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Drawing Large Pictures at Random
Oberwolfach Lecture 2007

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# Drawing large pictures at random

Wendelin Werner*

Oberwolfach Lecture 2007

## 1 Drawing at random?

This lecture is of very introductory nature. The goal will be to describe specific concrete questions, and to use them as a tool to convey some general ideas. Let me therefore skip the general introduction and immediately start with a first simple question: Is there a way to choose at random and uniformly among all possible choices a continuous curve (a $d$-dimensional curve, say)?

We know what it means to choose uniformly a point in a finite set $\Omega$ with $N$ elements: Each one has a probability $1/N$ to be selected. When the set $\Omega$ is infinite, the notion of "uniformity" is not canonical. In fact, when the set $\Omega$ is infinite and countable, it is easy to see that it is not possible to choose "uniformly" a point in this set (each one has the same probability to be chosen, and if this probability is 0, then one chooses no point...). So, one needs some additional structure on this infinite set in order to make sense of this "uniform measure".

One natural way to proceed is to discretize $\Omega$ in some "appropriate" way. For instance, we find consistent subdivisions of $\Omega$ into $N = 2^n$ pieces for $n = 1, 2, \ldots$ and we say that our uniformly chosen point in $\Omega$ will have a probability $2^{-n}$ to be in each one of these pieces.

In our question, a natural way to start goes as follows: Choose a grid that approximates the space. For instance, consider the square lattice with mesh-size $\delta$ to approximate the plane. Instead of choosing a continuous curve at random, choose a nearest-neighbor path started from the origin. If we fix the number $K$ of steps, the set of such nearest-neighbor paths has $4^K$ elements. So, we may say that (for small $\delta$), the law of this random nearest-neighbor path is "close" to the "uniform measure on paths". One can define our discrete measure on paths dynamically: It is the sum of independently chosen increments (at each step, the walk chooses one out of the four possible directions). This is just a random walk on the lattice.

We know that when $\delta \to 0$, and the time $K = K(\delta)$ is chosen appropriately (i.e. one can choose $K = 1/\delta^2$), then the random walk converges in law to a random continuous curve: Planar Brownian motion. So, in some sense, it makes

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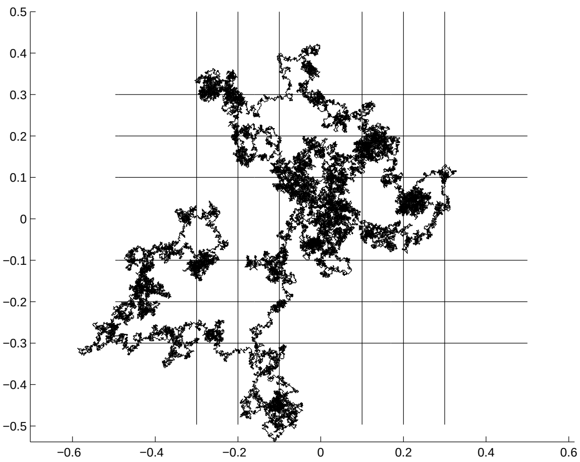


Figure 1: A long random walk

sense to state that *the uniform measure on continuous* 2*-dimensional curves is planar Brownian motion* (and the same statement in other dimensions).

It is worthwhile to insist on the following points:

- Our choice of the square lattice was arbitrary, so that one could wonder if the scaling limit keeps this arbitrary feature. We could for instance have chosen another regular lattice, such as the triangular lattice or the honeycomb lattice. But the fact that random walk converges to Brownian motion holds for these other lattices as well. In a way, Brownian motion is more universal than random walk.

- The law of Brownian paths are scale-invariant (if one forgets about their time-parametrization) and invariant under rotations. In fact, in two dimensions, a much stronger property holds, as described more than fifty years ago by Paul Lévy [11]: It is conformally invariant. Loosely speaking, if one distorts the picture of a planar Brownian path under any angle-preserving planar transformation, one still sees the picture of the sample of a Brownian path. Here is a more precise version of the conformal invariance statement: Consider an open domain $D$ and a planar Brownian motion $B$ started from a point $z$ stopped at its first exit time $T$ of the domain $D$. Suppose that $\Phi$ is a one-to-one angle-preserving map from $D$ onto $D'$ (we know from Rieman's mapping theorem that when $D$ and $D'$ are two given bounded simply connected domains, such maps exist), then the law of the image of $B$ under $\Phi$ is that of a planar Brownian motion started from $\Phi(z)$ and stopped at its first exit of $D'$. This fact can for instance be proved using stochastic calculus tools such as Itô's formula.

