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LUTZ ANGERMANN, YURY V. SHESTOPALOV, YURY G. SMIRNOV AND VASYL V. YATSYK

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Nonlinear multi-parameter eigenvalue problems for systems of nonlinear ordinary differential equations arising in electromagnetics

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Abstract. We investigate a generalization of one-parameter eigenvalue problems arising in the theory of nonlinear waveguides to a more general nonlinear multi-parameter eigenvalue problem for a nonlinear operator. Using an integral equation approach, we derive functional dispersion equations whose roots yield the desired eigenvalues. The existence and distribution of roots are verified.

Keywords: Nonlinear multi-parameter eigenvalue problem, coupled eigenvalues, dispersion equation

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Introduction

In this paper we investigate nonlinear multi-parameter eigenvalue problems. This is a new type of eigenvalue problems for nonlinear operators.

Such problems arise, for instance, in mathematical models of the coupled wave propagation in nonlinear media [2] – [4], in particular for coupled waves at different frequencies. This problem is a generalization of one-parameter eigenvalue problems arising in the theory of nonlinear waveguides [5] – [6]. The eigenvalue problem is formulated in unbounded domains, in particular on the real axis, and with transmission-type conditions as well as conditions at infinity that contain the spectral parameters – for example, if the coefficient in the equation preceding the nonlinear term is nontrivial within a finite interval $(0, a)$, $a > 0$, and additional conditions are required at the point a (continuity), at the origin (boundedness), and at infinity (rate of decay). An example considered in [2] – [4] is $L(\lambda_1, \lambda_2)u + \alpha B(u; \lambda_1, \lambda_2) = 0$, where L is a linear differential operator w.r.t. u and B is a nonlinear operator in all its arguments. We should point out that the operators considered in [2] – [4] do not depend explicitly on the independent variable of the problem.

In the generalization of this concept, the system of dispersion relations obtained w.r.t. several spectral parameters can be considered as a problem w.r.t. a tuple of eigenvalues for nonlinear operator equations and forms in this way a nonstandard eigenvalue problem. It is worth to be mentioned that the theory of *linear* multi-parameter eigenvalue problems is quite well developed (see, e.g., [1]) but to the knowledge of the authors there is no comparable understanding of the nonlinear situation.

The method of solution employs the transition to nonlinear integral equations [5] – [9] by the help of the Green's functions of the linear differential operators. After this, the eigenvalue problems are replaced by the determination of characteristic numbers of the integral operator-valued functions that are nonlinear both with respect to the solution and to the spectral parameters. The latter problems are reduced to functional dispersion equations, and their roots give the desired eigenvalues. The existence and distribution of roots are verified.

The objective of the present work is to develop the appropriate technique to study nonlinear multi-parameter eigenvalue problems and to prove the existence of coupled eigenvalues (eigentuples) for a generalization of the inhomogeneous nonlinear waveguide problem, to obtain the eigentuples – also as functions of the problem parameters (first of all, of the nonlinearity parameter). In a next step (not yet contained here) we aim to develop and investigate a numerical method to determine the eigentuples and eigenfunctions of the problem.

1. Statement of the Problem

We will consider an n-dimensional nonlinear eigenvalue problem for the system of differential equations

$$\begin{cases} u_i''(x) + (k_i^2\varepsilon - \lambda_i)u_i(x) = 0, \quad -h < x < h, \\ i = 1, 2, \ldots, n, \end{cases} \tag{1}$$

where ε is given by

$$\varepsilon(\mathbf{u}) := 1 + a|\mathbf{u}(x)|^2$$

and $\mathbf{u}(x) := (u_1(x), \ldots, u_n(x))^\top$, $|\mathbf{u}|^2 := \sum_{i=1}^n |u_i|^2$. In this system, the constants $a > 0$ and $k_i \in \mathbb{R}$ are given, whereas the vector $\boldsymbol{\lambda} := (\lambda_1, \ldots, \lambda_n)^\top \in \mathbb{C}^n$ and the vector-field $\mathbf{u} : [-h, h] \to \mathbb{C}^n$ are both unknown.

The boundary conditions are

$$\begin{cases} u_i'(-h) - \alpha_i u_i(-h) &= 0, \\ u_i'(h) + \beta_i u_i(h) &= 0, \\ i = 1, 2, \ldots, n, \end{cases} \tag{2}$$

where the coefficients

$$\alpha_i = \alpha_i(\lambda_i), \quad \beta_i = \beta_i(\lambda_i)$$

are real-valued, continuous and satisfy the conditions

$$\alpha_i(\lambda_i) \geq 0, \quad \beta_i(\lambda_i) \geq 0, \quad \alpha_i(\lambda_i) + \beta_i(\lambda_i) > 0 \quad \text{for all} \quad \lambda_i \in \Lambda_i$$

whith certain sets Λ_i specified below.

