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Arbeitsgemeinschaft: Cluster Algebras

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Abstract. Cluster algebras, invented by Sergey Fomin and Andrei Zelevinsky around the year 2000, are commutative algebras endowed with a rich combinatorial structure. Fomin–Zelevinsky's original motivations came from Lie theory but in the past two decades, cluster algebras have had strikingly fruitful interactions with a large array of other subjects including Poisson geometry, discrete dynamical systems, (higher) Teichmüller spaces, commutative and non-commutative algebraic geometry, representation theory, ... In this Arbeitsgemeinschaft, we have focused on 1) basic definitions and theorems, 2) cluster structures on algebraic varieties and 3) the recent connection between cluster algebras and symplectic topology, with its recent application to the construction of cluster structures on braid varieties.

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Introduction by the Organizers

The Arbeitsgemeinschaft Cluster Algebras, organised by Roger Casals, Bernhard Keller and Lauren Williams, attracted excellent researchers of various backgrounds from all over the world, including many graduate students and postdocs. It was organized with 48 on-site and 12 online participants. As usual for an Arbeitsgemeinschaft, the organisers had provided a detailed program and had distributed the talks to the participants. We had a total of 16 talks of one hour each with ample time for discussion and additional sessions for recaps, questions and answers, discussions and software demonstrations from eight to ten in the evenings. On Wednesday afternoon, we made an excursion to St. Roman and on Thursday evening, Andreas Thom moderated the discussion and vote on the next Arbeitsgemeinschaft in this series.

In this Arbeitsgemeinschaft, we focused on three main subjects:

A. the basic theory of cluster algebras (5 talks)
B. the most important classical examples of cluster structures on varieties (5 talks) and
C. the recent interaction between cluster algebras and symplectic topology and its application to the construction of cluster structures on braid varieties (6 talks).

The talks in part A were devoted to the definition and first examples of cluster algebras, the classification of the cluster-finite cluster algebras (parametrized by the finite root systems), the basic techniques for constructing cluster structures on (homogeneous) coordinate algebras of varieties with the example of the Grassmannian, additional notions and results on cluster combinatorics and the family of cluster algebras constructed from marked surfaces.

Part B started with a talk on more advanced techniques for constructing cluster structures on varieties followed by talks on the combinatorics of plabic graphs and the associated positroid cells, on webs and the cluster structure on the Grassmannian of 3-dimensional subspaces, on double Bruhat cells and generalizations and finally on Fock–Goncharov's cluster ensembles, which provide a more symmetric, geometric framework for the whole theory.

Part C focused on developments in symplectic geometry that have either used cluster algebras or been used to study them. In particular, this last series of lectures aimed at developing the intuitions and techniques from symplectic geometry (following Casals, Weng, Pascaleff–Tonkonog, Gao–Shen–Weng, ...) and the microlocal theory of sheaves (Kashiwara–Schapira, ...) to complement the more algebraic and combinatorial methods often used to study cluster algebras. On the one hand, these lectures explained new results in the study of Lagrangian surfaces, including the detection of infinitely many Lagrangian fillings, via techniques from cluster algebras (after Casals–Gao and Casals–Weng). On the other, the combinatorics of weaves were also presented from their original symplectic geometric viewpoint and then applied to prove new results in the study of cluster algebras. To wit, the lectures showed that the coordinate rings of braid varieties, which arise as certain moduli of Lagrangian fillings and generalize Richardson varieties, are indeed cluster algebras (after Casals–Gorsky–Gorsky–Simental and Casals–Gorsky–Gorsky–Le–Shen–Simental). For lack of time, we did not cover the alternative, more combinatorial construction of such cluster structures on braid varieties due to Galashin–Lam–Sherman-Bennett–Speyer.

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Arbeitsgemeinschaft: Cluster Algebras

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A1–Introduction to cluster algebras : Definition and first examples

Théo Pinet

The principal references for this note are the pioneering work of Fomin–Zelevinsky [1, 3], their book [4] and Keller's paper [5]. The main goal of the note is to introduce the notion of *cluster algebra* associated to a valued/ice quiver and to illustrate this notion on examples, with in particular the example of the homogeneous coordinate algebra of the Grassmannian of planes in $(n+3)$-dimensional space. Informally, the cluster algebra associated to a quiver Q with n vertices is a subalgebra of the field of rational functions $\mathbb{F} = \mathbb{Q}(x_1, \dots, x_n)$ whose generators, the cluster variables, are grouped in clusters of size n and are constructed recursively, starting from the initial seed $(Q, (x_1, \dots, x_n))$, using mutations. Let us now make this more precise.

Given a *good quiver* $Q = (Q_0, Q_1, s, t)$ (i.e. a finite directed graph with no loops or 2-cycles) and a vertex $k \in Q_0$, we define another good quiver $\mu_k(Q)$ from Q by

(1) adding an arrow $i \to j$ for all paths of the form $i \to k \to j$ in Q,
(2) inverting all arrows of the form $i \to k$ and $k \to j$ in Q, and
(3) removing all 2-cycles created from steps (1) and (2).

The good quiver $\mu_k(Q)$ is called *the mutation of* Q *at* k. Note that $\mu_k(\mu_k(Q)) = Q$. For example, mutating the *Markov quiver* Q_M below at vertex 1, gives us a quiver isomorphic to Q_M. We thus say that the *mutation class* of Q_M is $\{Q_M\}$.

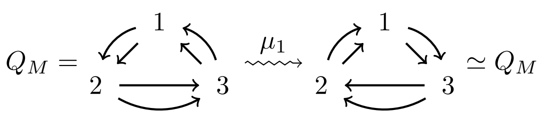

Figure 1. Example of quiver mutation with the Markov quiver Q_M.

Fix $n \in \mathbb{Z}_{\geq 0}$. A *seed* is a pair (Q, u) with Q a good quiver having n vertices and with $u = (u_1, \dots, u_n) \in \mathbb{F}^n$ a sequence satisfying $\mathbb{F} = \mathbb{Q}(u_1, \dots, u_n)$. Starting from a seed (Q, u) and a vertex $k \in Q_0$, the *mutated seed* $\mu_k(Q, u)$ *in direction* k is

$$\mu_k(Q, u) = (\mu_k(Q), u')$$

where $u' = (u_1, \dots, u_{k-1}, u'_k, u_{k+1}, \dots, u_n)$ with u'_k given by the *exchange relation*

$$u_k u'_k = \prod_{\substack{\alpha \in Q_1 \\ t(\alpha) = k}} u_{s(\alpha)} + \prod_{\substack{\alpha \in Q_1 \\ s(\alpha) = k}} u_{t(\alpha)}. \tag{1}$$

Fix now a good quiver Q with n vertices. A *cluster associated to* Q is a sequence $u' \in \mathbb{F}^n$ occuring in a seed (Q', u') that is linked to the *initial seed* $(Q, (x_1, \dots, x_n))$ by a finite sequence of mutations. We call *cluster variables* the components of the clusters associated to Q and define the *cluster algebra $\mathcal{A}_Q$ corresponding to Q* as the subalgebra of $\mathbb{F}$ generated by all cluster variables. In other words,

$$\mathcal{A}_Q = \mathbb{Q}[\text{cluster variables associated to } Q] \subseteq \mathbb{F}.$$

Most cluster algebras, like the one associated to the Markov quiver, have infinitely many cluster variables. These algebras can thus be quite hard to describe explicitly. However, their complexity is somewhat limited by the theorem below, which is one of the most remarkable results proven during the early study of cluster algebras.

Theorem 1 (Laurent phenomenon, [1, 2])**.** *Fix $u' = (u'_1, \ldots, u'_n)$ a cluster of $\mathcal{A}_Q$. Then, the cluster variables of $\mathcal{A}_Q$ all live inside the ring $\mathbb{Z}[(u'_1)^{\pm 1}, \ldots, (u'_n)^{\pm 1}]$.*

In particular, the cluster algebra $\mathcal{A}_Q$ is contained in the algebra $\mathbb{Q} \otimes_{\mathbb{Z}} \mathcal{U}_Q$ where

$$\mathcal{U}_Q = \bigcap_{\substack{u'=(u'_1,\ldots,u'_n) \\ \text{cluster of } \mathcal{A}_Q}} \mathbb{Z}[(u'_1)^{\pm 1}, \ldots, (u'_n)^{\pm 1}]$$

is the *upper cluster algebra corresponding to Q*. Note nevertheless that $\mathcal{A}_Q \neq \mathcal{U}_Q$ in general since, for $Q = Q_M$ the Markov quiver, the Laurent polynomial

$$f(x_1, x_2, x_3) = \frac{x_1^2 + x_2^2 + x_3^2}{x_1 x_2 x_3}$$

belongs to $\mathcal{U}_Q$, but not to $\mathcal{A}_Q$ (see e.g. [4]).

Now, let us add *frozen nodes* $\{n+1, \ldots, m\}$ (with $m \geq n$) to our good quiver Q in order to obtain an *iced quiver of type* (n, m). These frozen vertices can connect to the original (i.e. unfrozen) vertices of our quiver Q in any way that do not create 2-cycles, but cannot be connected to another frozen vertex. Here is an example:

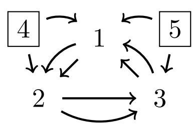

FIGURE 2. Example of iced quiver with 2 frozen nodes (indicated with a box) and an unfrozen part equal to the Markov quiver.

Given an iced quiver Q, we can define a cluster algebra $\mathcal{A}_Q$ exactly as above from the *initial seed* $(Q, (x_1, \ldots, x_m))$ by mutating at unfrozen vertices $\{1, \ldots, n\}$ (and at these vertices only). In this situation, the variables $x_{n+1}, \ldots, x_m$ belong to all clusters of $\mathcal{A}_Q$ and are called *coefficients* (instead of cluster variables). This slight generalization allows us to state the result below, again due to Fomin–Zelevinsky.

Theorem 2 ([2])**.** *Let X be a rational quasi-affine irreducible m-dimensional complex variety such that $\dim X = m$. Fix moreover Q an iced quiver of type (n, m). Suppose given functions φ_v and φ_{x_i} in the coordinate ring $\mathbb{C}[X]$ for all choices of cluster variables v of $\mathcal{A}_Q$ and all $n < i \leq m$. Suppose also that*

(i) *these functions altogether generate the coordinate ring $\mathbb{C}[X]$ and that*

(ii) *the map sending a cluster variable or a coefficient to the associated function sends exchange relations in* $\mathcal{A}_Q$ *to equalities in* $\mathbb{C}[X]$.

Then, the latter map extends to an algebra isomorphism $\mathbb{C} \otimes_{\mathbb{Q}} \mathcal{A}_Q \simeq \mathbb{C}[X]$.

When the conditions in the above theorem are satisfied, we say that the coordinate ring $\mathbb{C}[X]$ *carries a cluster structure of type* Q *with initial seed* $\{\varphi_{x_i}\}_{i=1}^m$. For an example of such a situation, fix $m = n+3$ with $n \geq 1$ and denote by A the algebra of polynomial functions on the cone over the Grassmannian $\mathrm{Gr}_{2,m}(\mathbb{C})$ of planes in $\mathbb{C}^m$. Then, A is generated by the *Plücker coordinates* x_{ij} (with $1 \leq i < j \leq m$) which are subject to the *Plücker relations*

$$x_{ik}x_{j\ell} = x_{ij}x_{k\ell} + x_{i\ell}x_{jk} \tag{2}$$

whenever $1 \leq i < j < k < \ell \leq m$. Let now P be a m-gon with a fixed triangulation T. Then a well-known procedure (see e.g. [2, 4]) produces an iced quiver Q of type (n, m) from P and T. Here is an example with $m = 6$:

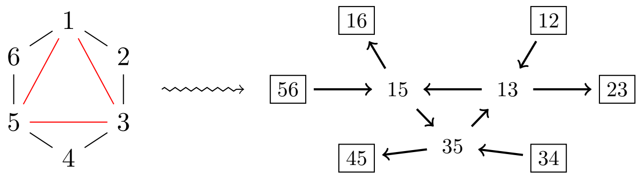

Figure 3. Iced quiver Q associated to hexagon P with triangulation T. Sides of P (diagonals of T) give frozen (resp. unfrozen) nodes, while arrows are obtained by turning in a counter-clockwise manner inside the triangles bounded by T (see e.g. [4]).

