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OWP 2008 - 04

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Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

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Imprint:

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The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

Interpolation in Bernstein and Paley–Wiener spaces

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Abstract

We construct closed sets S of arbitrarily small measure with the property: given any discrete set Λ, every $l^\infty-$function on Λ can be interpolated by an $L^\infty-$function with spectrum on S. This should be contrasted against Beurling–Landau type theorems for compact spectra.

1 Introduction

1. **Spaces**. Let S be a closed set in $\mathbb{R}$. We say that a function f defined on (another copy of) $\mathbb{R}$ belongs to Bernstein space B_S if

(i) $f \in L^\infty(\mathbb{R})$;

(ii) f is the Fourier transform of a Schwartz distribution F (notation: $f = \hat{F}$), supported by S.

The support of F is called the spectrum of f.

In other words, a function f belongs to B_S if and only if it is bounded and we have

$$\int_{\mathbb{R}} f(x)\hat{\varphi}(x)\,dx = 0,$$

for every smooth function $\varphi(x)$ whose support is compact and disjoint from S. Endowed with the $L^\infty-$norm, B_S is a Banach space.

We shall also consider the Paley–Wiener spaces

$$PW_S := \{f \in L^2(\mathbb{R}); f = \hat{F},\ F = 0 \text{ on } \mathbb{R} \setminus S\},$$

where the spectrum S can be any measurable set.

Classical Paley–Wiener (Bernstein) spaces are defined when S is a bounded (compact) set. The elements of these spaces are entire functions of finite exponential type. We shall focus on the situation when S is an unbounded closed set, and so the elements of the corresponding spaces do not necessarily admit analytic continuation into the complex plain.

2. **Interpolation**. Let $\Lambda = \{\lambda_j, j \in \mathbb{Z}\} \subset \mathbb{R}$ be a uniformly discrete (u.d.) set, that is

$$\inf_{j \neq k} |\lambda_j - \lambda_k| > 0.$$

Given a bounded data $\mathbf{c} = \{c_j, j \in \mathbb{Z}\}$, one wishes to find a function $f \in B_S$ such that

$$f(\lambda_j) = c_j,\ \ j \in \mathbb{Z}. \tag{1}$$

If this is possible for every $\mathbf{c} \in l^\infty(\mathbb{Z})$, one says that the interpolation problem is solvable, and Λ is called an *interpolation set* for B_S.

Similarly, if for arbitrary data $\mathbf{c} \in l^2(\mathbb{Z})$ there is a function $f \in PW_S$ satisfying (1), one says that Λ is a set of interpolation for PW_S.

General principles of functional analysis imply that if Λ is a interpolation set for B_S, then one can solve interpolation problem (1) with an additional estimate:

$$\|f\|_{L^\infty} \leq C\|\mathbf{c}\|_{l^\infty}, \tag{2}$$

where the constant C does not depend on data $\mathbf{c}$. A similar result is true for the Paley–Wiener spaces (see [11], p. 129).

3. **Density**. For every u.d. set Λ one can define its upper uniform density

$$D^+(\Lambda) := \lim_{l \to \infty} \max_{a \in \mathbb{R}} \frac{\#(\Lambda \cap (a, a+l))}{l}.$$

A fundamental role of this quantity in the interpolation problem, in the case when S is a single interval, was found by A. Beurling and J-P. Kahane: Beurling proved ([2]) that Λ is an interpolation set for B_S if and only if

$$D^+(\Lambda) < \frac{1}{2\pi} \operatorname{mes} S.$$

Even earlier, Kahane proved in [5] that for Λ to be an interpolation set for PW_S it is necessary that

$$D^+(\Lambda) \leq \frac{1}{2\pi}\,\mathrm{mes}\,S, \tag{3}$$

and it is sufficient that

$$D^+(\Lambda) < \frac{1}{2\pi}\,\mathrm{mes}\,S.$$

Both results are based on the theory of entire functions.

4. **Disconnected spectra**. The situation becomes much more delicate for disconnected spectra, in particular when S is a union of two intervals. For the sufficiency part, not only the size but also the arithmetics of Λ is important in that case. On the other hand, using a new approach Landau [6] succeeded to extend the necessity part to the general case:

Theorem 1.1 [6] *Let S be a bounded set. If a u.d. set Λ is an interpolation set for PW_S then condition (3) is fulfilled.*

We shall prove some versions of this result for both Paley–Wiener and Bernstein spaces. In fact, it already suffices to know that the "delta-functions" on Λ admit interpolation by B_S- or PW_S- functions (with not very fast growing norms) in order to obtain an estimate from below for the measure of S (see section 2 below).

However, for unbounded spectra the situation is completely different. The contrast is most striking for Bernstein spaces: Not only condition (3) is no longer necessary, but there exist "universal" spectra of arbitrary small measure which deliver positive solution to the interpolation problem for every u.d. Λ:

Theorem 1.2 *For every $\delta > 0$ there is a closed (unbounded) set S, $\mathrm{mes}\,S < \delta$, such that every u.d. set Λ is an interpolation set for the space B_S.*

This theorem will be proved in sec. 3.

Such a result cannot hold for Paley–Wiener spaces (see Propositions 4.1 and 4.2 in sec. 4). However, we show that for a generic Λ there are arbitrarily small spectra S such that Λ is "almost" a set of interpolation for PW_S (Theorem 4.1).

Results of this paper concerning Paley–Wiener spaces were announced in our note [8].

Acknowledgment. Part of the present work was done at the Mathematisches Forschungsinstitut Oberwolfach during a stay within the Research in Pairs Programme from April 1 to April 15, 2007. The authors appreciate the hospitality of the institute.

2 Compact spectra: Weak interpolation

The aim of this section is to show that even a much weaker interpolation property of Λ ensures Beurling–Landau type estimates on the measure of S.

In what follows we shall denote by $\|f\|_2$ and $\|f\|_\infty$ the L^2- and $L^\infty-$norm of f, respectively.

2.1. **Concentration.**

Definition: Given a number $c, 0 < c < 1$, we say that a linear subspace W of PW_S is *c-concentrated* on a set Q if

$$\int_Q |f(x)|^2\, dx \geq c\|f\|_2^2,\ f \in W.$$

Following Landau we estimate the dimension of a concentrated subspace.

Lemma 2.1 *Given sets $S, Q \subset \mathbb{R}$ and a number $0 < c < 1$, let W be a linear subspace of PW_S which is c-concentrated on Q. Then*

$$dim\, W \leq \frac{(mes\, Q)\,(mes\, S)}{2\pi c}.$$

Proof ([6], see (iii) and (iv) on p. 41). Let $Q, S \subset \mathbb{R}$ be two sets of finite positive measure. Let A_Q and B_S denote the orthogonal projections of $L^2(\mathbb{R})$ onto $L^2(Q)$ and PW_S, respectively:

$$A_Q f = \mathbf{1}_Q f,\ B_S f = \mathfrak{F}\, \mathbf{1}_S\, \mathfrak{F}^{-1} f,$$

where $\mathbf{1}_Q$ denotes the characteristic function of Q, and $\mathfrak{F} f$ is the Fourier transform of f:

$$\mathfrak{F} f(x) = \hat{f}(x) := \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} e^{itx} f(t)\, dt.$$

