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On the Geometry of the Space of Fibrations

Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

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The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

On the geometry of the space of fibrations

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May 27, 2009

Abstract

This research was done at the Mathematisches Forschungsinstitut Oberwolfach during a stay within the Research in Pairs Programme from February 1 to February 14, 2009.

We study geometrical aspects of the space of fibrations between two given manifolds M and B, from the point of view of Fréchet geometry. As a first result, we show that any connected component of this space is the base space of a Fréchet-smooth principal bundle with the identity component of the group of diffeomorphisms of M as total space. Second, we prove that the space of fibrations is also itself the total space of a smooth Fréchet principal bundle with structure group the group of diffeomorphisms of the base B.

1 Introduction and results

The aim of this paper is to study some geometrical properties of the space $\mathrm{Fib}(M, B)$ of all smooth fibrations $\pi : M \to B$, with M and B smooth finite-dimensional manifolds. By "fibration" we always mean a locally trivial fiber bundle. According to Ehresmann Theorem [1], $\mathrm{Fib}(M, B)$ is nothing but the space of all smooth surjective submersions from M to B. This space is known to be an open subset of the Fréchet manifold of all smooth maps $C^\infty(M, B)$, provided M is closed (see e.g. [2], p. 85). Throughout the paper, M (and B) will always be assumed to be closed and we will study $\mathrm{Fib}(M, B)$ in the framework of Fréchet differential geometry (see [4] or [2] for comprehensive introductions).

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This work is actually part of a larger project dealing with the study of the space of Lagrangian fibrations of symplectic manifolds, in order to derive applications to the theory of Hamiltonian completely integrable systems (this is the topic of subsequent papers [7, 3]). The present article consists of a preliminary study of the general (non-Lagrangian) case.

Suppose $\mathrm{Fib}(M, B)$ is non empty and let $\pi_0 \in \mathrm{Fib}(M, B)$ be a fibration. A π_0-*vertical* diffeomorphism ϕ is a diffeomorphism of M which lifts the identity of B, i.e., which satisfies

$$\pi_0 \circ \phi = \pi_0.$$

We denote by $\mathrm{Diff}(M)$ the group of diffeomorphisms of M, by $\mathrm{Diff}_0(M)$ its identity component and by $\mathcal{G}_{\pi_0}$ the subgroup of all π_0-vertical diffeomorphisms. We prove two independent theorems on the geometry of $\mathrm{Fib}(M, B)$. The first one relates the diffeomorphism group of M and $\mathrm{Fib}(M, B)$.

Theorem 1. *Let* $\pi_0 \in \mathrm{Fib}(M, B)$.

1. *The action of* $\mathcal{G}_{\pi_0}$ *(resp.* $\mathcal{G}_{\pi_0} \cap \mathrm{Diff}_0(M)$*) on* $\mathrm{Diff}(M)$ *(resp.* $\mathrm{Diff}_0(M)$*) by right composition gives* $\mathrm{Diff}(M)$ *(resp.* $\mathrm{Diff}_0(M)$*) the structure of a Fréchet principal bundle.*
2. *The connected component of* π_0 *in* $\mathrm{Fib}(M, B)$ *is naturally Fréchet diffeomorphic to the quotient space* $\mathrm{Diff}_0(M)/(\mathcal{G}_{\pi_0} \cap \mathrm{Diff}_0(M))$.

The first and second parts of this theorem are proved respectively in Sections 3 and 4.

Remark. The action of $\mathcal{G}_{\pi_0}$ by left composition leads also to a principal bundle structure, naturally obtained from the previous one by conjugation by the inversion of diffeomorphisms.

Remark. The diffeomorphism that we construct to prove the second part of Theorem 1 is induced by the map

$$\mathrm{Diff}_0(M) \to \mathrm{Fib}(M, B), \quad \phi \mapsto \pi_0 \circ \phi^{-1}.$$

Note that in general, the analogous map defined on $\mathrm{Diff}(M)$ is not surjective onto $\mathrm{Fib}(M, B)$, so that there is no similar result without the connected components assumptions. Indeed, for some manifolds M, B one can find two fibrations $\pi_1, \pi_2 : M \to B$ with non-diffeomorphic fibers, which therefore can not satisfy $\pi_1 = \pi_2 \circ \phi$ for any $\phi \in \mathrm{Diff}(M)$. For instance, let $M = \mathbb{S}_3 \times \mathbb{S}_2$, $B = \mathbb{S}_2$, π_1 be the canonical projection onto its second factor and π_2 be the composition of the Hopf fibration

$\mathbb{S}_3 \to \mathbb{S}_2$ with the projection onto the first factor. The fiber of π_1 is then $\mathbb{S}_3$ while the fiber of π_2 is $\mathbb{S}_1 \times \mathbb{S}_2$.

One immediate consequence of Theorem 1 is that $\mathrm{Diff}_0(M)$ acts transitively on each connected component of $\mathrm{Fib}(M,B)$. This was already proved by Michor in [6] using the Nash-Moser implicit function Theorem. On the contrary, our proof is based on explicit constructions.

A corollary of this transitivity property is the following lemma, for which we can also give a very simple and direct proof:

Lemma 2. *Two fibrations π_0, π_1 lying in the same connected component of* $\mathrm{Fib}(M,B)$ *have diffeomorphic fibers.*

Proof. Since $\mathrm{Fib}(M,B)$ is a smooth Fréchet manifold, one can find a smooth loop Π in $\mathrm{Fib}(M,B)$ going through π_0 and π_1. This loop then defines a map

$$\hat{\Pi} : \mathbb{S}^1 \times M \to \mathbb{S}^1 \times B, \quad (s,x) \mapsto (s, \Pi(s)(x))$$

which is a submersion. Indeed, its differential is everywhere upper-triangular with submersive diagonal blocks. Then, according to Ehresmann Theorem [1], $\hat{\Pi}$ is a fibration and in particular its fibers are all diffeomorphic to each other. In particular the fibers of π_0 are diffeomorphic to those of π_1. ☐

The second theorem concerns the action of $\mathrm{Diff}(B)$ on $\mathrm{Fib}(M,B)$ by left composition. It was inspired by Michor's article [5] about the principal bundle structure of the space of embeddings.

