, P.T. Nam, M. Napiórkowski, B. Schlein. *Fluctuations of N-particle quantum dynamics around the nonlinear Schrödinger equation.* Annales de l'Institut Henri Poincaré C, Analyse non linéaire, **36**, (5) (2019) 1201–1235.

[5] C. Brennecke, B. Schlein. *Gross-Pitaevskii dynamics for Bose-Einstein condensates.* Analysis & PDE **12** (6), (2019) 1513–1596.

[6] C. Brennecke, B. Schlein, S. Schraven. *Bogoliubov Theory for Trapped Bosons in the Gross-Pitaevskii Regime.* Ann. Henri Poincaré **23**, (2022)1583–1658.

[7] S. Buchholz, C. Saffirio, B. Schlein. *Multivariate central limit theorem in quantum dynamics.* J. Stat. Phys. **154** no. 1 (2014), 113–152.

[8] C. Caraci, J. Oldenburg, B. Schlein. *Quantum Fluctuations of Many-Body Dynamics around the Gross-Pitaevskii Equation.* arxiv preprint 2308.11687

[9] L. Erdős, B. Schlein, H.-T. Yau. *Derivation of the Gross-Pitaevskii equation for the dynamics of Bose-Einstein condensate,* Ann. of Math. **172**, no. 1, (2010), 291-370.

[10] L. Erdős, B. Schlein and H.-T. Yau. *Rigorous derivation of the Gross-Pitaevskii equation with a large interaction potential.* J. Amer. Math. Soc. **22** , (2009),1099–1156.

[11] M. Lewin, P. T. Nam, S. Serfaty, J.P. Solovej. *Bogoliubov spectrum of interacting Bose gases.* Comm. Pure Appl. Math. **68** (3), (2014), 413 – 471.

[12] P.T. Nam, A. Triay. *Bogoliubov excitation spectrum of trapped Bose gases in the Gross-Pitaevskii regime.* To appear J. Math. Pures Appl. (2023)

[13] P. Pickl. *Derivation of the time dependent Gross Pitaevskii equation with external fields.* Reviews in Mathematical Physics **27**, no. 01 (2015): 1550003.

[14] S. Rademacher. *Central limit theorem for Bose gases interacting through singular potentials.* Letters in Mathematical Physics **110**, no. 8 (2020) 2143-2174.

[15] S. Rademacher, B. Schlein. *Central limit theorem for Bose-Einstein condensates.* Journal of Mathematical Physics **60**, no. 7: (2019) 071902.

[16] I. Rodnianski and B. Schlein. *Quantum fluctuations and rate of convergence towards mean field dynamics.* Commun. Math. Phys., **291**, (2009) pp. 31–61.

The Fröhlich Polaron at Strong Coupling

Morris Brooks

(joint work with Robert Seiringer)

We study the Fröhlich Hamiltonian, which is a model describing the interaction between an electron and the optical modes of a polar crystal, acting on the space $L^2\left(\mathbb{R}^3\right) \otimes \mathcal{F}\left(L^2\left(\mathbb{R}^3\right)\right)$, given by the expression

$$\mathbb{H} := -\Delta_x - a\left(w_x\right) - a^\dagger\left(w_x\right) + \mathcal{N}, \tag{1}$$

where the annihilation and creation operators satisfy the rescaled canonical commutation relations $\left[a(f), a^\dagger(g)\right] = \alpha^{-2}\langle f|g\rangle$ for $f, g \in L^2\left(\mathbb{R}^3\right)$ with $\alpha > 0$ being the coupling strength, the interaction is given by $w_x(x') := \pi^{-\frac{3}{2}}|x'-x|^{-2}$ and $\mathcal{N}$ is the corresponding (rescaled) particle number operator, i.e. $\mathcal{N} := \sum_{n=1}^{\infty} a^\dagger(\varphi_n)a(\varphi_n)$ where $\{\varphi_n : n \in \mathbb{N}\}$ is an orthonormal basis of $L^2\left(\mathbb{R}^3\right)$.

In a series of two papers [2] and [3], we derived asymptotic formulas for the ground state energy

$$E_\alpha = e^{\mathrm{Pek}} - \frac{1}{2\alpha^2}\operatorname{Tr}\left[1 - \sqrt{H^{\mathrm{Pek}}}\right] + O_{\alpha\to\infty}\left(\alpha^{-(2+\epsilon)}\right), \tag{2}$$

where $\epsilon > 0$, as well as an asymptotically sharp lower bound on the ground state energy $E_\alpha(P)$ as a function of the total momentum

$$E_\alpha(P) \geq e^{\mathrm{Pek}} - \frac{1}{2\alpha^2}\operatorname{Tr}\left[1 - \sqrt{H^{\mathrm{Pek}}}\right] + \min\left\{\frac{|P|^2}{2\alpha^4 m}, \alpha^{-2}\right\} + O_{\alpha\to\infty}\left(\alpha^{-(2+\epsilon)}\right),$$

where m is an explicit constant. The corresponding upper bound has previously been derived in [6]. This especially means that the interacting electron behaves like a free particle having an effectively increased mass given by the celebrated Landau-Pekar formula $M_{\mathrm{eff}} = m\alpha^4$, at least in the regime of strong coupling α and momenta in the region $\alpha^{1-\frac{\epsilon}{2}} \ll |P| \leq \sqrt{2m}\alpha$.

The method we use is inspired by the sophisticated framework that has been developed in the study of Bose gases, especially the recent result [1], where the Bogoliubov approximation for a translation-invariant Bose gas has been established. Applying this framework allows us to break the translation-invariance of the quantum system defined in Eq. (1), which effectively introduces a confinement. The confined problem is then analysed in a similar way as in [5] and [4], where Eq. (2) has been verified for a system confined to a ball in $\mathbb{R}^3$, respectively the three dimensional torus.

References

[1] M. Brooks and R. Seiringer. Validity of Bogoliubov's approximation for translation-invariant Bose gases. *arXiv*: 2111.13864. *Prob. Math. Phys.* (in press).

[2] M. Brooks and R. Seiringer. The Fröhlich Polaron at Strong Coupling – Part I: The Quantum Correction to the Classical Energy. *Communications in Mathematical Physics*: To appear.

[3] M. Brooks and R. Seiringer. The Fröhlich Polaron at Strong Coupling – Part II: Energy-Momentum Relation and Effective Mass. *arXiv*: 2211.03353.

[4] D. Feliciangeli and R. Seiringer. The Strongly Coupled Polaron on the Torus: Quantum Corrections to the Pekar Asymptotics. *Archive for Rational Mechanics and Analysis* 242:1835–1906, 2021.

[5] R. Frank and R. Seiringer. Quantum Corrections to the Pekar Asymptotics of a Strongly Coupled Polaron. *Communications on Pure and Applied Mathematics* 74:544–588, 2021.

[6] D. Mitrouskas, K. Myśliwy, and R. Seiringer. Optimal parabolic upper bound for the energy-momentum relation of a strongly coupled polaron. *arXiv*: 2203.02454.

Counting eigenvalues using the landscape function

Sven Bachmann
(joint work with Richard Froese and Severin Schraven)

Given a non-negative potential V, a *landscape function* is a solution u of

$$(-\Delta + V(x))u(x) = 1, \qquad (x \in \mathbb{R}^d). \tag{1}$$

It was introduced in [1] in the context of Anderson localization for random Schrödinger operators. The initially numerical results, which have seeded a long series of further numerical and analytical ones indicate that the inverse $\frac{1}{u(x)}$ plays the role of an effective potential for spectral properties of the Schrödinger operator: In the case of the Anderson model, minima of $\frac{1}{u}$ match the localization centres of the localized eigenfunctions, the potential $\frac{1}{u}$ defines an Agmon distance [2], and the properly coarse-grained volume of the sublevel sets of $\frac{1}{u}$ match the integrated density of states [3].

Our work focusses on the last aspect of the list, and without disorder. The earliest estimates on the counting function for Schrödinger operators are the exact asymptotics due to Weyl [4]. In a similar vein but with negative potentials, the CLR bound [5, 6, 7] is an upper bound on the number of negative eigenvalues. Finally, the Fefferman-Phong strategy [8] yields both upper and lower bounds on the counting function by a coarse-grained volume of the sublevel sets of V, which are valid all the way to the bottom of the spectrum. Numerics suggests that the same holds for the effective potential, but without coarse-graining, and for a very general class of potentials in all dimensions.

A famous example is the potential

$$V_S(x, y) = x^2 y^2$$

in two dimensions, which was studied in details by Simon in [9]. On the one hand, the volume of the sublevel sets below μ is infinite for all $\mu > 0$, but on the other hand the spectrum is discrete. This is due to the increasingly steep and narrow 'valleys' of the potential along the coordinate axes. While a Weyl-type asymptotics is therefore not valid, coarse-graining yields a finite volume and the Fefferman-Phong bounds yield a finite number of eigenvalues below any μ.

In [10], we show that the function $\frac{1}{u}$ provides an appropriate smoothing of the potential V to allow for both upper and lower bounds on the counting function that do not require coarse-graining. In particular, the volumes of the sublevel sets of the effective potential for V_S are finite for all $\mu \in \mathbb{R}$. We first construct the landscape function in the whole space as an appropriate limit of finite volume approximations and show that the limit is a weak solution of (1). We then prove the following:

Theorem 1. *Assume that $V \geq 0, V \not\equiv 0$ satisfies the following conditions:*

(i) There exists $C_{\mathrm{K}}, \delta > 0$, such that

$$\frac{1}{r^{d-2+\delta}} \int_{B(x,r)} V(y)dy \leq C_{\mathrm{K}} \frac{1}{R^{d-2+\delta}} \int_{B(x,R)} V(y)dy$$

for all $x \in \mathbb{R}^d$ and all r, R with $0 < r < R$.

(ii) There exists $C_{\mathrm{D}} > 0$ such that

$$\int_{B(x,2r)} V(y)dy \leq C_{\mathrm{D}} \left(\int_{B(x,r)} V(y)dy + r^{d-2} \right)$$

for all $x \in \mathbb{R}^d$ and all $r > 0$.

We denote by H the Friedrichs extension of the positive symmetric operator

$$-\Delta + V$$

defined on $C_c^\infty(\mathbb{R}^d)$. Let $\mathcal{N}^V(\mu)$ be the rank of the spectral projection $\chi_{(\infty,\mu]}(H)$. Then there exist constants $c, C > 0$ such that

$$(c\mu)^{\frac{d}{2}} \mathcal{V}(c\mu) \leq \mathcal{N}^V(\mu) \leq (C\mu)^{\frac{d}{2}} \mathcal{V}(C\mu)$$

for all $\mu \in \mathbb{R}$, where

$$\mathcal{V}(\mu) = \int_{\left\{x \in \mathbb{R}^d : \frac{1}{u(x)} \leq \mu\right\}} dx.$$

The constants c, C depend only on $C_{\mathrm{K}}, C_{\mathrm{D}}, \delta$ and the spatial dimension d.

This immediatly yields a criterion for the discreteness of the spectrum:

Corollary 2. *Let V be as in Theorem 1. Then H has discrete spectrum if and only if $\lim_{R\to\infty} \|u\|_{L^\infty(\mathbb{R}^d \setminus B(0,R))} = 0$.*

When applied to the special case of polynomial potentials, we obtain the following result that was conjectured in [9].

Corollary 3. *Let V be a polynomial that is bounded from below. Then H has discrete spectrum if and only if none of the partial derivatives of V vanishes identically.*

In particular, this yields another proof of discreteness of the spectrum for V_S.

References

[1] Marcel Filoche and Svitlana Mayboroda. Universal mechanism for Anderson and weak localization. *Proceedings of the National Academy of Sciences*, 109(37):14761–14766, 2012.

[2] Douglas N Arnold, Guy David, Marcel Filoche, David Jerison, and Svitlana Mayboroda. Localization of eigenfunctions via an effective potential. *Communications in Partial Differential Equations*, 44(11):1186–1216, 2019.

[3] Guy David, Marcel Filoche, and Svitlana Mayboroda. The landscape law for the integrated density of states. *Advances in Mathematics*, 390:107946, 2021.

[4] Hermann Weyl. Über die asymptotische Verteilung der Eigenwerte. *Nachrichten von der Gesellschaft der Wissenschaften zu Göttingen, Mathematisch-Physikalische Klasse*, 1911:110–117, 1911.

[5] Michael Cwikel. Weak type estimates for singular values and the number of bound states of Schrödinger operators. *Annals of Mathematics*, pages 93–100, 1977.

[6] Elliott Lieb. Bounds on the eigenvalues of the Laplace and Schrödinger operators. *Bulletin of teh Americal Mathematical Society*, 82(5), 1976.

[7] Grigorii Vladimirovich Rozenblum. Distribution of the discrete spectrum of singular differential operators. *Izvestiya Vysshikh Uchebnykh Zavedenii. Matematika*, (1):75–86, 1976.

[8] Charles L Fefferman. The uncertainty principle. *Bulletin of the American Mathematical Society*, 9(2), 1983.
[9] Barry Simon. Some quantum operators with discrete spectrum but classically continuous spectrum. *Annals of Physics*, 146(1):209–220, 1983.
[10] Sven Bachmann, Richard Froese, and Severin Schraven. Counting eigenvalues of Schrödinger operators using the landscape function. *arXiv preprint arXiv:2306.03936*, 2023.

Infrared scattering states in the massless Nelson model

Gian Michele Graf
(joint work with Vincent Beaud and Wojciech Dybalski)

The Nelson model describes a particle coupled to a (bosonic) field. In the case where the field is massless, the model is a simple caricature of QED, as both theories share some traits in the infrared, and in particular the difficulty of defining scattering states. A possible solution, to be discussed, is that the particle emits an ever growing number of ever softer bosons. A precise and asymptotically (in time) exact description of the process will be given, motivated, and compared to earlier work by others.

References

[1] V. Beaud, W. Dybalski, G.M. Graf, *Infraparticle States in the Massless Nelson Model: Revisited*, Annales Henri Poincaré (2023).

Adiabatic Evolution of Low-Temperature Many-Body Systems

Marcello Porta
(joint work with Rafael L. Greenblatt, Markus Lange, Giovanna Marcelli)

We consider interacting fermions on a lattice $\Lambda_L = \mathbb{Z}^d/L\mathbb{Z}^d$, exposed to an external, slowly varying time-dependent perturbation. The time-dependent Hamiltonian of the system is, in second quantization:

$$\mathcal{H}(\eta t) = \sum_{x,y\in\Lambda_L} a_x^* H(x;y) a_y + \lambda \sum_{x,y\in\Lambda_L} n_x n_y v(x;y) + \varepsilon e^{\eta t}\mathcal{P} \,, \tag{1}$$

with a_x, a_x^* the usual fermionic creation and annihilation operators, acting on the fermionic Fock space; $n_x = a_x^* a_x$ is the density operator; and $\mathcal{P}$ is a sum of bounded local terms, *e.g.* $\mathcal{P} = \sum_{x\in\Lambda_L} \mu(x) n_x$. The first term in (1) is the non-interacting Hamiltonian, and it describes the hopping of the fermions on Λ_L, while the second term is the many-body interaction. We shall assume that both $H(x;y)$ and $v(x;y)$ are finite ranged.

Let us denote by $\mathcal{H}$ the Hamiltonian $\mathcal{H}(\eta t)$ for $\varepsilon = 0$. The equilibrium state of the system with Hamiltonian $\mathcal{H}$ is described by the usual Gibbs state:

$$\rho_{\beta,\mu,L} = \frac{e^{-\beta(\mathcal{H}-\mu\mathcal{N})}}{\operatorname{Tr} e^{-\beta(\mathcal{H}-\mu\mathcal{N})}} \,, \qquad \langle\mathcal{O}\rangle_{\beta,\mu,L} := \operatorname{Tr}\mathcal{O}\rho_{\beta,\mu,L} \,, \tag{2}$$

where $\mathcal{N}$ is the number operator, μ is the chemical potential and β is the inverse temperature. We will be interested in the evolution of the equilibrium state, after switching on the time-dependent perturbation. The evolution equation of the system is:

$$\begin{aligned} & i\partial_t \rho(t) = [\mathcal{H}(\eta t), \rho(t)] \,, \qquad t \leq 0 \,, \\ & \rho(-\infty) = \rho_{\beta,\mu,L} \,, \end{aligned} \tag{3}$$

and we shall suppose that $\eta > 0$. We shall be interested in the time-evolution under (3) of the expectation value of local observables $\mathcal{O}_X$, with $X \subset \Lambda_L$ independent of L, in the adiabatic limit $\eta \to 0^+$.

