Snapshots of modern mathematics
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$4 = 2 \times 2$, or the power of even integers in Fourier analysis

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We describe how simple observations related to vectors of length 1 recently led to the proof of an important mathematical fact: the sharp Stein–Tomas inequality from Fourier restriction theory, a pillar of modern harmonic analysis with surprising applications to number theory and geometric measure theory.

1 Introduction

We use simple geometric objects from our everyday (mathematical) life in order to give a complete proof of a recent mathematical breakthrough: the sharp three-dimensional Stein–Tomas inequality. The latter links the geometric notion of how curved a surface is to properties of the Fourier transform. It finds applications in many different areas of mathematics, including number theory and geometric measure theory, as we will illustrate in the last section.

2 Unit vectors

Most of this snapshot takes place on the two-dimensional sphere, that is, the set of all vectors of length 1 in 3-dimensional space. We call a vector x of

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length $|x| = 1$ a *unit vector*. In mathematical notation, the sphere can be expressed as

$$\mathbb{S}^2 := \{x \in \mathbb{R}^3 : |x| = 1\}. \tag{1}$$

The sphere $\mathbb{S}^2$ is a curved surface which is the stage for a number of interesting phenomena.

Archimedes showed that the surface area of $\mathbb{S}^2$ equals 4π. To do so, he inscribed $\mathbb{S}^2$ in a cylinder of the same radius and twice the height, and proved that the orthogonal projection from the lateral face of the cylinder onto the sphere is *area-preserving*; see Figure 1. This can of course be done with calculus, but it is worth pointing out that Archimedes died (ca. 212 BC) roughly 1850 years before Newton, one of the founders of calculus, was born (1642).

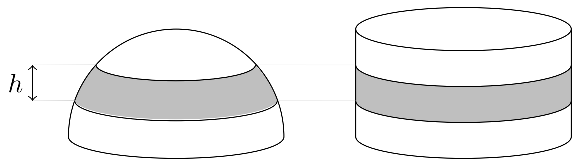

Figure 1: Archimedes' approach: The surface area of a spherical strip is the same as the surface area of a strip on a cylinder with the same height.

Let us now prove two identities for unit vectors. If $\omega_1, \omega_2, \omega_3 \in \mathbb{S}^2$ are three unit vectors whose sum is zero, then the squared lengths of their distinct pairwise sums satisfy

$$|\omega_1 + \omega_2|^2 + |\omega_2 + \omega_3|^2 + |\omega_3 + \omega_1|^2 = 3.$$

We can check this using the observation that $\omega_1 + \omega_2 = -\omega_3$, as in Figure 2 (left), so $|\omega_1 + \omega_2| = |\omega_3| = 1$, and the same happens for the other two summands.

If $\omega_1, \omega_2, \omega_3, \omega_4 \in \mathbb{S}^2$ are *four* unit vectors which sum to zero, as in Figure 2 (right), the above argument does not apply, but it still holds that

$$|\omega_1 + \omega_2|^2 + |\omega_2 + \omega_3|^2 + |\omega_3 + \omega_1|^2 = 4. \tag{2}$$

To see why this is the case, note that $\omega_1 + \omega_2 + \omega_3 = -\omega_4$ is a unit vector, so

$$1 = |\omega_1 + \omega_2 + \omega_3|^2 = 3 + 2(\omega_1 \cdot \omega_2 + \omega_2 \cdot \omega_3 + \omega_3 \cdot \omega_1), \tag{3}$$

where $\omega_i \cdot \omega_j$ denotes the usual inner product[2] between the unit vectors ω_i and ω_j. Expanding $|\omega_i + \omega_j|^2 = |\omega_i|^2 + 2\omega_i \cdot \omega_j + |\omega_j|^2 = 2 + 2\omega_i \cdot \omega_j$ for each $i \neq j$, adding them up and using identity (3) yields (2).

[2] For two vectors $\omega = (\omega_x, \omega_y, \omega_z)$ and $\nu = (\nu_x, \nu_y, \nu_z)$, we have $\omega \cdot \nu = \omega_x\nu_x + \omega_y\nu_y + \omega_z\nu_z$.