In addition we need to consider the scaling conditions

$$\mathbf{u}(h) = \mathbf{C}^{(h)}, \tag{3}$$

where $\mathbf{C}^{(h)} \in \mathbb{R}^n$ is prescribed such that $C_i^{(h)} \neq 0$, $i = 1, 2, \ldots, n$.

Definition 1 $\boldsymbol{\lambda}$ *is called a (scaled)* eigentuple *of the problem (1) – (3) if there exist nontrivial functions $u_i \in C^1[-h, h] \cap C^2(-h, h)$, $i = 1, 2, \ldots, n$, satisfying the system (1), the boundary conditions (2) and the scaling condition (3). The components λ_i of $\boldsymbol{\lambda}$ are also called* coupled eigenvalues*, and the functions u_i are called the* eigenfunctions *of the problem (1) – (3).*

Remark 1 *In contrast to the linear case (i.e. a=0 in (1)), where the eigenfunctions are unique up to multiplicative constants, here we have to fix the numbers $C_i^{(h)}$. Both the eigenvalues and the eigenfunctions depend on these numbers. It can be shown that the property $C_i^{(h)} = 0$ for some index i implies that $u_i = 0$, since the boundary condition (2) at $x = h$ leads to a Cauchy problem with vanishing initial conditions. Therefore we have to assume in (3) that $C_i^{(h)} \neq 0$, $i = 1, 2, \ldots, n$.*

Proposition 1 *If the problem (1) – (3) has a solution, then $\boldsymbol{\lambda} \in \mathbb{R}^n$ and $\mathbf{u} : [-h, h] \to \mathbb{R}^n$.*

Proof: Indeed, multiplying for any fixed index i the i-th equation of (1) by $\overline{u_i(x)}$ and integrating by parts, we obtain

$$\begin{gathered}\int_{-h}^{h} (-|u_i'(x)|^2 + k_i^2 \varepsilon(\mathbf{u})|u_i(x)|^2 \, dx \\ -\alpha_i |u_i(-h)|^2 - \beta_i |u_i(h)|^2 - \lambda_i \int_{-h}^{h} |u_i(x)|^2 \, dx = 0.\end{gathered} \tag{4}$$

Separating the imaginary part of (4) yields

$$-(\operatorname{Im} \lambda_i) \int_{-h}^{h} |u_i(x)|^2 \, dx = 0.$$

Hence $\operatorname{Im} \lambda_i = 0$ if u_i is a nontrivial function. Then it follows from the i-th equations of (1) and (3) that u_i is real. ◀

This result shows that it is sufficient to consider only the case of $\mathbb{R}^n$-valued $\boldsymbol{\lambda}$ and $\mathbf{u}$.

The main task of the study referred to as *problem P* is *to prove the existence of eigentuples.*

2. Integral Equations and Dispersion Equations

Using the notation

$$f_i := |\mathbf{u}|^2 u_i, \quad i = 1, \ldots, n,$$

the system (1) takes the form

$$u_i'' + (k_i^2 - \lambda_i) u_i = -a k_i^2 f_i, \quad -h < x < h, \quad i = 1, \ldots, n. \tag{5}$$

Next we will invert the differential operators

$$L_i := \frac{d^2}{dx^2} + \tau_i^2, \quad \tau_i^2 := k_i^2 - \lambda_i,$$

on the set $\{v \in C^1[-h, h] \cap C^2(-h, h) : \ v'(-h) = v'(h) = 0\}$ provided that

$$\tau_i \neq \frac{\pi m}{2h} \quad \text{for all} \quad m \in \mathbb{N} \cup \{0\}. \tag{6}$$

Under this assumption we can construct the Green's functions for the n boundary value problems

$$\begin{cases} L_i G_i = -\delta\left(x - s\right), \\ \partial_x G_i|_{x=-h} = \partial_x G_i|_{x=h} = 0. \end{cases}$$

It is well-known that the Green's functions have the form

$$G_i(x,s) = \begin{cases} -\dfrac{\cos\tau_i(x+h)\cos\tau_i(s-h)}{\tau_i\sin 2\tau_i h}, & x < s \le h \\ -\dfrac{\cos\tau_i(x-h)\cos\tau_i(s+h)}{\tau_i\sin 2\tau_i h}, & s < x \le h. \end{cases} \tag{7}$$

