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LINKING AND CLOSED ORBITS

STEFAN SUHR AND KAI ZEHMISCH

Abstract. We show that the Lagrangian of classical mechanics on a Riemannian manifold of bounded geometry carries a periodic solution of motion with prescribed energy, provided the potential satisfies an asymptotic growth condition, changes sign, and the negative set of the potential is non-trivial in the relative homology.

1. Introduction

In Hamiltonian mechanics one considers integral curves $u = u(t)$ of the Hamiltonian vector field X_H for the **Hamiltonian**

$$H(u) = \frac{1}{2}|u|^2 + V\big(\pi(u)\big)$$

on the phase space T^*Q. The motion of particles in the configuration space Q is described by $q(t) = \pi\big(u(t)\big)$ for the projection π of the cotangent bundle. The kinetic energy $\frac{1}{2}|\,.\,|^2$ is defined via the (dual) norm of a Riemannian metric g on Q. The potential energy V is a smooth function on Q. For the Liouville 1-form λ on T^*Q the Hamiltonian vector field X_H is determined by $i_{X_H}\mathrm{d}\lambda = -\mathrm{d}H$, i.e. the trajectories locally solve the Hamilton equations.

The total energy $H(u)$ along a solution $u = u(t)$ is preserved. So in order to understand the dynamics of the Hamiltonian vector field one can restrict to hypersurfaces $M = \{H = E\}$ of fixed energy E. An important question is the existence of periodic solutions $u = u(t)$ on a regular **energy surface** M. If the energy E is greater than $\sup_Q V$ the Hamiltonian flow on M is conjugate to the geodesic flow of the Jacobi metric $(E - V)^{-1}g$ according to the Euler-Maupertuis-Jacobi principle. If additionally Q is compact the existence of periodic orbits follows from the existence of closed geodesics for the Jacobi metric, see [20, Section 4.4]. Existence of closed geodesics on compact Riemannian manifolds is well known, cf. [22, 3]. In the case that the energy surface M intersects the zero section Q of T^*Q the Jacobi metric becomes singular along $\{V = E\}$. As explained in [20, Section 4.4] periodic orbits on compact energy surfaces M can be found via so-called brake orbits of a perturbed Jacobi metric, see [7, 16, 4].

Alternative approaches to periodic orbits on a compact energy surfaces via symplectic geometry are given in [33, 19, 20, 34, 12, 14]. Rabinowitz applied the minmax method to the Lagrangian multiplier functional for the symplectic action $\int \mathrm{d}\lambda$ on

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1-periodic curves $u = u(t)$ constraint by $\int H\big(u(t)\big)\mathrm{d}t = 0$, see [30], which led to the Rabinowitz-Floer theory [9].

Periodic orbits do not always exists if the energy surfaces M is non-compact. For example the Hamiltonians $\frac{1}{2}p^2 - q$ and $\frac{1}{2}p^2 - \frac{1}{2}q^2$ on $T^*\mathbb{R}$ do not allow any closed orbit. Moreover, if the potential equals $-E < 0$ periodic orbits correspond precisely to closed geodesics via π, cf. [13]. On the other hand if $\{V < E\}$ is connected with disconnected boundary and the potential satisfies asymptotic growths conditions periodic solutions are obtained in [28] using the Dirichlet principle for the energy functional of the Jacobi metric.

As it was shown in [5] in order to obtain periodic orbits the Lagrangian action functional can be used directly. For non-compact energy surfaces M periodic solutions do exists if the potential function on $\mathbb{R}^n$ satisfies certain topological and asymptotic conditions. In [6] the authors generalized the existence result to Riemannian manifolds with flat ends that satisfy a vanishing condition on the free loop space homology. The aim of these notes is to generalize the results obtained in [5, 6] to Riemannian manifolds of bounded geometry without restrictions on the topology of the loop space. As we will discuss in Section 1.2 all the above mentioned classical existence results with a sign changing potential are consequences of our theorem.

1.1. **The theorem.** We consider a connected manifold Q of dimension n together with a Riemannian metric $g = \langle\, .\, ,\, .\, \rangle$ of **bounded geometry** in the following sense: We require that the **injectivity radius**

$$\text{(INJ)} \qquad \operatorname{inj} g > 0,$$

which by definition is the infimum of the injectivity radii of all points of Q. In particular, the Riemannian manifold (Q, g) is complete, i.e. the geodesic flow is global. Moreover, we require that there exists a positive constant C such that the Riemannian curvature tensor R satisfies

$$\text{(CB)} \qquad |R|,\ |\nabla R| < C,$$

where ∇ denotes the covariant derivative w.r.t. the Levi-Civita connection and $|\,.\,|$ the norm on the tensor bundles induced by the metric.

We replace the **potential** V by $V - E$ and require that

$$\text{(REG)} \qquad 0 \text{ is a regular value of } V \text{ and } \{V = 0\} \text{ is not empty.}$$

This is equivalent to the requirement that 0 is a regular value of H and that the energy surface $M = \{H = 0\}$ intersects the zero section non-trivially. We call $N = \{V < 0\}$ the **negative set**. Notice, that the closure equals $\pi(M)$. We require that the relative homology group of the negative set

$$\text{(LNK)} \qquad H_*(N, \partial N) \neq 0$$

is non-trivial for some degree $* = 1, \ldots, n$.

In addition, the potential is required to satisfy the following **asymptotic conditions**: We denote by $\operatorname{grad} V$ the gradient of the potential V, the dual vector field of the differential TV w.r.t. the metric. The second covariant derivative ∇TV, the so-called **Hessian**, is a symmetric bilinear form which we denote by $\operatorname{Hess} V$. We require that there exist a positive constant K and a compact subset $\hat{Q}$ of Q such that

$$(\mathrm{AC}_1^a) \qquad |\operatorname{grad} V| \geq \frac{1}{K} \quad \text{on } Q \setminus \hat{Q}$$

and

$$(\mathrm{AC}_1^b) \qquad \frac{|\operatorname{Hess} V|}{|\operatorname{grad} V|} \leq K \qquad \text{on } Q \setminus \hat{Q}.$$

To formulate the second asymptotic condition we distinguish a base point o in $\hat{Q}$ and denote by $\operatorname{dist}(o, q)$ the distance to $q \in Q$, i.e. the minimal **length**

$$\operatorname{length}(c) = \int |\dot{c}| \mathrm{d}t$$

of piecewise C^1-curves c connecting o with q. We require that

$$(\mathrm{AC}_2) \qquad \frac{|\operatorname{Hess}_q V|}{|\operatorname{grad}_q V|} \longrightarrow 0 \qquad \text{if } \operatorname{dist}(o, q) \to \infty.$$

Of course (AC_2) implies (AC_1^b). If Q is compact the asymptotic conditions are automatically satisfied.

Assuming the above requirements the aim of these notes is to find periodic critical points $q = q(t)$ of the action with the **Lagrangian**

$$L(v) = \frac{1}{2}|v|^2 + U\big(\pi(v)\big)$$

constraint to the energy condition $\frac{1}{2}|\dot{q}|^2 + V(q) = 0$. Here we set $U = -V$ and π denotes the projection of the tangent bundle of Q. Notice, that the period of $q = q(t)$ is free to vary but in view of (REG) it has to be non-zero. The Euler-Lagrange equation for the critical points are given by $\ddot{q} = \operatorname{grad}_q U$, where $\ddot{q} = \nabla_{\dot{q}}\, \dot{q}$ is the covariant derivative along the curve $q = q(t)$, see Section 2.4.

Theorem. *Under the assumptions* (INJ) – (AC_2) *the equation of motion*

$$\ddot{q} = \operatorname{grad}_q U$$

has a contractible periodic solution $q = q(t)$ *such that* $\frac{1}{2}|\dot{q}|^2 = U(q)$.

