

# Oberwolfach Preprints



OWP 2009 - 11

DIETER HAPPEL AND DAN ZACHARIA

Homological Properties of Piecewise Hereditary  
Algebras

Mathematisches Forschungsinstitut Oberwolfach gGmbH  
Oberwolfach Preprints (OWP) ISSN 1864-7596

## Oberwolfach Preprints (OWP)

Starting in 2007, the MFO publishes a preprint series which mainly contains research results related to a longer stay in Oberwolfach. In particular, this concerns the Research in Pairs-Programme (RiP) and the Oberwolfach-Leibniz-Fellows (OWLF), but this can also include an Oberwolfach Lecture, for example.

A preprint can have a size from 1 - 200 pages, and the MFO will publish it on its website as well as by hard copy. Every RiP group or Oberwolfach-Leibniz-Fellow may receive on request 30 free hard copies (DIN A4, black and white copy) by surface mail.

Of course, the full copy right is left to the authors. The MFO only needs the right to publish it on its website *www.mfo.de* as a documentation of the research work done at the MFO, which you are accepting by sending us your file.

In case of interest, please send a **pdf file** of your preprint by email to *rip@mfo.de* or *owlf@mfo.de*, respectively. The file should be sent to the MFO within 12 months after your stay as RiP or OWLF at the MFO.

There are no requirements for the format of the preprint, except that the introduction should contain a short appreciation and that the paper size (respectively format) should be DIN A4, "letter" or "article".

On the front page of the hard copies, which contains the logo of the MFO, title and authors, we shall add a running number (20XX - XX).

We cordially invite the researchers within the RiP or OWLF programme to make use of this offer and would like to thank you in advance for your cooperation.

## Imprint:

Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)  
Schwarzwaldstrasse 9-11  
77709 Oberwolfach-Walke  
Germany

Tel +49 7834 979 50  
Fax +49 7834 979 55  
Email [admin@mfo.de](mailto:admin@mfo.de)  
URL [www.mfo.de](http://www.mfo.de)

The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.  
Copyright of the content is held by the authors.

# HOMOLOGICAL PROPERTIES OF PIECEWISE HEREDITARY ALGEBRAS

DIETER HAPPEL AND DAN ZACHARIA

ABSTRACT. Let  $\Lambda$  be a finite dimensional algebra over an algebraically closed field  $k$ . We will investigate homological properties of piecewise hereditary algebras  $\Lambda$ . In particular we give lower and upper bounds of the strong global dimension, show the behavior of the strong global under one point extensions and tilting. Moreover we show that the pieces of  $\text{mod } \Lambda$  have Auslander-Reiten sequence.

Let  $\Lambda$  be a finite dimensional algebra over an algebraically closed field  $k$ . We denote by  $\text{mod } \Lambda$  the category of finitely generated left  $\Lambda$ -modules. Recall that  $\Lambda$  is said to be piecewise hereditary, if there exists a hereditary, abelian category  $\mathcal{H}$  such that the bounded derived categories  $D^b(\Lambda)$  and  $D^b(\mathcal{H})$  are equivalent as triangulated categories [H1], [HRS2]. The category  $\mathcal{H}$  is called the type of  $\Lambda$ , but observe that the type is only determined up to derived equivalence. The categories  $\mathcal{H}$  occurring in this situation have been described in [H2], but we do not have to make use of these investigations. We refer to [H3] for the internal derived equivalences of these categories. In [HZ] we obtained a characterization of piecewise hereditary algebras in terms of the strong global dimension, a notion which was proposed by Ringel over twenty years ago. The definition will be recalled in Section 1. We refer to the references in [HZ] for further articles on the strong global dimension.

An equivalent approach to piecewise hereditary algebras uses tilting complexes in the sense of [Ri]. We recall that a tilting complex is an object  $T^\bullet$  in the derived category of  $\Lambda$  such that

- (1)  $\text{Hom}_{D^b(\Lambda)}(T^\bullet, T^\bullet[n]) = 0$  for all  $n \neq 0$ , and
- (2)  $\text{add}(T^\bullet)$ , the additive category of direct summands of finite direct sums of copies of  $T^\bullet$ , generates the homotopy category of bounded complexes of projective  $\Lambda$ -modules  $K^b(\Lambda\mathcal{P})$  as a triangulated category.

---

2000 *Mathematics Subject Classification.* 16E05, 16E10, 16G10.

The second author is supported by a grant from NSA. Most of these results were obtained during a stay in Oberwolfach within the Research in Pairs program from February 15 - February 28, 2009. We thank the Mathematical Institute in Oberwolfach for making this stay possible.

Then it follows from [Ri] that a finite dimensional  $k$ -algebra is piecewise hereditary if and only if there is a hereditary, abelian  $k$ -category  $\mathcal{H}$  such that  $\text{Hom}(X, Y)$  and  $\text{Ext}^1(X, Y)$  are finite dimensional  $k$ -vector spaces for all  $X, Y \in \mathcal{H}$ , and there is a tilting complex  $T^\bullet \in D^b(\mathcal{H})$  with  $\Lambda = \text{End}_{D^b(\mathcal{H})} T^\bullet$ . Note that tilting complexes arise in the following way. If  $\Lambda$  is piecewise hereditary and  $F : D^b(\Lambda) \rightarrow D^b(\mathcal{H})$  is a triangle equivalence, then  $T^\bullet = F({}_\Lambda \Lambda)$  is a tilting complex with  $\Lambda = \text{End}_{D^b(\mathcal{H})} T^\bullet$ .

If  $\mathcal{H}$  is a hereditary abelian  $k$ -category and  $X \in D^b(\mathcal{H})$  is indecomposable, then it is well known that  $X \in \mathcal{H}[i]$  for some  $i \in \mathbb{Z}$ , where we denote by  $[-]$  the shift functor on  $D^b(\mathcal{H})$ .

Given a piecewise hereditary algebra  $\Lambda$  and a triangle equivalence  $F : D^b(\Lambda) \rightarrow D^b(\mathcal{H})$  we may assume that  $F$  is normalized in the sense that there exists  $r \geq 0$  such that for every indecomposable  $\Lambda$ -module  $X$  we have  $F(X) \in \bigcup_{i=0}^r \mathcal{H}[i]$ , and that there exist indecomposable  $\Lambda$ -modules  $X, Y$  such that  $F(X) \in \mathcal{H}[0]$  and  $F(Y) \in \mathcal{H}[r]$ . Note that such normalized equivalences are not unique, and that the value of  $r$  may depend on the choice of  $\mathcal{H}$ . These normalized equivalences will be investigated in more detail in Section 2. For  $0 \leq i \leq r$  we denote by  $\mathcal{U}_i$  the full subcategory of  $\text{mod } \Lambda$  with objects the indecomposable  $\Lambda$ -modules  $X$  such that  $F(X) \in \mathcal{H}[i]$ . So by definition we have that  $\mathcal{U}_0 \neq \emptyset \neq \mathcal{U}_r$ . Usually we will assume that the algebra  $\Lambda$  is connected. It then follows that also  $\mathcal{U}_i \neq \emptyset$  for each  $1 \leq i \leq r-1$ . The additive closure of  $\mathcal{U}_i$  in  $\text{mod } \Lambda$  is denoted by  $\tilde{\mathcal{U}}_i$ . The subcategories  $\tilde{\mathcal{U}}_i$  are called the *pieces* of  $\text{mod } \Lambda$ , and we say that  $\Lambda$  has  $r+1$  pieces. The integer  $r$  depends of course, on the choice of  $F$ . We refer to an extreme example in [H3] where it is shown that even a hereditary algebra may have three pieces. Let  $\mathcal{V}_i \subset \mathcal{H}$  be the subcategory such that  $F$  induces an equivalence  $\mathcal{U}_i \rightarrow \mathcal{V}_i[i]$ . We denote again by  $\tilde{\mathcal{V}}_i$  the additive closure of  $\mathcal{V}_i$  in  $\mathcal{H}$ .

The paper is organized as follows. In the first section, we briefly recall some notation and elementary facts about derived categories and strong global dimension of a finite dimensional algebra which will be used in the later sections.

In Section 2 we will recall some homological properties of piecewise hereditary algebras. We will also show that we may choose for a given piecewise hereditary algebra  $\Lambda$  a normalized equivalence  $F$  yielding pieces  $\tilde{\mathcal{U}}_i$  for  $0 \leq i \leq r$  such that  $\tilde{\mathcal{U}}_r$  does not contain an indecomposable projective  $\Lambda$ -module, unless we are in the trivial case, that  $\Lambda$  is a finite dimensional hereditary algebra. This gives an upper bound, namely that  $\text{s.gl. dim } \Lambda \leq r+1$ , improving the one obtained in [HZ]. We

include an example that this bound is optimal. We will also obtain some lower bounds, which are not optimal, and also show more specific assertions on the structure of indecomposable complexes of maximal length.

In Section 3 we investigate the behavior of the strong global dimension under one point extensions. A piecewise hereditary algebra  $\Lambda$  is directed, so can be written as a one point extension algebra  $\Gamma[M]$ . We will show that there is always a presentation of  $\Lambda = \Gamma[M]$ , where  $\text{s. gl. dim } \Gamma \geq \text{s. gl. dim } \Lambda - 2$ . Again we will provide examples that this bound is optimal.

Section 4 studies the possible change of the strong global dimension under the tilting process. We will show there that for a piecewise hereditary algebra  $\Lambda$  and a tilting module  ${}_{\Lambda}T$  with  $\text{proj. dim}_{\Lambda} T = t$  and  $\Gamma = \text{End}_{\Lambda} T$  we have  $\text{s. gl. dim } \Lambda - t \leq \text{s. gl. dim } \Gamma \leq \text{s. gl. dim } \Lambda + t$ .

As one of the main results of this article we will show in Section 5 that the pieces  $\mathcal{U}_i$  of a piecewise hereditary algebra  $\Lambda$  have Auslander-Reiten sequences in the sense of [AS]. This follows from the fact that the subcategories  $\tilde{\mathcal{V}}_i$  are functorially finite in  $\mathcal{H}$ . For an explicit description of the subcategories  $\tilde{\mathcal{V}}_i$  in terms of the tilting complex realizing  $\Lambda$  we refer to Section 5.

We denote the composition of morphisms  $f : X \rightarrow Y$  and  $g : Y \rightarrow Z$  in a given category  $\mathcal{K}$  by  $fg$ . The notation and terminology introduced here will be fixed throughout this article. For unexplained representation-theoretic and derived category terminology, we refer to [ARS], [H1] and [R].

## 1. PRELIMINARIES

In this section we briefly recall some of the notation we will use for derived categories and state some useful facts involving triangles in triangulated categories involving nonzero and noninvertible maps between indecomposable objects.

