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Gerhard Huisken
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# Geometric Flows and 3-Manifolds

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The current article arose from a lecture[1] given by the author in October 2005 on the work of R. Hamilton and G. Perelman on Ricci-flow and explains central analytical ingredients in geometric parabolic evolution equations that allow the application of these flows to geometric problems including the Uniformisation Theorem and the proof of the Poincare conjecture. Parabolic geometric evolution equations of second order are non-linear extensions of the ordinary heat equation to a geometric setting, so we begin by reminding the reader of the linear heat equation and its properties. We will then introduce key ideas in the simpler equations of curve shortening and 2-d Ricci-flow before discussing aspects of three-dimensional Ricci-flow.

## 1 The Heat Equation

A function $u : \mathbb{R}^n \times [0,T] \to \mathbb{R}$ is a solution of the heat equation if

$$\frac{d}{dt}u = \Delta u = \sum_i^n D_i D_i u \tag{1.1}$$

holds everywhere on $[0,T]$. The heat equation is the gradient flow of the Dirichlet energy

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^n} |Du|^2 \, d\mu \tag{1.2}$$

with respect to the $L^2$-norm and has the fundamental property of smoothing out all derivatives of a solution even for non-smooth initial data in a precise quantitative way: For example, a uniform bound on the absolute value of $|u|$ on some time interval $[0,t)$ implies a corresponding bound

$$\sup |D^k u(\cdot,t)| \leq \frac{C_k}{t^{k/2}} \sup_{\mathbb{R}^n \times [0,t)} |u| \tag{1.3}$$

[1]Oberwolfach Lecture delivered in connection with the general meeting of the Gesellschaft für mathematische Forschung e.V. (GMF) in Oberwolfach on October 16th, 2005.

for each higher derivative $D^k u$ after some waiting time. The scaling behavior of the equation arises from the fact that with $u$ for each positive $\lambda$ also the function

$$u_\lambda(x,t) = u(\lambda x, \lambda^2 t) \tag{1.4}$$

is a solution of the heat equation. With respect to this scaling the heat-kernel

$$\rho_(x,t) = \frac{1}{(4\pi t)^{n/2}} \exp\Big(-\frac{|x|^2}{4t}\Big) \tag{1.5}$$

is a selfsimilar solution of the heat equation and thus connects the geometry of $\mathbb{R}^n$ with important analytical properties of heat-flow such as the Harnack inequality: For any positive solution $u > 0$ of the heat equation on $\mathbb{R}^n \times (0,T)$ one has the differential inequality

$$\Delta u - \frac{|Du|^2}{u} + \frac{n}{2t} u \geq 0, \tag{1.6}$$

with equality being valid on the heat-kernel. This inequality is equivalent to

$$\frac{d}{dt} \log u \geq |D \log u|^2 - \frac{n}{2t} \tag{1.7}$$

and implies by integration a sharp Harnack inequality for the values of $u$. General differential Harnack inequalities in the context of smooth Riemannian manifolds $(\mathcal{M}, g)$ were established in the work of Li and Yau, compare [9] . In the Euclidean case the inequality reads

$$u(x_2,t_2) \geq u(x_1,t_1)\Big(\frac{t_1}{t_2}\Big)^{n/2} \exp\left(-\frac{|x_2-x_1|^2}{4(t_2-t_1)}\right) \tag{1.8}$$

for all $x_1, x_2 \in \mathbb{R}$ and $t_2 > t_1 > 0$ since the expression

$$\frac{|x_2-x_1|^2}{4(t_2-t_1)} \tag{1.9}$$

minimizes the action

$$\frac{1}{4}\int_\gamma |\dot\gamma|^2 d\tau \tag{1.10}$$

among curves $\gamma$ connecting the events $(x_1,t_1),(x_2,t_2)$. It turns out that sharp inequalities of this type that hold with equality on selfsimilar solutions are one key to the understanding of the behavior of the Ricci-flow and its singularities.

