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Mini-Workshop: Flavors of Rabinowitz Floer and Tate Homology

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Abstract. Rabinowitz Floer homology originated 15 years ago in symplectic geometry. Recent developments have related it to algebraic topology via string toplogy and Tate homology, and to mirror symmetry via Fukaya categories. This mini-workshop brought together researchers from these different communities, in order to foster exchange and collaborations across research fields.

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Introduction by the Organizers

The mini-workshop *Flavors of Rabinowitz Floer and Tate homology* was organized by Kai Cieliebak, Alexandru Oancea and Nathalie Wahl. Its goal was to bring together specialists from symplectic geometry, topology, and algebra, in order to discuss recent algebraic structures that emerged in parallel in these different fields.

The workshop was structured as follows. The mornings were generally dedicated to individual talks by the participants, with an intended duration time of 30 min. and 30 min. discussion time for each talk. The afternoons were generally dedicated to discussions on topics that arose during the morning talks. This ensured an intense atmosphere of exchange during the whole duration of the workshop. Each of the participants in the mini-workshop gave a talk, and we also had a special guest talk by Peter Kropholler, who was participating in a parallel mini-workshop during the same week. The 16 participants in the workshop covered a large age spectrum and included 3 Ph.D. students, and also a large geographic area with participants from 8 countries. We had also a reasonable gender balance with 5 women and 11 men.

The first day started with an Introductory opening talk by Kai Cieliebak. He outlined the various symplectic, algebraic, and topological constructions that we were planning to discuss during the week, as well as the interactions between them, both those that are already understood and those that are not yet understood and that formed part of the discussion material for the workshop. The abstract of his talk can serve as a concise guide to the topics that were subsequently discussed during the week. Then followed two talks by Ph.D. students. Shuaipeng Liu explained in his talk *Introduction to symplectic homology* the fundamentals of Floer homology, with the specific goal of building common mathematical ground for the participants in the workshop. Zhen Gao explained in his talk *Calabi–Yau algebras* the fundamentals of Calabi–Yau structures. The talk was geared towards the participants that were not specialists in algebra, and it had a similar goal of building common mathematical ground. The afternoon of the first day was dedicated to discussing various flavors of Calabi–Yau algebras, arising both in the context of Hochschild homology and in the context of string topology. In the afternoon, Inbar Klang gave a talk on *String topology category as a Calabi–Yau category*, which put the previous notions in a topological context.

During the second day we steered towards various version of the Tate construction. Alexandru Oancea's first talk in the morning on *Cone perspective on Rabinowitz–Floer homology* explained a Tate-type construction in Floer theory. More specifically, he focused on algebraic structures arising from mixing together at chain level a product and a coproduct. Alice Hedenlund followed with a talk on the *Tate construction*, both in the classical setting of finite groups, and in the much more general setting of spectra. In the afternoon, Peter Kropholler gave a guest talk on *Tate homology without complete resolutions*, and Alex Takeda explained during the first discussion session various other flavors of Tate constructions arising in the context of Hochschild homology. The second part of the afternoon featured an intense discussion on the precise interpretation of Rabinowitz Floer homology as a Tate construction. Also during the second day we had an evening talk by Mohammed Abouzaid on *Symplectic cohomology with supports and framed* E_2*-algebra structures*, in which he explained how to build suitable models for symplectic cochains, strictly compatible both with the structure of framed E_2-algebra, and with Viterbo restriction maps.

The third day of the workshop started with a talk by Noémie Legout on *Rabinowitz Floer homology/category from the perspective of Symplectic Field Theory.* Her construction uses pseudo-holomorphic curves in symplectizations and complements geometrically the constructions inspired by the wrapped Fukaya category, as the one presented by Hanwool Bae later in the day. The second talk that morning was given by Urs Frauenfelder, on *Spectral jumps in Rabinowitz–Floer–Tate homology.* He emphasized certain spectral jump phenomena akin to ones encountered in quantum mechanics, which become visible once the classical Rabinowitz–Floer homology is enhanced by an additional construction of Tate flavor with respect to the natural circle action. The third talk in the morning was given by Hanwool Bae, on *Calabi–Yau structures on Rabinowitz Fukaya categories*. His talk complemented the one by Noémie Legout by presenting a construction of that category that relies heavily on methods from symplectic homology and wrapped Floer homology.

On Wednesday afternoon the whole group went on the traditional hike to Sankt Roman. It was a beautiful sunny afternoon.

The fourth day started with the talk by Amanda Hirschi, in which she presented the recent *Counterexamples to Donaldson's 4-6 question* that she discovered in joint work with Luya Wang. This is a classical question in 4-manifold topology, asking whether the diffeomorphism type of symplectic 4-manifolds is detected by the symplectic deformation class after stabilization with S^2. The second talk in the afternoon by Ph.D. student Colin Fourel was on *Sheaf and singular models for ∞-groupoid cohomology*. He explained how to prove the equivalence between singular cohomology of a space X and sheaf cohomology of the constant sheaf, based on the analogy between X and BG, with $G = \Omega X$. This point of view was directly relevant to the theme of the workshop, during which the based loop space played a prominent role. The third talk was given by Andrea Bianchi, on *String topology and graph cobordisms*, in which he explained how to generalize the fundamental operations from string topology to spaces of maps in a functorial way. The afternoon was dedicated to phrasing some key questions and discussing possible answers and future directions of research. We discussed the construction of operations in Floer theory, the interpretation of the Rabinowitz Floer chain complex as a classifying space for Tate homology, explicit computations for free loop spaces of spheres, as well as S^1- and $O(2)$-equivariant aspects. Koszul duality between $C_*(\Omega M)$ and $C^*(M)$ for a simply connected manifold M was discussed in several instances.

The last half-day of the workshop featured three talks. Alex Takeda explained the *Categorical formal punctured neighborhood at ∞*, a categorical construction that gives an open-string description of Rabinowitz Floer homology. Nathalie Wahl gave a talk on *Spaces of operations*, in which she explained Sullivan diagrams as a model for moduli spaces of Riemann surfaces, and how these give rise to algebraic operations on Hochschild complexes. The last talk of the workshop was by Georgios Dimitroglou Rizzell on *Relative Calabi–Yau structure from acyclic Rabinowitz–Floer complexes of Legendrians*. This was a beautiful conclusion to the workshop, putting to work all the algebraic structures that had been seen over the week in a geometric context.

At the end of the week all the participants were exhausted, but happy. One Ph.D. student said: "I learnt more mathematics this week than during a whole semester!" We would like to interpret that as a sign of success for the workshop.

Mini-Workshop: Flavors of Rabinowitz Floer and Tate Homology

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Abstracts

Three flavors of Rabinowitz Floer and Tate homology

Kai Cieliebak

Rabinowitz Floer homology appears in three flavours: symplectic, topological, and algebraic. The goal of this talk is to describe these aspects and discuss their relationships, with an emphasis on open problems. The talk has three parts.

Part 1 is devoted to the following isomorphisms for a closed oriented n-dimensional manifold M:

$$SH_*(D^*M) \cong H_*(\Lambda M) \cong HH_*\big(C_*(\Omega M)\big) \overset{\pi_1 M=0}{\cong} HH^*\big(C^*(M)\big).$$

Here the first term is the symplectic homology of the unit disk cotangent bundle D^*M. This is a purely symplectic invariant which is defined more generally for any Liouville domain V. The second term is of topological nature and denotes the singular homology of the free loop space ΛM. The third and fourth terms are of algebraic nature. Here $HH_*(A)$ and $HH^*(A)$ denote the Hochschild homology and cohomology of a differential graded algebra A (or, more generally, an A_∞-algebra or an A_∞-category). These are applied, respectively, to the singular chains on the based loop space ΩM and the singular cochains on M. All (co)homology groups are taken with $\mathbb{R}$-coefficients, and for the last isomorphism we assume that M is simply connected.

All four (co)homology groups are BV algebras in a natural way, and the isomorphisms are expected to respect this structure. In the first two groups the BV structure arises from the pair-of-pants product resp. the Chas-Sullivan loop product and the circle action. In the last two groups it arises from the cup product on Hochschild cohomology and the Connes operator, using the fact that $C_*(\Omega M)$ is smooth Calabi–Yau and (a suitable Poincaré model of) $C^*(M)$ is proper Calabi–Yau. The first isomorphism is known as the Viterbo isomorphism, and the last isomorphism arises from the fact that $C^*(M)$ is the Koszul dual of $C_*(\Omega M)$. The isomorphism $SH_*(D^*M) \cong HH_*\big(C_*(\Omega M)\big)$ is a special case of the isomorphism

$$SH_*(V) \cong HH_*\big(\mathcal{W}\mathrm{Fuk}(V)\big)$$

for any Weinstein domain V, where $\mathcal{W}\mathrm{Fuk}(V)$ denotes its wrapped Fukaya category.

Part 2 concerns the extension of the previous structures by coproducts. On the symplectic side, this requires the passage to reduced symplectic homology $\overline{SH}_*(V) = \mathrm{coker}(c_*)$ with respect to the canonical chain map

$$c : SC^{-*}(V) \to SC_*(V)$$

from symplectic cochains to chains. For a suitable class of Weinstein domains (including unit disk cotangent bundles), $\overline{SH}(V)$ carries a secondary pair-of-pants coproduct defining together with the pair-of-pants product the structure of a unital infinitesimal antisymmetric bialgebra. On the topological side, such a structure exists on reduced loop homology $\overline{H}_*(\Lambda M) = H_*(\Lambda M)/\chi(M)[\mathrm{pt}]$, the quotient of loop homology by the Euler characteristic times the point class. Here the coproduct is an extension of the Sullivan-Goresky-Hingston coproduct and the Viterbo isomorphism descends to an isomorphism of bialgebras

$$\overline{SH}_*(D^*M) \cong \overline{H}_*(\Lambda M).$$

The relation of this structure to similar structures on the algebraic side appears not yet to be fully understood.