Identity (2) is a cute observation which recently found a striking application that relates to the 22-century old approach of Archimedes, and allowed D. Foschi [1] to crack a famous open problem in an exciting research field of contemporary mathematics called *sharp restriction theory*.[3] The proof that Foschi found is deep but, perhaps surprisingly, simple enough that we can present it here in detail.

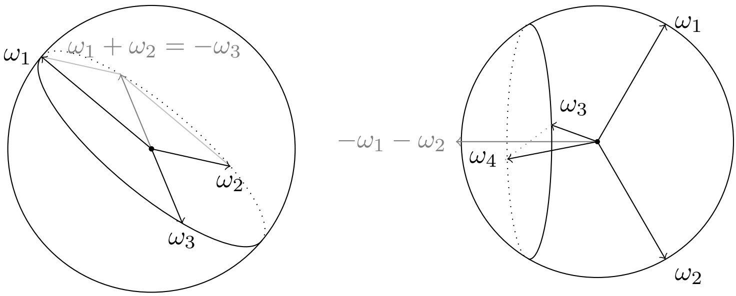

Figure 2: Examples of configurations of three (left) and four (right) unit vectors on $\mathbb{S}^2$ which sum to zero.

3 Spherical convolutions

From unit vectors, we move to *functions* on the sphere $\mathbb{S}^2$. For simplicity, we restrict attention to *non-negative* functions $f\colon \mathbb{S}^2 \to [0,\infty)$, that is, to functions which assign to each unit vector a non-negative number. Given two functions $f, g\colon \mathbb{S}^2 \to [0,\infty)$, we can consider their product fg, usually defined via $(fg)(\omega) \coloneqq f(\omega)g(\omega)$ for all unit vectors $\omega \in \mathbb{S}^2$. A less obvious way to "multiply" the functions f and g is via their *convolution product*,[4] denoted $f\sigma * g\sigma$, which defines a function from $\mathbb{R}^3$ to $[0,\infty)$. Here, σ is the notation for the surface measure on the sphere. We also define the Dirac delta[5] $\boldsymbol{\delta}(\cdot)$ which

[3] *Restriction theory* is the study of cancellation induced by restricting the Fourier transform to a curved surface. One of the aims of *sharp restriction theory* is to discover the optimal constant in all sorts of restriction inequalities; see the articles [2, 4] for recent surveys.

[4] Terry Tao's blog [6] has a nice discussion of the computational aspects of convolution measures which contains (5) as a special case.

[5] The Dirac delta is a "generalized function" that equals 0 everywhere, except at those points where its argument vanishes. It is defined as fulfilling the following property for every function F:

$$\int_{\mathbb{R}^3} F(x)\boldsymbol{\delta}(x-a)\,\mathrm{d}x = F(a).$$

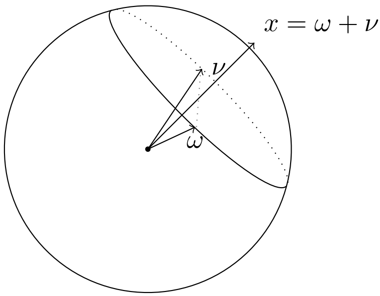

Figure 3: For each fixed x, the region of integration Γ_x is the set of pairs $(\omega, \nu) \in (\mathbb{S}^2)^2$ for which $\omega + \nu = x$.

ensures that integration does not take place over the whole product space $(\mathbb{S}^2)^2$ but only over the one-dimensional subset

$$\Gamma_x := \{(\omega, \nu) \in (\mathbb{S}^2)^2 : \omega + \nu = x\}, \tag{4}$$

which of course depends on x; see Figure 3.

Then we can define the convolution product as

$$(f\sigma * g\sigma)(x) := \int_{(\mathbb{S}^2)^2} f(\omega) g(\nu)\, \boldsymbol{\delta}\big(x - \omega - \nu\big)\, \mathrm{d}\sigma(\omega)\, \mathrm{d}\sigma(\nu). \tag{5}$$

To compute the integral in (5), the following question becomes relevant: In how many ways can a given $x \in \mathbb{R}^3$ be expressed as the sum of two unit vectors? If the length of x is greater than 2, then x can never be the sum of two unit vectors, and so $(f\sigma * g\sigma)(x) = 0$ if $|x| > 2$. But what happens if $|x| \leq 2$?

