Snapshots of modern mathematics
from Oberwolfach

№ 1/2014

Dirichlet Series

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Mathematicians are very interested in prime numbers. In this snapshot, we will discuss some problems concerning the distribution of primes and introduce some special infinite series in order to study them.

1 Convergent Series

A *series* is a sequence of terms added together, *e.g.*

$$1 + 2 + 3 + \ldots + 100,$$

a series with 100 terms.[2] Series may have a finite number of terms, in which case one wants to find a formula that gives the sum, either exactly or approximately; or they may be what we call an infinite series, which means they have an infinite number of terms. We use the notation a_n for the n^{th} term in a series, so we shall be considering series of the form

$$a_1 + a_2 + a_3 + \ldots . \tag{1}$$

How can one make sense of (1)? How can one sum an infinite number of terms? Consider the example $a_n = \frac{1}{2^n}$. This gives the series

$$\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \ldots . \tag{2}$$

[1] Partially supported by National Science Foundation Grant DMS 1300280

[2] This particular sum was supposedly given to Carl Friedrich Gauß (1777–1855) as a schoolboy to keep him occupied; but he found a clever way to find the sum quickly. Can you see how to find it without too much work?

Imagine you have a stick of height 1 stuck in the ground. Now you pile up bricks beside it. The first one has height $\frac{1}{2}$, the second one height $\frac{1}{2^2}$, the third height $\frac{1}{2^3}$, and so on. You see that each time you add a brick, you raise the total height of the stack exactly half the remaining distance to the top of the stick. So if you go on forever, the stack of bricks will never be higher than 1. On the other hand, for any height less than 1, the stack of bricks will eventually be higher than it. So we say the series (2) *converges* to the value 1. This means that if you take enough terms, you can get as close to 1 as you want, though you never quite reach it.[3]

2 Divergent Series

But not all infinite series converge. For example, the series $1+1+1+\ldots$ clearly grows without bound; we say then that this series *diverges*. More subtly, the harmonic series, which is given by $1+\frac{1}{2}+\frac{1}{3}+\ldots$, also diverges. We can see this by grouping:

$$\begin{aligned} & 1+\frac{1}{2}+\left(\frac{1}{3}+\frac{1}{4}\right)+\left(\frac{1}{5}+\frac{1}{6}+\frac{1}{7}+\frac{1}{8}\right)+\ldots \\ \geq\ & 1+\frac{1}{2}+\left(\frac{1}{4}+\frac{1}{4}\right)+\left(\frac{1}{8}+\frac{1}{8}+\frac{1}{8}+\frac{1}{8}\right)+\ldots \\ =\ & 1+\frac{1}{2}+\left(\frac{1}{2}\right)+\left(\frac{1}{2}\right)+\ldots \\ =\ & \infty. \end{aligned}$$

The series $1^2+\frac{1}{2^2}+\frac{1}{3^2}+\ldots$ converges; we can see this by showing that it is term by term smaller than another series, whose sum we can compute and is finite. We have, for every natural number $n \geq 2$:

$$\begin{aligned} & 1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots+\frac{1}{n^2} \\ <\ & 1+\frac{1}{(1)(2)}+\frac{1}{(2)(3)}+\frac{1}{(3)(4)}+\ldots+\frac{1}{(n-1)(n)} \qquad (3) \\ =\ & 1+\left(1-\frac{1}{2}\right)+\left(\frac{1}{2}-\frac{1}{3}\right)+\left(\frac{1}{3}-\frac{1}{4}\right)+\ldots+\left(\frac{1}{n-1}-\frac{1}{n}\right) \\ =\ & 1+1-\frac{1}{n} \qquad (4) \\ \leq\ & 2. \end{aligned}$$

[3] What does $\frac{1}{3}+\frac{1}{3^2}+\frac{1}{3^3}+\ldots$ converge to?