# 2 Harmonic Map Heat-flow

When considering more general maps $u : (\mathcal{M}^m, g) \to (N^n, h)$ between two Riemannian manifolds one studies again the $L^2$- gradient flow of the Dirichlet energy

$$E(u) = \frac{1}{2}\int_{\mathcal{M}^m} ||du||^2_{g,h}\, d\mu_g, \tag{2.1}$$

where the energy density $||du||^2$ is now computed with respect to the metrics $g, h$. The resulting parabolic system of equations

$$\frac{d}{dtu} = \Delta_{g,h} u \tag{2.2}$$

is linear in the second derivatives of $u$ but has a quadratic nonlinearity in $du$ due to the dependence of the target metric on the map $u$. The stationary points of this system are harmonic maps and it was the first triumph of geometric evolution equations when Eells and Sampson were able to show that in the case of negatively curved target manifolds the flow has a long time solution that converges to a harmonic map in the same homotopy class, see [1]. This striking result motivated the search for other evolution equations that would be able to deform a given geometric object into some canonical representative of its class. On the other hand the harmonic map heat-flow also provided the first examples of singularitie s in solutions of a geometric evolution equation when positive curvature in the target manifold forced so called bubbling phenomena.

# 3 Mean Curvature Flow

For $F_0 : \mathcal{M}^n \to \mathbb{R}^{n+1}$ a smooth immersion of an $n$–dimensional hypersurface in Euclidean space, $n \geq 1$, the evolution of $\mathcal{M}_0 = F_0(\mathcal{M})$ by mean curvature flow is the one–parameter family of smooth immersions $F : \mathcal{M}^n \times [0, T[ \to \mathbb{R}^{n+1}$ satisfying

$$\frac{\partial F}{\partial t}(p,t) = -H(p,t)\nu(p,t), \qquad p \in \mathcal{M}^n, t \geq 0, \tag{3.1}$$

$$F(\cdot, 0) = F_0. \tag{3.2}$$

Here $H(p,t)$ and $\nu(p,t)$ are the mean curvature and the outer normal respectively at the point $F(p,t)$ of the surface $\mathcal{M}_t^n = F(\cdot,t)(\mathcal{M}^n)$. The signs are chosen such that $-H\nu = \vec{H}$ is the mean curvature vector and the mean curvature of a convex surface is positive. In case $n = 1$ this flow is called the curve shortening flow. It is the gradient flow of $n$-dimensional area with respect to the $L^2$-norm and has similar scaling and smoothing properties as the heat equation and the harmonic map heat-flow. However, since now the righthand side of the equation,

$$-H(p,t)\nu(p,t) = \Delta_t F(p,t), \tag{3.3}$$

is computed from the Laplace-Beltrami operator with respect to the evolving induced metric on the hypersurface, the system of equations depends on first derivatives of the solution and is only quasilinear. It leads to a system of reaction-diffusion equations for the evolving second fundamental form of the solution surface,

$$\frac{d}{dt} h^i_j = \Delta h^i_j + |A|^2 h^i_j. \tag{3.4}$$

The nonlinear term can cause blowup of the curvature, an example is a shrinking sphere with $\mathcal{M}_t^n = S^n_{R(t)}(0)$ and $R(t) = \sqrt{R_0^2 - 2nt}$. Here the curvature $H^2$ blows up with the rate $(T-t)^{-1}$ at the singular time $T$. It is a remarkable fact that despite the singularity the diffusion part of the equation is strong enough to enforce a selfsimilar structure in the formation of the singularity.

The following uniformisation theorem for curves in the plane proven by Grayson [2] following work of Gage and Hamilton is a highly nontrivial one-dimensional demonstration of how a geometric evolution equation can straighten out a given geometry:

**Theorem 3.1** (M. Grayson) *If $F_0 : S^1 \to \mathbb{R}^2$ is an embedded initial curve, then the solution of the curve shortening flow (3.1) remains embedded and becomes convex after some finite time. It then converges smoothly to a point while its shape approaches a selfsimilar shrinking circle.*

One successful strategy for the proof of this result that we would like to briefly sketch consists in classifying all possible singularities of the curve shortening flow in the plane together with arguments that rule out all such possibilities accept the known shrinking circle.