Solutions are contained in the closure of the negative set and, moreover, are in one to one correspondences with closed integral curves $u = u(t)$ of the Hamiltonian vector field X_H on the energy surface M via the Legendre transformation. The correspondences is given by $q(t) = \pi\big(u(t)\big)$, cf. [2]. If Q is a line or a circle periodic solutions exists if and only if the negative set has compact closure. In this case the Hamiltonian lift $u = u(t)$ of $q = q(t)$ is always non-contractible.

Remark 1.1. If $\pi_1(N)$ injects into $\pi_1(Q)$ and if the dimension of Q is greater than one the contractibility of $q = q(t)$ implies the contractibility of $u = u(t)$ in M. Indeed, a perturbation of $u = u(t)$ can be brought into general position to be disjoint from $\partial N \subset M$. The projection $q = q(t)$ is contained in a contractible neighbourhood over which the unit cotangent bundle $ST^*Q \simeq M$ is trivial so that $u = u(t)$ is homotopic to $\big(q_*, p(t)\big)$ for a point q_* in Q. Connecting q_* with the boundary of N shows that the fibre over q_* bounds a n-disc inside M, which can be used to contract $u = u(t)$.

In order to prove the theorem we consider the Lagrangian action

$$\int_0^T L\big(\dot{q}(t)\big) \mathrm{d}t$$

for positive real numbers T and T-periodic curves $q = q(t)$ in Q. Critical points are the solutions claimed in the theorem. Following [5] we reparametrize the curves by setting

$$x(t) = q\big(\mathrm{e}^{\tau} t\big)$$

with $T = \mathrm{e}^{\tau}$. This results in the functional

$$\frac{\mathrm{e}^{-\tau}}{2} \int_0^1 |\dot{x}|^2 \mathrm{d}t + \mathrm{e}^{\tau} \int_0^1 U(x) \mathrm{d}t$$

for real numbers τ and 1-periodic curves $x = x(t)$ in Q. The critical points are solutions of $\ddot{x} = \mathrm{e}^{2\tau} \operatorname{grad}_x U$ such that $\frac{1}{2}|\dot{x}|^2 = \mathrm{e}^{2\tau} U(x)$, see Section 2.4. Periodic solutions are in one to one correspondence with T-periodic solutions in the theorem. We will prove that solutions exist provided the conditions (INJ) – (AC_2) are satisfied.

In Section 2 we describe the variational problem. In Section 3 we study compactness properties of the penalized action. Notice, that during the depenalization process in Lemma 3.3 we use a comparison argument in contrast to the asymptotic flow box in [5, 6]. This allows us to weaken the assumption on the metric. In Section 4 we prove existence of critical points of the penalized problem. We use a linking construction as in [5, 6] but in a different way. A rearrangement of Morse handles of the negative set allows us to apply gradient flow methods directly. Therefore, no requirements on the homology of the loop space as in [6] are necessary, cf. Section 1.2.

1.2. **Reeb orbits.** The kernel of the canonical symplectic form $\mathrm{d}\lambda$ on T^*Q restricted to TM defines a 1-dimensional distribution the so-called **characteristic distribution** on the energy surface M. A closed leaf of the foliation is called a **closed characteristic**. Because the characteristic foliation is integrated by the Hamiltonian vector field X_H the theorem implies:

Corollary 1. *Assuming* (INJ) – (AC_2) *the energy surface M carries a closed characteristic.*

We remark that the energy surface M of the Hamiltonian H is of restricted contact type in T^*Q. The Liouville form λ in the direction of X_H is twice the kinetic energy $|\,.\,|^2$. This follows because X_H is the sum of -1 times the cogeodesic vector field and the $\mathrm{d}\lambda$-dual of $-\pi^*V$, cf. [13]. Therefore, if M is disjoint from the zero section the restriction of λ to the characteristic distribution is positive, i.e. M is of **restricted contact type**, see [20, 8]. In the alternative case it suffices to find a function f on T^*Q such that $\lambda + \mathrm{d}f$ is positive along the restriction of X_H to M. Because X_H is nowhere tangent to Q along ∂N we can use flow box neighbourhoods for X_H on M which cover $M \cap Q$, a partition of unity, and the fundamental theorem of calculus to construct a function h on M such that $\mathrm{d}h(X_H) > -\lambda(X_H)$. An extension of h to T^*Q yields f as desired. In particular, the restriction α of $\lambda + \mathrm{d}f$ to TM defines a contact form on M together with its **Reeb vector field** R which is defined by $i_R \mathrm{d}\alpha = 0$ and $\alpha(R) = 1$. In particular, R is tangent to the characteristic foliation. The integral curves are called **Reeb orbits**. Therefore, the corollary ensures existence of closed Reeb orbits in the present situation. For more on closed Reeb orbits the reader is referred to, cf. [20, 15, 29].

Notice, that if a component of the negative set N has compact closure the existence of a periodic orbit follows from the classical results mentioned above. To see this replace Q with the double of the manifold with boundary $\bar{N}$. On the complement of $\bar{N}$ define the metric and the potential suitably such that the new negative set coincides with N. Alternatively, on can glue $\partial N \times [0,\infty)$ to N along the boundary using a collar neighbourhood induced by $\operatorname{grad} V/|\operatorname{grad} V|^2$. The result can be provided with a metric which interpolates to the product metric on the cylindrical end. Further the potential is the projection to the $\mathbb{R}$-factor on $\partial N \times [0,\infty)$. Then the classical existence results follow from the theorem because $(N, \partial N)$ carries a fundamental class with coefficients modulo 2.

Corollary 2. *If the energy surface M is compact then there exists a closed characteristic on M which is contractible provided the dimension of M is at least two.*

Remark 1.2. More generally, the gluing of $\partial N \times [0,\infty)$ to N as described above yields a Riemannian manifold Q with bounded geometry provided ∂N is compact. Consequently, the connected components of M for which the intersection with Q is compact and whose image under π satisfy the homological condition (LNK) carry a (contractible) closed characteristic. Therefore, *a priori* we can assume that those components N' are replaced with $\partial N' \times (-\infty, 0]$ where the potential is interpolated to the projection to the $\mathbb{R}$-component in a bounded neighbourhood of $\partial N'$.

We claim that the theorem implies the main result of [6]. This is of interest only if all connected components of $\bar{N}$ are non-compact. Assuming this we claim that

$$H_k(N, \partial N) \cong H_{k+n-1}(M)$$

naturally for $2 \leq k \leq n$ provided Q is orientable or the coefficients are taken modulo 2. As Geiges pointed out to us a proof can be given as follows: Denote by D' the unit codisc bundle of Q restricted to ∂N and its boundary sphere bundle by S'. Similarly, we denote the induce sphere bundle over N by S. W.l.o.g. we can assume that M equals $S \cup_{S'} D'$. By the excision property of homology we have that $H_{k+n-1}(S, S')$ is isomorphic to $H_{k+n-1}(M, \partial N)$. Therefore, it suffices to prove $H_k(N, \partial N) \cong H_{k+n-1}(S, S')$, which is obtained with the Gysin sequence for the bundle pair (S, S') over $(N, \partial N)$, see [10, Proposition 12.1]. Because $\bar{N}$ is not compact this holds for $k = 1$ as well. Therefore, the above isomorphism is correct for $k = 1$ provided ∂N is not compact. In the compact case we observe that $H_n(M)$ injects into $H_n(M, \partial N)$ and hence into $H_1(N, \partial N)$. In conclusion, if the homology of M is non-trivial for some degree $* = n, \ldots, 2n-1$ the condition (LNK) is satisfied.