Theorem 3. *Let $\pi_0 \in \mathrm{Fib}(M,B)$. Suppose that π_0 admits a global section. Then, the action of* $\mathrm{Diff}_0(B)$ *by left composition gives the connected component of π_0 in* $\mathrm{Fib}(M,B)$ *the structure of a Fréchet* $\mathrm{Diff}_0(B)$*-principal bundle.*

We will prove this theorem in Section 5.

Remark. Intuitively, two fibrations in the same orbit under the action of $\mathrm{Diff}_0(B)$ define the same foliation of M, so that the quotient space can be viewed as the space of bundle-like foliations.

Remark. Theorem 3 still holds when one replaces $\mathrm{Diff}_0(B)$ by $\mathrm{Diff}(B)$ and the connected component of π_0 by the subset of $\mathrm{Fib}(M,B)$ consisting in all fibrations admitting a global section. This subset is a union of connected components as follows easily from Theorem 1.

Remark. What happens when π_0 does not admit any global section remains an open problem.

Strangely enough, when M is symplectic and we consider the smaller space of Lagrangian fibrations, we have been able to prove the existence of the principal bundle structure (under the action of the whole $\mathrm{Diff}(B)$) without any assumption on the existence of a global section [3]. The methods used therein involve the Nash-Moser Theorem but unfortunately do not apply here.

Acknowledgements

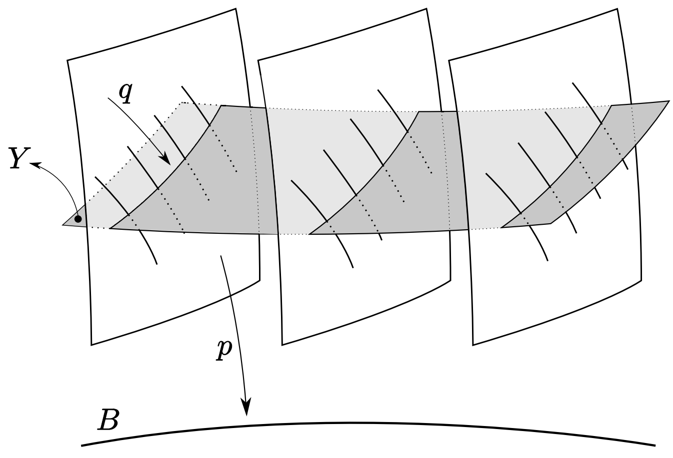
A large part of this work has been done during our "Research in Pair" stay at the MFO research center in Oberwolfach in February 2009. We thank the MFO for its warm hospitality and for the perfect working conditions it provided to us.

2 Preparation Lemmas

In this section, we prove two independent lemmas that will be useful in the proof of both Theorems 1 and 3.

Lemma 4. *Let $p : X \to B$ be a fiber bundle and $Y \subset X$ a subbundle, i.e., a submanifold of X such that the restriction of p to it defines a fiber bundle over B. Then, there exists a tubular neighbourhood U of Y, whose associated projection $q : U \to Y$ is p-vertical, i.e.,*

$$p \circ q = p.$$



Remark. Notice that in the case where $X \to B$ is a vector bundle, one can construct this tubular neighbourhood with the help of “local additions”, as defined in [5].

Proof. The usual construction of tubular neighbourhoods is the following. We first consider the normal bundle N_Y to Y, which is a vector bundle over Y. Then, with the help of a Riemannian metric on X, we can construct a diffeomorphism ϕ from a neighbourhood of the 0-section in N_Y to a neighbourhood of Y in X. Namely, one identifies any element of N_Y with a tangent vector on X which is orthogonal to Y, and the exponential map provides a point in X. Then, the projection q corresponds through the diffeomorphism ϕ to the projection $N_Y \to Y$. Unfortunately, constructed in this way, q has no reason to be p-vertical.

Instead of fixing a metric on X and using the corresponding exponential map to construct ϕ, we rather take a smooth family of metrics $(g_b)_{b\in B}$ on the fibers $X_b = p^{-1}(b)$. Such a family can be obtained for example by restricting a given metric g on X to each fiber X_b. Then, at each point $x \in Y$, we denote $b = p(x)$ and we identify $N_Y(x)$ with the vector space W_x orthogonal to $T_x(Y \cap X_b)$ in T_xX_b, with respect to the metric g_b on X_b, simply by

$$\begin{array}{ccl} W_x & \to & N_Y(x) = T_xX/T_xY \\ V & \to & [V]. \end{array}$$

This is injective since the intersection of W_x with T_xY is trivial. It is also surjective because of dimension matching. Indeed, the dimension of $N_Y(x)$ equals $\dim X - \dim Y$. On the other hand, the dimension of W_x is just $\dim X_b - \dim Y_b$, where $Y_b = Y \cap X_b$. But this is equal to $(\dim X - \dim B) - (\dim Y - \dim B)$, hence to $\dim N_Y(x)$. This defines therefore an isomorphism between W_x and $N_Y(x)$.

The construction of the diffeomorphism ϕ is as follows. For any element in $N_Y(x)$ at some point $x \in Y$, we take the corresponding vector in W_x. Then, we use the exponential map associated to the metric g_b, where $b = p(x)$, and obtain a point which lies by construction in the same fiber X_b as x does. The smoothness of this map is clear. The fact that it is a diffeomorphism from a neighbourhood of the 0-section in N_Y to a neighbourhood of Y in X, follows from the property that the linearisation at x of the exponential map is simply the identity on T_xX_b. Finally, the fact that q is p-vertical follows directly from the construction of ϕ. □

Lemma 5. *Let $\mathcal{F}$ be a Fréchet manifold together with an action of a Fréchet Lie group $\mathcal{G}$. Suppose that there exists a Fréchet space $\mathcal{E}$, such that for any $f \in \mathcal{F}$, there exist a $\mathcal{G}$-invariant neighbourhood $\mathcal{W}_f$ of f in $\mathcal{F}$, an open set $\mathcal{V}_f$ in $\mathcal{M}$ and a Fréchet diffeomorphism*

$$\Phi_f : \mathcal{W}_f \to \mathcal{G} \times \mathcal{V}_f,$$

which is equivariant under the action of $\mathcal{G}$, i.e., for all $g \in \mathcal{G}$, $\varphi \in \mathcal{W}_f$,

$$\Phi_f(g \cdot \varphi) = (g\Phi_f^1(\varphi), \Phi_f^2(\varphi)),$$

where Φ_f^1, Φ_f^2 denote the respective components of Φ_f.