Main ingredient for this approach is a *monotonicity formula* involving surface area with a heat-kernel as a weighting function: Let $u(x,t) = \sqrt{2(T-t)}\rho(x, T-t)$ be the backward heat-kernel adapted to the $n$-dimensional hypersurface, then

$$\frac{d}{dt}\int_{\mathcal{M}^n} u\, d\mu = -\int_{\mathcal{M}^n} |H + \nabla_\nu \log u|^2 u\, d\mu, \tag{3.5}$$

compare [7]. Since the zeros of the RHS are exactly the selfsimilar shrinking solutions of mean curvature flow, one can deduce with parabolic rescaling that all singularities of the flow with the natural blowup rate $1/\sqrt{2(T-t)}$ for the second fundamental form are asymptotically selfsimilar - and among these the shrinking circle is the only embedded possibility. The proof of Grayson's result is then complete if one can show that there is no singularity with a higher blowup rate of curvature than observed in the case of the shrinking circle. By a rescaling argument again this can be done if all convex translating solutions of mean curvature flow can be ruled out as the profile of the singularity. The key ingredient in this last step is a Harnack inequality for the curvature of convex curves moving by the curve shortening flow, that is fully analogous to the differential Harnack inequality (1.6) in the linear heat equation:

$$\kappa_{ss} - \frac{|\kappa_s|^2}{\kappa} + \frac{1}{2t}\kappa \geq 0. \tag{3.6}$$

It can be deduced that the "grim reaper" curve $y(x,t) = -\log\cos(x) + t$ is the only convex translating solution of the flow and this singularity profile can then be ruled out using the global embeddedness assumption for the initial curve in a quantitative way.

In summary, it is not necessary to control the shape of the evolving curves at intermediate times, the classification of all relevant singularities is sufficient. And this classification is achieved with the help of a sharp Harnack inequality and sharp integral estimates that characterize selfsimilar solutions of the flow. We will find exactly the same ingredients when studying the approach to the classical uniformisation theorem and the Poincare conjecture by Ricci-flow.

# 4 Ricci-Flow

Given a Riemannian manifold $(\mathcal{M}^n, g_0)$ in 1982 Richard Hamilton proposed to solve the evolution equation [3]

$$\frac{d}{dt}g = -2Rc(g) \tag{4.1}$$

with the given metric $g_0$ as initial datum. Here $\mathrm{Rc}(g(t))$ is the Ricci curvature of the evolving metric and the manifold is assumed to be compact without boundary. The equation is a quasilinear parabolic system of equations which is in many ways analogous to mean curvature flow. E.g. the evolution equation for the curvature resulting from Ricci-flow is again a system of reaction-diffusion equations that have the Laplace-Beltrami operator as their leading part in all components of the system:

$$\frac{d}{dt}\mathrm{Riem} = \Delta\mathrm{Riem} + \mathrm{Quad}(\mathrm{Riem}). \tag{4.2}$$

The algebraic properties of the quadratic reaction term determine the interplay of the various curvature quantities during the flow. In three dimensions the Riemann curvature tensor can be fully expressed in terms of the Ricci tensor, allowing a concentration of attention on the behavior of the three eigenvalues of this symmetric tensor field during the evolution. In [3] Hamilton was able to fully understand the 3-dimensional case with positive Ricci curvature:

**Theorem 4.1** *If the initial Riemannian metric $g_0$ on a closed 3-manifold has positive Ricci curvature then the solution of Ricci flow contracts smoothly to zero volume in finite time and appropriate rescalings of the metric converge to a smooth metric of constant sectional curvature. In particular, any 3-manifold carrying a metric of positive Ricci curvature is diffeomorphic to a spherical space-form.*

In three dimensions the algebraic properties of the reaction terms in (4.2) strongly favor metrics of constant positive sectional curvature while the second Bianchi identities allow control of the gradient of the scalar curvature through the gradient of the tracefree part of the Ricci tensor. It was then possible to prove the theorem with the maximum principle being the essential tool.

On two dimensional spheres the maximum principle seems not to be enough to control the size of the curvature and an appeal to integral estimates and a Harnack estimate becomes an alternative avenue just like in the curve shortening of embedded curves. Hamilton [4] proved that a Li-Yau type Harnack inequality holds for the scalar curvature $R$ in 2-d Ricci flow that is analogous to the ordinary heat equation, we only state the scalar version

$$\Delta R - \frac{|DR|^2}{R} + \frac{n}{2t}R \geq 0. \tag{4.3}$$

Using this inequality it is possible in a fashion analogous to the case of curve shortening to prove that the only possible singularities of 2-spheres are either asymptotic to shrinking selfsimilar spheres or to a "cigar"-type capped cylinder that satisfies Ricci-flow by moving under diffeomorphisms along a radial gradient vectorfield on $\mathbb{R}^2$. The metric of such a "translating soliton solution" is explicitly given by

$$ds^2 = \frac{dx^2 + dy^2}{1 + x^2 + y^2} \tag{4.4}$$

and satisfies the corresponding soliton equation

$$\frac{d}{dt} g_{ij} = (L_X g)_{ij} = 2 D_i D_j f = -2 R_{ij}, \tag{4.5}$$

where $X = Df$ is the gradient vector field driving the diffeomorphism.