Remark 2 *If we look at the Green's function G_i from (7) as a function depending also on the argument τ_i, then the numbers $\dfrac{\pi m}{2h}$ defined in (6) are the poles of G_i w.r.t. this argument.*

Using the second Green's formula we obtain

$$\int_{-h}^{h} (G_i L_i u_i - u_i L_i G_i) dx = u_i'(x) G_i(x,s)\big|_{x=-h}^{x=h}, \tag{8}$$

and then, by (2), an integral representation

$$\begin{aligned} u_i(s) &= ak_i^2 \int_{-h}^{h} G_i(x,s) f_i(x) dx + u_i'(h) G_i(h,s) - u_i'(-h) G_i(-h,s) \\ &= ak_i^2 \int_{-h}^{h} G_i(x,s) f_i(x) dx \\ &\quad + \ u_i(-h)\alpha_i \frac{\cos\tau_i(s-h)}{\tau_i \sin 2\tau_i h} + u_i(h)\beta_i \frac{\cos\tau_i(s+h)}{\tau_i\sin 2\tau_i h}\,. \end{aligned} \tag{9}$$

Note that for the the case $a = 0$ and for given values $u_i(-h)$, $u_i(h)$ we arrive at the solution for the linear equations (1):

$$u_i(s) = \frac{u_i(-h)\alpha_i + u_i(h)\beta_i}{2\tau_i \sin\tau_i h}\cos\tau_i s + \frac{u_i(-h)\alpha_i - u_i(h)\beta_i}{2\tau_i\cos\tau_i h}\sin\tau_i s\,.$$

Setting $s = \pm h$ in (9), we obtain

$$\begin{aligned} u_i(-h) &= ak_i^2 \int_{-h}^{h} G_i(x,-h) f_i(x) dx \\ &\quad + \ u_i(-h)\frac{\alpha_i \cos 2\tau_i h}{\tau_1 \sin 2\tau_i h} + u_i(h)\frac{\beta_i}{\tau_i \sin 2\tau_i h}\,, \end{aligned} \tag{10}$$

and

$$\begin{aligned} u_i(h) &= ak_i^2 \int_{-h}^{h} G_i(x,h) f_i(x) dx \\ &\quad + \ u_i(-h)\frac{\alpha_i}{\tau_i \sin 2\tau_i h} + u_i(h)\frac{\beta_i\cos 2\tau_i h}{\tau_i \sin 2\tau_i h}\,. \end{aligned} \tag{11}$$

From (10) and (3) we see that

$$u_i(-h) = \frac{ak_i^2 \int\limits_{-h}^{h} G_i(x,-h) f_i(x)dx + C_i^{(h)} \dfrac{\beta_i}{\tau_i \sin 2\tau_i h}}{1 - \dfrac{\alpha i \cos 2\tau_i h}{\tau_i \sin 2\tau_i h}}. \qquad (12)$$

Substituting the expressions (12) for $u_i(-h)$ into the formulas (11) we obtain a system of the so-called *dispersion equations*

$$\left\{ \begin{aligned} & u_i(h)\left[(\tau_i^2 - \alpha_i\beta_i)\sin 2\tau_i h - \tau_i(\alpha_i + \beta_i)\cos 2\tau_i h\right] \\ = \; & ak_i^2 \tau_i \left(\tau_i \sin 2\tau_i h - \alpha_i \cos 2\tau_i h\right) \int\limits_{-h}^{h} G_i(x,h) f_i(x)dx \\ & + ak_i^2 \tau_i \alpha_i \int\limits_{-h}^{h} G_i(x,-h) f_i(x)dx, \quad i = 1,\ldots,n. \end{aligned} \right. \qquad (13)$$

Since the values of Green's functions at $s = \pm h$ are known from (7), we can rewrite the dispersion equations (13) in the final form

$$C_i^{(h)} g_i(\lambda_i) = \frac{ak_i^2 Q_i(\boldsymbol{\lambda})}{\sin 2\tau_i h}, \qquad (14)$$

where

$$\begin{aligned} g_i(\lambda_i) &:= (\tau_i^2 - \alpha_i\beta_i)\sin 2\tau_i h - \tau_i(\alpha_i + \beta_i)\cos 2\tau_i h, \\ Q_i(\boldsymbol{\lambda}) &:= (\alpha_i \cos 2\tau_i h - \tau_i \sin 2\tau_i h) \int\limits_{-h}^{h} \cos \tau_i (x+h) f_i(x)dx \\ & \quad - \alpha_i \int\limits_{-h}^{h} \cos \tau_i (x-h) f_i(x)dx. \end{aligned} \qquad (15)$$

The dispersion equations (14) are considered w.r.t. the eigentuples.