2. The Lagrangian action

2.1. **Admissible curves.** We identify the circle S^1 with the 1-dimensional torus $\mathbb{R}/\mathbb{Z}$ and fix an isometric embedding of Q into a Euclidean space $\mathbb{R}^N$ as it is possible by a theorem of Nash [27, Theorem 3], cf. also [17, 18]. The Hilbert manifold H^1 of absolutely continuous maps $S^1 \to Q$ with square integrable derivative can be obtained as a submanifold of $H^1(S^1, \mathbb{R}^N)$, cf. [22]. The tangent space $T_x H^1$ is spanned by vector fields $\xi \in H^1(x^*TQ)$ along the loop $x \in H^1$. The Riemannian metric $\langle\, . \,, . \,\rangle$ on Q defines a Riemannian structure on H^1 via

$$\langle \xi, \eta \rangle_1 = \int_0^1 \langle \xi, \eta \rangle \mathrm{d}t + \int_0^1 \langle \dot{\xi}, \dot{\eta} \rangle \mathrm{d}t,$$

where we denote with $\dot{\xi}$ the covariant derivative $\nabla_{\dot{x}}\,\xi$ along the curve x. The distance between x and y in H^1 equals the minimal length of curves connecting x and y. This gives H^1 the structure of a complete metric space. This follows with [22, Theorem 1.4.5] where only the completeness of Q is used.

Denote by $\mathcal{M} \subset H^1$ the submanifold which consists of contractible loops x in Q. Notice that $\mathcal{M}$ is the connected component of the isometrically embedded and totally geodesic submanifold of point curves again denoted by Q in H^1.

2.2. **The parametrized Lagrangian action functional.** The **energy**

$$\mathcal{E}(x) = \frac{1}{2}\int_0^1 |\dot{x}|^2 \mathrm{d}t$$

and the **potential integral**

$$\mathcal{U}(x) = \int_0^1 U(x)\mathrm{d}t$$

of a curve x define smooth functions on H^1. For real parameters τ the **Lagrangian action** is defined by

$$\mathscr{L}(x,\tau) = \mathrm{e}^{-\tau}\mathcal{E}(x) + \mathrm{e}^{\tau}\mathcal{U}(x).$$

Its restriction to $\mathcal{M}$ is also denoted by $\mathscr{L}$.

2.3. **The penalty term.** For $\varepsilon > 0$ and

$$P(\tau) = \mathrm{e}^{-\tau} + \mathrm{e}^{\tau/2}$$

we add εP to the Lagrangian action to obtain

$$\mathscr{L}_\varepsilon(x,\tau) = \mathscr{L}(x,\tau) + \varepsilon P(\tau).$$

2.4. **Critical points.** Let (x,τ) be a critical point of $\mathscr{L}_\varepsilon$, i.e. a point on which the linearization $T_{(x,\tau)}\mathscr{L}_\varepsilon$ vanishes. Observe that

$$T_{(x,\tau)}\mathscr{L}_\varepsilon(\xi,0) = \mathrm{e}^{-\tau}\int_0^1 \langle \dot{x},\dot{\xi}\rangle \mathrm{d}t + \mathrm{e}^{\tau}\int_0^1 \langle \operatorname{grad}_x U, \xi\rangle \mathrm{d}t.$$

With integration by parts we obtain a solution of the **Euler-Lagrange equation**

$$\ddot{x} = \mathrm{e}^{2\tau}\operatorname{grad}_x U,$$

where $\ddot{x} = \nabla_{\dot{x}}\,\dot{x}$. It follows from elliptic regularity that the solutions are smooth.

Taking

$$T_{(x,\tau)}\mathscr{L}_\varepsilon(0,1) = -\mathrm{e}^{-\tau}\mathcal{E}(x) + \mathrm{e}^{\tau}\mathcal{U}(x) + \varepsilon P'(\tau)$$

we obtain the sum of the negative of the **parametrized Hamiltonian action** and (ε times) the derivative

$$P'(\tau) = -\mathrm{e}^{-\tau} + \frac{1}{2}\mathrm{e}^{\tau/2}.$$

Because (x,τ) is a critical point the Euler-Lagrange equation shows that the **parametrized Hamiltonian**

$$\frac{1}{2}\mathrm{e}^{-\tau}|\dot{x}|^2 - \mathrm{e}^{\tau}U(x) = \varepsilon P'(\tau)$$

is an integral of motion. We refer to this also as the **energy identity**.

Taking sum and difference of the critical value $c_\varepsilon = \mathscr{L}_\varepsilon(x,\tau)$ and $T_{(x,\tau)}\mathscr{L}_\varepsilon(0,1)$ we obtain

$$c_\varepsilon = 2\mathrm{e}^{\tau}\mathcal{U}(x) + \varepsilon\frac{3}{2}\mathrm{e}^{\tau/2},$$

resp.,

$$c_\varepsilon = 2\mathrm{e}^{-\tau}\mathcal{E}(x) + \varepsilon\left(2\mathrm{e}^{-\tau} + \frac{1}{2}\mathrm{e}^{\tau/2}\right).$$

3. Compactness

3.1. **Palais-Smale property.** We consider a Palais-Smale sequence (x_ν, τ_ν) of $\mathscr{L}_\varepsilon$, i.e. the sequence of linear operators $T_{(x_\nu,\tau_\nu)}\mathscr{L}_\varepsilon$ converges to zero and $\mathscr{L}_\varepsilon(x_\nu, \tau_\nu)$ converges to a real number c_ε as ν tends to infinity.

Proposition 3.1. *(x_ν, τ_ν) has a convergent subsequence.*

Proof. The difference of $\mathscr{L}_\varepsilon(x_\nu, \tau_\nu)$ and $T_{(x_\nu,\tau_\nu)}\mathscr{L}_\varepsilon(0, 1)$, which can be estimated by $\varepsilon\left(2\mathrm{e}^{-\tau} + \frac{1}{2}\mathrm{e}^{\tau/2}\right) \geq \frac{3}{2}\varepsilon$ from below, tends to c_ε. Therefore, the sequences $|\tau_\nu|$ and hence $\mathcal{E}(x_\nu)$ are bounded. We can assume that τ_ν converges to τ_*.

We claim that

$$\sup_{t\in S^1} \operatorname{dist}\big(o, x_\nu(t)\big)$$

is bounded. We argue by contradiction. Because of the bound on the energy $\mathcal{E}(x_\nu)$ we obtain a bound on the length of x_ν. Therefore, we can assume that $x_\nu(S^1)$ is contained in $Q \setminus \hat{Q}$. We consider the vector field

$$\xi_\nu = \frac{\operatorname{grad}_{x_\nu} U}{|\operatorname{grad}_{x_\nu} U|^2}$$

along x_ν, which is well defined by (AC_1^a). With $\operatorname{Hess} U = \langle \nabla \operatorname{grad} U, \,.\,\rangle$ and (AC_1^b) we obtain that $\|\xi_\nu\|_1^2$ is bounded by a positive constant times $1 + \mathcal{E}(x_\nu)$. Therefore, $T_{(x_\nu,\tau_\nu)}\mathscr{L}_\varepsilon(\xi_\nu, 0)$ tends to zero because (x_ν, τ_ν) is a Palais-Smale sequence. But a direct computation using (AC_1^a) and (AC_2) shows that the limit equals e^{τ_*}. This is a contradiction.

We claim that a subsequence of (x_ν, τ_ν) converges in $C^0(S^1, Q) \times \mathbb{R}$. Observe that

$$\operatorname{dist}\big(x_\nu(t_0), x_\nu(t_1)\big) \leq \operatorname{length}\big(x_\nu|_{t_0}^{t_1}\big) \leq \sqrt{|t_1 - t_0|}\sqrt{2\mathcal{E}(x_\nu)}.$$

The bound on $\mathcal{E}(x_\nu)$ shows that the sequence x_ν is equicontinuous. Because the Riemannian manifold Q is complete the theorem of Arzelà-Ascoli applies.