Again in parallel to the case of curve shortening this last model for a singularity can then be ruled out for example by an isoperimetric inequality satisfied by 2-d Ricci-flow. Gathering all cases and using an extra argument due to Chow it follows from the work of Hamilton that the Ricci flow has the best possible behavior on all 2-d surfaces:

**Theorem 4.2** *For any initial metric on a two-dimensional Riemann surface the solution of Ricci flow converges to a metric of constant Gauss curvature in the same conformal class (when appropriately rescaled).*

We note that in the spherical case the shrinking surface has to be scaled up while in the case of negative Gauss curvature the Ricci flow is expanding the surface such that it has to be scaled down; in the case of the torus the volume stays constant and no rescaling is necessary. As a final comment we note that the eternal selfsimilar solutions of curve shortening flow ("grim reaper") and Ricci-flow respectively ("cigar") that appeared in our analysis of singularities are of independent interest in the theory of Renormalisationgroup flows where they are known as the "hairpin" solution of curve shortening flow and the "2-d Riemannian black hole" respectively.

# 5 Ricci-Flow singularities in 3 dimensions

If the assumption of positive Ricci curvature is dropped in dimension $n = 3$ new singularities become possible, in particular when a long thin neck of type $S^2 \times [a, b]$ connecting two larger pieces of the three-manifold begins to pinch off under the influence of the large sectional curvature present in the small $S^2$. Such neckpinch singularities will inevitably happen for general initial data and cannot be avoided.

The great idea driving the work of Hamilton and Perelman in three dimensions is that this is essentially the only singularity that the Ricci flow can develop in finite time and that this singular behavior of the flow can in fact be turned to advantage: According to the geometrisation conjecture of Thurston every closed three-manifold admits a decomposition into irreducible pieces of eight different types along spherical and toroidal necks. The expectation is that standard neckpinch singularities at finitely many space-time instances of Ricci flow will happen at just the right places to effect the desired decomposition along spherical necks automatically and that the pinching of toroidal necks only happens in infinite Ricci flow time in a controllable way. One hopes to construct a standard surgery procedure for thin necks which replaces a piece of a spherical neck close to some cylinder $S^2 \times [a, b]$ by two positively curved spherical caps while keeping track of all curvature quantities -this procedure should then be carefully used each time a neckpinch forms. In particular, on a simply connected closed 3-manifold it is expected that after finitely many spherical neckpinch singularities which cut up the 3-manifold into disjoint pieces there will be only finitely many singularities asymptotic to selfsimilar shrinking 3-spheres analogous to the 2-d case: If this were true, we could retrace our flow backwards and conclude that the original surface was a finite connected sum of standard 3-spheres, hence a 3-sphere, thus proving the Poincare conjecture.

There are several major analytical obstacles to this program: There will be degenerate spherical neckpinches where some small 3-dimensional bubble doesn't pinch off properly and instead a 3-dimensional hemisphere gets squashed into a long thin horn developing rapidly towards a cusp. Again, this behavior is expected and cannot be avoided. A further difficulty lies in the need to demonstrate that these singularities are almost exactly axisymmetric in a precise quantitative way: Only then it will be possible to devise a detailed quantitative surgery algorithm for both cutting off the horns and cutting out the spherical necks.