Arguably the simplest non-zero function on $\mathbb{S}^2$ is the constant function $\mathbf{1}$, which assigns the value 1 to every unit vector. Specializing (5) to $f = g = \mathbf{1}$, it follows from Archimedes' approach that $(\mathbf{1}\sigma * \mathbf{1}\sigma)(x)$ goes to infinity as x tends to 0; more precisely:

$$|x|(\mathbf{1}\sigma * \mathbf{1}\sigma)(x) = 2\pi, \text{ for all } |x| \leq 2. \tag{6}$$

To see this, start by noting that the function $\mathbf{1}\sigma * \mathbf{1}\sigma$ is radial, that is, constant on spheres around the origin, so it suffices to multiply it by an arbitrary radial function $\Phi\colon \mathbb{R}^3 \to \mathbb{C}$ and integrate. Given such a $\Phi\colon \mathbb{R}^3 \to \mathbb{C}$, there exists a function $\phi\colon \mathbb{R} \to \mathbb{C}$ such that $\Phi(x) = \phi(|x|)$ for all $x \in \mathbb{R}^3$. Using also the invariance of the surface measure σ under arbitrary rotations and working in spherical coordinates,

$$\begin{aligned}\int_{\mathbb{R}^3} \phi(|x|)(\mathbf{1}\sigma * \mathbf{1}\sigma)(x)\,\mathrm{d}x &= \int_{\mathbb{R}^3} \phi(|x|) \int_{(\mathbb{S}^2)^2} \boldsymbol{\delta}\big(x-\omega-\nu\big)\,\mathrm{d}\sigma(\omega)\,\mathrm{d}\sigma(\nu)\,\mathrm{d}x \\ &= \int_{(\mathbb{S}^2)^2} \phi(|\omega+\nu|)\,\mathrm{d}\sigma(\omega)\,\mathrm{d}\sigma(\nu) \\ &= \int_{\mathbb{S}^2}\int_0^{2\pi}\int_0^{\pi} \phi(\sqrt{2+2\cos\varphi})\sin\varphi\,\mathrm{d}\varphi\,\mathrm{d}\theta\,\mathrm{d}\sigma(\nu) \\ &= 2\pi \int_{\mathbb{S}^2}\int_0^2 \phi(r) r\,\mathrm{d}r\,\mathrm{d}\sigma(\nu) \\ &= \int_{|x|\le 2} \phi(|x|)\frac{2\pi}{|x|}\,\mathrm{d}x,\end{aligned}$$

where the change of variables $r=\sqrt{2+2\cos\varphi}$ was used to pass to the fourth line, and Archimedes' approach helps us recognize that $r^2\,\mathrm{d}r\,\mathrm{d}\sigma(\nu)=\mathrm{d}x$. This implies

$$\int_{|x|\le 2} \phi(|x|)\left((\mathbf{1}\sigma * \mathbf{1}\sigma)(x) - \frac{2\pi}{|x|}\right)\mathrm{d}x = 0, \text{ for all } \phi.$$

For $\phi(|x|) = (\mathbf{1}\sigma * \mathbf{1}\sigma)(x) - \frac{2\pi}{|x|}$, we obtain $\int_{|x|\le 2}(\phi(|x|))^2\,\mathrm{d}x = 0$, so $\phi(|x|)$ must be 0 for every $|x|\le 2$ and identity (6) follows at once.

What happens if f and g are not constant?

4 A sharp inequality for positive functions

The map $f \mapsto f\sigma * f\sigma$ takes non-negative square-integrable functions on $\mathbb{S}^2$ to non-negative square-integrable functions on $\mathbb{R}^3$. A generalization of (6) is then still true, even if f is not constant. Quantitatively, the inequality

$$\int_{\mathbb{R}^3} (f\sigma * f\sigma)^2(x)\,\mathrm{d}x \le 2\pi\left(\int_{\mathbb{S}^2} f^2(\omega)\,\mathrm{d}\sigma(\omega)\right)^2 \tag{7}$$

holds for every function f for which the right-hand integral is finite. If 2π is replaced by any strictly smaller real number, then (7) no longer holds in general. Inequality (7) is thus said to be *sharp*, and the constant 2π is *optimal* for (7). The goal of this section is to confirm these assertions.