Then, $\mathcal{F}$ has the structure of a Fréchet principal $\mathcal{G}$-bundle.

Proof. Let $\mathcal{Q}$ denote the quotient space of $\mathcal{F}$ under the action of $\mathcal{G}$, endowed with the quotient topology. Let us check that $\mathcal{Q}$ is a Fréchet manifold.

For any $f \in \mathcal{F}$, the set $\mathcal{V}_f$ is then homeomorphic to some open set $\mathcal{U}_f$ in $\mathcal{Q}$ and the family $(\mathcal{U}_f)_{f\in\mathcal{F}}$ covers $\mathcal{Q}$. Let us check that $\mathcal{Q}$ is Haussdorf. Let q, q' be distinct elements of $\mathcal{Q}$. Let $\mathcal{U}$ denote one element of the family $(\mathcal{U}_f)_{f\in\mathcal{F}}$ containing q. If q' is in $\mathcal{U}$, since the Fréchet manifold $\mathcal{M}$ is haussdorf, there are two disjoint open subsets in $\mathcal{U}$ (and thus in $\mathcal{Q}$), containing respectively q and q'. If q' is not in $\mathcal{U}$, then we can also find two disjoint open sets containing q and q' by taking any open neighbourhood of q in $\mathcal{U}$ whose closure is included in $\mathcal{U}$ and the complement of its closure in $\mathcal{Q}$. This is possible since a Fréchet topology is metrizable.

Now, let $q = \Phi_f^2(f)$ and $q' = \Phi_{f'}^2(f')$ be two distinct elements in $\mathcal{Q}$, with two neighbourhoods $\mathcal{U}_f, \mathcal{U}_{f'}$ such that $\mathcal{U}_f \cap \mathcal{U}_{f'} \neq \emptyset$, and such that there exist two homeomorphisms $\phi : \mathcal{U}_f \to \mathcal{V}_f$, $\phi' : \mathcal{U}_{f'} \to \mathcal{V}_{f'}$. Then, for any fixed $g \in \mathcal{G}$, the transition map can be written

$$\phi' \circ \phi^{-1}|_{\phi(\mathcal{U}_f\cap\mathcal{U}_{f'})} = \Phi_{f'}^2 \circ (\Phi_f)^{-1}|_{\{g\}\times\phi(\mathcal{U}_f\cap\mathcal{U}_{f'})}$$

and hence is smooth. Therefore, the family of sets $(\mathcal{U}_f)_{f\in\mathcal{F}}$ is a smooth Fréchet atlas for $\mathcal{Q}$.

Finally, we see that the family of maps $(\Phi_f)_{f\in\mathcal{F}}$ are smooth local equivariant trivializations whose second coordinate correspond to the natural projection $\mathcal{F} \to \mathcal{Q}$. We thus have a principal bundle structure. □

3 The principal bundle structure of $\mathrm{Diff}(M)$

In this section, we prove that given a fixed fibration $\pi_0 : M \to B$ the action

$$\begin{array}{ccc} \mathcal{G}_{\pi_0} \times \mathrm{Diff}\,(M) & \longrightarrow & \mathrm{Diff}\,(M) \\ (\psi, \phi) & \longmapsto & \phi \circ \psi \end{array}$$

gives $\mathrm{Diff}\,(M)$ the structure of a Fréchet principal bundle with structure group $\mathcal{G}_{\pi_0}$, as claimed in the first point of Theorem 1. We leave to the reader to check that the proof works if one replaces $\mathrm{Diff}(M)$ by $\mathrm{Diff}_0(M)$ and $\mathcal{G}_{\pi_0}$ by $\mathcal{G}_{\pi_0} \cap \mathrm{Diff}(M)$.

The proof is divided in three steps:

- In Section 3.1 we show that the orbits of the $\mathcal{G}_{\pi_0}$-action are Fréchet submanifolds of $\mathrm{Diff}\,(M)$ and in particular that $\mathcal{G}_{\pi_0}$ is indeed a Fréchet Lie group.
- Then, in Section 3.2 we construct a $\mathcal{G}_{\pi_0}$-invariant neighbourhood $\mathcal{U}$ of $\mathrm{Id} \in \mathrm{Diff}\,(M)$ together with a Fréchet submanifold $\mathcal{S} \subset \mathcal{U}$, transverse to the $\mathcal{G}_{\pi_0}$-orbits.
- Finally, Section 3.3 provides the construction of the local charts of the Fréchet principal bundle

$$\mathrm{Diff}\,(M) \supset \mathcal{U} \longrightarrow \mathcal{S} \times \mathcal{G}_{\pi_0}.$$

  Thanks to the transitive action of $\mathrm{Diff}\,(M)$ onto itself by left composition, we then obtain a chart near any $\phi \in \mathrm{Diff}\,(M)$.

Throughout the proof, we will use intensively the standard identification of smooth maps on M with smooth sections of $M \times M$, namely

$$\begin{array}{ccc} C^\infty(M,M) & \xrightarrow{\cong} & \Gamma(M, M\times M) \\ \phi & \longmapsto & \hat{\phi} = (\mathrm{Id}, \phi)\,. \end{array}$$

Here and always except when stated explicitly, $M \times M$ is viewed as a trivial bundle over the first factor. Notice also that through this identification, diffeomorphisms of M correspond to an open subset of $\Gamma(M, M\times M)$, which we denote by $\widehat{\mathrm{Diff}}\,(M)$.