Part 3 concerns the partly conjectural isomorphisms

$$RFH_*(S^*M) \cong \widehat{H}_*(\Lambda M) \cong \widehat{HH}_*\big(C_*(\Omega M)\big) \overset{\pi_1 M=0}{\cong} \widehat{HH}^*\big(C^*(M)\big).$$

Here the first term is the Rabinowitz Floer homology of the unit sphere cotangent bundle S^*M. This is a purely symplectic invariant which is defined more generally for the boundary ∂V of any Liouville domain V. One of its definitions is as the homology of the cone of the map $c : SC^{-*}(V) \to SC_*(V)$ from above. Similarly, Rabinowitz loop homology $\widehat{H}_*(\Lambda M)$ can be defined as the homology of the cone of the map $c : C^{-*}(\Lambda M) \to C_*(\Lambda M)$ multiplying the point class by $\chi(M)$. On the algebraic side, $\widehat{HH}_*(A)$ and $\widehat{HH}^*(A)$ denote the Tate Hochschild homology and cohomology of a differential graded Frobenius algebra A, applied to suitable models for $C_*(\Omega M)$ and $C^*(M)$ (where to our knowledge only the latter has been defined so far).

All four (co)homology groups are expected to be graded Frobenius algebras and the isomorphisms are expected to respect this structure. However, this appears to be proved only for the first two groups and their isomorphism. The isomorphism $RFH_*(S^*M) \cong \widehat{HH}_*\big(C_*(\Omega M)\big)$ is expected to generalize to isomorphisms

$$RFH_*(\partial V) \cong \widehat{HH}_*\big(\mathcal{W}\mathrm{Fuk}(V)\big) \cong HH_*\big(\mathcal{RW}\mathrm{Fuk}(V)\big)$$

for certain Weinstein domains V, where $\mathcal{RW}\mathrm{Fuk}(V)$ denotes the Rabinowitz (wrapped) Fukaya category.

Further open questions concern the description of the above groups in terms of symplectic field theory, their S^1-equivariant versions, the underlying chain-level structures, their relation to Varolgunes' version of symplectic homology, and their role in semiclassical quantization.

Symplectic Homology and Rabinowitz Floer Homology Revisited

Shuaipeng Liu

Symplectic homology is defined in a Liouville domain $(W, d\lambda, \partial W = M, \lambda)$ with symplectic completion $\widehat{W} = W \sqcup_M ([1,\infty) \times M, d(r\lambda))$ by attaching the positive symplectization along the boundary M, where the Liouville vector field defined by $\iota_X d\lambda = \lambda$ transversally points outwards along M.

As a filtered version of Floer homology, the filtered chain group $CF_*^{<a}(H)$ is the $\mathbb{Q}-$vector space generated by the critical points of the Hamiltonian action functional of a loop $x : S^1 \to \widehat{W}$,

$$\mathcal{A}_H(x) := \int_{S^1} x^*\lambda - \int_0^1 H(t, x(t))dt,$$

and graded by the Conley-Zehnder index $CZ(x)$, where the filtration is given by the bounded action $\mathcal{A}_H(x) < a$. The differential operators $\partial_k : CF_k(H) \to CF_{k-1}(H)$ are defined by algebraically counting the number of the unparametrized moduli space of Floer trajectories,

$$\partial_k x := \sum_{CZ(y)=k-1} \#\mathcal{M}(y, x)y,\ x \in CF_k(H)$$

decreasing the action. By setting an action window $-\infty \leq a < b \leq \infty$, they restrict to differential operators $\partial_*^{(a,b)}$ on $CF_*^{(a,b)} := CF_*^{<b}/CF_*^{<a}$. Then the filtered Floer homology group is

$$FH_*^{(a,b)}(H) := H_*(CF^{(a,b)}, \partial^{(a,b)}).$$

The symplectic homology is defined as the direct limit of filtered Floer homologies with respect to Hamiltonians of an admissible class, which will be non-negative on W and grow linearly for r large enough in the completion $\widehat{W}$ with only nondegenerate 1-periodic orbits. By standard argument about the continuation map, i.e. a monotone increasing homotopy $\widehat{H} : H_- \to H_+$, one can define the symplectic homology by taking direct limit via the monotone homotopy,

$$SH_k^{(a,b)} := \varinjlim FH_k^{(a,b)}(H).$$

Likewise, one can dually define the symplectic cohomology by taking the inverse limit.

Symplectic homology and symplectic cohomology cannot be related directly by an isomorphism. The main result to describe the relationship is by a new version of Floer homology constructed via a new class of admissible Hamiltonians required addionally to be non-negative in some tubular neighborhood of the boundary M and positive elsewhere in the Liouville domain W, thus they vividly look like with the shape $\bigvee$. The new version is called the Rabinowitz Floer Homology, denoted by $\check{SH}$. And the main result is as follows

Theorem [1] There exists a long exact sequence

$$\cdots \longrightarrow SH^{-*}(W) \longrightarrow SH_*(W) \longrightarrow \check{SH}_*(W) \longrightarrow SH^{-*+1}(W) \longrightarrow \cdots$$

In the talk, I will briefly explain the notions mentioned in the definition of symplectic homology as a revisit to Floer-like theory.

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Flavors of Calabi–Yau structures

Zhen Gao

The notion of Calabi–Yau structure in the algebraic context was initially noticed by Maxim Kontsevich and formally introduced by Victor Ginzburg [1], then further studied and developed by many others, e.g. Michel Van den Bergh, Bertrand Toën, and Bernhard Keller. Calabi–Yau structures have played a prominent role in algebraic geometry, noncommutative geometry, and representation theory. Recently, Calabi–Yau structures are emerging in string topology and symplectic topology on the relevant homological algebraic invariants.

Let $\mathcal{A}$ be an DG/A_∞-category over a field $\mathbb{K}$. Denote by $\mathcal{A}_\Delta$ the diagonal $\mathcal{A}$-bimodule, and $\mathcal{A}^! := \mathbf{R}\mathrm{Hom}_{\mathcal{A}^e}(\mathcal{A}, \mathcal{A}^e)$ the inverse dualizing bimodule of $\mathcal{A}$ regarded as diagonal bimodule. There are following absolute Calabi–Yau structures:

Smooth n-CY: Suppose $\mathcal{A}$ is (homologically/locally) smooth, i.e. $\mathcal{A}$ is a perfect $\mathcal{A}$-bimodule, a *weak smooth Calabi–Yau structure of dimension n* on $\mathcal{A}$ is a Hochschild class $[\xi_{\mathcal{A}}] \in HH_n(\mathcal{A})$ such that the induced map $[\hat{\xi}_{\mathcal{A}} \circ \Sigma^{-n}] : A^![n] \to A$ is an isomorphism in the derived category $\mathcal{D}(\mathcal{A}^e)$. A *strong smooth Calabi–Yau structure of dimension n* on $\mathcal{A}$ is a negative cyclic class $[\tilde{\xi}_{\mathcal{A}}] \in HC_n^-(\mathcal{A})$ whose underlying Hochschild class $[\xi_{\mathcal{A}}] := h([\tilde{\xi}_{\mathcal{A}}]) \in HH_n(\mathcal{A})$ is a weak smooth n-Calabi–Yau structure.

Proper n-CY: Suppose $\mathcal{A}$ is (locally) proper, i.e. $\mathcal{A}_\Delta$ is a proper $\mathcal{A}$-bimodule, a *weak proper Calabi–Yau structure of dimension n* on $\mathcal{A}$ is a degree n chain map $\mathrm{tr} : C_\bullet(\mathcal{A}, \mathcal{A}) \to \mathbb{K}[n]$ inducing an isomorphism $\mathcal{A} \xrightarrow{\sim} (\mathcal{A}^{\mathrm{op}})^*[-n]$ in derived category $\mathcal{D}(\mathcal{A}^e)$. A *strong proper Calabi–Yau structure of dimension n* on $\mathcal{A}$ is a factorization of weak proper n-Calabi–Yau structure through the projection to the cyclic chain complex $\widetilde{\mathrm{tr}} : CC_\bullet(\mathcal{A}) \to \mathbb{K}[-n]$.

One of the significant consequences of the presence of smooth Calabi–Yau structure is the Poincaré duality between Hochschild cohomology and homology first observed by Michel Van den Bergh [2], see also [3], and naturally from which a Batalin–Vilkovisky algebra structure on Hochschild cohomology follows, e.g. [4].

Examples.

- In string topology, let X be a topological space, the mostly concerned DG/A_∞ algebras are chains of based loop space $C_\bullet(\Omega X; \mathbb{K})$ and singular cochains $C^\bullet(X; \mathbb{K})$. When X is a n-dimensional Poincaré duality space over characteristic 0 field $\mathbb{K}$, then $C_\bullet(\Omega X; \mathbb{K})$ resp. $C^\bullet(X; \mathbb{K})$ is strong smooth resp. proper n-CY.

- In symplectic topology, let X be a Liouville manifold, e.g. cotangent bundle T^*Q of a closed oriented smooth manifold Q, the relevant algebraic invariant is some A_∞-category called Fuakaya category, e.g. wrapped Fukaya category $\mathcal{W}(X)$ and its proper full subcategory $\mathcal{F}(X)$, introduced and established by Mohammend Abouzaid and Paul Seidel. When X is non-degenerate Liouville manifold, there is geometric strong smooth Calabi–Yau structure on $\mathcal{W}(X)$ resp. strong proper Calabi–Yau structure on $\mathcal{F}(X)$. C.f. [5],[6].