Theorem 3 ([2, 4]). *The algebra* A *carries a cluster structure with type the iced quiver* Q *above and with cluster variables (coefficients) the Plücker coordinates* x_{ij} *associated to diagonals (resp. sides) of* P. *Also, the clusters of* A *are the* n*-tuples of diagonals of* P *forming a triangulation and the exchange relations for the cluster algebra* A *(see* (1)*) are exactly the Plücker relations* (2).

Let us at last finish this note by recalling that iced quivers of type (n, m) are in bijection with integral $m \times n$ matrices with skew-symmetric $n \times n$ top submatrix. Using this bijection, we can define the notion of *matrix mutation* which can in turn be generalized to the setting of integral $m \times n$ matrices having skew-symmetrizable $n \times n$ top submatrix. This then leads to *mutation for valued iced quivers* [4, 5].

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A2–Classification of cluster-finite cluster algebras

Kaveh Mousavand

This talk was a summary of the seminal work of Fomin and Zelevinsky [1] on the classification of those cluster algebras which admit only finitely many clusters. Such algebras are traditionally called of “finite type”, and they are treated up to a suitable notion of isomorphism for cluster algebras. Before we recall the main ingredients and state the results, let us remark that there are other notions of finiteness in the study of cluster algebras (e.g. finite mutation type, or finitely generated cluster algebras, etc.) that are different from the problem considered in the talk. Also, we observe that in some textbooks, a commutative algebra is said to be of finite type if it is a quotient of a polynomial algebra in finitely many indeterminates. Unfortunately, this notion is different from the finiteness phenomenon treated in [1]. That is, there are examples of cluster algebras which are finite type as commutative algebras, but they admit infinitely many clusters (For instance, the coordinate algebras of maximal unipotent subgroups in [5], or any finitely generated cluster algebra with infinitely many clusters.). To avoid any confusion caused by the discrepancy in terminology, henceforth we adopt a less ambiguous term proposed by Benrhard Keller– one of the organizers of this Arbeitsgemeinschaft– and say that a cluster algebra is *cluster-finite* if it admits only finitely many clusters.

1. Notations, main ingredients and background

Here we only recall some standard terminology and notations that allow us to articulate the main problem and results. For detailed study of root systems, we refer to [4]. Moreover, all the required materials from cluster algebras that are used below can be found in [1].

Throughout, let Φ denote a finite irreducible crystallographic root system in the Euclidean space $\mathbb{R}^n$. It is known that, up to isometry and simultaneous rescaling of the vectors, Φ is uniquely determined by its Cartan matrix C_Φ, to which one can associate a unique Dynking graph. In particular, the Dynkin graphs of all finite irreducible crystallographic root systems are often denoted by A_n $(n \geq 1)$, B_n $(n \geq 2)$, C_n $(n \geq 3)$, D_n $(n \geq 4)$, E_6, E_7, E_8, F_4, or G_2 (for details, see [2] and [4]). For Φ, and a fixed simple system Δ in Φ, by Φ^+ we denote the set of positive roots. Furthermore, the set of *almost positive roots* is defined as $\Phi_{\geq -1} := \Phi^+ \cup -\Delta$, where $-\Delta := \{-\alpha | \alpha \in \Delta\}$.

Now, we briefly recall the main ingredients of the most general construction of cluster algebras, as in [1]. Let $\mathbb{P}$ be a semifield, and by F denote the field of rational functions in n indeterminates with cooeficients in $\mathbb{ZP}$. This will be the ambient field containing the cluster algebra $\mathcal{A}$ of rank n, described below. Every *seed* in

F is a triple $\Sigma = (\underline{x}, \underline{p}, B)$, where $\underline{x}$ is called a *cluster*, consisting of n elements in F. These elements are known as the *cluster variables* and form a free generating set for a field extension over the field of fractions of $\mathbb{ZP}$ in F. Moreover, the coefficient $\underline{p} = (p_x^{\pm})_{x \in \underline{x}}$ is a 2-tuple of elements in $\mathbb{P}$ satisfying the normalization condition $p_x^+ \oplus p_x^- = 1$. Here, $\oplus$ denotes the auxiliary addition in the semifield $\mathbb{P}$. Finally, $B = (b_{xy})_{x,y \in \underline{x}}$ denotes a sign-skew-symmetric matrix whose rows and columns are indexed by the cluster variables in $\underline{x}$. Namely, for all $x, y \in \underline{x}$, either $b_{xy} = b_{yx} = 0$, or else $b_{xy} b_{yx} < 0$. Through the explicit formulas in section 1 of [1], one can mutate the seed $\Sigma = (\underline{x}, \underline{p}, B)$ in all n directions, that is, to simultaneously mutate the cluster $\underline{x}$, the coefficient $\underline{p}$, as well as the matrix B.

Starting from an initial seed Σ, perform all possible mutations on Σ, and then iterate this procedure at every output obtained in each step. This iteration may terminate after only finitely many steps, that is, we get no new seeds after a finite number of mutations, or else one can mutate and produce infinitely many different seeds. Let $\mathcal{S}$ denote the set of all seeds in F obtained via all possible iterations of mutations starting from Σ. By $\mathcal{X}$ and $\mathcal{P}$, respectively denote the set of all cluster variables and the set of all coefficients in the seeds belonging to $\mathcal{S}$. Let $\mathbb{Z}[\mathcal{P}]$ denote the subring of F generated by $\mathcal{P}$. Then, the normalized *cluster algebra* $\mathcal{A}$ is the $\mathbb{Z}[\mathcal{P}]$-subalgebra of F generated by $\mathcal{X}$. As shown in [1], $\mathcal{A}$ can be studied up to strong isomorphism of cluster algebras. More precisely, over a fixed semifield $\mathbb{P}$, if F and F' are two ambient fields as above, and $\mathcal{A} \subset F$ and $\mathcal{A}' \subset F'$ are two cluster algebras, then $\mathcal{A}$ and $\mathcal{A}'$ are *strongly isomorphic* if there exists a $\mathbb{Z}[\mathcal{P}]$-algebra isomorphism between F and F' which additionally transports any seed in F to F'. Such an isomorphism induces an algebra isomorphism between $\mathcal{A}$ and $\mathcal{A}'$ which preserves the cluster structure. We remark that, even over a fixed semifield, an arbitrary $\mathbb{Z}[\mathcal{P}]$-algebra isomorphism between two cluster algebras is not necessarily a strong isomorphism. In fact, there exist $\mathbb{Z}[\mathcal{P}]$-algebras which admit two different cluster structures that are not strongly isomorphic (for explicit examples, see [3]).

2. Main results

Before we state the first theorem, let us recall that for an arbitrary $n \times n$ integer square matrix $B = (b_{ij})$, the *Cartan counterpart* of B, which we denote by $C_B = (c_{ij})$, is defined by putting $c_{ij} := 2$, if $i = j$, and $c_{ij} := -|b_{ij}|$, otherwise. Observe that C_B is not necessarily a Cartan matrix, but it is a generalized Cartan matrix. Now, we are ready to state the first main result. Throughout, we use the notations and terminology introduced above.

Theorem 1. *Fomin-Zelevinsky [1]: Let $\Sigma = (\underline{x}, \underline{p}, B)$ be a seed in F such that $b_{xy} b_{xz} \geq 0$, for all x, y and z in $\underline{x}$. If the Cartan counterpart of B is the Cartan matrix C_Φ of a finite root system Φ, then $\mathcal{A}$ is cluster-finite. Conversely, up to strong isomorphism, every cluster-finite cluster algebra is of the above form, that is, it admits a seed with the aforementioned properties.*

By the preceding theorem, if the cluster algebra $\mathcal{A}$ of rank n is cluster-finite, a unique finite root system Φ in $\mathbb{R}^n$ is associated to $\mathcal{A}$. Consequently, $\mathcal{A}$ is called of type Φ, and has the corresponding Dynkin graph with n vertices. For a more detailed treatment of cluster-finite cluster algebras from this viewpoint, see [2].

The second main result is the following theorem which gives equivalent characterizations of cluster-finite cluster algebras, and further describes the connection between their cluster variables and certain roots in the corresponding root system.

Theorem 2. *Fomin-Zelevinsky [1]: For any cluster algebra $\mathcal{A}$, the following are equivalent:*

(1) *$\mathcal{A}$ is cluster-finite;*
(2) *$\mathcal{A}$ admits finitely many cluster variables, that is, $\mathcal{X}$ is a finite set;*
(3) *In every seed $\Sigma = (\underline{x}, \underline{p}, B)$ in $\mathcal{S}$, we have $|b_{xy}b_{yx}| \leq 3$, for all $x, y \in \underline{x}$.*

That being the case, let Φ be the root system of $\mathcal{A}$ and $\underline{x}_0 = (x_1, \cdots, x_n)$ the initial cluster. Then, there is a unique bijection between the almost positive roots in Φ and the cluster variables in $\mathcal{X}$, expressed in terms of $\underline{x}_0$. More precisely, if $\Delta = \{\alpha_1, \cdots, \alpha_n\}$ is a simple system in Φ, for each $\alpha \in \Phi_{\geq -1}$, the corresponding cluster variable is $x[\alpha] = \frac{P_\alpha(\underline{x}_0)}{x^\alpha}$, where $P_\alpha(\underline{x}_0)$ is a polynomial over $\mathbb{ZP}$ in terms of cluster variables in $\underline{x}_0$ and has a non-zero constant term, and x^α is the monomial defined as $x^\alpha = x_1^{c_1} \cdots x_n^{c_n}$, where $\alpha = c_1\alpha_1 + \cdots + c_n\alpha_n$. In particular, $x[-\alpha_i] = x_i$.

We end with some remarks on the above theorem and the more recent results on the cluster-finite cluster algebras obtained after their original treatment in [1].

First, observe that the implication (1) $\to$ (2) in the preceding theorem follows from the definition, but the converse is far from trivial. In particular, a finite set of cluster variables could *a priori* appear in infinitely many clusters that belong to different seeds in $\mathcal{S}$. However, the above theorem says this never happens. Second, note that part (3) gives an explicit condition in terms of entries of the matrices of each seed. However, we remark that one should verify this condition for all seeds in $\mathcal{S}$ to conclude that $\mathcal{A}$ is cluster-finite. In fact, there are cluster algebras which are not cluster-finite, but they admit a seed which satisfies condition (3). Third, with regard to the correspondence between the almost positive roots and the cluster variables of cluster-finite cluster algebras, we remark that an elegant construction is given by Keller [7], where one can begin from the initial cluster variables and through a concrete knitting algorithm recover the aforementioned bijection between the almost positive roots and all cluster variables. Finally, we note that some other conceptual characterizations of cluster-finite cluster algebras have been achieved after their first appearance in [1]. In particular, in [6] it is shown that a cluster algebra $\mathcal{A}$ is cluster-finite if and only if the set of cluster monomials forms an additive basis for $\mathcal{A}$.

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A3–The cluster structure of the Grassmannian coordinate algebra

Liana Heuberger

One of the first examples where a coordinate ring admits a cluster algebra structure in a non-trivial way arises in the case of the affine cone over the Grassmannian. During the proof of this result, we encounter a fundamental tool in cluster algebra theory: the celebrated *starfish lemma*. This talk showcases the power of the lemma by applying it to a familiar, yet nontrivial context.

The Grassmannian of a-subspaces of a b-dimensional $\mathbb{C}$-vector space is one of the first projective varieties one encounters in geometry beyond projective spaces themselves. Its homogeneous coordinate ring, also known as the *Plücker ring* has been extensively studied and is known to be generated by *Plücker coordinates*. Expressing this ring in terms of $\mathrm{SL}_a(\mathbb{C})$-invariant polynomials allows us to understand the Plücker coordinates as $a \times a$ minors of an $a \times b$ matrix.