We get (4) by observing that in the previous line there is a lot of cancellation — the term $-\frac{1}{2}$ cancels with $+\frac{1}{2}$, the term $-\frac{1}{3}$ cancels with $+\frac{1}{3}$, and so on until in the second to last term there is a $-\frac{1}{n-1}$ that cancels with $\frac{1}{n-1}$ in the last term. Such a sum is called a *telescoping series*, as it resembles the act of pushing together a telescope, leaving only the first and the last term.

It follows that the infinite series $1^2 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots$ must converge, and moreover that it converges to some number that is less than 2.

Leonhard Euler (1707–1783) calculated the sum exactly in the year 1735. He showed that

$$1^2 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots = \frac{\pi^2}{6}. \tag{5}$$

He also showed that

$$1^4 + \frac{1}{2^4} + \frac{1}{3^4} + \ldots = \frac{\pi^4}{90}. \tag{6}$$

The Riemann zeta function, named in honor of Bernhard Riemann (1826–1866), is defined for all $s > 1$ by

$$\zeta(s) = 1 + \frac{1}{2^s} + \frac{1}{3^s} + \ldots.$$

Euler found a formula for $\zeta(s)$ whenever s is an even positive number. He showed that it is always a rational number times π^s, just as in equations (5) and (6). This raises a natural question: is $\zeta(3)$, the sum of the reciprocals of the cubes, a rational number times π^3? Nobody knows.

3 Dirichlet Series

The Riemann zeta function is the first and most important example of what is now called a *Dirichlet series*, named after the mathematician Johann Peter Gustav Lejeune Dirichlet (1805–1859). A Dirichlet series is obtained by choosing a sequence $a_1, a_2, a_3, \ldots$, and then considering the function f defined by the infinite series

$$f(s) = a_1 + \frac{a_2}{2^s} + \frac{a_3}{3^s} + \ldots.$$

The Riemann zeta function is the Dirichlet series for the sequence $a_1 = 1, a_2 = 1, a_3 = 1, \ldots$.

Dirichlet series arise when studying the distribution of prime numbers. Euclid gave a proof 2300 years ago that there are an infinite number of primes. As 2 is the only even prime, there must be an infinite number of odd primes. But are there an infinite number whose remainder is 1 when you divide them by 3? Dirichlet in 1837 used the theory of what we now call Dirichlet series, which he invented to solve this and similar problems, to prove that if n is any natural number bigger than 1, and r is any number between 1 and $n-1$ that has no factor (except 1) in common with n, then there are an infinite number of prime numbers whose remainder, when divided by n, is r. In formulas, this reads that for any $r \in \{1, \ldots, n-1\}$ with $\gcd(r, n) = 1$, there are infinitely many primes p such that

$$p \equiv r \bmod n,$$

where $\gcd(r, n)$ stands for the greatest common divisor of r and n.

We would like to know, when x is a large number, how many primes there are that are smaller than x. The prime number theorem, which was conjectured by Gauß in 1793 and proved by Jaques Hadamard (1865–1963) and Charles-Jean de la Vallée-Poussin (1866–1962) in 1896, says that this number is approximately $\frac{x}{\ln x}$, where $\ln x$ is the natural logarithm of x.

How good is this approximation? This is one of the greatest unsolved problems in mathematics. It is known that the accuracy of the approximation depends on properties of the Riemann zeta function. The Riemann hypothesis, conjectured by Riemann in 1859, would state that in some sense the approximation is very good. However, despite dedicated efforts of mathematicians all over the world for nearly two centuries, we still don't know if the Riemann hypothesis is true.

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Mathematical subjects
Analysis

DOI
10.14760/SNAP-2014-001-EN

Snapshots of modern mathematics from Oberwolfach are written by participants in the scientific program of the Mathematisches Forschungsinstitut Oberwolfach (MFO). The snapshot project is designed to promote the understanding and appreciation of modern mathematics and mathematical research in the general public worldwide. It is part of the mathematics communication project "Oberwolfach meets IMAGINARY" funded by the Klaus Tschira Foundation and the Oberwolfach Foundation. All snapshots can be found on www.imaginary.org and on www.mfo.de/snapshots.

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