This is by far the most technical part of the snapshot, and the heavy computations that follow in the rest of this section may be skipped on a first reading.

It follows from the definition in (5) with $f = g$ that

$$
\begin{aligned}
&\int_{\mathbb{R}^3} (f\sigma * f\sigma)^2(x)\,\mathrm{d}x \\
&\qquad = \int_{\mathbb{R}^3}\int_{(\mathbb{S}^2)^4} \boldsymbol{\delta}\big(x-\omega_1-\omega_2\big)\,\boldsymbol{\delta}\big(x-\omega_3-\omega_4\big)\prod_{j=1}^{4} f(\omega_j)\,\mathrm{d}\sigma(\omega_j)\,\mathrm{d}x \\
&\qquad = \int_{(\mathbb{S}^2)^4} \boldsymbol{\delta}\big(\omega_1+\omega_2-\omega_3-\omega_4\big)\prod_{j=1}^{4} f(\omega_j)\,\mathrm{d}\sigma(\omega_j).
\end{aligned}
\tag{8}
$$

To go from the second to the third line, we switched the order of integration and used an integral version of the following observation: if $x = \omega_1 + \omega_2$ and $x = \omega_3 + \omega_4$, then $\omega_1 + \omega_2 = \omega_3 + \omega_4$.

Another simple but useful remark is that, given arbitrary real numbers A and B, we always have $(A - B)^2 \geq 0$, which can be rewritten as

$$AB \leq \frac{A^2 + B^2}{2}, \text{ for all } A, B \in \mathbb{R}, \tag{9}$$

with equality if and only if $A = B$. In turn, this implies

$$\left|\frac{ac+bd}{2}\right| \leq \sqrt{\frac{a^2+b^2}{2}}\sqrt{\frac{c^2+d^2}{2}}, \text{ for all } a, b, c, d \in \mathbb{R}. \tag{10}$$

This can be proven by squaring both sides of (10), multiplying them out, and applying (9) with $A = ad$ and $B = bc$.

Specializing inequality (10) to $(a, b, c, d) = (f(\omega_2), f(-\omega_2), f(\omega_4), f(-\omega_4))$, changing variables ω_2 to $-\omega_2$ and ω_4 to $-\omega_4$, and repeating this procedure for ω_1 and ω_3 reveals that the term in (8) cannot decrease if f is replaced by a more symmetric version $f_\sharp$, defined as[6]

$$f_\sharp(\omega) := \sqrt{\frac{f^2(\omega) + f^2(-\omega)}{2}}.$$

Thus we may further assume f to be *antipodal*, that is, $f(-\omega) = f(\omega)$ for all $\omega \in \mathbb{S}^2$, and we can continue to estimate (8). For non-negative antipodal functions f, we have

$$
\begin{aligned}
\int_{\mathbb{R}^3} (f\sigma * f\sigma)^2(x)\,\mathrm{d}x &= \int_{(\mathbb{S}^2)^4} \boldsymbol{\delta}\Big(\sum_{j=1}^{4}\omega_j\Big)\prod_{j=1}^{4} f(\omega_j)\,\mathrm{d}\sigma(\omega_j) \\
&= \frac{3}{4}\int_{(\mathbb{S}^2)^4} |\omega_1+\omega_2|^2\,\boldsymbol{\delta}\Big(\sum_{j=1}^{4}\omega_j\Big)\prod_{j=1}^{4} f(\omega_j)\,\mathrm{d}\sigma(\omega_j),
\end{aligned}
$$

[6] Note that the squared integral of f equals that of $f_\sharp$.