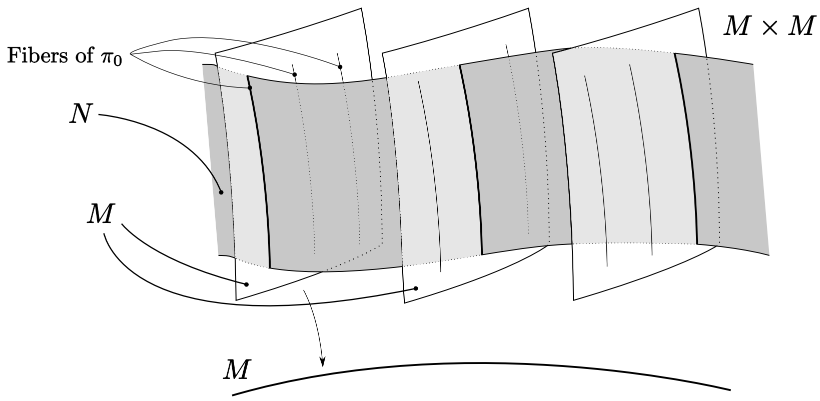
3.1 The structure group $\mathcal{G}_{\pi_0}$

We construct a subbundle $N \subset M \times M$ over M which will provide later a suitable principal bundle chart for $\mathrm{Diff}\,(M)$ around Id. We define the subset N by

$$N = \{(x,y) \in M \times M \mid \pi_0(x) = \pi_0(y)\}\,.$$

This is nothing but the pullback bundle of $M \xrightarrow{\pi_0} B$ by the (same) smooth map $\pi_0 : M \to B$, hence a subbundle of $M \times M$ over M. This subbundle N provides a nice parametrisation of the $\mathcal{G}_{\pi_0}$-orbits because of the following equivalence.

Lemma 6. *Let $\phi \in \mathit{Diff}\,(M)$ be a diffeomorphism. Then the corresponding $\hat{\phi} \in \Gamma(M, M \times M)$ lies in $\Gamma(M, N)$ if and only if $\phi \in \mathcal{G}_{\pi_0}$.*



Proof. If $\phi \in \mathcal{G}_{\pi_0}$, then for each x, $\pi_0 \circ \phi(x)$ equals $\pi_0(x)$ since ϕ is vertical with respect to π_0. But this precisely means that $(x, \phi(x))$ lies in N for all $x \in M$, i.e., $\hat{\phi} \in \Gamma(M, N)$. Conversely, if $(x, \phi(x)) \in N$ for all $x \in M$, this means by definition that $\pi_0(x) = \pi_0 \circ \phi(x)$ for all x, hence $\phi \in \mathcal{G}_{\pi_0}$. $\square$

It follows from this lemma that $\mathcal{G}_{\pi_0}$ is identified through the correspondance $\phi \mapsto \hat{\phi}$ with the intersection of the open subset $\widehat{\mathrm{Diff}}(M) \subset \Gamma(M, M \times M)$ and $\Gamma(M, N)$. On the other hand, it is well-known [2, Exp 4.2.2] that the set of sections of a subbundle is a Fréchet submanifold of the set of sections of the bundle. Therefore $\mathcal{G}_{\pi_0}$ is a Fréchet submanifold of $\mathrm{Diff}(M)$. Moreover $\mathcal{G}_{\pi_0}$ is also a subgroup of $\mathrm{Diff}(M)$, which is a Fréchet Lie group. This proves the following.

Lemma 7. *The group $\mathcal{G}_{\pi_0}$ is a Fréchet Lie group.*

Notice that the corresponding Lie algebra is $\Gamma(M, V_{\pi_0}) \subset \mathfrak{X}(M)$, where $V_{\pi_0} \subset TM$ is the π_0-vertical tangent bundle of M, i.e., $V_{\pi_0}(x) = \ker D\pi_0(x)$.

Notice also that the orbit of any $\phi_0 \in \mathrm{Diff}(M)$ is also a Fréchet submanifold of $\mathrm{Diff}(M)$. Indeed, this orbit is simply the image of $\mathcal{G}_{\pi_0}$ by the left composition map $L_{\phi_0} : \mathrm{Diff}(M) \to \mathrm{Diff}(M)$ which sends any ϕ to $\phi_0 \circ \phi$. This map is smooth [2, Exp. 4.4.5] and it inverse $L_{\phi_0^{-1}}$ as well. It is therefore a Fréchet diffeomorphism of $\mathrm{Diff}(M)$, and the result follows.

3.2 The local section $\mathcal{S}$

We now need a tubular neighbourhood of N in $M \times M$ with special properties, reflecting the fact that fibers of N over two different points $x, x' \in M$ satisfying $\pi_0(x) = \pi_0(x')$ are identified, since both are naturally identified with $\pi_0^{-1}(\pi_0(x))$.

Lemma 8. *There exists a tubular neighbourhood $U \subset M \times M$ of N, which is invariant under the action of $\mathcal{G}_{\pi_0}$ on the second factor and whose projection $P : U \to N$ has the form*

$$P(x,y) = (x, P_2(x,y))$$

with $P_2 : U \to M$ satisfying $P_2(\psi(x), y) = P_2(x,y)$ for any $\psi \in \mathcal{G}_{\pi_0}$.

Proof. In order to get the required $\mathcal{G}_{\pi_0}$-invariance property of our tubular neighbourhood inside $M \times M$, we will first make a construction in $B \times M$ and then lift it to $M \times M$.

Let us consider the trivial bundle $B \times M$ over B. Similarly as above, one defines

$$\tilde{N} = \{(b,y) \in B \times M \mid b = \pi_0(y)\},$$

which is a subbundle of $B \times M$ over B. Its fiber over b is simply $\{b\} \times \pi_0^{-1}(b)$. Now, according to Lemma 4, we can construct a tubular neighbourhood $\tilde{U} \subset B \times M$ of $\tilde{N}$ such that the fibers of its associated projection $\tilde{P} : \tilde{U} \to \tilde{N}$ are included in the fibers of $B \times M$, i.e., $\tilde{P}$ has the form $\tilde{P}(b,y) = \left(b, \tilde{P}_2(b,y)\right)$.

