Snapshots of modern mathematics
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On the containment problem

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Mathematicians routinely speak two languages: the language of geometry and the language of algebra. When translating between these languages, curves and lines become sets of polynomials called "ideals". Often there are several possible translations. Then the mystery is how these possible translations relate to each other. We present how geometry itself gives insights into this question.

1 Introduction

Computers dominate our everyday life. They handle finite data efficiently, but even though data sets can be huge, they are always finite. For example, computers cannot perform any exact calculation involving the number π, which has no finite decimal representation. In other words, no matter how many digits of the number π we write down, we are always just *approximating* π. Along the same lines, scientists describe the physical world by equations, or rather by their solutions which are functions. These functions, in their exact form, can be very complicated. However, thanks to the celebrated Stone–Weierstrass Theorem, any function appearing in the real world can be nicely approximated by *polynomials*.[1] These mathematical objects are the main heroes of this story.

[1] For details, see for example http://en.wikipedia.org/wiki/Stone-Weierstrass_theorem.

2 Polynomials and ideals

Polynomials are powerful objects in mathematics. They are put together (like Lego bricks) from simple building blocks called *monomials*. For example,

$$\begin{array}{ll} x & x^2y \\ x_1^{17}x_2^2x_3^5 & 1 \end{array}$$

are four different monomials. The first one, x, is very simple. It contains just one variable, namely x, and this variable appears there with power 1 (remember that $x = x^1$). The sum of all powers in a monomial is called the *degree* of the monomial. Thus x is a monomial of degree 1 in one variable x. Similarly x^2y is a monomial of degree $2 + 1 = 3$ in two variables x and y. Sometimes it is useful to enumerate variables by indices (especially when there are thousands of variables around, which easily happens in actual applications like modeling car motors). Our third example, $x_1^{17}x_2^2x_3^5$, is a monomial of degree $17 + 2 + 5 = 24$ in the variables x_1, x_2, and x_3. The last example, 1, is also a monomial and its degree is 0 by definition.

We can multiply a monomial by a *coefficient*, which is just a number. For example, $5x^2$ is the monomial x^2 multiplied by the coefficient 5. Polynomials are sums of monomials with coefficients. We encounter simple polynomials in school, such as

$$2x - 3.$$

Indeed $2x - 3 = 2x + (-3) \cdot 1$ is the sum of two monomials with coefficients: x and 1 with coefficients 2 and -3, respectively.

This polynomial can be considered as a function and then its graph is a straight line, as shown in Figure 1.

The point where the line intersects the horizontal axis is of particular interest. It is the *zero* of the polynomial. We also say – somewhat colloquially – that the polynomial *vanishes* at that point.

There are many polynomials with a zero at the same point, for example

$$\begin{aligned} f(x) &= x^2 - \frac{9}{4}, \\ g(x) &= 2x^3 - x^2 - x - 3. \end{aligned}$$

Computing $f(\frac{3}{2}) = 0$ and $g(\frac{3}{2}) = 0$ shows that f and g vanish at $\frac{3}{2}$.

On the other hand, not all polynomials vanish at that point. For example, none of the polynomials

$$\begin{aligned}&1,\\&x+7,\\&\frac{3}{10}x^3-\frac{3}{5}x^2-\frac{3}{2}x+\frac{9}{5}\end{aligned}$$

vanishes at $x=\frac{3}{2}$. The graphs of two of these examples are shown in Figure 2.

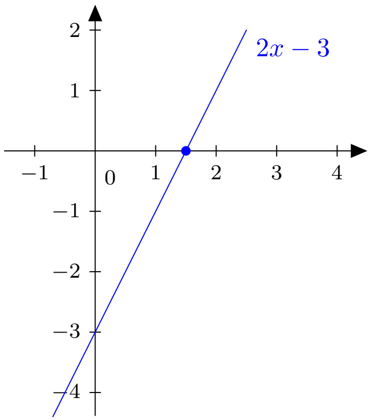

Figure 1: The graph of the polynomial $2x-3$ with its zero $\frac{3}{2}$ marked.

