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Waves and incidences

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The wave equation in Euclidean spaces describes many natural phenomena such as sound, light, or water waves. We explore how its solutions are related to the geometric problem of how long thin cylinders can intersect each other and discuss a related open problem.

1 Building solutions to the wave equation

Wave propagation is a fundamental phenomenon in science and engineering with applications to diverse fields such as telecommunication, signal processing, and medical imaging. It is a way of transferring energy through matter or space via oscillations or vibrations. In this snapshot, we explore a fascinating question regarding the various patterns that can arise as waves propagate through space. Despite much recent progress, said question remains open, and we will try to understand why.

For concreteness, let us focus on waves propagating through three spatial dimensions (say, sound waves). We denote by $u(x,t)$ the air pressure at a point $x \in \mathbb{R}^3$ and time $t \in \mathbb{R}$. A sound wave is described by the equation

$$\partial_t^2 u = \Delta u,$$

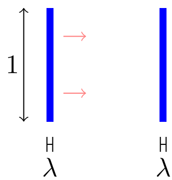
where Δ is the Laplacian, given by

$$\Delta u = (\partial_{x_1}^2 + \partial_{x_2}^2 + \partial_{x_3}^2) u.$$

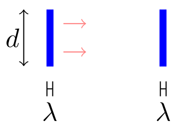
[1] I am partially supported by a Future Fellowship FT200100399 from the Australian Research Council. The presentation here is influenced in many ways by lectures given by Larry Guth. I would like to thank him for allowing some of his ideas to be reproduced here.

Intuitively, the Laplacian Δ measures the average difference of a function over small circles with respect to their center. It will be instructive to picture some solutions to the wave equation for $0 \le t \le 2$. In fact, we will start with some simple solutions, from which more complicated solutions can be built via the *principle of superposition.*

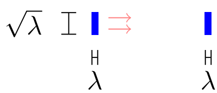
Let $0 < \lambda \ll 1$, meaning that λ is much smaller than 1. Imagine a loudspeaker with a diaphragm that takes the shape of a disc of unit diameter. If the diaphragm oscillates once over a distance of λ, it can produce a high pressure zone that is concentrated along a thin cylindrical slab of diameter ~ 1 and thickness $\sim \lambda$ (similar in shape to a coin). This creates a plane wave that propagates along a direction perpendicular to the diaphragm (at unit speed). The wave's *amplitude* encodes the maximum value of the signal when we measure the wave. In this example, the amplitude at time t is measured by the maximum pressure at that time. If our wave's initial amplitude at $t = 0$ is 1, then the amplitude remains ~ 1 for $0 \le t \le 2$.



In this example, the diaphragm took the shape of a disc of unit diameter. What if its diameter d is smaller? Then it should produce a plane wave concentrated along a thin cylindrical slab of diameter $\sim d$ and thickness $\sim \lambda$, *as long as d is not too small*, since otherwise the plane source becomes approximately a point source and we get a spherical rather than a plane wavefront.



It turns out that the threshold is at $d = \sqrt{\lambda}$: if the diaphragm is a disc of diameter $\sqrt{\lambda}$ and oscillates over a distance λ, then it produces, for $0 \le t \le 2$, a plane wave concentrated along a thin cylindrical slab of diameter $\sim \sqrt{\lambda}$ and thickness λ. This is a solution to the wave equation.



The initial data to such a solution of the wave equation is called a wave packet. A *wave packet* always travels in a direction perpendicular to the plane along which it is concentrated. The amplitude of the wave packet that we use below will be ~ 1, so that, as it travels through space, the amplitude will be ~ 1 for all $0 \le t \le 2$.

2 Local smoothing in three spatial dimensions

The plane waves in the examples above are *superpositions* of simple plane waves of the form $\cos(x \cdot \xi \pm t|\xi|)$ and $\sin(x \cdot \xi \pm t|\xi|)$ with frequency $|\xi| \sim 1/\lambda$. This means that the plane waves can be decomposed as sums of simple plane waves of the aforementioned form; this is a special feature of the wave equation. An outstanding open question concerns the possible concentration in space of waves with frequency $\sim 1/\lambda$ over the time interval $1 \le t \le 2$.