where the passage between lines follows from identity (2). Indeed, the Dirac delta ensures that integration takes place over the set of 4-tuples $(\omega_1,\omega_2,\omega_3,\omega_4)\in(\mathbb{S}^2)^4$ *which sum to zero.* An application of the Cauchy-Schwarz inequality then yields

$$\begin{aligned}
&\int_{\mathbb{R}^3}(f\sigma * f\sigma)^2(x)\,\mathrm{d}x\\
&\quad\le \frac{3}{4}\int_{(\mathbb{S}^2)^4}|\omega_1+\omega_2|^2\,\boldsymbol{\delta}\Big(\sum_{j=1}^4\omega_j\Big)f^2(\omega_1)f^2(\omega_2)\mathbf{1}(\omega_3)\mathbf{1}(\omega_4)\prod_{j=1}^4\mathrm{d}\sigma(\omega_j)\\
&\quad= \frac{3}{4}\int_{(\mathbb{S}^2)^2}|\omega_1+\omega_2|^2(\mathbf{1}\sigma*\mathbf{1}\sigma)(\omega_1+\omega_2)f^2(\omega_1)f^2(\omega_2)\,\mathrm{d}\sigma(\omega_1)\mathrm{d}\sigma(\omega_2)\\
&\quad= \frac{3\pi}{2}\int_{(\mathbb{S}^2)^2}|\omega_1+\omega_2|f^2(\omega_1)f^2(\omega_2)\,\mathrm{d}\sigma(\omega_1)\mathrm{d}\sigma(\omega_2),
\end{aligned}$$

where the latter two identities respectively follow from (5) and (6). For the final step, we need some more advanced mathematics. We will now verify that the quadratic form

$$Q(g) := \int_{(\mathbb{S}^2)^2}|\omega+\nu|g(\omega)g(\nu)\,\mathrm{d}\sigma(\omega)\,\mathrm{d}\sigma(\nu)$$

satisfies for any antipodal function g the following sharp inequality:

$$\frac{Q(g)}{\big(\int_{\mathbb{S}^2}g(\omega)\,\mathrm{d}\sigma(\omega)\big)^2}\le\frac{Q(\mathbf{1})}{\big(\int_{\mathbb{S}^2}\mathbf{1}(\omega)\mathrm{d}\sigma(\omega)\big)^2}=\frac{4}{3}. \tag{11}$$

Indeed, when we replace $g=f^2$ and plug (11) into the previous computation, we obtain (7). The most important part of (11) is the inequality step, since the exact value $4/3$ on the right-hand side follows from a routine computation which we do not carry out here.

The idea[7] is to exploit the maximal *symmetry* of the function $\mathbf{1}$. Symmetry is the property of an object to be left invariant by certain transformations: the larger the number of transformations that leave an object invariant, the more symmetric it is. For the case at hand, for a fixed $\tilde{\nu}\in\mathbb{S}^2$, we consider the *reflection* $\pi_{\tilde{\nu}}\omega := \omega - 2(\omega\cdot\tilde{\nu})\tilde{\nu}$ (see Figure 4) and define

$$g_{\tilde{\nu}}(\omega) := \frac{1}{2}\left(g(\omega)+g(\pi_{\tilde{\nu}}\omega)\right).$$

[7] Here we take a detour from Foschi's original approach and present a different and more elementary argument.

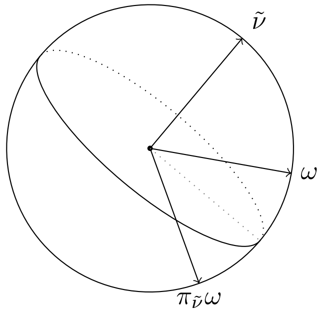

Figure 4: Reflection around the plane perpendicular to $\tilde{\nu}$ through the origin.