We can assume that $x_\nu \to x_*$ in C^0. Approximating x_* by a smooth loop y in Q we can further assume that the sequence x_ν is contained in a chart of $\mathcal{M}$ about y, w.l.o.g. $\mathcal{M} = H^1(y^*TQ)$. Because y is contractible we find an orthogonal trivialization of y^*TQ. For the following computations we further assume $Q = \mathbb{R}^n$ with the Euclidean metric by uniform equivalence of the Riemannian metrics on a compact set. Cf. also with [22, p. 26-27]. In other words we can assume that $\mathcal{M}$ equals the Hilbert space $H^1(S^1, \mathbb{R}^n)$.

Using Fourier series representations as in [1, 20] the norm of an element x of $H^1(S^1, \mathbb{R}^n)$ can be estimated from above by $\|x\|_\infty^2 + 2\mathcal{E}(x)$. In order to show that x_ν is a Cauchy sequence it suffices to show that $\mathcal{E}(x_\nu - x_\mu)$ tends to zero for $\nu, \mu \to \infty$. Because (x_ν, τ_ν) is a Palais-Smale sequence $T_{(x_\nu,\tau_\nu)}\mathscr{L}_\varepsilon(x_\nu - x_\mu, 0) \to 0$. Hence, the integral $\int\langle \dot{x}_\nu, \dot{x}_\nu - \dot{x}_\mu\rangle$ equals $\mathrm{e}^{2\tau_\nu}\int\langle \operatorname{grad}_{x_\nu} U, x_\nu - x_\mu\rangle$ up to a term which tends to zero. Because x_ν converges in C^0 (and a symmetry argument) $\mathcal{E}(x_\nu - x_\mu)$ tends to zero as well. Hence, x_ν is a Cauchy sequence in $\mathcal{M}$, which converges to x_*. ☐

3.2. **Depenalization.** As we will show in Section 4 there exist positive constants $K_1 < K_2$ and a sequence $\varepsilon \searrow 0$ such that $\mathscr{L}_\varepsilon$ carries a critical point $(x,\tau) = (x_\varepsilon, \tau_\varepsilon)$ whose critical value c_ε is contained in the interval $[K_1, K_2]$.

Lemma 3.2. *The sequence τ_ε is bounded above.*

Proof. The sum

$$c_\varepsilon = \mathscr{L}_\varepsilon(x,\tau) + T_{(x,\tau)}\mathscr{L}_\varepsilon(\xi, -1)$$

equals the following expression

$$\int_0^1 \Big(\mathrm{e}^{-\tau}\big(|\dot{x}|^2 + \langle \dot{x}, \dot{\xi}\rangle\big) + \mathrm{e}^{\tau}\langle \operatorname{grad}_x U, \xi\rangle\Big)\mathrm{d}t + \varepsilon\left(2\mathrm{e}^{-\tau} + \frac{1}{2}\mathrm{e}^{\tau/2}\right).$$

Removing the ε-term and plugging in

$$\xi = \delta\frac{\operatorname{grad}_x U}{1 + |\operatorname{grad}_x U|^2}$$

for some $\delta > 0$ yields

$$c_\varepsilon > \int_0^1 \Big(\mathrm{e}^{-\tau}\big(|\dot{x}|^2 + \langle \dot{x}, \dot{\xi}\rangle\big) + \mathrm{e}^{\tau}\delta\frac{|\operatorname{grad}_x U|^2}{1 + |\operatorname{grad}_x U|^2}\Big)\mathrm{d}t.$$

Due to (AC_1^a) and (AC_1^b) $|\dot{\xi}|$ is bounded by an ε-independent positive constant times $\delta|\dot{x}|$. We choose δ such that $|\dot{\xi}| \leq \frac{1}{2}|\dot{x}|$. This implies $|\dot{x}|^2 + \langle \dot{x}, \dot{\xi}\rangle \geq \frac{1}{2}|\dot{x}|^2$ and therefore,

$$c_\varepsilon > \mathrm{e}^{\tau}\int_0^1 \Big(\mathrm{e}^{-2\tau}\frac{1}{2}|\dot{x}|^2 + \delta\frac{|\operatorname{grad}_x U|^2}{1 + |\operatorname{grad}_x U|^2}\Big)\mathrm{d}t.$$

It suffices to bound the integrand I_ε from below.

If the curves x_ε stay outside the compact set $\hat{Q}$ a lower bound is given by $\delta(1+K^2)^{-1}$ using (AC_1^a). In the alternative case we find $t_\varepsilon \in S^1$ such that $x_\varepsilon(t_\varepsilon) \in \hat{Q}$. We can assume that $\tau_\varepsilon \geq 0$ because otherwise there is nothing to show. This implies that the multiplied energy identity

$$\frac{1}{2}\mathrm{e}^{-2\tau_\varepsilon}|\dot{x}_\varepsilon|^2 - U(x_\varepsilon) = \varepsilon\left(\frac{1}{2}\mathrm{e}^{-\tau_\varepsilon/2} - \mathrm{e}^{-2\tau_\varepsilon}\right) \longrightarrow 0$$

tends to zero independently of t because it takes values in the interval $(-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})$. Consider the set T_ε of $t \in S^1$ for which the first term of the multiplied energy identity is bounded away from zero by a small constant. The set T_ε is measurable. The integrand I_ε restricted to T_ε is bounded below as desired. On the complement $S^1 \setminus T_\varepsilon$ we can assume that $U(x_\varepsilon)$ is uniformly close to zero. In other words, $x_\varepsilon\big(S^1 \setminus T_\varepsilon\big)$ is contained in a small neighbourhood of $\partial N \cap \hat{Q}$ on which $|\operatorname{grad} U|$ is uniformly positive. This implies a lower bound of I_ε on the complement as well. □

Lemma 3.3. *The sequence τ_ε is bounded below.*

Proof. Arguing by contradiction we assume that $\tau_\varepsilon \to -\infty$ as ε tends to zero. Because $K_2 > c_\varepsilon > 2\mathrm{e}^{-\tau_\varepsilon}\mathcal{E}(x_\varepsilon)$, see Section 2.4, we infer $\mathcal{E}(x_\varepsilon) \to 0$. Therefore,

$$\operatorname{length}(x_\varepsilon) \to 0.$$

Again with Section 2.4 we get $K_1 < c_\varepsilon = 2\mathrm{e}^{\tau_\varepsilon}\mathcal{U}(x_\varepsilon) + \varepsilon\frac{3}{2}\mathrm{e}^{\tau_\varepsilon/2}$. Hence, the sequence $\mathcal{U}(x_\varepsilon)$ is unbounded. This implies

$$\inf_{t\in S^1} \operatorname{dist}\big(o, x_\varepsilon(t)\big) \longrightarrow \infty.$$

Consequently, we can assume that $x_\varepsilon(S^1)$ is contained in the intersection of $Q \setminus \hat{Q}$ and the geodesic ball $B(q)$ of radius $\frac{1}{2}\operatorname{inj} g$ about a point q on the curve x_ε. Moreover, with (AC_1^a) the solution x_ε of the Euler-Lagrange equation is not constant. The following arguments will lead to a contradiction.

