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How to choose a winner: the mathematics of social choice

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Suppose a group of individuals wish to choose among several options, for example electing one of several candidates to a political office or choosing the best contestant in a skating competition. The group might ask: what is the best method for choosing a winner, in the sense that it best reflects the individual preferences of the group members? We will see some examples showing that many voting methods in use around the world can lead to paradoxes and bad outcomes, and we will look at a mathematical model of group decision making. We will discuss *Arrow's impossibility theorem*, which says that if there are more than two choices, there is, in a very precise sense, **no** good method for choosing a winner.

This snapshot is an introduction to *social choice theory*, the study of collective decision processes and procedures. Examples of such situations include

- voting and election,
- events where a winner is chosen by judges, such as figure skating,
- ranking of sports teams by experts,
- groups of friends deciding on a restaurant to visit or a movie to see.

Let's start with two examples, one from real life and the other made up.

Example 1. In the 1998 election for governor of Minnesota, there were three candidates: Norm Coleman, a Republican; Skip Humphrey, a Democrat; and independent candidate Jesse Ventura. Ventura had never held elected office and was, in fact, a professional wrestler. The voting method used was what is called *plurality*: each voter chooses one candidate, and the candidate with the highest number of votes wins. Ventura was elected governor with 37% of the vote; Coleman received 35%, and Humphrey received 28%. On the other hand, based on exit polls, it seems that almost all of the voters who voted for Coleman preferred Humphrey second, and almost all who voted for Humphrey preferred Coleman second. In other words, Ventura won even though almost two-thirds of the voters liked him the least. Furthermore, among those who voted for Ventura, about half ranked Coleman second and about half ranked Humphrey second. This means that if the election had been between only Coleman and Humphrey or between only Coleman and Ventura, Coleman would have won in both cases. The voters preferred Colemen to both of the other two candidates, and yet he lost the election!

A problem with plurality is that it provides limited information about voters' preferences. By allowing voters to rank candidates rather than choosing just one preferred candidate, we can obtain more information. One voting method that uses voters' preferences is known as *Borda count*, named after the French mathematician, physicist, political scientist, and sailor Jean-Charles de Borda (1733–1799). Here is an example.

Example 2. Johannes, Christoph, and Monika are students in a school and are competing for the title of Best Mathematics Student. The 24 teachers in the school all rank them from best to worst, and the students receive points based on these rankings: 2 points for a first-place ranking, 1 point for a second place ranking, and 0 points otherwise. Here are the rankings:

Ranking			Number of teachers
1. Monika	2. Christoph	3. Johannes	11
1. Christoph	2. Johannes	3. Monika	7
1. Johannes	2. Christoph	3. Monika	3
Other rankings			0

Let's tally the points for each student. Monika receives $2 \times 11 = 22$ points, Christoph receives $(2\times 7)+11+3 = 28$ points, and Johannes receives $(2\times 3)+7 = 13$ points. Christoph is declared the winner, even though more than half of the teachers ranked Monika as the best student. Notice that if they had only counted first place votes, that is, used plurality, Monika would have won.

1 The social choice model

We are interested in situations where a group makes a choice based on the preferences of the individuals in the group. We assume that there is a finite set of choices for the group, and that each person in the group has a preference among these choices, so that the individuals rank the choices from first (most preferred) to last (least preferred). We assume that individuals have strict preferences, meaning that no ties are allowed in their ranked list. A *social choice method* is a procedure for determining a group preference from the individual preferences. We will allow ties in the group preference. Let's make this precise.

- We have a finite set of n voters and a finite set X of k *choices* or *candidates*.
- Let $L(X)$ be the set of all *preference lists*, that is, the set of all possible strict linear orderings[1] of the choices X. Let $O(X)$ be the set of all possible linear orderings of X (ties allowed).
- A *profile* or *election* is an element of the Cartesian product $L(X)^n$, that is, a profile is a set of n preference lists, one from each voter.
- A *social choice function* or *voting method* is a function $F : L(X)^n \to O(X)$. For a given profile $R \in L(X)^n$, the image $F(R)$ is sometimes called the *social choice* or *societal ranking*.