Let us assume that the chemical potential μ is chosen in a spectral gap of H: $\mathrm{dist}(\sigma(H), \mu) \geq \delta$ for some $\delta > 0$ uniform in L. The coupling λ is chosen small enough, $|\lambda| < \lambda_0$ with λ_0 dependent on δ and independent of β, L. In this setting, the well-known convergence of the fermionic cluster expansion implies the analyticity of the Gibbs state of $\mathcal{H}$ in λ, uniformly in β and L, and the exponential space-time decay of Euclidean (imaginary time) correlations functions. In this talk, I will discuss how to use cluster expansion methods to investigate the real-time quantum dynamics generated by (3). The result I will present is contained in [3]. Let $\eta_\beta \in \frac{2\pi}{\beta}\mathbb{N}$ with $\eta_\beta \geq \eta$. In [3] we prove the following representation formula for the expectation values of local observables:

$$\begin{aligned} & \mathrm{Tr}\, \mathcal{O}_X \rho(t) = \langle \mathcal{O}_X \rangle_{\beta,\mu,L} \\ & \quad + \sum_{n \geq 1} \frac{(-\varepsilon)^n}{n!} \int_{[0,\beta]^n} d\underline{s} \Big[\prod_{i=1}^n e^{\eta_\beta (t - i s_j)} \Big] \langle \mathbf{T} \gamma_{s_1}(\mathcal{P}); \gamma_{s_2}(\mathcal{P}); \cdots ; \gamma_{s_n}(\mathcal{P}); \mathcal{O}_X \rangle_{\beta,\mu,L} \\ & \quad + R_{\beta,\mu,L}(\varepsilon, \eta, t) \,, \end{aligned} \tag{4}$$

where:

(i) $\gamma_s(\mathcal{A}) = e^{s(\mathcal{H} - \mu\mathcal{N})} \mathcal{A} e^{-s(\mathcal{H} - \mu\mathcal{N})}$ is the imaginary-time evolution of $\mathcal{A}$;
(ii) $\mathbf{T}$ is the fermionic time-ordering;
(iii) in the correlation functions, the semicolon denotes trunctation: the series in the right-hand side of (4) involves connected Euclidean correlation functions, or Euclidean cumulants;
(iv) the error term $R_{\beta,\mu,L}$ is bounded as $|\varepsilon|(1/\beta\eta^{d+2})$ uniformly in L.

Under the above assumptions on the many-body Hamiltonian, the series in (4) turns out to be absolutely summable in ε, for $|\varepsilon|$ small enough uniformly in β, L and η. The proof of convergence is based on suitable decay estimates for the Euclidean correlation functions of the equilibrium Gibbs state $\langle \cdot \rangle_{\beta,\mu,L}$, which hold true for $\lambda = 0$ thanks to the spectral gap of H and to Wick's rule, and which are proven for $\lambda \neq 0$ via fermionic cluster expansion, whose convergence relies on the Brydges-Battle-Federbush formula for fermionic cumulants. The first two terms appearing in (4) coincide with the real-time Duhamel series for the dynamics generated by the time-dependent Hamiltonian:

$$\mathcal{H}_{\beta,\eta}(t) = \mathcal{H} + \varepsilon e^{\eta_\beta t} \mathcal{P} \,, \tag{5}$$

after a complex deformation argument at all orders (Wick rotation), inspired by the proof of stability of KMS states, see *e.g.* [2]. The term $R_{\beta,\mu,L}$ takes into account the error introduced by replacing $\mathcal{H}(\eta t)$ with $\mathcal{H}_{\beta,\eta}(t)$ in (3), and it is estimated using Lieb-Robinson bounds. Here we focused on the special case of the exponential switch function, but a wider class of switch functions can be considered [3].

The representation (4) allows to prove a many-body adiabatic theorem. We have:

$$\operatorname{Tr} \mathcal{O}_X \rho(t) = \frac{\operatorname{Tr} \mathcal{O}_X e^{-\beta(\mathcal{H}(\eta t)-\mu\mathcal{N})}}{\operatorname{Tr} e^{-\beta(\mathcal{H}(\eta t)-\mu\mathcal{N})}} + \widetilde{R}_{\beta,\mu,L}(\varepsilon,\eta,t) \ , \tag{6}$$

where $|\widetilde{R}_{\beta,\mu,L}(\varepsilon,\eta,t)| \leq C_1|\varepsilon|/(\beta\eta^{d+2}) + C_2|\varepsilon|\eta_\beta$. Making sure that this error term is small introduces a smallness requirement on the temperature of the system, which has to vanish as $\eta \to 0^+$ however uniformly in the system's size. In particular, in this small temperature regime the expected renormalization of the temperature in the instantaneous Gibbs state due to the heating introduced by the perturbation is not visible. Another application of (4) is the proof of validity of linear response theory: the first term in the expansion (4) reproduces Kubo formula, after Wick rotation, up to smaller contributions bounded uniformly in the size of the system. Previous results on the many-body adiabatic theorem and on the proof of linear response theory have been obtained in [1, 4], for the dynamics of zero temperature gapped systems.

The work [3] introduces new methods in the study of quantum dynamics for lattice models. In particular, the convergence of the expansion (4) actually only relies on suitable integrability properties of imaginary-time correlation functions, which hold true for many-body perturbations of gapped systems; it is an interesting open problem to understand how to apply this framework to study gapless quantum systems, already in the absence of interactions.

References

[1] S. Bachmann, W. de Roeck and M. Fraas. *The adiabatic theorem and linear response theory for extended quantum systems.* Comm. Math. Phys. **361**, 997-1027 (2018)

[2] O. Bratteli and D. W. Robinson. *Operator Algebras and Quantum Statistical Mechanics 2: Equilibrium States, Models in Quantum Statistical Mechanics.* Springer (1997)

[3] R. L. Greenblatt, G. Marcelli, M. Lange and M. Porta. *Adiabatic Evolution of Low-Temperature Many-Body Systems*, `arXiv:2211.16836`

[4] D. Monaco and S. Teufel. *Adiabatic currents for interacting electrons on a lattice.* Rev. Math. Phys. **31**, 1950009 (2019)

Upper bound for the grand canonical free energy of the Bose gas in the Gross–Pitaevskii limit

Andreas Deuchert

(joint work with Chiara Boccato and David Stocker)

The dilute Bose gas, that is, a bosonic system with rare but strong collisions, is one of the most fundamental and interesting models in quantum statistical mechanics. Its prominence is mostly due to the occurrence of the Bose–Einstein condensation (BEC) phase transition and its numerous phenomenological consequences. Triggered by the experimental realization of BEC in ultra cold alkali gases in 1995, see [1, 5], and by the subsequent experimental progress, in the past two decades there have been numerous mathematical investigations of dilute Bose gases in different parameter regimes. I refer to [11, 10, 12, 14, 15, 2, 3, 4, 6, 7, 8, 9] and to references therein for examples concerning questions in equilibrium statistical mechanics.

In the article I was presenting in my talk in Oberwolfach, we consider a system of bosons confined to a three-dimensional flat torus Λ with side length L in the grand canonical ensemble. The Hamiltonian of the system is given by

$$\mathcal{H}_N = \int_\Lambda \nabla a_x^* \nabla a_x \mathrm{d}x + \frac{1}{2}\int_{\Lambda^2} a_x^* a_y^* v_N(d(x,y)) a_y a_x \mathrm{d}(x,y) \tag{1}$$

and acts on a dense domain in the bosonic Fock space. By a_x^*, a_x I denote the usual bosonic creation and annihilation operators (actually operator-valued distributions) of a particle at the point $x \in \Lambda$ that satisfy the canonical commutation relations $[a_x, a_y^*] = \delta(x-y), [a_x, a_y] = 0 = [a_x^*, a_y^*]$. Here $\delta(x)$ is Dirac's delta distribution with unit mass at the origin. The interaction potential is of the form $v_N(d(x,y)) = N^2 v(Nd(x,y))$ with a nonnegative function v and a parameter $N > 0$ that we choose as the expected number of particles in the system. By $d(x,y)$ I denote the distance between x and y in Λ. The scattering length a_N of v_N, which is a combined measure for its range and strength, scales as $a_N/L \sim N^{-1}$. This assures that the interaction energy per particle is of the same order as the spectral gap of the Laplacian in Λ.

The quantity we are interested in is the grand canonical free energy at inverse temperatue β related to the Hamiltonian $\mathcal{H}_N$. It is defined as

$$F(\beta, N, L) = -\frac{1}{\beta}\ln\left(\mathrm{Tr}[\exp(-\beta(\mathcal{H}_N - \mu\mathcal{N}))]\right) + \mu N, \tag{2}$$

where the chemical potential μ is chosen such that the grand canonical Gibbs state

$$G = \frac{\exp(-\beta(\mathcal{H}_N - \mu\mathcal{N}))}{\mathrm{Tr}[\exp(-\beta(\mathcal{H}_N - \mu\mathcal{N}))]} \tag{3}$$

satisfies $\mathrm{Tr}[\mathcal{N}G] = N$ ($\mathcal{N}$ is the particle number operator).

The main result in our article is the following theorem. For the sake of simplicity I state it only in the special case $\beta = \kappa\beta_\mathrm{c}$ with $\kappa \in (1, \infty)$ and the inverse critical temperature for BEC β_c in the ideal gas (condensed phase).

Theorem 1. *Assume that the function $v : [0,\infty) \to [0,\infty]$ is nonnegative, compactly supported, satisfies $v(|\cdot|) \in L^3(\Lambda)$, and is strictly positive on a set of positive*

measure. By $\varrho = N/L^3$ we denote the particle density. We consider the combined limit $N \to \infty$, $\beta = \kappa\beta_c$ with $\kappa \in (1,\infty)$. The free energy in (2) satisfies the upper bound

$$F(\beta,N,L) \leq F_0^+(\beta,N,L) + 4\pi\mathfrak{a}_N L^3\left(2\varrho^2 - \varrho_0^2(\beta,N,L)\right) + \frac{\ln\left(4\beta\mathfrak{a}_N/L^3\right)}{2\beta} - \frac{1}{2\beta}\sum_{p\in(2\pi/L)\mathbb{Z}^3\setminus\{0\}}\left[\frac{16\pi\mathfrak{a}_N\varrho_0(\beta,N,L)}{p^2} - \ln\left(1+\frac{16\pi\mathfrak{a}_N\varrho_0(\beta,N,L)}{p^2}\right)\right] + O(L^{-2}N^{7/12}). \tag{4}$$

In the above formula $F_0^+(\beta,N,L)$ denotes the free energy of the non-condensed particles in the ideal gas with the chemical potential $\mu_0(\beta,N,L)$, and $\varrho_0(\beta,N,L)$ is the related condensate density. The first two terms in (4) had been justified for the first time for the dilute Bose gas in the thermodynamic limit, see [13, 15]. They also appeared in [7], where the asymptotics of the canonical free energy in the GP limit has been established with a remainder of the size $o(L^{-2}N)$ (The aim of this article was to give a proof of the BEC phase transition.). It is, however, expected that the canonical and the grand canonical free energies agree on that level of accuracy. The main novelty of the upper bound in (4) is therefore the appearance of the last two terms on the r.h.s., which are of the order $N^{2/3}\ln(N)$ and $N^{2/3}$, respectively. We highlight that the first two terms in (4) scale as $L^{-2}N^{5/3}$ and $L^{-2}N$, respectively. In the following I briefly discuss the origin of our new terms.

The third term on the r.h.s. of (4) is the free energy of the fluctuations of the interacting BEC. It originates from the following effective free energy:

$$F^{\mathrm{BEC}}(\beta,N_0,L,\mathfrak{a}_N) = -\frac{1}{\beta}\ln\left(\int_{\mathbb{C}}\exp\left(-\beta\left(4\pi\mathfrak{a}_N L^{-3}|z|^4 - \mu|z|^2\right)\right)\mathrm{d}z\right) + \mu\varrho_0(\beta,N,L)L^3. \tag{5}$$

Here $\mathrm{d}z = \mathrm{d}x\mathrm{d}y/\pi$, where x and y denote the real and imaginary part of the complex number z, respectively. The chemical potential μ in (5) is chosen such that the Gibbs distribution

$$g(z) = \frac{\exp\left(-\beta\left(4\pi\mathfrak{a}_N L^{-3}|z|^4 - \mu|z|^2\right)\right)}{\int_{\mathbb{C}}\exp\left(-\beta\left(4\pi\mathfrak{a}_N L^{-3}|z|^4 - \mu|z|^2\right)\right)\mathrm{d}z} \tag{6}$$

satisfies $\int_{\mathbb{C}}|z|^2 g(z)\mathrm{d}z = \varrho_0(\beta,N,L)L^3$ ($|z|^2$ should be interpreted as a particle number). Under the assumption of Theorem 1 we have

$$F^{\mathrm{BEC}}(\beta,N_0,L,\mathfrak{a}_N) = 4\pi\mathfrak{a}_N L^3\varrho_0^2 + \frac{\ln\left(4\beta\mathfrak{a}_N/L^3\right)}{2\beta} + O\left(L^{-2}\exp\left(-cN^{1/6}\right)\right). \tag{7}$$

In combination with $\int|z|^2 g(z)\mathrm{d}z = \varrho_0 L^3$, this implies

$$4\pi\mathfrak{a}_N L^{-3}\left(\int_{\mathbb{C}}|z|^4 g(z)\mathrm{d}z - \left(\int_{\mathbb{C}}|z|^2 g(z)\mathrm{d}z\right)^2\right) - \frac{1}{\beta}S(g) = \frac{\ln\left(16\beta\mathfrak{a}_N/L^3\right)}{2\beta} + O\left(L^{-2}\exp\left(-cN^{1/6}\right)\right), \tag{8}$$

where $S(g) = -\int g(z)\ln(g(z))\mathrm{d}z$ denotes the classical entropy of g. This explains my claim about the physical interpretation of the term on the r.h.s. from above.

The last term in (4) is related to the free energy of the Bogoliubov Hamiltonian

$$\mathcal{H}^{\mathrm{Bog}} = \sum_{p\neq 0} p^2 a_p^* a_p + 4\pi\mathfrak{a}_N \varrho_0(\beta, N, L) \sum_{p\neq 0} \left(2a_p^* a_p + a_p^* a_{-p}^* + a_p a_{-p}\right), \tag{9}$$

which depends on β via $\varrho_0(\beta, N, L)$ (From a physics point of view this Hamiltonian can be motivated by the c-number substitution $a_0^*, a_0 \mapsto \sqrt{\varrho_0(\beta, N, L)L^3}$.). The operators a_p^* and a_p create and annihilate a particle with momentum $p \in 2\pi\mathbb{Z}^3/L$, respectively. To see the relation between $\mathcal{H}^{\mathrm{Bog}}$ and the last term in (4), we note that

$$\begin{aligned}
-\frac{1}{\beta}\ln \mathrm{Tr}\exp(-\beta\mathcal{H}^{\mathrm{Bog}}) &= \frac{1}{\beta}\sum_{p\neq 0}\ln\left(1-\exp\left(-\beta\sqrt{p^2-\mu_0}\sqrt{p^2-\mu_0+16\pi\mathfrak{a}_N\varrho_0}\right)\right)\\
&= \frac{1}{\beta}\sum_{p\neq 0}\ln\left(1-\exp\left(-\beta(p^2-\mu_0)\right)\right) + 8\pi\mathfrak{a}_N L^3(\varrho-\varrho_0)\varrho_0\\
&\quad - \frac{1}{2\beta}\sum_{p\neq 0}\left[\frac{16\pi\mathfrak{a}_N\varrho_0}{p^2} - \ln\left(1+\frac{16\pi\mathfrak{a}_N\varrho_0}{p^2}\right)\right] + o(L^{-2}N^{2/3}).
\end{aligned}$$

The first and the second term on the r.h.s. contribute to F_0^+ and to the second term on the r.h.s. of (4), respectively. The third term is the novel contribution in the second line of (4).