On the other hand, one can assume that $\tilde{U}$ is invariant under the action of $\mathcal{G}_{\pi_0}$ on the second factor of $B \times M$. Indeed, for any neighbourhood $V \subset B \times B$ of the diagonal, the set $\rho^{-1}(V)$, where $\rho : B \times M \to M$ is defined by $\rho(b,x) = (b, \pi_0(x))$, is a neighbourhood of $\tilde{N}$ in $B \times M$. Then, we can take V so small that $\rho^{-1}(V)$ is contained in $\tilde{U}$. To see this, fix a metric on $B \times M$ and consider the distance δ between $\tilde{N}$ and the boundary of $\tilde{U}$. It is non-vanishing by compactness of M and B. Then one can take V with a diameter smaller than δ, implying $\rho^{-1}(V) \subset \tilde{U}$. In other word, up to taking $\tilde{U}$ smaller, we can assume it has the form $\tilde{U} = \rho^{-1}(V)$, which is by construction invariant under the action of $\mathcal{G}_{\pi_0}$ on the second factor of $B \times M$.

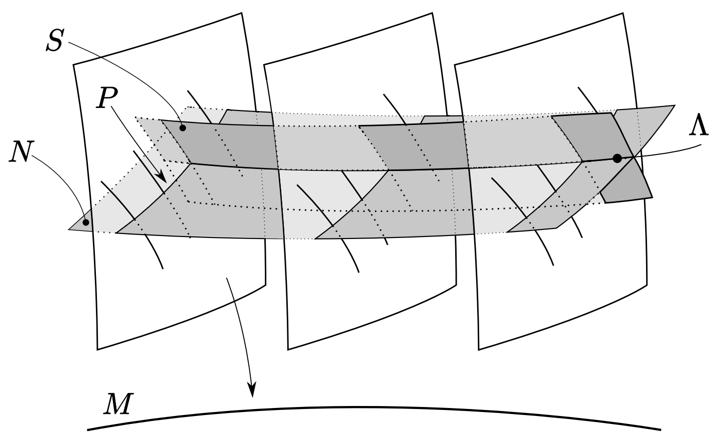
Now, if we define $\hat{\pi}_0 : M \times M \to B \times M$ by $\hat{\pi}_0(x,y) = (\pi_0(x), y)$, it follows that $\hat{\pi}_0^{-1}\left(\tilde{N}\right)$ is precisely N and that $U = \hat{\pi}_0^{-1}\left(\tilde{U}\right)$ is a neighbourhood of N. Then we check easily that the map

$$P : (x,y) \longmapsto \left(x, \tilde{P}_2 \circ \hat{\pi}_0(x,y)\right)$$

is indeed a projection from U onto N. Moreover, by construction for any $\psi \in \mathcal{G}_{\pi_0}$ and any point $x \in M$, one has $\pi_0(x) = \pi_0(\psi(x))$. This implies that

$$\tilde{P}_2 \circ \hat{\pi}_0(\psi(x), y) = \tilde{P}_2 \circ \hat{\pi}_0(x,y)$$

and therefore $P_2(\psi(x), y) = P_2(x,y)$. On the other hand, it is straightforward to check that U is invariant under the action of $\mathcal{G}_{\pi_0}$ on the second factor of $B \times M$, because $\tilde{U}$ is so. □



The next step is to construct a submanifold $\mathcal{S} \subset \mathrm{Diff}(M)$ which is transverse to the $\mathcal{G}_{\pi_0}$-orbits. We consider the subbundle $N \subset M \times M$ defined at the beginning of the section, and the associated tubular neighbourhood $U \overset{P}{\to} N$ of Lemma 8. Then we denote by $\Delta \subset M \times M$ the diagonal and by ι_Δ the associated inclusion. Clearly, Δ lies actually in N. Then we define $S = P^{-1}(\Delta)$. It is a submanifold of $M \times M$, but in fact will see that it is even a subbundle of $M \times M$ transverse to N.

To prove that it is a subbundle, first notice that S is nothing but the total space of the induced bundle $\iota_\Delta^* P$ over Δ. Then, since the restriction to Δ of the first projection $P_M : M \times M$ is a diffeomorphism $\Delta \to M$, one gets by composition a bundle $S \to M$. Finally, since $P_M \circ P = P_M$, the projection of this last bundle is nothing but the restriction of P_M to S, so that it is indeed a subbundle of $M \times M$. Since TS contains the vertical direction of the bundle P over N, S is transverse to N.

Consequently, the set of sections of S or more precisely

$$\mathcal{S} = \left\{ \phi \in \mathrm{Diff}(M) \mid \hat{\phi} \in \Gamma(M, S) \right\},$$

is a Fréchet submanifold of $\mathrm{Diff}(M)$. The following characterisation will be useful.

Lemma 9. *A diffeomorphism ϕ lies in $\mathcal{S}$ if and only if $P \circ \hat{\phi} = \hat{\mathrm{Id}}$.*

Proof. First, the left composition of a section $\hat{\phi} \in \Gamma(M, M \times M)$ by P is still a section. Therefore, $P \circ \hat{\phi}$ equals $\hat{\mathrm{Id}}$ if and only if its image $P \circ \hat{\phi}(M)$ coincides with the image of $\hat{\mathrm{Id}}$, namely Δ. But this happens precisely when the image of $\hat{\phi}$ lies in S, by definition of S. □

3.3 The principal bundle charts

We now have all the technical tools for defining the principal bundle charts on $\mathrm{Diff}(M)$. We first define a chart near Id. We consider the open set U from Lemma 8 and define the set

$$\mathcal{U} = \left\{\phi \in \mathrm{Diff}(M) \mid \mathrm{im}\left(\hat{\phi}\right) \subset U\right\}$$

which is an open neighbourhood of Id in $\mathrm{Diff}(M)$. It is also $\mathcal{G}_{\pi_0}$-invariant because of the corresponding property for U.