The operator $C := A_Q B_S A_Q$ acts from $L^2(\mathbb{R})$ into itself. Clearly, C is self-adjoint and positive. It can be written explicitly as

$$Cf(y) = \frac{1}{\sqrt{2\pi}} \int_{\mathbb{R}} \mathbf{1}_Q(x) \mathbf{1}_Q(y) \hat{\mathbf{1}}_S(y-x) f(x)\, dx.$$

Since the kernel is square-integrable, C is a compact operator. Denote by l_j its eigenvalues arranged in non-increasing order (counting multiplicities). The trace $\operatorname{Tr} C$ is equal to the integral of kernel along the "diagonal":

$$\operatorname{Tr} C := \sum_j l_j = \frac{(\operatorname{mes} Q)\,(\operatorname{mes} S)}{2\pi}. \tag{4}$$

2. One can easily show that the spectrum of operator $D := B_S A_Q B_S$ is identical to the spectrum of operator C.

Let W be a linear subspace of PW_S of dimension k. The quadratic form (Df, f) on W is given by $(A_Q f, f) = \|\mathbf{1}_Q f\|_2^2$. If W is $c-$ concentrated on Q, then

$$\inf_{f \in W, \|f\|_2 = 1} (Df, f) \geq c.$$

It is well-known that, among all subspaces of dimension k, the greatest value of $\inf(Df, f)$ on the unit sphere is achieved on a subspace spanned by the first k eigenvectors of D. This means that $c \leq l_k \leq \operatorname{Tr} D/k = \operatorname{Tr} C/k$. This and (4) prove the lemma. One can show that a similar result holds in several dimensions, too.

Remark. The uncertainty principle in Fourier analysis is a statement that a function and its Fourier transform cannot both be concentrated on small sets. A simple corollary of Lemma 2.1 is the following variant of uncertainty principle due to Amrein and Berthier [1]: Let S and Q be any sets of finite measure. There exists $C = C(S, Q)$ such that

$$\int_{\mathbb{R}} |f(x)|^2\, dx \leq C \left(\int_{\mathbb{R} \setminus S} |f(x)|^2\, dx + \int_{\mathbb{R} \setminus Q} |\widehat{f}(t)|^2\, dt \right), \; f \in L^2(\mathbb{R}).$$

2.2. **Interpolation of delta–functions.** We start with the following

Theorem 2.1 *Let S be a compact set and $\Lambda = \{\lambda_j, j \in \mathbb{Z}\}$ be a u.d. set. Suppose that for every $j \in \mathbb{Z}$ there is a function $f_j \in PW_S$ such that*

$$f_j(\lambda_k) = \begin{cases} 1, & k = j \\ 0, & k \neq j \end{cases}, \ k \in \mathbb{Z}; \tag{5}$$

$$\sup_{j \in \mathbb{Z}} \|f_j\|_2 < \infty. \tag{6}$$

Then (3) holds.

Proof. 1. Recall that if S is a compact set, then the corresponding Paley–Wiener space consists of entire functions of exponential type. The following lemma is well known (see [11], p. 82):

Lemma 2.2 *Suppose $\Lambda = \{\lambda_j, j \in \mathbb{Z}\}$ is a u. d. set, and $S \subset \mathbb{R}$ is a compact set. Then there exists $C > 0$ such that*

$$\|f\|_2^2 \geq C \sum_{j \in \mathbb{Z}} |f(\lambda_j)|^2, \text{ for every } f \in PW_S.$$

2. Fix a number $\delta > 0$, and set

$$S(\delta) := S + [-\delta, \delta]. \tag{7}$$

Take functions $f_j \in PW_S$ satisfying (5) and (6), and set

$$g_j(x) := f_j(x) \left(\frac{\sin \delta(x - \lambda_j)}{\delta(x - \lambda_j)} \right)^2, \ j \in \mathbb{Z}.$$

Clearly, functions g_j satisfy (5), (6) and belong to $PW_{S(2\delta)}$.

3. Fix any number R, and denote by $\#(\Lambda \cap (R - r, R + r))$ the number of $\lambda_j \in \Lambda$ in the interval $(R - r, R + r)$. Since Λ is u.d. we have:

$$\#\left(\Lambda \cap (R - r, R + r)\right) \leq Cr, \quad \text{for all } r > 1, \tag{8}$$

where $C > 0$ is a constant which does not depend on R.

Let us introduce a linear space of functions

$$W_r := \left\{ g(x) = \sum_{k: |\lambda_k - R| < r} c_k g_k(x), \ c_k \in \mathbb{C} \right\}.$$

Clearly, $\dim W_r = \#(\Lambda \cap (R - r, R + r))$.

4. We would like to estimate the concentration of W_r on the interval $(R-r-r\delta, R+r+r\delta)$. Choose any function $g=\sum_j c_j g_j \in W_r$. Then $g(\lambda_j)=c_j$ when $|\lambda_j - R|<r$, and $g(\lambda_j)=0$ when $|\lambda_j - R|\ge r$. By Lemma 2.2, there is a constant $C=C(S,\Lambda)>0$ such that

$$\|g\|_2^2 \ge C \sum_{\lambda_j\in\Lambda} |g(\lambda_j)|^2 = C \sum_{|\lambda_j-R|<r} |c_j|^2. \tag{9}$$

On the other hand, by Parseval's identity and (6), we have

$$|f_j(x)|^2 \le \left(\frac{1}{\sqrt{2\pi}}\int_S |\hat{f}_j(t)|\,dt\right)^2 \le \frac{\operatorname{mes} S}{2\pi}\|\hat{f}_j\|_2^2 \le C,\ j\in\mathbb{Z},$$

for some $C>0$.

Observe that $|x-\lambda_j|\ge \delta r$ whenever $\lambda_j\in(R-r,R+r)$ and $|x-R|\ge r+\delta r$. Hence, the last inequality and (8) give

$$\int_{|x-R|\ge r+\delta r} |g(x)|^2\,dx =$$

$$\int_{|x-R|\ge r+\delta r}\left|\sum_{|\lambda_j-R|<r} c_j f_j(x)\left(\frac{\sin\delta(x-\lambda_j)}{\delta(x-\lambda_j)}\right)^2\right|^2 dx \le$$

$$Cr\left(\sum_{|\lambda_j-R|<r}|c_j|^2\right)\int_{|x|>\delta r}\left(\frac{1}{\delta x}\right)^4 dx \le \frac{C}{\delta^7 r^2}\left(\sum_{|\lambda_j-R|<r}|c_j|^2\right).$$

This and (9) show that for every $\epsilon>0$ there exists r_ϵ such that g is $(1-\epsilon)-$concentrated on $(R-r-\delta r, R+r+\delta r)$ for all $r\ge r_\epsilon$.

4. It now follows from Lemma 2.1, that

$$\#\left(\Lambda\cap(R-r,R+r)\right) \le \frac{(\operatorname{mes} S(2\delta))(\operatorname{mes}(R-r-\delta r, R+r+\delta r))}{2\pi(1-\epsilon)},$$

whenever $r\ge r_\epsilon$. Since this is true for every R, we obtain:

$$D^+(\Lambda)\le \frac{1+\delta}{1-\epsilon}\frac{\operatorname{mes} S(2\delta)}{2\pi}.$$

Letting $\delta\to 0$ and $\epsilon\to 0$, we conclude that $D^+(\Lambda)\le(2\pi)^{-1}\operatorname{mes} S$.