We would like to distinguish those polynomials that vanish on a certain set of points from those that do not, but at first, we make two fundamental observations. Take three polynomials f, g, and h, such that α is a common zero of f and g, that is $f(\alpha)=0$ and $g(\alpha)=0$. Then calculating

$$(f+g)(\alpha)=f(\alpha)+g(\alpha)=0+0=0 \tag{1}$$

shows that α is also a zero of $f+g$, and

$$(hf)(\alpha)=h(\alpha)f(\alpha)=h(\alpha)\cdot 0=0 \tag{2}$$

shows that α is also a zero of hf. As an example of the latter case, take

$$\begin{aligned}f(x)&=2x-3\\h(x)&=x+7.\end{aligned}$$

Then $(hf)(x)=(2x-3)(x+7)$ vanishes at $x=\frac{3}{2}$ even though $x+7$ alone does not.

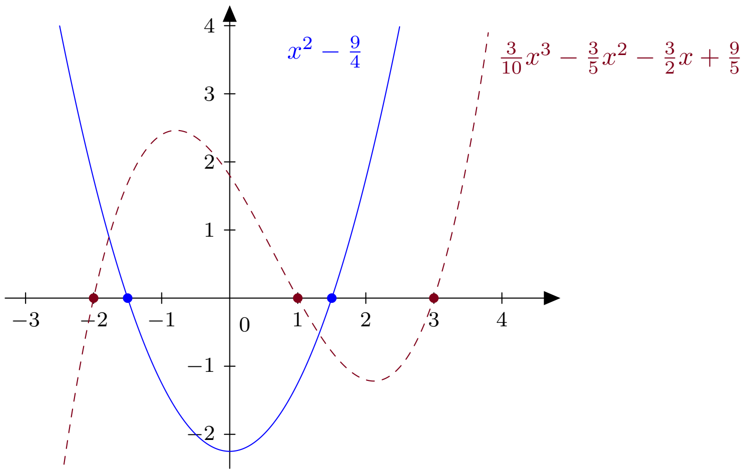

Figure 2: The graphs of two polynomials: $x^2 - \frac{9}{4}$ with zeroes $-\frac{3}{2}$ and $\frac{3}{2}$, and $\frac{3}{10}x^3 - \frac{3}{5}x^2 - \frac{3}{2}x + \frac{9}{5}$ with zeroes -2, 1, and 3.

Motivated by our observations (1) and (2), algebraists[2] introduced the concept of *ideals*: an *ideal of polynomials* $\mathcal{I}$ is a set such that

1. Every element of $\mathcal{I}$ is a polynomial.
2. The sum of any two elements of $\mathcal{I}$ is again an element of $\mathcal{I}$.
3. The product of an element of $\mathcal{I}$ with another arbitrary polynomial is again an element of $\mathcal{I}$.

Now let $\mathcal{I}$ be the set of all polynomials vanishing at $\frac{3}{2}$,

$$\mathcal{I} = \left\{ f \text{ is a polynomial} \mid f\left(\tfrac{3}{2}\right) = 0 \right\}. \tag{3}$$

It follows from (1) and (2) that $\mathcal{I}$ is an ideal of polynomials![3] Here are some

[2] Algebraists are mathematicians working in "algebra", a branch of mathematics that deals, for example, with polynomials.

[3] Mathematicians define ideals in the more general context of "rings", but in this snapshot, only ideals of polynomials are important. For more information about rings, see for example `http://en.wikipedia.org/wiki/Ring_(mathematics)`.

examples of elements of $\mathcal{I}$

$$
\begin{aligned}
f(x) &= 2x - 3, \\
g(x) &= x - \frac{3}{2}, \\
h(x) &= 4x^2 - 9, \\
j(x) &= 2x^3 - x^2 - x - 3.
\end{aligned}
$$

Here, f and g have degree 1, while h has degree 2 and j has degree 3. One can check that all elements of $\mathcal{I}$ are products of f with another polynomial. In particular, we have

$$
\begin{aligned}
h(x) &= (2x - 3)(2x + 3) \\
j(x) &= (2x - 3)(x^2 + x + 1).
\end{aligned}
$$

We say that $f(x) = 2x - 3$ *generates* $\mathcal{I}$.