- After $N$ steps, the simple random walk $S_N = X_1 + \cdots + X_N$ on $\mathbb{Z}^2$ is roughly at distance $\sqrt{N}$ of the starting point. One way to see this is to

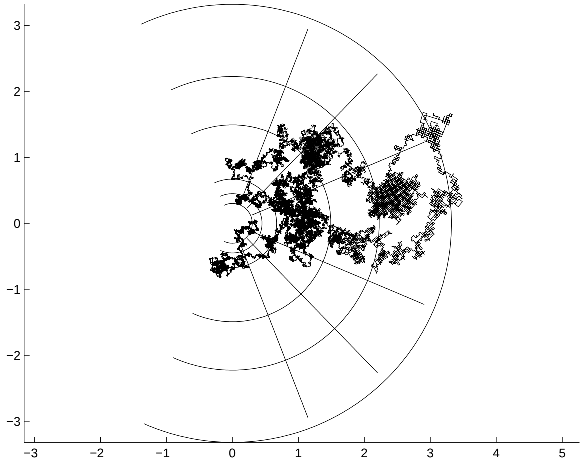

Figure 2: Image of the previous figure under a conformal map

compute the mean value of $S_N^2$ and to note that

$$E(S_N^2) = E((\sum_{j=1}^{N} X_j)^2) = \sum_{i,j=1}^{N} E(X_i X_j) = \sum_{i=1}^{N} E(X_i^2) = N.$$

In fact, the law of $S_N/\sqrt{N}$ converges to a normal distribution (this is the *central limit Theorem*). This square root was the reason for our $K = \delta^{-2}$ choice of the length. One consequence (that would require more justification in order to be rigorous) is that the Hausdorff dimension of planar (and higher-dimensional) Brownian motion is 2. It does not quite fill the plane, but it is quite "fat", as shown in the picture...

Note that the geometry of a two-dimensional Brownian path is extremely complicated. To convince you of this fact, here is a striking result [10]: There exists points in the plane that it has visited an uncountable set of times before time 1!

## 2 Use an eraser

So, a uniformly chosen random picture is a Brownian motion? But for many purposes, we may require our curve to be self-avoiding and not a clumsy cluster like Brownian motion. Around us, many self-avoiding paths look random (rivers, coastlines, level lines etc.) and we may wonder what sort of mathematical object one could use to describe them.

There are two approaches. The first one is *dynamical*: Take a pen and start to draw following some local rules for the evolution of the pen on the paper

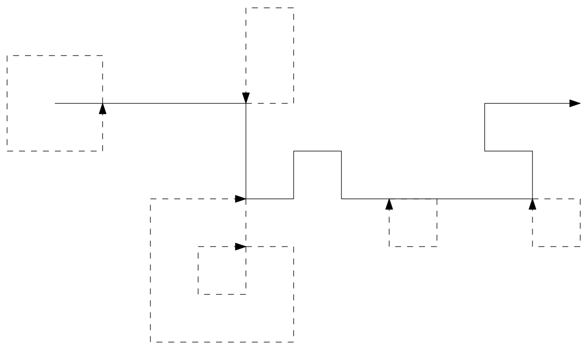

Figure 3: The loop-erasing procedure

that ensure that in the end, one has created a (random) self-avoiding path. The second approach is to look at the path as a whole, and to try to define a "uniform measure" on the space of all self-avoiding paths using a discretization procedure. In the next three sections, we shall try to describe three concrete and natural ways to proceed.

The first way has been introduced by Greg Lawler in [4]. It may look at first slightly artificial, but it turns out to be very natural and to have many nice properties. The idea is to use a usual (non self-avoiding) random walk and to find a self-avoiding subpath $(L_l)$ of the random walk $(S_n)$, by erasing its loops. More precisely, we choose to erase loops in chronological order. In other words, if a nearest-neighbor path $S_0 = x_0, \ldots, S_N = x_1$ is given, we define inductively $L_0 = x_0$, $j_0 = \max\{j \leq N \ : \ X_j = x_0\}$ and for $l \geq 0$,

$$L_{l+1} = X_{j_l+1} \text{ and } j_{l+1} = \max\{j \leq N \ : \ X_j = L_{l+1}\}.$$

In other words, $L$ follows the track of $S$, but as soon as it creates a loop, we erase the loop. In the end, we get a self-avoiding nearest-neighbor path that joins $x_0$ and $x_1$. Note that the number of steps of the loop-erasure of a $N$-step random walk varies. For a usual very long random walk, the distance between $S_0$ and $S_N$ is of the order of $N^{1/2}$, but we expect that most of the $N$ steps of the walk will be erased by the loop-erasing procedure.