Remark. The assumptions of Theorem 2.1 admit a geometric interpretation: Set

$$E(\Lambda) := \{e^{i\lambda t}, \lambda\in\Lambda\}.$$

Conditions (5) and (6) are equivalent to the property that the exponential system $E(\Lambda)$ is *uniformly minimal* in $L^2(S)$, that is the L^2-distance from any element of the system to the span of other elements is greater then some positive constant.

Observe that assumptions (5) and (6) are less restrictive then the requirement for Λ to be an interpolation set for PW_S. Indeed, consider a classical example: $S = [-\pi, \pi]$ and $\Lambda = \{\lambda_j, j \in \mathbb{Z}\}$,

$$\lambda_j := \begin{cases} j + \frac{1}{4}, & j = 1, 2, ... \\ 0, & j = 0 \\ j - \frac{1}{4}, & j = -1, -2, ... \end{cases}$$

The corresponding exponential system $E(\Lambda)$ is uniformly minimal in $L^2(-\pi, \pi)$ (see [7], Theorem 5). However, Λ is not an interpolation set for $L^2(-\pi, \pi)$ (see the remark following Theorem 5 in [7]).

An analog of Theorem 2.1 for Bernstein spaces is also true:

Theorem 2.2 *Let S be a compact set, and $\Lambda = \{\lambda_j\}$ a u.d. set. Suppose that for every $j \in \mathbb{Z}$ there is a function $f_j \in B_S$ satisfying (5) and*

$$\sup_{j \in \mathbb{Z}} \|f_j\|_\infty < \infty. \tag{10}$$

Then (3) holds.

Indeed, suppose functions $f_j \in B_S$ satisfy assumptions of Theorem 2.2. Take any positive number δ, and set

$$g_j(x) := f_j(x) \frac{\sin \delta(x - \lambda_j)}{\delta(x - \lambda_j)}.$$

Then the functions g_j belong to $PW_{S(\delta)}, j \in \mathbb{Z}$, where $S(\delta)$ is defined in (7), and satisfy assumptions of Theorem 2.1. Hence, $D^+(\Lambda) \leq (2\pi)^{-1} \text{mes}\, S(\delta)$. Letting $\delta \to 0$, we obtain Theorem 2.2.

An immediate corollary of Theorem 2.2 is a variant of Landau's Theorem 1.1 for Bernstein spaces:

Corollary *Let S be a compact set. If a u.d. set Λ is an interpolation set for B_S, then condition (3) is fulfilled.*

2.3. **Growth of $\|f_j\|_2$ and upper density.**

An estimate similar to (3) holds even if the norms of functions satisfying (5) grow, but not very fast:

$$\|f_j\|_2 \leq C e^{|\lambda_j|^\alpha}, \ j \in \mathbb{Z}, \tag{11}$$

where $C > 0$ and $0 < \alpha < 1$. We shall show that this and (5) imply estimate (3) in which D^+ is replaced by the upper density D^*.

Definition: Let Λ be a u.d. set. The upper density of Λ is defined as follows:

$$D^*(\Lambda) := \limsup_{a\to\infty} \frac{\#(\Lambda \cap (-a,a))}{2a}.$$

Clearly, we have $D^*(\Lambda) \le D^+(\Lambda)$. Observe that for regularly distributed Λ, for example if $\Lambda = \{j + O(1), j \in \mathbb{Z}\}$, these two densities are equal.

Theorem 2.3 *Let $S \subset \mathbb{R}$ be a compact set, and $\Lambda = \{\lambda_j, j \in \mathbb{Z}\}$ a u.d. set. Suppose there exist functions $f_j \in PW_S, j \in \mathbb{Z}$, satisfying (5) and (11) with some $C > 0$ and $0 < \alpha < 1$. Then $D^*(\Lambda) \le (2\pi)^{-1} \operatorname{mes} S$.*

Proof. 1. We shall need

Lemma 2.3 *Set*

$$\varphi(z) := \prod_{k\ge k_0} \frac{\sin(k^{-1-\beta} z)}{k^{-1-\beta} z},$$

where $k_0 \in \mathbb{N}$ and $0 < \beta < 1$ are some numbers. Then $\varphi \in PW_{[-\sigma,\sigma]}$, where $\sigma = \sum_{k\ge k_0} k^{-1-\beta}$, and

$$|\varphi(x+iy)| \le C \exp\left(-\frac{1}{C}|x|^{\frac{1}{1+\beta}}\right),\ x \in \mathbb{R},\ |y| \le 1, \tag{12}$$

where C is a constant.

Proof. For every $a > 0$, the function $\sin az/az$ is the Fourier transform of a positive constant function on $[-a,a]$. Hence, $\varphi(z)$ is the Fourier transform of the convolution of such functions. This convolution is concentrated on $[-\sigma,\sigma]$, $\sigma = \sum_{k\ge k_0} k^{-1-\gamma}$, and so $\varphi \in PW_{[-\sigma,\sigma]}$.

Let us denote by δ_1 and δ_2 positive constants such that

$$e^{-\delta_1|z|^2} \le \left|\frac{\sin z}{z}\right| \le e^{\frac{1}{\delta_2}y^2 - \delta_2 x^2},\ z = x+iy \in \mathbb{C},\ |z| \le 1. \tag{13}$$

Since $|\sin(a+ib)| \le \exp|b|$, for a, b real, we have:

$$\prod_{k_0\le k<|z|^{\frac{1}{1+\beta}}} \left|\frac{\sin(k^{-1-\beta}z)}{k^{-1-\beta}z}\right| \le \prod_{k_0\le k<|z|^{\frac{1}{1+\beta}}} \left|\sin(k^{-1-\beta}z)\right| \le$$

$$\exp\left(|y|\sum_{k_0\le k<|z|^{\frac{1}{1+\beta}}}\frac{1}{k^{1+\beta}}\right)\le e^{C|y|},\ z=x+iy,$$

where C is a constant. Now, using the right-hand inequality in (13), we get for every $z\in\mathbb{C}, |z|\ge 1+k_0^{1+\beta}$, that

$$\prod_{k\ge|z|^{\frac{1}{1+\beta}}}\left|\frac{\sin(k^{-1-\gamma}z)}{k^{-1-\beta}z}\right|\le C\exp\left(-\delta_2x^2\sum_{k\ge|z|^{\frac{1}{1+\beta}}}\frac{1}{k^{2+2\beta}}\right)$$

$$\le C\exp\left(-\frac{1}{C}|x|^{\frac{1}{1+\beta}}\right),\ |y|\le 1,$$

where C is a constant. This and the previous inequality give (12).

2. Fix a small number $\delta>0$ and pick up k_0 in Lemma 2.3 so large that $\sigma\le\delta$, i.e. the function $\varphi\in PW_{[-\delta,\delta]}$. Also, assume that $\beta<1$ satisfies $\alpha<1/(1+\beta)$.

The rest of the proof is pretty similar to the proof of Theorem 2.1.

Without loss of generality, we may assume that $\lambda_0=0$. Take functions $f_j\in PW_S$ satisfying (5) and (11), and set

$$g_j(x):=f_j(x)\varphi(x-\lambda_j),\ j\in\mathbb{Z},$$

where φ is a function from Lemma 2.3. It is clear that $g_j\in PW_{S(\delta)}$, where $S(\delta)$ is defined in (7), and that (5) is true for every $j\in\mathbb{Z}$.