To quantify this, let u be a superposition of plane waves of frequency $\sim 1/\lambda$ and consider the family of energies

$$E_p u(t) := \int_{\mathbb{R}^3} |u|^p + |\lambda\, \partial_t u|^p dx \quad \text{for } p \ge 2.$$

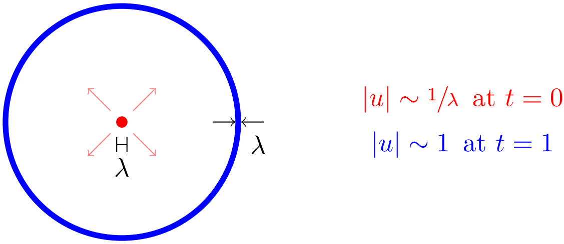
It can be shown that $E_2 u(t)$ remains approximately constant as t varies (this is essentially equivalent to the physical principle of conservation of energy). However, as p grows, $E_p u(t)$ detects the concentration of u in space at time t: if the solution u is concentrated in a very small region in space at a given time t_0, then $E_p u(t_0)$ becomes huge as p grows. To see this heuristically, notice first that $E_p u(t_0)$ is roughly $E_2 u(t_0)$ times the $(p-2)^{\text{th}}$ power of the amplitude of u at time t_0. Now, u concentrating strongly at time t_0 just means that its amplitude is huge at time t_0. Therefore, since $E_2 u(t_0)$ is essentially a constant independent of t_0, the value of $E_p u(t_0)$ becomes huge when p is much bigger than 2!

One way of phrasing our earlier open question is to ask how large $\int_1^2 E_p u(t) dt$ can possibly be relative to $E_p u(0)$. A famous conjecture of Sogge [26], known as the *local smoothing conjecture* for the wave equation, suggests that for every arbitrarily small but positive parameter $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ such that

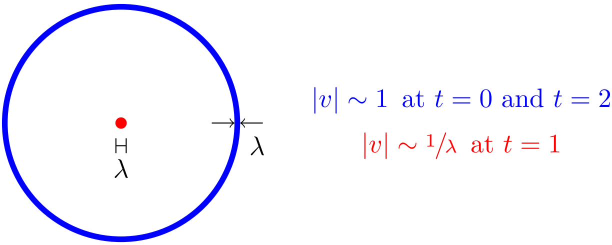
$$\int_1^2 E_p u(t) dt \le C_\varepsilon \lambda^{-(p-3+\varepsilon)} E_p u(0) \quad \text{for } p \ge 3 \tag{1}$$

whenever u is a wave of frequency $\sim 1/\lambda$ in three spatial dimensions.

To shed some light on the difficulty of this conjecture, it may be illuminating to know that $E_p u(t)$ can be huge at any given time $1 \le t \le 2$. For instance, imagine a small explosion in the air that produces an initial high pressure zone in a small ball of diameter λ at time $t = 0$. This produces a single wavefront with "thickness" of one wavelength λ, travelling away from the source at the center (at unit speed) with an amplitude that decreases over time. As such, it is also a wave of frequency $\sim {}^1\!/\!_\lambda$. If the initial amplitude at $t = 0$ is of height $\sim {}^1\!/\!_\lambda$, then conservation of energy shows that the amplitude at $t = 1$ is of height ~ 1. Let us temporarily denote this solution by $u(x,t)$.