Relative Calabi–Yau structures are introduced by Christopher Brav and Tobias Dyckerhoff in [7] for a DG functor between DG categories $F : \mathcal{A} \to \mathcal{B}$. Definitions are similar to absolute cases hence generalizing the notions to relative sense, despite in addition the relevant Hochschild classes in $HH_\bullet(\mathcal{A})$ should also induces isomorphisms in derived category $\mathcal{D}(\mathcal{A}^e)$ between some distinguished triangles for the homotopy cofiber and fiber of certain induced maps $\gamma_F^!$ and γ_F from the functor F. In very recent work, Christopher Brav and Nick Rozenblyum have shown that in the compactly generated DG categories setting, there is framed E_2-algebra structure on the chain-level of Hochschild cohomology given a relative Calabi–Yau structure.

Examples.

- In string topology, key examples are $C_\bullet(\Omega\partial Q) \hookrightarrow C_\bullet(\Omega Q)$ and $C_\bullet(\partial Q) \hookrightarrow C_\bullet(Q)$ where Q is taken to be compact oriented smooth manifold with boundary ∂Q.
- In symplectic topology, Gergeois Dimitroglou Rizell and Noémie Legout reveal that Chekanov-Eliashberg DG algebra with coefficient as based loop DG algebra for some Legendrian submanifold carries relative smooth Calabi–Yau structure.

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The string topology category as a Calabi–Yau category

Inbar Klang

The string topology category $\mathcal{S}_M$ of a connected, closed, oriented manifold M was defined by Blumberg–Cohen–Teleman in [1]. The objects of this category are some collection of connected, closed, oriented submanifolds $N \subseteq M$ that all contain some chosen point $q \in M$. The morphisms between N_1 and N_2 consist of chains on the space of paths P_{N_1,N_2} that start in N_1 and end in N_2, shifted by the dimension of N_1.

The composition in this category is given by intersecting and concatenating, similar to the Chas–Sullivan product on homology of a free loop space of a manifold. To define this more precisely, one uses Poincaré duality on N_1 to rewrite the morphism complexes as a derived hom,

$$Rhom_{C_*(\Omega M)}(C_*(P_{q,N_1}), C_*(P_{q,N_2}))$$

Here $C_*(\Omega M)$ denotes the based loop space, $Map_*(S^1, M)$. This embeds the string topology category as a full subcategory of $Perf_{C_*(\Omega M)}$, the category of perfect modules over $C_*(\Omega M)$. In fact, if $\{q\} \in M$, the string topology category includes the generator of $Perf_{C_*(\Omega M)}$, and is Morita equivalent to it.

The category $Perf_{C_*(\Omega M)}$, and in the above case also the string topology category, are smooth Calabi–Yau categories. This comes from the fact that $C_*(\Omega M)$ is a smooth Calabi–Yau algebra. Roughly, A is a smooth Calabi–Yau algebra over k if it is smooth (a perfect $A \otimes A^{op}$ module) and has an S^1-invariant "fundamental class" in the Hochschild chains of A, evaluation on which gives an equivalence between the Hochschild chains and cochains of A (with an appropriate shift.)

In the case $A = C_*(\Omega M)$, this fundamental class comes from the fundamental class of M in $C_*(M)$, which can then be mapped to the Hochschild chains of $C_*(\Omega M)$, which agree with $C_*(Map(S^1, M))$. Since Hochschild cochains always have a shuffle product, this gives a product on (a shift of) $C_*(Map(S^1, M))$, which agrees with the Chas–Sullivan product.

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Cone perspective on Rabinowitz Floer homology

Alexandru Oancea
(joint work with Kai Cieliebak, Nancy Hingston)

In joint work with Kai Cieliebak and Nancy Hingston [1] we studied Rabinowitz Floer homology and cohomology $RFH_*(V)$ of a Liouville domain V. One of our key results is that they both carry the structure of graded Frobenius algebras and that they are related by a Poincaré duality isomorphism.

In the particular case when the Liouville domain is the unit disc cotangent bundle D^*Q of a closed smooth manifold Q, its symplectic topology is related to the topology of the free loop space $\Lambda = \mathrm{Map}(S^1, Q)$. Let $\Lambda_0 \subset \Lambda$ be the subspace of constant loops. We define the *Rabinowitz loop homology of Q* to be

$$\widehat{H}_*\Lambda = RFH_*(D^*Q).$$

Let $H_*\Lambda$ denote the homology of the free loop space of Q, and $H^*\Lambda$ its cohomology. Consider the map $\varepsilon : H^{-*}\Lambda \to H_*\Lambda$ that is everywhere zero, except in degree 0 in the component of contractible loops, where it is multiplication by the Euler characteristic $\chi(Q)$. The *reduced loop homology and cohomology groups*

$$\overline{H}_*\Lambda := \operatorname{coker}\varepsilon, \qquad \overline{H}^*\Lambda := \ker\varepsilon$$

therefore differ from $H_*\Lambda$ and $H^*\Lambda$ only by $\chi(M)$ times the point class.

Theorem [1, 2, 3]. *(i) The Chas-Sullivan product on $H_*\Lambda$ descends to $\overline{H}_*\Lambda$. The Goresky-Hingston product on $H^*(\Lambda, \Lambda_0)$ extends (canonically if $H_1Q = 0$) to $\overline{H}^*\Lambda$.*

(ii) We have a short exact sequence in which ι is a ring map

$$0 \to \overline{H}_*\Lambda \xrightarrow{\iota} \widehat{H}_*\Lambda \xrightarrow{\pi} \overline{H}^{1-*}\Lambda \to 0, \tag{1}$$

which splits (canonically if $H_1Q = 0$) via a ring map $\overline{H}^{1-*}\Lambda \xrightarrow{\bar{i}} \widehat{H}_*\Lambda$. The product on $\widehat{H}_*\Lambda$ restricts to the Chas-Sullivan product on $\overline{H}_*\Lambda$, and to the extended Goresky-Hingston product on $\overline{H}^{1-*}\Lambda$.

To prove this theorem, we developed in joint work with K. Cieliebak [2] a theory of multiplicative structures on cones. Indeed, the previous theorem can be proved by describing Rabinowitz loop homology at chain level as the "cone of ε".

The general setup for multiplicative structures on cones is that of a chain complex $(\mathcal{A}, \partial)$ and a chain map $c : \mathcal{A}^\vee \to \mathcal{A}$. We prove in [2] that a multiplicative structure on $\mathrm{Cone}(c) = \mathcal{A} \oplus \mathcal{A}^\vee[-1]$ can be obtained from the data of an A_2^+-*structure* on $\mathcal{A}$, a notion that we define. This consists of the chain map c, a homotopy between $c^\vee$ and c, a degree 0 product on $\mathcal{A}$ and a degree 1 coproduct on $\mathcal{A}$, satisfying certain relations. The product on the cone is obtained by dualizing the product and coproduct in all possible ways at their inputs and outputs. The axioms of an A_2^+-structure ensure that the resulting operation is a chain map. It is an open problem to develop the theory of A_3^+-structures (associativity), and indeed A_∞^+-structures (associativity up to homotopy).

In the talk I have explained the notion of an A_2^+-structure, how it determines a product structure one the cone, and how that articulates with Rabinowitz Floer homology. I have also argued that this construction can be interpreted as a chain level counterpart of a classical construction called *the Drinfeld double* [4].

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Tate Cohomology of Finite Groups and the Tate Construction

Alice Hedenlund

1. Tate Cohomology of Finite Groups

Let k be some commutative ring. Classically, *Tate cohomology* of the finite group G with coefficients in the G-module M is defined as

$$\hat{H}^*(G;M) = \widehat{\mathrm{Ext}}^*_{kG}(k, M)$$

where $\widehat{\mathrm{Ext}}$ denotes the *complete Ext* of Mislin [3]. In practice, the Tate cohomology groups are computed via the *complete resolution*

$$\hat{P}_* := (\cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\quad\quad\quad} P_0^\vee \longrightarrow P_1^\vee \longrightarrow \cdots), \qquad P_0 \xrightarrow{\epsilon} k \cong k^\vee \xrightarrow{\epsilon^\vee} P_0^\vee$$

where $\epsilon : P_* \to k$ is a projective resolution of k as a G-module and $P_i^\vee$ denotes the k-linear dual of P_i as in [1]. We note that this splices together group homology and cohomology together via the *norm map*

$$\mathrm{Nm}_G : M_G \longrightarrow M^G\,, \quad m \mapsto \sum_{g\in G} gm\,.$$

2. The Tate Construction

In this section, G will be a topological group. We denote by BG a fixed classifying space. Letting $\mathcal{D}k$ denote the derived ∞-category over k, we consider objects in the category $\mathrm{Fun}(BG, \mathcal{D}k)$ which we call *chain complexes with G-action.* If G is a finite group, then a G-module M can be viewed as an object in this category, and we have that

$$M_{hG} = \operatorname*{colim}_{BG} M \simeq k \otimes^L_{kG} M \quad \text{and} \quad M^{hG} = \lim_{BG} M \simeq R\mathrm{Hom}_{kG}(k, M)\,,$$

whose homology groups recover group homology and cohomology, respectively.