On the other hand, in the definition of the principal bundle chart below, we will need that the composition $P \circ \hat{\phi}$ lies in $\widehat{\mathrm{Diff}}(M)$, or equivalently that the second factor $P_2 \circ \hat{\phi}$ is a diffeomorphism. We thus have to restrict to the smaller set

$$\tilde{\mathcal{U}} = \left\{\phi \in \mathcal{U} \mid P_2 \circ \hat{\phi} \in \mathrm{Diff}(M)\right\},$$

which is open since the left composition by P_2 is a Fréchet smooth map. The set $\tilde{\mathcal{U}}$ turns out to be also $\mathcal{G}_{\pi_0}$-invariant. Indeed, for any $\phi \in \tilde{\mathcal{U}}$ and any $\psi \in \mathcal{G}_{\pi_0}$, we compute $P_2 \circ \widehat{\phi \circ \psi}$. This gives

$$P_2 \circ (\mathrm{Id}, \phi \circ \psi) = P_2 \circ (\psi, \phi \circ \psi)$$

because of the second property of P given in Lemma 8. But this is equal to $P_2 \circ \hat{\phi} \circ \psi$ which is a diffeomorphism of M since both $P_2 \circ \hat{\phi}$ and ψ are so.

Lemma 10. *The map*

$$\begin{array}{rcl} \Phi : \tilde{\mathcal{U}} & \longrightarrow & \mathcal{S} \times \mathcal{G}_{\pi_0} \\ \phi & \longmapsto & (\phi_{\mathcal{S}}, \psi), \end{array}$$

where $\hat{\psi} = P \circ \hat{\phi}$ and $\phi_{\mathcal{S}} = \phi \circ \psi^{-1}$, is well-defined and is a Fréchet smooth diffeomorphism, whose inverse is

$$\Phi^{-1} : (\phi_{\mathcal{S}}, \psi) \longmapsto \phi_{\mathcal{S}} \circ \psi.$$

Proof. First we check that this definition makes sense. Since the image of P lies in N, it follows that $\hat{\psi}$ is a section of N and, thanks to Lemma 6, that $\psi \in \mathcal{G}_{\pi_0}$. It remains to check that $\phi_{\mathcal{S}}$ lies indeed in $\mathcal{S}$. According to Lemma 9, we only need to check that $P \circ \hat{\phi}_{\mathcal{S}} = \hat{\mathrm{Id}}$, or equivalently that $P_2 \circ \hat{\phi}_{\mathcal{S}} = \mathrm{Id}$. The composition $P_2 \circ \hat{\phi}_{\mathcal{S}}$ equals $P_2 \circ \left(\mathrm{Id}, \phi \circ \psi^{-1}\right)$. Now, if we use the property of P_2 given in Lemma 8, we obtain $P_2 \circ \left(\psi^{-1}, \phi \circ \psi^{-1}\right)$ and thus $P_2 \circ \hat{\phi} \circ \psi^{-1}$. But this is exactly $\psi \circ \psi^{-1}$ and thus Id. The map Φ is therefore well defined. It is also smooth since it is made of compositions and inversion of diffeomorphisms, which are both Fréchet smooth [2, Exp. 4.4.5 and 4.4.6].

Now, the image of the claimed inverse Φ^{-1} is included in $\tilde{U}$. Indeed, since S is included in U, then $\mathcal{S}$ is included in $\mathcal{U}$. It is moreover in $\tilde{\mathcal{U}}$ thanks to Lemma 9. Finally, $\tilde{\mathcal{U}}$ is $\mathcal{G}_{\pi_0}$-invariant, hence the image of Φ^{-1} is included in $\tilde{U}$.

Then, it is straightforward to see that the claimed inverse Φ^{-1} is a left inverse of Φ. Let us check that it is also a right inverse. For any $\phi_{\mathcal{S}} \in \mathcal{S}$ and any $\psi \in \mathcal{G}_{\pi_0}$ we have to compute $(\phi'_{\mathcal{S}}, \psi') = \Phi(\phi_{\mathcal{S}} \circ \psi)$. First, we have $\psi' = P_2 \circ \widehat{\phi_{\mathcal{S}} \circ \psi}$, i.e.,

$$\psi' = P_2 \circ (\mathrm{Id}, \phi_{\mathcal{S}} \circ \psi)\,.$$

But the property of P_2 given in Lemma 8 shows that this equals $P_2 \circ \hat{\phi}_{\mathcal{S}} \circ \psi$. On the other hand, since $\hat{\phi}_{\mathcal{S}}$ lies in $\mathcal{S}$, one has $P_2 \circ \hat{\phi}_{\mathcal{S}} = \mathrm{Id}$, hence $\psi' = \psi$. Therefore, $\Phi(\phi_{\mathcal{S}} \circ \psi)$ is equal to $(\phi'_{\mathcal{S}}, \psi)$ with $\phi'_{\mathcal{S}}$ given by $\phi_{\mathcal{S}} \circ \psi \circ \psi^{-1}$ which thus coincides with $\phi_{\mathcal{S}}$. We have thus proved that the map $(\phi_{\mathcal{S}}, \psi) \mapsto \phi_{\mathcal{S}} \circ \psi$ is indeed the (double-sided) inverse of Φ. □

Therefore, near $\mathrm{Id} \in \mathrm{Diff}(M)$, we have constructed a diffeomorphism Φ from a $\mathcal{G}_{\pi_0}$-invariant neighbourhood $\tilde{\mathcal{U}}$ of Id to $\mathcal{S} \times \mathcal{G}_{\pi_0}$. It is actually $\mathcal{G}_{\pi_0}$-equivariant, as one can easily check on the inverse Φ^{-1}.

