Snapshots of modern mathematics
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Arrangements of lines

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We discuss certain open problems in the context of arrangements of lines in the plane.

1 Introduction

Imagine a finite set of lines in the plane. These lines may intersect each other in certain points – at some points only two lines might meet, at other places three or even more lines might meet. Such a set of lines is called an arrangement A. The arrangement may, for example, just consist of three lines joining vertices P, Q and R of a triangle, see Figure 1.

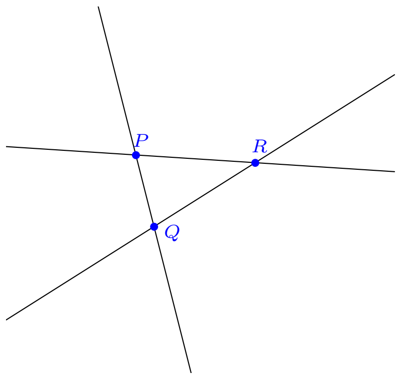

Figure 1: An arrangement of three lines

One can calculate the *self-crossing count* c of the arrangement by counting the number of lines through each intersection point, squaring it, and adding these squares up. In our case, $c = 2^2 + 2^2 + 2^2 = 12$. By subtracting this self-crossing number from the square of the number of lines d, one obtains a number $d^2 - c$, which may be positive, negative, or zero. In our example, $d^2 - c = 3^2 - 12 = -3$. It is easy to find arrangements for which this number $d^2 - c$ is very large, very negative, or close to zero, so the really interesting question is what happens if we divide $d^2 - c$ by the number n of points where two or more lines cross. That is, one is interested in the number $H(A) = (d^2 - c)/n$. In our example, $H(A) = (3^2 - 12)/3 = -1$. It is not hard to find arrangements A for which $H(A)$ is positive, even very large. What is of interest is finding arrangements A for which $H(A)$ is very negative.

It is not known what the minimum value of $H(A)$ is for arrangements of lines A, but it is known (as we discuss below) that $H(A)$ must be greater than -4 and that values of $H(A)$ occur with $H(A) < -2$. Yet it is still an open problem to determine the largest r such that $H(A) \geq r$ for all arrangements A. In Section 2, we will describe some examples of arrangements A with $H(A) < -2$.

This question is a special case of a current research problem in algebraic geometry. More generally, one is interested not only in arrangements of lines, but in algebraic curves C in the plane (i.e., zero sets of polynomials in two variables). For example, the curve could be the union of a circle and a line intersecting each other in two points, see Figure 2. While c and n can be thought of as rough analogs of the case of an arrangement of lines, one has to find a more general definition of the number d: it is the so-called "degree" of the algebraic curve C; in our example $d = 3$ (1 for the line and 2 for the circle). The line crosses the circle in two places, giving $n = 2$ as the number of points where self-crossings occur. The self-crossing count is $2^2 + 2^2 = 8$ since two "branches" of the curve C cross at each of the two crossing points. This gives $H(C) = (3^2 - 8)/2 = 0.5$.

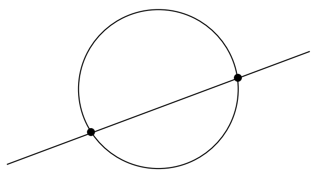

Figure 2: An example of an algebraic curve

Just like for arrangements of lines, it is not known how negative $H(C)$ can be for a curve C. Thus the conceptually simplest simplest case, when C is an arrangement of lines, is already interesting, even more so since currently the most negative known values of $H(C)$ occur in this case. This case is also a nice starting point for the general problem since the technical background needed in this case (such as what an algebraic curve is, and what its degree is) is minimal, yet the problem is still interesting and challenging. So now we look in more detail at it.

2 Arrangements of lines

Figure 3 shows a configuration of 12 lines with 9 double points (i.e., points where exactly 2 lines meet) and 19 triple points (where 3 lines meet). The triple points are indicated by fattened dots in Figure 3.