The analogue of the Tate construction is obtained by considering a generalization of the norm map. If X is equipped with a G-action, we can consider it as equipped with a $G\times G$-action by adding a trivial right action. Consider the chains $C_*(G;k)$, which is a chain complex with $G \times G$-action by the natural action of G on itself from the right and the left. Under the appropriate identifications the *norm map* is simply the colimit-limit exchange map

$$\begin{array}{ccc} \mathrm{colim}_{BG}\lim_{BG}(X\otimes C_*(G;k)) & \xrightarrow{\kappa} & \lim_{BG}\mathrm{colim}_{BG}(X\otimes C_*(G;k)) \\ \downarrow\simeq & & \downarrow\simeq \\ (X\otimes D_{BG})_{hG} & \xrightarrow{\mathrm{Nm}_{BG}} & X^{hG} \end{array}$$

where D_{BG} is the dualizing spectrum of G by Klein [2]. The Tate construction on X is defined as the cofibre

$$X^{tG} = \mathrm{cofib}(\mathrm{Nm}_{BG} : (X \otimes D_{BG})_{hG} \to X^{hG})\,.$$

If G is a finite group and M is a G-module, then the homology groups of the Tate construction in the above sense recover the Tate cohomology groups of G with coefficients in M, as in the previous section.

Let us finally outline how the Tate construction is related to Poincaré duality. If $G = \Omega Q$ where Q is an closed n-dimensional manifold, then D_Q can be identified with the Spivak normal bundle of Q. The norm map on homology groups is then

$$H_{*+n}(M;\omega_M) \longrightarrow H^{-*}(M;k)\,,$$

where ω_M is the orientation bundle associated to M. This is the same map that appears in the statement of twisted Poincaré duality. In this case, the norm map is an equivalence, so that the Tate construction vanishes.

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Tate Cohomology — with or without Complete Resolutions

Peter H. Kropholler

Fix an associative ring with one [8] and denoted R. Let ${}_R\mathbf{Mod}$ be the category of left R-modules. A *cohomological functor* H^* with domain ${}_R\mathbf{Mod}$ consists of a family $(H^n)_{n\in\mathbb{Z}}$ of functors $H^n\colon {}_R\mathbf{Mod} \to \mathbf{Ab}\,(= {}_{\mathbb{Z}}\mathbf{Mod})$ such that there are natural connecting homomorphisms $\delta : H^n(N') \to H^{n+1}(N'')$ associated to any short exact sequence $0 \to N'' \xrightarrow{\iota} N \xrightarrow{\pi} N' \to 0$ yielding a long exact sequence

$$\cdots \to H^{n-1}(N') \xrightarrow{\delta} H^n(N'') \xrightarrow{\iota_*} H^n(N) \xrightarrow{\pi_*} H^n(N') \xrightarrow{\delta} H^{n+1}(N'') \to \cdots\,.$$

Chain complexes of projective modules provide a source of cohomological functors. Let P_* be a chain complex of projective modules. That the assignment $N \mapsto H^n(\hom_R(P^*, N))$ defines a cohomological functor rests on two key properties P^*: firstly it is a chain complex so that $\hom_R(P^*, N)$ is a cochain complex; secondly each P_n is projective ensuring the existence of long exact sequences. There is no requirement that the chain complex P^* be exact (i.e. having homology everywhere zero) or acyclic (i.e. having the homology of a point). Any chain complex of projective modules will serve as a foundation for a cohomological functor.

We define a *Tate cohomological functor* to be a cohomological functor which vanishes on projective modules in all dimensions. Mislin [7] shows that to any cohomological functor H^* there is a Tate cohomological functor $\widehat{H}^*$ together with a map $H^* \to \widehat{H}^*$ so that the following universal property holds: for any map ν from H^* to a Tate cohomological functor K^* there is a unique map $\widehat{H}^* \to K^*$ so that ν factorises as the composite $H^* \to \widehat{H}^* \to K^*$.

Projective modules. The importance of projective modules here is paramount. Recall the classical definition that a module P is *projective* if every map from P to the codomain of an epimorphism factors through the domain. Since ${}_R\mathbf{Mod}$ is an abelian category we can reformulate this definition: the modern definition might read: a module P is *projective* when the functor $\hom(P, \ ) : {}_R\mathbf{Mod} \to \mathbf{Ab}$ commutes with finite colimits. This is equivalent to the classical definition: in effect the classical definition tells us that $\hom(P, \ )$ commutes with coequalisers but it automotically commutes with all limits and since finite coproducts in an abelian category are naturally identified with finite products we deduce that a classically project P yields a functor $\hom(P, \ )$ that commutes with finite coproducts and with coequalisers and therefore with all finite colimits. This philosophy holds for many abelian categories including categories of sheaves over a space or site. The category ${}_R\mathbf{Mod}$ admits a forgetful functor to set which has a left adjoint: the free module on a set. A map of modules is an epimorphism if it surjective on the underlying sets (to put this in modern language we may say that the forgetful functor reflects epimorphisms) and so free modules are projective. This leads to the characterisation that a module is projective if and only if it is a direct summand of some free module.

Mislin's approach to Tate Cohomology via Satellites. For a fixed R-module N choose any projective resolution $P_* \to N \to 0$. Let $\Omega^0 N$ denote N and let $P_{-1} = 0$. For $n \geq 1$, let $\Omega^n N$ denote the kernel of the map $P_{n-1} \to P_{n-2}$. Then we have short exact sequences $\Omega^n N \to P_{n-1} \to \Omega^{n-1} N$ for $n \geq 1$. We have a sequence of connecting homomorphisms

$$H^n(N) \xrightarrow{\delta} H^{n+1}(\Omega N) \xrightarrow{\delta} H^{n+2}(\Omega^2 N) \xrightarrow{\delta} H^{n+3}(\Omega^3 N) \xrightarrow{\delta} H^{n+4}(\Omega^4 N) \xrightarrow{\delta} \cdots .$$

The colimit of this sequence is the nth Tate cohomology group $\widehat{H}^n(N)$.

There are two other accounts by Goichot–Vogel and by Benson–Carlson published at around the same time and based on chain maps and chain homotopies. See the work of Cornick (some joint with the author) [3, 4, 5, 2] for further information.

Benson–Carlson [1]**, Goichot–Vogel** [6]**.** Tate cohomology can be defined using almost chain maps modulo almost chain homotopies. Let $C_\bullet$ and $C'_\bullet$ be chain complexes. An *almost chain map* $\phi : C_\bullet \to C'_\bullet$ *of degree* j is a family of maps $(\phi_* : C_* \to C'_{*+j})$ such that for all sufficiently large n the square starting at C_n commutes. An *almost chain homotopy* from an almost chain map ϕ of degree j to an almost chain map ψ of degree j is a map s of degree $j+1$ such that $ds+sd = \psi-\phi$ at the square starting at C_n for sufficiently large n. Using this approach we define the Tate Ext groups $\widehat{\mathrm{Ext}}^j_R(M,N)$ where M and N are two R-modules by first choosing two projective resolutions $P^* \to M \to 0$ and $Q^* \to N \to 0$ and then defining two projective chain complexes $P_\bullet$ and $Q_\bullet$ by removing M and N and defining P_n and Q_n to be zero when $n < 0$. Using these projective chain complexes we can define a cohomology theory by considering almost chain maps modulo almost chain homotopies from $P_\bullet$ to $Q_\bullet$. This is essentially the treatment advocated by Goichot [6] and attributed to Vogel and it is described in these terms by Benson–Carlson [1]. Crucially this definition produces a cohomological functor isomorphic to Mislin's construction when applied to $\mathrm{Ext}^*_R(M, \quad)$. As an elegant consequence we have the

Lemma. *For any R-module M, $\widehat{\mathrm{Ext}}^0_R(M,M) = 0$ if and only if M has finite projective dimension over R.*

Tate and Farrell Cohomology. Historically, Tate cohomology was introduced first for finite groups having its origins in algebraic number theory. It concerns a finite Galois group G and for any G-module N there are isomorphisms

$$\widehat{H}^n(G,N) \simeq \begin{cases} H^n(G,N) & n \geq 1 \\ H_{-n-1}(G,N) & n \leq -1 \end{cases}$$

showing that the Tate cohomology conveniently records the ordinary cohomology in positive degrees and the ordinary homology in degrees ≤ -2. In dimension 0 there is the norm map $H_0(G,N) \to H^0(G,N)$ and the Tate cohomology groups $\widehat{H}^{-1}(G,N)$ and $\widehat{H}^0(G,N)$ are the kernel and cokernel of this map. Farrell, interested in generalising Tate cohomology to a wider class of groups, used the idea of virtual cohomological dimension. This conveniently applies to arithmetic groups such as $\mathrm{GL}_n(\mathbb{Z})$ that have torsion free subgroups of finite index. His theory of Tate cohomology produces a theory which coincides with ordinary cohomology in dimensions greater that the cohomological dimension.

Complete projective resolutions. It turns out the for Tate cohomology of finite groups one can take a projective resolution of the trivial module and then extend to the right to make a complete resolution that computes the Tate cohomology in all degrees. The same conclusion holds for Farrell's generalization but one has to perform the surgery a little way along the resolution beyond the virtual cohomological dimension. A study of when there is a complete resolution can be found in [4]. There is a connection between the existence of complete resolutions and the presence of certain finiteness conditions of which finite virtual cohomological dimension is a special case. But as remarked at the outset, one really just needs a projective chain complex P_* and to have one that determines the Tate cohomology does not in general require exactness. At first sight, all that is really needed in order to define a Tate cohomology is a projective chain complex P_* such that for any projective module Q, $\hom_R(P_*, Q)$ is exact. This holds in situations that go far beyond the Tate–Farrell cases. For example Richard Thompson's group F has cohomology that vanishes everywhere on projective modules so an ordinary projective resolution computes the Tate cohomology (the ordinary cohomology and the Tate completion of it coincide everywhere for this group, including in degrees 0 and -1). Another simpler example where this happens is a free abelian group of infinite rank.