References

[1] M.H. Anderson, J.R. Ensher, M.R. Matthews, C.E. Wieman, E.A. Cornell, *Observation of bose-einstein condensation in a dilute atomic vapor*, Science **269**, 198 (1995)

[2] G. Basti, S. Cenatiempo, B. Schlein, *A new second order upper bound for the ground state energy of dilute Bose gases*, Forum Math. Sigma **9**, e74 (2021)

[3] C. Boccato, C. Brennecke, S. Cenatiempo, B. Schlein, *Optimal Rate for Bose–Einstein Condensation in the Gross–Pitaevskii Regime*, Commun. Math. Phys. 1–85, (2019)

[4] C. Boccato, C. Brennecke, S. Cenatiempo, B. Schlein, *Bogoliubov Theory in the Gross-Pitaevskii Limit*, Acta Mathematica **222**, 219 (2019)

[5] K.B. Davis, M.-O. Mewes, M.R. Andrews, N.J. van Druten, D.S. Durfee, D.M. Kurn, W. Ketterle, *Bose-Einstein Condensation in a Gas of Sodium Atoms*, Phys. Rev. Lett. **75**, 3969 (1995)

[6] A. Deuchert, R. Seiringer, J. Yngvason, *Bose-Einstein Condensation for a Dilute Trapped Gas at Positive Temperature*, Commun. Math. Phys. **368**, 723 (2019)

[7] A. Deuchert, R. Seiringer, *Gross–Pitaevskii Limit of a Homogeneous Bose Gas at Positive Temperature*, Arch. Ration. Mech. **236**(3), 1217 (2020)

[8] S. Fournais, J.P. Solovej, *The energy of dilute Bose gases*, Ann. Math. **192**, No. 3, 893 (2020)

[9] S. Fournais, J.P. Solovej, *The energy of dilute Bose gases II: The general case*, Invent. math. **232**, 863 (2023)

[10] E.H. Lieb, R. Seiringer, J. P. Solovej, J. Yngvason, *The Mathematics of the Bose Gas and its Condensation*, Birkhäuser, Basel (2005)

[11] E.H. Lieb, J. Yngvason, *Ground State Energy of the Low Density Bose Gas*, Phys. Rev. Lett. **80**, 2504 (1998)

[12] N. Rougerie, *Scaling limits of bosonic ground states, from many-body to nonlinear Schrödinger*, arXiv:2002.02678 [math.AP] (2020)
[13] R. Seiringer, *Free Energy of a Dilute Bose Gas: Lower Bound*, Comm. Math. Phys. **279**, 595 (2008)
[14] H.-T. Yau, J. Yin, *The Second Order Upper Bound for the Ground Energy of a Bose Gas*, J. Stat. Phys. **136**, 453 (2009)
[15] J. Yin, *Free Energies of Dilute Bose Gases: Upper Gound*, J. Stat. Phys. **141**, 683 (2010)

The Random Interchange Model and Quantum Spin Systems

Daniel Ueltschi

On the graph $(\Lambda, \mathcal{E})$ one considers a random sequence of i.i.d. edges, $(e_1, \dots, e_n)$, and one looks at the permutation

$$\sigma = \tau_{e_n} \circ \cdots \circ \tau_{e_1},$$

where τ_{e_i} is the transposition of the endpoints of the edge e_i. The number n is typically random, $n \sim \text{Poisson}(\beta|\mathcal{E}|)$, with $\beta > 0$ a parameter. The question is whether a transition (in β) occurs where the permutation contains cycles of diverging length. Namely, there should exist β_0 such that cycle lengths remain finite when $\beta < \beta_0$, and become "infinite" when $\beta > \beta_0$.

Among several motivations, the random interchange model can represent the quantum Heisenberg model [12]. There are interesting results on trees [3, 7], on the complete graph [11, 4, 2, 5], on the hypercube [9], and further (random) graphs [1, 8, 10].

Very recently, Dor Elboim and Allan Sly have produced a remarkable analysis of the model on $\mathbb{Z}^d$ with $d \geq 5$ [6]. It uses the "cyclic-time random walk" representation [3] and shows that it is transient with positive probability. The strategy is rather natural but its implementation is difficult. The authors prove that the walk satisfies several properties (in essence, that it diffuses sufficiently), which they can prove for time intervals of the form $(4^{m-1}, 4^m]$, inductively on m. An essential step is to establish an "escape lemma" that guarantees that the walk does not get trapped by its past.

I reviewed the interchange process, its relation with the Heisenberg model and also with the Bose-Hubbard model, and I explained some of the ideas of Elboim and Sly.

References

[1] R. Adamczak, M. Kotowski, P. Miłoś, *Phase transition for the interchange and quantum Heisenberg models on the hamming graph*, Ann. Inst. Henri Poincaré 57, 273–325 (2021)
[2] G. Alon, G. Kozma, *The probability of long cycles in interchange processes*, Duke Mathematical Journal 162(9), 1567–1585 (2013)
[3] O. Angel, *Random infinite permutations and the cyclic time random walk*, Discr. Math. Theoret. Comp. Sci. (2003)
[4] N. Berestycki, Emergence of giant cycles and slowdown transition in random transpositions and k-cycles, Electr. J. Probab. 16, 152–173 (2011)
[5] N. Berestycki, G. Kozma, *Cycle structure of the interchange process and representation theory*, Bull. Soc. Math. France 143, 265–281 (2015)

[6] D. Elboim, A. Sly, *Infinite cycles in the interchange process in five dimensions*, arXiv:2211.17023 (2023)
[7] A. Hammond, *Sharp phase transition in the random stirring model on trees*, Probab. Th. Rel. Fields 161, 429–448 (2015)
[8] J. Hermon, J. Salez, *The interchange process on high-dimensional products*, Ann. Appl. Probab. 31, 84–98 (2021)
[9] R. Kotecký, P. Miłoś, D. Ueltschi, *The random interchange process on the hypercube*, Electr. Commun. Probab. 21, 1–9 (2016)
[10] R. Poudevigne, *Macroscopic cycles for the interchange and quantum Heisenberg models on random regular graphs*, arXiv:2209.13370 (2022)
[11] O. Schramm, *Compositions of random transpositions*, Israel J. Math. 147, 221–243 (2005)
[12] B. Tóth, *Improved lower bound on the thermodynamic pressure of the spin* 1/2 *Heisenberg ferromagnet*, Lett. Math. Phys. 28, 75–84 (1993)

Long persistence of localization in a disordered anharmonic chain beyond the atomic limit

Wojciech de Roeck
(joint work with François Huveneers, Oskar A. Prośniak)

We consider the nonlinear Klein-Gordon chain with disorder, given by the classical Hamiltonian

$$\mathsf{H}(q,p) \;=\; \sum_{x\in\Lambda_L}\left(\frac{p_x^2}{2}+\frac{\omega_x^2 q_x^2}{2}+\frac{\eta}{2}(q_x-q_{x+1})^2+\frac{\gamma}{4}q_x^4\right) \tag{1}$$

where Λ_L is the discrete torus $\mathbb{Z}/(L\mathbb{Z})$, η,γ are positive parameters, and (q_x,p_x) are canonically conjugate variables. The frequencies ω_x are i.i.d. random variables, with a compact density. Let us write $\rho(x)$ for the expression between brackets in (1), so that $\rho(x)$ is the energy density, and $\sum_x \rho(x)$ is conserved in time.

Numerical motivation. We study the evolution of ρ in time, i.e. under the Hamiltonian evolution. Our interest in this system mainly stems from the following numerical observation, made first in [1]: If ρ is originally compactly supported, say around $x=0$, then the width w of the ρ-distribution, defined as

$$w^2=\sum_x x^2\rho(x)$$

grows as $t^{1/6}$ for large times (but of course small enough so that the periodic boundary condition is not yet relevant). This power-law growth of w is very surprising from a theoretical point of view, as we explain next.

Harmonic limit $\gamma\to 0$. Let us first introduce the Anderson Schrodinger operator $\mathcal{H}$ acting on $\ell^2(\Lambda_L)$ and given by

$$\mathcal{H}f(x)=\omega_x^2 f(x)-\eta(f(x+1)+f(x-1)-2f(x)).$$

Since $\mathcal{H}$ is nonnegative, we can write its eigenvalues as ν_k^2 and we denote the corresponding normalized eigenfunctions as $\psi_k\in\ell^2(\Lambda_L)$. If the anharmonicity γ

in the Hamiltonian (1) is set to 0, then H is quadratic and it can be cast as a sum over modes:

$$H = \sum_k E_k, \qquad E_k = \frac{1}{2}\left(P_k^2 + \nu_k^2 Q_k^2\right)$$

Here, (Q_k, P_k) are conjugate variables given by $P_k = \sum_x \psi_k(x) p_x$ and $Q_k = \sum_x \psi_k(x) q_x$ with ψ_k and ν_k^2 as introduced above. As is well-known, the operator $\mathcal{H}$ exhibits Anderson localization: the functions ψ_k are exponentially concentrated around a k-dependent localization center. The consequence of this is that the density ρ does not spread. More precisely, the width w is uniformly bounded (by a disorder dependent constant) in time.

When $\gamma > 0$, the above reasoning does of course no longer apply. However, as the width w grows, the density ρ decreases locally, since its total sum has to be conserved. Inspection of (1) shows that the anharmonic term $\frac{\gamma}{4} q_x^4$ becomes comparatively weaker when ρ tends to zero. In fact, a scaling argument allows to recast the dynamics to make it apparent that the *effective anharmonicity* is $\rho\gamma$. Hence, as the packet spreads, the system gets ever closer to the harmonic limit, i.e. closer to a system that does not spread at all.

Several theoretical [2, 3] and even mathematical works [4, 5, 6, 7] have suggested that a well localized system as the one in the $\gamma = 0$ limit, responds very weakly to perturbations. In particular, we expect transport and dissipation to manifest themselves by non-perturbative effects, in particular slower than any power law in the perturbation parameter. We call this ubiquitous phenomenon "Quasi-localization", see e.g. [8].

In the light of such results, the power law spreading $w(t) \propto t^{1/\kappa}$ for some $\kappa > 0$ seems too rapid. One would expect instead a law of the non-perturbative form $w(t) \propto (\log t)^{\kappa}$.

The most constructive contribution to this tension between numerics and theory would be to prove a time-dependent upper bound on $w(t)$ of the above non-perturbative form, but that seems currently out of reach.

Our mathematical contribution. Let, for any observable A,

$$\langle A \rangle = (1/Z) \int_{\mathbb{R}^L \times \mathbb{R}^L} dqdp\, e^{-H} A, \qquad Z = \int_{\mathbb{R}^L \times \mathbb{R}^L} dqdp\, e^{-H}$$

denote the Gibbs state at unit temperature. For any mode k and time $t \geq 0$, let us define the *decorrelation indicator*

$$\eta_k(t) = \frac{1}{2} \frac{\langle (E_k(t) - E_k(0))^2 \rangle}{\langle E_k(t)^2 \rangle - \langle E_k(t) \rangle^2}.$$

At $t = 0$, we obviously have $\eta_k(0) = 0$, whereas, if we pretend that the Hamiltonian dynamics is mixing, i.e. $\langle E_k(t) E_k(0) \rangle \to \langle E_k \rangle^2$, then $\eta_k(t) \to 1$ as $t \to \infty$. The mixing assumption is not realistic due to conservation of energy, but the relevance of this obstruction disappears as $L \to \infty$. The decorrelation indicator allows us to quantify how fast the thermal system returns to equilibrium. In accordance with the intuition about the non-perturbative origin of dissipation explained above, we expect that the approach of η_k to 1 takes a time longer than any power law in

γ. This is indeed what we prove in [9], though after an additional averaging over modes k:

Theorem 1. *For any natural number n, there exist deterministic constants $C_n < +\infty$ and $c > 0$ so that*

$$\limsup_{L\to\infty} \frac{1}{L}\sum_{k=1}^{L} \eta_k(t) \;\le\; C_n\left(\gamma^c + (\gamma^n t)^2\right) \qquad a.s. \quad \forall\gamma \ge 0, \quad \forall t \ge 0.$$

Our numerical contribution. We perform numerics to determine the mode-averaged decorrelation indicators

$$\eta_\gamma(t) = \frac{1}{L}\sum_{k=1}^{L} \eta_k(t),$$

where we made explict the dependence on the anharmonicity γ. We find [10] that the growth in time of $\eta_\gamma(t)$, for small anharmonicities, seems to depend on the anharmonicity γ via a scaling relation

$$\eta_\gamma(t) = \eta(\gamma^4 t) \tag{2}$$

A theoretical (though non-rigorous) analysis reveals that this scaling relation is equivalent to the numerically observed scaling relation $w \propto t^{1/6}$. In this sense we confirm the numerical findings that motivated our work. However, and this is the crucial point, *we know that the scaling relation* (2) *cannot be correct for very small γ, as it would contradict Theorem 1.* Therefore, we conclude that current numerics is not capable of probing the asymptotic regime of vanishing anharmonicity. This makes it plausible that also the numerics on the growth of the width w did not reach the asymptotic regime yet. We conclude hence that this system is indeed likely described by "Quasi-localization" and the true asymptotic growth of $w(t)$ is slower than a power law in time.

References

[1] S. Flach, D. O. Krimer, and C. Skokos, "Universal spreading of wave packets in disordered nonlinear systems," *Phys. Rev. Lett.*, vol. 102, p. 024101, Jan 2009.

[2] S. Fishman, Y. Krivolapov, and A. Soffer, "The nonlinear schrödinger equation with a random potential: results and puzzles," *Nonlinearity*, vol. 25, no. 4, p. R53, 2012.

[3] D. M. Basko, "Weak chaos in the disordered nonlinear schrödinger chain: Destruction of anderson localization by arnold diffusion," *Annals of Physics*, vol. 326, pp. 1577–1655, Jul 2011.

[4] J. Fröhlich, T. Spencer, and C. E. Wayne, "Localization in disordered, nonlinear dynamical systems," *Journal of Statistical Physics*, vol. 42, pp. 247–274, 1986.

[5] G. Benettin, J. Fröhlich, and A. Giorgilli, "A nekhoroshev-type theorem for hamiltonian systems with infinitely many degrees of freedom," *Communications in Mathematical Physics*, vol. 119, pp. 95–108, Mar 1988.

[6] W.-M. Wang and Z. Zhang, "Long time anderson localization for the nonlinear random schrödinger equation," *Journal of Statistical Physics*, vol. 134, pp. 953–968, Mar 2009.

[7] F. Huveneers, "Drastic fall-off of the thermal conductivity for disordered lattices in the limit of weak anharmonic interactions," *Nonlinearity*, vol. 26, p. 837, Feb 2013.

[8] W. De Roeck and F. Huveneers, "Asymptotic localization of energy in nondisordered oscillator chains," *Communications on Pure and Applied Mathematics*, vol. 68, no. 9, pp. 1532–1568, 2015.

[9] W. De Roeck, F. Huveneers, and O. A. Prośniak, "Long persistence of localization in a disordered anharmonic chain beyond the atomic limit," *arXiv preprint arXiv:2308.11243*, 2023.

[10] W. De Roeck, F. Huveneers, and O. A. Prośniak, "Slow dissipation and spreading in disordered classical systems: A direct comparison between numerics and mathematical bounds," *arXiv preprint arXiv:2308.10572*, 2023.

Bose Metaphysics

Manfred Salmhofer

Georg Christoph Lichtenberg describes metaphysics as the art of catching, in a dark room, a black cat that is not actually in there. Replacing the dark room with a quantum many-boson system and the cat by reflection positivity motivates the title of this talk: the standard quantum mechanical models for interacting bosons have been known not to be reflection-positive, yet this talk gives an account of a search for reflection positivity for bosons on a lattice and for closely related models.

To this end, the effect of spatial and temporal reflections is studied in a discrete-time functional integral representation, which takes the form of a lattice spin system with a complex action, and which converges to the grand canonical partition function for quantum bosons in the time-continuum limit. Tracing the effect of iterated reflections motivates the definition of a class of lattice field theories in external $\mathbb{Z}_2$ gauge fields. One of them is reflection-positive, and a variant of Gaussian domination is shown.