There exist two known constructions of the cluster algebra structure of this ring, the first of which appeared in the work of Scott [1]. Scott chooses a seed whose cluster variables are themselves Plücker coordinates, and such that the one-step mutation at each variable yields a cluster variable which is again a Plücker coordinate. The combinatorial setup of this method, involving *alternating strand diagrams*, is less self-contained than that of the alternative construction of Fomin, Williams and Zelevinsky [2], whose proof we chose to present throughout this talk.

The seed chosen in [2] is formed of distinguished Plücker coordinates whose respective Young tableaux are *rectangles*. More precisely, one can associate a Plücker coordinate to any sub-rectangle of an $a \times (b-a)$ rectangle in a unique way, and we choose this set of coordinates as the seed of our cluster algebra. The frozen variables correspond to those coordinates with consecutive indices, while the remainder are cluster variables.

The proof involves a double inclusion: one has to prove that each mutation of this distinguished seed remains in the Plücker ring (as opposed to its fraction field), and conversely that every Plücker coordinate is generated by a subsequent mutation.

The first implication relies on the starfish lemma, which roughly guarantees that if one starts from a polynomial seed whose one-step mutations produce polynomial cluster variables, then the same holds for all subsequent mutations. For this distinguished seed, we no longer obtain Plücker coordinates after one-step mutations, yet we are still able to control the behaviour of the new cluster variables:

this achieved by combining well-known Plücker relations between the variables of the distinguished seed, and the exchange relations of the mutations. We show that the one-step cluster variables are indeed polynomial, thereby concluding the first half of the proof.

For the second implication, Fomin, Williams and Zelevinsky have an inductive approach via the *Muir embedding*. More specifically, one can embed rectangular quivers of smaller size inside a fixed rectangular quiver and use the inductive hypothesis to obtain some (but not all) Plücker coordinates. They then use *cyclic shifts*, shown to be mutations of the distinguished seed, to obtain the outstanding coordinates and the proof concludes.

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C1–Introduction to Lagrangian fillings

Yu Pan

Symplectic and contact geometry, rooted from classical mechanics, has experienced a rapid development in the last forty years. It mainly concerns manifolds with additional geometrical structures called symplectic and contact manifolds and special knots and surfaces in them called Legendrian knots and exact Lagrangian fillings.

A symplectic manifold is an even dimensional manifold with a non-degenerate closed 2-form. An example is the cotangent space $\mathbb{R}^4 = T^*\mathbb{R}^2, \omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$. Note that this symplectic manifold is also exact, i.e., $\omega = d\lambda$ (in the example $\lambda = -q_1 dp_1 - q_2 dp_2$). An odd dimensional counterpart is called contact manifold, which is an odd dimensional manifold with a contact structure given by the kernel of a 1-form α such that $\alpha \wedge d\alpha^n \neq 0$. An example of a 3-dimensional contact manifold is $\mathbb{R}^3_{std} = (\mathbb{R}^3, ker\ \alpha)$ where $\alpha = dz - ydx$. Darboux theorem shows that every symplectic (contact) manifold locally are the same. Therefore it is more interesting to explore the global geometrical (i.e., topological) properties of symplectic/contact manifolds.

For similar reason as the one for knots and surfaces being essential in low dimensional topology, it is also important to consider special knots and surfaces in contact and symplectic manifolds that cooperate well with the additional geometrical structures. These knots and surfaces are called Legendrian knots and exact Lagrangian surfaces.

In particular, a Legendrian knot $\Lambda \in \mathbb{R}^3_{std}$ in $(\mathbb{R}^3, ker\ \alpha)$ is a knot in $\mathbb{R}^3$ such that α vanishes on it. An important way to visualize it is through front projection $\Pi_F : \mathbb{R}^3 \to \mathbb{R}^2_{xz}$. Note that we do not loose information in the front projection of a Legendrian knot Λ since the y-coordinate can be recovered through $y = \frac{dz}{dx}$ (since the 1-form α vanishes on Λ). One can see the example of front projections of an unknot and a trefoil in Figure 1. As a generalization of the trefoil, the (-1)

closure of a positive braid β is sketched in the Figure 1 (c). This will be the main example of Legendrian links we will focus on in the latter C-lectures.

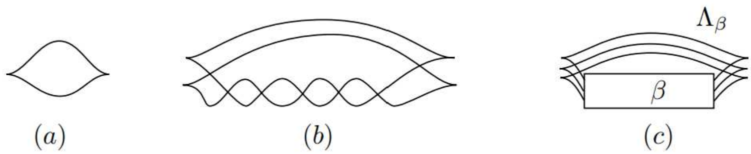

FIGURE 1. Front projections of unknot (a), trefoil (b) and (-1)-closure of positive braid β (c).

An exact Lagrangian filling L of $\Lambda \in \mathbb{R}^3_{std}$ in $(\mathbb{R}^4, \omega = d\lambda)$ is an embedded surface L bounded by Λ such that $\lambda|_{TL}$ is exact. The exact Lagrangian condition imposes strong rigidity on exact Lagrangian fillings. One evidence is that once a Legendrian knot has an exact Lagrangian filling, then the genus of the filling is fixed (differently compared with topological fillings, in which case the genus can increase freely), which is the 4-ball genus of the knot.

An essential question in symplectic geometry is that given a Legendrian knot in $\mathbb{R}^3_{std}$, how many exact Lagrangian fillings does it have in $\mathbb{R}^4$. Currently, the only known case is the maximum Thurston-Bennequin number (max-tb) unknot. By Eliashberg and Polterovich, the max-tb unknot has a unique exact Lagrangian filling. Note that the max tb condition is a necessary condition for a Legendrian to bound an exact Lagrangian filling. For the next easiest example, which is the Legendrian max-tb trefoil, which is also the (-1)-closure of a positive $(2,5)$ braid, we introduce a way to build exact Lagrangian fillings of it through concatenating elementary blocks together. The construction gives 5 exact Lagrangian fillings that are smoothly isotopic but are not Hamiltonian isotopic. This will match with the A_2 cluster structure will introduce in latter lectures for the positive $(2,3)$ braid.

As to other Legendrians, Casals and Gao in 2020 showed that (-1) closure of positive $(m, n+m)$ braids (which is a topological (m,n)-torus link), for $n \geq 3, m \geq 6$ or $(m,n) = (4.4), (4,5), (5,5)$, all have infinitely many exact Lagrangian fillings. This is essentially because of the fact that the positive (m,n) braid correspond to some cluster algebra of infinite type.

The goal of the C-lectures is to build connection of "the space of exact Lagrangian fillings of the Legendrian (-1)-closure of a positive braid β" with a cluster algebra so that we can use the cluster algebra structure to understand the geometrical space better. In particular, each exact Lagrangian filling has an $\mathbb{L}$-compressing disk system that corresponds to a quiver. The Lagrangian surgery operation that changes one exact Lagrangian filling to another corresponds to a mutation.

A4–More Cluster Combinatorics: g-vectors, c-vectors, F-polynomials

Merik Niemeyer

The goal of this talk was to deepen our understanding of cluster combinatorics by introducing c- and g-vectors, as well as F-polynomials. These come from a certain choice for the frozen part of the quiver, but contain enough information to reconstruct both the cluster variables and the y-variables of any cluster algebra associated to an ice quiver with the same mutable part. Moreover, we looked at some tropical dualities due to Nakanishi-Zelevinsky, which establish remarkable connections between c- and g-vectors. The talk largely followed Keller's survey paper [3].

1. Preparation

In the previous talks we have seen quivers and their corresponding exchange matrices, as well as ice quivers, which contain some frozen nodes, and can be described by extended exchange matrices. If the cluster variables of the initial seed are $x_1, ..., x_n$, and the frozen variables are $x_{n+1}, \dots, x_m$, every cluster variable will be a Laurent polynomial in $x_1, \dots, x_n$ with coefficients in $\mathbb{Z}[x_{n+1}, \dots, x_m]$. In order to phrase some of the results in the language of our reference material, let us slightly change perspective and set

$$y_j = \prod_{i=n+1}^{m} x_i^{b_{ij}} \in \operatorname{Trop}(x_{n+1}, \dots, x_m)\,,$$

for $1 \leq j \leq n$, where $\operatorname{Trop}(x_{n+1}, \dots, x_m)$ denotes a certain tropical semifield. These y-variables follow a 'tropical' mutation rule and capture how the frozen nodes are attached to the mutable nodes of the quiver. Therefore instead of keeping track of the extended exchange matrix and the cluster variables as we mutate, we can take the (principal part of the) exchange matrix, the cluster variables and the y-variables. This data constitutes a *seed*. Now, pick a vertex t_0 of the labeled n-regular tree $\mathbb{T}_n$, assign the initial seed to it, and then assign the seed mutated according to the edge labelling to the neighbouring vertices. Inductively, we obtain the *seed pattern*.

2. c-vectors, g-vectors and F-polynomials

2.1. **Definitions.** Let Q be a quiver (without frozen nodes), with nodes labelled $1, ..., n$. We first add frozen nodes in a particular way:

Definition 1. The *principal extension* Q_{pr} of Q is the quiver obtained from Q by adding nodes i' for $1 \leq i \leq n$ and arrows $i' \to i$.
The *cluster algebra with principal coefficients* associated to Q is the cluster algebra associated to Q_{pr}.

Let B be the exchange matrix of Q, then the extended exchange matrix of Q_{pr} is given by

$$\tilde{B} = \begin{pmatrix} B \\ \mathrm{Id}_n \end{pmatrix}.$$

This mutates according to the rules of matrix mutation, and thus we assign a matrix $\tilde{B}(t)$ to every vertex $t \in \mathbb{T}_n$, which has the form

$$\tilde{B}(t) = \begin{pmatrix} B(t) \\ C(t) \end{pmatrix}.$$

Definition 2. The matrix $C(t)$ is the *matrix of c-vectors*, its columns are the *c-vectors* $c_j(t)$, $1 \leq j \leq n$.

Theorem 3. *Every c-vector is non-zero and its entries are either all non-negative or all non-positive.*

This appeared implicitly as a conjecture in [1] and was proved in full generality in [2].

Next, we define F-polynomials: Recall again, that every cluster variable $x_j(t)$, for $1 \leq j \leq n$ and $t \in \mathbb{T}_n$, is a Laurent polynomial in the initial cluster variables, with coefficients in $\mathbb{Z}[x'_1, ..., x'_n]$, where x'_i denotes the (frozen) variable associated to the node i'.

Definition 4. Let $1 \leq j \leq n$ and $t \in \mathbb{T}_n$. The *F-polynomial* $F_j(t) \in \mathbb{Z}[x'_1, ..., x'_n]$ is obtained by specializing $x_j(t)$ to $x_1 = ... = x_n = 1$.

In the original paper [1], Fomin and Zelevinsky prove that any F-polynomial is a ratio of two polynomials with positive integer coefficients, which implies that it can be evaluated in any semifield (we now know that every F-polynomial is in fact a polynomial with positive integer coefficients [2]). Moreover, they conjectured the following theorem, which is equivalent to the sign property of c-vectors given above.

Theorem 5. *Every F-polynomial has constant term 1.*

The final object we need to introduce are the g-vectors, which we obtain by endowing the Laurent ring $\mathbb{Z}[x_1^{\pm 1}, ..., x_n^{\pm 1}, x'_1, ..., x'_n]$ with the following $\mathbb{Z}^n$-grading:

$$\deg(x_i) = e_i,$$
$$\deg(x'_i) = -Be_i,$$

for $1 \leq i \leq n$, where e_i denotes the i-th standard vector. Fomin and Zelevinsky proved that any cluster variable $x_j(t)$ is homogeneous with respect to this grading, allowing us to define:

Definition 6. Let $t \in \mathbb{T}_n, 1 \leq j \leq n$. The *$g$-vector* $g_j(t)$ is defined as

$$g_j(t) = \deg(x_j(t)).$$

The g-vectors are the columns of the *matrix of g-vectors*, denoted $G(t)$.

Again, we have a theorem which is equivalent to the two we gave previously:

Theorem 7. *The g-vectors are sign-coherent, meaning that for any $t \in \mathbb{T}_n$ every row of the matrix $G(t)$ is non-zero and either has only non-negative or only non-positive entries.*

As we had seen in previous talks, the cluster variables and entries of the exchange matrix are obtained recursively from the initial data via mutation. Consequently, one can deduce recursive formulas for all the above objects, and we gave an idea of how to do that.