Using the expressions (12), the system (9) can be rewritten as

$$\begin{aligned} u_i(s) = \; & ak_i^2 \int\limits_{-h}^{h} G_i(x,s) f_i(x)dx \\ & + a\frac{k_i^2 \alpha_i \cos \tau_i(s-h)}{\tau_i \sin 2\tau_i h - \alpha_i \cos 2\tau_i h} \int\limits_{-h}^{h} G_i(x,-h) f_i(x)dx \\ & + C_i^{(h)} \left[\frac{\alpha_i \cos \tau_i(s-h)}{\tau_i \sin 2\tau_i h - \alpha_i \cos 2\tau_i h} + \cos \tau_i(s+h)\right] \frac{\beta_i}{\tau_i \sin 2\tau_i h}, \\ & i = 1,\ldots,n. \end{aligned} \qquad (16)$$

Next, we rewrite the system (16) in the operator form.

Let $K(x,s)$ be the diagonal kernel matrix

$$K(x,s) := \{K_{ij}(x,s)\}_{i,j=1}^{n}, \quad K_{ij}(x,s) := \delta_{ij} k_i^2 G_i(x,s).$$

Introduce the matrix integral operator $\mathbf{Kg} := \int\limits_{-h}^{h} K(x,s)\mathbf{g}(x)dx$ where $\mathbf{g} := (g_1, \dots, g_n)^\top$. Set

$$\begin{aligned} \tilde{K}(x,s) &:= \{\tilde{K}_{ij}(x,s)\}_{i,j=1}^n, \\ \tilde{K}_{ij}(x,s) &:= k_i^2 p_i \cos\tau_i(s-h) G_i(x,-h), \end{aligned}$$

where

$$p_i := \frac{\alpha_i}{\tau_i \sin 2\tau_i h - \alpha_i \cos 2\tau_i h}\,.$$

Define one more matrix integral operator $\tilde{\mathbf{K}}\mathbf{g} := \int\limits_{-h}^{h} \tilde{K}(x,s)\mathbf{g}(x)dx$. Introduce a vector $\mathbf{h} := (h_1, \dots, h_n)^\top$ where

$$h_i := C_i^{(h)} \frac{\beta_i \left[p_i \cos\tau_i(s-h) + \cos\tau_i(s+h)\right]}{\tau_i \sin 2\tau_i h}.$$

Then the system (16) can be rewritten in the form

$$\mathbf{u} = a\mathbf{K}\left(|\mathbf{u}|^2\mathbf{u}\right) + a\tilde{\mathbf{K}}\left(|\mathbf{u}|^2\mathbf{u}\right) + \mathbf{h}. \tag{17}$$

It should be noted that $\mathbf{K}$ and $\tilde{\mathbf{K}}$ are linear operators.

Introduce also two linear operators $\mathbf{N} := a(\mathbf{K} + \tilde{\mathbf{K}})$ and $\mathbf{N}_1 := \mathbf{K} + \tilde{\mathbf{K}}$.

In what follows we will study the equation (17) in the space $\mathbf{C}[-h,h] := C[-h,h] \times \ldots \times C[-h,h]$ equipped with the norm $\|\mathbf{u}\|_{\mathbf{C}}^2 := \sum_{i=1}^n \|u_i\|_C^2$, where $\|u\|_C := \max\limits_{x\in[-h,h]} |u(x)|$.

3. Solvability of the problem P

In order to formulate a theorem about the existence of eigentuples it is necessary to formulate some auxiliary statements about the equation (17) (all proofs can be found in [2]). Since the equation (17) has the same form as the analogous equation in [2], all the statements and theorems in [2] remain valid for (17).

The following theorem provides existence and uniqueness of solution to equation (17).