The first model considered here is a spin system with a complex action on a space-time lattice $\mathbb{X}$, defined as follows. Let $\mathbb{X} = \mathbb{T} \times X$ where $\mathbb{T} = \epsilon(\mathbb{Z}/\ell\mathbb{Z})$ is a discrete torus in Euclidian time with spacing $\epsilon = \frac{\beta}{\ell}$ and sidelength β, and $X = \eta\mathbb{Z}^d/L\mathbb{Z}^d$ is a discrete torus of lattice spacing $\eta > 0$ and sidelength $L = \eta\hat{L} > 0$ with $\hat{L} \in \mathbb{N}$ large. Spin configurations are maps $a : \mathbb{X} \to \mathbb{C}$, $(\tau, x) \mapsto a_{\tau,x}$. With the notations $(f \mid g)_{\mathbb{X}} = \int_{\tau,x} f_{\tau,x}\, g_{\tau,x}$ (here integral notation is used for the Riemann sums over lattice points), the partition function is

$$Z_{\mathbb{X}} = \left\langle e^{-S_{0,\mathbb{X}}(a)} \right\rangle_{1,\mathbb{X}} . \tag{1}$$

The action S_0 is quadratic in a and given by

$$S_0(a) = (\bar{a} \mid -\partial_\tau^+ a)_{\mathbb{X}} + (\bar{a} \mid -\Delta a)_{\mathbb{X}} \tag{2}$$

where ∂_τ^+ denotes the forward discretized derivative with respect to τ, $-\Delta \geq 0$ denotes the graph Laplacian on X, and

$$\langle F(a) \rangle_{1,\mathbb{X}} = \int_{\mathbb{C}^{\mathbb{X}}} F(a) \prod_{(\tau,x)\in\mathbb{X}} \mathrm{d}\mu_1(a_{\tau,x}) \tag{3}$$

with

$$\mathrm{d}\mu_1(a) = e^{-\frac{\mathrm{v}}{2}\,(|a|^2-\rho)^2 - I_1}\, \frac{\mathrm{d}\bar{a}\wedge a}{2\pi i} , \qquad \rho = \frac{\mu}{\mathrm{v}} . \tag{4}$$

μ_1 is a positive measure on $\mathbb{C}$, and the constant I_1 is chosen such that μ_1 is normalized.

It is proven in [5, 6] that the limit $\ell \to \infty$ of $Z_{\mathbb{X}}$ is the partition function of the grand canonical ensemble at inverse temperature β and chemical potential μ, for the Bose-Hubbard model, i.e. a quantum system of bose particles on a lattice with the graph Laplacian $-\Delta$ as the kinetic term and an on-site repulsion v. In the limit $\mathrm{v} \to \infty$, μ_1 concentrates on the circle of radius $\sqrt{\rho}$, and in this limit, ρ is equal to the particle density. The action is invariant under the global $U(1)$-transformation $a(\tau, x) \to e^{i\phi} a(\tau, x)$. Spontaneous breaking of this symmetry is equivalent to Bose-Einstein condensation. In the present statistical-mechanical setup, it can occur only in the thermodynamic limit $\hat{L} \to \infty$ (η is kept fixed throughout).

Reflection positivity has allowed to prove spontaneous breaking of continuous symmetries in statistical-mechanical systems in a very elegant way [1]. An alternative strategy is a renormalization group method, used to treat $O(N)$ vector models of classical statistical mechanics by Bałaban [3] and currently being applied to the many-boson system in [4]. It is much more generally applicable and yields very detailed information, but also much more demanding on a technical level. Its application to the Bose condensation is still work in progress [4].

Spatial and temporal reflections r at planes in-between the lattice planes [1] can be lifted to the natural algebra of functions of a, by setting $(\theta F)(a) = \overline{F(ra)}$, where $(ra)_{\tau,x} = a_{r(\tau,x)}$. The expectation value functional $\langle \cdot \rangle_{1,\mathbb{X}}$, is reflection-positive, i.e. $\langle F\, \theta F \rangle_{1,\mathbb{X}} \geq 0$ for all F that depend only on the a variables on one side of the reflection plane [1, 2]. However, the time derivative term in the action (2) is not invariant under spatial reflections. Alternative definitions of reflections that leave this term invariant do not leave the term involving the Laplacian invariant.

Writing out the integral for $|Z_{\mathbb{X}}|^2 = Z_{\mathbb{X}} \overline{Z_{\mathbb{X}}}$ motivates a "doubling" of the system: if the integration variable in $\overline{Z_{\mathbb{X}}}$ is called a', it is natural to combine a and a' into a two-component vector $\mathtt{a}$ and the action is $\mathbb{S}_0(\mathtt{a}) = S_0(a) + S_0(\bar{a}')$. This allows for other definitions of reflections, in particular the reflection $(\Theta F)(\mathtt{a}) = \overline{F(\sigma_1 r \mathtt{a})}$, where $\sigma_1 = \left(\begin{smallmatrix} 0 & 1 \\ 1 & 0 \end{smallmatrix} \right)$. The reflection Θ now also involves an exchange $a \leftrightarrow a'$, and it leaves the time derivative term of $\mathbb{S}_0$ invariant. Its action on the spatial derivative part of $\mathbb{S}_0$ motivates a natural generalization of $\mathbb{S}_0$ that includes an external $\mathbb{Z}_2$ gauge field, where the local gauge transformation is $\mathtt{a}_{\tau,x} \mapsto \sigma_{h_{\tau,x}} \mathtt{a}_{\tau,x}$, with $h_{\tau,x} \in \mathbb{Z}_2$ (and σ_0 the 2×2 identity matrix) and the spatial derivative terms are changed to gauge-covariant derivatives by including $\mathbb{Z}_2$ link variables $\sigma_{g_{\tau,x,\mu}}$ (here $\mu \in \{1, \ldots, d\}$ indexes the spatial components of vectors in X). Studying the effect of reflections for this class of models reveals that the model with all gauge fields $\sigma_{g_{\tau,x,\mu}} = 1$ is reflection-positive, a variant of Gaussian domination holds, and it implies a generalization of the infrared bound, corresponding to the action $\mathbb{S}_0$, for the two-point correlation function of $\mathtt{a}$. Moreover, in the limit $\mathrm{v} \to \infty$, the lower bound on a local spin expectation value, required in proofs of symmetry breaking using reflection positivity [1], follows from the constraint $|a_{\tau,x}|^2 = \rho$. Details will appear in [6] and [7].

References

[1] J. Fröhlich, B. Simon, T. Spencer, *Infrared Bounds, Phase Transitions and Continuous Symmetry Breaking*, Commun. Math. Phys. **50** (1976) 79–85

[2] J. Fröhlich, R.B. Israel, E.H. Lieb, B. Simon, *Phase Transitions and Reflection Positivity. I. General Theory and Long Range Lattice Models*, Commun. Math. Phys. **62** (1978) 1–34

[3] T. Bałaban, *A low temperature expansion for classical N-vector models I,II,III*, Commun. Math. Phys. **167** (1995) 103–154; **182** (1996) 675–721, **196** (1998) 485–521

[4] T. Bałaban, J. Feldman, H. Knörrer, E. Trubowitz, *Complex Bosonic Many-Body Models: Overview of the Small Field Parabolic Flow*, Ann. Henri Poincaré **18** (2017) 2873, and references therein

[5] M. Salmhofer, *Functional Integral and Stochastic Representations for Ensembles of Identical Bosons on a Lattice*, Commun. Math. Phys. **385** (2021) 1163–1211

[6] M. Salmhofer, *Analyticity and Decay of Bose Correlations*, to appear

[7] M. Salmhofer, *Reflections on Bose Systems*, to appear

The Spectral Gap and Low-Energy Spectrum in Mean-Field Quantum Spin Systems

Chokri Manai

(joint work with Simone Warzel)

Mean-field quantum spin systems are ubiquitous in effective descriptions of a variety of phenomena. A popular example is the family of Lipkin-Meshkov-Glick (LMG) Hamiltonians, which were originally conceived in [1, 2, 3] to explain shape transitions in nuclei. This family includes the quantum Curie-Weiss (CW) Hamiltonian, whose simplicity continues to draw the attention of many communities [4, 5, 6, 7, 8, 9, 10]. In particular, in [11] such models were used to test conjectures related to quantum annealing for which information about the spectral gap is crucial.

We consider mean-field Hamiltonians which are defined in terms of a symmetric polynomial $P : \mathbb{R}^3 \to \mathbb{R}$ of fixed degree on which the three components of the total spin-vector $\mathbf{S} = \sum_{n=1}^{N} \mathbf{S}(n)$ is evaluated:

$$H = N \, \mathrm{P}\Big(\tfrac{2}{N}\mathbf{S}\Big), \tag{1}$$

where the scaling ensures that Hamiltonian is extensive. For a system of N interacting qubits, the Hilbert space on which these operators act is the tensor product $\mathcal{H}_N = \bigotimes_{n=1}^{N} \mathbb{C}^2$. The vectors $\mathbf{S}(n) = \mathbb{1} \otimes \cdots \otimes \mathbf{S} \otimes \cdots \otimes \mathbb{1}$ stand for the natural lift of the spin vectors $\mathbf{S} = (S_x, S_y, S_y)$ to the n-th component of the tensor product. On each copy of $\mathbb{C}^2$ the spin vector coincides with the three generators of $SU(2)$:

$$S_x = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad S_y = \frac{1}{2}\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad S_z = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

For example, the Quantum Curie-Weiß model is described by $P(\mathbf{m}) = -Jm_3^2 - \gamma m_1$ with $J, \gamma \in \mathbb{R}$. An important observation is that H is a function of the total spin $\mathbf{S}$,

so H is block diagonal with respect to the decomposition irreducible representations of the total spin:

$$\mathcal{H}_N \equiv \bigoplus_{J=\frac{N}{2}-\lfloor\frac{N}{2}\rfloor}^{N/2} \bigoplus_{\alpha=1}^{M_{N,J}} \mathbb{C}^{2J+1}, \qquad M_{N,J} = \frac{2J+1}{N+1}\binom{N+1}{\frac{N}{2}+J+1}.$$

Here $J = N/2, N/2-1, \ldots$ denotes the total spin number and $M_{N,J}$ is the corresponding degeneracy [12].

Let us briefly discuss the thermodynamics of the corresponding mean-field model. The free energy is determined by minimizing a variational functional involving the classical energy P on the unit ball and the (shifted) binary entropy

$$I(r) := \begin{cases} -\frac{1+r}{2}\ln\frac{1+r}{2} - \frac{1-r}{2}\ln\frac{1-r}{2}, & r \in (0,1). \\ 0, & r \in \{0,1\} \end{cases}$$

Proposition 1. *For a mean-field Hamiltonian $H = N\ P\Big(\frac{2}{N}\mathbf{S}\Big)$ with a symmetric polynomial P, the pressure for any $\beta > 0$ is given by:*

$$p(\beta) := \lim_{N\to\infty} N^{-1} \ln \mathrm{Tr} \exp\Big(-\beta N P\Big(\tfrac{2}{N}\mathbf{S}\Big)\Big) = \max_{r\in[0,1]} \Big\{ I(r) - \beta \min_{\Omega\in S^2} P(r\mathbf{e}(\Omega)) \Big\}$$

with the unit vector $\mathbf{e}(\Omega)$ in the angular direction Ω.

As a special case with constant field $P(\mathbf{m}) = -\lambda m_3$, $\lambda \geq 0$, one obtains the Legendre relation

$$\ln 2\cosh(\beta\lambda) = \max_{r\in[0,1]} \left[I(r) + \beta\lambda r \right].$$

By inverting this Legendre transform, one obtains the slightly more familiar form

$$p(\beta) = \max_{r\in[0,1]} \min_{\lambda\geq 0} \left\{ \ln 2\cosh(\beta\lambda) - \beta \left(\min_{\Omega\in S^2} \mathrm{P}\left(r\mathbf{e}(\Omega)\right) + \lambda r \right) \right\}.$$

As far as we know, the limiting formula for the pressure was first rigorously established by Fannes, Spohn and Verbeurre [13]. Their approach exploits the exchange symmetry of the mean-field Hamiltonian using a version of the quantum de Finetti theorem and easily extends to higher spin numbers. An alternative, for our purposes more insightful, proof strategy is based on coherent states and the Berezin-Lieb inequality.

It is well known that the study systems of large spin quantum number J are to be facilitated by Bloch coherent states $\big|\Omega, J\rangle$ on the Hilbert space $\mathbb{C}^{2J+1}$ [14, 15, 16, 17, 18]. The state $\big|\Omega, J\rangle \in \mathbb{C}^{2J+1}$ is the normalized eigenstate of the spin operator $\langle \mathbf{e}(\Omega), \mathbf{S}$ corresponding to (maximal) eigenvalue J on the Hilbert space $\mathbb{C}^{2J+1}$. More explicit formulas can be found in [14]. The Bloch coherent states form an overcomplete set of vectors as expressed through the resolution of unity $\frac{2J+1}{4\pi}\int \big|\Omega, J\rangle\langle\Omega, J\big|\ d\Omega = \mathbb{1}_{\mathbb{C}^{2J+1}}$. Every linear operator G on $\mathbb{C}^{2J+1}$ is associated with a lower and upper symbol: the lower symbol is $G(\Omega, J) := \langle\Omega, J\big|G\big|\Omega, J\rangle$, and the upper symbol is characterized through $G = \frac{2J+1}{4\pi}\int g(\Omega, J)\ \big|\Omega, J\rangle\langle\Omega, J\big|\ d\Omega$.

The Berezin and Lieb's semiclassical bounds [14, 19] allow to bound the partition in terms of the symbols g and G:

$$\frac{2J+1}{4\pi}\int e^{-\beta G(\Omega,J)}d\Omega \leq \mathrm{Tr}_{\mathbb{C}^{2J+1}}\, e^{-\beta G} \leq \frac{2J+1}{4\pi}\int e^{-\beta g(\Omega,J)}d\Omega.$$

In the semiclassical limit of large spin quantum number J, these bounds are known to asymptotically coincide [14, 20]. In our situation, we have a more explicit bound using the corresponding classical polynomial function on the unit ball B_1, which parametrises the Hilbert space semiclassically. With spherical coordinates $\mathbf{e}(\Omega) = (\sin\theta\cos\varphi, \sin\theta\sin\varphi, \cos\theta) \in S^2$, one has

$$\sup_{0\leq J\leq N/2}\left\| \mathrm{P}\Big(\tfrac{2}{N}\mathbf{S}\Big)\Big|_{\mathbb{C}^{2J+1}} - \frac{2J+1}{4\pi}\int \mathrm{P}\Big(\tfrac{2J}{N}\mathbf{e}(\Omega)\Big)\,|\Omega,J\rangle\langle\Omega,J|\,d\Omega\right\| \leq \mathcal{O}(N^{-1})$$

for the operator norm $\|\cdot\|$ on $\mathbb{C}^{2J+1}$. This statement is a quantitative version of Duffield's theorem [20].

From here one immediately obtains the result of the proposition above. However, we note that the semiclassical estimates are correct up to order $\mathcal{O}(1)$, so one might hope that the semiclassics allows us to deduce more precise characteristics of the mean-field Hamiltonian. Indeed, our main result is an explicit formula for the spectral gap in the thermodynamic limit, which we will describe next. Our method also allows to determine the whole low energy spectrum and the corresponding eigenstates.

We will denote by $E_0(H) \leq E_1(H) \leq E_2(H) \leq \dots$ the ordered sequence of its eigenvalues counted with multiplicities. In particular, the existence and leading asymptotic value of the spectral gap

$$\mathrm{gap}\, H = E_1(H) - E_0(H)$$

can be read of from the location of the minimum $\mathbf{m}_0$ of the polynomial $P : \mathbb{R}^3 \to \mathbb{R}$ restricted to B_1. In case the minimum is unique and located on the surface S^2, the operator (1) generically has a spectral gap. The situation with multiple minima is more involved, but our methods can be extended to this situation. To leading order in N, the value of this gap is in fact completely determined by the coefficients of the quadratic polynomial which is uniquely associated with P. In view of the notorious difficulty of determining the spectral gap in quantum lattice systems [21, 22, 23, 24], this simplicity might be somewhat surprising.