2.2. **Separation formulas.** With all of this in place, we can reobtain both cluster and y-variables. These formulas are due to Fomin-Zelevinsky [1].

Theorem 8. *Let $t \in \mathbb{T}_n$, $\mathbb{P}$ any (coefficient) semifield, and $\mathcal{F} = \mathbb{Q}(\mathbb{P})(x_1, \ldots, x_n)$ the ambient field.*

(a) $y_j(t) = y_1^{c_{1j}(t)} \cdots y_n^{c_{nj}(t)} \prod_{i=1}^n F_i(t)|_{\mathbb{P}}(y_1, \ldots, y_n)^{b_{ij}(t)}$,

(b) $x_j(t) = x_1^{g_{1j}(t)} \cdots x_n^{g_{nj}(t)} \frac{F_j(t)|_{\mathcal{F}}(\hat{y_1}, \ldots, \hat{y_n})}{F_j(t)|_{\mathbb{P}}(y_1, \ldots, y_n)}$, *where* $\hat{y_j} = y_j \prod_{i=1}^n x_i^{b_{ij}}$.

Let us stress that this allows us to compute the cluster variables and coefficients for any cluster algebra just using the data obtained from the corresponding cluster algebra with principal coefficients.

3. Tropical dualities

Finally, we saw some tropical dualities, due to Nakanishi and Zelevinsky [4], which relate c- and g-vectors in various ways. To state these, we need to upgrade our notation slightly. We write $C(B, t_0, t)$ for the matrix of c-vectors obtained by starting with the exchange matrix B at $t_0 \in \mathbb{T}_n$ and mutating to t, and analogously for the matrix of g-vectors.

Theorem 9. *Let B be a skew-symmetrizable exchange matrix, $t_0, t \in \mathbb{T}_n$. Then:*

(a) $G(B, t_0, t)^T = C(-B^T, t_0, t)^{-1}$,

(b) $C(B, t_0, t) = C(-B(t), t, t_0)^{-1}$,

(c) $G(B, t_0, t) = G(-B(t), t, t_0)^{-1}$.

The c-vectors appearing in formula (a) belong to the *Langlands-dual* quiver which is obtained by replacing the exchange matrix B with $-B^T$.

In the last five minutes of the talk, we defined the notion of maximal green and reddening mutation sequences, notions which were further discussed in the evening session.

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A5–Cluster algebras from surfaces

Kayla Wright

Endowing mathematical objects with a cluster structure boomed after the axiomatization of cluster algebras by Fomin and Zelevnisky in the early 2000's. In this talk, we will explain how to endow a topological marked surface (S, M) with a cluster structure. Namely, we take a Riemannian orientable surface S with nonempty boundary and a finite set of marked points M on the boundary of S such that each boundary component contains at least one marked point. We triangulate (S, M) by drawing arcs between the marked points so that they are maximally non-crossing up to isotopy relative to the boundary. For example, if we take S to be a hexagon and M to be its 6 vertices, we triangulate S by drawing three non-crossing diagonals.

With this topological set up, we see beautiful bijections between arcs and cluster variables, triangulations and clusters, and skein relations and cluster mutation. This story can be further enhanced when incorporating the geometry of Teichmüller theory. Namely, if we look at the space of certain hyperbolic metrics on (S, M) and properly define lengths of geodesics on the surface, we are able to see a cluster structure on Teichmüller space, denoted $\mathcal{T}(S, M)$. More specifically, if we fix a metric in Teichmüller space and a choice of small circle around each marked point $m \in M$, we can define the length of a geodesic between marked points m, m' on (S, M) as the signed distance between the circles around m and m'. These small circles are called horocycles and the choice of horocycle at each marked point gives the data of decorated $\widetilde{\mathcal{T}(S, M)}$. We coordinatize this decorated version of Teichmüller space with Penner coordinates, also known as λ-lengths, which are an exponential version of the above defined length. These λ-lengths satisfy Ptolemy's Theorem which is the geometric version of the skein relations from the topological set-up. Altogether, this means that decorated Teichmüller space has a cluster structure, wheres cluster variables are in bijection with geodesics and cluster mutation is given by this hyperbolic version of Ptolemy's Theorem.

C2–Fronts and Lagrangian fillings of Legendrian links

Agniva Roy

The references for this talk are Section 4 of [1], and Sections 2 and 7 of [2].

1. Demazure weave fillings of positive braid closures

Definition 1 (Demazure Product)**.** Given a positive braid word β, the Demazure product of β, denoted $\delta(\beta)$, is the braid that corresponds to quotienting out the braid word using the relations $\sigma_i^2 = \sigma_i$, and also braid relations.

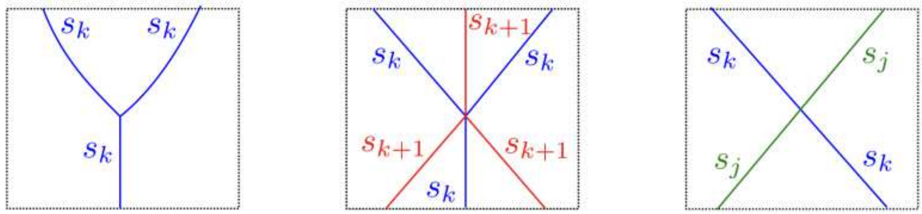

FIGURE 1. The figure is courtesy of the authors of [1].

Example. Given the word $\sigma_1^2\sigma_2^2$ representing a 3-stranded braid, the Demazure product is the braid $\sigma_1\sigma_2$.

In this section we describe an algebraic procedure that takes as input a positive braid β and outputs the braid corresponding to $\delta(\beta)$, the Demazure product of β. We will encode the braid purely by its crossings, as follows, and the three allowable moves will be *braid commutations*, *pinching a crossing* and a *braid move*, as shown in Figure 1.

A positive braid will be represented by encoding each Artin generator by a colour; thus an N-stranded braid with Demazure product w_0 will need $N-1$ colours. Then, the algorithm to build a Demazure weave proceeds by using commutations, braid moves and pinch moves to eliminate all powers of generators till we are left with just the Demazure product. Typically, we will use the moves to isolate the Demazure product on one side and then use pinch and braid moves successively to remove the powers of generators one by one.

The result of this procedure is called the **Demazure weave**. In Section 2, we will show how this algebraic procedure builds an exact Lagrangian filling for the (-1)-closure of the braid $\beta\delta(\beta)$.

Example. We give an example, see Figure 1, of the procedure using a 3-stranded braid $\beta = \sigma_1\sigma_2^2\sigma_1^2\sigma_2$. This example will not see any commuting relations being used. In this picture, we use blue to represent σ_1 and red for σ_2. The Demazure product of β is $\delta(\beta) = \sigma_1\sigma_2\sigma_1$.

We will interpret these diagrams as being properly embedded in a 2-disk, and call them N-graphs.

2. Legendrian surfaces from weaves

Given an N-graph G on D^2, one can construct an immersed surface in $\mathbb{R}\times D^2$, which is the front projection of a Legendrian surface $\Lambda(G)$ in $J^1(D^2)$ by weaving as follows. The objective is to create an immersed surface that projects to D^2, whose singularities are encoded by the N-graph:

- start with N sheets over D^2
- for every $(i, i+1)$-edge, introduce a line A_1^2-singularities between the corresponding two sheets

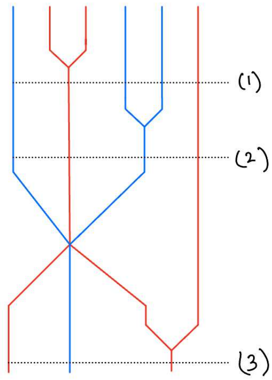

FIGURE 2. At level (1), we see the Demazure product on the left. At levels (2) and (3) respectively, we see the blue and red generators on the right being pinched so that at the end, we are left with $\delta(\beta)$.

- for every hexavalent vertex, introduce an A_1^3-singularity between the corresponding triple of consecutive sheets
- for every trivalent vertex, introduce a D_4^--singularity between the corresponding two sheets

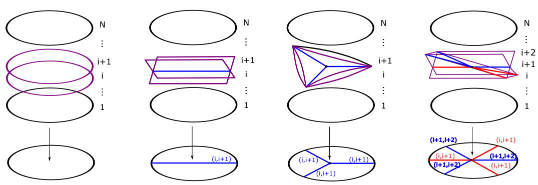

FIGURE 3. The weaving of singularities of fronts along the edges of the N-graph (courtesy of Roger Casals and Eric Zaslow). Gluing these local models according to the N-graph Γ yields the weave $\Lambda(\Gamma)$.

Some topological properties of the resulting surface: $\Lambda(G)$ is an N-fold branched cover over D^2 simply branched over the trivalent vertices of G.

(1) Euler characteristic $-\chi(\Lambda(G)) = N\chi(D^2) - v(G)$ where v is the number of trivalent vertices
(2) 1-cycles correspond to Y-trees. This is indicated in Figure 4.

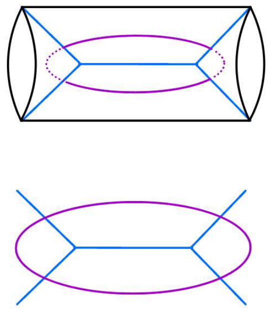

Figure 4. An I-tree corresponds to a cycle in the Legendrian surface, called an I-cycle.

Definition 2. An N-graph is called $free$ if the corresponding Legendrian weave has no Reeb chords.

The Demazure weaves built in Section 1 are free, hence the projection from $\mathbb{R}^5$ to $\mathbb{R}^4$ is embedded, as the only double points that could show up are due to Reeb chords. Also, by construction, the surfaces in $\mathbb{R}^4$ are exact Lagrangian, hence this procedure now produces an exact Lagrangian filling of the (-1)-closure of $\beta\delta(\beta)$ for any positive braid word β.

3. Quivers from Weaves

Associated to a Demazure weave, we can build a quiver that encodes the 1-cycles on the graph and their pairwise intersections. Further, there is a mutation operation one can do on the 1-cycles that show up in the weave to create another exact Lagrangian filling for the same braid, which may or may not be equivalent (up to Hamiltonian isotopy) to the previous one. We show how to do this in case of 2-weaves, i.e. weaves corresponding to 2-graphs, i.e. with only one colour. Firstly, given any 2-graph, encode all the I-trees as vertices on the quiver. Then, add arrows from every cycle to cycles that share a vertex with them, with arrows going from a cycle to one that is counter-clockwise of them.

Example. Consider the trefoil knot $T(2,3)$. It is the (-1)-closure of the braid σ_1^5, which we can consider to be $\beta\delta(\beta)$ for $\beta = \sigma^4$. We can see two I-cycles in the Demazure weave, and can build the A_2-quiver from them as shown in Figure 5. Mutating at an I-cycle corresponds to a local $I - H$ move.

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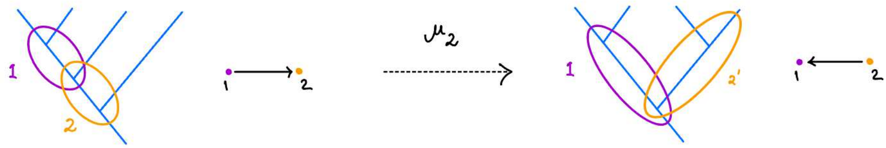

FIGURE 5. The Demazure weave and its mutation along the cycle denoted by 2, for the trefoil $T(2,3)$. These correspond to distinct exact Lagrangian fillings of the trefoil knot.

B1–Techniques for constructing cluster structures on varieties

COLIN KRAWCHUK

In recent years, cluster structures have been discovered on the coordinate rings of many varieties, including open positroid varieties [8, 9, 4], double Bott-Samelson varieties [10] and braid varieties [5, 2]. The presence of a cluster structure has important implications for the geometry of an algebraic variety, including the existence of canonical linearly independent sets of regular functions.

It is therefore natural to ask how one might determine if a given variety inherits a cluster algebra structure. Identifying such a structure involves constructing an initial seed of regular functions, showing each cluster variable in the associated algebra is indeed a regular function, and showing that the cluster variables generate the coordinate ring of the variety. While there is no general method for this procedure, we recount several useful techniques that have been successfully applied to construct cluster structures on varieties.