Notice that the social choice is a ranked list of choices (with ties possible). Much of the time we care only about the choice or choices on the top of this list of societal preferences, which we'll call the *winner* (or *winners*).

In Example 2, the social choice method used is Borda count, and we have $n = 24$ (teachers), $k = 3$ (candidates), and $L(X)$ is the set of six possible preference lists. The function F returns the candidate with the most total points (Christoph).

Examples of social choice methods:

1. *Plurality.* As described in Example 1. Candidates are ranked by the number of first-place rankings they have, so that the winner(s) is/are the candidate(s) with the most first-place rankings. This method is used in many elections, including many local and state elections in the US. It is used to elect half the seats in the German Bundestag.
2. *Antiplurality.* The candidate with the least last-place rankings wins. In general, candidates are ranked from last to first by the number of last-place rankings they receive.

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[1] A *linear ordering* is a generalization of the "lesser-than/greater-than" relation in real numbers. *Strict* means no ties allowed.

3. *Borda count.* This is the method used in Example 2. With k candidates, $k-1$ points are given for a first place ranking, $k-2$ points for a second place ranking, etc. Candidates are ranked by the total number of points they receive; the candidate(s) with the most points win(s). This method is used frequently for sports-related polls.
4. *Instant runoff.* The candidate(s) with the least first-place rankings is/are removed from each preference list, yielding a new set of preference lists for a smaller set of candidates. This process is repeated until all candidates are eliminated. The social choice is formed by listing the candidates in the reverse order in which they were eliminated. This method is used for elections in Australia and for presidential elections in Ireland.

We illustrate these methods with an example:

Example 3. Anne (A), Brigitte (B), Claus (C), and David (D) are all running for the president of a club with 27 members. The preference lists of the 27 voters are as follows. Note that there are 24 possible preference lists, but for the purpose of this example we are only using 4.

Preference list	**Number of voters**
A B C D	12
B C D A	7
C D A B	5
D C B A	3
Other preferences	0

Using plurality, Anne is the winner since she has the most first-place votes. Under antiplurality, Claus has the least last-place votes and is the winner. If we use Borda count, Anne has 41 points, Brigitte has 48 points, Claus has 47 points, and David has 26 points, hence Brigitte is the winner. Using instant runoff, David is eliminated in the first round, followed by Brigitte, and finally Anne, hence Claus wins.

This example shows that the winner of the election might depend on which voting method we choose! Does this seem reasonable?

2 Condorcet's method and Condorcet winners

An important notion in social choice theory is that of a head-to-head contest. Suppose we have a set of preference lists. For two candidates A and B, we say that A beats B in a *head-to-head contest* if more voters rank A above B than rank B above A. In other words: if the voters had to choose only between A and B, then A would win. In Example 1, Coleman would win a head-to-head contest with both of the other candidates.

The Marquis de Condorcet (1743–1794), a French philosopher, mathematician, and political scientist, wrote about voting methods and published a paper in 1785 describing what has become known as Condorcet's paradox. He wrote about the idea of head-to-head contests and related notions.

Definition 1. Suppose we have an election, that is, a set of preference lists.

- A candidate who would beat all other candidates in head-to-head contests is called a *Condorcet winner*.
- A candidate who would lose to all other candidates in head-to-head contests is called a *Condorcet loser*.
- A voting method satisfies the *Condorcet winner criterion* if, whenever there is a Condorcet winner, that candidate is the unique winner of the election.

In Example 1, Coleman is a Condorcet winner, and Ventura is a Condorcet loser. This example shows that plurality does not satisfy the Condorcet winner criterion. In Example 2, Monika is a Condorcet winner. This example shows that Borda count does not satisfy the Condorcet winner criterion.

3 Voting methods that satisfy the Condorcet winner criterion

The Condorcet winner criterion seems to be a very desirable property for voting systems: if the candidate A were to win all head-to-head contests, then it would seem reasonable that A should win. We saw that plurality and Borda count don't satisfy the Condorcet winner criterion. Are there reasonable methods that do satisfy it?