First we recall the definition of the strong global dimension of a finite dimensional algebra  $\Lambda$ . We define first the length of a complex. For this let  $\mathfrak{a}$  be an additive  $k$ -category which is Krull-Schmidt (see [R]). Let  $C^b(\mathfrak{a})$  be the category of bounded complexes over  $\mathfrak{a}$ . This is a Frobenius category in the sense of [H1]. Recall that the indecomposable projective objects in  $C^b(\mathfrak{a})$  are given by shifts of complexes of the form  $Y^{\bullet} = (Y^i, d^i)$  with  $Y^0 = Y = Y^1$ ,  $d^0 = \text{id}_Y$  and zero otherwise for  $Y \in \mathfrak{a}$  indecomposable. We denote by  $K^b(\mathfrak{a})$  the corresponding stable (or homotopy) category. If  $X^{\bullet} = (X^i, d^i) \in K^b(\mathfrak{a})$  is a complex we may consider a preimage  $\overline{X}^{\bullet} = (\overline{X}^i, \overline{d}^i)$  of  $X^{\bullet}$  in  $C^b(\mathfrak{a})$  without

indecomposable projective direct summands. Clearly  $\overline{X^\bullet}$  is uniquely determined by  $X^\bullet$  up to isomorphism of bounded complexes in  $C^b(\mathfrak{a})$ . Thus the following is well-defined: if  $0 \neq X^\bullet \in K^b(\mathfrak{a})$ , there exists  $r \leq s$  such that  $\overline{X^r} \neq 0 \neq \overline{X^s}$  and  $\overline{X^i} = 0$  for  $i < r$  and  $i > s$ . Then by definition, the length of  $X^\bullet$  is defined as  $\ell(X^\bullet) = s - r$ . Throughout this paper we will always identify  $X^\bullet$  with  $\overline{X^\bullet}$ .

If  $\Lambda$  is a finite dimensional algebra, we denote by  ${}_\Lambda\mathcal{P}$  the full subcategory of  $\text{mod } \Lambda$  consisting of the finitely generated projective  $\Lambda$ -modules. Then, we define the strong global dimension of  $\Lambda$  by

$$\text{s. gl. dim } \Lambda = \sup \{ \ell(P^\bullet) \mid P^\bullet \in K^b({}_\Lambda\mathcal{P}) \text{ indecomposable} \}$$

The following characterization of piecewise hereditary algebras was proved in [HZ].

**Theorem 1.1.** *A finite dimensional algebra  $\Lambda$  is piecewise hereditary if and only if  $\text{s. gl. dim } \Lambda < \infty$ .*

The following is an easy consequence of 1.1.

**Corollary 1.2.** *Let  $P$  be a projective module over a piecewise hereditary algebra  $\Lambda$ . Then  $\Gamma = \text{End}_\Lambda P$  is piecewise hereditary.*

*Proof.* In fact, it is easily seen that  $\text{s. gl. dim } \Gamma < \infty$ .  $\square$

Let  $\mathcal{A}$  be an abelian  $k$ -category and let  $D^b(\mathcal{A})$  be its bounded derived category. For  $X^\bullet \in D^b(\mathcal{A})$  we denote by  $X^\bullet[1]$  the shift in the triangulated category  $D^b(\mathcal{A})$ . We have an embedding of  $\mathcal{A}$  into  $D^b(\mathcal{A})$  by sending  $X \in \mathcal{A}$  to the stalk complex concentrated in degree 0 with stalk  $X$ . We denote by  $\mathcal{A}[0]$  the image of this embedding. Then for each  $i \in \mathbb{Z}$  we also have  $\mathcal{A} \simeq \mathcal{A}[i] \subset D^b(\mathcal{A})$ . For a complex  $X^\bullet = (X^i, d^i) \in D^b(\mathcal{A})$  we denote by  $H^i(X^\bullet) = \ker d^i / \text{im } d^{i-1}$  the  $i$ -th cohomology space.

We will also need the following simple fact, whose proof is left to the reader. For this it is helpful to recall that for an algebra of finite global dimension the embedding  $K^b({}_\Lambda\mathcal{P}) \rightarrow D^b(\Lambda)$  is a triangle equivalence.

**Lemma 1.3.** *Let  $\Lambda$  be a finite dimensional algebra of finite global dimension and let  $P^\bullet = (P^i, d^i) \in K^b({}_\Lambda\mathcal{P})$  be an indecomposable complex. Let  $S$  be a simple  $\Lambda$ -module and let  $P(S)$  be its projective cover and  $P^\bullet(S)$  be its minimal projective resolution. If  $P(S)$  is a direct summand of  $P^s$  for some  $s$ , then*

$$0 \neq \text{Hom}_{D^b(\Lambda)}(P^\bullet, S[-s]) \simeq \text{Hom}_{K^b({}_\Lambda\mathcal{P})}(P^\bullet, P^\bullet(S)[-s]).$$

For the rest of this section we denote by  $\mathcal{C}$  a Krull-Schmidt triangulated  $k$ -category, with the property that for all  $X, Y \in \mathcal{C}$  the dimension

of  $\text{Hom}_{\mathcal{C}}(X, Y)$  is finite. For a map  $f: X \rightarrow Y$  in  $\mathcal{C}$  we have a triangle  $X \rightarrow Y \rightarrow C_f \rightarrow X[1]$  in  $\mathcal{C}$ . The object  $C_f$  is uniquely determined up to isomorphism and is called the cone of  $f$ . The following two results will be needed later. The first is contained in [H1] and the second, which is very useful in constructing indecomposable complexes from given ones, is from [HZ].

**Lemma 1.4.** *The following are equivalent for a triangle*

$$X \xrightarrow{f} Y \xrightarrow{u} Z \xrightarrow{v} X[1]$$

- (a)  $f$  is split mono.
- (b)  $u$  is split epi.
- (c)  $v = 0$ .

**Proposition 1.5.** *Let  $f: X \rightarrow Y$  be nonzero and not invertible for  $X, Y$  indecomposable, and let*

$$X \xrightarrow{f} Y \xrightarrow{u} C_f \xrightarrow{v} X[1]$$

*be a triangle. If the induced map  $f^*: \text{Hom}_{\mathcal{C}}(Y, X[1]) \rightarrow \text{Hom}_{\mathcal{C}}(X, X[1])$  is injective, then  $C_f$  is indecomposable. In particular, if*

$$\text{Hom}_{\mathcal{C}}(Y, X[1]) = 0$$

*then  $C_f$  is indecomposable.*

As an application of 1.5. we show now that if  $\Lambda$  is a finite dimensional algebra with  $\text{s. gl. dim } \Lambda = d < \infty$ , then the indecomposable complexes  $P^\bullet = (P^i, e^i) \in K^b(\Lambda\mathcal{P})$  of maximal length must have a special form. Applying the shift functor if necessary, we may assume without loss of generality that  $P^i = 0$  for all  $i < 0$  and  $i > d$ .

**Lemma 1.6.** *Using the notation above, the map  $e^0$  is injective and  $\text{Hom}_{\Lambda}(\text{Coker } e^{d-1}, \Lambda) = 0$ .*

*Proof.* Assume that  $e^0$  is not injective. Let  $X_0 = \text{Ker } e^0$ . Let  $P$  be an indecomposable projective  $\Lambda$ -module and  $0 \neq f: P \rightarrow X_0$ . Then  $f$  induces a nonzero morphism  $\bar{f}: P \rightarrow P^\bullet$  in  $K^b(\Lambda\mathcal{P})$  such that  $C_{\bar{f}}$  is indecomposable by 1.5. and  $\ell(C_{\bar{f}}) = d + 1 > d$ , a contradiction.

Similarly, assume that  $\text{Hom}_{\Lambda}(\text{Coker } e^{d-1}, \Lambda) \neq 0$ . Let  $Q$  be an indecomposable projective  $\Lambda$ -module and  $0 \neq g: \text{Coker } e^{d-1} \rightarrow Q$ . Then  $g$  induces a nonzero morphism  $\bar{g}: P^\bullet \rightarrow Q[-d]$  in  $K^b(\Lambda\mathcal{P})$  such that  $C_{\bar{g}}$  is indecomposable by 1.5. and  $\ell(C_{\bar{g}}) > d$  yielding again a contradiction.  $\square$

Let  $\Lambda$  be a finite dimensional algebra of finite strong global dimension. Let  $n = \text{gl. dim } \Lambda$  and  $d = \text{s. gl. dim } \Lambda$ . Then, we always have

$d \geq n$ . If  $d = n$  then there are indecomposable complexes  $P^\bullet \in K^b({}_\Lambda \mathcal{P})$  of maximal length such that  $H^j(P^\bullet)$  is nonzero for exactly one  $j$ . Namely, one can take  $P^\bullet$  as any shift of a minimal projective resolution of an indecomposable  $\Lambda$ -module of maximal projective dimension. We do not know whether all indecomposable complexes of maximal length are of this form. Assume now that  $d > n$  and let  $P^\bullet = (P^i, d^i) \in K^b({}_\Lambda \mathcal{P})$  be an indecomposable complex of maximal length  $d$ . Applying the shift functor if necessary, we may assume that  $P^i = 0$  for all  $i < 0$  and  $i > d$ . Clearly,  $H^d(P^\bullet) \neq 0$ , since  $\ell(P^\bullet) = d$ . By Lemma 1.6. we have that  $H^0(P^\bullet) = 0$ . The following is an example of a piecewise hereditary algebra of global dimension 3 and strong global dimension 4, such that there do not exist indecomposable complexes

$$\cdots 0 \rightarrow P^0 \rightarrow P^1 \rightarrow P^2 \rightarrow P^3 \rightarrow P^4 \rightarrow 0 \cdots$$

of maximal length 4, with  $H^i(P^\bullet) = 0$  for  $0 \leq i < 3$ .

**Example 1.7.** Consider the following quiver  $\overrightarrow{\Delta}$

$$\begin{array}{ccccccccc} \circ & \xrightarrow{\alpha} & \circ & \xrightarrow{\beta} & \circ & \xrightarrow{\gamma} & \circ & \xrightarrow{\delta} & \circ & \xrightarrow{\eta} & \circ \\ 1 & & 2 & & 3 & & 4 & & 5 & & 6 \end{array}$$

Let  $I$  be the two sided ideal of the path algebra  $k\overrightarrow{\Delta}$  generated by  $\alpha\beta, \beta\gamma$  and  $\delta\eta$ , and set  $\Lambda = k\overrightarrow{\Delta}/I$ . We denote the simple  $\Lambda$ -modules corresponding to the vertices of  $\overrightarrow{\Delta}$  by  $S_1, \dots, S_6$  and their projective covers by  $P_1, \dots, P_6$ . It is easy to see that  $\Lambda$  is piecewise hereditary of type  $\mathbb{A}_6$ , say with linear orientation. Clearly we have that  $\text{gl.dim } \Lambda = 3$  and that  $\text{s.gl.dim } \Lambda = 4$ . Up to shift, there exists a unique indecomposable complex  $P^\bullet \in K^b({}_\Lambda \mathcal{P})$  of maximal length. It is given as follows, where the differentials are nonzero maps between indecomposable projective  $\Lambda$ -modules, which are uniquely determined up to multiplication by scalars:

$$P^\bullet = \cdots 0 \rightarrow P_6 \rightarrow P_5 \rightarrow P_3 \rightarrow P_2 \rightarrow P_1 \rightarrow 0 \cdots$$

where  $P_6$  is in degree zero. Then  $H^0(P^\bullet) = H^1(P^\bullet) = H^3(P^\bullet) = 0$ ,  $H^2(P^\bullet) = S_4$  and  $H^4(P^\bullet) = S_1$ . Note that  $S_1$  is the unique indecomposable  $\Lambda$ -module  $X$  with  $\text{proj.dim}_\Lambda X = 3$ , but  $\text{Hom}_\Lambda(S_1, \Lambda) = 0$ .