One of the most useful criteria for showing that the cluster variables arising from a candidate seed are regular functions is the Starfish Lemma:

Lemma 1. [1, Starfish Lemma] Let $\mathcal{R} = \mathbb{C}[X]$ be the coordinate ring of an irreducible normal affine complex algebraic variety X. Let $(Q, \tilde{x})$ be seed of rank n in $\mathbb{C}(X)$ with $\tilde{x} = (x_1, \dots, x_m)$ for $n \leq m$ whose variables lie in $\mathcal{R}$ such that

(1) the cluster variables in $\tilde{x}$ are pairwise coprime,
(2) for each cluster variable $x_k \in \tilde{x}$, the seed mutation μ_k replaces x_k with an element x'_k that lies in $\mathcal{R}$ and is coprime to x_k.

Then $\mathcal{A}(Q, \tilde{x}) \subset \mathcal{R}$.

The proof of the Starfish lemma relies on Hartogs' principle (showing that a function on X which is regular outside a subset of codimension 2 is regular everywhere). Under the conditions of the lemma, this property is satisfied not just for cluster variables but for elements of the upper cluster algebra of $\mathcal{A}(Q, \tilde{x})$.

To demonstrate the converse, that the cluster variables generate the coordinate ring of the variety, a frequent strategy is to first show that $\mathcal{A}(Q, \tilde{x})$ coincides with its upper cluster algebra. There are several reasons why this approach is beneficial.

Often it is easier to show that regular functions on the variety are generated by elements of the upper cluster algebra than arbitrary cluster variables. Moreover, if we wish to apply the Starfish Lemma then this equality must hold in order for $\mathcal{A}(Q, \tilde{x})$ to be a cluster structure on $\mathbb{C}[X]$. On the other hand, cluster algebras that do not equal their upper cluster algebra are often unwieldy, and it can be challenging to show containment in these cases.

For these reasons, criteria for $\mathcal{A}(Q, \tilde{x})$ to be equal to its upper cluster algebra have been introduced by several authors. In [6] Muller introduced the class of locally acyclic cluster algebras, which admit a finite cover by certain simpler cluster algebras (called acyclic cluster localisations). A consequence of this definition is that any local property of acyclic cluster algebras is true of locally acyclic cluster algebras. In particular, we have the following useful result:

Theorem 2. [6] *If a cluster algebra is locally acyclic, then it coincides with its upper cluster algebra.*

Any locally acyclic cluster algebra $\mathcal{A}$ also inherits a covering of $Spec(\mathcal{A})$ by open subvarieties corresponding to cluster localisations. In [7] Muller and Speyer refined this idea by defining Louise cluster algebras that have the additional property that the cluster localisations associated to this covering satisfy a Mayer-Vietores decomposition of cluster algebras. As an application, they showed the following:

Theorem 3. [7] *Cluster algebras associated to Postnikov diagrams in the disk are Louise.*

Unfortunately, the definition of locally acyclic cluster algebras does not suggest a method to check whether a given cluster algebra possesses this property. However, if the quiver of a seed $(Q, \tilde{x})$ belongs to a class of quivers called Banff quivers, then the corresponding cluster algebra $\mathcal{A}(Q, \tilde{x})$ is locally acyclic [6]. Moreover, a recursive algorithm is given in [6] for checking if a quiver is indeed Banff. Similarly, the class of sink-recurrent quivers is defined in [5] and seeds with sink-recurrent quivers are shown to give rise to locally acyclic cluster algebras. Notably, this fact was used by the authors to prove that cluster algebras arising from 3D-Plabic graphs are locally acyclic.

A final strategy for showing that $\mathcal{A}$ coincides with its upper cluster algebra relies on quasi-homorphisms between cluster algebras in the sense of Fraser [3]. In particular, if the elements of a generating set for the upper cluster algebra belong to either $\mathcal{A}$ or a quasi-equivalent cluster algebra, then $\mathcal{A}$ coincides with its upper cluster algebra. This approach was taken in [2] where it is shown that cyclic rotations of braid words induce quasi-cluster transformations.

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B2–Combinatorics of plabic graphs

Peter Spacek

We introduced *plabic graphs*: *planar, bicolored graphs* properly embedded into the closed disk with b (uncolored) vertices on the boundary. (Loops and multiple edges are allowed.) We also defined *move-equivalence* of plabic graphs, i.e. two graphs are move-equivalent if they are related by the *square move* (exchanging colors on a square with alternatingly colored vertices), the *creative/destructive move* (inserting a colored vertex on an edge or removing a bivalent vertex), and finally the *(de)construction move* (merging two vertices of the same color connected by an edge or splitting a colored vertex into two connected by an edge).

We then discussed how to construct a quiver associated to a given plabic graph: a vertex for each face (if a face borders the boundary of the disk, the corresponding vertex is frozen), and an arrow between vertices for each edge of the plabic graph with a white vertex on the left and a black on the right (cancel out any 2-cycles arising from this). We noted that the square move leads to mutation of the associated quiver, as long as every two consecutive faces bordering the square are distinct.

Next, we related plabic graph to combinatorial objects that appeared before: we showed how to construct a plabic graph from a triangulation of a polygon and from (double) wiring diagrams. We quickly discussed how the quiver of a triangulation coincides with the quiver of the plabic graph arising from a triangulation, and mentioned that the same holds for (double) wiring diagrams.

We then defined *reduced plabic graphs*: namely, plabic graphs that are *not* move-equivalent to a plabic graph containing the "forbidden configurations", namely the *hollow digon* (two vertices with two edges connecting them), and an internal leaf connected to a trivalent vertex of the other color that is not move-equivalent to a bivalent vertex. To obtain a more direct characterization, we introduced *trip permutations*: a *trip* is a path through the plabic graph following the "rules of the road", turning to the right at black and to the left at white vertices; trips either start and end at a boundary vertex, or are *round trips* in the interior; the *trip* *permutation* (associated to a plabic graph G) is the permutation π_G of b elements that sends i to j if the trip in G starting at i ends at j.

We mentioned that move-equivalent plabic graphs have the same trip permutations, and that in reduced plabic graphs a fixed point i of the permutation implies that the component connected to the boundary vertex i is move-equivalent to a lollipop. This led to the definition of *decorated trip permutations* of reduced plabic graph: each fixed point i of the trip permutation of the reduced plabic graph is decorated with $\overline{i}$ or $\underline{i}$ if the lollipop attached to i is white resp. black. This allowed us to state the *fundamental theorem of reduced plabic graphs*: two reduced plabic graphs are move-equivalent if and only if their decorated trip permutations coincide. This in particular led to the observation that reduced plabic graphs are exactly those plabic graphs with a given trip permutation that have the minimal number of faces.

We continued by discussing the relation between reducedness and normalcy: a *normal plabic graph* is a bipartite plabic graph with trivalent white vertices and only black vertices connected to the boundary vertices. We say that a plabic graph has a *bad feature* if it contains either a round trip, a *essential self-intersection* (a trip that pass through the same edge twice), or a *bad double crossing* (two trips both crossing two given edges in the same order). We then stated the theorem that a normal plabic graph is reduced if and only if it contains *no* bad features. Afterwards, we sketched an algorithm that uses move-equivalences to turn a plabic graph into a normal plabic graph (or results into a non-reduced plabic graph), allowing the previous theorem to be applied to general plabic graphs. We also mentioned the existence of the *resonance property* to check reducedness.

Finally, we defined *source* and *target face labelings* of reduced plabic graphs: a face is labeled by the set of those i such that the trips starting (resp. ending) at i have the given face to the left of the trip. (This works due to the fact that trips in a reduced plabic graph bisect the disk.) We mentioned that the labels of the faces of a given reduced plabic graph all have the same cardinality. Finally, we defined the *positroid* associated to a reduced plabic graph given by the face labels of the boundary faces.

The main reference for this talk was Chapter 7 of [1]. The seminal reference for plabic graphs is [2].

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C3–Constructible sheaves on Legendrian knots

Yoon Jae Nho

Given a Legendrian knot $\Lambda \subset \mathbb{R}^3$, one can construct a D^--stack $\mathfrak{R}(\Lambda)$ which is a Legendrian isotopy invariant of Λ. If Λ is a positive braid knot, this stack can be identified with the open Bott-Samelson variety associated with β. One interpretation of $\mathfrak{R}(\Lambda)$ is that it is the "moduli" of exact Lagrangian fillings of Λ. Indeed, an exact Lagrangian filling L of Λ gives rise to an open toric chart $(\mathbb{C}^*)^{b_1(L)}$ of $\mathfrak{R}(\Lambda)$, which can be verified by direct calculation in the case of free Legendrian weaves, using the machinery of [3].

Building $\mathfrak{R}(\Lambda)$ is a two-step process. First, one considers the category of constructible sheaves on $\mathbb{R}^2_{x,y}$ supported on Λ. These categories admit combinatorial descriptions, but they are not Legendrian isotopy invariants. Then, one can further restrict to sheaves with *singular support* on Λ. The theorem of GKS[4] then states that the category of such sheaves is indeed Legendrian isotopy invariant. Then, we can further restrict to "microlocal rank 1" sheaves with singular support on Λ with vanishing stalks at $y = -\infty$. The moduli of such sheaves then yield $\mathfrak{R}(\Lambda)$.

As a concrete example, in the case $\Lambda = \Lambda_\beta$ for the (-1)-closure of a positive braid-knot β with reduced word expression $\beta = s_{i_1}...s_{i_n}$, where s_i is the transposition of the ith strand with the $i+1$th strand, one can show that the moduli $\mathfrak{R}(\Lambda_\beta)$ is given by the moduli of tuples of complete flags $(F_1, ..., F_{n+1})$ with relative position conditions $F_j \sim_{s_{i_j}} F_{j+1}$, and $F_{n+1} = F_1$, which is indeed the open Bott-Samelson variety.

In this talk, we address the first part of the problem. Given a (regular cell refinement) of stratification induced by the front-projection of Λ on $\mathbb{R}^2_{x,y}$, we introduce the notion of *constructible sheaves*, i.e. sheaves whose restriction to each stratum are locally constant sheaves. Then, we compute constructible sheaves supported on the local model for the arc, the cusp and the crossing. We then use the local-to-global principle to express constructible sheaves supported on more general Legendrian knots as functors from the poset category induced by the stratification to the category of k-modules.

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B3–Webs and $Gr(3, n)$

Emine Yildirim

The goal of this talk to understand the cluster algebra structure in the homogeneous coordinate ring of Grassmannian $\mathbb{C}[\widehat{Gr(3,n)}]$ of 3-planes in $\mathbb{C}^n$ from Fomin-Pylyvaskyy perspective using Kuberberg's web basis. We mainly follow the following references: [4] and [1, Section 9.1]. We start the talk by explaining the definition of a tensor diagram. Then, we show how a tensor diagram encodes an element in the homogeneous coordinate ring of Grassmannian. Fomin and Pylyvaskyy show that if a tensor diagram is a planar tree, then the corresponding web invariant is a cluster or coefficient variable. We give a complete example of the cluster algebra structure in the case of $n = 6$. The cluster algebra for $Gr(3, 6)$ is of Dynkin type D_4; it has 22 cluster variables - six of which are frozen variables. Since Plücker coordinates are cluster variables, we have 20 Plücker cluster variables and two non-Plücker cluster variables in this case. We explicitly compute these two non-Plücker cluster variables using skein relations.

A tensor diagram is called a web if it is planar. Non-elliptic webs give rise to web invariants which form a linear basis in the ring of invariants. Let us now state some of Fomin-Pylyvaskyy's conjectures.

Conjectures.

(1) The set of cluster (and coefficient) variables coincide with the set of indecomposable arborizable web invariants.
(2) Two cluster variables lie in the same cluster if and only if they are compatible web invariants.
(3) If $n \geq 9$, there are infinitely many indecomposable non-arborizable web invariants.