Since the Condorcet winner criterion is based on head-to-head contests, one way to make sure the Condorcet winner criterion is satisfied is to choose a winner based on such contests. In *sequential pairwise voting*, we fix an (arbitrary) ordering of the candidates and then hold rounds of head-to-head contests between the candidates following the fixed ordering. The winner of the contest between the first two goes up against the third candidate and so on until one candidate survives. It is easy to see that this method satisfies the Condorcet winner criterion, since a Condorcet winner will beat everyone else on the list.

Example 4. Let's take another look at Example 3.

Preference list	**Number of voters**
$A \quad B \quad C \quad D$	12
$B \quad C \quad D \quad A$	7
$C \quad D \quad A \quad B$	5
$D \quad C \quad B \quad A$	3
Other preferences	0

Notice that there is no Condorcet winner in this case. Let's try sequential pairwise voting with fixed ordering $A\,B\,C\,D$. In a head-to-head contest between A and B, 17 voters prefer A to B while 10 voters prefer B to A, so A wins this contest. Continuing, we see that C beats A (15 to 12) and finally C beats D (15 to 12); C is declared the winner.

If we use the fixed ordering $A\,C\,B\,D$, it is easy to check that B is the winner. If we use $B\,C\,A\,D$, then D is the winner, and if we use $B\,C\,D\,A$, then A is the winner.

This example shows that it is possible that any of the candidates can win, depending on the fixed ordering we choose! Perhaps sequential pairwise voting is not such a good idea, even though it satisfies the Condorcet winner criterion.

Another example of a method that satisfies the Condorcet winner criterion was proposed by economist Duncan Black in 1958. In Black's method, if a Condorcet winner exists, then that candidate is the winner. If there is no Condorcet winner, then we use Borda count to pick a winner. We could define other "hybrid" methods, where we choose the Condorecet winner if one exists and use some other method if there is no Condorcet winner. Perhaps a hybrid method is the way to go in order to ensure that reasonable properties hold for our chosen method.

4 Independence of irrelevant alternatives

Example 5. In 1995, women's figure skating was judged using a voting method called *best of majority*. Although the judges awarded points to each skater, these points were used mainly to create a preference list from each judge. In the 1995 Women's Figure Skating World Championship the following happened: with one skater left to skate, the top three places were: 1. Chen Lu (China), 2. Nicole Bobek (US), 3. Suraya Bonaly (France). The last skater was Michelle Kwan of the US, who ended up finishing in 4th place. However, the final results had Nicole Bobek and Suraya Bonaly switching places, so that Bonaly earned the silver medal and Bobek the bronze. This means that even though Kwan was behind both Bobek and Bonaly, her scores caused Bonaly to move ahead of Bobek in the final rankings. Intuitively, it seems that the question of whether

Bobek beat Bonaly or not should be independent of the performance of another skater. Because of this incidence the method used for scoring figure skating was changed.

This illustrates another property that seems reasonable for voting systems. We say that a voting method satisfies *independence of irrelevant alternatives* if it fulfills the following: whenever candidate A is ranked higher than candidate B in the social choice and some voters change their preference lists, but no voter changes their preference between A and B, then A should remain higher ranked than B. In other words, the societal preference between two candidates should depend only on the voters' preferences between A and B.

Example 5 shows that the method that was used for judging figure skating in the 1995 world championship does not satisfy independence of irrelevant alternatives. The reader may check that plurality and Borda count satisfy independence of irrelevant alternatives.

5 Monotonicity

Example 6. Suppose 17 members of a club are trying to decide what type of restaurant they will choose for their end of the year dinner. The choices are Thai, Chinese, Italian, and German. They decide to use instant runoff to choose the restaurant. The preferences of the members of the club are below.

Preference list	**Number of voters**
Thai Chinese Italian German	6
Chinese Thai Italian German	5
Italian German Chinese Thai	4
German Italian Thai Chinese	2
Other preferences	0

Using instant runoff, German is eliminated first, then Chinese, followed by Italian, so that Thai food would be the winner. However, before the ballots are cast, there is a heated debate about whether Thai food is better than Italian food. The two voters corresponding to the last row decide to move Thai ahead of Italian so that their preference lists are now: German Thai Italian Chinese. The vote is held with these new preference lists, and, as is easily checked, Italian food now wins.