## 2. NORMALIZED EQUIVALENCES

In this section we investigate in more detail normalized equivalences for piecewise hereditary algebras. We refer to the introduction for the definition, but first we want to recall some homological properties from [H1]. We point out that the proofs given there in the case that  $\mathcal{H} = \text{mod } H$  for a finite dimensional hereditary algebra  $H$  also apply to



the more general situation considered here. We will also give a bound of the strong global dimension of a piecewise hereditary algebra which improves the bound given in [HZ]. We start by collecting some homological properties from [H1]. Note that we will give some alternative proofs for some of the assertions at the end of this section. We recall first the definition of a cycle. We say that a sequence

$$X_0 \xrightarrow{f_0} X_1 \rightarrow \cdots \rightarrow X_{r-1} \xrightarrow{f_{r-1}} X_r$$

of maps through indecomposable  $\Lambda$ -modules  $X_0, \dots, X_r$  is a cycle if  $r \geq 1$ ,  $X_0 \simeq X_r$  and all the  $f_i: X_i \rightarrow X_{i+1}$  are nonzero, and nonisomorphisms.

**Theorem 2.1.** *Let  $\Lambda$  be a piecewise hereditary algebra with pieces  $\tilde{\mathcal{U}}_i$  for  $0 \leq i \leq r$ . Then*

- (i)  $\tilde{\mathcal{U}}_i$  is closed under extensions for all  $0 \leq i \leq r$ ,  $\tilde{\mathcal{U}}_0$  is closed under submodules and  $\tilde{\mathcal{U}}_r$  is closed under factor modules.
- (ii) If  $X \in \tilde{\mathcal{U}}_i$ , then  $\text{proj. dim } X \leq i + 1$  and  $\text{inj. dim } X \leq r - i + 1$ .
- (iii) Let  $X \in \tilde{\mathcal{U}}_i$  and  $Y \in \tilde{\mathcal{U}}_j$ . If  $t < i - j$  and  $t > i - j + 1$ , then  $\text{Ext}_\Lambda^t(X, Y) = 0$ .
- (iv) If  $X \in \tilde{\mathcal{U}}_i$  is indecomposable and satisfies  $\text{Ext}_\Lambda^1(X, X) = 0$ , then  $\text{End}_\Lambda X = k$ . Moreover,  $\text{Ext}_\Lambda^i(X, X) = 0$  for all  $i \geq 2$ .
- (v) Given two simple  $\Lambda$ -modules  $S, S'$  then there is at most one  $t \geq 0$  such that  $\text{Ext}_\Lambda^t(S, S') \neq 0$ .
- (vi) Each  $\tilde{\mathcal{U}}_i$  contains a simple  $\Lambda$ -module.
- (vii) If  $\mathcal{C}$  is a subcategory of  $\tilde{\mathcal{U}}_i$  for some  $i$  which is closed under extensions and direct summands and contains a cycle, then  $\mathcal{C}$  contains an indecomposable module  $X$  such that  $\text{End}_\Lambda X \neq k$ .
- (viii) Let  $X \in \tilde{\mathcal{U}}_i$  for some  $i$  be indecomposable. If  $\text{End}_\Lambda X \neq k$ , then  $X$  has a submodule  $U \in \tilde{\mathcal{U}}_i$  and a factor module  $V \in \tilde{\mathcal{U}}_i$  such that  $\text{Ext}_\Lambda^1(U, U) \neq 0 \neq \text{Ext}_\Lambda^1(V, V)$ . In this case  $\tilde{\mathcal{U}}_i$  contains infinitely many pairwise nonisomorphic indecomposable  $\Lambda$ -modules.

**Remark 2.2.** Let  $\vec{\Delta}$  be the linearly oriented quiver of type  $\mathbb{A}_{12}$  and let  $I$  be the two sided ideal of  $k\vec{\Delta}$  generated by all paths of length three. Let  $\Lambda = k\vec{\Delta}/I$ . It is easy to check that  $\text{mod } \Lambda$  admits a decomposition in pieces such that all the conditions of 2.1 hold. But  $\Lambda$  is not piecewise hereditary as shown in [HS].

We point out the following trivial consequence of 2.1(ii).

**Corollary 2.3.** *Let  $\Lambda$  be a piecewise hereditary algebra with exactly one piece. Then  $\Lambda$  is hereditary. If  $\Lambda$  has two pieces, then it is quasitilted.*

*Proof.* In fact, assume that  $\text{mod } \Lambda = \widetilde{\mathcal{U}}_0$ . Then by 2.1(ii) we have that  $\text{proj.dim}_\Lambda X \leq 1$  for all  $X \in \text{mod } \Lambda$ , or equivalently that  $\Lambda$  is a hereditary algebra. The second part of the corollary follows easily along the same lines.  $\square$

Recall from [H1] that for a finite dimensional algebra  $\Lambda$  of finite global dimension there is an equivalence  $\tau : D^b(\Lambda) \rightarrow D^b(\Lambda)$  which serves as the Auslander-Reiten translation on  $D^b(\Lambda)$ . We recall its construction. First, denote by

$$\nu : {}_\Lambda \mathcal{P} \rightarrow {}_\Lambda \mathcal{I}$$

the Nakayama transformation. It is defined as follows: if  $P$  is an indecomposable projective  $\Lambda$ -module with simple top  $S$ , then  $\nu P = I$  where  $I$  is the indecomposable injective  $\Lambda$ -module with simple socle  $S$ . It is easy to see that  $\nu = D \text{Hom}_\Lambda(-, \Lambda)$  where  $D$  is the standard duality  $\text{Hom}_k(-, k)$  on  $\text{mod } \Lambda$ , and that  $\nu$  takes bounded complexes of projective modules into bounded complexes of injective modules. If  $P^\bullet$  is a bounded complex of projective  $\Lambda$ -modules, then  $\tau$  is defined at the level of the derived category by

$$\tau P^\bullet = \nu P^\bullet[-1].$$

The derived categories of the hereditary, abelian categories occurring in our situation thus inherit an Auslander-Reiten translation via the triangle equivalence  $F : D^b(\Lambda) \rightarrow D^b(\mathcal{H})$ . Moreover,  $F$  will commute with the Auslander-Reiten translations. At the same time, hereditary categories with tilting objects have Auslander-Reiten sequences, see for example [HRS1], hence there exists an Auslander-Reiten translation  $\tau : \mathcal{H} \rightarrow \mathcal{H}$ . It is easy to see that, on nonprojective indecomposable objects, this translation coincides with the induced Auslander-Reiten translation on  $D^b(\mathcal{H})$ . We use the same symbol  $\tau$  to denote all these translations.

**Lemma 2.4.** *Let  $\Lambda$  be a piecewise hereditary algebra and let*

$$F : D^b(\Lambda) \rightarrow D^b(\mathcal{H})$$

*be a normalized equivalence with  $F({}_\Lambda \Lambda) = \bigoplus_{i=0}^r T_i[i]$ . Then*

$$F(D\Lambda_\Lambda) = \bigoplus_{i=0}^r \tau T_i[i+1].$$

*Proof.* By the previous remark we have in  $D^b(\Lambda)$  that  $D(\Lambda_\Lambda) = \tau(\Lambda[1])$ . Hence  $F(D(\Lambda_\Lambda)) = F\tau(\Lambda[1])$ . Since  $F$  commutes with  $\tau$  and with the shift functor the assertion follows.  $\square$

**Proposition 2.5.** *Let  $\Lambda$  be a connected piecewise hereditary algebra with  $r + 1 \geq 2$  pieces. Then there exists a hereditary, abelian category  $\mathcal{H}$  and a normalized equivalence*

$$F: D^b(\Lambda) \rightarrow D^b(\mathcal{H})$$

*with pieces  $\tilde{\mathcal{U}}_i$  for  $0 \leq i \leq r$ , such that  $\tilde{\mathcal{U}}_r$  does not contain any indecomposable projective  $\Lambda$ -module.*

*Proof.* We know that there exists a hereditary, abelian category  $\mathcal{H}'$  and a normalized equivalence  $F: D^b(\Lambda) \rightarrow D^b(\mathcal{H}')$  with pieces  $\mathcal{U}_i$  for  $0 \leq i \leq r$ . Assume that  $\tilde{\mathcal{U}}_r$  contains an indecomposable projective  $\Lambda$ -module. Then  $F(\Lambda) = \bigoplus_{i=0}^r T_i[i] = T^\bullet$  and  $T_r \neq 0$ . We claim first that  $T_r$  must be a projective object in  $\mathcal{H}'$ . If not, we have that  $\tau T_r \neq 0$ , so  $\tau T_r[r+1] \neq 0$ . By 2.4.,  $F(D\Lambda) = \bigoplus_{i=0}^r \tau T_i[i+1]$ . By assumption we have that  $F$  takes the indecomposable  $\Lambda$ -modules into  $\bigcup_{i=0}^r \mathcal{H}'[i]$ . So we conclude that  $\tau T_r[r+1] = 0$ , hence  $T_r$  is a projective object in  $\mathcal{H}'$ . Since  $\Lambda$  is connected, we infer by [H3] that  $\mathcal{H}' \simeq \text{mod } H'$  for some finite dimensional hereditary algebra  $H'$ . We have two possibilities depending on whether  $T_0$  has a projective summand or not. If  $T_0$  has no indecomposable projective direct summand, then  $\tau T_0 \in \mathcal{H}'[0]$ . At the same time,  $\tau T_r[r] = \nu T_r[r-1] \in \mathcal{H}'[r-1]$ , since  $\nu T_r$  is injective in  $\mathcal{H}'[0]$ . Consider the tilting complex

$$\tau T^\bullet = \tau T_0 \oplus \tau T_1[1] \oplus \cdots \oplus \tau T_{r-1}[r-1] \oplus \tau T_r[r].$$

The resulting normalized equivalence yields pieces  $\tilde{\mathcal{U}}_i$  for  $0 \leq i \leq r$  such that  $\tilde{\mathcal{U}}_r$  contains no indecomposable projective  $\Lambda$ -module. We turn to the case where  $T_0$  contains an indecomposable projective direct summand. We may change the orientation of the underlying quiver of  $H'$  to obtain a hereditary algebra  $H$  and a tilting complex  $\bar{T}^\bullet = \bigoplus_{i=0}^r \bar{T}_i[i]$ , again with  $\bar{T}_r$  a projective  $H$ -module, such that all simple projective  $H$ -modules are direct summands of  $\bar{T}_0$ . But then  $\text{Hom}_H(\bar{T}_0, \bar{T}_r) \neq 0$ , in contrast to  $\bar{T}^\bullet$  being a tilting complex, since  $r \geq 1$ .  $\square$

Proposition 2.5. yields now an upper bound for the strong global dimension of a piecewise hereditary algebra.