Note that $\mathbf{1}_{\tilde{\nu}} = \mathbf{1}$ for all $\tilde{\nu} \in \mathbb{S}^2$, and all functions g that satisfy $g_{\tilde{\nu}} = g$ for all $\tilde{\nu} \in \mathbb{S}^2$ are constant multiples of $\mathbf{1}$. Moreover, since

$$\int_{\mathbb{S}^2} g(\pi_{\tilde{\nu}}\omega)\,\mathrm{d}\sigma(\omega) = \int_{\mathbb{S}^2} g(\omega)\,\mathrm{d}\sigma(\omega),$$

we can check that $\int_{\mathbb{S}^2} g\,\mathrm{d}\sigma = \int_{\mathbb{S}^2} g_{\tilde{\nu}}\,\mathrm{d}\sigma$. So if we verify

$$Q(g) \leq Q(g_{\tilde{\nu}}) \tag{12}$$

for arbitrary $\tilde{\nu}$, it will follow that the left-hand side of (11) must necessarily be maximized by a function g that satisfies $g = g_{\tilde{\nu}}$ for all $\tilde{\nu}$. As this implies that the maximizer is a multiple of $\mathbf{1}$, we have proven (11).

To prove (12), we compute that, for an arbitrary g, $Q(g_{\tilde{\nu}}) - Q(g)$ equals

$$\begin{aligned}
&\int_{(\mathbb{S}^2)^2} |\omega+\nu| \left(\frac{1}{4}\Big(g(\omega)+g(\pi_{\tilde{\nu}}\omega)\Big)\Big(g(\nu)+g(\pi_{\tilde{\nu}}\nu)\Big) - g(\omega)g(\nu)\right) \mathrm{d}\sigma(\omega)\mathrm{d}\sigma(\nu)\\
=&\frac{1}{2}\int_{(\mathbb{S}^2)^2} |\omega+\nu| g(\pi_{\tilde{\nu}}\omega)g(\nu)\,\mathrm{d}\sigma(\omega)\mathrm{d}\sigma(\nu) - \frac{1}{2}\int_{(\mathbb{S}^2)^2} |\omega+\nu| g(\omega)g(\nu)\,\mathrm{d}\sigma(\omega)\mathrm{d}\sigma(\nu)\\
=&\frac{1}{2}\int_{(\mathbb{S}^2)^2} \left(|\pi_{\tilde{\nu}}\omega+\nu| - |\omega+\nu|\right) g(\omega)g(\nu)\,\mathrm{d}\sigma(\omega)\mathrm{d}\sigma(\nu), \qquad (13)
\end{aligned}$$

where we change the variable ω to $\pi_{\tilde{\nu}}\omega$ in the first integral in the second line. Now notice that $|\omega+\nu| = \sqrt{|\omega+\nu|^2} = \sqrt{2+2t}$ for $t = \omega\cdot\nu$. Since ω and ν are unit vectors, $|t| \leq 1$, and we may apply the generalized binomial theorem:

$$\sqrt{2}\sqrt{1+t} = \sqrt{2}\sum_{n=0}^{\infty}\binom{\frac{1}{2}}{n}t^n, \qquad \text{for } |t| \leq 1. \tag{14}$$

Here the important property of the generalized binomial coefficient $\binom{1/2}{n}$ is that

$$\binom{\frac{1}{2}}{n} := \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\dots(\frac{1}{2}-n+1)}{n!} < 0 \text{ for } n = 2, 4, 6, \dots \tag{15}$$

Plugging (14) into (13), we obtain

$$Q(g_{\tilde{\nu}}) - Q(g) = \frac{\sqrt{2}}{2}\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{2k}\int_{(\mathbb{S}^2)^2}\left((\pi_{\tilde{\nu}}\omega\cdot\nu)^{2k} - (\omega\cdot\nu)^{2k}\right) g(\omega)g(\nu)\, \mathrm{d}\sigma(\omega)\mathrm{d}\sigma(\nu).$$

All summands corresponding to odd n vanished because of the antipodal property $g(-\nu) = g(\nu)$.