Note that the loop-erasing procedure is non-symmetric: One erases the loops in chronological order from the starting point of the random walk to its end-point. However, there is a hidden symmetry: Let us fix the two points $x_0$ and $x_1$. Start a random walk from $x_0$ and stop it at its first hitting time of $x_1$ and loop-erase it chronologically. This gives a random way to generate a self-avoiding curve from $x_0$ to $x_1$. We could have also have interchanged the role of $x_1$ and $x_0$ (start a random walk from $x_1$, stop it at its first hitting of $x_0$ and loop-erase it) and this would have created a random self-avoiding curve from $x_1$ to $x_0$. It turns out that in fact these two algorithm generate the same law on self-avoiding curves that join these two points. One way to generalize

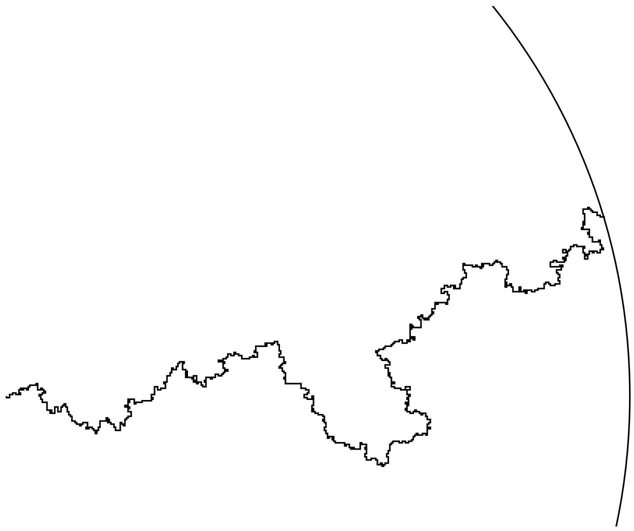

Figure 4: A very long loop-erased random walk

this symmetry goes as follows [16] in the case where the graph $G$ is finite (but this can be generalized to infinite graphs as well): Choose uniformly among all choices, a subgraph of $G$ with one connected component (it is "spanning") but without loop (it is a tree). On this random tree (called the "uniform spanning tree"), there exists a unique (self-avoiding) path that joins $x_0$ to $x_1$. The law of this path is exactly that of a loop-erased walk that we have just described (and indeed the role of $x_0$ and $x_1$ can be interchanged). For some planar graphs, such as the square grid, there is a one-to-one mapping from the set of spanning trees onto the set of domino tilings. These domino tilings can be enumerated thanks to determinants using ideas of Kasteleyn and others. This has led Rick Kenyon [3] to the following difficult result: The mean number of steps of the loop-erased random walk joining two points at distance $\sqrt{N}$ from each other is of the order of $(\sqrt{N})^{5/4}$. So, loosely speaking, one can say that the long loop-erased random walks have a "mean" dimension equal to 5/4.

In fact, one can even prove that there exists a scaling limit for the law of loop-erased random walk in a domain when one lets the mesh of the grid vanish. The following results can be rigorously proved [6]. Note that they parallel the statements for planar random walks:

- Considers a simply connected domain $D$, a starting point $z$ and random walks $S$ starting from $z$ on fine-mesh $\delta$ approximations of $D$, stopped at their exit of $D$. Then, when $\delta \to 0$, the law of the loop-erasure of $S$ converges to that of a continuous random self-avoiding path from $z$ to $\partial D$.

- The law of this scaling limit does not depend on the lattice that we used to approximate the domain (as long as random walk on this lattice converges to Brownian motion).

- The law of this scaling limit is supported on fractal curves with Hausdorff dimension equal to 5/4.
- One has asymptotical conformal invariance, with the same statement as for Brownian motion (in two conformally equivalent domains, the law of the respective scaling limits of loop-erased random walks are conformally equivalent).

In some way, one can interpret this random continuous curve as "loop-erased brownian motion". But this can not simply be defined directly. Indeed, the planar Brownian motion creates immediately infinitely many small loops, so that deciding what loop to erase first (and deciding it in a conformally invariant way) is more than delicate... We shall describe later the construction of this random curve.

# 3 Short memory

The previous strategy of drawing self-avoiding curves was rather "careless". One never foresees anything and just wanders around at random in the plane and a posteriori erases the loops that one has created. One could be slighlty more cautious and simply decide not to close a loop when one is in the situation of doing so. In other words, we choose at random at each step one of the "allowed steps" where we know that we do not trap the tip of the curve. In this way, one will be able to continue to grow the self-avoiding curve forever. This is sometimes called a myopic self-avoiding walk or a self-avoiding true walk. More precisely, the law of the walk can be described as follows. At each step $j$, we let $R$ denote the infinite connected component of the infinite graph obtained by removing $S_0, \ldots, S_j$. We assume (and this will be the case) that $R$ contains some neighbors of $S_j$ (in other words, there exists an infinite path from $S_j$ that avoids $S_0, \ldots, S_{j-1}$). Then, conditionally on $S_0, \ldots, S_j$, we choose uniformly one of these neighbors that are in $D$, and call it $S_{j+1}$.

This procedure is particularly simple on the honeycomb lattice. Indeed, because each site has only three neighbors, only two cases can occur at the $j$-th step (recall that one of the neighbors of $S_j$ is $S_{j-1}$ and therefore not allowed for $S_{j+1}$): Either $S_j$ has one just neighbor in $R$ and then the move to $S_{j+1}$ is forced, or it has has two neighbors in $R$ and then one tosses a fair coin to choose which one is $S_{j+1}$. Hence, on this lattice, one can construct this myopic random walk dynamically using a sequence of fair coin tosses.