Imagine making a video of this solution and playing it backwards in time. What we see is, then, the time-reversed solution $v(x,t) := u(x, 1-t)$; it still solves the wave equation and it can be set up so that $\partial_t v(x,1) = 0$, in which case $v(x,t) = v(x, 2-t)$ holds. Thus, the solution first concentrates into a tiny ball of diameter λ as t goes from 0 to 1, and subsequently disperses back again into its original position at $t = 0$ as t goes from 1 to 2.[2]



It turns out that $E_p v(1) \sim \lambda^{-(p-2)} E_p v(0)$ for all $p \geq 2$, which is big compared to $E_p v(0)$ (recall ${}^1\!/\!_\lambda \gg 1$).[3] Fortunately, $E_p v(t)$ does not stay big for a very

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[2] Indeed, since a two-dimensional sphere of radius 1 can be covered by $\sim {}^1\!/\!_\lambda$ discs of diameter $\sqrt{\lambda}$, we can think of the initial data on the blue shell around the unit sphere as the superposition of $\sim {}^1\!/\!_\lambda$ many wave packets, and these $\sim {}^1\!/\!_\lambda$ many wave packets would interfere constructively at the red ball at time $t = 1$, producing a peak of height $\sim {}^1\!/\!_\lambda$.

[3] In fact, a classic result of Peral [23] and Miyachi [22] says that this is essentially the worst possible case: they independently showed that for any $0 < \lambda \ll 1$ and any solution u to the wave equation in three spatial dimensions with frequency $\sim {}^1\!/\!_\lambda$, one has

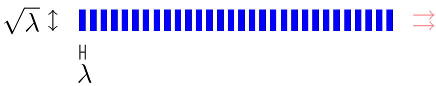
$$E_p u(1) \lesssim \lambda^{-(p-2)} E_p u(0)$$

for $p \geq 2$. This was extended by Seeger, Sogge and Stein [24] to a more general class of wave equations (e.g. waves in a smooth inhomogeneous medium), and much more recently long time as t varies from 1 to 2; as a result, there is still a constant C such that

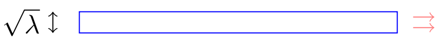
$$\int_1^2 E_p v(t)dt \leq C E_p v(0)$$

for $p \geq 3$, which is consistent with the local smoothing conjecture (1). But to prove the conjecture (1), we must show that we are *always* fortunate; this is part of the reason why conjecture (1) is hard.

Indeed, some solutions to the wave equation can remain concentrated for a longer time than in the previous example (despite the fact that such solutions are less focussing than the previous example at any given time). To see one such solution, let us first build a so-called wave train by having initial data with many parallel wave packets, all sent to move in the same direction. Such a wave train, consisting of waves of frequency $\sim {}^1/_\lambda$ with wavelength $0 < \lambda \ll 1$, is depicted in the following picture.



For simplicity, we represent a wave train by the long thin cylinder containing it.



A wave u of frequency $\sim {}^1/_\lambda$ can be formed by putting together many wave trains, initially concentrated on disjoint cylinders T of diameter $\sqrt{\lambda}$ and height ~ 2, as illustrated in the following picture, travelling into a common unit square. For any time $1 \leq t \leq 2$, each wave train occupies a fixed cylinder $\widetilde{T}$ of height ~ 1 in the said unit square.

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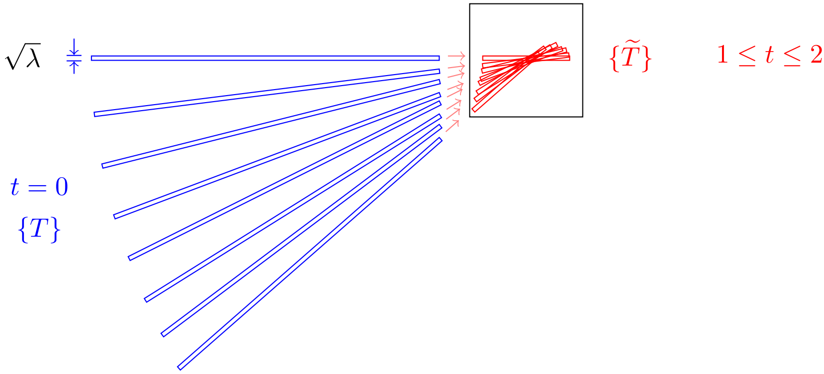
by Frey and Portal [11] and Hassell and Rozendaal [17] to two even more general classes of wave equations (concerning waves in certain rough inhomogeneous media). These latest advances were, in turn, based on a new function space, introduced by Hassell, Portal and Rozendaal [15] (building on earlier work by Smith [25]; see also [16]), that is particularly suited to study solutions to various wave equations.