By rotating the coordinate axes, we can assume that $\tilde{\nu} = (0, 0, 1)$. Write $\omega = (\omega_x, \omega_y, \omega_z)$, so that $\pi_{\tilde{\nu}}\omega = (\omega_x, \omega_y, -\omega_z)$, and similarly for ν. Then $\pi_{\tilde{\nu}}\omega \cdot \nu = \omega_x\nu_x + \omega_y\nu_y - \omega_z\nu_z$, and with the multinomial generalization of the binomial theorem, we have

$$(\pi_{\tilde{\nu}}\omega \cdot \nu)^{2k} - (\omega \cdot \nu)^{2k} = \sum_{|\alpha|=2k}\left((-1)^{\alpha_z} - 1\right)\binom{2k}{\alpha}\omega^{\alpha}\nu^{\alpha}. \tag{16}$$

Here, we use the *multi-index notation*: the letter α denotes a triple $(\alpha_x, \alpha_y, \alpha_z)$ of non-negative integers, and $|\alpha| = \alpha_x + \alpha_y + \alpha_z$; moreover,

$$\binom{2k}{\alpha} := \frac{(2k)!}{\alpha_x!\alpha_y!\alpha_z!}, \quad \omega^{\alpha} := \omega_x^{\alpha_x}\omega_y^{\alpha_y}\omega_z^{\alpha_z}, \quad \nu^{\alpha} := \nu_x^{\alpha_x}\nu_y^{\alpha_y}\nu_z^{\alpha_z}.$$

Notice that $(-1)^{\alpha_z} - 1 = 0$ if α_z is even, otherwise it equals -2. We conclude

$$\begin{aligned} Q(g_{\tilde{\nu}}) - Q(g) &= \frac{\sqrt{2}}{2}\sum_{k=1}^{\infty}\binom{\frac{1}{2}}{2k}\cdot(-2)\sum_{\substack{|\alpha|=2k \\ \alpha_z \text{ odd}}}\binom{2k}{\alpha}\int_{\mathbb{S}^2}\omega^{\alpha}g(\omega)\mathrm{d}\sigma(\omega)\int_{\mathbb{S}^2}\nu^{\alpha}g(\nu)\,\mathrm{d}\sigma(\nu) \\ &= \sum_{k=1}^{\infty}\sum_{\substack{|\alpha|=2k \\ \alpha_z \text{ odd}}} -\sqrt{2}\binom{\frac{1}{2}}{2k}\binom{2k}{\alpha}\left(\int_{\mathbb{S}^2}\omega^{\alpha}g(\omega)\mathrm{d}\sigma(\omega)\right)^2. \end{aligned}$$

By (15), the right-hand side is a sum of non-negative numbers. Therefore $Q(g_{\tilde{\nu}}) - Q(g) \ge 0$, which proves (12) as we wanted.

This concludes the verification of (7). The inequality is indeed sharp and the constant 2π is optimal since the choice $f = \mathbf{1}$ turns every single step in the proof into an equality.

5 On the other side of the (Fourier) mirror

The *Fourier transform* of a function $F\colon \mathbb{R}^3 \to \mathbb{C}$ is a function $\widehat{F}\colon \mathbb{R}^3 \to \mathbb{C}$, defined by

$$\widehat{F}(\xi) := \int_{\mathbb{R}^3} e^{ix\cdot\xi} F(x)\,\mathrm{d}x. \tag{17}$$

The Fourier transform has a number of nice properties that turn it into a powerful tool to tackle problems in mathematical analysis and beyond. For instance, the Fourier transform of a square-integrable function is still square-integrable, and the corresponding integrals are a constant multiple of each other:

$$\int_{\mathbb{R}^3} |\widehat{F}(\xi)|^2\,\mathrm{d}\xi = (2\pi)^3 \int_{\mathbb{R}^3} |F(x)|^2\,\mathrm{d}x. \tag{18}$$

For functions $f\colon \mathbb{S}^2 \to \mathbb{C}$ like the ones we considered in Section 4, we define the Fourier transform of the measure $f\sigma$ at a given $\xi \in \mathbb{R}^3$ as

$$\widehat{f\sigma}(\xi) := \int_{\mathbb{S}^2} e^{i\omega\cdot\xi} f(\omega)\,\mathrm{d}\sigma(\omega). \tag{19}$$