Again, one is now able to prove that there exists a scaling limit for the law of this myopic walk in a domain when one lets the mesh of the honeycomb lattice vanish. The following results can be rigorously established [22, 5, 1, 24]:

- If one considers a simply connected domain $D$ and a starting point $z$ and a myopic walk $M$ starting from $z$ on a fine-mesh $\delta$ approximation of $D$

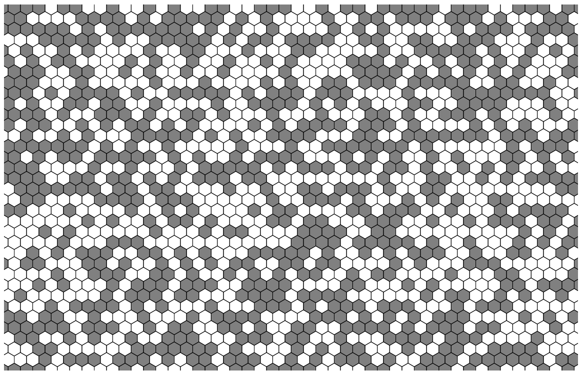

Figure 5: Percolation

stopped at its exit of $D$. Then, when $\delta \to 0$, the law of $M$ converges to that of a continuous random path $\mathcal{M}$ from $z$ to $\partial D$.

- The law of this scaling limit is supported on fractal curves with Hausdorff dimension equal to 7/4. In the discrete case, the typical number of steps needed to reach distance $N$ increases like $N^{7/4+o(1)}$ as $N \to \infty$.
- One has asymptotical conformal invariance, with the same statement as for Brownian motion (in two conformally equivalent domains, the law of the respective scaling limits of myopic walks are conformally equivalent).

Some differences with the loop-erased walk should be however stressed:

- This is proved (at this point) only for this particular lattice.
- In the scaling limit, this curve is not self-avoiding anymore (even if it is self-avoiding on the lattice) because it has long loops ($M_j$ and $M_{j+l}$ can be neighbors for very large $l$).

The clue to these results is that this model can be embedded in a nice and simple lattice model: Critical percolation. If one colors each hexagonal cell of the honeycomb lattice in black or white with probability 1/2. We can decide to color progressively each cell that the walk meets for the first time according to the direction of the turn that the walk makes when it "bumps" into that cell. If it decides to turn right, color this cell in black and if it turns left color it in white (unless the move is forced). In this way, we see that the walk follows an interface between black and white clusters. Conformal invariance of this percolation model has been established by Smirnov [22] and together with the computations involving the SLE paths that we will describe a little later, it is now well-understood mathematically.

Models such as percolation have been of interest to physicists, and the theoretical physics community has succeeded to develop tools and ideas that enabled

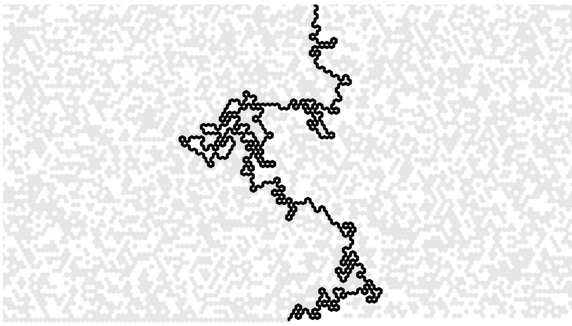

Figure 6: First steps of a myopic curve

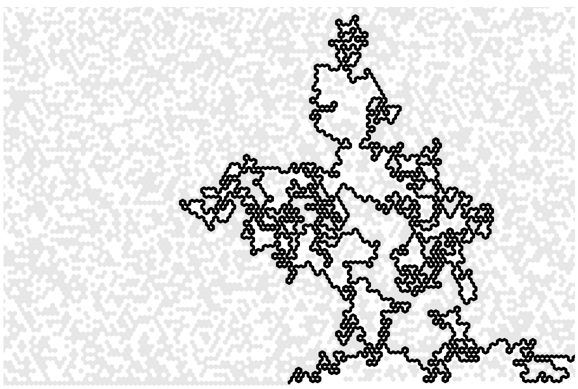

Figure 7: The same path a little later

them to predict many features of these random two-dimensional curves. Most of the results that we have stated and that we will state in the paper have been conjectured in the physics literature before: The loop-erased random walk exponent 5/4 appears in a paper of Satya Majumdar [13] and the 7/4 exponent of the myopic walk appears for instance in papers of Duplantier and Saleur [18]. Smirnov's proof of conformal invariance is done by proving a formula predicted by John Cardy [2].

# 4 Optimal strategy

Our third attempt will be to try to define the "uniform" measure on self-avoiding curves via a discretization procedure. Consider a regular planar lattice (such as the square lattice, the triangular lattice, the honeycomb lattice etc.) and define the set $\Omega_N$ of all $N$-step self-avoiding nearest-neighbor paths on this lattice that start from the origin. Choose one at random and uniformly in this set. Note that this definition is not "dynamic"; we first fix $N$ and then choose a path uniformly in $\Omega_N$.