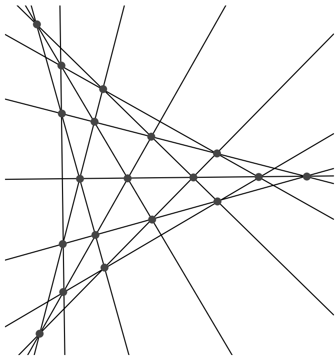

Figure 3: An arrangement with $H(A) < -2$

Here $d = 12$, $n = 9 + 19 = 28$, and $c = 2^2 \cdot 9 + 3^2 \cdot 19 = 207$, so $H(A) = (12^2 - 207)/28 \approx -2.25$. The arrangement of Figure 3 is thus an example of an arrangement with $H(A) < -2$. To study arrangements of lines in general, it is helpful to use the symbol t_k to denote the number of points where k lines meet. The set of data consisting of the number of lines d together with the numbers $t_2, t_3, t_4, \dots, t_d$ is the basic combinatorial data attached to an arrangement. In the example above, we have

$$d = 12,\ t_2 = 9,\ t_3 = 19,\ \text{ and } t_4 = t_5 = t_6 = \cdots = t_{12} = 0.$$

With this notation $n = t_2 + t_3 + t_4 + \cdots + t_d$ and $c = 2^2 t_2 + 3^2 t_3 + 4^2 t_4 + \cdots + d^2 t_d$, so that

$$H(A) = \frac{d^2 - (2^2 t_2 + 3^2 t_3 + 4^2 t_4 + \cdots + d^2 t_d)}{t_2 + t_3 + t_4 + \cdots + t_d}.$$

Our next example of an arrangement is created by taking all lines joining vertices P, Q, R, S, and T of a regular pentagon and, in addition, the axes of symmetry of the pentagon, see Figure 4.

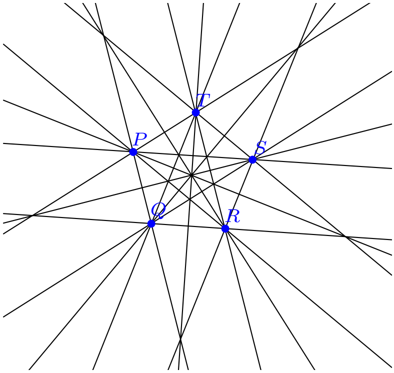

Figure 4: An arrangement with some parallel lines

There are $t_2 = 10$ double points in this configuration. Furthermore, there are also $t_3 = 10$ triple points, $t_5 = 6$ points with multiplicity 5 (the vertices and the center of the pentagon), and no points of other multiplicity. In this case $d = 15$, $n = t_2 + t_3 + t_5 = 10 + 10 + 6 = 26$ and $c = 2^2 \cdot t_2 + 3^2 \cdot t_3 + 5^2 \cdot t_5 = 2^2 \cdot 10 + 3^2 \cdot 10 + 5^2 \cdot 6 = 280$, so $H(A) = (15^2 - 280)/26 \approx -2.12$. This is less negative than our previous example. However, notice that there are five sets of pairs of parallel lines. If we regard a set of parallel lines as "meeting at infinity", this gives five more double "points".[1] Taking into account these "points at infinity" gives

$$d = 15,\ t_2 = 15,\ t_3 = 10, \ \text{ and } \ t_5 = 6.$$

Thus $H(A) \approx -2.42$, a number even lower than in the previous example.[2]

It is natural for mathematicians to wonder if there are any constraints on the values of t_k which can occur for an arrangement of lines. In fact, there are. The first constraint is purely combinatorial. Given an arrangement of d lines, there are $d \cdot (d-1)/2$ pairs of lines, since there are d lines and each line can be paired with $d - 1$ other lines, giving $d \cdot (d - 1)$ pairs, counting each pair twice. Alternatively, each pair of lines determines a crossing point (at infinity if the lines are parallel). When k lines cross at a single point, there are $k \cdot (k - 1)/2$ pairs which cross at that point. Adding up the number of pairs for each crossing

[1] This technique of also considering points at infinity is related to projective geometry.

