

# The Kadison-Singer problem

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In quantum mechanics, unlike in classical mechanics, one cannot make precise predictions about how a system will behave. Instead, one is concerned with mere probabilities. Consequently, it is a very important task to determine the basic probabilities associated with a given system. In this snapshot we will present a recent uniqueness result concerning these probabilities.

## 1 The mathematical formulation of quantum mechanics

Before the invention of quantum mechanics by Erwin Schrödinger (1887 – 1961) and Werner Heisenberg (1901 – 1976), physicists were describing motion by means of classical mechanics.<sup>□</sup> In that framework, the state of a particle at a given instant of time is completely described by its position (given by 3 spatial coordinates) and its momentum (3 more coordinates). So, from a classical point of view, all information you can possibly have about a particle is represented by a point in a 6-dimensional vector space, the so-called *phase space*. If you are interested in, say, the position of the particle, you just have to take all the information available and “forget” the momentum of the particle, that is, you *project* the 6-dimensional phase space onto a 3-dimensional subspace.

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<sup>□</sup> The following formulation of classical mechanics was developed by the Anglo-Irish physicist, astronomer, and mathematician Sir William Rowan Hamilton (1805 – 1865).

Mathematically, such a projection is a so-called *linear operator*<sup>[2]</sup>. In the context of physics, it is called an *observable (quantity)*. In the classical perspective, there is in principle no obstacle to measuring both position and momentum very precisely.<sup>[3]</sup>

In quantum mechanics, however, we view things differently. The information about a system is usually represented by a point in an infinite-dimensional vector space! The observables are still linear operators, which we now require to be “self-adjoint” (that is, they fulfil a certain symmetry property).<sup>[4]</sup> Some of these observables are still compatible, meaning that they can be measured simultaneously arbitrarily precisely (remember that in the classical scenario all our observables were compatible). Others, however, are not, for example position and momentum: if you want to make simultaneous predictions about position and momentum, the more precise your results for the position of a point mass are, the less precise get your results for its momentum, and vice versa. This is known as Heisenberg’s *Uncertainty Principle*. It was named after the German physicist and Nobel laureate Werner Heisenberg (1901 – 1976) and limits in a fundamental way the precision with which observables like position and momentum of a particle can be known simultaneously.

How is this encoded in the mathematics of the model? Mathematically, compatibility of observables means that the linear operators commute (that is, it does not matter which operator you apply first to your set of data), while operators of incompatible observables do not commute (you get different results if you reverse the order in which you apply the two linear operators).<sup>[5]</sup>

Does this mean that it is impossible to come to grips with a system in quantum mechanics? No, but quantum mechanics is probabilistic in nature, meaning that what actually *can* be measured are probabilities! That is, while you might not be able to predict the outcome of a measurement of an observable, you can calculate the probability that the observable is in a certain “quantum state”.

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[2] More about linear operators can be found in the following snapshot: Greg Knese, *Operator theory and the singular value decomposition*, Snapshots of modern mathematics (2014), no. 9, 1–7.

[3] Of course the precision of your measurement is limited by the precision of the instruments you use. But this is a different matter.

[4] If you are familiar with matrices, you may think of the operators as infinite-dimensional matrices. A self-adjoint operator is then a symmetric matrix.

[5] In terms of matrices, commutativity means  $A \cdot B = B \cdot A$ .

## 2 An extension problem...

How can we determine the basic probabilities associated with the quantum states of a system? The British physicist Paul Dirac (1902 – 1984) addressed this question in Section 18 of his book [1]. He explicitly gives a procedure:

- Start with a commuting set of observables.
- Enlarge it to a maximal set of commuting observables. Nowadays, we call this a *MASA*<sup>[6]</sup>.
- Specify the probability distributions associated to the commuting observables in the MASA. That is, for each observable in the MASA you need to choose how likely it should be to measure a given value for the observable. Mathematically, this amounts to defining a linear functional<sup>[7]</sup> on the MASA that satisfies some positivity properties; such a functional is called a *pure state*.
- Extend this pure state to *all* observables.