Then, near any $\phi \in \mathrm{Diff}(M)$, we can construct a similar diffeomorphism

$$\Phi_{\phi_0} : \tilde{\mathcal{U}}_{\phi_0} \longrightarrow \mathcal{S}_{\phi_0} \times \mathcal{G}_{\pi_0},$$

where $\tilde{\mathcal{U}}_{\phi_0}$ and $\mathcal{S}_{\phi_0}$ are obtained respectively from $\tilde{\mathcal{U}}$ and $\mathcal{S}$ by left composition with ϕ_0. The map Φ_{ϕ_0} is simply defined by

$$\Phi_{\phi_0}(\phi) = \left(\phi_0 \circ \Phi^{\mathcal{S}}\left(\phi_0^{-1} \circ \phi\right), \Phi^{\mathcal{G}_{\pi_0}}\left(\phi_0^{-1} \circ \phi\right)\right),$$

where $\Phi^{\mathcal{S}}$ and $\Phi^{\mathcal{G}_{\pi_0}}$ are respectively the $\mathcal{S}$ and $\mathcal{G}_{\pi_0}$ component of Φ.

Finally, it follows from Lemma 5 that the maps Φ_{ϕ_0} form a principal bundle atlas for $\mathrm{Diff}(M)$ with structure group $\mathcal{G}_{\pi_0}$.

4 The quotient space $\mathrm{Diff}_0(M)/(\mathcal{G}_{\pi_0} \cap \mathrm{Diff}_0(M))$

In this section, we prove the second part of Theorem 1.

Proof. First recall that, as already mentioned in the introduction, $\mathrm{Fib}(M, B)$ is an open subset of $C^\infty(M, B)$. Let $\pi_0 \in \mathrm{Fib}(M, B)$. The smooth map

$$\Psi : \mathrm{Diff}_0(M) \to \mathrm{Fib}(M, B), \quad \phi \mapsto \pi_0 \circ \phi^{-1}$$

induces a map

$$\tilde{\Psi} : \mathrm{Diff}_0(M)/(\mathcal{G}_{\pi_0} \cap \mathrm{Diff}_0(M)) \to \mathrm{Fib}(M, B).$$

In order to prove that $\tilde{\Psi}$ is a diffeomorphism onto the component of π_0, we will prove that it is injective, smooth, that it admits a local smooth inverse near any point and finally that its image is the connected component of π_0.

Proving the injectivity is easy: for any two diffeomorphisms $\phi_1, \phi_2 \in \mathrm{Diff}_0(M)$ satisfying $\pi_0 \circ \phi_1^{-1} = \pi_0 \circ \phi_2^{-1}$, then $\phi_1^{-1} \circ \phi_2$ obviously belongs to $\mathcal{G}_{\pi_0}$ and thus ϕ_1 and ϕ_2 represent the same element in $\mathrm{Diff}_0(M)/(\mathcal{G}_{\pi_0} \cap \mathrm{Diff}_0(M))$.

The fact that $\tilde{\Psi}$ is smooth follows from the first part of Theorem 1. Indeed, for any φ in the quotient $\mathrm{Diff}_0(M)/\mathcal{G}_{\pi_0}$, the first part of Theorem 1 implies that there exists a smooth section σ of the bundle $\mathrm{Diff}_0(M) \to \mathrm{Diff}_0(M)/\mathcal{G}_{\pi_0}$ defined on a neighbourhood of φ. On this neighbourhood, $\tilde{\Psi}$ can be written $\tilde{\Psi} = \Psi \circ \sigma$ and therefore is smooth.

To construct a local inverse to $\tilde{\Psi}$, we will make use of the following lemma which relates the spaces of sections of two different fibrations of the same manifold.

Lemma 11 (Roy [7])**.** *Let X be a smooth manifold and $p_1, p_2 : X \to M$ be two fibrations over M. Suppose there exists a common section $s_0 : M \to X$, i.e., a smooth map satisfying*

$$p_1 \circ s_0 = p_2 \circ s_0 = \mathrm{Id}_M.$$

Then, there exists open subsets $V_1 \subset \Gamma(p_1)$ and $V_2 \subset \Gamma(p_2)$ containing s_0 and such that the map

$$V_1 \to V_2, \quad \alpha \mapsto \alpha \circ (p_2 \circ \alpha)^{-1}$$

is a diffeomorphism.

For the sake of completeness, we briefly recall its proof.

Proof. First of all, the map $A : \Gamma(p_1) \to C^\infty(M, M)$, $\alpha \mapsto p_2 \circ \alpha$ is smooth and in particular continuous so that the pre-image V_1 of the open subset $\mathrm{Diff}(M) \subset C^\infty(M, M)$ is an open subset of $\Gamma(p_1)$. Moreover, $s_0 \in V_1$ since $A(s_0) = \mathrm{Id}$. The map $\alpha \mapsto \alpha \circ (p_2 \circ \alpha)^{-1}$ is well-defined on V_1 and smooth, because built of compositions and inversion of diffeomorphisms. On the other hand, the image of α is indeed a section of p_2, since $p_2 \circ \alpha \circ (p_2 \circ \alpha)^{-1} = \mathrm{Id}$.

Reversing the roles of p_1 and p_2 one construct similarly a map $\beta \mapsto \beta \circ (p_1 \circ \beta)^{-1}$, defined on some open set $V_2 \subset \Gamma(p_2)$ containing s_0. It is then easily checked that, up to taking smaller V_1 and V_2, this map and the previous one are inverse to each other. □

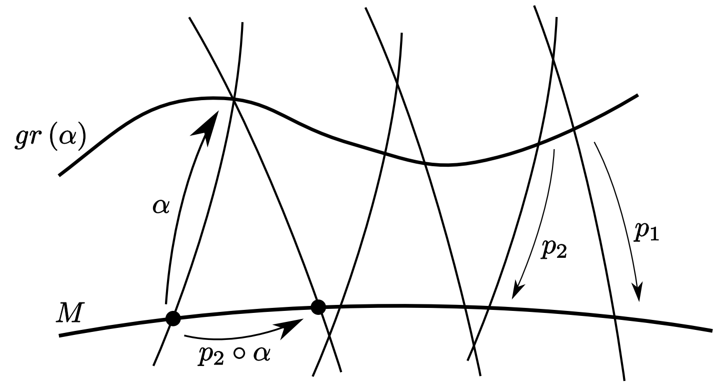


Figure 1: On this picture, the graph of s_0 is identified with M

Let us now use Lemma 11 to construct a local inverse to $\tilde{\Psi}$ around π_0. Denote by $p_M : M \times B \to M$ and $p_B : M \times B \to B$ the natural projections on the first and the second factor respectively, and $s_0 : M \to M \times B$, $x \mapsto (x, \pi_0(x))$. It turns out that the graph of s_0 is a subbundle of p_B precisely because $\pi_0 : M \to \times B$ is a submersion. Therefore, according to Lemma 4, there exists a tubular neighbourhood X of the graph of s_0 whose projection q is p_B-vertical.