[2] There were no parallel lines in figure 3 or in the triangle considered in the introduction, so our "new method" does not change $H(A)$ for these arrangements.

point just gives the number of pairs of lines. Thus

$$\frac{d\cdot(d-1)}{2}=\frac{2\cdot 1}{2}\cdot t_2+\frac{3\cdot 2}{2}\cdot t_3+\frac{4\cdot 3}{2}\cdot t_4+\cdots+\frac{d\cdot(d-1)}{2}\cdot t_d. \quad (1)$$

3 The Orchard Problem: are there other constraints?

Other constraints of a more qualitative asymptotic nature have been conjectured, which have only recently been verified. In particular, the number of double points cannot be too small, which means that the number of triple points cannot be too big. A specific manifestation of this is known as the "Orchard Problem", posed by James Joseph Sylvester (1814-1897) in 1893 [5].

Sylvester himself was motivated by an article by John Jackson [4] dating back to the early 19th century and apparently forgotten since then. We quote here the original formulation of the problem due to Jackson:

> *Your aid I want, nine trees to plant*
> *In rows just half a score;*
> *And let there be in each row three*
> *Solve this: I ask no more.*

It is this formulation that led to the problem being named the Orchard Problem, generalized by Sylvester in the following manner:

> *Prove that it is not possible to arrange any finite number of real points so that a right [straight] line through every two of them shall pass through a third, unless they all lie in the same right line.*

It is a common trick to interchange the roles of lines and points: if P is a point and L is a line, just call P from now on a line and L a point and replace statements like "P lies on L" by "P contains L", and "L contains P" by "L lies on P", thus obtaining the *dual statement*. It is a useful fact that a theorem about points and lines in the real plane holds if and only if its dual theorem holds. [3] In our case, the corresponding dual problem reads as follows: prove that it is not possible to arrange any finite number of lines so that each point on any two of the lines shall also be on a third line, unless all of the lines go through the same point. I.e., prove there is no configuration of $d \geq 3$ lines in which $t_2 = 0$ (unless $t_d \neq 0$). Many proofs of this statement have appeared in the literature over the years. Excluding the case where all lines meet in one point, there is no configuration of lines whose intersection points are all at least triple points, there must always be some double points!

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[3] Remember that we are taking intersection points at infinity into account in case there are parallel lines in an arrangement.

Confronted with a result of this kind, mathematicians ask further-reaching questions: Given a configuration of d lines, what is the maximal number of triple points which can be achieved (equivalently, what is the minimal number of double points which must appear)? Surprisingly this problem has been solved only recently, and only for sufficiently large values of d, by Green and Tao [2], who show there must be at least $d/2$ double points when d is large enough.

4 The Bounded Negativity Conjecture

The problem of finding the most negative value of $H(A)$ addressed in the introduction is closely related to the so-called "Bounded Negativity Conjecture". This conjecture was the subject of an Oberwolfach workshop in 2014. It turns out that one can choose values for t_k consistent with the combinatorial constraint (1) such that $H(A)$ becomes arbitrarily negative, and this would be a counterexample to the Bounded Negativity Conjecture if there really were arrangements with the given data.

Luckily there is another constraint on the combinatorial data. This constraint is an inequality obtained in the mid 1980s by German mathematician Friedrich Hirzebruch (1927-2012) by applying highly non-trivial and sophisticated methods of algebraic geometry. Here is Hirzebruch's inequality (but in order to be sure that it holds, one has to assume $t_d = t_{d-1} = 0$) [3, Theorem p. 132]:

$$t_2 + \frac{3}{4}t_3 \geq d + \sum_{k \geq 5}^{d-2} (2k-9)t_k. \tag{2}$$

It is by using this inequality that one can show (with some extra effort) that $H(A) > -4$ for any arrangement A of lines. Thus bounded negativity holds for any arrangement of lines.

It remains an open and intriguing problem to decide which combinatorial data allowed by the constraints (1) and (2) come from actual arrangements of lines.

Arrangements A with $H(A) < -2$ also unexpectedly often seem to give counterexamples to a conjecture in commutative algebra (involving containment of symbolic powers of certain ideals in ordinary powers of the ideals), which was also a topic of discussion at the 2014 Oberwolfach workshop, and some of these counter-examples were previously unknown (see [1] for the first known counter-example).

In view of the importance of such configurations in various branches of contemporary mathematics, we consider understanding them better to be a challenging problem of high importance.

References

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[2] B. Green and T. Tao, *On sets defining few ordinary lines*, Discrete Comput. Geom. **50** (2013), 409–468.

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