**Corollary 2.6.** *Let  $\Lambda$  be a connected piecewise hereditary algebra. with pieces  $\tilde{\mathcal{U}}_i$  for  $0 \leq i \leq r$ . Then  $\text{s. gl. dim } \Lambda \leq r + 1$ .*

*Proof.* If one uses the above proposition, the proof given in [HZ] shows that  $\text{s. gl. dim } \Lambda \leq r + 1$ .  $\square$

**Example 2.7.** We point out that the bound given in 2.6. is optimal. Consider a linearly oriented quiver  $\vec{\Delta}$  of type  $\mathbb{A}_{r+2}$ . Let  $k\vec{\Delta}$  be the path

algebra of  $\overrightarrow{\Delta}$  over  $k$ , and let  $I$  be the two sided ideal of  $k\overrightarrow{\Delta}$  generated by all paths of length two. Let  $\Lambda = k\overrightarrow{\Delta}/I$ . Then  $\Lambda$  is piecewise hereditary with  $\text{s. gl. dim } \Lambda = r + 1$ . Also it is straightforward to see that  $\Lambda$  can be realized with  $r + 1$  pieces.

**Example 2.8.** In general there is no good relationship between the number of pieces of a piecewise hereditary algebra  $\Lambda$ , and its global dimension as the following example shows. For this consider the linearly oriented quiver  $\overrightarrow{\Delta}$  of type  $\mathbb{A}_n$  for  $n$  odd. We label the arrows by  $\alpha_1, \dots, \alpha_{n-1}$ . Let  $I$  be the two sided ideal of  $k\overrightarrow{\Delta}$  generated by  $\alpha_1\alpha_2, \alpha_3\alpha_4, \dots, \alpha_{n-2}\alpha_{n-1}$ . Let  $\Lambda = k\overrightarrow{\Delta}/I$ . Then  $\Lambda$  is piecewise hereditary with  $\text{gl. dim } \Lambda = 2$ . It is easily checked that  $\Lambda$  can be realized with  $n - 1$  pieces and that  $\text{s. gl. dim } \Lambda = n - 1$ .

In the following proposition we give a criterion for the global dimension to be as large as possible.

**Proposition 2.9.** *Let  $\Lambda$  be a piecewise hereditary algebra given as the endomorphism algebra of a tilting complex  $T^\bullet = \bigoplus_{i=0}^{r-1} T_i[i] \in D^b(\mathcal{H})$  with  $r \geq 1$ . Then  $\text{gl. dim } \Lambda = r + 1$ , if and only if  $\text{Hom}(T_0, \tau^2 T_{r-1}) \neq 0$ .*

*Proof.* Let  $F : D^b(\Lambda) \rightarrow D^b(\mathcal{H})$  be the normalized equivalence induced by  $T^\bullet$ . So  $F(\Lambda) = T^\bullet$ , and from 2.4., we know that  $F(D\Lambda_\Lambda) = \tau T^\bullet[1]$ . By 2.6., we have that  $\text{gl. dim } \Lambda \leq r + 1$ , since the global dimension is bounded by the strong global dimension. If  $r = 1$ ,  $\Lambda$  has only two pieces, and so is quasitilted by 2.3., and so the global dimension equals 2. Thus we may assume without loss of generality that  $r \geq 2$ . Clearly  $\text{gl. dim } \Lambda = r + 1$  if and only if  $\text{Ext}_\Lambda^{r+1}(D\Lambda_\Lambda, \Lambda) \neq 0$ . The assertion follows now from the following sequence of isomorphisms:

$$\begin{aligned} \text{Ext}_\Lambda^{r+1}(D\Lambda_\Lambda, \Lambda) &\simeq \text{Hom}_{D^b(\Lambda)}(D\Lambda_\Lambda, \Lambda[r + 1]) \\ &\simeq \text{Hom}_{D^b(\mathcal{H})}(F(D\Lambda_\Lambda), F(\Lambda[r + 1])) \\ &\simeq \text{Hom}_{D^b(\mathcal{H})}(\tau T^\bullet[1], T^\bullet[r + 1]) \\ &\simeq \text{Hom}_{D^b(\mathcal{H})}(\tau T_{r-1}[r], T_0[r + 1]) \\ &\simeq \text{Ext}_{\mathcal{H}}^1(\tau T_{r-1}, T_0) \\ &\simeq D\text{Hom}_{\mathcal{H}}(T_0, \tau^2 T_{r-1}) \end{aligned}$$

where the 4th isomorphism follows from 2.1. □

Next we determine a lower bound for the strong global dimension of a piecewise hereditary algebra  $\Lambda$ . We denote by  $\text{ind } \Lambda$  the full subcategory of  $\text{mod } \Lambda$  containing one indecomposable from each isomorphism class.

**Proposition 2.10.** *Let  $\Lambda$  be a piecewise hereditary algebra. Then  $\text{s. gl. dim } \Lambda \geq \max_{X \in \text{ind } \Lambda} (\text{proj. dim}_\Lambda X + \text{inj. dim}_\Lambda X - 1)$ .*

*Proof.* Suppose that there exists an indecomposable  $\Lambda$ -module  $X$  with  $\text{proj. dim}_\Lambda X = t$  and  $\text{inj. dim}_\Lambda X = s$ . Clearly, we may assume that  $s \geq 2$ . Since  $\text{inj. dim}_\Lambda X = s$ , there is a simple  $\Lambda$ -module  $S$  such that  $\text{Ext}_\Lambda^s(S, X) \neq 0$ , hence  $\text{proj. dim}_\Lambda S \geq s$ . Let  $P^\bullet$  be the minimal projective resolution of  $X$  and let  $Q^\bullet$  be the minimal projective resolution of  $S$ . Then the fact that  $\text{Ext}_\Lambda^s(S, X) \neq 0$  shows that there exists  $0 \neq f \in \text{Hom}_{K^b(\Lambda\mathcal{P})}(Q^\bullet, P^\bullet[s])$ . Now

$$\text{Hom}_{K^b(\Lambda\mathcal{P})}(P^\bullet[s], Q^\bullet[1]) = \text{Ext}_\Lambda^{1-s}(X, S) = 0$$

shows that the mapping cone  $C_f$  is indecomposable by 1.5. The assertion follows, since  $\ell(C_f) \geq t + s - 1$ .  $\square$

**Remark 2.11.** Let  $\Lambda$  be the piecewise hereditary algebra given in the example following 2.6. The lower bound determined in 2.10 yields 3, whereas  $\text{s. gl. dim } \Lambda = n - 1$ . So the lower bound is far from being optimal. We believe that an optimal lower bound should be  $r - 1$ , if  $r + 1$  is number of pieces of a piecewise hereditary algebra  $\Lambda$ .

The following immediate corollary generalizes a result previously obtained for  $d = 2$  in [HZ].

**Corollary 2.12.** *Let  $\Lambda$  be a piecewise hereditary algebra such that*

$$d = \text{gl. dim } \Lambda = \text{s. gl. dim } \Lambda.$$

*Then  $\text{proj. dim}_\Lambda X + \text{inj. dim}_\Lambda X \leq d + 1$  for each indecomposable  $\Lambda$ -module  $X$ .*

In the remainder of this section, we will show that the indecomposable complexes in  $K^b(\Lambda\mathcal{P})$  for a piecewise hereditary algebra  $\Lambda$  are quite restricted. In fact, this will yield a different homological characterization of piecewise hereditary algebras.

**Proposition 2.13.** *The following are equivalent for a finite dimensional algebra  $\Lambda$ .*

- (i)  $\text{s. gl. dim } \Lambda < \infty$
- (ii) *For all  $P^\bullet \in K^b(\Lambda\mathcal{P})$  indecomposable and all simple  $\Lambda$ -modules  $S$  there are at most two degrees  $i$  and  $j$  such that  $P(S)$  is a direct summand of  $P^i$  and  $P^j$  and if  $i \neq j$ , then  $|i - j| = 1$ .*

*Proof.* It is immediate that (ii) implies (i). Applying the shift functor if necessary we may assume that  $P^s = 0$  for  $s > 0$ . Let

$$F: D^b(\Lambda) \rightarrow D^b(\mathcal{H})$$

be a normalized equivalence, and let  $m \in \mathbb{Z}$  such that  $F(P^\bullet) \in \mathcal{H}[m]$ . If  $P(S)$  is a direct summand of  $P^i$  for some  $i \leq 0$ , we conclude from 1.3 that  $\text{Hom}_{D^b(\Lambda)}(P^\bullet, S[-i]) \neq 0$ . Then also  $\text{Hom}_{D^b(\mathcal{H})}(F(P^\bullet), F(S)[-i])$  is nonzero. Since  $F(S) \in \mathcal{H}[t]$  for some  $0 \leq t \leq r$ ,  $F(S)[-i] \in \mathcal{H}[t-i]$ , and we obtain that  $F(P^\bullet) \in \mathcal{H}[t-i-1] \cup \mathcal{H}[t-i]$ , so  $t-i-1 \leq m \leq t-i$ . This means that  $i$  can take only two possible values:  $t-m-1$  and  $t-m$  which proves the assertion.  $\square$

Using this proposition and 1.1 we obtain immediately the following homological characterization of piecewise hereditary algebras coming from hereditary algebras of finite representation type.