With help of the Euler-Lagrange equation for x_ε we observe

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2}U\big(x_\varepsilon(t)\big) = \Big(\operatorname{Hess}_{x_\varepsilon(t)} U\Big)\big(\dot{x}_\varepsilon(t), \dot{x}_\varepsilon(t)\big) + \mathrm{e}^{2\tau_\varepsilon}|\operatorname{grad}_{x_\varepsilon(t)} U|^2.$$

For the maximum t_ε of the function $t \mapsto U\big(x_\varepsilon(t)\big)$ on the circle this yields

$$\mathrm{e}^{2\tau_\varepsilon}|\operatorname{grad}_{x_\varepsilon(t_\varepsilon)} U|^2 \leq \big|\operatorname{Hess}_{x_\varepsilon(t_\varepsilon)} U\big|\,|\dot{x}_\varepsilon(t_\varepsilon)|^2.$$

Invoking (AC_1^b) the estimate implies

$$\mathrm{e}^{2\tau_\varepsilon}|\operatorname{grad}_{x_\varepsilon(t_\varepsilon)} U| \leq K|\dot{x}_\varepsilon(t_\varepsilon)|^2.$$

Moreover, the multiplied energy identity

$$\frac{1}{2}|\dot{x}_\varepsilon|^2 - \mathrm{e}^{2\tau_\varepsilon}U(x_\varepsilon) = \varepsilon\left(\frac{1}{2}\mathrm{e}^{3\tau_\varepsilon/2} - 1\right) \longrightarrow 0$$

plugged in gives

$$\mathrm{e}^{2\tau_\varepsilon}|\operatorname{grad}_{x_\varepsilon(t_\varepsilon)} U| \leq K\Big(o(1) + \mathrm{e}^{2\tau_\varepsilon}U\big(x_\varepsilon(t_\varepsilon)\big)\Big).$$

In order to show that the right hand side tends to zero we use the following **mean value argument**. By Section 2.4 there exists t_0 such that

$$\mathrm{e}^{2\tau_\varepsilon}U\big(x_\varepsilon(t_0)\big) < \frac{1}{2}K_2\mathrm{e}^{\tau_\varepsilon}.$$

We denote by c the unit speed geodesic from $c(0) = x_\varepsilon(t_0)$ to $c(s_0) = x_\varepsilon(t_\varepsilon)$ inside the geodesic ball $B\big(x_\varepsilon(t_\varepsilon)\big)$. With the fundamental theorem of calculus we obtain $U\big(x_\varepsilon(t_\varepsilon)\big) \leq U\big(x_\varepsilon(t_0)\big) + \int_0^{s_0}|\operatorname{grad}_{c(s)} U|\mathrm{d}s$. An application of Grönwall's lemma to the function $s \mapsto |\operatorname{grad}_{c(s_0-s)} U|$ gives $|\operatorname{grad}_{c(s)} U| \leq e^{Ks_0}|\operatorname{grad}_{x_\varepsilon(t_\varepsilon)} U|$ for all $s \in [0, s_0]$ using (AC_1^a) and (AC_1^b). Therefore,

$$\mathrm{e}^{2\tau_\varepsilon}|\operatorname{grad}_{x_\varepsilon(t_\varepsilon)} U| < K\Big(o(1) + \frac{1}{2}K_2\mathrm{e}^{\tau_\varepsilon} + s_0\mathrm{e}^{2\tau_\varepsilon + Ks_0}|\operatorname{grad}_{x_\varepsilon(t_\varepsilon)} U|\Big).$$

Because the length of x_ε (and hence s_0) tends to zero we can choose ε such that

$$\mathrm{e}^{2\tau_\varepsilon}|\operatorname{grad}_{x_\varepsilon(t_\varepsilon)} U| < K\Big(o(1) + K_2\mathrm{e}^{\tau_\varepsilon}\Big).$$

Invoking Grönwall's lemma along geodesics connecting $x_\varepsilon(t_\varepsilon)$ with the boundary of $B\big(x_\varepsilon(t_\varepsilon)\big)$ we obtain as above

$$\mathrm{e}^{2\tau_\varepsilon}|\operatorname{grad} U| \longrightarrow 0$$

uniformly on $B\big(x_\varepsilon(t_\varepsilon)\big)$.

Remark 3.4. Consider the geodesic ball B^ε of radius $\frac{1}{4}\operatorname{inj} g$ with center $x_\varepsilon(t_0)$. If $\varepsilon > 0$ is sufficiently small B^ε is contained in $B\big(x_\varepsilon(t_\varepsilon)\big)$ and contains $x_\varepsilon(S^1)$. Integrating along geodesics which start at $x_\varepsilon(t_0)$ shows

$$U\big(x_\varepsilon(t)\big) \leq U\big(x_\varepsilon(t_0)\big) + \frac{1}{4}\operatorname{inj} g\ \sup_{B^\varepsilon}|\operatorname{grad} U|$$

for all $t \in S^1$. Hence,

$$\mathrm{e}^{2\tau_\varepsilon}U(x_\varepsilon) \leq \frac{1}{2}K_2\mathrm{e}^{\tau_\varepsilon} + o(1)$$

uniformly on S^1. By the multiplied energy identity $\frac{1}{2}|\dot{x}_\varepsilon|^2 \leq o(1)$ tends uniformly to zero. Therefore, we can assume that $|\dot{x}_\varepsilon|^2 < 1$ on S^1.

We continue with the proof of the lemma. With (AC_2) we obtain the stronger estimate

$$\mathrm{e}^{2\tau_\varepsilon} |\operatorname{grad}_{x_\varepsilon(t_\varepsilon)} U| \leq o(1)\, |\dot{x}_\varepsilon(t_\varepsilon)|^2$$

as ε tends to zero. The aim is to find a similar estimate for all $t \in S^1$ with a variation of the above mean value argument. We consider a unit speed geodesic c inside $B\big(x_\varepsilon(t)\big)$ connecting $c(0) = x_\varepsilon(t)$ with $c(s_0) = x_\varepsilon(t_\varepsilon)$. With the fundamental theorem of calculus and the Grönwall's lemma applied to the function $s \mapsto |\operatorname{grad}_{c(s)} U|$ we obtain

$$U\big(x_\varepsilon(t_\varepsilon)\big) - U\big(x_\varepsilon(t)\big) \leq s_0 \mathrm{e}^{K s_0} |\operatorname{grad}_{x_\varepsilon(t)} U|$$

using (AC_1^a) and (AC_1^b). Combining this with the difference of the multiplied energy identities gives

$$|\dot{x}_\varepsilon(t_\varepsilon)|^2 - |\dot{x}_\varepsilon(t)|^2 \leq 2 s_0 \mathrm{e}^{2\tau_\varepsilon + K s_0} |\operatorname{grad}_{x_\varepsilon(t)} U|.$$

We will use this to estimate $\mathrm{e}^{2\tau_\varepsilon} |\operatorname{grad}_{x_\varepsilon(t)} U|$ from above. The Grönwall's lemma gives a bound by $\mathrm{e}^{2\tau_\varepsilon + K s_0} |\operatorname{grad}_{x_\varepsilon(t_\varepsilon)} U|$, which with the initial estimate is bounded by $o(1)\, \mathrm{e}^{K s_0} |\dot{x}_\varepsilon(t_\varepsilon)|^2$. Therefore,

$$\mathrm{e}^{2\tau_\varepsilon} |\operatorname{grad}_{x_\varepsilon(t)} U| \leq o(1) \Big(|\dot{x}_\varepsilon(t)|^2 + 2 s_0 \mathrm{e}^{2\tau_\varepsilon + K s_0} |\operatorname{grad}_{x_\varepsilon(t)} U| \Big).$$

Choosing ε (and hence s_0 and $|\dot{x}_\varepsilon(t)|^2$, see Remark 3.4) sufficiently small we obtain the linear estimate

$$\mathrm{e}^{2\tau_\varepsilon} |\operatorname{grad}_{x_\varepsilon(t)} U| \leq o(1)\, |\dot{x}_\varepsilon(t)|$$

uniformly for all $t \in S^1$.