Now we consider polynomials with more than one variable. For example

$$
\begin{aligned}
f(x, y) &= x^2 + y^2 - 1, \\
g(x, y, z) &= x^3 - zy^3 - \frac{3}{4}z^3, \\
h(x_1, x_2, x_3, x_4, x_5) &= x_1^7 - x_2^5 x_3 x_4 + 207 x_1 x_2 x_5^3 - 900.
\end{aligned}
$$

An important class of polynomials is given by *homogeneous* polynomials. These are polynomials which contain only monomials of the same degree, for example the polynomials

$$
\begin{aligned}
j(x, y, z) &= x^2 + 2y^2 - 3z^2 \\
k(x, y, z) &= 6x^2yz - 7xz^3 + yz^3 - z^4.
\end{aligned}
$$

Here, j is homogeneous of degree 2 and k is homogeneous of degree 4, whereas none of the polynomials f, g, or h above is homogeneous. When there are many variables, ideals become more complicated, in particular they typically have more than one generator. However, no matter how complicated an ideal is, it always has a *finite* number of generators. This was proved by David Hilbert in 1890 and is very important in applications.[4]

We write $\mathcal{I} = \langle f_1, f_2, \ldots, f_k \rangle$ to indicate that the polynomials $f_1, \ldots, f_k$ generate the ideal $\mathcal{I}$. Thus in example (3) on page 4 we have $\mathcal{I} = \langle 2x - 3 \rangle$, with just one generator.

[4] For a proof of Hilbert's Basis Theorem, see for example [7, Theorem 1.2].

To be clear, saying that $f_1, \ldots, f_k$ *generate* $\mathcal{I}$ means that *every* polynomial g in $\mathcal{I}$ can be written as the sum of products of the generators with some other polynomials. For example, if $\mathcal{I} = \langle x, y \rangle$, then every polynomial f in $\mathcal{I}$ can be written in the form

$$f(x, y) = xg(x, y) + yh(x, y), \tag{4}$$

for some polynomials g and h. Thus we see that $f_1(x, y) = x^3 - 2xy + 7$ is not an element of $\mathcal{I}$, because 7 is divisible by neither x nor y. In contrast, $f_2(x, y) = x^3 - 2xy + 7y^4$ is contained in $\mathcal{I}$ since we can write $f_2(x, y)$ as

$$f_2(x, y) = x(x^2 - 2y) + y(7y^3).$$

But we can as well write $f_2(x, y)$ as

$$f_2(x, y) = x(x^2 - y) + y(-x + 7y^3).$$

Thus the presentation in (4) is *not unique*.

3 The order of vanishing

There is a natural interplay between algebra and geometry which plays a central role in many branches of mathematics. This allows translating from the world of polynomials to the world of lines, planes and other geometrical objects. So how does this interplay work?

Consider again the ideal $\mathcal{I} = \langle 2x - 3 \rangle$. It determines the point $\frac{3}{2}$ on the real line, given as the solution of the equation $2x - 3 = 0$. In fact $\frac{3}{2}$ is the *common zero* of *all* polynomials in $\mathcal{I}$, because as we saw all elements of $\mathcal{I}$ are multiples of $2x - 3$.

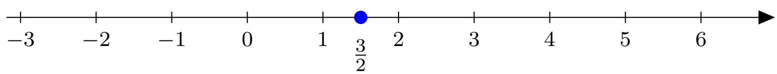

Figure 3: The zero set of a polynomial.

Similarly $\mathcal{J} = \langle x \rangle$ determines the point 0, since this is the unique solution to the equation $x = 0$. But an ideal may determine more than one point. For example, $\mathcal{K} = \langle x(x - 1) \rangle$ determines the set $\{0, 1\}$.

Any *finite* subset

$$\{\alpha_1, \dots, \alpha_s\}$$

of the line can easily be represented as the common zeros of an ideal, namely

$$\langle (x - \alpha_1) \cdot \ldots \cdot (x - \alpha_s) \rangle \,.$$

Does this mean that there is a perfect correspondence between points on the line and ideals? No!