3. Since Λ is u.d., we have

$$\#\left(\Lambda\cap(-r,r)\right)\le Cr, \tag{14}$$

for some $C>0$ and all $r>1$. Set

$$W_r:=\left\{g(x)=\sum_{|\lambda_k|<r}c_kg_k(x):c_k\in\mathbb{C}\right\}.$$

Clearly, $\dim W_r=\#(\Lambda\cap(-r,r))$.

4. Let us estimate the concentration of W_r on the interval $(-r-r\delta, r+r\delta)$. Take any function $g\in W_r$. Since $g(\lambda_j)=0$ when $|\lambda_j|\ge r$, by Lemma 2.2, we have:

$$\|g\|_2^2\ge C\sum_{|\lambda_j|<r}|c_j|^2. \tag{15}$$

Further, by (11) and Parseval's identity, we have

$$|f_j(x)|^2 \leq \left(\frac{1}{\sqrt{2\pi}} \int_S |\hat{f}_j(t)|\, dt \right)^2 \leq \frac{\operatorname{mes} S}{2\pi} \|\hat{f}_j\|_2^2 \leq C e^{r^\alpha},$$

for some $C > 0$ and all $j, |\lambda_j| < r$. Observe also that $|x - \lambda_j| \geq \delta r$ whenever $|\lambda_j| < r$ and $|x| \geq r + \delta r$. This, (14) and the last inequality give

$$\int_{|x| \geq r+\delta r} |g(x)|^2 \, dx = \int_{|x| \geq r+\delta r} \left| \sum_{|\lambda_j| < r} c_j f_j(x) \varphi(x - \lambda_j) \right|^2 dx \leq$$

$$C r e^{r^\alpha} \left(\sum_{|\lambda_j| < r} |c_j|^2 \right) \int_{|x| > \delta r} |\varphi(x)|^2 \, dx.$$

Recall that $\alpha < 1/(1+\beta)$, so that (12) gives:

$$r e^{r^\alpha} \int_{|x| > \delta r} |\varphi(x)|^2 \, dx \to 0, \ r \to \infty.$$

Hence, by (15), we see that for every $\epsilon > 0$ there exists r_ϵ such that g is $(1-\epsilon)-$concentrated on $(-r - \delta r, r + \delta r)$ for all $r \geq r_\epsilon$.

4. It now follows from Lemma 2.1, that

$$\#\left(\Lambda \cap (-r, r)\right) \leq \frac{(\operatorname{mes} S(\delta))(\operatorname{mes}(-r - \delta r, r + \delta r))}{2\pi(1-\epsilon)}, \ r \geq r_\epsilon.$$

This gives:

$$D^*(\Lambda) = \limsup_{r \to \infty} \frac{\#\left(\Lambda \cap (-r, r)\right)}{2r} \leq \frac{1+\delta}{2\pi(1-\epsilon)} \operatorname{mes} S(\delta).$$

Letting $\delta \to 0$ and $\epsilon \to 0$, we conclude that $D^*(\Lambda) \leq (2\pi)^{-1} \operatorname{mes} S$.

By using the same argument which deduces Theorem 2.2 from Theorem 2.1, one can easily extend Theorem 2.3 to Bernstein spaces:

Theorem 2.4 *Let $S \subset \mathbb{R}$ be a compact set, and $\Lambda = \{\lambda_j, j \in \mathbb{Z}\}$ a u.d. set. Suppose that there exist functions $f_j \in B_S, j \in \mathbb{Z}$, satisfying (5) and*

$$\|f_j\|_\infty \leq C e^{|\lambda_j|^\alpha}, \ j \in \mathbb{Z}, \tag{16}$$

for some $C > 0$ and $0 < \alpha < 1$. Then $D^*(\Lambda) \le (2\pi)^{-1}\, mes\, S$.

Remark. The "growth restrictions" (11) and (16) can be replaced by any "non quasi-analytic growth". However, we do not know if they can be omitted at all.

2.4. A counterexample. The results of last subsection are not true for the density D^+. We shall show this only for the Paley–Wiener case.

Theorem 2.5 *For every $\rho > 1$ there exist a u.d. set $\Lambda, D^+(\Lambda) \ge \rho$, and functions $f_j \in PW_{[-\pi,\pi]}, j \in \mathbb{Z}$, which satisfy (5) and (11) with some $C > 0, 0 < \alpha < 1$.*

Proof. 1. Fix any numbers $k_0 \in \mathbb{N}$ and $0 < \beta < 1$, and let φ be the function in Lemma 2.3. Let us show that for every $r \neq 0$ we have

$$|\varphi(x + ri)| \ge C \exp\left(-\frac{1}{C}|x|^{\frac{1}{1+\beta}} \log(1 + |x|)\right), \; x \in \mathbb{R}, \tag{17}$$

where $C > 0$ depends only on r. For simplicity, we shall assume that $r = 1$.

From the left-hand inequality of (13), we obtain:

$$\prod_{k \ge |x+i|^{\frac{1}{1+\beta}}} \left|\frac{\sin(k^{-1-\beta}(x+i))}{k^{-1-\beta}(x+i)}\right| \ge \exp\left(-\delta_1|x+i|^2 \sum_{k \ge |x+i|^{\frac{1}{1+\beta}}} \frac{1}{k^{2+2\beta}}\right)$$

$$\ge \exp\left(-C|x+i|^{\frac{1}{1+\beta}}\right),$$

where C is a constant. Further, since

$$|\sin(a + ib)| \ge \frac{e^b - e^{-b}}{2} \ge b, \; a \in \mathbb{R}, b > 0,$$

we have

$$\prod_{k_0 \le k < |x+i|^{\frac{1}{1+\beta}}} \left|\frac{\sin(k^{-1-\beta}(x+i))}{k^{-1-\beta}(x+i)}\right| \ge$$

$$\prod_{k_0 \ge k < |x+i|^{\frac{1}{1+\beta}}} \frac{\exp\left(\frac{1}{k^{1+\beta}}\right) - \exp\left(-\frac{1}{k^{1+\beta}}\right)}{2|x+i|k^{-1-\beta}} \ge \left(\frac{1}{|x+i|}\right)^{|x+i|^{1/(1+\beta)}}.$$

This and the previous estimate prove (17).

2. Set

$$\psi(z) := \eta(z)\eta(-z),\ \eta(z) := \prod_{j=1}^{\infty}\prod_{k=0}^{j-1}\left(1-\frac{z}{4^j+k}\right).$$

Observe that

$$\prod_{k=0}^{j-1}\left|1-\frac{z}{4^j+k}\right| \le \exp\left(|z|\sum_{k=0}^{j-1}\frac{1}{4^j+k}\right) \le \exp\left(\frac{|z|j}{4^j}\right) < e^{j2^{-j}},$$

whenever $2^j > |z|$. This gives:

$$|\eta(z)| \le \prod_{2^j>|z|} e^{j2^{-j}} \prod_{2^j\le|z|}(1+|z|)^j \le C\,(1+|z|)^{\frac{\log|z|}{\log 2}} \le Ce^{C\log^2(1+|z|)},$$

from which is follows that

$$|\psi(z)| \le Ce^{C\log^2(1+|z|)}, \tag{18}$$

where C is a constant.