The desired contradiction will be achieved with the following comparison argument: We assume that $Q = \mathbb{R}^n$ using geodesic normal coordinates on $B(q)$ for a point q on the curve x_ε. By [21, 11] and (CB) the metric $\langle\, .\, ,\, .\, \rangle$ is uniformly equivalent to the Euclidean metric $\langle\, .\, ,\, .\, \rangle_0$ and for the Christoffel symbols we have $\Gamma(x) = O(|x|)$ uniformly. In particular, with Remark 3.4

$$\Big| \Gamma_{x_\varepsilon(t)}\big(\dot{x}_\varepsilon(t), \dot{x}_\varepsilon(t)\big) \Big|_0 \leq o(1)\, |\dot{x}_\varepsilon(t)|_0$$

uniformly in $t \in S^1$. Consequently, we have using the Euler-Lagrange equation

$$\big|\dot{x}_\varepsilon(t_1) - \dot{x}_\varepsilon(t_2)\big|_0 \leq \int_{t_1}^{t_2} \Big| \frac{\mathrm{d}}{\mathrm{d}t} \dot{x}_\varepsilon(t) \Big|_0 \mathrm{d}t \leq o(1) \int_0^1 |\dot{x}_\varepsilon(t)|_0 \mathrm{d}t,$$

for all $t_1, t_2 \in [0,1]$. Let t_1 be the maximum of $t \mapsto |\dot{x}_\varepsilon(t)|_0$. Let t_2 be a point such that $\dot{x}_\varepsilon(t_2)$ vanishes or is perpendicular to $\dot{x}_\varepsilon(t_1)$ w.r.t. the Euclidean metric. With the Pythagorean theorem

$$|\dot{x}_\varepsilon(t_1)|_0 \leq \big|\dot{x}_\varepsilon(t_1) - \dot{x}_\varepsilon(t_2)\big|_0 \leq o(1)\, |\dot{x}_\varepsilon(t_1)|_0.$$

This is a contradiction because the curve x_ε is not constant. □

The above lemmata ensure bounds on the sequence τ_ε of Lagrangian multiplier. A repetition of the arguments from Proposition 3.1 proves:

Proposition 3.5. *The sequence of critical points $(x_\varepsilon, \tau_\varepsilon)$ has a convergent subsequence as ε tends to zero.*

In particular the limit curve is a critical point of $\mathscr{L}$ with vanishing parameterized Hamiltonian energy.

4. Mountain pass

The aim of this section is to prove the following existence statement, which in view of Section 3.2 and Proposition 3.5 proves Theorem 1.1:

Proposition 4.1. *There exist positive constants K_1 and K_2 with $K_1 < K_2$ such that for all $\varepsilon \in (0, K_1)$ there exist $\varepsilon_0 \in (0, \varepsilon)$ and a critical point (x, τ) of $\mathscr{L}_{\varepsilon_0}$ such that $K_1 \leq \mathscr{L}_{\varepsilon_0}(x, \tau) \leq K_2$.*

4.1. **Begin of the proof.** Arguing by contradiction we find a sequence ε_ν of positive real numbers such that for all $\varepsilon \in (0, \varepsilon_\nu)$ the interval $[1/\nu, \nu]$ contains no critical value of $\mathscr{L}_\varepsilon$. We will lead this assumption to a contradiction in several steps organized as separate sections.

4.2. **A deformation of the negative set.** By (AC_1^a) the negative potential function U has no critical point in the complement of the compact set $\hat{Q}$. Hence, in view of (REG) the level sets $\{U = \pm\delta\}$ are isotopic to $\{U = 0\}$ for $\delta > 0$ sufficiently small. An isotopy is given by following the (negative) gradient flow lines of U. We set

$$N_{\pm\delta} = \{U > \pm\delta\}.$$

Notice that $N = N_0$.

Lemma 4.2. *There exist $\delta > 0$ and an open subset $Q_B \subset Q$ such that the pairs $(Q_B, \partial Q_B)$ and $(N_\delta, \partial N_\delta)$ are isotopic and the minimal distance* $\mathrm{dist}(\partial N_\delta, Q_B)$ *is positive.*

Proof. Consider the function

$$f = \frac{U}{\sqrt{1 + |\operatorname{grad} U|^2}}.$$

The set Q_B is defined by

$$Q_B = \{f > \sqrt{2}\delta\} \subset N_\delta.$$

Notice that $\partial N = \{f = 0\}$. Invoking (AC_1^a) and (AC_1^b) there exists $\delta' > 0$, which only depends on K, such that $|\operatorname{grad} f|$ is uniformly positive on $\{|f| < \delta'\}$. Because the metric on Q is complete we can assume by shrinking $\delta' > 0$ that there exists a complete vector field X on Q which coincides with $|\operatorname{grad} f|^{-2} \operatorname{grad} f$ on $\{|f| < \delta'\}$. The flow of X brings N to Q_B provided we choose $\delta < \delta'/\sqrt{2}$. This yields the desired isotopy.

In order to show positivity of $\mathrm{dist}(\partial N_\delta, Q_B)$ consider a point q in Q_B. Notice that

$$U(q) > \delta\big(1 + |\operatorname{grad}_q U|\big).$$

Choose $r \in (0, \operatorname{inj} g)$ and consider the geodesic ball $B_r(q)$ of radius r about q. We assume that $B_r(q)$ is contained in $Q \setminus \hat{Q}$. Each point p on the boundary $\partial B_r(q)$ can be connected with the center q by a radial unit speed geodesic. By a mean value argument analogously to the application of Grönwall's lemma in Lemma 3.3 we obtain

$$U(q) - U(p) \leq r\mathrm{e}^{Kr} |\operatorname{grad}_q U|$$

using (AC_1^a) and (AC_1^b). Combining both estimates yields

$$U(p) > \delta + |\operatorname{grad}_q U|(\delta - r\mathrm{e}^{Kr}).$$

Choose r such that $r\mathrm{e}^{Kr} \leq \delta$. Hence, $U(p) > \delta$. In other words $U > \delta$ on any geodesic ball of radius r in $Q \setminus \hat{Q}$ about points in Q_B. Because $\hat{Q}$ is compact the shortest length of a curve connecting points of ∂N_δ with those of Q_B is positive. □

4.3. **The linked set.** We define a subset

$$\mathcal{B} = \mathcal{B}' \times \mathbb{R}$$

of $\mathcal{M} \times \mathbb{R}$ via

$$\mathcal{B}' = \big\{x \in \mathcal{M} \,|\, \mathcal{E}(x) = r \text{ and } x(0) \in Q_B\big\}.$$

Observe that the length of the curves $x \in \mathcal{B}'$ is bounded by $\sqrt{2r}$. In view of Lemma 4.2 we choose $r > 0$ such that

$$\operatorname{length}(x) \leq \sqrt{2r} < 2\operatorname{dist}(\partial N_\delta, Q_B).$$

Therefore, $x(S^1) \subset N_\delta$. In other words $\mathcal{U}(x) \geq \delta$ so that the restriction of $\mathscr{L}$ to $\mathcal{B}$ is bounded from below by $\mathrm{e}^{-\tau} r + \mathrm{e}^{\tau}\delta$. Consequently,

$$\mathscr{L} \geq 2\sqrt{r\delta} \qquad \text{on} \quad \mathcal{B}.$$

Remark 4.3. We shrink $r > 0$ further such that the energy functional $\mathcal{E}$ has no critical points on $\{0 < \mathcal{E} \leq r\}$. In view of the positivity assumption on the injectivity radius this is not a restriction.