A first question is how far from the origin a self-avoiding walk with a large number of steps lies. In other words, if the path $(W(0), \ldots, W(N))$ is chosen uniformly in $\Omega_N$, how large is $W(N)$ (typically i.e. with a high probability – in the same sense as $S(N)$ was of the order of $\sqrt{N}$ for simple random walk)? This time, this is an open question. But physicists (this prediction is due to Bernard Nienhuis [15]) tell us what we should expect:

- It is conjectured that typically $W(N)$ will be at distance $N^{3/4}$ of the starting point.
- It is conjectured that the law of the long planar self-avoiding walk has a continuous scaling limit, that does not depend on the chosen lattice, that it is conformally invariant (in some appropriate sense) and supported on the set of curves with fractal dimension 4/3 (this corresponds to the 3/4 exponent in the first conjecture).

At this point, we are still quite far from proving this conjecture, but we know the candidate for this conformally invariant scaling limit, and understand why the numbers 3/4 (and 4/3) should appear [7]. Let us briefly describe what we can say to describe this continuous candidate for the scaling limit.

Suppose that for each simply connected domain $D$ with two boundary points $a \neq b$, we have a law $P_{D,a,b}$ on self-avoiding curves that join $a$ to $b$ in $D$. Suppose furthermore that:

- We have conformal invariance: The image of the measure $P_{D,a,b}$ under a conformal map $\Phi$ is the measure $P_{\Phi(D),\Phi(a),\Phi(b)}$.
- We have restriction. If $D' \subset D'$ is any other simply connected connected domain that still has $a$ and $b$ on its boundary, then the probability $P_{D,a,b}$ conditioned on the event that the curve stays in $D'$ is exactly $P_{D',a,b}$.

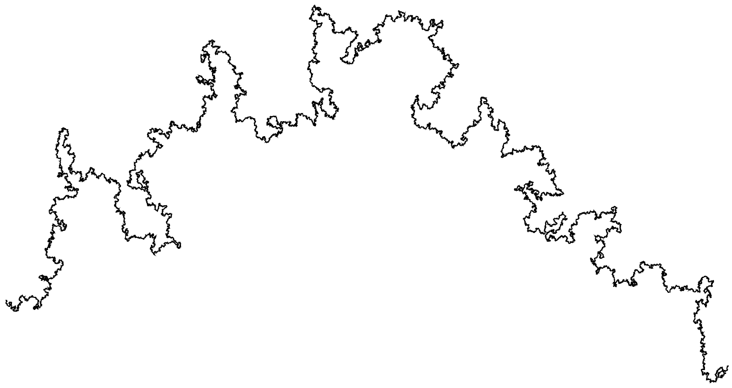

Figure 8: Part of a very long self-avoiding path (simulated by Tom Kennedy)

Note that this second condition is natural if we look for the "uniform" measure on self-avoiding curves: Conditioning the uniform measure defined on a larger set $A_1$ (the curves that stay in $D$) on the event that one is in a smaller set $A_2$ (here the curves that stay in $D'$), one obtains the uniform measure on $A_2$.

One can show [8] that:

- The measures $(P_{D,a,b})$ are fully characterized by this properties.
- They exist, are unique and supported on the set of curves with Hausdorff dimension 4/3.

This shows that very loosely speaking, "in some conformally invariant way" the uniform measure on self-avoiding curves is supported on the set of paths with dimension 4/3 and that "most" self-avoiding paths have dimension 4/3.

# 5 Random growth and conformal invariance

We now describe the idea that lead Oded Schramm to the definition of the Schramm-Loewner Evolution (SLE) in [19]. This is the clue to define directly all these continuous (conjectured or proved) scaling limits for our discrete self-avoiding random curves.

Recall from Riemann's mapping theorem that any two simply connected domains $D$ and $D'$ (that are not equal to the entire plane) are conformally equivalent: One can fin a one-to-one map from $D$ onto $D'$. Furthermore, one can impose that the image of a specified boundary point $a$ of $D$ is a specified boundary point $a'$ of $D'$, and that the image a given point $z \in D$ is a specified point $z' \in D'$. Hence, if one thinks "modulo conformal invariance", all simply connected domains with a fixed inner point look the same seen from a boundary point.