**Corollary 2.14.** *Let  $\Lambda$  be a piecewise hereditary algebra of type mod  $H$  for a finite dimensional hereditary algebra  $H$  of finite representation type. Let  $P^\bullet \in K^b(\Lambda\mathcal{P})$  be an indecomposable complex. Let  $S$  be a simple  $\Lambda$ -module and  $P(S)$  its projective cover. Then there is at most one  $i$  such that  $P(S)$  is a direct summand of  $P^i$ .*

*Proof.* By 2.13. we know  $P(S)$  occurs as a direct summand in at most two consecutive degrees of  $P^\bullet$ . Assume that  $P(S)$  occurs as a direct summand in two consecutive degrees of  $P^\bullet$  and let  $F : D^b(\Lambda) \rightarrow D^b(H)$  be an equivalence. By writing  $F(P^\bullet) = X[s]$  and  $F(S) = Y[t]$  for some  $X, Y \in \text{ind } H$ , we easily obtain that

$$\text{Hom}_H(X, Y) \neq 0 \neq \text{Ext}_H^1(X, Y)$$

contradicting the fact that  $H$  is representation directed.  $\square$

### 3. ONE POINT EXTENSIONS

If  $\Gamma$  is a finite dimensional algebra and  $M$  a  $\Gamma$ -module, let  $\Lambda = \Gamma[M]$  be the one point extension of  $\Gamma$  by the module  $M$ . Recall that  $\Gamma[M]$  is defined to be the triangular matrix ring

$$\Lambda[M] = \begin{bmatrix} \Gamma & M \\ 0 & k \end{bmatrix}$$

with multiplication given by

$$\begin{pmatrix} \gamma & m \\ 0 & \alpha \end{pmatrix} \begin{pmatrix} \gamma' & m' \\ 0 & \alpha' \end{pmatrix} = \begin{pmatrix} \gamma\gamma' & \gamma m' + m\alpha' \\ 0 & \alpha\alpha' \end{pmatrix}$$

for  $\gamma, \gamma' \in \Gamma, m, m' \in M$  and  $\alpha, \alpha' \in k$ . We refer to [R] for details. Since a piecewise hereditary algebra  $\Lambda$  is directed,  $\Lambda$  can always be written as a one point extension algebra  $\Gamma[M]$ . Clearly,  $\Gamma$  is again piecewise hereditary, and  $\text{s. gl. dim } \Gamma \leq \text{s. gl. dim } \Lambda$ , since a complex in  $K^b(\Gamma\mathcal{P})$  is a complex in  $K^b(\Lambda\mathcal{P})$ . Note that this presentation will usually be not unique, since there may be several simple injective  $\Lambda$ -modules. In

this section we show that there is always a presentation  $\Lambda = \Gamma[M]$  such that  $\text{s. gl. dim } \Gamma \geq \text{s. gl. dim } \Lambda - 2$ . We include an example showing that this bound is optimal.

**Lemma 3.1.** *Let  $\Lambda$  be a piecewise hereditary algebra with normalized equivalence  $F: D^b(\Lambda) \rightarrow D^b(\mathcal{H})$  and pieces  $\mathcal{U}_t$  for  $0 \leq t \leq r$  with  $r \geq 1$  such that  $\mathcal{U}_r$  does not contain an indecomposable projective  $\Lambda$ -module. Let  $P^\bullet \in K^b({}_\Lambda \mathcal{P})$  be an indecomposable complex with  $P^i = 0$  for  $i < 0$  and  $P^i = 0$  for  $i > s$  such that  $\ell(P^\bullet) = s \geq 1$ . Let  $S$  be a simple  $\Lambda$ -module with  $S \in \mathcal{U}_r$  and  $P(S)$  an indecomposable direct summand of  $P^j$ . Then  $j \geq s - 1$ .*

*Proof.* Let  $T^\bullet = \bigoplus_{i=0}^{r-1} \in D^b(\mathcal{H})$  be a tilting complex with  $\text{End } T^\bullet = \Lambda$ , which we know to exist by 2.5. Let  $P^\bullet \in K^b({}_\Lambda \mathcal{P})$  be an indecomposable complex of length  $s$  with  $P^i = 0$  for  $i < 0$  and  $P^i = 0$  for  $i > s$ . Then  $F(P^\bullet) \in \mathcal{H}[m]$  for some  $m \in \mathbb{Z}$ . Assume that there exists a simple  $\Lambda$ -module  $S \in \mathcal{U}_r$  such that  $P(S)$  is a direct summand of  $P^j$  for  $j \leq s - 2$ . Then, since  $\text{Hom}_{D^b(\Lambda)}(P^\bullet, S[-j]) \neq 0$ , we can conclude that  $F(S) \in \mathcal{H}[m+j] \cup \mathcal{H}[m+j+1]$ , and so  $m+j \leq r \leq m+j+1$ , since we also have that  $F(S) \in \mathcal{H}[r]$ . Let now  $S'$  be a simple  $\Lambda$ -module such that  $P(S')$  is an indecomposable direct summand of  $P^s$ . We have that  $F(S') \in \mathcal{H}[t]$  for some  $0 \leq t \leq r$ . Then, just as above,  $\text{Hom}_{D^b(\Lambda)}(P^\bullet, S'[-s]) \neq 0$  shows that  $F(S') \in \mathcal{H}[m+s] \cup \mathcal{H}[m+s+1]$ , so  $m+s \leq t \leq m+s+1$ . Thus  $m+s \leq t \leq r \leq m+j+1 \leq m+(s-2)+1 = m+s-1$ , a contradiction. So we must have  $j \geq s - 1$ .  $\square$

In the proof of the next lemma we will work with truncations of complexes. Let  $X^\bullet = (X^i, d^i)$  be a bounded complex over some additive Krull-Schmidt  $k$ -category  $\mathfrak{a}$ , so we may assume that there exist integers  $s \leq s'$  such that  $X^i = 0$  for  $i < s$  and  $i > s'$ . If there is an integer  $m$ , with  $s \leq m < s'$  we denote by  $X_m^\bullet = (X_m^i, d_m^i)$  the complex with  $X_m^i = X^i$  for  $s \leq i \leq m$  and zero otherwise, and  $d_m^i = d^i$  for  $s \leq i < m$  and zero otherwise. Note that we obtain a morphism  $\pi: X^\bullet \rightarrow X_m^\bullet$  of complexes. We call  $X_m^\bullet$  a truncation of  $X^\bullet$ . Somehow surprisingly, it turns out that truncating indecomposable complexes gives rise to new indecomposable complexes having the “right” length.

**Lemma 3.2.** *Let  $\mathfrak{a}$  be an additive Krull-Schmidt category. Let  $X^\bullet$  be an indecomposable complex in  $K^b(\mathfrak{a})$  such that  $X^i = 0$  for  $i < s$  and  $i > s'$ , and  $\ell(X^\bullet) = s' - s \geq 1$ . Let  $X_m^\bullet$  be a truncation of  $X^\bullet$  for some  $s \leq m < s'$ . Then  $X_m^\bullet$  has an indecomposable direct summand of length  $m - s$ .*

*Proof.* Since  $\ell(X^\bullet) = s' - s$  we clearly have that  $X^s \neq 0 \neq X^{s'}$ . Suppose that all indecomposable direct summands  $Y^\bullet$  of  $X_m^\bullet$  have lengths  $\ell(Y^\bullet) < m - s$ . Then there exists an indecomposable direct summand  $Y^\bullet$  of  $X_m^\bullet$  such that  $Y^m = 0$ . Let  $f : Y^\bullet \rightarrow X_m^\bullet$  be the canonical split mono. Since  $Y^m = 0$  there is  $\tilde{f} : Y^\bullet \rightarrow X^\bullet$  such that  $\tilde{f}\pi = f$ , where  $\pi : X^\bullet \rightarrow X_m^\bullet$  is the canonical projection map. So we obtain the following commutative diagram of triangles in  $K^b(\mathbf{a})$ . Note that  $\varphi$  exists, since  $K^b(\mathbf{a})$  is a triangulated category.

$$\begin{array}{ccccccc} Y^\bullet & \xrightarrow{\tilde{f}} & X^\bullet & \xrightarrow{\tilde{g}} & C_{\tilde{f}} & \xrightarrow{\tilde{h}} & Y^\bullet[1] \\ \parallel & & \pi \downarrow & & \varphi \downarrow & & \parallel \\ Y^\bullet & \xrightarrow{f} & X_m^\bullet & \xrightarrow{g} & C_f & \xrightarrow{h} & Y^\bullet[1]. \end{array}$$

Since  $f$  is split mono we have by 1.4. that  $h = 0$ , hence  $\tilde{h} = 0$  too. Again using 1.4. we see that  $\tilde{f}$  is a split mono, so  $Y^\bullet$  is a proper indecomposable direct summand of  $X^\bullet$ , in contrast to  $X^\bullet$  being indecomposable. Thus there exists an indecomposable direct summand  $Y^\bullet$  of  $X_m^\bullet$  of length  $m - s$ .  $\square$

We can prove now the main result of this section.

**Theorem 3.3.** *Let  $\Lambda$  be piecewise hereditary algebra. Then there exists an indecomposable projective  $\Lambda$ -module  $P(\omega)$ , and a piecewise hereditary algebra  $\Gamma$  such that  $\Lambda = \Gamma[M]$  with  $M = \text{rad } P(\omega)$  and such that  $\text{s. gl. dim } \Gamma \leq \text{s. gl. dim } \Lambda \leq \text{s. gl. dim } \Gamma + 2$ .*

*Proof.* Let  $\{\tilde{\mathcal{U}}_t\}$  for  $0 \leq t \leq r$  be the pieces of  $\text{mod } \Lambda$ . We clearly may assume by 2.3. that  $r \geq 1$ . Since  $\Lambda$  is directed,  $\tilde{\mathcal{U}}_r$  contains a simple injective  $\Lambda$ -module  $S(\omega)$ . Let  $P(\omega)$  be its projective cover. Note, that if  $\Lambda$  is given by a tilting complex  $T^\bullet = \bigoplus_{i=0}^{r-1} T_i[i]$ , then  $P(\omega)$  corresponds under the normalized equivalence induced by  $T^\bullet$  to an indecomposable direct summand of  $T_{r-1}[r-1]$ . Let  $M = \text{rad } P(\omega)$ . Let  $\Gamma = \text{End}_\Lambda(\bigoplus_{S \neq S(\omega)} P(S))$ . It follows that  $\Lambda = \Gamma[M]$  and it is easy to show that  $\text{s. gl. dim } \Gamma \leq \text{s. gl. dim } \Lambda$ . Therefore  $\Gamma$  is also piecewise hereditary.

Let  $d = \text{s. gl. dim } \Lambda$ . To show the other inequality it is enough to construct an indecomposable complex  $P^\bullet \in K^b({}_\Gamma \mathcal{P})$  with  $\ell(P^\bullet) \geq d-2$ . For this, let  $P^\bullet \in K^b({}_\Lambda \mathcal{P})$  be an indecomposable complex with length  $\ell(P^\bullet) = d$ . Applying the shift functor if necessary we may assume that  $P^i = 0$  for  $i < 0$ . By 3.1. we know that  $P(\omega)$  is not a direct summand of  $P^j$  for  $0 \leq j \leq d-2$ . Thus the truncation  $P_{d-2}^\bullet \in K^b({}_\Gamma \mathcal{P})$ . By



3.2. we have that  $P_{d-2}^\bullet$  contains an indecomposable direct summand of length  $d - 2$ . In particular we obtain that  $\text{s. gl. dim } \Gamma \geq d - 2$ .  $\square$

The following example illustrates that the bound given in 3.3. is optimal.