Broadly speaking, our results are in accordance with the general belief of fluctuation theory that the second-order approximation to P, which involves the gradient $\nabla P(\mathbf{m}_0)$ and the Hessian $D_P(\mathbf{m}_0) = (\partial_j\partial_k P(\mathbf{m}_0))_{j,k=1}^3$, yields the description of the low-energy spectra. Related statements have been proven in the context of mean-field Bose systems (see e.g. [25, 26]). For the precise formulation of such a result for quantum spin systems and in order to point out a subtlety caused by the geometry, we need some basic geometric facts on functions on B_1.

If $\mathbf{m}_0 \in S^2$ is a minimum of P on B_1, the gradient either vanishes or points towards the center of the ball, $\nabla P(\mathbf{m}_0) = -|\nabla P(\mathbf{m}_0)|\,\mathbf{m}_0$. The quadratic approximation of the polynomial is then given by $D_P(\mathbf{m}_0)$ projected on the directions

perpendicular to $\mathbf{m}_0$. In terms of the normalized directional vector $\mathbf{e}_{\mathbf{m}_0} = \frac{\mathbf{m}_0}{|\mathbf{m}_0|}$ we set $Q_\perp := \mathbb{1}_{\mathbb{R}^3} - \mathbf{e}_{\mathbf{m}_0}^T \mathbf{e}_{\mathbf{m}_0}$, which is understood as a linear projection map on $\mathbb{R}^3$. Introducing a local chart $\Phi : \mathbb{R}^2 \to T_{\mathbf{m}_0} S^2$, the linear map on ran $Q_\perp \equiv T_{\mathbf{m}_0} S^2$ given by

$$D_P^\perp(\mathbf{m}_0) := Q_\perp D_P(\mathbf{m}_0) Q_\perp + |\nabla P(\mathbf{m}_0)|\, Q_\perp$$

is then the quadratic approximation to $P \circ \Phi$ at $\mathbf{m}_0$. The shift of the Hessian in cartesian coordinates by the norm of the gradient $|\nabla P(\mathbf{m}_0)|$ is thus an effect of the constraint due to the spherical geometry.

Theorem 2. *Let H be a self-adjoint operator on $\mathcal{H}_N$ of the form (1) with a symmetric polynomial $P : \mathbb{R}^3 \to \mathbb{R}$ of fixed degree. Suppose that the minimum of P restricted to the unit ball B_1 is unique and located at a point $\mathbf{m}_0 \in S^2$ on the unit sphere. Then,*

$$\operatorname{gap} H = 2 \min \left\{ |\nabla P(\mathbf{m}_0)|, \sqrt{\det D_P^\perp(\mathbf{m}_0)} \right\} + o(1).$$

is the spectral gap above the unique ground state in case the rhs is strictly positive.

References

[1] H. Lipkin, N. Meshkov, and A. Glick, "Validity of many-body approximation methods for a solvable model: (i). exact solutions and perturbation theory," *Nuclear Physics*, vol. 62, no. 2, pp. 188–198, 1965.

[2] N. Meshkov, A. Glick, and H. Lipkin, "Validity of many-body approximation methods for a solvable model: (ii). linearization procedures," *Nuclear Physics*, vol. 62, no. 2, pp. 199–210, 1965.

[3] A. Glick, H. Lipkin, and N. Meshkov, "Validity of many-body approximation methods for a solvable model: (iii). diagram summations," *Nuclear Physics*, vol. 62, no. 2, pp. 211–224, 1965.

[4] R. Brout, K. A. Müller, and H. Thomas, "Tunnelling and collective excitations in a microscopic model of ferroelectricity," *Solid State Communications*, vol. 4, pp. 507–510, 1966.

[5] P. Ribeiro, J. Vidal, and R. Mosseri, "Exact spectrum of the Lipkin-Meshkov-Glick model in the thermodynamic limit and finite-size corrections," *Physical Review E*, vol. 78, no. 2, p. 021106, 2008.

[6] L. Chayes, N. Crawford, D. Ioffe, and A. Levit, "The phase diagram of the quantum Curie-Weiss model," *Journal of Statistical Physics*, vol. 133, no. 1, pp. 131–149, 2008.

[7] K. Landsman, V. Moretti, and C. J. F. van de Ven, "Strict deformation quantization of the state space of $M_k(\mathbb{C})$ with applications to the Curie–Weiss model," *Reviews in Mathematical Physics*, vol. 32, no. 10, p. 2050031, 2020.

[8] C. J. F. van de Ven, G. C. Groenenboom, R. Reuvers, and N. P. Landsman, "Quantum spin systems versus Schrödinger operators: a case study in spontaneous symmetry breaking," *SciPost Phys.*, vol. 8, p. 022, 2020.

[9] J. E. Björnberg, J. Fröhlich, and D. Ueltschi, "Quantum spins and random loops on the complete graph," *Communications in Mathematical Physics*, vol. 375, no. 3, pp. 1629–1663, 2020.

[10] A. Bulekov, "Semiclassical approximation for the Curie–Weiss model," *Journal of Physics: Conference Series*, vol. 1740, no. 1, p. 012069, 2021.

[11] V. Bapst and G. Semerjian, "On quantum mean-field models and their quantum annealing," *Journal of Statistical Mechanics: Theory and Experiment*, vol. 2012, no. 06, p. P06007, 2012.

[12] V. V. Mihailov, "Addition or arbitrary number of identical angular momenta," *Journal of Physics A: Mathematical and General*, vol. 10, no. 2, p. 147, 1977.

[13] M. Fannes, H. Spohn, and A. Verbeure, "Equilibrium states for mean field models," *Journal of Mathematical Physics*, vol. 21, no. 2, pp. 355–358, 1980.
[14] E. H. Lieb, "The classical limit of quantum spin systems," *Commun. Math. Phys.*, vol. 31, pp. 327–340, 1973.
[15] K. Hepp and E. H. Lieb, "Equilibrium statistical mechanics of matter interacting with the quantized radiation field," *Physical Review A*, vol. 8, pp. 2517–2525, 1973.
[16] T. Michoel and B. Nachtergaele, "Central limit theorems for the large-spin asymptotics of quantum spins," *Probab. Theory Relat. Fields*, vol. 130, pp. 493–517, 2004.
[17] T. Michoel and B. Nachtergaele, "The large-spin asymptotics of the ferromagnetic XXZ chain," *Markov Processes and Related Fields*, vol. 11, pp. 237–266, 2005.
[18] M. Biskup, L. Chayes, and S. Starr, "Quantum spin systems at positive temperature," *Communications in Mathematical Physics*, vol. 269, no. 3, pp. 611–657, 2007.
[19] F. Berezin, "Convex functions of operators.," *Mat. Sb.*, vol. 88, pp. 268–276, 1972.
[20] N. G. Duffield, "Classical and thermodynamic limits for generalised quantum spin systems," *Communications in Mathematical Physics*, vol. 127, pp. 27–39, 1 1990.
[21] F. D. M. Haldane, "Continuum dynamics of the 1-D Heisenberg antiferromagnet: Identification with the O(3) nonlinear sigma model," *Physics Letters A*, vol. 93, no. 9, pp. 464–468, 1983.
[22] I. Affleck and E. H. Lieb, "A proof of part of Haldane's conjecture on spin chains," *Letters in Mathematical Physics*, vol. 12, no. 1, pp. 57–69, 1986.
[23] I. Affleck, "Quantum spin chains and the Haldane gap," *Journal of Physics: Condensed Matter*, vol. 1, no. 19, p. 3047, 1989.
[24] T. Koma, B. Nachtergaele, and S. Starr, "The spectral gap for the ferromagnetic spin-J XXZ chain," *Adv. Theor. Math. Phys.*, vol. 5, pp. 1047–1090, 2001.
[25] P. Grech and R. Seiringer, "The excitation spectrum for weakly interacting bosons in a trap," *Communications in Mathematical Physics*, vol. 322, no. 2, pp. 559–591, 2013.
[26] M. Lewin, P. T. Nam, S. Serfaty, and J. P. Solovej, "Bogoliubov spectrum of interacting Bose gases," *Communications on Pure and Applied Mathematics*, vol. 68, no. 3, pp. 413–471, 2015.

Emergent Quasi-Bosonicity in Interacting Fermi Gases

Martin Ravn Christiansen
(joint work with Christian Hainzl, Phan Thành Nam)

We consider a Fermi gas in the mean-field setting. The Hamiltonian of such a system can be written in the form of a quadratic Hamiltonian with respect to "quasi-bosonic" operators. By applying the theory of Bogolubov transformations the Hamiltonian can be approximately diagonalized to obtain the leading contribution to the correlation energy. In this talk we will see how this quasi-bosonic behaviour emerges, and how to efficiently carry out such a diagonalization procedure in the non-exact case.

Semi-classical limit of Hartree's equation in large magnetic fields

Nicolas Rougerie
(joint work with Denis Périce)

We study the dynamics of two dimensional fermionic particles submitted to a perpendicular magnetic field. We start from the Hartree equation for the first reduced density matrix, describing the mean field behaviour of a large fermionic system, and derive a gyrokinetic transport equation for the particle density. This derivation has been considered previously as a combination of a "semi-classical limit on the position/momentum phase-space" and a "gyrokinetic limit of Vlasov's equation". Here we consider the truly 'large magnetic field regime' where the gap between Landau levels is of the same order as the other energy contributions. The aforementioned approach does not apply and a real "semi-classical limit on the Landau level index/position of cyclotron orbit center phase-space" is called for, as considered in works of Lieb-Solovej-Yngvason and Fournais-Madsen at the level of ground states. We rely on vortex coherent states and the associated Husimi functions to derive the effective dynamics.

Low energy spectrum of the XXZ model coupled to a magnetic field

Alessandro Pizzo
(joint work with Simone Del Vecchio, Jürg Fröhlich, Alessio Ranallo)

I outline a recent development concerning the control of short-range perturbations of the Hamiltonian of an Ising chain. An example covered by our analysis is the celebrated XXZ chain. This model plays an important role also in MBL (many-body localisation) since in fermionic variables represents fermions hopping along the one-dimensional lattice and interacting through a Hubbard-type potential. To our knowledge, the antiferromagnetic XXZ chains in an external magnetic field have not been studied previously with mathematically rigorous techniques, except for some results within the range of Bethe ansatz. Indeed, the unperturbed Hamiltonian of the antiferromagnetic XXZ chain, i.e., the sum of the Ising Hamiltonian and the interaction term with the external magnetic field, has a ground-state subspace which is two-dimensional under natural assumptions on the size of the magnetic field and on the parity of the number of sites in the chain. But, in contrast to models such as the celebrated "AKLT model", no "local quantum topological order condition" holds.

References

[1] Affleck, Ian and Kennedy, Tom and Lieb, Elliott H. and Tasaki, Hal. *Rigorous results on valence-bond ground states in antiferromagnets* Physical Review Letters. 59(7): 799-802 (1987).

[2] S. Bachmann, B. Nachtergaele. *On gapped phases with a continuous symmetry and boundary operators* J. Stat. Phys. 154(1–2): 91-112 (2014)

[3] S. Bravyi and M.B. Hastings. *A short proof of stability of topological order under local perturbations* Commun. Math. Physics. **307**, 609 (2011).

[4] S. Bravyi, M.B. Hastings and S. Michalakis. *Topological quantum order: stability under local perturbations* J. Math. Phys. **51**, 093512, (2010).

[5] N. Datta, R. Fernandez, J. Fröhlich, L. Rey-Bellet. *Low-Temperature Phase Diagrams of Quantum Lattice Systems. II. Convergent Perturbation Expansions and Stability in Systems with Infinite Degeneracy* Helvetica Physica Acta 69, 752–820 (1996)

[6] S. Del Vecchio, J. Fröhlich, A. Pizzo, A. Ranallo. *Boundary effects and the stability of the low energy spectrum of the AKLT model* (in preparation).

[7] S. Del Vecchio, J. Fröhlich, A. Pizzo, S. Rossi. *Lie-Schwinger block-diagonalization and gapped quantum chains with unbounded interactions* Comm. Math. Phys. https://link.springer.com/article/10.1007/s00220-020-03878-y

[8] S. Del Vecchio, J. Fröhlich, A. Pizzo, S. Rossi. *Lie-Schwinger block-diagonalization and gapped quantum chains: analyticity of the ground-state energy* J. Funct. Anal. V. 279 Issue 8 https://www.sciencedirect.com/science/article/abs/pii/S0022123620302469

[9] S. Del Vecchio, J. Fröhlich, A. Pizzo, S. Rossi. *Local* iterative block-diagonalization of gapped Hamiltonians: a new tool in singular perturbation theory J. Math. Phys. 63 (2022) 073503, https://doi.org/10.1063/5.0084552.

[10] S. Del Vecchio, J. Fröhlich, A. Pizzo. *Block-diagonalization of infinite-volume lattice Hamiltonians with unbounded interactions* J. Funct. Anal. V. 284, Issue 1, 1 January 2023, 109734

[11] W. De Roeck, M. Salmhofer. *Persistence of Exponential Decay and Spectral Gaps for Interacting Fermions* Comm. Math. Phys. https://doi.org/10.1007/s00220-018-3211-z

[12] F. Franchini. *An Introduction to Integrable Techniques for One-Dimensional Quantum Systems* Lecture Notes in Physics (LNP, volume 940)

[13] J. Fröhlich, A. Pizzo. *Lie-Schwinger block-diagonalization and gapped quantum chains* Comm. Math. Phys. 375, 2039-2069 (2020)

[14] M.B. Hastings. *The Stability of Free Fermi Hamiltonians* J. Math. Phys. 60, 042201 (2019)

[15] T. Koma. *Stability of the Spectral Gap for Lattice Fermions* arXiv:2005.04548

[16] T. Koma, B. Nachtergaele. *The Spectral Gap of the Ferromagnetic XXZ-Chain* Letters in Mathematical Physics (40), 1-16 (1997)

[17] T. Koma, B. Nachtergaele., S. Starr. *The Spectral Gap for the Ferromagnetic Spin-J XXZ Chain* Adv. Theor. Math. Phys, 5, 1047-1090 (2001)

[18] T. Kennedy, H. Tasaki. *Hidden symmetry breaking and the Haldane phase in $S = 1$ quantum spin chains* Comm.. Math. Phys. 147, 431-484 (1992)

[19] A. Lucia, A. Moon, A. Young. *Stability of the spectral gap and grounds state indistinguishability for a decorated AKLT model.* preprint arXiv:2209.01141

[20] E. H. Lieb, D. W. Robinson. *The finite group velocity of quantum spin systems* Communications in Mathematical Physics. 28 (3) 251-257 (1972)

[21] A. Moon, B. Nachtergaele. *Stability of Gapped Ground State Phases of Spins and Fermions in One Dimension* J. Math. Phys. 59, 091415 (2018)

[22] B. Nachtergaele, R. Sims. *Locality Estimates for Quantum Spin Systems* in New Trends in Mathematical Physics 591-614 (2009). Spinger Netherlands.