The last step is the interesting one. An extension of a pure state to all observables is always possible, but can there be different extensions? It seems that Dirac was convinced that this final step could be done in a unique way. At the end of the 1950s, it was realized that this was not obvious, and in 1959 Richard Kadison and Isadore Singer tackled the mathematical question whether the extension of a pure state from a MASA to all observables is unique [2].<sup>[8]</sup> As it turns out, the answer depends on what the MASA is like. If, for example, the set of operators you start with contains only projections onto one-dimensional subspaces, then the MASA you obtain is not very large; we call it *atomic*.<sup>[9]</sup> There are also (in a certain sense) more complicated MASAs called *diffuse*. Kadison and Singer showed that if the MASA is diffuse, there are pure states not extending uniquely to all observables, thus proving Dirac’s intuition to be wrong. For atomic MASAs, however, they left the question open.

The question “*Does any pure state of an atomic MASA extend uniquely to all observables?*” became known as the *Kadison-Singer problem*. Over the years, it was proved to be equivalent to a number of other open questions in various fields of mathematics: linear algebra, harmonic analysis, signal theory...<sup>[10]</sup>

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<sup>[6]</sup> This is short for “maximal abelian self-adjoint algebra of observables”.

<sup>[7]</sup> A linear functional is a linear function on a vector space whose values are (real or complex) numbers.

<sup>[8]</sup> This pertains the study of so-called *C\*-algebras*.

<sup>[9]</sup> Thinking of matrices, you may imagine an atomic MASA as containing only diagonal matrices.

<sup>[10]</sup> Names to be quoted here are C. A. Akemann, J. Anderson, N. Weaver, P. Casazza and collaborators.

### 3 ... and its solution!

By an interesting twist in history, the Kadison-Singer problem was solved in June 2013... by three computer scientists [3]! Actually, the Kadison-Singer problem has a positive answer! Adam Marcus, Dan Spielman and Nikhil Srivastava<sup>[11]</sup> proved an equivalent form of the Kadison-Singer problem, which involves only basic linear algebra:

*Given  $\alpha > 0$  and vectors  $v_1, \dots, v_m \in \mathbb{R}^n$  satisfying  $\sum_{i=1}^m \langle v_i, x \rangle^2 = 1$  for every  $\|x\| = 1$  and  $\|v_i\|^2 \leq \alpha$ , there exists a partition  $T_1 \cup T_2$  of  $\{1, 2, \dots, m\}$  satisfying*

$$\left| \sum_{i \in T_j} \langle v_i, x \rangle^2 - \frac{1}{2} \right| \leq 5\sqrt{\alpha}$$

*for every  $\|x\| = 1$  and  $j = 1, 2$ .*

The proof that this statement is equivalent to a positive answer to the Kadison-Singer problem is due to Nik Weaver, see [4]. Marcus, Spielman, and Srivastava actually prove that if you choose the partition in a certain random way, then the probability that  $\left| \sum_{i \in T_j} \langle v_i, x \rangle^2 - \frac{1}{2} \right| \leq 5\sqrt{\alpha}$  for every  $\|x\| = 1$  and  $j = 1, 2$  is larger than zero. So there must exist a partition that fulfils the desired inequality. They deduce Weaver's statement from results on the characteristic polynomial (viewed as a random polynomial) of the sum of  $m$  independent random variables taking values in rank 1 matrices.

It is remarkable that their proof only involves linear algebra, elementary probability theory, differential calculus in several variables and (at one place) a dash of complex analysis.

Summarizing, we know that Dirac's intuition was wrong about diffuse MASAs in general but right for atomic ones.

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<sup>[11]</sup> Marcus is working four days a week doing R&D in a software company, and one day per week at Yale University. Spielman is a professor for theoretical computer science at Yale and a Nevanlinna laureate. Srivastava is working for Microsoft India, and will present the solution of the Kadison-Singer problem at ICM 2014 in Seoul.

## References

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