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The framed E_2 structure on symplectic cohomology

Mohammed Abouzaid

The construction fo operations on symplectic cohomology has so far relied on adhoc methods relying on inductive choices of Floer data to define operations corresponding to moduli spaces of pseudo-holomorphic curves with increasing number of inputs, or with increasing energy. With Groman and Varolgunes, we developed a method that does not rely on choices, yielding a completely functorial invariant at the chain level. The key idea of the construction is to incorporate all possible choices of Floer data required to define an operation in an algebraic package, which takes the form of a topologically enriched multicategory, and the essential lemma to prove is that the choice of such data, lying over an abstract Riemann surface, is contractible.

We apply this method to construct a chain model for symplectic cohomology with support, which carries an algebra structure over an operad weakly equivalent to the framed E_2 operad. Our construction associates such an algebra to each compact subset of a symplectic manifold which is tame at infinity, and should in particular specialise to Rabinowitz Floer homology. The resulting invariant is defined over the Novikov ring, is strictly functorial under inclusions, and satisfies the Mayer-Vietoris property for Varolgunes covers.

An A_∞-category of Lagrangian cobordisms

Noémie Legout

Using techniques of Symplectic Field Theory, we define a Floer complex $RFC(\Sigma_0, \Sigma_1)$ associated to a pair of exact Lagrangian cobordisms in the symplectization of a contact manifold (Y, α). We describe higher order operations on this complex, leading to the definition of a cohomologically unital A_∞-category: the Fukaya category of $\mathbb{R} \times Y$. Namely, we have:

Theorem 0.1. *There exists a unital A_∞-category $\mathcal{F}uk(\mathbb{R} \times Y)$ whose objects are exact Lagrangian cobordisms equipped with augmentations of its negative ends and whose morphism spaces in the cohomological category satisfy*

$$H^*(\hom_{\mathcal{F}uk(\mathbb{R}\times Y)}(\Sigma_0, \Sigma_1)) \cong H^*(RFC(\Sigma_0, \Sigma_1)),$$

whenever Σ_0 and Σ_1 are transverse.

1. The Rabinowitz bimodule

Let $\Lambda_0^\pm, \Lambda_1^\pm \subset Y$ be Legendrian submanifolds of (Y, α) and denote $\mathcal{C}_0, \mathcal{C}_1$ the Chekanov-Eliashberg DGA over $\mathbb{Z}_2$ of Λ_0^- and Λ_1^- respectively, i.e.

$$\mathcal{C}_i = \left(\mathbb{Z}_2\langle \text{Reeb chords of } \Lambda_i^- \rangle = \mathbb{Z}_2 \oplus C_i \oplus C_i^{\otimes 2} \oplus C_i^{\otimes 3} \oplus \dots, \partial\right),$$

where C_i is the $\mathbb{Z}_2$-vector space generated by Reeb chords of Λ_i^-.

Given two transverse exact Lagrangian cobordisms $\Sigma_0, \Sigma_1 \subset (\mathbb{R} \times Y, d(e^t\alpha))$ from Λ_0^- to Λ_0^+ and Λ_1^- to Λ_1^+ respectively, the Rabinowitz complex denoted $(RFC(\Sigma_0, \Sigma_1), \mathfrak{m}_1)$ is a DG $(\mathcal{C}_1, \mathcal{C}_0)$-bimodule generated by three types of generators, namely:

$$RFC(\Sigma_0, \Sigma_1) = C(\Lambda_1^+, \Lambda_0^+) \oplus CF(\Sigma_0, \Sigma_1) \oplus C(\Lambda_0^-, \Lambda_1^-)$$

where $C(\Lambda_1^+, \Lambda_0^+), CF(\Sigma_0, \Sigma_1)$ and $C(\Lambda_0^-, \Lambda_1^-)$ are $(\mathcal{C}_1, \mathcal{C}_0)$-bimodules generated respectively by Reeb chords from Λ_0^+ to Λ_1^+, intersection points in $\Sigma_0 \cap \Sigma_1$, and Reeb chords from Λ_1^- to Λ_0^-.

The differential $\mathfrak{m}_1$ is defined by a count of pseudo-holomorphic discs with boundary on Σ_0 and Σ_1, and with punctures asymptotic to Reeb chords and intersection points. See Figure 1, where each disc can have extra negative asymtptotics to Reeb chords of $\Lambda_0^\pm$ and $\Lambda_1^\pm$, which all become bimodule coefficients (using the functoriality of the Chekanov-Eliashberg DGA via cobordism).

Transversality and compactness results on the moduli spaces imply that $\mathfrak{m}_1^2 = 0$, i.e. $\mathfrak{m}_1$ is a differential.

At first glance, the Rabinowitz complex looks similar to the Cthulhu complex $Cth(\Sigma_0, \Sigma_1)$ defined by Chantraine, Dimitroglou Rizell, Ghiggini and Golovko [1], but it has actually different properties. For example, it is not always acyclic when the contact manifold Y is the contactization of a Liouville manifold. Moreover, it admits a product structure and a *continuation element* with respect to this product.

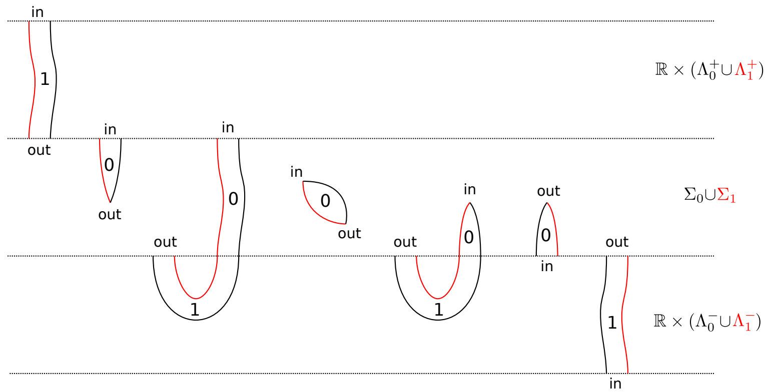

FIGURE 1. Pseudo-holomorphic discs contributing to the differential $\mathfrak{m}_1$, where "in" stands for "input" and "out" for "output".

2. THE PRODUCT STRUCTURE

Given a triple $\Sigma_0, \Sigma_1, \Sigma_2$ of transverse exact Lagrangian cobordisms from Λ_i^- to Λ_i^+, $i = 0, 1, 2$, such that the Chekanov-Eliashberg DGAs $\mathcal{C}_i$ of Λ_i^- admit augmentations (in such a way, the bimodules $RFC(\Sigma_i, \Sigma_j)$ can be turned into $\mathbb{Z}_2$-vector spaces), we define a map

$$\mathfrak{m}_2 \colon RFC(\Sigma_1, \Sigma_2) \otimes RFC(\Sigma_0, \Sigma_1) \to RFC(\Sigma_0, \Sigma_2)$$

by a count of pseudo-holomorphic discs, and show that it satisfies the Leibniz rule $\mathfrak{m}_1 \circ \mathfrak{m}_2 + \mathfrak{m}_2(\mathfrak{m}_1 \otimes 1) + \mathfrak{m}_2(1 \otimes \mathfrak{m}_1) = 0$. We then show:

Theorem 2.1. *When Σ_1 is a negative perturbed copy of Σ_0, there exists an element $e_{01} \in RFC(\Sigma_0, \Sigma_1)$ such that the map*

$$\mathfrak{m}_2(\cdot, e_{01}) \colon RFC(\Sigma_1, \Sigma_2) \to RFC(\Sigma_0, \Sigma_2)$$

is a quasi-isomorphism.

We construct more generally a family of maps $\{\mathfrak{m}_d\}_{d \geq 1}$ satisfying the A_∞-equations, which together with Theorem 2.1 are used to construct the category $\mathcal{F}uk(\mathbb{R} \times Y)$ by localization. It is expected (but not proved) that this category is equivalent to the Rabinowitz wrapped Fukaya category defined recently by Ganatra, Gao, Venkatesh [2] using Hamiltonian techniques (and under the hypotheses that the contact manifold Y is fillable.)

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Spectral Jumps in Rabinowitz Tate Homology

Urs Frauenfelder

We consider m exact symplectic manifolds $(M_i, \omega_i = d\lambda_i)$ with $1 \leq i \leq m$. Each loop space $C^\infty(S^1, M_i)$ is endowed with a circle action obtained by reparametrisation of its domain. This endows the loop space of the product manifold $M = M_1 \times \ldots \times M_m$

$$C^\infty(S^1, M) = C^\infty(S^1, M_1) \times \ldots \times C^\infty(S^1, M_m)$$

with an action of the m-dimensional torus T^m. We further assume that each symplectic manifold is endowed with a smooth function

$$H_i \colon M_i \to \mathbb{R}.$$

This gives rise to a smooth function

$$H \colon M \to \mathbb{R}^m, \quad (x_1, \ldots, x_m) \to \big(H(x_1), \ldots, H(x_m)\big).$$

There is further given a smooth function

$$f \colon \mathbb{R}^m \to \mathbb{R}.$$

so that the composition leads to a smooth function

$$f \circ H \colon M \to \mathbb{R}.$$

Abbreviating

$$\lambda = \lambda_1 \oplus \ldots \oplus \lambda_m \in \Omega^1(M)$$

we have two Rabinowitz action functionals

$$\mathcal{A} \colon C^\infty(S^1, M) \times \mathbb{R}, \quad (v, \tau) \mapsto \int v^*\lambda - \tau \int f \circ H(v) dt$$

and

$$\widetilde{\mathcal{A}} \colon C^\infty(S^1, M) \times \mathbb{R}, \quad (v, \tau) \mapsto \int v^*\lambda - \tau f \circ \int H(v) dt.$$

Both functionals have the same critical points on which they attain the same critical values. However, the second one is invariant under the action of the torus T^m, while the first one not necessarily is. Hence for the second functional we can consider Rabinowitz Tate homology for the torus action. Already for the case of a harmonic oscillator its chain complex is extremely rich and has nonvanishing homology classes. We discuss how the double filtration in Rabinowitz Tate homology leads to the phenomenon that their spectral numbers can jump from minus infinity to plus infinity.