**Example 3.4.** Let  $H$  be a wild hereditary algebra and let  $M$  be an indecomposable  $H$ -module such that the one point extension  $\Lambda = H[M]$  is quasitilted [HRS1]. For example one may think of the canonical algebras of Ringel [R]. It is shown in [KL] (see also [L]), that there exists an  $i > 0$  such that the algebra  $\Lambda_i = H[\tau^{-i}M]$  is piecewise hereditary and not quasitilted. So  $\text{s. gl. dim } \Lambda_i > 2$ , see [HZ]. By 3.3. we obtain  $\text{s. gl. dim } \Lambda_i = 3$ , and clearly  $\text{s. gl. dim } H = 1$ , so the bound given in 3.3 is optimal.

#### 4. BEHAVIOR UNDER TILTING

In this section we investigate the behavior of the strong global dimension under tilting. Recall that if  $\Lambda$  is a finite dimensional algebra, then a finitely generated left module  $T$  is a tilting module, if

- (i)  $\text{proj. dim}_\Lambda T = t$ ,
- (ii)  $\text{Ext}_\Lambda^i(T, T) = 0$  for  $i > 0$  and
- (iii) there is an exact sequence of left  $\Lambda$ -modules

$$0 \rightarrow {}_\Lambda \Lambda \rightarrow T^0 \rightarrow \dots \rightarrow T^t \rightarrow 0$$

with  $T^i \in \text{add } T$  for all  $i \geq 0$ , where  $\text{add } T$  denotes the full subcategory of  $\text{mod } \Lambda$  containing the direct sums of direct summands of  $T$ .

Let  $\Gamma = \text{End}_\Lambda T$ . There is a nice relationship between the global dimensions of  $\Lambda$  and of  $\Gamma$ , namely

$$\text{gl. dim } \Lambda - t \leq \text{gl. dim } \Gamma \leq \text{gl. dim } \Lambda + t.$$

see [H1]. We prove in this section that a similar relationship exists between the strong global dimensions of  $\Lambda$  and  $\Gamma$ . It is clear that  $\Lambda$  is piecewise hereditary if and only if  $\Gamma$  is piecewise hereditary, since  $\Gamma$  and  $\Lambda$  are derived equivalent. Thus using 1.1., we have that  $\text{s. gl. dim } \Lambda < \infty$  if and only if  $\text{s. gl. dim } \Gamma < \infty$ . So we may assume that  $\text{s. gl. dim } \Lambda < \infty$  to investigate the possible change under tilting. We start with a preliminary lemma whose proof is straightforward, but we include it for the convenience of the reader.

**Lemma 4.1.** *Let  $\Lambda$  be a finite dimensional algebra and let  ${}_{\Lambda}T$  be a tilting module with  $\text{proj. dim } T = t$*

- (i) *If  $T^{\bullet} \in K^b(\text{add } T)$  is a complex with length  $\ell(T^{\bullet}) = s$ , then there exists a complex  $P^{\bullet} \in K^b({}_{\Lambda}\mathcal{P})$  with  $\ell(P^{\bullet}) \leq s + t$ , and a quasi isomorphism  $P^{\bullet} \rightarrow T^{\bullet}$ .*
- (ii) *If  $P^{\bullet} \in K^b({}_{\Lambda}\mathcal{P})$  is a complex with length  $\ell(P^{\bullet}) = s$ , then there exists a complex  $T^{\bullet} \in K^b(\text{add } T)$  with  $\ell(T^{\bullet}) \leq s + t$ , and a quasi isomorphism  $P^{\bullet} \rightarrow T^{\bullet}$ .*

*Proof.* We first show (i). Let  $T^{\bullet} = (T^i, d^i)$  with length  $\ell(T^{\bullet}) = s$ . We proceed by induction on  $s$ . By applying the shift functor if necessary, we may assume that  $T^0 \neq 0$  and  $T^m = 0$  for  $m < 0$ . For  $s = 0$  this follows from  $\text{proj. dim } T = t$ , by simply choosing  $P^{\bullet}$  to be a minimal projective resolution of  $T^0$ . Assume now that  $s > 0$ . Consider the truncated complex  $T_{\geq 1}^{\bullet}$  with  $T_{\geq 1}^i = 0$  for  $i \leq 0$  and  $T_{\geq 1}^i = T^i$  for  $i > 0$ , with the differentials induced by the differentials of  $T^{\bullet}$ . Observe that  $d^0$  induces a morphism of complexes  $\gamma: T^0[-1] \rightarrow T_{\geq 1}^{\bullet}$  whose cone is  $T^{\bullet}$ . Now,  $\ell(T_{\geq 1}^{\bullet}) = s - 1 < s$ , and  $\ell(T^0[-1]) = 1$ . So by induction we obtain quasi isomorphisms  $\alpha: P_1^{\bullet} \rightarrow T^0[-1]$  and  $\beta: P_2^{\bullet} \rightarrow T_{\geq 1}^{\bullet}$ , where  $P_1^{\bullet}$  and  $P_2^{\bullet}$  are perfect complexes of lengths  $\ell(P_1^{\bullet}) \leq t$  and  $\ell(P_2^{\bullet}) \leq t + s - 1$ . Thus  $\gamma$  induces a morphism  $\delta: P_1^{\bullet} \rightarrow P_2^{\bullet}$ . Consequently, we obtain the following commutative diagram of triangles, the existence of the morphism  $\eta$  follows from the axioms of a triangulated category.

$$\begin{array}{ccccccc}
P_1^{\bullet} & \xrightarrow{\delta} & P_2^{\bullet} & \longrightarrow & C_{\delta} & \longrightarrow & P_1^{\bullet}[1] \\
\alpha \downarrow & & \beta \downarrow & & \eta \downarrow & & \alpha[1] \downarrow \\
T^0[-1] & \xrightarrow{\gamma} & T^{\bullet} & \longrightarrow & T^{\bullet} & \longrightarrow & T^0.
\end{array}$$

Clearly  $C_{\delta} \in K^b({}_{\Lambda}\mathcal{P})$  with length  $\ell(C_{\delta}) \leq s + t$ , and  $\eta$  is also a quasi isomorphism.

The second assertion follows similarly using the coresolution of  ${}_{\Lambda}\Lambda$  in  $\text{add } T$  given in part (iii) of the definition of a tilting module.  $\square$

**Theorem 4.2.** *Let  $\Lambda$  be a piecewise hereditary algebra and let  $T$  be a tilting module  $\Lambda$ -module with  $\text{proj. dim } T = t$ . Let  $\Gamma = \text{End}_{\Lambda} T$ . Then  $\text{s. gl. dim } \Lambda - t \leq \text{s. gl. dim } \Gamma \leq \text{s. gl. dim } \Lambda + t$ .*

*Proof.* It suffices to show the right hand side inequality, since the other inequality follows by tilting symmetry. By tilting theory we have a triangle equivalence  $K^b(\text{add } T) \rightarrow K^b({}_{\Gamma}\mathcal{P})$  given by  $\text{Hom}_{\Lambda}(T, -)$ . Let  $P^{\bullet} \in K^b({}_{\Lambda}\mathcal{P})$  be an indecomposable perfect complex with maximal length  $\ell(P^{\bullet}) = s = \text{s. gl. dim } \Lambda$ . So by 4.1(ii) there exists an indecomposable complex  $T^{\bullet} \in K^b(\text{add } T)$  with  $\ell(T^{\bullet}) \leq s + t$ . Now

$\text{Hom}_\Lambda(T, T^\bullet) \in K^b({}_\Gamma\mathcal{P})$  and all indecomposable complexes in  $K^b({}_\Gamma\mathcal{P})$  are of this form. Thus  $\text{s.gl.dim } \Gamma \leq t + s$ , which shows the assertion.  $\square$

**Remark 4.3.** Let  $T$  be a tilting  $\Lambda$ -module of  $\text{proj.dim } T = 1$ . Then there is an associated torsion pair  $(\mathcal{T}(T), \mathcal{F}(T))$  in  $\text{mod } \Lambda$ , where

$$\mathcal{T}(T) = \{X \in \text{mod } \Lambda \mid \text{Ext}_\Lambda^1(T, X) = 0\}$$

and

$$\mathcal{F}(T) = \{X \in \text{mod } \Lambda \mid \text{Hom}_\Lambda(T, X) = 0\}$$

We say that the torsion pair  $(\mathcal{T}(T), \mathcal{F}(T))$  splits, if for each indecomposable  $\Lambda$ -module  $X$  we have that  $X \in \mathcal{T}(T) \cup \mathcal{F}(T)$ , or equivalently  $\text{Ext}_\Lambda^1(X, Y) = 0$  for all  $X \in \mathcal{F}(T)$  and all  $Y \in \mathcal{T}(T)$ .

It is easy to construct examples of piecewise hereditary algebras  $\Lambda$  with  $\text{gl.dim } \Lambda = d > 1$ , such that there is no tilting module  $T$  with  $\text{proj.dim } T = 1$  and such that the associated torsion pair  $(\mathcal{T}(T), \mathcal{F}(T))$  on  $\text{mod } \Lambda$  splits and  $\text{gl.dim, End}_\Lambda T < d$ . A concrete example is provided by the algebra considered in 2.8. But we do not know such an example if we look instead at the strong global dimension.

## 5. FUNCTORIAL FINITENESS

Let  $\Lambda$  be a piecewise hereditary algebra. We show in this section that the pieces of  $\text{mod } \Lambda$  have Auslander-Reiten sequences. Note that this trivially holds by 2.3. in the case of one piece. So we will assume for the rest of this section that the number of pieces of  $\text{mod } \Lambda$  is at least two. We have shown in 2.5. that there is a hereditary, abelian category  $\mathcal{H}$  and a normalized equivalence  $F: D^b(\Lambda) \rightarrow D^b(\mathcal{H})$  such that  $F({}_\Lambda\Lambda) = \bigoplus_{i=0}^{r-1} T_i[i]$ . Let  $\tilde{\mathcal{U}}_i$  for  $0 \leq i \leq r$  be the pieces of  $\text{mod } \Lambda$ . Note that  $T^\bullet = \bigoplus_{i=0}^{r-1} T_i[i]$  is a tilting complex.