We also need to estimate from below the derivative ψ' at the zero points $\pm(4^j+k)$ of ψ. Pick up numbers $J\in\mathbb{N}$ and $K, 0\le K<J$. We have

$$\prod_{0\le k<J,k\ne K}\left|1-\frac{4^J+K}{4^J+k}\right| = \prod_{0\le k<J,k\ne K}\frac{|K-k|}{4^J+k} \ge \left(\frac{1}{4^J+J}\right)^J \ge e^{-CJ^2}.$$

Further, since

$$\frac{4^J}{4^j+k}\ge 2,\ 1\le j<J,\ \frac{4^J+j}{4^j}\le 1,\ j>J, 0\le k<j,$$

we get:

$$\prod_{j\ge1,j\ne J}\prod_{k=0}^{j-1}\left|1-\frac{4^J+K}{4^j+k}\right| \ge \prod_{j>J}\left(1-\frac{4^J+J}{4^j}\right)^j \ge C.$$

This and the previous estimate show that

$$|\eta'(4^J+K)| \ge Ce^{-CJ^2},\ J\in\mathbb{N},\ 0\le K<J,$$

where C is a constant. Since $\eta(-x)\ge 1$ for $x\ge 0$, the same estimate holds for $\psi'(4^J+K)$. Since ψ is even, we obtain:

$$|\psi'(\lambda)| \ge Ce^{-C\log^2|\lambda|},\quad \text{for all } \lambda,\ \psi(\lambda)=0.$$

Now, choose a number $\rho > 1$ and set

$$f(z) := \psi(\rho z)\varphi(\frac{\pi}{\sigma}(z+i)),$$

where φ is a function from Lemma 2.3, and σ is its type. The last inequality and and (17) imply:

$$|f'(\lambda)| \geq Ce^{-\frac{1}{C}|\lambda|^{\frac{1}{1+\beta}}\log(1+|\lambda|)}, \text{ for all } \lambda, \ \psi(\rho\lambda) = 0. \qquad (19)$$

Moreover, from (18) and (12), we see the type of f is equal to π and that $f \in L^2(\mathbb{R})$. Hence, $f \in PW_{[-\pi,\pi]}$, by the Paley–Wiener theorem.

Let Λ be the zero set of $\psi(\rho x)$:

$$\Lambda := \Gamma \bigcup (-\Gamma), \ \Gamma := \bigcup_{j=1}^{\infty} \{\frac{4^j}{\rho}, ..., \frac{4^j + j - 1}{\rho}\}.$$

It is clear that $D^+(\Lambda) = \rho$. Set $\lambda_0 = 4$, and numerate the elements of $\Lambda = \{\lambda_j, j \in \mathbb{Z}\}$ in increasing order. Set

$$f_j(z) := \frac{f(z)}{f'(\lambda_j)(z - \lambda_j)}, \ j \in \mathbb{Z}.$$

Then functions $f_j \in PW_{[-\pi,\pi]}$ and satisfy (5).

We have:

$$\|f_j\|_2^2 \leq \frac{\|f\|_2^2}{|f'(\lambda_j)|^2} + \int_{\lambda_j - 1}^{\lambda_j + 1} \left| \frac{f(x)}{f'(\lambda_j)(x - \lambda_j)} \right|^2 dx.$$

Using the Cauchy integral formula, one can estimate from above the integral in the right hand-side by

$$2|f'(\lambda_j)|^{-2} \left(\sup_{|z-\lambda_j|=1} |f(z)| \right)^2.$$

We conclude that

$$\|f_j\|_2^2 \leq \frac{C}{|f'(\lambda_j)|^2},$$

where C depends only on f. Now, by (19), we see that assumption (11) is also fulfilled for each $\alpha < 1$ satisfying $\alpha > 1/(1+\beta)$.

3 Unbounded spectra: interpolation in Bernstein spaces

The goal of this section is to construct universal interpolation spectra S of arbitrarily small measure:

Theorem 3.1 *For every $\delta > 0$ there is a closed (unbounded) set S, $\operatorname{mes} S < \delta$, such that every set Λ which has no finite limit points is an interpolation set for the space B_S.*

This result is a refinement of Theorem 1.2 in which we relax assumption of uniform discreteness of Λ.

3.1. **Lemmas.**

Lemmas 3.1 *For every $N \geq 2$ there exists a set $S(N) \subset (-N, N)$, $\operatorname{mes} S(N) = \frac{2}{N}$, such that*

$$\left| \frac{N}{2} \int_{S(N)} e^{itx}\, dt - \frac{\sin Nx}{Nx} \right| \leq \frac{C}{N}, \ x \in \mathbb{R}, \tag{20}$$

where $C > 0$ is an absolute constant independent on N.

Proof. 1. Fix an integer $N \geq 2$, and let $M_j, j = 1, ..., N$, be any even numbers satisfying

$$M_1 \geq N^4, M_{j+1} \geq N^2 M_j, \ j = 1, ..., N-1. \tag{21}$$

Set

$$\Omega(j,k) := j - 1 + \frac{2k-1}{M_j} + \frac{1}{M_j}(-1,1], \ \Omega(-j,k) := -\Omega(j,k),$$

$$S(j,k) := j - 1 + \frac{2k-1}{M_j} + \frac{1}{N^2 M_j}(-1,1], \ S(-j,k) := -S(j,k),$$

where $j = 1, ..., N$ and $k = 1, ..., M_j/2$. Set

$$S(N) := \bigcup_{|j|=1}^{N} \bigcup_{k=1}^{M_j/2} S(j,k).$$

One can check that

$$\bigcup_{k=1}^{M_j/2} \Omega(j,k) = (j-1, j], \ j = 1, ..., N,$$

and that

$$\operatorname{mes} S(N) = \frac{2}{N}.$$

2. For simplicity, throughout the proof we denote by C different positive constants.

Since

$$\frac{\sin Nx}{Nx} = \frac{1}{2N} \int_{-N}^{N} e^{itx}\, dt,$$

to prove (20) it suffices to show that

$$\left| \int_{-N}^{N} e^{itx}\, dt - N^2 \int_{S(N)} e^{itx}\, dt \right| < C, \;\; x \in \mathbb{R}. \tag{22}$$

3. Assume first that $|x| \leq M_l$, for some $1 \leq l \leq N$. Using (21), we have for $j \geq l, j \leq N$, that

$$\left| \int_{\Omega(j,k)} e^{itx}\, dt - N^2 \int_{S(j,k)} e^{itx}\, dt \right| =$$

$$2 \left| \frac{\sin(x/M_j)}{x} - \frac{N^2 \sin(x/N^2 M_j)}{x} \right| \leq \frac{Cx^2}{M_j^3} \leq \frac{CM_l^2}{M_j^3}, \;\; k = 1, ..., \frac{M_j}{2}.$$

This and (21) give:

$$\left| \int_{\{l-1 \leq |t| \leq N\}} e^{itx}\, dt - N^2 \int_{\{l-1 \leq |t| \leq N\} \cap S(N)} e^{itx}\, dt \right|$$

$$\leq \sum_{|j|=l}^{N} \sum_{k=1}^{M_j/2} \frac{CM_l^2}{M_j^3} \leq C. \tag{23}$$

Clearly, this proves (22) for $|x| \leq M_1$.

4. Assume now that $|x| > M_{l-1}$, for some $2 \leq l \leq N$. Then, clearly, we have

$$\left| \int_{\{|t| \leq l-1\}} e^{itx}\, dt \right| = 2 \left| \frac{\sin(l-1)x}{x} \right| \leq \frac{2}{M_{l-1}}. \tag{24}$$

Also,

$$\left| N^2 \int_{\{l-2 \leq |t| \leq l-1\} \cap S(N)} e^{itx}\, dt \right|$$

$$\leq N^2 \operatorname{mes}(\{l-2 \leq |t| \leq l-1\} \cap S) = 2. \tag{25}$$

These estimates and (23) prove (22) for $M_1 < |x| \leq M_2$.