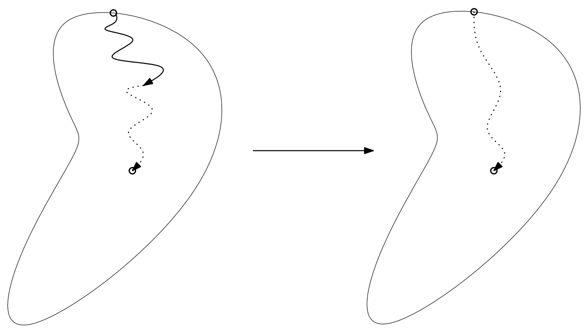

Figure 9: The conformal map $g_t$

Now, we would like to grow from $a \in \partial D$ a random (self-avoiding) curve towards $z \in D$. Let it grow a little bit, say from $\gamma(0) = a$ to $\gamma(t) \in D$. We still have to grow a curve $\tilde{\gamma}$ from $\gamma(t)$ to $z$. The self-avoidance constraint imposes that the remaining path $\tilde{\gamma}$ stays in the domain $D \setminus \gamma[0,t]$. This is a simply connected domain (call it $D_t$) and we have to construct a path from its boundary point $\gamma(t) \in \partial D_t$ to an inside point $z \in D_t$. So, we still have a similar problem as the one that we have started with. In fact, we have the same problem "modulo conformal invariance". It is therefore natural to define the (unique) conformal map $g_t$ from $D_t$ back onto $D$ such that $g_t(\gamma(t)) = a$ and $g_t(z) = z$. Then $g_t(\tilde{\gamma})$ is a random curve from $a$ to $z$ in $D$.

Note that the knowledge of all maps $(g_t, t \geq 0)$ defines the curve $\gamma$. Indeed, $\gamma_t = g_t^{-1}(a)$ for each $t \geq 0$.

We assume (and this is natural for the models that we are considering because the discrete models satisfy an analogous property) that conditionally on $\gamma[0,t]$, the law of $g_t(\tilde{\gamma})$ is identical to that of $\gamma$. In other words, modulo conformal invariance, once we have have traced $\gamma[0,t]$, the law of $\tilde{\gamma}$ is still the same. In particular, we see that the first part $\gamma[t,2t] = \tilde{\gamma}[0,t]$ has the same law as $g_t^{-1}(\hat{\gamma}[0,t])$ where $\hat{\gamma}$ is an independent copy of $\gamma$. So, $g_{2t}$ is obtained as the iteration of two independent conformal maps: The one that removes $\hat{\gamma}[0,t]$ and the map $g_t$. Hence, the "independence" is nicely phrased in terms of iterations of conformal maps: The map $g_{2t}$ is the iteration of two independent copies of $g_t$. Similarly, the map $g_{nt}$ is the iteration of $n$ independent copies of $g_t$, and $g_t$ itself can be constructed as the iteration of $n$ independent copies of $g_{t/n}$. This shows that the entire law of $\gamma$ is encoded in the behavior of $g_\epsilon$ of infinitesimal $\epsilon$.

Charles Loewner has studied such deterministic families $g_t$ of conformal maps corresponding to slit domains in the 1920's. He has shown that they can be simply encoded by a one-dimensional real-valued curve $(\zeta(t), t \geq 0)$. The fact that a two-dimensional curve can be encoded by a one-dimensional curve is not surprising: Recall that smooth curve $Z(s)$ parametrized by arclength is fully determined by its starting point and its derivative $(Z'(s), s \geq 0)$. But this latter function takes its value in the (one-dimensional) unit circle. In the present case, one can intuitively understand $\zeta(t)$ as follows: The increments of $\zeta$ correspond to the fact that the infinitesimal slit growing at time $t$ "turns right or left" seen from $z$ in $D_t$. Anyway, the fact that $g_t$ is obtained by iteration of independent identically distributed conformal maps is translated in terms of $\zeta$ into the fact that $\zeta$ is a process with independent increments. But we know that the only scale-invariant real-valued continuous process with independent increments is Brownian motion and its multiples. Hence, for some $\kappa > 0$, $(\zeta(t/\kappa), t \geq 0)$ is a standard Brownian motion.

Finally, we can conclude this heuristic argument with the fact that if $P$ is the law of a random curve satisfying the previous property, then there exists a $\kappa > 0$, such if one defines $\zeta(t) = \beta(\kappa t)$ when $\beta$ is a one-dimensional Brownian motion, then one can construct $(g_t)$ starting from $(\zeta)$ and then, the law of the corresponding curve $\gamma$ is exactly $P$. Hence, there exists only a one-dimensional family of candidates for the law of $P$.

In fact, one can prove [17, 1, 5, 6, 22, 8]:

- For each $\kappa \in [0, 8]$, this procedure constructs indeed a random curve $\gamma$ called the SLE with parameter $\kappa$.
- For each $d \in [1, 2]$, there exists exactly one such law that is supported on the set of curves with dimension $d$. The relation between $d$ and $\kappa$ is $d = 1 + \kappa/8$ ("the larger $\kappa$, the wigglier the curve $\gamma$").
- In fact, when $d \leq 3/2$, the random curve $\gamma$ is indeed self-avoiding, whereas when $d > 3/2$, then the random curve has double points (like the scaling limit of the myopic walk). The dimension 3/2 is in a way the maximal possible dimension of random conformally invariant self-avoiding curves.
- The random curve with dimension 5/4 is the scaling limit of loop-erased random walk.
- The random curve with dimension 4/3 is closely related to the law on self-avoiding curves satisfying conformal restriction described in the previous section.
- The random curve with dimension 7/4 is the scaling limit of the myopic walk/percolation interfaces.