Fomin-Pylyvaskyy [4] verify these conjectures in the finite type examples:

$Gr(2, n+3)$	$Gr(3, 6)$	$Gr(3, 7)$	$Gr(3, 8)$
A_n	D_4	E_6	E_8

Note that this talk is a restrictive setting of Fomin-Pylyvaskyy paper - keep in mind the theorems and conjectures we mention in this abstract can be stated in a more general set up for $SL(V)$ invariant rings that is Fomin-Pylyvaskyy's main object in their paper [4]. Furthermore, C. Fraser [1] proves that for the cluster algebra in the homogeneous coordinate ring of Grassmannian $Gr(3, 9)$:

(1) Every cluster variable is an indecomposable arborizable web invariant.
(2) Every cluster monomial is a web invariant.
(3) There are infinitely many indecomposable non-arborizable web invariants.

These results are strong evidences for the validity of the conjectures. Finally, we would like to mention that webs may seem similar to dimers; [2] is a reference to see how they are related. Also, we refer curious audience to the paper [3] for further reading and a general view on this topic.

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C4–Microsupport and Legendrian fronts

Laurent Côté

Summary. Given Legendrian Λ in the cosphere bundle of $\mathbb{R}^2$, one can associate to it a variety $\mathcal{M}(\Lambda)$ whose properties carry useful information about Λ. This variety is defined as the moduli space of objects of the category of constructible sheaves *microsupported* along Λ. The purpose of this talk was to introduce the notions which enter into this construction.

1. Microsupport of sheaves

Conventions 1. Throughout this report, all sheaves are implicitly assumed to be constructible (with perfect stalks) and valued in the dg derived category of chain complexes over $\mathbb{C}$. All functors are implicitly derived. All stratifications are assumed to be Whitney. Finally, for consistency with some of the literature (e.g. [3]) we work throughout in the real analytic category.

Let M be a manifold. Fix a stratification $\mathcal{S}$ on M and a point $x \in M$. Let S_x be the stratum containing x. A function $f : Op(x) \to \mathbb{R}$ is said to be *stratified Morse* at $x \in M$ if either (a) $f|_{S_x}$ is non-critical at x or (b) $f|_{S_x}$ has a Morse critical points at x *and* $df_x(\tau) \neq 0$ for any $\tau \subset T_xM$ which is equal to a limit of tangent vectors of a larger stratum $Y > S_x$.

Construction-Definition 2. Given $\mathcal{F} \in sh(M)$, fix $x \in M$ and a function $f : Op(x) \to \mathbb{R}$ such that $f(x) = 0$. Fix $\epsilon, \delta > 0$ and set

$$M_{(x,f,\epsilon,\delta)}(\mathcal{F}) := cone\left(\mathcal{F}(B_\epsilon(x) \cap f^{-1}(-\infty, \delta)) \to \mathcal{F}(B_\epsilon(x) \cap f^{-1}(-\infty, -\delta))\right)$$

If f is stratified Morse at $x \in M$, then it can be shown that $M_{(x,f,\epsilon,\delta)}(\mathcal{F})$ stabilizes as $\epsilon, \delta \to 0$. In fact, the output only depends on $(x, df_x) \in T_x^*M$.

Definition 3. For $(x, \xi) \in S_x^*M$ and f stratified Morse at $x \in M$, we define the *Morse group* $M_{(x,\xi)}(\mathcal{F}) := M_{(x,f,\epsilon,\delta)}(\mathcal{F})$ for ϵ, δ small enough.

A covector $(x, \xi) \in S^*M$ is said to be *characteristic* if $M_{(x,\xi)}(\mathcal{F}) \neq 0$. Note that this notion depends on the stratification $\mathcal{S}$.

The characteristic covectors correspond precisely to the (co)directions along which the restriction map of $\mathcal{F}$ is non-trivial. This suggests that the set of characteristic co-vectors is a useful invariant of $\mathcal{F}$.

Definition 4 (Microsupport)**.** The *microsupport* (or *singular support*) of $\mathcal{F}$ is the set

$$SS(\mathcal{F}) := \overline{\{(x, \xi) \in S^*M \mid (x, \xi) \text{ is characteristic } \}}. \tag{1}$$

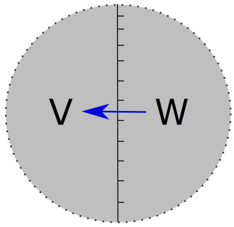

FIGURE 1. Here the vertical arrow is the front of a Legendrian arc $\Lambda \subset S^*\mathbb{R}^2$. The "hair" specifies a unique lift of the front.

While the notion of a characteristic vector depends on the stratification, it can be shown that the microsupport does not depend on this choice. In fact, the microsupport can be defined without choosing a stratification and appealing to the theory of stratified Morse functions; see [2, Sec. 5.1]. However, the Morse-theoretic viewpoint is useful for intuition and computations.

2. The category $Sh_\Lambda(M)$

Let $\Lambda \subset S^*M$ be a Legendrian.

Definition 5. We let $sh_\Lambda(M) \subset sh(M)$ be the full subcategory on objects whose microsupport is contained in Λ.

To get a handle on this definition, let us suppose that $\pi(\Lambda) \subset M$ is a front. Then we can consider the category of sheaves $sh_{\mathcal{S}}(M)$ constructible with respect to any stratification $\mathcal{S}$ containing the front. According to the exit-path definition of a constructible sheaf, this is the same thing as a module over the exist path category. In other words, a constructible sheaf is the data of a stalk on each stratum and restriction maps from lower dimensional strata to higher dimensional strata.

The microsupport condition picks out a full subcategory $sh_\Lambda(M) \subset sh_{\mathcal{S}}(M)$ by forcing some of the restriction maps to be isomorphisms. This is illustrated in the following example.

Example. Suppose that Λ is a lift of the front drawn in Figure 2. Then the category of sheaves constructible with respect to the induced stratification is equivalent to the category of representations of the quiver ($\bullet \xleftarrow{\alpha} \bullet \xrightarrow{\beta} \bullet$). However, for a constructible sheaf to lie in $sh_\Lambda(D^2)$, it must have the property that the restriction map corresponding to β is an isomorphism: indeed, the failure of this map to be an isomorphism would be witnessed by a point in the microsupport. But by definition of $sh_\Lambda(D^2)$, the microsupport of $\mathcal{F}$ in $S^{*,-}D^2$ is empty (the "hair" points in the $+$ direction). We conclude that the category $sh_\Lambda(D^2)$ is equivalent to the category of representations of the A_2 quiver.

The great virtue of the category $sh_\Lambda(M)$, as opposed to $sh_{\mathcal{S}}(M)$, is that it is an invariant of Λ. This is the content of the following theorem:

Theorem 6 (Fundamental theorem [1] (Guillermou–Kashiwara–Schapira))**.** *A Legendrian isotopy $\Lambda \rightsquigarrow \Lambda' \subset S^*M$ induces an equivalence of categories*

$$sh_\Lambda(M) \to sh_{\Lambda'}(M).$$

In general, $sh_\Lambda(M)$ can be very complicated. However, when $M = \mathbb{R}^2$, then the front projection of a Legendrian generically only has cusps and crossings. Hence the study of $sh_\Lambda(\mathbb{R}^2)$ can be reduced to local models. The simplest local model was computed in Example 2; the other two (cusp and crossing) were also computed in the talk. See [3, Sec. 3.3].

3. The moduli space of rank 1 objects

In order to access the category $sh_\Lambda(M)$, it is often useful to consider categorical invariants associated to it. The main class of invariants which were discussed in the talk are so-called "moduli spaces" of objects.

Definition 7. Suppose $\Lambda \subset S^*M$ is connected. The *microlocal rank* of $\mathcal{F} \in sh_\Lambda(M)$ is the rank of $M_{(x,\xi)}(\mathcal{F})$ for any $(x, \xi) \in \Lambda$.

Theorem 8 ([5] Toën–Vaquié)**.** *There exists a "derived stack" $\mathcal{M}^r(\Lambda)$ whose points are in bijection with isomorphism classes of objects of $sh_\Lambda(M)$ having microlocal rank r.*

This theorem is an abstract result valid for categories satisfying a certain finiteness assumption. Our standing assumption that constructible sheaves have perfect stalks is essential in order to appeal to it.

For many Legendrians which arise in practice, the output of this theorem (a priori a derived stack) is an ordinary variety which can be explicitly described.

Example. In the talk, we explicitly computed the moduli space of rank 1 objects where Λ is the (lift of the) front drawn in Figure 3. The answer is as follows. We first consider the moduli space

$$\widetilde{\mathcal{M}}^1(\Lambda) := \{(\ell_0, \dots, \ell_4) \in \mathrm{Mat}_{2\times 5}(\mathbb{C}) \mid \ell_i \in \mathrm{Mat}_{2\times 1}(\mathbb{C}), \ell_i \pitchfork \ell_{i+1}, i \in \mathbb{Z}/5\}$$

Then the moduli space of rank 1 objects is the quotient

$$\mathcal{M}^1(\Lambda) = \widetilde{\mathcal{M}}^1(\Lambda)/(GL_2(\mathbb{C}) \times \mathrm{Diag}_5(\mathbb{C})).$$

One can also consider a *framed* variant

$$\mathcal{M}^1_{fr}(\Lambda) = \widetilde{\mathcal{M}}^1(\Lambda)/GL_2(\mathbb{C}).$$

The main idea for performing such computations is to restrict ourselves to local models, for which (as explained above) the category of microlocal sheaves is fully understood. We refer to [3, Sec. 6] and [4, Sec. 3] for related computations.

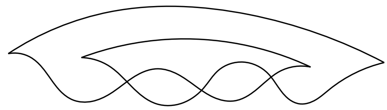

Figure 2. The front projection of a trefoil

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B4–Double Bruhat cells and generalisations

Matthew Pressland

1. Double Bruhat cells

One of the earliest results equipping the coordinate ring of an algebraic variety with a cluster algebra structure is due to Berenstein, Fomin and Zelevinsky [2], who achieve this for double Bruhat cells. Before describing their construction, we give the necessary set-up and definitions.

Fix a connected, simply connected, semisimple algebraic group G over $\mathbb{C}$, with opposite Borel subgroups B_+ and B_-. This determines a maximal torus $T = B_+ \cap B_- \cong (\mathbb{C}^\times)^n$, a Weyl group $W = \mathrm{Norm}_G(T)/T$, and a Dynkin diagram Δ. The Weyl group is generated by n simple reflections s_i, for $i \in \Delta_0$.

Each node $i \in \Delta_0$ determines a homomorphism $\varphi_i \colon \mathrm{SL}_2(\mathbb{C}) \to G$, taking upper triangular matrices into B_+ and lower triangular matrices into B_-. We may lift W to a subset (but not a subgroup) of G by identifying s_i with $\bar{s}_i = \varphi_i \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$, and a general element $w = s_{i_1} \cdots s_{i_\ell}$ with $\bar{s}_{i_1} \cdots \bar{s}_{i_\ell}$, where the expression for w as a product of simple reflections is chosen to be *reduced*, i.e. of minimal length $\ell = \ell(w)$. Viewing W as a subset of G in this way, we obtain a pair of *Bruhat decompositions*

$$G = \bigcup_{u \in W} B_+ u B_+ = \bigcup_{v \in W} B_- v B_-.$$

Definition 1. A *double Bruhat cell* is $G^u_v := B_+ u B_+ \cap B_- v B_-$, for $u, v \in W$.

To describe a cluster algebra structure on the coordinate ring $\mathbb{C}[G^u_v]$, we restrict for simplicity to the case that the Dynkin diagram Δ is simply-laced, that is, of type A, D or E. This will allow us to describe the initial seed via a quiver, rather

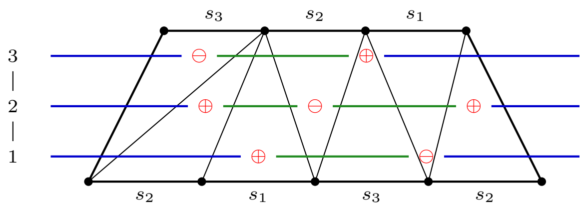


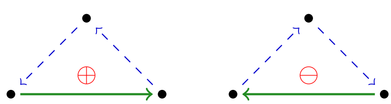
FIGURE 1. A triangulation of (u, v), with the associated string diagram overlaid, for $G = \mathrm{SL}_4(\mathbb{C})$, so $\Delta = \mathsf{A}_3$ (shown left), with $u = s_3 s_2 s_1$ and $v = s_2 s_1 s_3 s_2$. Closed strings are shown in green, and open strings in blue.

than a more general valued quiver or skew-symmetrisable matrix. We will also deviate from the original presentation in [2], and using instead a description of this seed derived from work of Shen and Weng [7], which we will return to shortly.