Assume that $|x| > M_{l-1}$, for some $l \geq 3$. Since

$$\left| N^2 \int_{S(j,k)} e^{itx}\, dt \right| = 2N^2 \left| \frac{\sin x/(N^2 M_j)}{x} \right| \leq \frac{2N^2}{M_{l-1}},\ |x| > M_{l-1},$$

by (21), we obtain:

$$\left| N^2 \int_{\{|t| \leq l-2\} \cap S(N)} e^{itx}\, dt \right| \leq \sum_{j=1}^{l-2} \sum_{k=1}^{M_j/2} \frac{2N^2}{M_{l-1}} \leq \frac{2N^2 M_{l-2}}{M_{l-1}} \leq C.$$

From this, (24) and (25) we get

$$\left| \int_{\{|t| \leq l-1\}} e^{itx}\, dt - N^2 \int_{\{|t| \leq l-1\} \cap S} e^{itx}\, dt \right| \leq C,\ |x| > M_{l-1}, l \geq 3.$$

Now, this and (23) imply (22) for $|x| > M_{l-1}, l \geq 3$, which completes the proof of Lemma 3.1.

Lemma 3.2 *For every $\epsilon > 0$ there is a compact $S = S_\epsilon$ and a function $g = g_\epsilon \in B_S$ such that:*

(i) $\operatorname{mes} S \leq \epsilon$;

(ii) $\|g\|_\infty = g(0) = 1$;

(iii) $|g(x)| < \epsilon$ for $|x| > \epsilon$.

In addition, S can be chosen disjoint from any given segment.

This follows from Lemma 3.1. Indeed, fix an $\epsilon > 0$, and choose N in Lemma 3.1 so large that $\operatorname{mes} S(N) < \epsilon$, $|\sin Nx/Nx| < \epsilon/2$ when $|x| > \epsilon$ and $C/N < \epsilon/2$, where C is the constant in (20). Given a segment I, set $S := R + S(N)$, where R is any number such that $R + S(N) \cap I = \emptyset$. Set

$$g(x) := \frac{N}{2} \int_S e^{itx}\, dt = \frac{N}{2} e^{iRx} \int_{S(N)} e^{itx}\, dt.$$

Conditions (ii) and (iii) follow immediately from Lemma 3.1.

3.2. **Proof of Theorem 3.1**. Suppose $0 < \delta < 1$, and take any sequence $\epsilon(j) > 0, j \in \mathbb{Z}$, such that

$$\sum_{j \in \mathbb{Z}} \epsilon(j) < \delta. \tag{26}$$

Fix a sequence of disjoint compacts $S_{\epsilon(j)}$, $\mathrm{mes}\, S_{\epsilon(j)} < \epsilon(j)$, tending to infinity and satisfying the conditions of Lemma 3.2, such that the set

$$S := \bigcup_{j \in \mathbb{Z}} S_{\epsilon(j)}.$$

is closed. Let $\Lambda = \{\lambda_j, j \in \mathbb{Z}\}$ be a set without finite limit points. Taking if necessary a subsequence of $\epsilon(j)$, we may assume that

$$d_j := \inf_{k: k \neq j} |\lambda_k - \lambda_j| > \epsilon(j), \ j \in \mathbb{Z}. \tag{27}$$

Conditions (ii) and (iii) of Lemma 3.2 allow one to define a sequence of functions $f_j \in B_S$ such that each function f_j has a compact support,

$$1 = \|f_j\|_\infty = f_j(l_j),$$

and

$$|f_j(x)| < \epsilon(j), \ \text{for } |x - l_j| \geq d_j. \tag{28}$$

By (27), we see that

$$|f_j(\lambda_k)| < \epsilon(j), \ j \neq k. \tag{29}$$

Consider a linear operator $A : l^\infty \to l^\infty$ defined as

$$A := (a(j,k))_{j,k \in \mathbb{Z}}, \ a(j,k) = \begin{cases} f_j(\lambda_k), & j \neq k \\ f_j(\lambda_j) - 1, & j = k \end{cases}$$

From (26) and (29) one can see that

$$\|A\| = sup_k \sum_j |a(j,k)| = sup_k \sum_{j \neq k} |f_j(\lambda_k)| < 1,$$

so that the operator $A + I, I$ is the identity operator, is invertible in l^∞. Take an arbitrary "data" $\mathbf{c} = \{c_j\} \in l^\infty$, and denote by $\mathbf{b} = \{b_j\} \in l^\infty$ the solution of the equation

$$(A + I)\mathbf{b} = \mathbf{c}.$$

Set

$$f(x) := \sum_{j \in \mathbb{Z}} b_j f_j(x).$$

By (26), (27) and (28), we see that the series converges uniformly on every finite interval, and

$$\sup_{x\in\mathbb{R}} |f(x)| \le (1+\delta)\|\mathbf{b}\|_{l^\infty} < \infty.$$

Also, it is clear that f satisfies the interpolation condition $f(l_j) = c(j),\ j \in \mathbb{Z}$.

Now let φ be any smooth test–function supported by a segment disjoint from S. Let $\epsilon(R) \to 0$ and $N(R) \to \infty$ as $R \to \infty$, be some functions satisfying

$$\left| f(x) - \sum_{|j|\le N(R)} b_j f_j(x) \right| < \epsilon(R),\ \ |x| \le R.$$

Since each f_j has a compact support which lies in S, then $(\hat{\varphi}, f_j) = 0, j \in \mathbb{Z}$, and so we have

$$|(\hat{\varphi}, f)| = |(\hat{\varphi}, f - \sum_{|j|\le N(R)} b_j f_j)| \le \epsilon(R) \int_{|x|\le R} |\varphi(x)|\, dx +$$

$$(1+\delta)\|\mathbf{b}\|_{l^\infty} \int_{|x|>R} |\hat{\varphi}(x)|\, dx \to 0,\ \ R \to \infty.$$

Hence, $(\hat{\varphi}, f) = 0$, which shows that $f \in B_S$.

3.3. **Remark on sampling sets**. We finish this section by the following remark.

Definition. A set Λ is called a *sampling set* for B_S if there is a constant $C > 0$ such that

$$\|f\|_\infty \le C \sup_{\lambda\in\Lambda} |f(\lambda)|,$$

for every $f \in B_S$.

Similarly, one may define sampling set for PW_S-spaces.