We recall the definition of the right orthogonal category in the sense of [GL] (see also [H4]). Let  $\mathcal{H}$  be a hereditary, abelian category, and let  $X \in \mathcal{H}$ . We define the right orthogonal category  $X^{\text{perp}}$  to be the full subcategory of  $\mathcal{H}$  containing those objects  $Y$  such that  $\text{Hom}_{\mathcal{H}}(X, Y) = 0$  and  $\text{Ext}_{\mathcal{H}}^1(X, Y) = 0$ . We also define the left orthogonal category  ${}^{\text{perp}}X$  to be the full subcategory of  $\mathcal{H}$  containing those objects  $Y$  such that  $\text{Hom}_{\mathcal{H}}(Y, X) = \text{Ext}_{\mathcal{H}}^1(Y, X) = 0$ . It is easy to see that  $X^{\text{perp}} = {}^{\text{perp}}(\tau X)$  if  $X$  does not have an indecomposable projective direct summand. We will also need the following notation. For  $X \in \mathcal{H}$  we denote by  $\mathcal{T}(X)$  be the full subcategory of  $\mathcal{H}$  containing those  $Y$  such that  $\text{Ext}_{\mathcal{H}}^1(X, Y) = 0$  and by  $\mathcal{F}(X)$  the full subcategory of  $\mathcal{H}$  containing those  $Y$  such that  $\text{Hom}_{\mathcal{H}}(X, Y) = 0$ . So  $X^{\text{perp}} = \mathcal{T}(X) \cap \mathcal{F}(X)$ .

For each  $0 \leq t \leq r$  we denote by  $\tilde{\mathcal{V}}_t$  the following subcategory of  $\mathcal{H}$ :

$$\tilde{\mathcal{V}}_t = \bigcap_{i \neq t, t-1} T_i^{\text{perp}} \cap \mathcal{T}(T_t) \cap \mathcal{F}(T_{t-1}).$$

The following well-known lemma, compare for example [S], gives a more explicit description of the pieces of  $\text{mod } \Lambda$  and of homological properties of  $T^\bullet$ .

**Lemma 5.1.** *Let  $\Lambda$  be a piecewise hereditary algebra given by a tilting complex  $T^\bullet$  and a normalized equivalence  $F$  as above. Then*

- (i)  $\text{Hom}_{\mathcal{H}}(T_i, T_j) = 0$  if  $i \neq j$ , and  $\text{Ext}_{\mathcal{H}}^1(T_i, T_j) = 0$  if  $j \neq i + 1$ .
- (ii) *The restriction of  $F$  to  $\tilde{\mathcal{U}}_t$  induces an equivalence between  $\tilde{\mathcal{U}}_t$  and  $\tilde{\mathcal{V}}_t[t]$ .*

We refer to [AS] for the notion of functorially finiteness of a subcategory and Auslander-Reiten sequences in subcategories. To show the functorial finiteness of the subcategories  $\tilde{\mathcal{V}}_t \subset \mathcal{H}$ , we begin with the following assertion.

**Lemma 5.2.** *Let  $\mathcal{H}$  be a hereditary, abelian and connected category such that both  $\text{Hom}_{\mathcal{H}}(X, Y)$  and  $\text{Ext}_{\mathcal{H}}^1(X, Y)$  are finite dimensional vector spaces for all  $X, Y \in \mathcal{H}$ . Let  $X \in \mathcal{H}$  with  $\text{Ext}_{\mathcal{H}}^1(X, X) = 0$ . Then, the subcategories  $\mathcal{T}(X)$ ,  $\mathcal{F}(X)$ ,  $X^{\text{perp}}$  and  ${}^{\text{perp}}X$ , are all functorially finite in  $\mathcal{H}$ .*

*Proof.* (i) To prove the functorial finiteness of  $\mathcal{T}(X)$ , we start by showing that  $\mathcal{T}(X)$  is covariantly finite in  $\mathcal{H}$ . For this let  $Z \in \mathcal{H}$ , and not in  $\mathcal{T}(X)$ . Consider the universal extension (see for example [B])

$$0 \rightarrow Z \xrightarrow{\alpha_Z} F_Z \rightarrow \tilde{X} \rightarrow 0$$

with  $\tilde{X} \in \text{add } X$ . By construction, the connecting homomorphism  $\text{Hom}(X, \tilde{X}) \rightarrow \text{Ext}^1(X, Z)$  is surjective. Thus we see that  $F_Z \in \mathcal{T}(X)$ . If  $Z' \in \mathcal{T}(X)$  and  $f : Z \rightarrow Z'$ , then clearly we obtain  $g : F_Z \rightarrow Z'$  such that  $f = \alpha_Z g$ , since  $\text{Ext}^1(X, Z') = 0$ . This proves that  $Z \xrightarrow{\alpha_Z} F_Z$  is a  $\mathcal{T}(X)$ -approximation, hence  $\mathcal{T}(X)$  is covariantly finite.

We show now that  $\mathcal{T}(X)$  is contravariantly finite. Let  $Z \in \mathcal{H}$ , and consider the minimal left  $\text{add } \tau X$ -approximation  $\beta_Z : Z \rightarrow \tau \tilde{X}$  of  $Z$ , and let  $G_Z = \text{Ker } \beta_Z$ . We claim that  $G_Z \xrightarrow{\mu_Z} Z$  is a right  $\mathcal{T}(X)$ -approximation of  $Z$ . We have a short exact sequence

$$0 \rightarrow G_Z \xrightarrow{\mu_Z} Z \rightarrow \text{im } \beta_Z \rightarrow 0.$$

Let  $Z' \in \mathcal{T}(X)$  and  $f : Z' \rightarrow Z$ . Since  $\text{Ext}^1(X, Z') = 0$ , we obtain  $\text{Hom}(Z', \tau X) = 0$ , hence  $\text{Hom}(Z', \text{im } \beta_Z) = 0$ , since  $\text{im } \beta_Z$  is cogenerated by  $\tau X$ . Thus there is  $g : Z' \rightarrow G_Z$  with  $f = g\mu_Z$ . Therefore  $\mathcal{T}(X)$  is also contravariantly finite.

(ii) We prove now that  $\mathcal{F}(X)$  is covariantly finite. For this let  $Z \in \mathcal{H}$ . Let  $\gamma_Z : \tilde{X} \rightarrow Z$  be a minimal right add  $X$ -approximation of  $Z$ , and let  $F_Z = \text{Coker } \gamma_Z$ . So  $F_Z \in \mathcal{F}(X)$  and we obtain a short exact sequence

$$0 \rightarrow \text{im } \gamma_Z \rightarrow Z \xrightarrow{\pi_Z} F_Z \rightarrow 0.$$

Let  $Z' \in \mathcal{F}(X)$  and  $f : Z \rightarrow Z'$ . Since  $\text{im } \gamma_Z$  is generated by  $X$ , we have  $\text{Hom}(\text{im } \gamma_Z, Z') = 0$ . This implies the existence of a map  $g : F_Z \rightarrow Z'$  such that  $f = \pi_Z g$ , and thus  $\mathcal{F}(X)$  is covariantly finite.

To show that  $\mathcal{F}(X)$  is contravariantly finite, let  $Z \in \mathcal{H}$ , and write  $X = P \oplus X'$  where  $P$  is projective, and  $X'$  has no indecomposable projective direct summands. If  $P \neq 0$  this implies, since  $\mathcal{H}$  is connected, that  $\mathcal{H} = \text{mod } H$  for a finite dimensional hereditary algebra  $H$  by [H3]. Let  $I = D \text{Hom}_H(P, H)$ , so  $I$  is an injective  $H$ -module. First we consider a minimal left add  $I$ -approximation  $\beta_Z : Z \rightarrow \tilde{I}$  of  $Z$ . Set  $E_Z = \text{Ker } \beta_Z$  and denote the inclusion  $E_Z$  to  $Z$  by  $\mu_Z$ . By construction we have that  $\text{Hom}(P, E_Z) = \text{Hom}(E_Z, I) = 0$ . Note that if  $Z' \in \mathcal{F}(X)$ , then  $0 = \text{Hom}(P, Z') = \text{Hom}(Z', I)$ . This implies that if we have a map  $f : Z' \rightarrow Z$ , then there is  $g : Z' \rightarrow E_Z$  such that  $f = g\mu_Z$ . Therefore, if  $E_Z \in \mathcal{F}(X)$ , then  $\mu_Z : E_Z \rightarrow Z$  is a right  $\mathcal{F}(X)$ -approximation of  $Z$ . So assume that  $\text{Hom}(X, E_Z) \neq 0$ , so that also  $\text{Ext}^1(E_Z, \tau X') \neq 0$ . Consider the universal extension

$$0 \rightarrow \widetilde{\tau X'} \rightarrow F_Z \xrightarrow{\gamma_Z} E_Z \rightarrow 0$$

with  $\widetilde{\tau X'} \in \text{add } \tau X'$ . We claim that  $F_Z \in \mathcal{F}(X)$ . To see this, note first that  $\text{Hom}(X', F_Z) = 0$ , since by construction we have  $\text{Ext}^1(F_Z, \tau X') = 0$ . Since  $\text{Ext}^1(X, X) = 0$ , we get that  $\text{Ext}^1(X', P) = 0$ , thus also by the Auslander-Reiten formula  $\text{Hom}(P, \tau X') = 0$ . Applying  $\text{Hom}(P, -)$  to the above universal extension yields  $\text{Hom}(P, F_Z) = 0$  hence  $F_Z \in \mathcal{F}(X)$ . Consider the composition  $\delta = \gamma_Z \mu_Z : F_Z \rightarrow Z$ . Let  $Z' \in \mathcal{F}(X)$  and  $f : Z' \rightarrow Z$ . We already know that there is  $g : Z' \rightarrow E_Z$  with  $f = g\mu_Z$ . Since  $Z' \in \mathcal{F}(X)$  we have that  $\text{Ext}^1(Z', \tau X') = 0$ , thus there is  $h' : Z' \rightarrow F_Z$  with  $g = h'\gamma_Z$ , hence  $f = h'\gamma_Z \mu_Z$ . Hence the homomorphism  $\delta : F_Z \rightarrow Z$  is a right  $\mathcal{F}(X)$ -approximation of  $Z$ . It remains to look at the case when  $X$  has no projective summands. Let  $Z \in \mathcal{H}$  such that  $\text{Hom}(X, Z) \neq 0$ . Then  $\text{Ext}^1(Z, \tau X) \neq 0$  so we may consider the universal extension

$$0 \rightarrow \widetilde{\tau X} \rightarrow F_Z \xrightarrow{\gamma_Z} Z \rightarrow 0$$

By construction,  $F_Z \in \mathcal{F}(X)$ , and it is easy to show that the map  $F_Z \xrightarrow{\gamma_Z} Z$  is a right  $\mathcal{F}(X)$ -approximation of  $Z$ . So  $\mathcal{F}(X)$  is contravariantly finite.

(iii) First note that  $X^{\text{perp}} = \mathcal{T}(X) \cap \mathcal{F}(X)$ . We start by showing that  $X^{\text{perp}}$  is covariantly finite. For this let  $Z \in \mathcal{H}$ . By (i) we have the minimal left  $\mathcal{T}(X)$ -approximation  $\alpha_Z : Z \rightarrow F$  and by (ii) the minimal left  $\mathcal{F}(X)$ -approximation  $\pi_F : F \rightarrow G$ . Since  $\pi_F$  is surjective and  $\mathcal{T}(X)$  is closed under factors we infer that  $G \in X^{\text{perp}}$ . Trivially  $\alpha_Z \pi_F$  is a left  $X^{\text{perp}}$ -approximation, so  $X^{\text{perp}}$  is covariantly finite.