By using first maximum principle estimates on the reaction-diffusion system for the Ricci curvature establishing that all singularities in 3-d are asymptotically non-negatively curved (Hamilton-Ivey estimates) and then proving a higher dimensional version of the Li-Yau type Harnack inequality for the Ricci tensor Hamilton gave a preliminary classification of finite time singularities of 3-d Ricci flow in [5]: The classification includes the selfsimilar shrinking 3-sphere, the expected selfsimilar spherical neckpinch singularity $S^2 \times \mathbb{R}$ and the translating Ricci solitons of strictly positive curvature modelling the tip of the horn in a degenerate spherical neckpinch. Unfortunately there is one more possible singularity model on the list, namely the "cigar" type translating 2-d soliton from (4.4) cross $\mathbb{R}$. The presence of such a singularity could indicate a degenerating cross-section of a pinching neck and would make the proposed surgery and continuation of the flow impossible. A further complication lay in the fact that Hamilton's classification only applied in regions where the curvature of the manifold is of comparable size to the maximum curvature in the region - making it inapplicable in certain situations. Just like in curve shortening or in 2-d Ricci flow a new estimate was needed to rule out the undesirable "cigar" singularity.

Perelman in his breakthrough contributions in [10] discovered new estimates for weighted volume distributions on the evolving three-manifold that enabled him to rule out the cigar type solitons since they have very little volume in large regions of small curvature, so called "collapsed regions", which are not present in the other singularity models.

One of the crucial new concepts developed by Perelman for this purpose is the concept of a new action on curves $\gamma : [\tau_1, \tau_2] \to (\mathcal{M}^3, g(t))$ in Ricci flow space-time, given by

$$\mathcal{L}(\gamma) := \int_\gamma \sqrt{\tau}(R(\gamma(\tau)) + |\dot{\gamma}(\tau)|^2)d\tau \tag{5.1}$$

that leads to the concept of a $\mathcal{L}$-shortest curve in a natural way. Fixing a point $p \in \mathcal{M}^3$ and $\tau_1 = 0$ we let $L(q,\tau)$ be the $\mathcal{L}$-length of the $\mathcal{L}$-shortest curve connecting $p$ and $q$ and denote by $l(q,\tau) = \frac{1}{2\sqrt{\tau}} L(q,\tau)$ the reduced distance. Then Perelman proves

**Theorem 5.1** *The reduced volume*

$$V(\tau) = \int_{\mathcal{M}} \tau^{-\frac{n}{2}} \exp\left(-l(q,\tau)\right) dq \tag{5.2}$$

*is monotonically increasing in* $\tau$ *if the metric satisfies* $\frac{d}{d\tau} g_{ij} = 2R_{ij}$.

This result following from a careful analysis of the variational behavior of the $\mathcal{L}-$ geodesics should be seen as related in spirit to the monotonicity formulae for mean curvature flow and harmonic map heat-flow discussed earlier in this article since the integrand resembles a nonlinear backward heat-kernel adapted to the space-time geometry of the Ricci flow. Also compare the $\mathcal{L}-$integral with the action appearing in the Li-Yau Harnack inequality (1.10).

The theorem just explained can be seen to exclude collapsing behavior on finite time intervals, thus ruling out the undesirable "cigar"-type singularity in 3-d Ricci flow at finite times. This is the starting point for a precise quantitative description of the remaining singularities involving spherical neckpinches and for encorporating the quantitative surgery procedure on necks and horns developed by Hamilton in [6].

Perelman sketches in [11] how to set up an algorithm from smooth Ricci flow and intermittent surgeries that maintains all the a priori estimates controlling the flow and keeps the total number of surgeries finite on finite time intervals. Finally, in [12] Perelman outlines that for a subclass of 3-manifolds containing all possible counterexamples of the Poincare conjecture the Ricci flow must stop at a finite time when the volume tends to zero. Simply connectedness and the classification of singularities described above ensures that there are only finitely many spheres left at this stage of the procedure which is the desired outcome implying the Poincare conjecture.

Since the time of the lecture in Oberwolfach three careful manuscripts have appeared that give details of the above results, by Kleiner-Lott [13], by Cao-Zhu [14] and Morgan-Tian [15]. The first two manuscripts include details also for the whole geometrisation conjecture while the last manuscript follows Perelman's arguments in [12] on finite time extinction providing a shorter route to the Poincare conjecture. Hamilton has announced an algorithm based on his previous work and Perelman's non-collapsing estimate that follows the strategy of [6]. Finally, in [8] Huisken and Sinestrari constructed a mean curvature flow with surgery for two-convex hypersurfaces that is inspired by Hamilton's surgery approach in [6].

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