Theorem 1 *Let $B_{r_0} := \{\mathbf{v} \in \mathbf{C}[-h,h] : \|\mathbf{v}\| \le r_0\}$ be a closed ball of radius $r_0 > 0$ with the origin at zero. Assume that the two conditions $q := 3ar_0^2\|\mathbf{K} - \tilde{\mathbf{K}}\| < 1$ and $ar_0^3\|\mathbf{K} - \tilde{\mathbf{K}}\| + \|\mathbf{h}\| \le r_0$ are satisfied. Then there exists a unique solution $\mathbf{u} \in B_{r_0}$ of the equation (17) (or, equivalently, of the system (16)). The sequence of approximate solutions $\mathbf{u}^{(p)} \in B_{r_0}$ of equation (17) (or system (16)) defined by the iteration process*

$$\mathbf{u}^{(p+1)} = a\mathbf{K}\left(\left|\mathbf{u}^{(p)}\right|^2 \mathbf{u}^{(p)}\right) + a\tilde{\mathbf{K}}\left(\left|\mathbf{u}^{(p)}\right|^2 \mathbf{u}^{(p)}\right) + \mathbf{h}, \quad p = 0,1,2,\ldots$$

converges in $\mathbf{C}[-h,h]$ to the (unique) exact solution $\mathbf{u} \in B_{r_0}$ of the equation (17) (or the system (16)) for any initial approximation $\mathbf{u}^{(0)} \in B_{r_0}$ with the rate of geometric progression q.

The following theorem gives a constraint on the parameter a.

Theorem 2 *Let* $A := \dfrac{2}{3}\dfrac{1}{\|\mathbf{h}\|\sqrt{3\|\mathbf{N}_1\|}}$ *with* $\|\mathbf{N}_1\| := \|\mathbf{K} + \tilde{\mathbf{K}}\|$ (> 0). *If* $a \le A^2$, *then the equation(17) has a unique solution* $\mathbf{u}$ *in the ball* $B_{r_*} := \{\mathbf{v} \in \mathbf{C}[-h, h] : \|\mathbf{v}\| \le r_*\}$, *where*

$$r_* := -\frac{2}{\sqrt{3\,\|\mathbf{N}\|}}\cos\left(\frac{1}{3}\arccos\left(\frac{3\sqrt{3}}{2}\,\|\mathbf{h}\|\,\sqrt{\|\mathbf{N}\|}\right) - \frac{2\pi}{3}\right)$$

is a root of the equation $r_0^3 - \|\mathbf{N}\|^{-1}\, r_0 + \|\mathbf{h}\| \cdot \|\mathbf{N}\|^{-1} = 0$.

It should be noted that $A > 0$ does not depend on a.

Now we rewrite equation (17) in the form $\mathbf{u} = \mathbf{N}\left(|\mathbf{u}|^2\mathbf{u}\right) + \mathbf{h}$.

The next theorem states that the solution of the equation (17) depends continuously on the parameter $\boldsymbol{\lambda}$.

Theorem 3 *Let the matrix integral operators* $\mathbf{N} = \mathbf{N}(\boldsymbol{\lambda})$ *and the right-hand side* $\mathbf{h} = \mathbf{h}(\boldsymbol{\lambda})$ *of the equation (17) depend continuously on the parameter* $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_0$ *for a certain real* n*-dimensional set* $\boldsymbol{\Lambda}_0 := \Lambda_0^{(1)} \times \ldots \times \Lambda_0^{(n)}$. *Assume also that* $\|\mathbf{h}\| \le \dfrac{2}{3}\dfrac{1}{\sqrt{3\,\|\mathbf{N}(\boldsymbol{\lambda})\|}}$. *Then a unique solution* $\mathbf{u} = \mathbf{u}(\boldsymbol{\lambda})$, $\boldsymbol{\lambda} \in \boldsymbol{\Lambda}_0$, *of the equation (17) exists and depends continuously on the parameter* $\boldsymbol{\lambda}$, *i.e.* $\mathbf{u} \in \mathbf{C}(\boldsymbol{\Lambda}_0)$.

Now let us prove the solvability of the problem P. To this end, we consider the dispersion equations (13).

If $a = 0$, we obtain the dispersion equations for the linear case of (1):

$$g_i(\lambda_i) := (\tau_i^2 - \alpha_i\beta_i)\sin 2\tau_i h - \tau_i(\alpha_i + \beta_i)\cos 2\tau_i h = 0. \tag{18}$$

Setting

$$\mu_i^{(m)} := k_i^2 - \frac{\pi^2 m^2}{4h^2}, \quad m \in \mathbb{N},$$

we have that

$$g_i(\mu_i^{(m)}) = (-1)^m \frac{\pi m}{2h}\left(\alpha_i(\mu_i^{(m)}) + \beta_i(\mu_i^{(m)})\right). \tag{19}$$

Remark 3 *From the formula (19) we see that the function* g_i *alternates its sign at the points* $\mu_i^{(m)}$, $m \in \mathbb{N}$, *since* $\alpha_i(\lambda_i) + \beta_i(\lambda_i) > 0$ *by assumption.*

It is possible to prove the following statement.