[23] B. Nachtergaele, R. Sims, A. Young. *Lieb-Robinson bounds, the spectral flow, and stability of the spectral gap for lattice fermion systems* Mathematical Problems in Quantum Physics, pp.93-115

[24] B. Nachtergaele, R. Sims, A. Young. *Stability of the bulk gap for frustration-free topologically ordered quantum lattice systems* preprint arXiv:2102.07209

[25] B. Nachtergaele, R. Sims, A. Young. *Quasi-Locality Bounds for Quantum Lattice Systems. Part II. Perturbations of Frustration-Free Spin Models with Gapped Ground States* Ann. Henri Poincaré, 23, 393-511 (2022)

[26] Y. Ogata. *An $H^3(G, \mathbb{T})$-valued index of symmetry-protected topological phases with on-site finite group symmetry for two-dimensional quantum spin systems* Forum of Mathematics, Pi, Vol. 9:e13 1-62 (2021)

[27] Rakov, Mykhailo V., Weyrauch Michael. *Spin-$\frac{1}{2}$ XXZ Heisenberg chain in a longitudinal magnetic field* Physical Review B 100.13 (2019): 134434

[28] S. B. Rutkevich *Kink confinement in the antiferromagnetic XXZ spin-(1/2) chain in a weak staggered magnetic field* Europhysics Letters, Volume 121, Number 3
[29] N. Sopenko. *An index for two-dimensional SPT states* J. Math. Phys. 62, 111901 (2021)

Pointwise bounds on eigenstates in non-relativistic quantum field theory

Marcel Griesemer
(joint work with Valentin Kußmaul)

States of an atom or molecule with energy distribution strictly below the ionization threshold are well localized in a neighborhood of the nuclei: both in models of non-relativistic quantum mechanics and non-relativistic quantum field theory (QFT), the wave function decays exponentially as the distance $|x|$ of the electronic configuration $x \in \mathbb{R}^{3N}$ from the nuclear positions grows. This decay implies an effective screening of the positive nuclear charges and it plays an important role in the mathematical analysis of many-particle quantum systems. While this exponential decay is well understood and well established in non-relativistic quantum mechanics [1], in atomic models from non- (or pseudo-) relativistic quantum field theory we know little more than an averaged decay of the form

$$\int e^{2(1-\varepsilon)\beta|x|}|\psi(x)|^2\,dx < \infty, \qquad \text{all } \varepsilon > 0, \tag{1}$$

for the wave function $\psi : \mathbb{R}^{3N} \to \mathscr{H}'$ with $|\psi(x)|$ the norm of $\psi(x) \in \mathscr{H}'$, $\mathscr{H}'$ being the tensor product of spin and Fock space [2, 5, 4, 12]. The rate of decay, $\beta > 0$, is explicit and depends on the difference between ionization threshold and upper bound on the energy distribution of ψ [5]. One expects that (1) implies the corresponding *pointwise* decay at the same rate, at least if ψ is an eigenvector, but a general result of this type was not yet known. In this talk we report on the progress from [6].

To pass from (1) to the corresponding pointwise bound, one needs a Harnack inequality or a subsolution estimate. Such an estimate, as the name suggests, requires ψ to be a subsolution, only, rather than a solution to an elliptic equation. We consider vector-valued functions $\psi \in H^2_{\mathrm{loc}}(\mathbb{R}^n; \mathscr{H}')$, with $\mathscr{H}'$ some separable Hilbert space, that are subsolutions in the sense that, pointwise in $x \in \mathbb{R}^n$,

$$\mathrm{Re}\,(\psi, -\Delta\psi) \leq |\nabla\psi|^2 + q_-|\psi|^2, \tag{2}$$

where $(\cdot,\cdot)$ and $|\cdot|$ denote inner product and norm of $\mathscr{H}'$, respectively. On the right of (2), $q_- : \mathbb{R}^n \to [0,\infty)$ denotes a measurable function that, for some $c > 0, \mu > 0, s \in (2,4)$, all $u \in H^1(\mathbb{R}^n)$ and all $\varepsilon > 0$ small enough, obeys

$$\|q_-^{1/2}u\|_2 \leq \varepsilon\|\nabla u\|_2 + c\,\varepsilon^{-\mu}\|u\|_2, \tag{3}$$

$$\int dx\, q_-|u|^s < \infty. \tag{4}$$

These conditions are satisfied, for example, if $q_- = V_- + K$, with $K > 0$ some constant and $V_- = \max(0, -V)$ the negative part of an N-particle potential

$$V(x) = \sum_{j=1}^{N} v_j(x_j) + \sum_{j<k} w_{jk}(x_j - x_k), \quad x = (x_1, ..., x_N) \in \mathbb{R}^{3N}, \tag{5}$$

with $(v_j)_-, (w_{jk})_- \in L^p(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$ for some $p > 3/2$. Our main abstract result reads:

Theorem 1. *If $\psi \in H^2_{\mathrm{loc}}(\mathbb{R}^n; \mathscr{H}')$ satisfies the pointwise bound (2), then for every $p_0 > 2$ and $0 < r < R$ there exists a constant C_0 such that for all $x \in \mathbb{R}^n$,*

$$\|\psi\|_{\infty, B(x,r)} \leq C_0 \|\psi\|_{p_0, B(x,R)}. \tag{6}$$

If (2) holds without the gradient term, then (6) holds with $p_0 = 2$.

Remarks: If $\mathscr{H}' = \mathbb{C}$ and ψ solves a Schrödinger equation $(-\Delta + V)\psi = E\psi$, then clearly

$$\mathrm{Re}\, \overline{\psi}(-\Delta\psi) \leq q_- |\psi|^2, \tag{7}$$

with $q_- = (E - V)_+$ (or $q_- = V_- + |E|$). In this case the theorem is due to Moser and Stampacchia [13, 14], and it traces back to work of De Giorgi and Nash. A proof based on Assumptions (3) and (4) is given in Agmon's book [1]. Notice that Hypothesis (2) is weaker than (7) due to the presence of the gradient term.

In view of Kato's distributional inequality, $-\Delta|\psi| \leq \mathrm{Re}\, \overline{\psi}(-\Delta\psi)/|\psi|$, (7) implies that $-\Delta|\psi| \leq q_-|\psi|$. In contrast, by the computation $\Delta|\psi|^2 = 2\mathrm{Re}\,(\psi, \Delta\psi) + 2|\nabla\psi|^2$, Hypothesis (2) means that $-\Delta|\psi|^2 \leq 2q_-|\psi|^2$. From this one would expect (6) with $p_0 = 4$, only.

Application to the Pauli-Fierz model. Theorem 1 can be applied to the Pauli-Fierz model of non-relativistic quantum electrodynamics [2, 3]. The Hamiltonian, in the unsual notation, reads

$$H = \sum_{j=1}^{N} \left[(-i\nabla_j + \sqrt{\alpha} A_j)^2 + \sqrt{\alpha}\, \sigma_j \cdot B_j \right] + V + H_f. \tag{8}$$

Suppose that V has the form (5) with $v_j, w_{jk} \in L^2(\mathbb{R}^3) + L^\infty(\mathbb{R}^3)$. Then, for all values of the coupling constant $\alpha > 0$, the Hamiltonian is self-adjoint on $D(H) = D(-\Delta + H_f)$ and bounded from below [10, 8].

For eigenvectors ψ of H Hypothesis (2) is easily verified and hence (6) holds by Theorem 1. This allows us to prove the following theorem:

Theorem 2. *Let ψ be a normalized eigenvector of (8) and suppose that*

$$\int e^{2(1-\varepsilon)f(x)} |\psi(x)|^2 \, dx < \infty, \qquad \text{all } \varepsilon > 0, \tag{9}$$

with some Lipschitz function $f : \mathbb{R}^{3N} \to [0, \infty)$. Then for every $\varepsilon > 0$ there exists a constant C_ε such that for a.e. $x \in \mathbb{R}^{3N}$,

$$|\psi(x)| \leq C_\varepsilon e^{-(1-\varepsilon)f(x)}. \tag{10}$$

Remarks: Suppose $w_{jk}(x) \to 0$ as $|x| \to \infty$, in addition to the previous assumptions. It is then well-known that H has a ground state ψ provided that $E := \inf \sigma(H) < \Sigma$, where Σ denotes the *ionization threshold* of H, see [7, 5]. Moreover, (9) (and hence (10)) holds with $f(x) = \sqrt{\Sigma - E}|x|$. Previously, similar pointwise exponential bounds - with some positive rate of decay - were established by Hiroshima and Hikada [11, 9].

A result similar to Theorem 2 holds for the Nelson model [6].

In the case of one electron in three-dimensional space (Hydrogen atom), the statement of Theorem 2 holds for all ψ from some spectral subspace $\operatorname{Ran} P_\lambda(H)$ of H. It is then not necessary for ψ to be an eigenvector. This is interesting because all vectors $\psi \in \operatorname{Ran} P_\lambda(H)$ with $\lambda < \Sigma$ decay exponentially in sense of (9) (and hence (10)) with $f(x) = \sqrt{\Sigma - \lambda}|x|$. A similar remark applies to the one-particle density of N-electron systems in three space dimensions [6].

References

[1] Shmuel Agmon. *Lectures on exponential decay of solutions of second-order elliptic equations: bounds on eigenfunctions of N-body Schrödinger operators*, volume 29 of *Mathematical Notes*. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1982.

[2] Volker Bach, Jürg Fröhlich, and Israel Michael Sigal. Quantum electrodynamics of confined nonrelativistic particles. *Adv. Math.*, 137(2):299–395, 1998.

[3] Volker Bach, Jürg Fröhlich, and Israel Michael Sigal. Spectral analysis for systems of atoms and molecules coupled to the quantized radiation field. *Comm. Math. Phys.*, 207(2):249–290, 1999.

[4] Volker Bach and Oliver Matte. Exponential decay of eigenfunctions of the Bethe-Salpeter operator. *Lett. Math. Phys.*, 55(1):53–62, 2001.

[5] M. Griesemer. Exponential decay and ionization thresholds in non-relativistic quantum electrodynamics. *J. Funct. Anal.*, 210(2):321–340, 2004.

[6] M. Griesemer and V. Kußmaul. Pointwise bounds on eigenstates in non-relativistic quantum field theory. arXiv:2307.14986.

[7] Marcel Griesemer, Elliott H. Lieb, and Michael Loss. Ground states in non-relativistic quantum electrodynamics. *Invent. Math.*, 145(3):557–595, 2001.

[8] D. Hasler and I. Herbst. On the self-adjointness and domain of Pauli-Fierz type Hamiltonians. *Rev. Math. Phys.*, 20(7):787–800, 2008.

[9] Takeru Hidaka and Fumio Hiroshima. Pauli-Fierz model with Kato-class potentials and exponential decays. *Rev. Math. Phys.*, 22(10):1181–1208, 2010.

[10] F. Hiroshima. Self-adjointness of the Pauli-Fierz Hamiltonian for arbitrary values of coupling constants. *Ann. Henri Poincaré*, 3(1):171–201, 2002.

[11] Fumio Hiroshima. Pointwise exponential decay of bound states of the Nelson model with Kato-class potentials. In *Analysis and operator theory*, volume 146 of *Springer Optim. Appl.*, pages 225–250. Springer, Cham, 2019.

[12] Oliver Matte and Edgardo Stockmeyer. Exponential localization of hydrogen-like atoms in relativistic quantum electrodynamics. *Comm. Math. Phys.*, 295(2):551–583, 2010.

[13] Jürgen Moser. A new proof of De Giorgi's theorem concerning the regularity problem for elliptic differential equations. *Comm. Pure Appl. Math.*, 13:457–468, 1960.

[14] Guido Stampacchia. Le problème de Dirichlet pour les équations elliptiques du second ordre à coefficients discontinus. *Ann. Inst. Fourier (Grenoble)*, 15:189–258, 1965.

Propagation bounds for lattice bosons

Marius Lemm
(joint work with Jérémy Faupin, Carla Rubiliani, Israel Michael Sigal, Jingxuan Zhang)

We consider a system of lattice bosons on a discrete box $\Lambda \subset \mathbb{Z}^d$ of side length L. On the bosonic Fock space over $\ell^2(\Lambda)$, we consider the Hamiltonian

$$H = -\sum_{x,y\in\Lambda} h_{xy} b_x^\dagger b_y + \sum_{x,y\in\Lambda} w_{xy} n_x n_y \tag{1}$$

where $b_x^\dagger$ and b_x are the usual bosonic creation and annihilation operators. The hopping matrix (h_{xy}) is Hermitian and the density-density interaction (w_{xy}) is real-valued. Moreover, we assume the power-law decay condition

$$|h_{xy}| \leq C(1+|x-y|)^{-\alpha}$$

for some $\alpha > 0$. The special case where h is the adjacency matrix of Λ shifted by a multiple of the identity and $w(x,y) = U\delta_{x,y}$ corresponds to the paradigmatic Bose-Hubbard model that arises in modern experiments with ultracold bosons in optical lattices [BDZ08].

We are interested in proving propagation bounds in the spirit of Lieb-Robinson bounds (LRBs) [LR72] for quantum spin systems which have been widely extended and applied starting with [Has04, HW05, NS06, Has07]. The standard LRB measures the dynamical growth of the support of two bounded local observables A and B whose supports are at some distance $d(A,B) > 0$ and so $[A,B] = 0$. (The support of an observable A is defined as the maximal lattice region X such that A acts as the identity away from X, i.e., $A = A'_X \otimes 1_{X^c}$.) For a quantum spin system with rapidly decaying and bounded interactions, the standard LRB says that the commutator remains small in norm if $A(t) = e^{-itH}Ae^{itH}$ is Heisenberg time evolved, i.e.,

$$\|[A(t),B]\| \leq C\|A\|\|B\|e^{\xi(vt-d(A,B))}. \tag{2}$$

The interpretation of the LRB (2) is that quantum propagation is exponentially suppressed outside of the spacetime "light cone" $vt = d(A,B)$. For quantum spin systems with long-range (power-law) interactions, the light cone deteriorates for power law exponents $\alpha \in (2d, 2d+1)$ until it becomes logarithmic (i.e., very weak) at $\alpha = 2d$ [TGB+21]. Crucially, the LR velocity v depends on the operator norm of the terms in the Hamiltonian and this prevents the standard bound from being useful for bosonic Hamiltonians such as (1). Moreover, there are examples of translation-invariant lattice bosons with exponentially accelerating information propagation which shows that the topic is subtle [EG09].

The purpose of this talk was to report on recent progress on bosonic propagation bounds. The propagation bounds control three types of quantities roughly of increasing precision: macroscopic particle transport, microscopic particle transport, and general local observables (i.e., the LRB).

We write P_A for the spectral projection associated to a self-adjoint operator A. Set $A(t) = e^{itH}Ae^{-itH}$ with H given by (1).

Theorem 1 (Macroscopic particle transport [FLS22b, VKS23]). *Suppose that $\alpha > d$. For all numbers $0 < \eta < \xi \leq 1$, we have for any initial state ρ_0 with $\mathrm{Tr}(\rho_0 P_{N_{X^c} \leq \eta N}) = 1$ that*

$$\mathrm{Tr}(\rho_0 P_{N_Y \geq \xi N}(t)) \leq C_{\epsilon,\eta,\xi} t d(X,Y)^{-\min\{1, \lfloor \alpha - d - \epsilon \rfloor\}}$$

This can be interpreted as saying that the transport of the macroscopic number of particles $(\xi - \eta)N$ between regions X and Y takes time proportional to their distance if $\alpha > d+1$ (and the corresponding light cone deteriorates as α approaches d). The work [FLS22b] used the ASTLO method (adiabatic spacetime localization observables) to obtain the result for $\alpha > d + 2$. This was improved to the above in [VKS23] by optimal transport theory together with an argument for optimality. Thus, the problem of macroscopic transport is essentially settled, including in the long-range case.

Theorem 2 (Microscopic particle transport [FLS22a, KVS22, LRZ23]). *Let $p \geq 1$. Suppose that $\alpha > 2dp + 1$. Then there exists $v > 0$ such that any initial state with $\mathrm{Tr}(\rho_0 n_x^p) \leq \lambda_p$ for all $x \in \Lambda$ satisfies for all $0 < r < R$ that*

$$\sup_{0 < t < \frac{R-r}{v}} \mathrm{Tr}(N_{B_r}^p(t)) \leq \left(1 + \frac{C}{R-r}\right) \mathrm{Tr}(N_{B_R}^p) + C\lambda_p$$

This statement says that, given a bounded-density initial state ρ_0, only particles that were initially in B_R can make it into B_r after time t. The long-range case was first obtained in [FLS22a] by the ASTLO approach but with a mild N-dependence, see also [LRSZ23]. The nearest-neighbor case was treated in [KVS22] by a hands-on Gronwall argument. The mild N-dependence in the long-range case was recently removed by a multiscale iteration [LRZ23].