The Fourier transform takes convolutions into products, as the following calculation reveals:

$$\begin{aligned}
(\widehat{f\sigma * g\sigma})(\xi) &= \int_{\mathbb{R}^3} e^{ix\cdot\xi}(f\sigma * g\sigma)(x)\,\mathrm{d}x\\
&= \int_{(\mathbb{S}^2)^2} \left(\int_{\mathbb{R}^3} e^{ix\cdot\xi}\,\boldsymbol{\delta}\big(x-\omega-\nu\big)\,\mathrm{d}x\right) f(\omega)g(\nu)\,\mathrm{d}\sigma(\omega)\,\mathrm{d}\sigma(\nu)\\
&= \int_{(\mathbb{S}^2)^2} e^{i(\omega+\nu)\cdot\xi} f(\omega)g(\nu)\,\mathrm{d}\sigma(\omega)\,\mathrm{d}\sigma(\nu)\\
&= \widehat{f\sigma}(\xi)\widehat{g\sigma}(\xi).
\end{aligned}$$

Here, we used definition (17) of the Fourier transform and definition (5) of the convolution product, switched the order of integration, and in the last step appealed to definition (19).

The basic Fourier theory just developed already has interesting consequences for the sharp inequality (7). Note that $\widehat{f\sigma * f\sigma} = (\widehat{f\sigma})^2$, and so by (18) the left-hand side of (7) can be rewritten as

$$(2\pi)^3 \int_{\mathbb{R}^3} (f\sigma * f\sigma)^2(x)\,\mathrm{d}x = \int_{\mathbb{R}^3} |\widehat{f\sigma * f\sigma}|^2(\xi)\,\mathrm{d}\xi = \int_{\mathbb{R}^3} |\widehat{f\sigma}|^4(\xi)\,\mathrm{d}\xi,$$

where the last identity follows from the first part of this snapshot's title: $4 = 2\times 2$. The conclusion is that inequality (7) is equivalent to

$$\int_{\mathbb{R}^3} |\widehat{f\sigma}|^4(\xi)\,\mathrm{d}\xi \le (2\pi)^4 \left(\int_{\mathbb{S}^2} f^2(\omega)\,\mathrm{d}\sigma(\omega)\right)^2, \tag{20}$$

which is the sharp version of the celebrated *Stein–Tomas inequality* on $\mathbb{S}^2$. The previous reasoning reveals that constant functions are the unique real-valued *maximizers*, that is, non-zero functions which turn (20) into an equality.

6 Other dimensions

A version of the Stein–Tomas inequality exists on each d-dimensional sphere,

$$\mathbb{S}^d := \{x \in \mathbb{R}^{d+1} : |x| = 1\},$$

see [5, 7]. However, the value of the corresponding optimal constant and the nature of maximizers (if they exist at all) remain a mystery which is the subject of ongoing mathematical research; see [4, §2].

7 Further connections

The Stein–Tomas inequality (20) can be rewritten in adjoint form as

$$\int_{\mathbb{S}^2} |\widehat{F}(\xi)|^2 \,\mathrm{d}\sigma(\xi) \leq (2\pi)^2 \left(\int_{\mathbb{R}^3} |F(x)|^{\frac{4}{3}} \,\mathrm{d}x \right)^{\frac{3}{2}}. \tag{21}$$

Then it becomes a statement about the possibility of *restricting* the Fourier transform of a p-integrable function $F \colon \mathbb{R}^3 \to \mathbb{C}$ to the unit sphere $\mathbb{S}^2 \subset \mathbb{R}^3$, at least when $p = 4/3$. This is only possible because the sphere is *curved*, which was perceived as a very surprising phenomenon when E. M. Stein first observed it in the 1960s, later giving rise to a fertile ground for research in modern harmonic analysis that goes by the name of *Fourier restriction theory*.

In turn, Fourier restriction theory has connections to many other central problems in mathematics. For instance, Roth's theorem in the primes (that is, the statement that any set containing a positive proportion of the primes contains a 3-term arithmetic progression) was proved in [3] by adapting the Stein–Tomas argument to establish a *restriction theorem for the primes*. As a second example, the so-called *restriction conjecture* in $\mathbb{R}^d$ implies that Kakeya sets (that is, sets containing a unit line segment in every possible direction) have full Hausdorff dimension equal to d; we refer to the Snapshot 6/2020 "Rotating needles, vibrating strings, and Fourier summation" [8] for further information on the Kakeya problem and its links to harmonic analysis, and bid farewell for now.

Image credits

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