A result from incidence geometry[4] says that the $\widetilde{T}$s can be arranged to overlap quite a lot more than the initial Ts (which are disjoint): we can have

$$\text{Volume}\Big(\bigcup T\Big) \gtrsim \log \lambda^{-1} \cdot \text{Volume}\Big(\bigcup \widetilde{T}\Big),$$

even though

$$\int_{\cup T} |u(x,0)|^2 dx \sim \int_{\cup \tilde{T}} |u(x,t)|^2 dx$$

for every $1 \leq t \leq 2$. Then, $E_3 u(t) \gtrsim \sqrt{\log \lambda^{-1}} E_3 u(0)$ for every $1 \leq t \leq 2$. As a result,

$$\int_1^2 E_3 u(t) dt \gtrsim \sqrt{\log \lambda^{-1}} E_3 u(0). \tag{2}$$

Now recall the conjecture (1) with $p = 3$. While (2) does not contradict (1), the example (2) does show that when $p = 3$, the conjecture (1) is essentially sharp: it cannot hold when $\varepsilon = 0$.

3 The issue of overlapping thin cylinders

The previous example begs the following question: What if we can produce another pattern of thin cylinders in $\mathbb{R}^3$ that point in separated directions but overlap even more than the previous example? This way, we might produce a counterexample to the local smoothing conjecture!

In other words, in order to prove the local smoothing conjecture, we will have to rule out the possibility of having lots of thin cylinders in $\mathbb{R}^3$ that point in separated directions but overlap significantly more than the previous

[4] See pioneering work of Fefferman [10] and Bourgain [4] for some early realizations of the relevance of incidence geometry in this circle of problems.

example. This is the content of the *Kakeya conjecture*. One form of it states that any collection of cylinders in $\mathbb{R}^3$ with diameter $\sqrt{\lambda}$ and height 1 that point in directions separated by angles that are larger than $\lambda^{1/2}$ cannot overlap too much, meaning that their union's volume is a constant multiple of λ^{ε} for every $\varepsilon > 0$ (with the constant given by an implicit function in ε).

The Kakeya conjecture is open and considered very difficult.[5] It explains partly why the local smoothing conjecture is difficult (because local smoothing implies Kakeya). There is a local smoothing conjecture for every spatial dimension $n \geq 2$, and this difficulty involving incidences is understood in dimension $n = 2$ (see [9]). But it still takes non-trivial effort to establish the local smoothing conjecture in two spatial dimensions: this was only achieved by Guth, Wang and Zhang [14] in 2020.

It is nevertheless possible to make partial progress on the local smoothing conjecture in three spatial dimensions. In fact, (1) was first shown to be true for $p > 74$ by Wolff [28] in 2000, and for $p \geq 4$ by Bourgain and Demeter [6] in 2015. The key was in understanding the possible ways that waves travelling in different directions may superimpose; the resulting interference patterns are quantified by certain so-called *Fourier decoupling inequalities*.[6] They have been in the focus of much recent active research because of their wide applicability to other areas of mathematics, such as analytic number theory, where one seeks to understand the number of solutions to certain Diophantine equations [7], and the Riemann ζ function [5, 8]. See also [12], where inspirations from number theory (particularly the work of Wooley [29] on efficient congruencing) was used to understand Fourier decoupling.

We have seen in this snapshot that the study of wave propagation reveals surprising connections to a vast array of fields, including incidence geometry and Fourier analysis. Tools from these topics have already been used successfully to study problems from number theory. In the case of wave propagation, however, some exciting and fundamental conjectures remain open. This makes wave propagation very much an active and exciting area of research.

[5] It remains open despite much recent progress, by Katz and Zahl [20, 21], Zahl [30], Hickman, Rogers and Zhang [18], building upon earlier work of Bourgain [4], Wolff [27], Katz and Tao [19], Guth and Katz [13], among others.

[6] These inequalities were proved using multilinear incidence estimates such as [3] and perturbed versions of [1, 2].

Image credits

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