For example: which set of points is determined by the ideal $\mathcal{L} = \langle x^2 \rangle$? The equation $x^2 = 0$ has only one solution $x = 0$. So $\mathcal{L}$ determines the same point as the ideal $\mathcal{J} = \langle x \rangle$!

However, the ideals $\mathcal{L}$ and $\mathcal{J}$ differ: all elements of $\mathcal{L}$ are also in $\mathcal{J}$, but not all elements of $\mathcal{J}$ are also in $\mathcal{L}$.

Expressed in formulas, the inclusion

$$\langle x^2 \rangle \subseteq \langle x \rangle$$

holds since

$$x^2 = xx,$$

where one x is the generator of $\mathcal{J}$ and the other x is a polynomial coefficient as in (4).

The reverse inclusion does not hold since it is impossible to write x in the form

$$x = f(x)x^2.$$

The reason is that $f(x)x^2$ has degree *at least* 2, whereas x has degree 1.

Taking a closer look at the equation $x^2 = 0$, we see that 0 is a *double* solution of this equation or equivalently: the polynomial $g(x) = x^2$ *vanishes at* 0 *to order* 2. We see that polynomials vanishing to the second order at a point are among those which just vanish there but not vice versa, that is, not all polynomials vanishing at a point vanish there to order two.

For example, divisibility by x^2 is clearly a more restrictive condition than divisibility by x. Similarly, divisibility by x^3 is more restrictive than divisibility by x^2 and so on. This leads to a sequence of containments

$$\ldots \subseteq \langle x^{n+1} \rangle \subseteq \langle x^n \rangle \subseteq \ldots \subseteq \langle x^3 \rangle \subseteq \langle x^2 \rangle \subseteq \langle x \rangle \subseteq \langle 1 \rangle \,.$$

Given a set V, we say that a polynomial f *vanishes along* V if $f(x) = 0$ for all elements x in V. If we denote the ideal of all polynomials vanishing along V to order m by $\mathcal{I}^{(m)}$, we have

$$\ldots \subseteq \mathcal{I}^{(n+1)} \subseteq \mathcal{I}^{(n)} \subseteq \ldots \subseteq \mathcal{I}^{(3)} \subseteq \mathcal{I}^{(2)} \subseteq \mathcal{I} \subseteq \langle 1 \rangle \,.$$

The ideals $\mathcal{I}^{(m)}$ are called *symbolic powers* of the ideal $\mathcal{I}$. This name is used in order to distinguish them from *ordinary* powers of $\mathcal{I}$, which are denoted simply by $\mathcal{I}^r$. These ordinary powers are defined by taking all products of r elements in $\mathcal{I}$, with repetitions allowed. It is a very convenient feature of ideals of polynomials that it suffices to take products of generators.[5]

For example, the second ordinary power of $\mathcal{I}$ is the ideal generated by products of any two generators of $\mathcal{I}$. More specifically, if $\mathcal{I} = \langle f, g \rangle$, then $\mathcal{I}^2 = \left\langle f^2, fg, g^2 \right\rangle$ and similarly $\mathcal{I}^3 = \left\langle f^3, f^2g, fg^2, g^3 \right\rangle$. We see that there is again a sequence of containments, as taking higher and higher powers, the ideals get smaller and smaller.

$$\ldots \subseteq \mathcal{I}^{n+1} \subseteq \mathcal{I}^n \subseteq \ldots \subseteq \mathcal{I}^3 \subseteq \mathcal{I}^2 \subseteq \mathcal{I} \subseteq \langle 1 \rangle .$$

In the example with $\mathcal{J} = \langle x \rangle$, it is now clear that

$$\mathcal{J}^n = \mathcal{J}^{(n)} \tag{5}$$

for any $n \geq 1$.