4.4. **The linking set.** By (LNK) and Section 4.2 there is a simplicial cycle c in $N_{-\delta}$ relative $\partial N_{-\delta}$ which is non-trivial in homology, cf. [26]. We identify Q with its image in $\mathcal{M}$. Observe,

$$\mathscr{L} = -\mathrm{e}^{\tau}\delta \qquad \text{on} \quad |\partial c| \times \{\tau\},$$

where we denote with $|\partial c|$ the geometric realization of the simplicial cycle ∂c (the boundary taken of the absolute chain c) as a subset of $\partial N_{-\delta}$ and τ is a real number. For $\tau_0 \ll -1$ we find

$$\mathscr{L} \leq \mathrm{e}^{\tau_0} \max_{|c|} U \qquad \text{on} \quad |c| \times \{\tau_0\}.$$

The following lemma will be proved in Section 4.7.

Lemma 4.4. *There exists a chain c' in $\mathcal{M}$ homotopic to c with boundary fixed such that $\mathcal{U}(x) \leq -\frac{1}{2}\delta$ for all $x \in |c'|$.*

Remark 4.5. Notice that all curves x in the chain $|c'|$ leave N_δ if $x(0)$ is in the closure of Q_B. This is because the negative potential integral $\mathcal{U}$ of x is negative by Lemma 4.5. Therefore,

$$r < \inf\{\mathcal{E}(x) \,|\, x \in |c'| \text{ and } x(0) \in Q_B\}$$

by the choice of r in Section 4.3.

By compactness of $|c'|$ the energy is bounded on $|c'|$. Therefore, we find $\tau_1 \gg 1$ such that

$$\mathscr{L} < 0 \qquad \text{on} \quad |c'| \times \{\tau_1\}$$

uniformly. Let $\mathcal{A}$ be the union

$$\mathcal{A} = \Big(|\partial c| \times [\tau_0, \tau_1]\Big) \cup \Big(|c| \times \{\tau_0\}\Big) \cup \Big(|c'| \times \{\tau_1\}\Big)$$

so that we obtain

$$\sup_{\mathcal{A}} \mathscr{L} \leq \mathrm{e}^{\tau_0} \max_{|c|} U.$$

In view of the assumption in Section 4.1 we choose $\nu \in \mathbb{N}$ such that $1/\nu < \sqrt{r\delta}$. We choose $\tau_0 \ll -1$ such that

$$\sup_{\mathcal{A}} \mathscr{L} < \frac{1}{\nu}.$$

This implies

$$\sup_{\mathcal{A}} \mathscr{L} < \inf_{\mathcal{B}} \mathscr{L}.$$

In particular $\mathcal{A}$ and $\mathcal{B}$ are disjoint.

4.5. **A chain.** Define

$$\mathcal{C} = \Big(|c| \times [\tau_0, 0]\Big) \cup \bigcup_{s\in[0,1]} \Big(|c_s| \times \{0\}\Big) \cup \Big(|c'| \times [0, \tau_1]\Big)$$

where c_s is the homotopy from $c_0 = c$ to $c_1 = c'$ in $\mathcal{M}$ relative ∂c which we will construct in Section 4.7, cf. Lemma 4.4. By construction $\mathcal{A}$ and $\mathcal{C}$ can be given the structure of simplicial chains such that $\partial\mathcal{C} = \mathcal{A}$. In particular $\mathcal{A}$ is a cycle. By compactness of $\mathcal{C}$ we can assume that additionally

$$\sup_{\mathcal{C}} \mathscr{L} < \nu.$$

Increasing ν amounts to decreasing τ_0. But this does not effect the above estimates.

4.6. **The minmax argument.** By compactness of $\mathcal{A}$ and $\mathcal{C}$ we find $\varepsilon \in (0, \varepsilon_\nu)$ such that

$$\sup_{\mathcal{A}} \mathscr{L}_\varepsilon < 1/\nu < \inf_{\mathcal{B}} \mathscr{L}_\varepsilon$$

and

$$\sup_{\mathcal{C}} \mathscr{L}_\varepsilon < \nu.$$

Lemma 4.6. *The action window set $\mathscr{L}_\varepsilon^{-1}\big([1/\nu, \nu]\big)$ is diffeomorphic to*

$$\{\mathscr{L}_\varepsilon = 1/\nu\} \times [0, \nu - 1/\nu].$$

Proof. Notice that by Section 4.1 there are no critical points in $\mathcal{W} = \mathscr{L}_\varepsilon^{-1}\big([1/\nu, \nu]\big)$. We define a vector field

$$\xi = \frac{\operatorname{grad} \mathscr{L}_\varepsilon}{|\operatorname{grad} \mathscr{L}_\varepsilon|_1^2}$$

on $\mathcal{W}$ and consider its flow, cf. [1, 23]. By the Palais-Smale property in Section 3.1 the vector field ξ is of bounded length $\sup_{\mathcal{W}} |\xi|_1 < \infty$. We can assume that ξ is extended to $\mathcal{M} \times \mathbb{R}$ via a partition of unity such that ξ has support in a slightly larger action window. By completeness of $\mathcal{M} \times \mathbb{R}$ the flow φ of ξ is global. The desired diffeomorphism is

$$\big((x, \tau), s\big) \longmapsto \varphi_s(x, \tau)$$

for $(x, \tau) \in \{\mathscr{L}_\varepsilon = 1/\nu\}$ and $s \in [0, \nu - 1/\nu]$. □

Therefore, the relative cycle $\mathcal{C}$ in $\big(\{\mathscr{L}_\varepsilon \leq \nu\}, \{\mathscr{L}_\varepsilon < 1/\nu\}\big)$ is homologically trivial. Notice, that by Remark 4.5 the intersection $\mathcal{C} \cap \mathcal{B}$ is non-empty and that $\mathcal{A} = \partial\mathcal{C}$ and $\mathcal{B}$ are disjoint. Moreover, $\mathcal{B} \subset \mathcal{M} \times \mathbb{R}$ is a hypersurface defined via a smooth function, see Remark 4.3. Similarly, $\partial\mathcal{B}$ is contained in the preimage of ∂Q_B under the surjective submersion

$$\begin{array}{rcl} \mathrm{ev} : \mathcal{M} \times \mathbb{R} & \longrightarrow & Q \\ (x, \tau) & \longmapsto & x(0) \end{array}$$

and ∂Q_B is the zero set of a smooth function. By a generic *a posteriori* choice of r and δ we can assume that $\mathcal{C} \cap \mathcal{B}$ defines a cycle in $(\mathcal{B}, \partial\mathcal{B})$, which is trivial by the above discussion.

Let $\mathcal{D}$ be the intersection of $\mathcal{C}$ with

$$\big\{(x, \tau) \,|\, \mathcal{E}(x) \leq r \;\text{ and }\; x(0) \in Q_B\big\}.$$

$\mathcal{D}$ has the structure of a simplicial chain with boundary in the union of

$$\mathcal{B} \;\text{ with }\; \mathrm{ev}^{-1}(\partial Q_B) \;\text{ and with }\; |c| \times \{0\}.$$

Therefore, the cycles $c \cap \bar{Q}_B$ and $\mathrm{ev}(\mathcal{C} \cap \mathcal{B})$ are homologous in $(Q_B, \partial Q_B)$ via $\mathrm{ev}(\mathcal{D})$. The triviality of $\mathcal{C} \cap \mathcal{B}$ implies the triviality of $\mathrm{ev}(\mathcal{C} \cap \mathcal{B})$ so that the cycle $c \cap \bar{Q}_B$ is trivial in relative homology of $(Q_B, \partial Q_B)$. Because $(N, \partial N)$ and $(Q_B, \partial Q_B)$ are isotopic the cycle c is trivial in $(N, \partial N)$. This contradicts the choice of c.

4.7. **Handles of the negative potential.** In order to finish the proof of Proposition 4.1 we prove Lemma 4.4.