Definition 2. Given $u, v \in W$, consider a trapezium with its upper edge cut into $\ell(u)$ segments, and lower edge cut into $\ell(v)$ segments. A *triangulation* of (u, v) is a choice of reduced expression for each of u and v, together with a decomposition of the trapezium into triangles such that exactly one edge of each triangle lies on the upper or lower edge of the trapezium. See Figure 1 for an example.

Given a triangulation of (u, v), we label the segments on the upper and lower edges via the chosen reduced expressions of u and v, reading from left to right. This labels exactly one edge of each triangle by a simple reflection, and hence induces a labelling of the triangles. A triangulation of (u, v) determines a string diagram in the following way. Draw $n = |\Delta_0|$ strands through the trapezium, indexed by the nodes of the Dynkin diagram. In a triangle labelled by s_i, cut strand i, and label the cut by $\oplus$ if the labelled edge of the triangle is on the bottom of the trapezium, and by $\ominus$ if the labelled edge is on the top. This process cuts the strands into *strings*, which can be either *closed* (incident with two cuts), or *open* (incident with at most one cut). Again, an example is shown in Figure 1.

Definition 3. The (ice) quiver $Q(t)$ of a triangulation t of (u, v) has as vertices the strings of the associated string diagram, with open strings frozen. At each cut, we see one of the following configurations in the quiver, depending on the sign.



Here the solid arrow connects the two strings from strand i meeting at the cut, and we draw a pair of dashed arrows as shown for each string passing through the triangle containing the cut and lying on a strand j with i and j joined by an edge of Δ. These dashed arrows are interpreted as 'half-arrows': in the final quiver, two half-arrows in the same direction add together to form a full (solid) arrow, while those in opposite directions cancel out. This process produces a natural collection

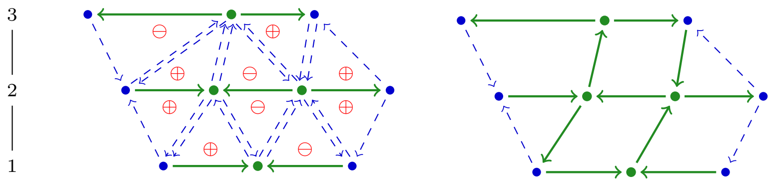


FIGURE 2. Constructing the quiver of the triangulation in Figure 1; the initial construction involving half-arrows (left), and the final quiver (right). Mutable vertices are green, and frozen vertices are blue.

of half-arrows between frozen vertices, but these play no role in defining the cluster algebra. See Figure 2 for the quiver associated to the triangulation in Figure 1.

Theorem 4 (Berenstein–Fomin–Zelevinsky [2])**.** *Let $u, v \in W$, let t be a triangulation of (u, v), and let $\mathscr{A}(t)$ be the cluster algebra associated to $Q(t)$, with invertible frozen variables. Then there is an isomorphism*

$$\mathscr{A}(t) \xrightarrow{\sim} \mathbb{C}[G_v^u]$$

sending the initial cluster variables to generalised minors.

Strictly speaking, the original result from [2] gives an isomorphism with the upper cluster algebra $\mathscr{U}(t)$ associated to $Q(t)$. However, Muller and Speyer [6] show that this cluster algebra is locally acyclic, and hence $\mathscr{A}(t) = \mathscr{U}(t)$.

We do not give the general definition of generalised minors here, but note that in type A, where $G = \mathrm{SL}_{n+1}(\mathbb{C})$, they are ordinary matrix minors. There is an explicit combinatorial recipe for computing which minors are the images of the initial cluster variables under the isomorphism of Theorem 4. For our running example, the result is

$$\begin{tikzcd}
D^{123}_{134} \arrow[dr, dashed] & & & D^{124}_{134} \arrow[lll] \arrow[rr] & & D^{124}_{123} \arrow[d] & & \\
& D^{12}_{23} \arrow[rr] & & D^{12}_{24} \arrow[u] \arrow[dl] & & D^{14}_{13} \arrow[ll] \arrow[rr] & & D^{14}_{12} \arrow[ull, dashed] \arrow[dl, dashed] \\
& & D^{1}_{3} \arrow[ul, dashed] \arrow[rr] & & D^{1}_{1} \arrow[ur] & & D^{4}_{1} \arrow[ll] &
\end{tikzcd}$$

where D^I_J denotes the minor on rows I and columns J.

2. Double Bott–Samelson varieties

Recall that the braid group $\mathrm{Br}(\Delta)$ is defined similarly to the Coxeter group of Δ (which is isomorphic to W), but excluding the relations $s_i^2 = e$. A *positive braid* is an element of $\mathrm{Br}(\Delta)$ expressible as a word in the letters s_i, $i \in \Delta_0$ (in contrast to a general braid, in which the letters s_i^{-1} may be necessary). Given $u, v \in \mathrm{Br}(\Delta)$, one can define a triangulation exactly as in Definition 2, replacing 'reduced expression' by 'positive braid word'. Given such a triangulation t, construct the associated

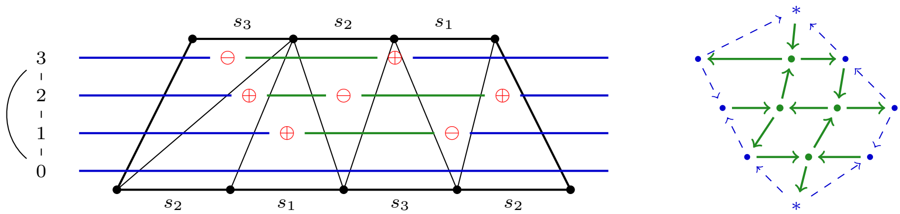


FIGURE 3. A triangulation of positive braid words (u, v), with the associated 'affine' string diagram overlaid (left), and a schematic of the associated quiver (right). To obtain the actual quiver, the two vertices labelled by $*$ should be identified.

string diagram as before, but viewing u and v as elements of $\mathrm{Br}(\widetilde{\Delta})$, for $\widetilde{\Delta}$ the associated affine diagram. Let $\widetilde{Q}(t)$ be the associated quiver, which differs from $Q(t)$ by adding a single frozen vertex, corresponding to the single open string labelled by the extending vertex of $\widetilde{\Delta}$, and its incident arrows. An example is given in Figure 3; while this reuses the reduced expressions for elements of W from the previous example, we emphasise that the general construction applies to arbitrary positive braid words.

The cluster algebra $\widetilde{\mathscr{A}}(t)$ with invertible frozen variables associated to $\widetilde{Q}(t)$ also turns out to have a geometric interpretation.

Definition 5 (Shen–Weng [7])**.** Let $u = s_{i_1} \cdots s_{i_\ell}$ and $v = s_{j_1} \cdots s_{j_m}$ be positive braids. Then the *double Bott–Samelson variety* BS^u_v consists of tuples of flags $(x_0B_+, \dots, x_\ell B_+, y_0B_-, \dots, y_mB_-) \in G\backslash\big((G/B_+)^\ell \times (G/B_-)^m\big)$ (that is, each tuple is considered up to the left action of G on the product of flag varieties) subject to the conditions that

(1) $x_{k-1}^{-1}x_k \in B_+s_{i_k}B_+$ for $k = 1, \dots, \ell$,
(2) $y_k^{-1}y_{k-1} \in B_-s_{j_k}B_-$ for $k = 1, \dots, m$,
(3) $x_0^{-1}y_0 \in B_+B_-$ and $x_\ell^{-1}y_m \in B_+B_-$.

Letting $U_\pm$ denote the unipotent radicals of $B_\pm$, the *decorated Bott–Samelson variety* $\widehat{\mathrm{BS}}^u_v$ consists of those tuples

$$(x_0U_+, x_1B_+, \dots, x_\ell B_+, y_0B_-, \dots, y_{m-1}B_-, y_mU_-)$$

in $G\backslash\big(G/U_+ \times (G/B_+)^{\ell-1} \times (G/B_-)^{m-1} \times G/U_-\big)$ which map to points of BS^u_v under the natural projection.

Shen and Weng [7] show that both of these varieties depend, up to isomorphism, only on the positive braids u and v, and not on the choice of braid words.

Theorem 6 (Shen–Weng [7])**.** *Let u and v be positive braids and let t be a triangulation of (u, v). Then there is an isomorphism*

$$\widetilde{\mathscr{A}}(t) \xrightarrow{\sim} \mathbb{C}[\widehat{\mathrm{BS}}^u_v].$$

Remark 7. The ordinary Bott–Samelson variety BS_w associated to $w \in W$ was introduced [1] to provide a desingularisation of the Schubert variety $\overline{BwB}/B$. Double Bott–Samelson varieties are special cases of braid varieties, and so Theorem 6 is an important precursor to the general result that all such varieties carry cluster algebra structures [3, 4, 5].

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C5–Cluster algebras and symplectic topology: Microlocal holonomies and the Bott-Samelson case

Mikhail Gorsky

This talk concerns a point of view on cluster algebra structures on coordinate rings of certain affine algebraic varieties by means of symplectic geometry. Several families of varieties appearing in talks B1 – B4, such as open positroid varieties, double Bruhat cells, and double Bott-Samelson cells, can be described as moduli spaces of decorated microlocal rank-1 constructible sheaves on $\mathbb{R}^2$ supported on front projections of Legendrian links in $\mathbb{R}^3$ with the standard contact structure ξ_{st}. This perspective connects talks from series A and B with the framework of series C.

Consider a positive braid β with the Demazure product $w_0 \in S_n$. The Legendrian (-1)-closure of the braid represented by β is a Legendrian link Λ_β in $(\mathbb{R}^3, \xi_{st})$. If $\beta = \Delta\beta'$ for some positive braid β', where Δ is the half-twist, the link Λ_β is Legendrian isotopic to the rainbow closure of β', as considered in [7, Section 6.5]. With a Legendrian link Λ taken with a collection of marked points T one can naturally associate a moduli stack $\mathfrak{M}(\Lambda, T)$ of decorated microlocal rank-1 constructible sheaves on $\mathbb{R}^2$ whose support is contained in the front projection of Λ. It turns out that for (Λ_β, T) with T containing at least one marked point per link component, $\mathfrak{M}(\Lambda, T)$ is in fact a smooth affine algebraic variety: it can be realized, up to a torus factor, as a braid variety $X(\beta)$ in type A which will be discussed in more detail in talk C6. The smoothness of braid varieties follows from work of Escobar [3].

Shen and Weng in [8] and in a joint work with Gao [5] introduced several versions of double Bott-Samelson (BS) varieties associated with pairs of positive braid words. In particular, half-decorated double BS varieties for pairs (e, β') were proved in [5] to be isomorphic to $\mathfrak{M}(\Lambda_{\Delta\beta'}, T)$ for T having precisely one marked point per strand of $\Delta\beta'$. For decorated double BS varieties, a cluster $\mathcal{A}$-structure was defined in [8] in terms of generalized minors. This was done by extending standard approaches to cluster structures on double Bruhat cells via amalgamation techniques of Fock-Goncharov [4]. This algebra structure was translated to the symplectic framework in [5], where $\mathfrak{M}(\Lambda_{\Delta\beta'}, T)$ was interpreted as the augmentation variety of the link $\Lambda_{\Delta\beta'}$. An undecorated variant of $\mathfrak{M}(\Lambda_{\Delta\beta'}, T)$, denoted by $\mathcal{M}_1(\Lambda_{\Delta\beta'}, T)$, is also isomorphic to a variant of a double BS variety, depending on the choice of T. The latter was proved in [8] to admit a cluster $\mathcal{X}$-structure, also known as a cluster Poisson structure, forming a cluster ensemble (as defined in talk B5) with $\mathfrak{M}(\Lambda_{\Delta\beta'}, T)$ interpreted as a (half-)decorated double BS variety.