Calabi–Yau structures on Rabinowitz Fukaya categories

Hanwool Bae
(joint work with Wonbo Jeong, Jongmyeong Kim)

Rabinowtiz Floer homology, introduced by Cieliebak and Frauenfelder [1], is a Floer homology associated to a symplectic manifold with contact boundary. As its open-string analogue, Rabinowitz Floer homology can be also associated to a Lagrangian submanifold with Legendrian boundary. It was shown in [5] that the Rabinowitz Floer homology can be defined as the homology of the mapping cone of a continuation map from the Floer complex for symplectic cohomology to that for symplectic homology. Following this idea, Ganatra-Gao-Venkatesh [6] introduced the Rabinowitz Fukaya category of a Liouville domain, which can be said to be a categorification of Rabinowitz Floer homology of Lagrangian submanifolds. Indeed, for given two Lagrangian submanifolds L_0 and L_1 of V, the morphism space $\mathrm{RFC}^*(L_0, L_1)$ is defined by the mapping cone of a continuation map from the Floer complex for wrapped Floer homology to that for wrapped Floer cohomology.

On the other hand, it has been shown by Cieliebak-Oancea([4, 5]) and Cieliebak-Hingston-Oancea([3]) that Rabinowitz Floer homology of a Louville domain (or a Lagrangian submanifold) has a duality that extends the classical Poincaré duality of its boundary. It was further shown that such a duality comes from a Frobenius algebra structure on Rabinowitz Floer homology. Consequently, it is natural to ask if the Frobenius nature of Rabinowitz Floer homology extends to the level of category.

As an answer to this question, I and collaborators(Jeong and Kim) proved that the Rabinowitz Fukaya category $\mathcal{RW}(V)$ of a Liouville domain (V, λ) of dimension $2n$ has a $(n-1)$-Calabi–Yau structure under a degree-wise finiteness assumption on Rabinowitz Floer homologies between generators. In particular, this means that, for every pair (X, Y) of objects of the derived Rabinowitz Fukaya category and every integer k, there is an isomorphism

$$\mathrm{RFH}^k(X, Y) \cong \mathrm{RFH}^{n-1-k}(Y, X)^\vee,$$

where RFH denotes the homology of the chain complex RFC and $^\vee$ is the linear dual.

To be more precise, we have shown that there is a $\mathcal{RW}(V)$-$\mathcal{RW}(V)$-bimodule quasi-isomorphism between $\mathcal{RW}(V)$ and $(\mathcal{RW}(V)^{\mathrm{op}})^\vee[1-n]$ if

- there are at most countable Lagrangian submanifolds $\{L_i\}_{i \in I}$ of V generating the wrapped Fukaya category of V and
- the dimension $\dim \mathrm{RFH}^k(L_i, L_j)$ is finite for all $i, j \in I$ and $k \in \mathbb{Z}$.

This can be proved by constructing a bimodule homomorphism from $\mathcal{RW}(V)$ to $(\mathcal{RW}(V)^{\mathrm{op}})^\vee[1-n]$ extending the natural Poincaré duality between Floer homologies.

For example, if a Liouville domain (V, λ) is given by the disk cotangent bundle $(D^*Q, \lambda_{\mathrm{can}})$ of a simply-connected smooth closed manifold Q, then the above two requirements are satisfied and therefore the corresponding Rabinowitz Fukaya category is Calabi–Yau.

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Some counterexamples to the Donaldson 4-6 question

Amanda Hirschi
(joint work with Luya Wang)

The following question, credited to Donaldson, concerns the uniqueness of symplectic structures and their relation to the smooth topology of the underlying manifold.

Conjecture 1. *Let (X_1, ω_1) and (X_2, ω_2) be two closed (simply connected) symplectic 4-manifolds such that X_1 and X_2 are homeomorphic. Then the product symplectic manifolds $(X_1 \times S^2, \omega_1 \oplus \omega_{\mathrm{std}})$ and $(X_2 \times S^2, \omega_2 \oplus \omega_{\mathrm{std}})$ are deformation equivalent if and only if X_1 and X_2 are diffeomorphic.*

Two symplectic structures σ_1 on X_1 and σ_2 on X_2 are *deformation equivalent* if there exists a diffeomorphism $\varphi\colon X_1 \to X_2$ and a path $\{\sigma'_t\}_{t\in[0,1]}$ of symplectic structures on X_1 with $\sigma'_0 = \varphi^*\sigma_2$ and $\sigma'_1 = \sigma_1$.

If the conjecture were to be true, it would be a symplectic analogy of the fact that given two smooth simply-connected homeomorphic 4-manifolds X_1 and X_2, the products $X_1 \times S^2$ and $X_2 \times S^2$ are diffeomorphic. It holds for certain classes of symplectic 4-manifolds by [2] and [1]. However, Smith, [3], and Vidussi, [4], constructed symplectic forms on the same smooth 4-manifold that are distinguished by their first Chern classes. We show that this difference is preserved after taking the product with S^2.

Theorem 1. *There exist closed simply connected symplectic 4-manifolds (X_1, ω_1) and (X_2, ω_2), such that X_1 is diffeomorphic to X_2, while $(X_1 \times S^2, \omega_1 \oplus \omega_{\mathrm{std}})$ and $(X_2 \times S^2, \omega_2 \oplus \omega_{\mathrm{std}})$ are deformation inequivalent.*

This shows that the symplectic geometry of a product remembers more about the symplectic geometry of the factors than is true for the smooth structures.

While the proof of this result only uses classical invariants, Gromov–Witten invariants can be used in combination with [1] to prove the following partial converse.

Theorem 2. *Given two simply connected symplectic* 4*-manifolds* (X_0, ω_0) *and* (X_1, ω_1) *so that* $(X_0, \omega_0) \times (S^2, \omega_{\text{std}})$ *and* $(X_1, \omega_1) \times (S^2, \omega_{\text{std}})$ *are deformation equivalent and* $\sigma(X_i) \neq 0$*, there exists a homeomorphism* $X_0 \to X_1$ *relating their Seiberg-Witten invariants.*

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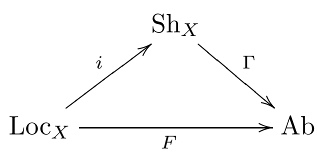
Sheaf and singular models for ∞-groupoïd cohomology

Colin Fourel

The goal of the talk was to explain how one can use group cohomology to prove that sheaf and singular cohomology are isomorphic on CW complexes.

Let G be a discrete group, then the cohomology of G coincides with the singular cohomology of any connected CW complex satisfying $\pi_1 = G$ and $\pi_i = 0$ for $i \geq 2$. Let X be such a CW complex, we also have the following commutative diagram of abelian categories



where Loc_X denotes the category of local systems over X, Sh_X that of sheaves over X, i the inclusion, Γ the global sections functor and $F(M) = M^G$. The statement that the sheaf cohomology groups with coefficients in local systems over X are isomorphic to the corresponding cohomology groups of G, is equivalent to the commutativity of this diagram at the level of derived functors.

Now, this is in turn equivalent to the fact that whenever I is an injective object of Loc_X, then $i(I)$ is an acyclic sheaf. Let us give an independent proof of that. Denote $\pi : \widetilde{X} \to X$ a universal cover of X. Since the total space of π is contractible, and its fiber are discrete, and since $\pi^*(i(I))$ is constant, the Leray spectral sequence implies that $\pi_*\pi^*(i(I))$ is acyclic. The unit of the adjunction between π_* and π^* gives an injective map of local systems $I \to \pi_*\pi^*(i(I))$ which, by injectivity of I, has a retract. Hence I is acyclic. We thus recover the isomorphism between the sheaf and singular cohomologies of X.

Let us now assume that X is any connected CW complex. Let us introduce the ∞-category of ∞-*local systems* over X, denoted $\infty\mathrm{Loc}_X$, which has the following three equivalent descriptions

(1) $D(C_*(\Omega X, \mathbb{Z}))$, the derived ∞-category of dg modules over chains over the based loop space of X with coefficients in $\mathbb{Z}$,
(2) $\mathrm{LC}(X; D(\mathbb{Z}))$, the ∞-category of locally constant sheaves on X with values in $D(\mathbb{Z})$,
(3) $\mathrm{Fun}(\Pi_\infty(X), D(\mathbb{Z}))$, the ∞-category of functors from the fundamental ∞-groupoïd of X to $D(\mathbb{Z})$.

The equivalence between (1) and (2) is proven in [2] (theorem 6.26), the equivalence between (2) and (3) is proven in [1] (theorem A.4.19).