Lemma 4.7. *There exists a smooth function $\tilde{U}$ on Q and a compact subset $\hat{N} \subset N$ such that*

- *$\tilde{U} \geq U$ and $\tilde{U} = U$ on $Q \setminus \hat{N}$ and*
- *the restriction of $\tilde{U}$ to N is a Morse function without local minima.*

Proof. By a local perturbation of U we find a function $\tilde{U}$ as in the lemma, see [24, Section 2], but eventually with positive local minima. With (AC_1^a) we find a vector field X on Q which equals

$$\frac{\operatorname{grad} U}{|\operatorname{grad} U|^2}$$

on $Q \setminus \hat{Q}$, has bounded length (is therefore complete), and is gradient-like for $\tilde{U}$ on N, [24, Lemma 3.2]. The aim is to remove all local minima of $\tilde{U}$ by a cancellation process as described in [24].

With [24, Lemma 2.8] we can assume that different critical points in N have different critical values. For regular values $0 \leq a < b$ the manifold with boundary

$$W_{ab} = \tilde{U}^{-1}\big([a, b]\big)$$

is called an **action window set**. W_{ab} is called **regular** if $\tilde{U}$ has no critical point in W_{ab}. Using the flow of X as in Lemma 4.6 W_{ab} is diffeomorphic to $\{\tilde{U} = a\} \times [0, b-a]$, cf. [24, Theorem 3.4]. W_{ab} is called **elementary** if $\tilde{U}$ has exactly one critical point q_0 on W_{ab}. The flow lines of X whose closure does not intersect q_0 connect $\{\tilde{U} = a\}$ with $\{\tilde{U} = b\}$. The intersection $S_L(q_0)$ of flow lines of X with $\{\tilde{U} = a\}$ which connect with q_0 in forward time is diffeomorphic to a sphere of dimension $\mathrm{ind}(q_0) - 1$, where $\mathrm{ind}(q_0)$ denotes the Morse index of q_0. The intersection $S_R(q_0)$ with $\{\tilde{U} = b\}$ in backward time is diffeomorphic to a sphere of dimension $n - \mathrm{ind}(q_0) - 1$. We call

$S_L(q_0)$ the **left-hand sphere** of q_0 and $S_R(q_0)$ the **right-hand sphere** of q_0, see [24, Definition 3.9].

In order to alter $\tilde{U}$ into a self-indexing-like Morse function on N we consider the composition

$$W_{ac} = W_{ab} \cup W_{bc}$$

of elementary action window sets. The critical points are denoted by $q_0 \in W_{ab}$ and $q_1 \in W_{bc}$. If $\mathrm{ind}(q_1) \leq \mathrm{ind}(q_0)$ a compactly supported diffeotopy of $\{\tilde{U} = b\}$ yields a gradient-like vector field of $\tilde{U}$ which coincides with X near ∂W_{ac} and outside a compact set such that the right- and left-hand spheres $S_R(q_0)$ and $S_L(q_1)$ in $\{\tilde{U} = b\}$ are disjoint, cf. [24, Theorem 4.4]. Therefore, the compact sets $K(q_0)$, resp., $K(q_1)$, of flow lines of the vector field (again denoted by) X connecting q_0, resp., q_1, in W_{ac} are disjoint. As in [24, Theorem 4.1] we can increase the function $\tilde{U}$ in a neighbourhood of $K(q_0)$ keeping the critical points q_0 and q_1 such that X is still gradient-like and the critical value of q_0 lies above the critical value of q_1. Moreover, near ∂W_{ac} the Morse function is not changed. In other words, after a rearrangement of the critical points on N we obtain a function $\tilde{U}$ on Q which coincides with U on $Q \setminus \hat{N}$ for a compact subset $\hat{N}$ of N such that $\tilde{U} \geq U$ and for all positive critical points q_0 and q_1 of $\tilde{U}$ we have

- if $\mathrm{ind}(q_0) = \mathrm{ind}(q_1)$ then $\tilde{U}(q_0) = \tilde{U}(q_1)$,
- if $\mathrm{ind}(q_0) < \mathrm{ind}(q_1)$ then $\tilde{U}(q_0) < \tilde{U}(q_1)$.

I.e. $\tilde{U}$ behaves like a self-indexing Morse function on N, see [24, Theorem 4.8].

If $\tilde{U}$ has no positive local minimum we are done. It remains to consider the alternative case. With (AC_1^a) and the flow of $-X$ Courants minmax argument as in [32, Theorem 4.2] applies to the set of paths c connecting a positive local minimum $c(0)$ with a point $c(1)$ outside N. Therefore, there exists a positive saddle point of Morse index 1. Because $\tilde{U}$ is self-indexing-like on N there exists an index 1 positive saddle point q_1 which is connected with a positive local minimum q_0 via exactly one flow line T of X.

Increasing $\tilde{U}(q_0)$ and $\tilde{U}(q)$ slightly for all index 1 positive saddle points $q \neq q_1$ we can assume that q_0 and q_1 are the critical points of the composition W_{ac} of the elementary action window sets W_{ab} and W_{bc}. We claim that the first cancellation theorem [24, Theorem 5.4] applies: Let $D_R(q_0)$ be the n-dimensional **right-hand disc** of q_0 which by definition is the union of all flow lines of X in W_{ab} starting at q_0. Let K_T be the compact neighbourhood of T which is the union of $D_R(q_0) \cup T$ with the set of flow lines of X in W_{bc} which ends in a small compact tubular neighbourhood of the hypersurface $S_R(q_1)$ in $\{\tilde{U} = c\}$. Notice, that if a flow line in W_{ac} leaves K_T once it never comes back. Following the arguments in [24, p. 51ff] we can alter the vector field X inside a neighbourhood of T in K_T such that the flow of the vector field (again denoted by) X yields a diffeomorphism from W_{ac} to $\{\tilde{U} = a\} \times [0, c-a]$. With the construction in [24, p. 54] and $X(\tilde{U}) = 1$ on $Q \setminus \hat{Q}$ there exists a new function $\tilde{U}$ which coincides with the old one outside a compact set and on ∂W_{ac} such that X is gradient-like for $\tilde{U}$. Because $\tilde{U}$ increases along the flow of X the new function has no critical point in W_{ac}. Further, it can be assumed to be greater or equal than the old.

Repeating this argument we can remove all positive local minima. This proves the lemma. □

Proof of Lemma 4.4. Let Y be a vector field on Q which does not vanish in a compact neighbourhood of $|c|$. As in the proof of Lemma 4.7 we consider a complete gradient-like vector field X for $\tilde{U}$. We can assume that the spaces of flow lines $\mathcal{N}$ of X connecting positive critical points of $\tilde{U}$ are manifolds of dimension $\leq n-1$, cf. [31]. We can perturb Y not to be tangent to $\mathcal{N}$ at the points of $|c|$.

Following the flow of Y on a small interval around 0 we find a chain $\tilde{c}$ in $\mathcal{M}$ which is homotopic to c relative ∂c. The chain $\tilde{c}$ is obtained from c by adding small loops induced by Y which start at points on $|c|$. We can assume that the loops starting on $|\partial c|$ are constant; those starting on $|c| \cap \{U > -\delta/2\}$ not. Additionally, the intersections of every loop with $\mathcal{N}$ are uniformly finite.

Let Z be a complete vector field on Q which coincides with $-X$ on N and vanishes on $\{U \leq -\delta\}$. Applying the flow of Z to the loops representing $\tilde{c}$ we get a 1-parameter family c_s in $\mathcal{M}$ starting at $c_0 = \tilde{c}$ with boundary ∂c_s fixed. Moreover, all $x \in |c_s|$ converge to arcs in $\{U \leq -\delta/2\}$ in C^∞_{loc} outside the intersections with $\mathcal{N}$. Therefore, we have $\mathcal{U}(x) \leq \int \tilde{U}(x)\mathrm{d}t \leq -\delta/2$ for all $x \in |c_s|$ and s sufficiently large. □

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