Finally, one can use microscopic particle propagation bounds to obtain LRBs for lattice bosons which we do not state explicitly here but which were presented in the talk. The idea is roughly that the microscopic particle propagation bound allows to control the accumulation of bosons and therefore the growth of the Lieb-Robinson velocity but implementing this intuition is subtle. According to the current state-of-the art, there appears to be a qualitative dichotomy of bosonic information propagation depending on the properties of the initial state:

(i) localized state of bounded density $\Rightarrow$ LRB with $v \sim 1$ (linear light cone) [FLS22a, LRSZ23]
(ii) general bounded-density state $\Rightarrow$ LRB with $v \sim t^{d-1}$ for n.n. (bent light cone) [KVS22]

The work [KVS22] contains a heuristic that (ii) is sharp. Understanding rigorously if and when this accelerated bosonic information propagation occurs, both for nearest-neighbor and long-range interactions is a fundamental open problem.

References

[BDZ08] Immanuel Bloch, Jean Dalibard, and Wilhelm Zwerger. Many-body physics with ultracold gases. *Reviews of Modern Physics*, 80(3):885, 2008.

[EG09] Jens Eisert and David Gross. Supersonic quantum communication. *Physical Review Letters*, 102(24):240501, 2009.

[FLS22a] Jérémy Faupin, Marius Lemm, and Israel Michael Sigal. Maximal speed for macroscopic particle transport in the Bose-Hubbard model. *Physical Review Letters*, 128(15):150602, 2022.

[FLS22b] Jérémy Faupin, Marius Lemm, and Israel Michael Sigal. On Lieb–Robinson Bounds for the Bose–Hubbard Model. *Communications in Mathematical Physics*, 394(3):1011–1037, 2022.

[Has04] Matthew B Hastings. Lieb-Schultz-Mattis in higher dimensions. *Physical Review B*, 69(10):104431, 2004.

[Has07] Matthew B Hastings. An area law for one-dimensional quantum systems. *Journal of Statistical Mechanics: Theory and Experiment*, 2007(08):P08024, 2007.

[HW05] Matthew B Hastings and Xiao-Gang Wen. Quasiadiabatic continuation of quantum states: The stability of topological ground-state degeneracy and emergent gauge invariance. *Physical Review B*, 72(4):045141, 2005.

[VKS23] Tan Van Vu, Tomotaka Kuwahara, and Keiji Saito. Optimal form of light cones for bosonic transport in long-range systems. *arXiv preprint arXiv:2307.01059*, 2023.

[KVS22] Tomotaka Kuwahara, Tan Van Vu, and Keiji Saito. Optimal light cone and digital quantum simulation of interacting bosons. *arXiv preprint arXiv:2206.14736*, 2022.

[LRSZ23] Marius Lemm, Carla Rubiliani, Israel Michael Sigal, and Jingxuan Zhang. Information propagation in long-range quantum many-body systems. *arXiv e-prints*, pages arXiv–2303, 2023.

[LRZ23] Marius Lemm, Carla Rubiliani, and Jingxuan Zhang. Microscopic propagation bounds for long-range bosons. *in preparation*, 2023.

[LR72] Elliott H Lieb and Derek W Robinson. The finite group velocity of quantum spin systems. *Communications in Mathematical Physics*, 28(3):251–257, 1972.

[NS06] Bruno Nachtergaele and Robert Sims. Lieb-Robinson bounds and the exponential clustering theorem. *Communications in Mathematical Physics*, 265:119–130, 2006.

[NRSS07] Bruno Nachtergaele, Hillel Raz, Benjamin Schlein, and Robert Sims. Lieb-Robinson bounds for harmonic and anharmonic lattice systems. *arXiv preprint arXiv:0712.3820*, 2007.

[TGB+21] Minh C Tran, Andrew Y Guo, Christopher L Baldwin, Adam Ehrenberg, Alexey V Gorshkov, and Andrew Lucas. Lieb-Robinson light cone for power-law interactions. *Physical Review Letters*, 127(16):160401, 2021.

Keller estimates for Dirac operators

David Gontier

(joint work with Jean Dolbeault, Fabio Pizzichillo, and Hanne Van Den Bosch)

In [1], Keller asked whether one can control the lowest eigenvalue of a Schrödinger operator of the form $-\Delta - V$ with the L^p–norm of V. More specifically, he was asking whether the minimization problem

$$\Lambda_S(\alpha, p) := \inf \left\{ \lambda_1(V),\ V \in L^p(\mathbb{R}^d),\ \|V\|_p = \alpha \right\}$$

is well-posed, where $\lambda_1(V)$ is the lowest eigenvalue of $-\Delta - V$, given by the minimization problem

$$\lambda_1(V) = \inf \left\{ \int_{\mathbb{R}^d} |\nabla u|^2 - V|u|^2,\ u \in H^1(\mathbb{R}^d), \quad \|u\|_2 = 1 \right\}.$$

It is known that an optimal potential V and eigenfunction u do exist whenever $p > \max\{1, \frac{d}{2}\}$. The Euler–Lagrange equations for this minimization problem shows that $V = |\widetilde{u}|^{\frac{2}{p-1}}$ with $\widetilde{u}$ an unormalized eigenfunction of $-\Delta - V$, and that $\widetilde{u}$ solves the non-linear Schrödinger equation (NLS)

$$-\Delta \widetilde{u} - |\widetilde{u}|^{\frac{2}{p-1}} \widetilde{u} = \Lambda_S(\alpha, p)\widetilde{u}, \qquad \text{with} \qquad \int_{\mathbb{R}^d} |\widetilde{u}|^{\frac{2p}{p-1}} = \int_{\mathbb{R}^d} V^p = \alpha^p.$$

In dimension $d = 1$, Keller solved this NLS equation, and finds the explicit solution $V(x) = A/\cosh^2(Bx)$ for some (undisplayed) $A, B > 0$.

By scaling, it is not difficult to see that, setting $\gamma = p - \frac{d}{2}$, the constant $L_S := \Lambda_S(\alpha = 1, d)^\gamma$ is optimal for the inequality

$$\forall V \in L^p(\mathbb{R}^d, \mathbb{R}^+), \qquad |\lambda_1(V)|^\gamma \leq L_S \int_{\mathbb{R}^d} V^{\gamma + \frac{d}{2}}.$$

This inequality was then extended by Lieb and Thirring in [2] to the sum of eigenvalues. Namely, for all $\gamma > \max\{0, 1 - \frac{d}{2}\}$, there is an optimal (lowest) constant $L_{\gamma,d}$ so that

$$\forall V \in L^p(\mathbb{R}^d, \mathbb{R}^+), \qquad \sum_{n=1}^{\infty} |\lambda_n(V)|^\gamma \leq L_S \int_{\mathbb{R}^d} V^{\gamma + \frac{d}{2}}.$$

In our work [3], we study a similar problem for the Dirac operator, and ask whether one can control the lowest eigenvalue of $\not{D}_m - V$ with an L^p–norm of V. Here $\not{D}_m$ is the massive Dirac operator

$$\not{D}_m := \sum_{j=1}^{d} \alpha_j(-\mathrm{i}\partial_{x_j}) + m\beta,$$

where $m > 0$ is the mass, and $\alpha_1, \cdots, \alpha_d, \beta$ are $N \times N$ hermitian matrices, with $N = 2^{\lfloor \frac{d+1}{2} \rfloor}$, chosen to satisfy some commutation rules. These rules ensure in particular that

$$\not{D}_m^2 = -\Delta + m^2 \qquad \text{and} \qquad \sigma(\not{D}) = (-\infty, -m] \cup [m, \infty).$$

Note that the spectrum of $\not{D}_m$ is no longer bounded from below, so the notion of *lowest eigenvalue* is unclear. Actually, our goal is to control the lowest eigenvalue in the gap $(-m, m)$, which we denote by $\lambda_D(V)$. We therefore study

$$\Lambda_D(\alpha, p) := \inf \left\{ \lambda_D(V),\ V \in L^p(\mathbb{R}^d, \mathbb{R}^+),\ \|V\|_p = \alpha \right\}.$$

Note that we require the potential V to be positive. Our main results can be stated as follows.

Theorem 1.

(i) For all $p \geq d \geq 1$, there is $\alpha_*(p) > 0$ such that, for all $V \in L^p(\mathbb{R}^d, \mathbb{R}^+)$ with $\|V\|_p < \alpha_*(p)$, $\lambda_D(V)$ is well-defined, and belongs to $(-m, m]$. In addition, $\Lambda_D(\alpha, p) > -m$.

(ii) The map $\alpha \mapsto \Lambda_D(\alpha, p)$ is continuous decreasing from $[0, \alpha_*(p))$ to $[m, -m)$.

(iii) If $(p, d) \neq (1, 1)$, there is an optimal potential $V = V_*$ and (unormalized) eigenstate $\Psi = \Psi_*$. They satisfies $V_* = |\Psi_*|^{\frac{2}{p-1}}$ and the non-linear Dirac (NLD) equation

$$\not{D}_m \Psi_* - |\Psi_*|^{\frac{2}{p-1}} \Psi_* = \Lambda_D(\alpha, p) \Psi_*, \qquad \int_{\mathbb{R}^d} |\Psi_*|^{\frac{2p}{p-1}} = \int_{\mathbb{R}^d} V_*^p = \alpha^p.$$

(iv) In dimension $d = 1$ and $p > 1$, the optimal potential (up to translations) is of the form:

$$V(x) = \frac{1}{\cosh^2(Bx) + C}, \quad \text{if} \quad \alpha < \alpha_*, \quad \text{and} \quad V(x) = \frac{A}{1 + Bx^2} \quad \text{if} \quad \alpha = \alpha_*,$$

for some (undisplayed) constants $A, B, C > 0$.

In particular, if the potential V is positive–valued (as we assume here), we expect the eigenvalues of $\not{D}_m - V$ to emerge from the upper essential spectrum, a fact that we indeed prove. This result is optimal in the sense that at $\alpha = \alpha_*$, there is a potential V with $\|V\|_p = \alpha_*$ so that the energy $-m$, corresponding to the lower essential spectrum, is an eigenvalue (or a resonance).

One main difference with the Schrödinger case is that we no longer have a scaling property for the map $\alpha \mapsto \Lambda_D(\alpha, d)$. Also, we do not have a min-max principle to characterize λ_D. The main idea of the proof is to study the inverse map of $\alpha \mapsto \Lambda_D(\alpha, d)$, that we denote by $\lambda \mapsto A_D(\lambda, d)$, and which is defined by

$$A_D(\lambda, d) := \inf \left\{ \|V\|_{L^p}, \quad \lambda \in \sigma(\not{D}_m - V), \qquad V \in L^p(\mathbb{R}^d, \mathbb{R}^+) \right\}.$$

This function is much easier to study. For instance, to detect whether λ is in the spectrum of $\not{D}_m - V$, one can use the Birmann-Schwinger principle, and detect whether 1 is in the spectrum of the (compact) operator $\sqrt{V}(\not{D}_m - \lambda)^{-1}\sqrt{V}$.

In dimension $d = 1$, we are able to solve the NLD equation in order to find the optimal potential (this is the fourth point of the Theorem). In higher dimension, and for the critical point $\alpha = \alpha_*$, we conjecture that the optimal potential is also of the Aubin–Talentin form $V(x) = \frac{A}{1+B|x|^2}$. If correct, it would imply that the critical value $\alpha_*(p, d)$ has the explicit formula

$$\alpha_*(p, d)^p = p^p \pi^{\frac{d}{2}} \left(\frac{2}{p - d} \right)^{p-d} \frac{\Gamma(p - \frac{d}{2})}{\Gamma(p)}.$$

This is confirmed by numerical simulations.

Finally, concerning Lieb–Thirring inequalities for Dirac operators, we have the following result. We denote by $-m < \lambda_1 \leq \lambda_2 \leq \cdots \leq m$ the eigenvalues in the gap, and set $e_k := e_k(m, V) := m - \lambda_k$ the distance between the k-th eigenvalue and the upper essential spectrum.

Theorem 2. *For all $\gamma > \frac{d}{2}$ and all $p \in (d, \gamma + \frac{d}{2}]$, there is a constant $L_{\gamma,d,p} > 0$ so that, for all $V \in L^p(\mathbb{R}^d, \mathbb{R}^+)$ and all $m > 0$, we have*

$$\sum_{k=1}^{\infty} e_k(m,V)^\gamma \leq L_{\gamma,d,p} m^{\frac{d}{2}} \int_{\mathbb{R}^d} V_m^{\gamma+\frac{d}{2}-p} V^p, \quad \textit{with} \quad V_m := \min\{m, V\}.$$

References

[1] J. B. Keller, *Lower bounds and isoperimetric inequalities for eigenvalues of the Schrödinger equation*, J. Mathematical Phys., 2 (1961), pp. 262–266.

[2] E. H. Lieb and W. E. Thirring, *Inequalities for the moments of the eigenvalues of the Schrödinger Hamiltonian and their relation to Sobolev inequalities*, Essays in Honor of Valentine Bargmann, E. Lieb, B. Simon, A. Wightman Eds. Princeton University Press, 1976, pp. 269–303.

[3] J. Dolbeault and D. Gontier and F. Pizzichillo and H. Van Den Bosch, *Keller and Lieb–Thirring estimates of the eigenvalues in the gap of Dirac operators*, Revista Matemática Iberoamericana (2023).

Upper bounds on the energy of dilute gases of hard–sphere bosons

Giulia Basti

(joint work with Serena Cenatiempo, Alessandro Giuliani, Alessandro Olgiati, Giulio Pasqualetti, Benjamin Schlein)

Consider N bosons moving in the three dimensional unitary torus Λ. In the Gross-Pitaevskii regime the Hamilton operator takes the form

$$H_N = \sum_{i=1}^{N} -\Delta_{x_i} + \sum_{i<j} N^2 V(N(x_i - x_j))$$

where V is a generic potential positive and with finite scattering length a. Let E_N be the ground state energy of H_N, it was proven in [16], [17], [18] that, to leading order in N, it is given by

$$E_N = 4\pi a N + o(N)$$

as $N \to +\infty$. More recently, it was shown in [1] (see also [7, 13, 19]) that, for $V \in L^3(\mathbb{R}^3)$,

$$E_N = 4\pi a(N-1) + e_\Lambda a^2 - \frac{1}{2} \sum_{p \in 2\pi\mathbb{Z}^3 \setminus \{0\}} \left[p^2 + 8\pi a - \sqrt{|p|^4 + 16\pi a p^2} - \frac{(8\pi a)^2}{2p^2} \right] + o(1) \tag{1}$$

where

$$e_\Lambda = 2 - \lim_{M \to +\infty} \sum_{\substack{p \in \mathbb{Z}^3 \setminus \{0\}: \\ |p_1|, |p_2|, |p_3| \leq M}} \frac{\cos(|p|)}{p^2}\,.$$

I will report on the result obtained in [4] where we proved an upper bound on E_N in agreement with (1) in the case of hard-sphere interaction *i.e.* when the potential V is formally replaced by

$$V^{\rm hc} = \begin{cases} +\infty & |x| < a \\ 0 & \text{otherwise} \end{cases}.$$

More precisely, se consider

$$E_N^{\rm hs} = \inf \left\langle \Psi, \sum_{j=1}^N -\Delta_{x_j} \Psi \right\rangle$$

where the infimum is taken over all normalized wave functions $\Psi \in L^2_s(\Lambda^N)$ satisfying the hard-core condition

$$\Psi(x_1, \dots, x_N) = 0$$

almost everywhere on the set

$$\bigcup_{i<j}^N \left\{ (x_1, \dots, x_N) \in \mathbb{R}^{3N} : |x_i - x_j| \leq a/N \right\}.$$

Unfortunately, a trial state inspired by Bogoliubov approximation as the one coming from the analysis in [1], doesn't seem to be well suited for hard-sphere interactions since Bogoliubov transformations are not in the right domain. Hence, in [4], we consider instead a trial state of the form

$$\Psi_N(x_1, \dots, x_N) = \prod_{\substack{i,j=1 \\ i \neq j}}^N f_\ell(x_i - x_j) \Phi(x_1, \dots, x_N) \tag{2}$$

thus taking the product of an N-wave function Φ with the Bijl-Dingle-Jastrow factor (see [6, 8, 14])

$$\prod_{\substack{i,j=1 \\ i \neq j}}^N f_\ell(x_i - x_j).$$

The advantage of the trial state in (2) is that it allows us to use the Bijl-Dingle-Jastrow factor to fulfill the hard-sphere condition and to implement correlations at small scale by choosing f_ℓ as the solution of the Neumann problem:

$$\begin{cases} -\Delta f_\ell(x) = \lambda_\ell f_\ell(x) \chi_\ell(x) & \text{for } \ a/N \leq |x| \leq \ell \\ f_\ell(x) = 0 & \text{if } |x| < a/N \\ f_\ell(x) = 1 & \text{if } |x| > \ell \\ \partial_r f_\ell(x) = 0 & \text{if } \ |x| = \ell \end{cases}.$$

On the other hand, since the Bijl-Dingle-Jastrow becomes difficult to be handled at large scales, we take $a/N \ll \ell \ll 1$ and use Φ to describe correlations up to length scales of order $\ell_0 \sim 1$ *i.e.* the healing length in this regime. In particular Φ will be chosen as the minimizer of an effective Hamiltonian (in a less singular

regime than Gross-Pitaevskii) using Bogoliubov transformations similarly to what is done in [2].