Let $K \in \infty\mathrm{Loc}_X$. Using description (3), we define the i^{th} ∞-*groupoïd cohomology group of* $\Pi_\infty(X)$ with coefficients in K as:

$$H^i(\Pi_\infty(X), K) = H^i(\lim K).$$

Consider the constant ∞-local system $\mathbb{Z}$ on X. Using description (1) we have $H^i(\Pi_\infty(X), \mathbb{Z}) = \mathrm{Ext}^i_{C_*(\Omega X)}(\mathbb{Z}, \mathbb{Z})$, which is isomorphic to $H^i_{\mathrm{sing}}(X, \mathbb{Z})$ (see [3], Theorem B and Proposition 11.7). On the other hand, using description (2) we have $H^i(\Pi_\infty(X), \mathbb{Z}) = H^i(\Gamma(\mathbb{Z}))$, which is isomorphic to the usual sheaf cohomology group $H^i(X, \underline{\mathbb{Z}})$ ([4], Proposition 10 and Corollary 11). We thus recover the isomorphism between the sheaf and singular cohomologies of X.

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String topology and graph cobordisms

Andrea Bianchi

String topology is broadly concerned with the study of invariants of mapping spaces of the form $M^X = \mathrm{map}(X, M)$, where X is a topological space and M is a smooth, closed manifold of some dimension $d \geq 1$. More specifically, we want to study, for a commutative ring R, the homology $H_*(M^X; R)$, in the assumption that M is R-oriented. A case of particular interest is $X = S^1$, recovering the free loop space LM: the homology $H_*(LM; \mathbb{Z})$ agrees with the (suitably twisted) symplectic homology of the Liouville domain T^*M, and the topology of LM can be used to study the existence and the number of closed geodesics on M, when we endow M with a Riemannian metric.

It is convenient to study the homology groups $H_*(M^X;R)$ for fixed M and varying X, as one can describe several *string operations* relating different homology groups. The most basic operations are:

(1) for a map $Y \to X$, we get a restriction map $H_*(M^X;R) \to H_*(M^Y;R)$;
(2) for $Y = X \sqcup *$, we get map $H_*(M^X;R) \overset{-\times[M]}{\to} H_{*+d}(M^X \times M;R) = H_{*+d}(M^Y;R)$ by cross product with the fundamental class of M;
(3) for $Y = X \sqcup_{\partial I} I$, i.e. Y is obtained from X by attaching a 1-cell, Chas and Sullivan [1] constructed a natural operation $H_*(M^X;R) \to H_{*-d}(M^Y;R)$.

The notion of graph cobordism gives a common denominator to (1)-(3). A *graph cobordism* between X and Y is a cospan of spaces $X \hookrightarrow W \leftarrow Y$, together with a finite cell structure of W relative to X consisting only of 0-cells and 1-cells. Each graph cobordism gives, by combining the above basic operations, an operation $H_*(M^X;R) \to H_{*+d\cdot\chi(W,X)}(M^Y;R)$.

I define a moduli space $\mathfrak{M}_{\mathrm{Gr}}(X,Y)$ of graph cobordisms from X to Y, by taking the classifying space of a suitable topological category $\mathrm{Gr}(X,Y)$ of graph cobordisms, with morphisms given by forest collapses. I also define a coefficient system ξ_d over $\mathfrak{M}_{\mathrm{Gr}}(X,Y)$, taking values in homologically graded R-modules, whose fibre over $X \hookrightarrow W \leftarrow Y$ is (non-canonically) isomorphic to $R[-d\cdot\chi(W,X)]$.

The main stated theorem is an extension of (1)-(3) to a chain map

$$C_*(M^X;R) \otimes_R C_*(\mathfrak{M}_{\mathrm{Gr}}(X,Y);\xi_d) \to C_*(M^Y;R).$$

The entire construction can in fact be generalised in the case in which R is an E_∞-ring spectrum: in this case ξ_d is a parametrised R-module of rank 1 over $\mathfrak{M}_{\mathrm{Gr}}(X,Y)$, and we obtain a map of R-modules

$$(R \otimes X) \otimes_R (\mathrm{colim}_{\mathfrak{M}_{\mathrm{Gr}}(X,Y)}\xi_d) \to (R \otimes Y).$$

The construction can be further generalised to the case in which M is an R-oriented Poincaré duality space; in particular all string operations arising in this way are invariant under homotopy equivalences of manifolds.

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Efimov's categorical formal punctured neighborhood of infinity, Rabinowitz Fukaya category, CY and pre-CY structures

Alex Takeda

The purpose of this talk is to propose the construction of the "categorical formal punctured neighborhood of infinity" [5] as an organizing principle to understand the relationship between duality structures on (usual) Floer theory and Rabinowitz Floer theory. This is a purely algebraic construction, that applied to a dg-category $\mathcal{C}$, produces a dg category $\widehat{\mathcal{C}}_\infty$, called the *categorical formal punctured neighborhood of $\mathcal{C}$*.

This category $\mathcal{C}$ should be thought of as an algebraic incarnation of the following geometric construction: for a smooth but non-compact algebraic variety Y, one chooses a compactification $\overline{Y} = Y \cup D$ and looks at perfect complexes (of coherent sheaves) supported on the punctured formal neighborhood of D. Suitably defined, this category $\mathrm{Perf}(\widehat{\overline{Y}}_D \setminus D)$ is independent of the choice of compactification $\overline{Y}$. Efimov's construction is a noncommutative version of this operation in the sense that if one takes $\mathcal{C}$ to be (a dg enhancement of) the bounded derived category of X, then $\widehat{\mathcal{C}}_\infty \cong \mathrm{Perf}(\widehat{\overline{Y}}_D \setminus D)$.

This construction has been extended to A_∞-categories and applied to symplectic topology by [7], where it is proven that, given any nondegenerate Liouville manifold X, there is an A_∞-equivalence $\mathcal{RW}(X) \to (\widehat{\mathcal{W}(X)})_\infty$, from the Rabinowitz Fukaya category of X, to the categorical formal punctured neighborhood of the wrapped Fukaya category. The former category has as morphism spaces the 'open string' version of Rabinowitz Floer homology [2], with composition maps as in [4]. As a result of this identification, together with some yet-unpublished work of Rezchikov, one gets an identification of Rabinowitz Floer cohomology $RFH^*(X)$ with

$$HH_*(\mathcal{W}(X), \mathcal{RW}(X)) = HH_*(\mathcal{W}(X), \widehat{\mathcal{W}(X)}_\infty),$$

that is, Hochschild homology of the wrapped Fukaya category with coefficients in the Rabinowitz Fukaya category. Moreover, this relation recovers the 'Tate construction' perspective [3] on RFH^*, since the complex calculating $HH_*(\mathcal{W}(X), \widehat{\mathcal{W}(X)}_\infty)$ is obtained by a cone construction.

After introducing these constructions and results, I explained in my talk a sketch of how this perspective could be used to understand the origin of products on RFH^*, as well as the 'Frobenius' property described by [1]. For example, in my own work with Rivera and Wang [8], we study products on $HH_*(\mathcal{C}, \widehat{\mathcal{C}}_\infty)$ constructed from a type of structure on some category $\mathcal{C}$ called a pre-Calabi–Yau structure, which in particular can be produced for the wrapped Fukaya category as a consequence of its smooth Calabi–Yau structure together with the results in [6]; it is reasonable to conjecture that the geometrically-defined product on RFH^* arises in such a way.

Lastly, the relation between such a description and the products constructed by [9] on 'singular Hochschild cohomology' should be given by some sort of Koszul duality. In some cases, where the wrapped Fukaya category $\mathcal{W}(X)$ has a 'proper Koszul-dualizing subcategory', as defined in [7], combining all the above results, one gets an equivalence between $\mathcal{RW}(X)$ and the derived category of singularities of a certain dg algebra. I ended my talk with the conjecture that all the product and duality structures above, should match under the many dualities and identifications. If proven, this would mean that they all encode the same data, given just by the smooth Calabi–Yau structure on the wrapped Fukaya category.

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Spaces of operations by example: two BV structures on the Hochschild homology of symmetric Frobenius algebras

Nathalie Wahl

Given a differential graded associative algebra, let $C_*(A,A)$ denote its Hochschild complex, $\overline{C}_*(A,A) = C_*(A,A)/C_{0,0}(A,A)$ the reduced complex where the copy of A_0 in Hochschild degree 0 has been killed, and $HH_*(A,A)$, $\overline{HH}_*(A,A)$ the corresponding homology groups.

A BV-algebra is a commutative differential graded algebra V_* equipped with an operator $\Delta : V_* \to V_{*+1}$ satisfying the BV-relation

$$\Delta(abc) = \Delta(ab)c+(-1)^{|a|}a\Delta(bc) + (-1)^{|a|(|b|+1)}b\Delta(ac) + \Delta(a)bc + (-1)^{|a|}a\Delta(b)c + (-1)^{|a|+|b|}ab\Delta(c).$$

A BV-algebra of dimension d is a BV-algebra with a product of degree $\pm d$ and appropriately modified signs in the commutativity and BV-relation; we refer to [8, Sec 6.3] for a systematic way to define a "dimension d" version of this type of algebraic structure.

Recall that the Hochschild complex is endowed with a degree 1 operator $B : C_*(A,A) \to C_{*+1}(A,A)$, the Connes-Rinehart operator. When A is a symmetric Frobenius algebra of dimension $d > 0$, the (long proven) cyclic Deligne conjecture states that this operator B, together with the dual of the cup product, defines a coBV-structure of dimension d on $H_*(A,A)$, induced from a chain-level structure. It corresponds to the string topology BV-structure of Chas-Sullivan when $A \simeq C^*M$, see [2]. From the papers [1, 4, 5], one can also deduce that the same operator B, together with a product corresponding to the dual of the Goresky-Hingston string topology coproduct when $A \simeq C^*M$ (see [6]), endows the reduced

Hochschild homology $\overline{HH}_*(A, A)$ with a 1-suspended BV–algebra structure of dimension d.