Next we show that  $X^{\text{perp}}$  is contravariantly finite. Again let  $Z \in \mathcal{H}$ . By (ii) we have the minimal right  $\mathcal{F}(X)$ -approximation  $\beta_Z : F \rightarrow Z$  and by (i) the minimal right  $\mathcal{T}(X)$ -approximation  $\mu_F : G \rightarrow F$ . Since  $\mu_F$  is injective and  $\mathcal{F}(X)$  is closed under subobjects we infer that  $G \in X^{\text{perp}}$ . Trivially  $\mu_F \beta_Z$  is a right  $X^{\text{perp}}$ -approximation, so  $X^{\text{perp}}$  is contravariantly finite.

(iv) This is analogous to (iii).  $\square$

**Theorem 5.3.** *Let  $\mathcal{H}$  be a hereditary, abelian category such that both  $\text{Hom}(X, Y)$  and  $\text{Ext}^1(X, Y)$  are finite dimensional  $k$ -vector spaces for all  $X, Y \in \mathcal{H}$ . Let  $T^\bullet = \bigoplus_{i=0}^{r-1} T_i[i] \in \mathcal{H}$ ,  $r \geq 1$  be a tilting complex. Then for each  $0 \leq t \leq r$ , the subcategories  $\tilde{\mathcal{V}}_t$  are functorially finite in  $\mathcal{H}$ .*

*Proof.* Let  $0 \leq t \leq r$  and consider  $\tilde{\mathcal{V}}_t \subset \mathcal{H}$ . We will use the description of  $\tilde{\mathcal{V}}_t$  given in 5.1, which can be rewritten as follows:

$$\tilde{\mathcal{V}}_t = \bigcap_{i \neq t-1} \mathcal{T}(T_i) \cap \bigcap_{i \neq t} \mathcal{F}(T_i).$$

We start by showing that  $\tilde{\mathcal{V}}_t$  is covariantly finite in  $\mathcal{H}$ . For this let  $Z \in \mathcal{H}$ , and set  $F_r = Z$ . If  $Z \in \mathcal{T}(T_{r-1})$ , we let  $F_{r-1} = F_r$  and define the map  $\alpha_r : F_r \rightarrow F_{r-1}$  as being the identity map. Assume that  $Z$  is not in  $\mathcal{T}(T_{r-1})$ . By 5.2(i) we have an exact sequence

$$0 \rightarrow F_r \xrightarrow{\alpha_r} F_{r-1} \rightarrow \tilde{T}_{r-1} \rightarrow 0$$

such that the map  $\alpha_r : F_r \rightarrow F_{r-1}$  is a minimal left  $\mathcal{T}(T_{r-1})$ -approximation. We continue in the same way for each  $1 \leq i \leq r-1$  and  $i \neq t-1$ . We keep the same notations as above. Namely, if  $F_i \in \mathcal{T}(T_{i-1})$ , we set  $F_i = F_{i-1}$  and the map  $\alpha_i : F_i \rightarrow F_{i-1}$  is the identity map. If  $F_i$  is not in  $\mathcal{T}(T_{i-1})$ , we have an exact sequence

$$(*) \quad 0 \rightarrow F_i \xrightarrow{\alpha_i} F_{i-1} \rightarrow \tilde{T}_{i-1} \rightarrow 0$$

where  $\tilde{T}_{i-1} \in \text{add } T_{i-1}$  and such that the map  $\alpha_i : F_i \rightarrow F_{i-1}$  is a minimal left  $\mathcal{T}(T_{i-1})$ -approximation. If  $i = t$ , we set  $\alpha_t$  as being

the identity map from  $F_t$  to itself if  $F_t \in \mathcal{T}(T_{t-2})$  and  $F_{t-2} = F_t$ . If  $F_t \notin \mathcal{T}(T_{t-2})$  we define  $\alpha_t$  using the same type of universal extensions as above. Set  $F_s = F_1$  if  $t = 1$  and  $F_s = F_0$  otherwise. By 5.1 and the construction in 5.2(i) we infer that  $F_s \in \bigcap_{i \neq t-1} \mathcal{T}(T_i)$  and trivially we have that the composition  $Z = F_{r+1} \rightarrow \cdots \rightarrow F_s$  is a left  $\bigcap_{i \neq t-1} \mathcal{T}(T_i)$ -approximation of  $Z$ . Set  $F_s = G_r$ . Using 5.2(ii) we have a sequence of surjections  $\beta_i : G_i \rightarrow G_{i-1}$  for  $1 \leq i \leq r, i \neq t$  (leapfrogging over  $G_t$  if needed) and such that  $\beta_i : G_i \rightarrow G_{i-1}$  is a minimal left  $\mathcal{F}(T_{i-1})$ -approximation. Set  $G_s = G_1$  if  $t = 0$  and  $G_s = G_0$  otherwise. By 5.1 and the construction in 5.2(ii) we infer that  $G_s \in \bigcap_{i \neq t} \mathcal{F}(T_i)$ , and since  $\bigcap_{i \neq t-1} \mathcal{T}(T_i)$  is closed under factors we see that  $G_s \in \tilde{\mathcal{V}}_t$ . Moreover, we have that the composition  $Z \rightarrow F_s \rightarrow \cdots \rightarrow G_s$  is a left  $\tilde{\mathcal{V}}_t$ -approximation of  $Z$ .

Using 5.2 we can show in a similar way that  $\tilde{\mathcal{V}}_t$  is contravariantly finite.  $\square$

**Example 5.4.** We point out that the pieces of a piecewise hereditary algebra need not be functorially finite in  $\text{mod } \Lambda$ , as the following example shows. Let  $\mathcal{H} = \text{coh } \mathbb{P}^1(k)$  be the category of coherent sheaves on the projective line. Denote by  $\mathcal{O}$  the structure sheaf. Then  $\mathcal{O} \oplus \mathcal{O}(1)$  is a tilting object in  $\mathcal{H}$ , and its endomorphism algebra  $\Lambda$  is the path algebra of the Kronecker quiver. Thus that  $\text{mod } \Lambda$  has two pieces  $\mathcal{U}_0$  and  $\mathcal{U}_1$ . Now  $\mathcal{U}_0$  consists of the indecomposable preprojective and indecomposable regular  $\Lambda$ -modules and  $\mathcal{U}_1$  consists of the indecomposable preinjective  $\Lambda$ -modules. However, it follows from [CH], or via a direct calculation that neither  $\tilde{\mathcal{U}}_0$  is contravariantly finite in  $\text{mod } \Lambda$ , nor  $\tilde{\mathcal{U}}_1$  is covariantly finite in  $\text{mod } \Lambda$ .

**Corollary 5.5.** *Let  $\Lambda$  be a piecewise hereditary algebra with pieces  $\tilde{\mathcal{U}}_t$  for  $0 \leq t \leq r$ . Then, for each  $0 \leq t \leq r$ ,  $\tilde{\mathcal{U}}_t$  has Auslander-Reiten sequences.*

*Proof.* By 5.3 we have that the subcategories  $\tilde{\mathcal{V}}_t$  are functorially finite for all  $0 \leq t \leq r$ . Thus they all have Auslander-Reiten sequences by [AS]. Since each  $\tilde{\mathcal{U}}_t$  is equivalent to  $\tilde{\mathcal{V}}_t[t]$ , we infer that each  $\tilde{\mathcal{U}}_t$  has Auslander-Reiten sequences.  $\square$

## REFERENCES

- [ARS] M. Auslander, I. Reiten, S. Smalø, *Representation theory of artin algebras*, Cambridge University Press 1995.
- [AS] M. Auslander, S. Smalø, *Almost split sequences in subcategories*, J. Algebra 69 (1981), no. 2, 426 - 454.

- [B] K. Bongartz, *Tilted algebras*, Representations of Algebras (Puebla 1980), Springer Lecture Notes in Mathematics 903 (1981), 26-38.
- [CH] J. Carlson, D. Happel, *Contravariantly finite subcategories and irreducible maps*, Proc. Amer. Math. Soc. 117 (1993), no. 1, 61 - 65.
- [GL] W. Geigel, H. Lenzing, *Perpendicular categories with applications to representations and sheaves*, J. Algebra 144 (1991), no. 2, 273 - 343.
- [H1] D. Happel, *Triangulated categories in the representation theory of finite dimensional algebras*, London Math. Soc. Lecture Notes Series 119 (1988).
- [H2] D. Happel, *A characterization of hereditary categories with tilting object*, Invent. Math. 144 (2001), no. 2, 381 - 398.
- [H3] D. Happel, *Quasitilted algebras*, Algebras and modules, I (Trondheim, 1996), 55 -82, CMS Conf. Proc., 23, Amer. Math. Soc., Providence, RI, 1998.
- [H4] D. Happel, *Partial tilting modules and recollement*, Proceedings of the International Conference of Algebra, Contemporary Mathematics 131, Providence 1992, 345-362.
- [HRS1] D. Happel, I. Reiten, S. Smalø, *Tilting in abelian categories and quasitilted algebras*, Mem. Amer. Math. Soc. 575 (1996).
- [HRS2] D. Happel, I. Reiten, S. Smalø, *Piecewise hereditary algebras*, Arch. Math. 66 (1996), 182 - 186.
- [HS] D. Happel, U. Seidel, *Piecewise hereditary Nakayama algebras*, preprint.
- [HZ] D. Happel, D. Zacharia, *A homological characterization of piecewise hereditary algebras*, Math. Z. 260 (2008), no. 1, 177 - 185.
- [KL] O. Kerner, F. Lukas, *Elementary modules*, Math. Z. 223 (1996), no. 3, 421 - 434.
- [L] S. Lache, *Piecewise hereditary one-point extensions*, J. Algebra 226 (2000), no. 1, 53 - 70.
- [R] C. M. Ringel, *Tame algebras and integral quadratic forms*, Springer Lecture Notes in Mathematics 1099, Heidelberg 1984.
- [Ri] J. Rickard *Morita theory for derived categories*, J. London Math. Soc. (2) 39 (1989), no. 3, 436 - 456.
- [S] U. Seidel, *On tilting complexes and piecewise hereditary algebras*, Dissertation Technische Universität Chemnitz, Chemnitz 2003.

FAKULTÄT FÜR MATHEMATIK, TECHNISCHE UNIVERSITÄT CHEMNITZ, 09107 CHEMNITZ, GERMANY. EMAIL: HAPPEL@MATHEMATIK.TU-CHEMNITZ.DE

DEPARTMENT OF MATHEMATICS, SYRACUSE UNIVERSITY, SYRACUSE, NY 13244-0001, USA. EMAIL: ZACHARIA@SYR.EDU