In this talk I will also address the natural question of whether the same method could be pushed to cover more singular regimes, and in particular whether it allows to treat the thermodynamic limit in which N bosons are confined to a torus Λ_L of side length L in the limit $N, L \to +\infty$ with particle density $\rho = N/L^3$ kept fixed. Assuming $\rho a^3 \ll 1$ the ground state energy density $e(\rho)$ has been predicted in [15] to satisfy the following expansion

$$e(\rho) = 4\pi a \rho^2 \Big(1 + \frac{128}{15\sqrt{\pi}} (\rho a^3)^{1/2} + \dots \Big). \tag{3}$$

The rigorous validity of (3) has been established at first order in [9, 17] while the second order correction was obtained as a lower bound in [10, 11] for a very general class of potentials also including the hard-sphere case (see also [12]). On the contrary, un upper bound in agreement with (3) was shown in [20] and later extended in [5] to cover potentials in $L^3(\mathbb{R}^3)$ thus leaving open the hard-sphere case.

While the strategy adopted in [4] can be expected to be extended to treat regimes in which the scattering goes scales as $N^{-1+\kappa}$ for small $\kappa \geq 0$ (Gross-Pitaevskii regime thus corresponding to $\kappa = 0$) a trial state as in (2) would certainly become substantially more difficult to be handled as soon as $\kappa > 1/3$, thus well before the thermodynamic limit corresponding (after appropriate rescaling) to $\kappa = 2/3$ but also smaller than the $\kappa = 1/2$: the threshold above which, through a well known box method also used in [5, 20], an upper bound on the ground state energy in the thermodynamic limit can be infered.

As a matter of fact the best available upper bound on the ground state energy density in the thermodynamic limit for hard-sphere interactions is the one recently obtained in [3] where we proved that there exists $C > 0$ such that

$$e(\rho) \leq 4\pi a \rho^2 \Big(1 + C(\rho a^3)^{1/2}\Big).$$

Remarkably, the trial state used in [3] is as simple as a Bijl-Dingle-Jastrow factor on a length scale $\ell = c(\rho a)^{-1/2}$ for a constant $c > 0$ small enough.

References

[1] C. Boccato, C. Brennecke, S. Cenatiempo, B. Schlein. Bogoliubov Theory in the Gross-Pitaevskii limit. *Acta Mathematica* **222**(2) (2019), 219–335.

[2] C. Boccato, C. Brennecke, S. Cenatiempo, B. Schlein. The excitation spectrum of Bose gases interacting through singular potentials. *J. Eur. Math. Soc.* **22**(7), 2331–2403 (2020)

[3] G. Basti, S. Cenatiempo, A. Giuliani, A. Olgiati, G. Pasqualetti, B. Schlein. Upper bound for the ground state energy of a dilute Bose gas of hard spheres. Preprint: arXiv: 2212.04431v2.

[4] G. Basti, S. Cenatiempo, A. Olgiati, G. Pasqualetti, B. Schlein. A second order upper bound for the ground state energy of a hard-sphere gas in the Gross-Pitaevskii regime. *Commun. Math. Phys.* **399** (2023), 1–55

[5] G. Basti. S. Cenatiempo, B. Schlein. A new second order upper bound for the ground state energy of dilute Bose gases. *Forum Math. Sigma* **9** (2021), no. e74.

[6] A. Bijl. The lowest wave function of the symmetrical many particles system. *Physica* **7** (1940), no. 9, 869-886.

[7] C. Boccato, C. Brennecke, B. Schlein, S. Schraven. Bogoliubov Theory for Trapped Bosons in the Gross-Pitaevskii Regime. *Ann. Henri Poincaré* **3** (2022)
[8] R. Dingle. The zero-point energy of a system of particles. *The London, Edinburgh, and Dublin Philosophical Magazine and Journal of Science* **40** (1949), no. 304, 573–578.
[9] F.J. Dyson. Ground-State Energy of a Hard-Sphere Gas. *Phys. Rev.* **106** (1957), 20–26
[10] S. Fournais, J.P. Solovej. The energy of dilute Bose gases. *Ann. Math.* **192**(3) (2020), 893–976
[11] S. Fournais, J.P. Solovej. The energy of dilute Bose gases II: the general case. *Invent. math.* **232** (2023), 863–994
[12] F. Haberberger, C. Hainzl, P. T. Nam, R. Seiringer, A. Triay. The free energy of dilute Bose gases at low temperatures. Preprint arXiv:2304.02405
[13] C. Hainzl, B. Schlein, A. Triay. Bogoliubov Theory in the Gross-Pitaevskii Limit: a Simplified Approach. Preprint arXiv:2203.03440.
[14] R. Jastrow. Manny-body problem with strong forces. *Phys. Rev.* **98** (1955), no. 5, 1479–1484.
[15] T. D. Lee, K. Huang, and C. N. Yang, Eigenvalues and eigenfunctions of a Bose system of hard spheres and its low-temperature properties. *Physical Review*, **106** (1957), 1135–1145.
[16] E. H. Lieb, R. Seiringer, and J. Yngvason. Bosons in a trap: A rigorous derivation of the Gross-Pitaevskii energy functional. *Phys. Rev. A* **61** (2000), 043602.
[17] E. H. Lieb, J. Yngvason. Ground State Energy of the low density Bose Gas. *Phys. Rev. Lett.* **80** (1998), 2504–2507.
[18] P. T. Nam, N. Rougerie, R. Seiringer. Ground states of large bosonic systems: The Gross-Pitaevskii limit revisited. *Analysis and PDE.* **9** (2016), no. 2, 459-485
[19] P. T. Nam, A. Triay. Bogoliubov excitation spectrum of trapped Bose gases in the Gross-Pitaevskii regime. *J. Math. Pure Appl.* **176** (2023), 18–101
[20] H.-T. Yau, J. Yin. The second order upper bound for the ground state energy of a Bose gas. *J. Stat. Phys.* **136**(3) (2009), 453–503.

Boundary Superconductivity in BCS Theory

Barbara Roos
(joint work with Robert Seiringer)

Bardeen-Cooper-Schrieffer (BCS) theory is a model of superconductivity. Recently, physicists discovered that BCS theory predicts *boundary superconductivity*, the phenomenon that a material is superconducting at the boundary but a normal metal in the bulk [9]. In particular, the Cooper-pair wave function appears to be localized at the boundary, in contrast to the results of Abrikosov [1] and de Gennes [2].

We would like to justify the results of Samoilenka and Babaev [9] in a mathematically rigorous way. However, BCS theory is non-linear and computing the Cooper-pair wave function is currently out of reach. Fortunately, there is a simpler way to estimate the critical temperature. We focus on proving that the presence of a boundary increases the critical temperature.

The BCS model is based on a functional defined on self-adjoint operators Γ of the form $\Gamma = \begin{pmatrix} \gamma & \alpha \\ \overline{\alpha} & 1-\overline{\gamma} \end{pmatrix}$ acting on $L^2(\Omega) \oplus L^2(\Omega)$, where Ω is the shape of the system and $0 \leq \Gamma \leq 1$. The BCS functional is given by

$$\mathcal{F}_T(\Gamma) = \operatorname{Tr}(-\Delta - \mu)\gamma + T \operatorname{Tr} \Gamma \ln \Gamma + \int_{\Omega\times\Omega} |\alpha(x,y)|^2 V(x-y) \mathrm{d}x \mathrm{d}y,$$

where $\alpha(x,y)$ is the integral kernel of α, T the temperature, μ the chemical potential, and V the effective interaction between the electrons in the metal. The system is superconducting iff $\alpha \not\equiv 0$ for the minimizer Γ of $\mathcal{F}_T$. For more details about the BCS functional we refer to [4].

The minimizer of $\mathcal{F}_T$ has to satisfy the Euler–Lagrange equation. However, this equation is non-linear and cannot be solved in general. There is one solution with $\alpha = 0$ though, the so-called normal state Γ_0. One can consider the Hessian $\mathcal{H}_T$ at the normal state. If the spectrum of the Hessian satisfies $\inf\sigma(\mathcal{H}_T) < 0$, the normal state is unstable and the system is in the superconducting state. On the other hand, if $\inf\sigma(\mathcal{H}_T) > 0$, the normal state is a local minimum of the functional, but we do not know whether it is also the global minimum, i.e. whether the system is in the normal state. An explicit computation shows that for variations $\tilde{\Gamma} = \begin{pmatrix} \tilde{\gamma} & \tilde{\alpha} \\ \overline{\tilde{\alpha}} & -\overline{\tilde{\gamma}} \end{pmatrix}$ one has $\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Big|_{t=0} \mathcal{F}_T(\Gamma_0 + t\tilde{\Gamma}) = 2\langle\tilde{\gamma}, G_T\tilde{\gamma}\rangle + 2\langle\tilde{\alpha}, (K_T+V)\tilde{\alpha}\rangle$, where we interpret the integral kernels of $\tilde{\gamma}$ and $\tilde{\alpha}$ as functions in $L^2(\Omega\times\Omega)$, G_T is a positive operator, and

$$K_T = \frac{-\Delta_x - \Delta_y - 2\mu}{\tanh\left(\frac{-\Delta_x-\mu}{2T}\right) + \tanh\left(\frac{-\Delta_y-\mu}{2T}\right)}$$

was computed in [3]. In particular, $\inf\sigma(\mathcal{H}_T) = 2\inf\sigma(K_T+V)$. We define the (local) critical temperature $T_c := \inf\{T>0 | \inf\sigma(K_T+V) \geq 0\}$. It follows from the above discussion that the system is superconducting for $T < T_c$ and in an unknown state for $T > T_c$. For translation invariant systems it was shown in [5] that the system is in the normal state for $T > T_c$.

We show that half-spaces in dimensions $d \in \{1,2,3\}$ with generic inter-particle interactions V have a strictly higher (local) critical temperature than $\mathbb{R}^d$, at least at weak enough coupling [7]. Since $\mathbb{R}^d$ is translation invariant, this shows that there is a temperature range where a half-space is superconducting while the system on $\mathbb{R}^d$ is in the normal state. This generalizes the results from [6], where a half-infinite wire with delta interaction was considered. Furthermore, we show that the (local) critical temperature is higher on a quadrant than on a half-space in two dimensions [8]. It would be interesting to know whether the half-space system is in the normal state above its (local) critical temperature and whether the quadrant can be superconducting while the half-space is in the normal state. This is one possible direction for future research, besides considering other shapes and external fields.

References

[1] A. A. Abrikosov. Concerning Surface Superconductivity in Strong Magnetic Fields. *J. Exptl. Theoret. Phys. (U.S.S.R)*, 47(2):720–733, 1964.

[2] P. G. De Gennes. Boundary Effects in Superconductors. *Reviews of Modern Physics*, 36(1):225–237, Jan. 1964.

[3] R. L. Frank, C. Hainzl, and E. Langmann. The BCS critical temperature in a weak homogeneous magnetic field. *Journal of Spectral Theory*, 9(3):1005–1062, Mar. 2019.

[4] C. Hainzl and R. Seiringer. The Bardeen–Cooper–Schrieffer functional of superconductivity and its mathematical properties. *Journal of Mathematical Physics*, 57(2):021101, Feb. 2016.
[5] C. Hainzl, E. Hamza, R. Seiringer, and J. P. Solovej. The BCS Functional for General Pair Interactions. *Communications in Mathematical Physics*, 281(2):349–367, July 2008.
[6] C. Hainzl, B. Roos and R. Seiringer. Boundary Superconductivity in the BCS Model. *Journal of Spectral Theory*, 12(4):1507–1540, 2022.
[7] B. Roos and R. Seiringer. BCS Critical Temperature on Half-Spaces. *arXiv:2306.05824*, June 2023.
[8] B. Roos and R. Seiringer. Enhanced BCS Superconductivity at a Corner. *arXiv:2308.07115*, Aug 2023.
[9] A. Samoilenka and E. Babaev. Boundary states with elevated critical temperatures in Bardeen-Cooper-Schrieffer superconductors. *Physical Review B*, 101(13):134512, Apr. 2020.

Concentration of measure and properties of typical quantum states

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(joint work with Roderich Tumulka and Cornelia Vogel)

Lévy's lemma states that a Lipschitz function defined on a D-dimensional sphere (equipped with the normalized uniform distribution) is close to its mean-value on a set that has measure close to one whenever D is large [1, 2]. In [3] we generalize Lévy's lemma to a much more general class of measures, so-called GAP measures [4, 5]. For any given density matrix ρ on a separable Hilbert space $\mathcal{H}$, $\mathrm{GAP}(\rho)$ is the most spread out probability measure on the unit sphere of $\mathcal{H}$ that has density matrix ρ and thus forms the natural generalization of the uniform distribution. We prove concentration-of-measure whenever the largest eigenvalue $\|\rho\|$ of ρ is small. We use this fact to generalize and improve well-known and important typicality results of quantum statistical mechanics to GAP measures, namely canonical typicality [7, 6] and dynamical typicality [8, 9]. Canonical typicality is the statement that for "most" pure states ψ of a given ensemble, the reduced density matrix of a sufficiently small subsystem is very close to a ψ-independent matrix. Dynamical typicality is the statement that for any observable and any unitary time-evolution, for "most" pure states ψ from a given ensemble the (coarse-grained) Born distribution of that observable in the time-evolved state ψ_t is very close to a ψ-independent distribution. So far, canonical typicality and dynamical typicality were known for the uniform distribution on finite-dimensional spheres, corresponding to the micro-canonical ensemble, and for rather special mean-value ensembles. Our result shows that these typicality results hold in general for systems described by a density matrix ρ with small eigenvalues. Since certain GAP measures are quantum analogs of the canonical ensemble of classical mechanics, our results can also be regarded as a version of equivalence of ensembles.

References

[1] P. Lévy, *Problèmes concrets d'analyse fonctionnelle*, Gauthier-Villars, 1951.
[2] V.D. Milman and G. Schechtman, *Asymptotic Theory of Finite Dimensional Normed Spaces: Isoperimetric Inequalities in Riemannian Manifolds*, LNM 1200, Springer Verlag, 1986.

[3] S. Teufel, R. Tumulka, and C. Vogel, *Canonical Typicality For Other Ensembles Than Micro-Canonical*, ArXiv:2307.15624, 2023.
[4] R. Jozsa, D. Robb, and W.K. Wootters, *Lower bound for accessible information in quantum mechanics*, Physical Review A, 49(2):668, 1994
[5] S. Goldstein, J.L. Lebowitz, R. Tumulka, and N. Zanghi, *On the Distribution of the Wave Function for Systems in Thermal Equilibrium*, Journal of Statistical Physics, 125(5/6), 2006
[6] S. Popescu, A.J. Short, and A. Winter. *Entanglement and the foundation of statistical mechanics*, Nature Physics, 2:754–758, 2006.
[7] S. Goldstein, J.L. Lebowitz, R. Tumulka, and N. Zanghi. *Canonical Typicality*, Physical Review Letters, 96:050403, 2006.
[8] C. Bartsch and J. Gemmer. *Dynamical typicality of quantum expectation values*, Physical Review Letters, 102:110403, 2009
[9] S. Teufel, R. Tumulka, and C. Vogel, *Time Evolution of Typical Pure States from a Macroscopic Hilbert Subspace*, Journal of Statistical Physics, 190:69, 2023.

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