# 6 Random shapes

We now wish to describe random "islands" i.e. random self-avoiding loops. In this context, it it natural to look for a measure on the set $\Omega$ of self-avoiding loops in the plane that is scale-invariant and translation invariant. Hence, we are looking for a measure with infinite mass. But this measure $\mu$ corresponds to a probability measure $P$ on self-avoiding loops in the following way: We define the product measure $(d\rho/\rho) \otimes d^2 z \otimes P(d\gamma)$ on $(0, \infty) \times \mathbb{R}^2 \times \Omega$ and define $\tilde{\gamma} = z + \rho\gamma$, and we say that $\mu$ is the measure under which $\tilde{\gamma}$ is defined. With this definition, the measure $\mu$ is scale-invariant. If $P$ is rotationally invariant, then so is $\mu$. We may wonder if there is a way to define a "conformally invariant" measure $\mu$. So, for each simply connected domain $D$, we define $\mu_D$ to be the measure $\mu$ restricted to the set of islands that stay in $D$.

A simple argument [25] shows that there exists (up to multiplication by a constant) a unique measure $\mu$ that is conformally invariant in the sense that for any two simply connected domains $D$ and $D'$, the measures $\mu_D$ and $\mu_{D'}$ are conformally equivalent. The argument goes as follows:

- Suppose that $\mu$ satisfies this property. Then, for any $D' \subset D$ that contains the origin, the mass of the set of loops that surround the origin, stay in $D$ but not in $D'$ is equal to a constant times $\log |\Phi'(0)|$, where $\Phi$ is a conformal map from $D$ onto $D'$ that preserves the origin.
- There exists at most one measure $\mu$ satisfying this formula.
- We can construct explicitly a measure $\mu$ that is conformally invariant [9]: Just start with the law $P$ of a standard planar Brownian loop (this is planar Brownian motion, started from the origin and conditioned to be back at the origin at time 1). Conformal invariance of $\mu$ basically follows from conformal invariance of planar Brownian motion.

However, it turns out that there are two other ways to construct such a conformally invariant measure $\mu$:

- The first one is to come back to the percolation model that we have briefly described, and to define $\mu$ as the fine-mesh limit of the shape of percolation clusters.
- The second one is to come back to our measure on self-avoiding paths of dimension 4/3 defined via SLE, and to use it to define a "uniform" measure on self-avoiding loops.

Hence, we end up with the fact that these three constructions leads to the same measure: Shapes of Brownian islands, of (the scaling limit of) percolation clusters, and self-avoiding loops defined via SLE with $\kappa = 8/3$ are identical! A consequence is that the dimension of the outer boundary of Brownian islands and that of the scaling limit of percolation clusters has dimension equal to 4/3. Note that Mandelbrot [14] had already predicted back in the early 1980's that the shape of Brownian islands was an interesting object.

# 7 Further pictures

We give an impressionistic overview of some related recent results (this short list is very far from exhaustive!):

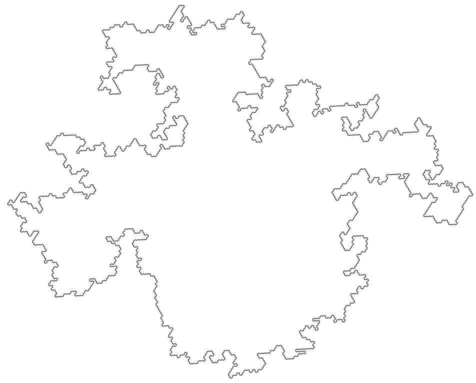

Figure 10: Is this the outer boundary of a percolation cluster or of a Brownian loop?

### 7.1 Random pointillism

We have just seen that there exist a "universal conformally invariant random shape" described by the measure $\mu$. It is tempting to use it as a building block in order to construct more complicated pictures. For instance, for each domain $D$, define a Poisson point process with intensity $c\mu_D$. In other words, draw at random shapes (they can overlap and intersect each other) described by the measure $\mu$. This is a little bit like if each shape appears independently on the picture with a probability proportional to its "intensity" according to $\mu$. This is the Brownian loop-soup [9].

Since $\mu$ is conformally invariant, the obtained random countable collection of shapes in $D$ is conformally invariant as well. If we fill in black the interior of all the shapes, we can wonder what the set of "uncovered points" in $D$ look like. It turns out that this will depend a lot on the quantity $c$. As long as $c$ is not large, this white set still has non-trivial connected components, and it turns out that their boundaries are described [21] via SLE loops of dimension $d = d(c)$ varying from $4/3+$ (for very small $c$) up to $3/2$ for the critical maximal $c = c_0$. When $c > c_0$, the white set has no non-trivial connected components anymore. This quantity $c$ corresponds to the "central charge" of the corresponding model in the language of representation theory and Conformal Field Theory.