Proposition 2 *Let* $k_i \ge \dfrac{\pi m}{2h}$, $m = 1, \ldots, l_i$ *for a certain* $l_i \in \mathbb{N}$, $l_i \ge 2$. *Then there exist at least* $l_i - 1$ *roots* $\lambda_i^{(j)}$ *of the equation (18) where* $\lambda_i^{(j)} \in \left(\mu_i^{(j+1)}, \mu_i^{(j)}\right)$.

Proof: Let $\delta_i^{(j)} > 0$, $j = 1, \ldots, l_i - 1$, be sufficiently small such that

$$\lambda_i^{(j)} \in \Lambda_i^{(j)} := \left[\mu_i^{(j+1)} + \delta_i^{(j)}, \mu_i^{(j)} - \delta_i^{(j)}\right], \quad j = 1, \ldots, l_i - 1,$$

and the values $g_i(\lambda_i)$ have different signs at the different endpoints of the segments $\Lambda_i^{(j)}$. Under these assumptions, taking into account (19), the function $g_i(\lambda_i)$ vanishes at the points $\lambda_i^{(j)}$, $j = 1, \dots, l_i - 1$. ◀

Now we define $\Lambda_i := \bigcup_{j=1}^{l_i-1} \Lambda_i^{(j)}$ and $\mathbf{\Lambda} := \Lambda_1 \times \ldots \times \Lambda_n$. The following estimates hold:

$$\left| \frac{k_i^2 Q_i(\boldsymbol{\lambda})}{\sin 2\tau_i h} \right| \le C_i r_{00}^3 \quad \text{where} \quad r_{00} := \max_{\boldsymbol{\lambda} \in \mathbf{\Lambda}} r_*.$$

Let $C := \max\{C_1, \dots, C_n\}$ and

$$M_i^{(1)} := \min_{1 \le j \le l_i - 1} \left| g_i \left(\mu_i^{(j+1)} + \delta_i^{(j)} \right) \right|,$$

$$M_i^{(2)} := \min_{1 \le j \le l_i - 1} \left| g_i \left(\mu_i^{(j)} - \delta_i^{(j)} \right) \right|.$$

Finally we set $\tilde{M} := \min_{1 \le i \le n} \left\{ M_i^{(1)}, M_i^{(2)} \right\}$ and $A_1 := \min_{\boldsymbol{\lambda} \in \mathbf{\Lambda}} A(\boldsymbol{\lambda})$, where $A(\boldsymbol{\lambda}) := \dfrac{2}{3 \|\mathbf{h}\| \sqrt{3 \|\mathbf{N}_1(\lambda)\|}}$.

The following theorem is the main result of this study.

Theorem 4 *Let $\alpha_i(\lambda_i) \ge 0$, $\beta_i(\lambda_i) \ge 0$, and $\alpha_i(\lambda_i) + \beta_i(\lambda_i) > 0$ for all $\lambda_i \in \Lambda_i$. Assume also that $0 < a \le a_0$, where $a_0 := \min \left\{ A_1^2, \dfrac{\tilde{M}}{C r_{00}^3} \right\}$, and that the conditions $k_i \ge \frac{\pi m}{2h}$, $m = 1, \dots, l_i$ hold for certain numbers $l_i \in \mathbb{N}$, $l_i \ge 2$. Then there exist at least $(l_1 - 1) \dots (l_n - 1)$ eigentuples $\left(\lambda_1^{[j]}, \dots, \lambda_n^{[j]} \right)^\top$, $j = 1, \dots, l_i - 1$, of the problem P, i.e. for $\lambda_i = \lambda_i^{[j]}$, $j = 1, \dots, l_i - 1$, $i = 1, \dots, n$, the problem (1) – (3) has a nontrivial solution.*

The proof of this theorem virtually repeats the proof of a similar statement from [4]. In spite of the fact that the formulas of this paper are similar to the analogous expressions in [4], we strengthen that their meaning is different because we consider here the case of arbitrary n. Therefore the conclusion can be made that Theorem 4 constitutes an essentially new result.

The proof of Theorem 4 employs the method of small parameters where the nonlinearity coefficient a is supposed to be small.

This approach can be applied for the analysis of nonlinear eigenvalue problems because in many physical models involving nonlinear eigenvalue problems of the type considered in this study it is known that a is sufficiently small.

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