From the point of view of symplectic geometry, results of [5, 8] indicated the existence of cluster $\mathcal{A}$- and $\mathcal{X}$-structures on moduli spaces of microlocal rank-1 sheaves associated with Legendrian links, but the construction presented in these works was fairly unsatisfactory. In the talk, a "symplectic" construction of cluster $\mathcal{A}$-structures on $\mathfrak{M}(\Lambda_{\Delta\beta'}, T)$ and of cluster $\mathcal{X}$-structures on $\mathcal{M}_1(\Lambda_{\Delta\beta'}, T)$ (the latter improving on earlier work [6]) was presented. This construction is due to Casals and Weng [1] who used technology of weaves introduced by Casals and Zaslow [2]. Weaves are certain coloured graphs representing Lagrangian fillings of Legendrian links, as explained in talk C2. The main result presented at the talk is the following.

Theorem 1. [1] *For a positive braid β' and a collection T of marked points on $\Lambda_{\Delta\beta'}$ with at least one point per component, the pair*

$$(\mathfrak{M}(\Lambda_{\Delta\beta'}, T), \mathcal{M}_1(\Lambda_{\Delta\beta'}, T))$$

forms a cluster ensemble, where the initial seeds of $(\mathfrak{M}(\Lambda_{\Delta\beta'}, T)$ *and* $\mathcal{M}_1(\Lambda_{\Delta\beta'}, T))$ *are described in terms of an exact embedded Lagrangian filling L of $\Lambda_{\Delta\beta'}$ described via a certain explicit weave.*

The construction and a sketch of the proof were presented. Cluster $\mathcal{A}$-variables are indexed by certain relative cycles $\eta \in H_1(L\backslash T, \Lambda\backslash T)$ and can be interpreted as so-called microlocal merodromies, which intuitively give parallel transport along η, while $\mathcal{X}$-variables are indexed by absolute cycles in $\gamma \in H_1(L)$ and can be interpreted as microlocal monodromies along γ. The language of weaves not only provided a symplectic interpretation of cluster algebra structures on the sheaf moduli spaces, but also allowed to simplify some of the proofs, compared to those in [8].

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C6–Cluster structures on braid varieties

Tonie Scroggin

Given a braid word β we may define an algebraic variety called a braide variety. In this talk we show that the coordinate ring of regular functions on any braid variety is a cluster algebra. By defining Lusztig cycles, intersections and functions on the Lusztig cycles we are able tio produce a quiver and cluster variables which constitutes the seed of the cluster algebra $\mathbb{C}[\chi(\beta)]$

Introduction to Cluster Ensembles and the Fock-Goncharov duality conjectures

Geoffrey Janssens

A geometric counterpart of Fomin-Zelevinsky's cluster algebras was introduced by Fock and Goncharov [2, 3] in which "seed tori" are glued together along cluster transformations, which are certain distinguished birational maps, to produce cluster varieties. These varieties come in pairs and form a so-called cluster ensemble.

In this talk we start by introducing the above concepts following the historical reference [3]. Secondely we explain the geometric gains of cluster ensembles. Hereby we emphasize the importance of some recent works, such as Gross-Hacking-Keel-Kontsevich [4, 5] and Argüz-Bousseau [1]. Finally, we give a brief introduction to Fock-Goncharov's duality conjectures. The recurrent example used during the talk is the one of (higher) Teichmüller theory. Indeed, cluster varieties have deep connections with several areas of mathematics, in particular in the study of the moduli space of local systems on topological surfaces [3].

1. Introduction to Cluster ensembles

In earlier talks cluster algebras associated to quivers with frozen vertices have been introduced and the translation to cluster algebras with coefficients was mentioned.

For this talk we consider the general setting. In other words, let $(\mathbb{P}, \oplus, \cdot)$ be some semi-field and $(\mathbf{x}, \mathbf{y}, B)$ a labeled seed with B a skew-symmetrizable $n \times n$ matrix. In particular, $\mathbf{y} \in \mathbb{P}^n$ and the coordinates of $\mathbf{x} = (x_1, \dots, x_n)$ form a free generating set, over $\mathbb{Q}[\mathbb{P}]$, of a given field $\mathcal{F}$ of rational functions in n variables.

In order to associate an algebraic variety to every seed obtained from $(\mathbf{x}, \mathbf{y}, B)$ by mutation a coordinate-free point of view is more natural. More precisely, a seed can be viewed as the data $\vec{i} = (\Lambda, I, F, E, D)$ where

- Λ is a lattice of rank n (i.e. $\Lambda \cong \mathbb{Z}^n$) equipped with a skew-symmetric $\mathbb{Q}$-bilinear form $(\cdot, \cdot)$,
- I an index set with $F \subset I$ the frozen indices,
- $E = \{e_i\}$ is a basis of Λ and $D = (d_i)$ the multipliers. In particular, in the skew-symmetric case $d_i = 1$ for all i.

Forming the matrix $(\epsilon_{i,j} := d_j(e_i, e_j))_{i,j}$ recovers the transpose of the mutation matrix considered in the previous talks. However, with the above notion of a seed, mutation at some $k \in I \setminus F$ is defined as Y-seed mutation. The usual cluster mutation is found by considering the dual lattice $\Lambda^* = \hom(\Lambda, \mathbb{Z})$ with dual basis $\{e_i^*\}$. More precisely one needs $\Lambda^\circ = \operatorname{span}\{f_i := d_i^{-1} e_i\}$.

Now with the seed $\vec{i}$, via Λ and Λ°, one can naturally associate tori:

$$\mathcal{X}_{\vec{i}} = \operatorname{spec} k[\Lambda] = \hom(\Lambda, \mathbb{G}_m)$$

and similarly $\mathcal{A}_{\vec{i}} = \operatorname{spec} k[\Lambda^\circ]$. These tori are called the *seed $\mathcal{X}$-torus*, respectively *seed $\mathcal{A}$-torus*. If $\vec{i}' = \mu_k(\vec{i})$ is another seed obtained by mutating at $k \in I \setminus F$, then there are birational morphisms

$$\mu_k^{\mathcal{X}} : \mathcal{X}_{\vec{i}} \to \mathcal{X}_{\vec{i}'} \text{ and } \mu_k^{\mathcal{A}} : \mathcal{A}_{\vec{i}} \to \mathcal{A}_{\vec{i}'}$$

connecting the associated tori. It is usual to define these morphisms explicitly by pullback formulas at level of characters which mimic Y and $X-$cluster mutation. Using these maps one can glue all the tori in order to obtain a scheme structure on $\bigcup_{\vec{i}} \mathcal{X}_{\vec{i}}$ and also on $\bigcup_{\vec{i}} \mathcal{A}_{\vec{i}}$. For algebraic geometrical (complete) details we refer to [4, Proposition 2.4]. By doing so one obtains the tuple $(\mathcal{X}, \mathcal{A})$ called the *cluster ensemble* and which was introduced by Fock-Goncharov. As was pointed out, $\mathcal{A}$ is an honest variety, i.e. it is separated. However, in general $\mathcal{X}$ is not seperated.

Subsequently we explained that considering global regular functions on $\mathcal{A}$ one recovers the *upper cluster algebra* which by the Laurent phenomenon contains the cluster algebra. At level of the $\mathcal{X}$-variety the global regular functions yield the so-called *Poisson cluster algebra*. However the Laurent phenomenon doesn't hold in this case.

2. The geometric structure and duality phenomenons

The name Poisson cluster algebras refers to the fact that the $\mathcal{X}$-variety has a Poisson structure. More precisely, using the bilinear form $(\cdot, \cdot)$ one writes down an explicit Poisson structure on each torus $\mathcal{X}_{\vec{i}}$, which moreover is invariant under mutation. In particular it induces a Poisson structure on $\mathcal{X}$. On his turn the torus $\mathcal{A}_{\vec{i}}$ carries a mutation well-behaved closed 2-form Ω which induces a symplectic structure on $\mathcal{A}$.

These structures are connected to each other. To be more precise one needs to introduce a crucial map connecting the both varieties. To start one defines the skew-symmetrizable form $[e_i, e_j] = d_j(e_i, e_j)$ and subsequentely considers the map

$$\Lambda \to \Lambda^\circ : v \mapsto \sum_j [v, e_j] f_j.$$

Associated is the morphism of seed tori

$$\mathcal{A}_{\vec{i}} = \operatorname{spec} k[\Lambda^\circ] \to \mathcal{X}_{\vec{i}} = \operatorname{spec} k[\Lambda].$$

These maps behave well with mutaton and hence one obtains a morphism

$$p : \mathcal{A} \to \mathcal{X}$$

called the *assembly map* which, crucially, is monomial and positive. A reassuring fact now is that the symplectic structure on $p(\mathcal{A}_{\vec{i}})$ induced by Ω coincides with the symplectic structure given by the restriction of the Poisson structure on $\mathcal{X}_{\vec{i}}$.

The interplay however doesn't stop there and in the talk a glimpse was given of two deeper connections between the both varieties, both of a duality nature. For the first one needs an alternate description of cluster varieties by Gross-Hacking-Keel [4] using log Calabi-Yau varieties. In brief, they have shown that $\mathcal{X}$ is up to codimension 2 a blow-up of some concrete toric variety. In particular, besides the Fock-Goncharov dual variety $\mathcal{A}$ of $\mathcal{X}$, one can associate the mirror dual of the log Calabi-Yau variety (constructed in the framework of the Gross-Siebert program). It was recently proven by Argüz-Bousseau [1] that the mirror to the $\mathcal{X}$ cluster variety is a degeneration of the $\mathcal{A}$ cluster variety and vice versa.

A second attractive conjectural duality between $\mathcal{X}$ and $\mathcal{A}$ is given by the Fock-Goncharov duality conjectures. During the talk we presented a short intuitive tropical path to the statement. This required to make the birational morphisms $\mu_k^{\mathcal{X}}$ and $\mu_k^{\mathcal{A}}$ explicit in terms of the coordinate functions z^{e_i}. With a slight abuse of notation, they are given by

$$\begin{array}{l} \left(\mu_k^{\mathcal{X}}\right)^* : z^v \mapsto z^v(1 + z^{e_k})^{-(v, e_k)} \\ \left(\mu_k^{\mathcal{A}}\right)^* : z^\gamma \mapsto z^\gamma(1 + z^{(e_k, \cdot)})^{-\gamma(e_k)} \end{array}$$

where $v \in \Lambda$ and $\gamma \in \Lambda^\circ$. Thus, the gluing maps are substraction-free. A wonderful by-product of this is that one can take $\mathbb{P}$-points for any semi-field $\mathbb{P}$. Now choosing for $\mathbb{P}$ the tropical integers $\mathbb{Z}^{tr} = (\mathbb{Z}, +\max)$, a direct computation yields an intriguing phenomena. Namely, denoting $\mathcal{A}^{tr}$ for the $\mathbb{Z}^{tr}$-points of $\mathcal{A}$, the morphism $\mu_k^{\mathcal{A}^{tr}}$ is up to a change of e_k to $-e_k$ and of ϵ_{ij} to $-\epsilon_{ji}$ given by the same formula as $\mu_k^{\mathcal{X}}$ on Λ. In other words, $\mu_k^{\mathcal{X}}$ is the tropicalization of the Laglands dual $\mu_k^{\mathcal{A}^\vee(\mathbb{Z}^{tr})}$. The Fock-Goncharov duality conjecture states that the duality is far more reaching. For example the basis conjecture predicts that $\Gamma(\mathcal{X}^\vee, \mathcal{O}_{\mathcal{X}^\vee})$ has a basis indexed naturally by $\mathcal{A}^{tr}$ and vice-versa.

To finish the talk, we mentioned that in [4] the authors showed that the original Fock-Goncharov conjecture do not hold without certain positivity assumptions. Nevertheless, they suggest that some formal version of the conjecture should hold.

In their seminal work Gross-Hacking-Keel-Kontsevich [5] proved the formal Fock-Goncharov conjecture, as well as the original Fock-Goncharov conjecture with the necessary positivity assumptions.

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