### 7.2 Random level lines

It is possible to define a Gaussian random field $(X(x), x \in D)$ in each domain $D$ via its covariance matrix $E(X(x)X(y)) = G(x,y)$, where $G$ is the Green's function in $D$. This is not a random function because $G(x,x) = \infty$, but it is well-defined as a random distribution. It inherits conformal invariance properties from the definition of the Green's function (recall that this function describes the mean occupation time of the neighborhood of $y$ by a Brownian motion started from $x$ and killed at its first exit of $D$). Sheffield, and Schramm-Sheffield [20] have shown that it is possible to give a rigorous meaning to the notion of level-lines of this random distribution, and to see that all these SLE curves as deterministic functionals of the Gaussian free field. This important facts seem closely related to the notions of Coulomb gas and of quantum gravity that had been developed by physicists.

### 7.3 Other lattice models

Physicists studying phase transition were first interested in models that for instance describe magnetism such as the Ising model, where as opposed to percolation, the states of the different cells are not independent. A major conjecture in the field is to establish their conformal invariance. Stas Smirnov [23] has recently succeeded in proving conformal invariance of several such models (and in particular for the Ising model), and to prove the convergence of the corresponding interfaces to SLE curves in the fine-mesh limit. The idea is to identify combinatorial features that one can interpret as a discrete analyticity of certain funtionals, and to prove that this discrete analyticity yields convergence to a continuous analytic function in the fine-mesh limit.

## References


[1] V. Beffara (2004), The dimension of the SLE curves, Ann. Probab., to appear.

[2] J.L. Cardy (1984), Conformal invariance and surface critical behavior, Nucl. Phys. **B 240**, 514-532.

[3] R. Kenyon (2000), The asymptotic determinant of the discrete Laplacian, Acta Math, **185**, 239-286.

[4] G. F. Lawler (1980), A self-avoiding random walk, Duke Mathematical Journal **47**, 655-694.

[5] G.F. Lawler, O. Schramm, W. Werner (2001), Values of Brownian intersection exponents II: Plane exponents, Acta Mathematica **187**, 275-308.

[6] G.F. Lawler, O. Schramm, W. Werner (2004), Conformal invariance of planar loop-erased random walks and uniform spanning trees, Ann. Prob. **32**, 939-995.

[7] G.F. Lawler, O. Schramm, W. Werner (2004), On the scaling limit of planar self-avoiding walks, in Fractal geometry and application, A jubilee of Benoit Mandelbrot, AMS Proc. Symp. Pure Math. **72**, Part 2, 339–364.

[8] G.F. Lawler, O. Schramm, W. Werner (2003), Conformal restriction properties. The chordal case, J. Amer. Math. Soc. **16**, 917-955.

[9] G.F. Lawler, W. Werner (2004), The Brownian loop-soup, Probab. Theory Rel. Fields **128**, 565-588.

[10] J.-F. Le Gall (1987), Le comportement du mouvement brownien entre les deux instants o'u il passe par un point double. J. Funct. Anal. **71**, 246-262.

[11] P. Lévy, *Processus Stochastiques et Mouvement Brownien*, Gauthier-Villars, Paris, 1948.

[12] K. Löwner (1923), Untersuchungen über schlichte konforme Abbildungen des Einheitskreises I., Math. Ann. **89**, 103-121.

[13] S.N. Majumdar (1992), Exact fractal dimension of the loop-erased random walk in two dimensions, Phys. Rev. Lett. **68**, 2329–2331.

[14] B.B. Mandelbrot, *The Fractal Geometry of Nature*, Freeman, 1982.

[15] B. Nienhuis (1982), Exact critical exponents for the $O(n)$ models in two dimensions, Phys. Rev. Lett. **49**, 1062-1065.

[16] R. Pemantle (1991), Choosing a spanning tree for the integer lattice uniformly. Ann. Probab. **19**, 1559 - 1574.

[17] S. Rohde, O. Schramm (2005), Basic properties of SLE, Ann. Math. **161**, 883–924.

[18] H. Saleur, B. Duplantier (1987), Exact determination of the percolation hull exponent in two dimensions, Phys. Rev. Lett. **58**, 2325.

[19] O. Schramm (2000), Scaling limits of loop-erased random walks and uniform spanning trees, Israel J. Math. **118**, 221–288.

[20] O. Schramm, S. Sheffield (2006), Contour lines of the two-dimensional discrete Gaussian free field preprint, math.PR/0605337

[21] S. Sheffield, W. Werner, in preparation.

[22] S. Smirnov (2001), Critical percolation in the plane: conformal invariance, Cardy's formula, scaling limits, C. R. Acad. Sci. Paris Sr. I Math. **333**, 239–24

[23] S. Smirnov (2007), Conformal invariance in random cluster models. I. Holomorphic fermions in the Ising model, preprint.

[24] S. Smirnov, W. Werner (2001), Critical exponents for two-dimensional percolation, Math. Res. Lett. **8**, 729-744.

[25] W. Werner (2008), The conformal invariant measure on self-avoiding loops, J. Amer. Math. Soc. **21**, 137-169.

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