We saw above that ideals $\mathcal{I}$ in one variable determine points on the real line, given as the common zeroes of all polynomials in $\mathcal{I}$. When we turn to the more interesting case of ideals in several variables, what do the common zeroes look like? So take an ideal $\mathcal{I}$ of polynomials in n variables $x_1, \ldots, x_n$. Now consider the set along which all polynomials in $\mathcal{I}$ vanish, which we denote by $V(\mathcal{I})$. In formulas,

$$V(\mathcal{I}) = \{(x_1, \ldots, x_n) \mid f(x_1, \ldots, x_n) = 0 \text{ for every } f \text{ in } \mathcal{I}\} .$$

We call $V(\mathcal{I})$ the *vanishing set* or *zero set* of $\mathcal{I}$. Fortunately, it suffices to check $f(x_1, \ldots, x_n) = 0$ just for the generators of $\mathcal{I}$. By design, all polynomials in $\mathcal{I}$ vanish along $V(\mathcal{I})$ – we just say that $\mathcal{I}$ vanishes along $V(\mathcal{I})$. The vanishing sets of ideals in more than one variable can have interesting shapes, two examples are displayed in Figure 4: The vanishing set of the ideal $\left\langle y^2 + x^2 - 4 \right\rangle$ is a circle with radius 2, and the vanishing set of the ideal $\left\langle y^2 - x^2(x+1) \right\rangle$ is a so-called "cubic curve"[6].

Take another example: the ideal $\mathcal{I} = \langle xy, xz, yz \rangle$. The set of zeroes $V(\mathcal{I})$ is the union of the coordinate axes in three-dimensional space. This example leads us back to symbolic powers, because unlike (5), already the second ordinary and symbolic powers differ:

$$\mathcal{I}^2 \neq \mathcal{I}^{(2)}.$$

[5] For a more advanced introduction to symbolic powers, see for example [7, Section 3.9].

[6] For more information on cubic curves, see for example `https://en.wikipedia.org/wiki/Cubic_plane_curve`.

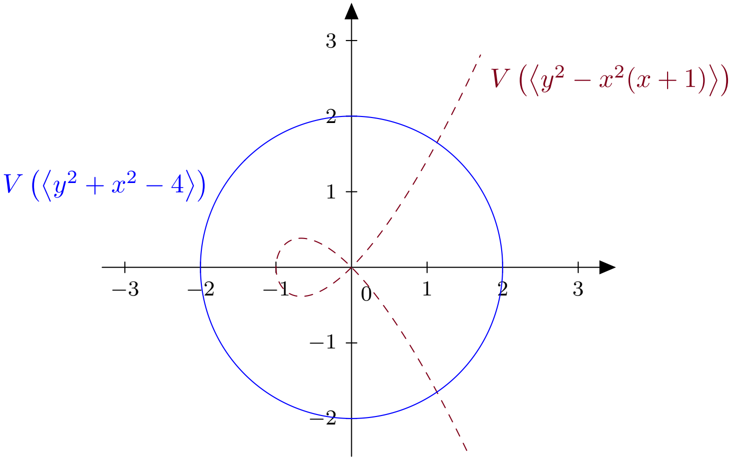

Figure 4: The vanishing sets of two ideals in two variables: a circle and a cubic curve.

Why is that the case? By what we have said above, it is clear that the least degree of a polynomial in $\mathcal{I}^2$ is 4, since all generators of $\mathcal{I}$ have degree 2. On the other hand the monomial xyz is contained in $\mathcal{I}^{(2)}$. Indeed, the set of zeroes of this monomial is the union of all three coordinate planes. Now, every coordinate axis is an intersection of two coordinate planes (for example, the z-axis is the intersection of the planes $x = 0$ and $y = 0$), hence xyz vanishes doubly along each axis (already xy vanishes doubly along the z-axis).

We saw above that $\left\langle x^2\right\rangle \subseteq \langle x\rangle$. But what is the important difference between $\langle x\rangle$ and $\left\langle x^2\right\rangle$? Both ideals vanish at 0, but $\langle x\rangle$ is special because it is *the largest ideal* that vanishes at 0, a property that we call *radical*: An ideal $\mathcal{I}$ is called *radical* if it contains all polynomials that vanish along $V(\mathcal{I})$, in other words, if $\mathcal{I}$ is *the largest* ideal vanishing along $V(\mathcal{I})$. For example, $V(\left\langle x^2\right\rangle) = \{0\}$. Since x vanishes at 0 but is not in $\left\langle x^2\right\rangle$, the ideal $\left\langle x^2\right\rangle$ is not radical.