When S is a single interval, the sampling sets for B_S were completely characterized by Beurling in terms of so-called "lower uniform density" $D^-(L)$ (see [3]), by the following condition:

$$D^-(L) > \frac{1}{2\pi} \operatorname{mes} S.$$

For the disconnected compacts S no such metrical characterization may exist. However, one can construct a set of critical density which serves as B_S-sampling set for every compact of given measure. More precisely, the following is true:

Theorem 3.2 *There is a set $\Lambda = \{j + O(1), j \in \mathbb{Z}\}$, which is a sampling set for the Bernstein space B_S, for every compact S of measure $< 2\pi$.*

This is a consequence of Theorem 3 in [9] (see details in [10]), where such a "universal" sampling set was constructed for Paley–Wiener spaces. In order to deduce the proposition above from Theorem 3 in [9], one needs

Lemma 3.3 *Suppose there exists $\delta_0 > 0$ such that a u.d. set Λ is a sampling set for $PW_{S(\delta)}, 0 < \delta \le \delta_0$, where $S(\delta)$ is defined in (7). Then Λ is a sampling set for B_S.*

Indeed, assume that the conclusion of Lemma 3.3 does not hold: for every $\epsilon > 0$ there exists $f \in B_S$ such that $\sup_x |f(x)| = 1$, and $|f(\lambda)| < \epsilon$ for every $\lambda \in \Lambda$. Pick up a point x_0 such that $|f(x_0)| \ge 1/2$, and set

$$g(x) := f(x)\frac{\sin\delta(x - x_0)}{\delta(x - x_0)} \in PW_{S(\delta)},$$

where $\delta \le \delta_0$. Let $\sigma > 0$ be such that $S(\delta) \subseteq [-\sigma, \sigma]$. Since $|g(x)| \le 1$ for all x, the Bernstein inequality says that $\sup |f'(x)| \le \sigma$. Using this and inequality $|g(x_0)| \ge 1/2$, one can easily deduce that $\|g\|_2^2 \ge C$, where C depends only on σ. On the other hand, since Λ is u.d., we have

$$\sum_{\lambda\in\Lambda} |g(\lambda)|^2 \le \epsilon^2 \sum_{\lambda\in\Lambda} \left|\frac{\sin\delta(\lambda - x_0)}{\delta(\lambda - x_0)}\right|^2 \le C\epsilon^2,$$

where C depends only on the infimum of distances between elements of Λ. However, since this can be done for every $\epsilon > 0$, we see that Λ is not a sampling set for $PW_{S(\delta)}$, which is a contradiction.

In a sharp contrast to Theorem 3.2, the following claim is true:

Corollary 3.1 *A sampling set for the closed (unbounded) spectrum S, constructed in the proof of Theorem 3.1, must be dense in $\mathbb{R}$.*

Indeed, suppose a set Λ satisfies $\Lambda \cap (a, b) = \emptyset$, for some $a < b$. By the construction in the proof of Theorem 3.1, for every $\epsilon > 0$ and $c \in \mathbb{R}$ there exists $f \in B_S$ such that $f(c) = 1$ and $|f(x)| < \epsilon$ for $|x - c| > \epsilon$. By choosing $c = (a+b)/2$ and $\epsilon < (b-a)/2$, we see that for every $\epsilon > 0$ there exists a function $f \in B_S$ satisfying $\|f\|_\infty > (1/\epsilon)\|f\|_{l^\infty(\Lambda)}$, so that Λ is not a sampling set for B_S.

4 Unbounded spectra: interpolation in Paley–Wiener spaces

In the previous section we have seen that there exist *universal* closed sets S of arbitrarily small measure such that for arbitrary Λ and $\mathbf{c}$ the interpolation problem (1) can be solved by functions $f \in B_S$. In this section we obtain somewhat similar results for the Paley–Wiener spaces. Observe, however, that the "universality" is not possible to achieve in this case:

Proposition 4.1 *No set of measure $< 2\pi$ admits interpolation of any $\delta-$function on $\mathbb{Z}$ by $f \in PW_S$.*

Proof. Given a set $S \subset \mathbb{R}$ of finite measure, set

$$S_\pi := \left(\bigcup_{j\in\mathbb{Z}} (S + 2\pi j)\right) \bigcap [-\pi, \pi].$$

Now, suppose there exists $f \in PW_S$ such that $f(0) = 1$ and $f(j) = 0, j \in \mathbb{Z} \setminus \{0\}$. Denote by $F \in L^2(S)$ the inverse Fourier transform of f, and set

$$F_\pi(t) := \sum_{j\in\mathbb{Z}} F(t + 2\pi j).$$

Observe that

$$\int_{-\pi}^{\pi} |F_\pi(t)|\, dt = \int_{\mathbb{R}} |F(t)|\, dt \leq \sqrt{\operatorname{mes} S}\, \|F\|_2 < \infty,$$

so that the function F_π is a.e. finite. Also, we see that

$$\int_{-\pi}^{\pi} e^{ijt} F_\pi(t)\, dt = \int_{\mathbb{R}} e^{ijt} F(t)\, dt = f(j) = \begin{cases} 1, & j = 0 \\ 0, & j \neq 0 \end{cases}$$

This implies $F_\pi(t) = 2\pi$ for a.e. $t \in [-\pi, \pi]$, and so $\operatorname{supp} F_\pi = [-\pi, \pi]$. However, clearly, $\operatorname{supp} F_\pi \subseteq S_\pi$. We conclude that $\operatorname{mes} S \geq 2\pi$.

Proposition 4.1 shows that some restrictions on Λ are necessary in order to get interpolation by $PW-$functions with small spectra. We consider a generic situation: elements of Λ are supposed to be rationally-independent. In this case, we construct closed spectra which admit interpolation of any function on Λ with a certain weak decay condition.

Theorem 4.1 *Let a u.d. set $\Lambda = \{\lambda_j, j \in \mathbb{Z}\}$ be linearly independent over rationales (mod π). Then for every $\delta > 0$ there is is a set S, a union of some of intervals centered at integers, such that:*
(i) $\operatorname{mes} S < \delta$;
(ii) for every sequence $\mathbf{c} = \{c_j\}$ satisfying

$$\sum_{j\in\mathbb{Z}} |c_j|^2(1 + |j|^\beta) < \infty$$

with some $\beta > 1$, there is a function $f \in PW_S$ satisfying (1).

Remark. One can see from the proof below that the assumption (ii) can be relaxed by replacing it with

$$\sum_{j\in\mathbb{Z}} \frac{|c_j|^2}{w_j} < \infty,$$

where the $\{w_j\} \in l^1(\mathbb{Z})$ is a fixed positive sequence.

However, assumption (ii) cannot be replaced by $\mathbf{c} \in l^2$. Indeed, consider a random sequence of interpolation knots:

$$\Lambda := \{n + \xi(n)\}, \tag{30}$$

where $\xi_j, j \in \mathbb{Z}$, are independent random variables uniformly distributed on $(-a, a), 0 < a < 1/2$. Clearly, Theorem 4.1 is applicable in this case: Almost surely, there is a closed set S of arbitrary small measure which admits interpolation in PW_S of every sequence $\mathbf{c}$ satisfying (ii). On the other hand, we have

Proposition 4.2 *Let Λ be a random set in (30). With probability 1, there is no set S of measure $< 2\pi$ such that Λ is an interpolation set for PW_S.*

Observe that we don't know if there exists $\Lambda = \{j + O(1), j \in \mathbb{Z}\}$ which is a set of interpolation for PW_S for some (unbounded) S, $\operatorname{mes} S < 2\pi$. Proposition 4.1 shows that if such a set Λ does exist then it should be an exception.

Proof of Proposition 4.2. It is well-known that the interpolation property in a Hilbert space is equivalent to certain inequalities (see [11], p.129). In our case, a set $\Lambda = \{\lambda_j, j \in \mathbb{Z}\}$ is a set of interpolation for PW_S if and only if there is a positive constant A such that

$$\int_S \left| \sum_j c_j e^{i\lambda_j t} \right|^2 dt \geq A \sum_j |c_j|^2, \tag{31}$$

for every finite sequence $\{c_j\}$.