One can in principle prove the above stated results in homology by direct computation, as the product, coproduct and the operator B have explicit descriptions, but it is very difficult to get the signs right when checking the relations! We explain here how these statements follow from a more general result, and comes from two different embeddings of the BV-operad in a prop acting on the Hochschild complex of symmetric Frobenius algebras.

Recall that there is an isomorphism of operads $BV \cong H_*(fE_2)$ between the operad BV governing BV–algebras and the homology of the framed E_2–operad, an operad that is also equivalent to the cactus operad. We will here denote by $Cact$ the chain operad of normalised cacti, as defined in [3], with $H_*(Cact) = BV$. The above BV and co-BV structures are a consequence of the following chain level statement:

Theorem 1.

(1) [7, 8] *The Hochschild complex $C_*(A, A)$ of a symmetric Frobenius dg algebra A of dimension d admits an action of the dg-prop SD_d of degree d–shifted Sullivan diagrams.*

(2) [4, 8] *There are inclusions $Cact(n) \hookrightarrow SD(1, n)$ and $Cact(n) \times \Delta^{n-1} \hookrightarrow SD(n, 1)$ compatible with composition.*

(3) [5] *The resulting action of $Cact(n) \times \Delta^{n-1}$ on $C_*(A, A)$ descends to an action of $Cact(n) \times \Delta^{n-1}/\partial\Delta^{n-1}$ on the reduced Hochschild chains $\overline{C}_*(A, A)$.*

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Relative Calabi–Yau structure from acyclic Rabinowitz–Floer complexes of Legendrians

Georgios Dimitroglou Rizell

(joint work with N. Legout)

1. Outline

In this joint work with Legout we establish a geometric incarnation of morphisms of distinguished triangles of bimodules, realised through the Legendrian invariant of Rabinowitz Floer complex. The morphism is a quasi-isomorphism if and only if this complex is acyclic, which is equivalent to the existence of a Calabi–Yau structure in the sense of Brav–Dyckerhoff [BD19].

2. The morphism of triangles

Consider the canonical inclusion $\iota\colon \mathcal{A}_* \hookrightarrow \mathcal{C}_* = \mathcal{C}_*(\Lambda;\mathcal{A})$ of the DGA of chains of the based loop space $\mathcal{A}_* = C_{-*}(\Omega\Lambda;\mathbf{k})$ into

$$\mathcal{C}_* = (\mathcal{A}_*\langle \text{Reeb chords of } \Lambda\rangle = \mathcal{A} \oplus Q \oplus (Q \otimes_{\mathcal{A}} Q) \oplus (Q \otimes_{\mathcal{A}} Q \otimes_{\mathcal{A}} Q) \oplus \ldots, \partial),$$

i.e. the Chekanov–Eliashberg DGA of a closed $\mathbf{k}$-oriented Legendrian submanifold $\Lambda^n \subset (Y^{2n+1},\alpha)$ of a contact manifold over the chains of the based loop space $\mathcal{A}_*$.

For a DGA $\mathcal{B}_*$, denote by $\mathcal{B}_\Delta$ the so-called diagonal (non-free) left $\mathcal{B}^e = \mathcal{B} \otimes_{\mathbf{k}} \mathcal{B}^{op}$-module (equivalently: left $\mathcal{B}-$bimodule) given by $\mathcal{B}$ endowed with the canonical bimodule structure coming from DGA-multiplication.

The DG-morphism ι induces a canonical map

$$\mu\colon \iota_!(\mathcal{A}_\Delta) \coloneqq \mathcal{C}^e \otimes^{\mathbb{L}}_{\mathcal{A}^e} \mathcal{A}_\Delta = \mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} \xrightarrow{\mu} \mathcal{C}_\Delta$$

of left $\mathcal{C}^e-$modules induced by the multiplication $\mathcal{C} \otimes_{\mathcal{A}} \mathcal{C} \xrightarrow{\mu} \mathcal{C}$ of the DGA.

Theorem 2.1. *There is a quasi-isomorphism of the distinguished triangles*

$$\begin{array}{ccccccc}
\iota_!(\mathcal{A}) & \xrightarrow{\mu} & \mathcal{C}_\Delta & \longrightarrow & \operatorname{cof}(\mu) & \dashrightarrow & \cdots \\
\downarrow\simeq & & \downarrow\simeq & & \downarrow\simeq & & \\
C_*(\Lambda;\mathcal{C}^e) & \hookrightarrow & LCC_*(\Lambda,\Lambda^+;\mathcal{C}^e) & \twoheadrightarrow & LCC_*(\Lambda,\Lambda^+;\mathcal{C}^e)/C_*(\Lambda;\mathcal{C}^e) & \dashrightarrow & \cdots,
\end{array}$$

The upper row is induced by $\mathcal{A}_* \hookrightarrow \mathcal{C}_*$, while the lower row is a short exact sequence induced by the action filtration in Legendrian contact homology.

Here the complex $LCC_*(\Lambda,\Lambda^+;\mathcal{C}^e)$ denotes the Legendrian contact homology complex with coefficients in $\mathcal{C}^e$, which is a projective left $\mathcal{C}^e-$module generated by the Reeb chords from Λ to its small Reeb push-off Λ^+. Note that this push-off creates also a small set of Reeb chords which are in bijection with the critical points of a small function; thus we get an inclusion of the Morse complex

$$C_*(\Lambda;\mathcal{C}^e) = \mathcal{C}^e \otimes_{\mathcal{A}^e} C_*(\Lambda;\mathcal{A}^e) \subset LCC_*(\Lambda,\Lambda^+;\mathcal{C}^e).$$

Recall that $C_*(\Lambda;\mathcal{A}^e) \simeq \mathcal{A}_\Delta$ is the Morse homolgoy of Λ with $\mathcal{A}^e$ as a two-sided derived local system.

The bimodule dual $(-)^! := R\mathrm{hom}_{\mathcal{C}^e}(-,\mathcal{C}^e)$ is an endofunctor $(-)^!\colon D^b(\mathcal{C}^e) \to D^b(\mathcal{C}^e)$ which preserves semi-free $\mathcal{C}^e$−modules. There is a chain map

$$b\colon LCC_*(\Lambda,\Lambda^+;\mathcal{C}^e) \to LCC_*(\Lambda^+,\Lambda;\mathcal{C}^e)^![n+1]$$

defined by counting "bananas" in the symplectisation with two positive punctures. The co-domain of b, which is a Legendrian contact cohomology complex generated by Reeb chords from Λ_+ to Λ (note the order!) can be seen to be isomorphic to $(LCC_*(\Lambda,\Lambda^+;\mathcal{C}^e)/C_*(\Lambda;\mathcal{C}^e))^![n+1]$ by invariance under Legendrian isotopy.

Theorem 2.2. *The map b extends to a morphism of distinguished triangles*

$$\begin{array}{ccccccc}
C_*(\Lambda;\mathcal{C}^e) & \xhookrightarrow{\mu} & LCC_*(\Lambda,\Lambda^+;\mathcal{C}^e) & \longrightarrow\!\!\!\!\!\rightarrow & \mathrm{cof}(\mu) & \overset{\Sigma}{\dashrightarrow} & \cdots \\
\simeq\downarrow \widetilde{\mathcal{CY}} & & \downarrow b & & \downarrow b^! & & \\
(C_*(\Lambda;\mathcal{C}^e))^![n] & \overset{\Sigma}{\dashrightarrow} & \mathrm{cof}(\mu)^![n+1] & \hookrightarrow & LCC_*(\Lambda,\Lambda^+;\mathcal{C}^e)^![n+1] & \twoheadrightarrow & \cdots
\end{array}$$

Here the leftmost vertical map is a quasi-isomorphism that is induced by the *absolute* n-Calabi–Yau structure

$$\mathcal{A}_\Delta \simeq C_*(\Lambda;\mathcal{A}^e) \xrightarrow{\mathcal{CY}} (C_*(\Lambda;\mathcal{A}^e))^![n] \simeq \mathcal{A}_\Delta^![n]$$

by tensoring $\mathcal{C}^e \otimes_{\mathcal{A}^e} (-)$. See [Gan13] or [Leg23] for the latter quasi-isomorphism. The Rabinowitz–Floer complex is the Legendrian isotopy invariant Cone(b), i.e.

$$RFC_*(\Lambda,\Lambda^+;\mathcal{C}^e) := \left(LCC_*(\Lambda,\Lambda^+;\mathcal{C}^e) \oplus LCC_*(\Lambda^+,\Lambda;\mathcal{C}^e)^![n], \begin{bmatrix} \partial & b \\ 0 & \partial^! \end{bmatrix} \right).$$

Theorem 2.3. *The Rabinowitz–Floer complex is acyclic when:*

- $Y = \partial_\infty(P\times\mathbb{C})$ *is the contact boundary of a subcritical Weinstein domain;*
- $\Lambda \subset Y$ *can be displaced from its Reeb trace by a contact isotopy; or*
- $Y = J^1S^2$ *with a non-trivial bulk-deformation by the H_2-class.*

The acyclicity of the Rabinowitz–Floer complex is equivalent to Theorem 2.2 being a quasi-isomorphism of triangles. This translates into the property that the morphism $\iota\colon \mathcal{A}_* \to \mathcal{C}_*$ of DGAs is a **relative** $(n+1)$**-Calabi–Yau pair** as defined by Brav–Dyckerhoff [BD19]. This can be seen as a generalisation of Sabloff duality [EES09] from augmentations to general DG-bimodules.

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