Notice what just happened here – Thai food moved up in some preference lists and went from winning to losing!

A voting system is *monotone* (or satisfies *monotonicity*) if the following is fulfilled: whenever some voters move candidate A up in their preference lists and no voters move A down, then A cannot move down in the final ranking (the social choice). The above example shows that instant runoff is not monotone. It's easy to see that plurality and Borda count are monotone.

6 Arrow's impossibility theorem

Kenneth Arrow (*1921) was an economist who in the early 1950s decided to look for a voting method that was "fair" in the sense that it satisfied properties like the ones we have been discussing. Instead of finding a suitable voting system, he ended up proving that no such voting system exists. Arrow received the Nobel prize in Economics in 1971, for this and other important work in the area. Details of his work can be found in a paper from 1951 [1].

Arrow gave a list of five conditions that voting systems should satisfy, most of which we have discussed above. Here are the conditions with the names that Arrow gave them:

1. *Universality.* Voters can choose any possible preference order.
2. *Positive association of social and individual values.* This is the monotonicity condition discussed in Section 5.
3. *Independence of irrelevant alternatives.* As in Section 4.
4. *Citizen sovereignty.* There should never be a pair of candidates A and B so that A ranks higher than B on the social choice order regardless of how the the voters choose preference lists. In other words, the ranking of A and B should not be imposed on the voters.
5. *Nondictatorship.* There should not be a dictator, that is, there should not be one voter whose preference list determines the societal ranking completely.

Arrow's impossibility theorem. *If there are more than two candidates, then any social choice method cannot satisfy all of Arrow's five conditions.*

Arrow's theorem says that it is impossible to find a social choice method which satisfies reasonable conditions – all voting methods are necessarily flawed!

7 Beyond Arrow's theorem

Arrow's Theorem says that all social choice methods, including most of the ones used throughout the world for political elections, have flaws. Since Arrow published his famous work, there has been much research by mathematicians, economists, political scientists, and others on many aspects of voting methods and social choice.

Methods of choosing winners that are not social choice methods have been proposed. One example is *approval voting*, in which voters choose whether or not to "approve" each candidate. The societal preference order is determined by the number of "approve" votes the candidate receives. This method is used by the American Mathematical Society to elect members of the governing council and other governing positions. Another method, proposed by M. Balinski and R. Laraki [2], is called *majority judgment*. In this method, voters grade candidates in some way, which can vary depending on the situation, and these grades are aggregated in a specific way. If the goal is to simply declare a winner, then the candidate with the highest median grade[2] is the winner. The authors argue that this system avoids some of the problems of traditional social choice methods.

D. Saari, a professor of mathematics and economics, has written a number of books and articles on voting systems. In [6], Saari uses geometry to explain and explore the complexities and paradoxes of voting. In [8], Saari proposed weakening the independence of irrelevant alternatives condition by taking into account the *intensity* of voters' preference between two candidates, which he defines as the number of other candidates listed between the two candidates. He defines the *intensity of binary independence* criterion as follow: if some voters change their preference lists, but no voter changes their preference between candidates A and B or the intensity of their preference, then the ranking of A and B in the social choice should not change. Saari shows that Borda count, for example, satisfies the conditions of Arrow's theorem with independence of irrelevant alternatives replaced by intensity of binary independence.

Other questions about social choice have been explored including questions such as the probability of a particular social choice method failing independence of irrelevant alternatives or monotonicity. Related to this is work on the *Condorcet efficiency* of a particular social choice method: Given that there is Condorcet winner, what is the probability that the Condorcet winner will be elected? This is one way to compare different methods. For more on Condorcet efficiency, see the book [3] by W. Gehrlein and F. Valgones.

8 Further reading

We have only touched on some aspects of the subject of social choice and voting systems. For those who would like to explore this subject more, there are numerous introductory books on the subject, for example [4] and [5]. In addition to the books by Saari mentioned above, his book [7] is a fun and interesting introduction to these ideas.

[2] When all grades are arranged in ascending order, the *median grade* is the one in the middle.

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