Turning to powers of ideals again, if $\mathcal{I}$ is a radical ideal in one variable, we have $\mathcal{I}^n = \mathcal{I}^{(n)}$. But we have seen that if we allow for more variables, the ideals become more complicated. In fact, it happens rarely that the equality $\mathcal{I}^n = \mathcal{I}^{(n)}$ holds. There is always the containment

$$\mathcal{I}^n \subseteq \mathcal{I}^{(n)}$$

for a radical ideal $\mathcal{I}$. This is evident since an n-fold product of polynomials that all vanish along a set V to order 1 vanishes there at least to order n. It might however happen (as we saw in the example with $\langle xy, xz, yz\rangle$) that there are polynomials which vanish along V to order n but are not products of n elements of $\mathcal{I}$. On the contrary,

$$\mathcal{I}^{(n)} \not\subseteq \mathcal{I}^n$$

is the typical behavior! Naturally enough, this situation inspired mathematicians to wonder about a more general containment problem:

Question 1 *Given an ideal $\mathcal{I}$, determine all integers m, r such that the containment*

$$\mathcal{I}^{(m)} \subseteq \mathcal{I}^r \tag{6}$$

holds.

This question has occupied algebraists for quite a number of years. It is clear that for a fixed m there is an r such that the containment in (6) holds, take $r = 1$ for example. The difficulty of the problem lies in finding the largest possible r.

A surprisingly uniform answer has been found independently by two teams of researchers: Lawrence Ein, Robert Lazarsfeld and Karen Smith [6] and Melvin Hochster and Craig Huneke [10]. A somewhat simplified form of their results is the following statement.

Theorem 1 *Let $\mathcal{I}$ be a polynomial ideal in $n+1$ variables. Then for all m and r satisfying $m \geq nr$ there is the containment*

$$\mathcal{I}^{(m)} \subseteq \mathcal{I}^r. \tag{7}$$

Examples show that this result cannot be improved in general, but these examples are somewhat artificial.

This has led Craig Huneke to ask if one can improve the constants in (7) under additional assumptions. In particular, he asked if the following containment holds:

$$\mathcal{I}^{(3)} \subseteq \mathcal{I}^2 \tag{8}$$

provided that $\mathcal{I}$ is a radical ideal in three variables with the following two properties:

1. The generating polynomials of $\mathcal{I}$ can be chosen to be homogeneous.
2. The set of common zeroes $V(\mathcal{I})$ consists of a finite number of lines through the origin.

Note that Theorem 1 implies the inclusion $\mathcal{I}^{(4)} \subseteq \mathcal{I}^2$. In this snapshot we saw that $\mathcal{I}^{(4)} \subseteq \mathcal{I}^{(3)}$, so indeed the hard question is if the ideal $\mathcal{I}^{(3)}$ also fits into $\mathcal{I}^2$.

This question has been studied by a number of authors [3, 8], who obtained partial results confirming the containment in (8). Recently, Marcin Dumnicki, Halszka Tutaj-Gasińska and Tomasz Szemberg constructed in [5] the first counterexample to the containment in (8). The points appearing in this counterexample come up as intersection points of a certain configuration of lines (see Snapshot 5/2014 *Arrangement of lines* by Brian Harbourne and Tomasz Szemberg for an introduction to arrangements of lines). Since then, a number of further counterexamples has been constructed [2, 9, 4, 11]. All these counterexamples revolve around configurations of lines, although a recent paper gives evidence that some counterexamples come only from configurations of curves other than lines [1]. One of the aims of a recent workshop in Oberwolfach was to explain how and why the two topics, containment of ideals and configurations of lines, are related. This is an ongoing research project with many possible variants and refinements, so that it presents a nice experimental field in algebra and geometry with potentially interesting and powerful results still waiting to be discovered.[7]

Acknowledgments We would like to thank Thomas Bauer and Brian Harbourne for helpful comments on the first draft of this snapshot.

[7] To learn about a different question related to zeroes of polynomials, see Snapshot 8/2015 *Ideas of Newton–Okounkov bodies* by Valentina Kiritchenko, Evgeny Smirnov, and Vladlen Timorin.

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