Let ξ_j be random sequence defined above. Fix an integer N. Clearly, we have with probability one that for every $\epsilon > 0$ there exists $k = k(\epsilon, N)$ such that

$$|\xi_{k+j} - \xi_k| < \epsilon, \text{ for all } j, \ |j| \leq N.$$

Fix an element of the underlying probability space such that the latter is true, and assume that $S \subset \mathbb{R}$ is such that the set Λ in (30) is a set of interpolation for PW_S. By (31), we have for every $c_{-N}, ..., c_N$ that

$$\int_S \left| \sum_{j=-N}^{N} c_j e^{ijt} \right|^2 dt = \lim_{\epsilon \to 0} \int_S \left| \sum_{j=-N}^{N} c_j e^{i(k+j+\xi_{k+j})t} \right|^2 dt \geq A \sum_{j=-N}^{N} |c_j|^2.$$

Since this is true for every N, we see that $\mathbb{Z}$ is also a set of interpolation for PW_S. By Proposition 2, we conclude that $\operatorname{mes} S \geq 2\pi$.

Proof of Theorem 4.1. Throughout the proof we shall denote by C different positive constants.

1. Without loss of generality we may assume that $\beta < 2$. Set

$$S := \bigcup_{j \in \mathbb{Z}} S_j, \ S_j := (-M_j - 4\beta_j, -M_j + 4\beta_j) \bigcup (M_j - 4\beta_j, M_j + 4\beta_j),$$

where

$$\beta_j := \frac{\gamma}{1 + |j|^\beta},$$

the sequence M_j will be specified in step 4, and γ is any small positive number such that mes $S < \delta$. In what follows we also assume that γ is so small that $S_j \cap S_k = \emptyset$, for $j \neq k$.

2. Set
$$\Lambda_k := (\Lambda - \lambda_k) \setminus \{0\}, \ k \in \mathbb{Z}.$$
The independence condition on Λ implies, by Kronecker's theorem, that for every $N > 0$ the subgroup
$$\{m\lambda \ (\mathrm{mod}\,\pi), \ \lambda \in \Lambda_k \cap [-N, N], m \in \mathbb{Z}\}$$
is dense in the $l-$dimensional torus, l being the number of elements in $\Lambda_k \cap [-N, N]$. Hence, the l numbers $|\cos(Mx)|$, $x \in \Lambda_k \cap [-N, N]$, can be made as small as we like by choosing appropriate $M \in \mathbb{N}$.

3. Set
$$g_j(x) := \cos\left(M_j(x - \lambda_j)\right) \left(\frac{\sin \gamma_j(x - \lambda_j)}{\gamma_j(x - \lambda_j)}\right)^4.$$
Clearly, the spectrum of g_j belongs to S_j, and we have
$$g_j(\lambda_j) = 1, \tag{32}$$
and
$$\|g_j\|_2^2 \leq \frac{C}{\gamma_j} \leq C\left(1 + |j|^\beta\right), \ j \in \mathbb{Z}. \tag{33}$$

4. Since Λ is uniformly discrete, there is a constant $C > 0$ such that
$$\left(\frac{\sin \gamma_j(\lambda_k - \lambda_j)}{\gamma_j(\lambda_k - \lambda_j)}\right)^4 \leq \frac{C}{\gamma_j^4 (k-j)^4}, \ k \neq j, \ k, j \in \mathbb{Z}.$$
Take any small number $\epsilon > 0$, and let N_j be so large that we have
$$\left(\frac{\sin \gamma_j(\lambda_k - \lambda_j)}{\gamma_j(\lambda_k - \lambda_j)}\right)^4 \leq \frac{\epsilon}{(1 + j^2)(k - j)^2}, \ |\lambda_k - \lambda_j| \geq N_j, \ k \neq j.$$
By Step 2, the first factor in the definition of g_j can be made arbitrarily small for $0 \neq |\lambda_k - \lambda_j| < N_j$. We shall choose $M_j \in \mathbb{N}$ such that
$$|\cos M_j(\lambda_k - \lambda_j)| \leq \frac{\epsilon}{(1 + j^2)\max\{(k - j)^2, |\lambda_k - \lambda_j| < N_j\}},$$

for all $k \neq j$ such that $|\lambda_k - \lambda_j| < N_j$. This and the previous estimate give

$$|g_j(\lambda_k)| \le \frac{\epsilon}{(1+j^2)(j-k)^2}, \; k \neq j, \; k, j \in \mathbb{Z}.$$

One may check that this estimate implies

$$\sum_{j\in\mathbb{Z}, j\neq k} |g_j(\lambda_k)|^2 \le \frac{C\epsilon^2}{1+|k|^3}. \tag{34}$$

5. Given a sequence $\mathbf{c} = \{c_j, \; j \in \mathbb{Z}\}$, set

$$\|\mathbf{c}\|_\beta^2 := \sum_{j=-\infty}^{\infty} |c_j|^2 \left(1+|j|^\beta\right).$$

Let l^2_β denote the weighted space of all sequences $\mathbf{c}$, $\|\mathbf{c}\|_\beta < \infty$. Define a linear operator $R : l^2_\beta \to l^2_\beta$ as follows:

$$R\mathbf{e}_j := \sum_{k=-\infty}^{\infty} g_j(\lambda_k)\mathbf{e}_k - \mathbf{e}_j, \; j \in \mathbb{Z}.$$

Using (32), we have

$$\|R\sum_{j\in\mathbb{Z}} c_j\mathbf{e}_j\|_\beta^2 = \|\sum_{k\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}, j\neq k} c_j g_j(\lambda_k)\right)\mathbf{e}_k\|_\beta^2 =$$

$$\sum_{k\in\mathbb{Z}} \left|\sum_{j\in\mathbb{Z}, j\neq k} c_j g_j(\lambda_k)\right|^2 (1+|k|^\beta) \le$$

$$\|\mathbf{c}\|_\beta^2 \sum_{k\in\mathbb{Z}} \left(\sum_{j\in\mathbb{Z}, j\neq k} \frac{|g_j(\lambda_k)|^2}{1+|j|^{2\beta}}\right)(1+|k|^\beta).$$

Since $\beta < 2$, we see from (34) that

$$\|R\sum_{j\in\mathbb{Z}} c_j\mathbf{e}_j\|_\beta^2 \le C\epsilon^2\|\mathbf{c}\|_\beta^2,$$

for some constant $C > 0$. Choose ϵ so small that the norm of operator R in l^2_β is less than 1. It follows that the operator $T := I+R$ is invertible in l^2_γ, where I is the identity operator. We conclude that for every $\mathbf{c} \in l^2_\beta$ the interpolation problem (1) has a solution f given by

$$f(x) := \sum_{j\in\mathbb{Z}} b_j g_j(x), \ \{b_j\} = T^{-1}\mathbf{c} \in l^2_\beta.$$

Recall, that the spectrum of g_j belongs to the set S_j defined in step 1. Since S_j and S_k are disjoint for $j \neq k$, it follows that the functions g_j and g_k are orthogonal in $L^2(\mathbb{R})$. Using this and (33), we see that

$$\|f\|_2^2 = \sum_{j\in\mathbb{Z}} |b_j|^2 \|g_j\|_2^2 \leq C\|\mathbf{b}\|^2_\beta < \infty.$$

Recall again that $g_j \in PW_{S_j} \subset PW_S, j \in \mathbb{Z}$. We conclude that $f \in PW_S$.

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