We now define the two necessary fibrations for applying Lemma 11. The first one $p_1 : X \to M$ is simply the restriction of p_M to X. The second one $p_2 : X \to M$ is given by $p_2 = p_1 \circ q$. Now, we see easily that s_0, which is obviously a section of p_1, is also a section of p_2. This is because the projection q acts as the identity on the graph of s_0. Therefore, we can apply Lemma 11 and deduce that for any section $s = (\mathrm{Id}, \pi)$ of p_1 close to s_0, there exists a diffeomorphism $f \in \mathrm{Diff}(M)$ (depending smoothly on s and hence on π), such that $s \circ f$ is a section of p_2. Note by the way that since $\mathrm{Fib}(M, B)$ is open in $C^\infty(M, B)$, if s is sufficiently close to s_0, then π is a fibration. On the other hand, by construction any section of p_2 has the form (g, π_0), with $g : M \to M$. Indeed, for any $x \in M$, the fibre $p_2^{-1}(x)$ is contained in $M \times \{b\}$, with $b = \pi_0(x)$. Thus, one has

$$(f, \pi \circ f) = s \circ f = (g, \pi_0).$$

This proves in particular the crucial property $\pi \circ f = \pi_0$.

We denote $[f]$ the image of a diffeomorphism $f \in \mathrm{Diff}_0(M)$ in the quotient $\mathrm{Diff}_0(M)/(\mathcal{G}_{\pi_0} \cap \mathrm{Diff}_0(M))$. The map $\chi : \pi \mapsto [f]$ is our candidate for being the local smooth inverse of $\tilde{\Psi}$ around π_0. Indeed, for any π near π_0, one has

$$\tilde{\Psi}\left(\chi(\pi)\right) = \tilde{\Psi}\left([f]\right) = \Psi(f) = \pi_0 \circ f^{-1} = \pi.$$

Conversely, for any $[f]$ close to $[\mathrm{Id}]$, $\chi\left(\tilde{\Psi}([f])\right)$ is given by

$$\chi(\Psi(f)) = \chi\left(\pi_0 \circ f^{-1}\right) = [g],$$

where $g \in \mathrm{Diff}(M)$ satisfies $\pi_0 \circ f^{-1} \circ g = \pi_0$. But this precisely means that $f^{-1} \circ g \in \mathcal{G}_{\pi_0}$, hence $[g] = [f]$. Now, around any element $\tilde{\Psi}(\phi) = \pi_0 \circ \phi^{-1}$ in the image of $\tilde{\Psi}$, we construct the local inverse of $\tilde{\Psi}$ by $\pi \mapsto \phi \circ [\chi(\pi \circ \phi)]$. Here, the left composition of ϕ with a class $[f]$ is defined to be $[\phi \circ f]$, which makes sense because $\phi \circ (f \circ \mathcal{G}_{\pi_0}) = (\phi \circ f) \circ \mathcal{G}_{\pi_0}$. We leave to the reader to check that this is indeed a local inverse of $\tilde{\Psi}$.

Let us now finish our proof. It remains to show that the image of $\tilde{\Psi}$ is exactly the connected component of π_0 in $\mathrm{Fib}(M, B)$. First, the image of $\tilde{\Psi}$ is connected since $\mathrm{Diff}_0(M)$ is. Then, let π_1 be an element in the connected component of π_0 in $\mathrm{Fib}(M, B)$, so that there exists a path $(\pi_t)_{t\in[0,1]}$ joining them. The compactness of $[0, 1]$ allows us to find $0 = t_0 < \ldots < t_k = 1$ such that for any $i = 1, \ldots, k$, π_{t_i} belongs to the domain of the local inverse as constructed previously around $\pi_{t_{i-1}}$ (just replace π_0 with $\pi_{t_{i-1}}$). Therefore, for any $i = 1, \ldots, k$, there exists an element ϕ_i in $\mathrm{Diff}_0(M)$ such that

$$\pi_{t_i} = \pi_{t_{i-1}} \circ \phi_i.$$

Thus $\pi_1 = \pi_0 \circ (\phi_k \circ \ldots \circ \phi_1)$ which implies that π_1 lies in the image of $\tilde{\Psi}$. □

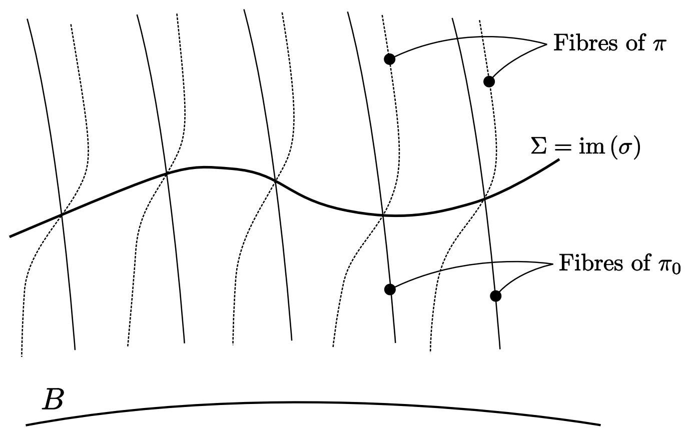
5 The principal bundle structure of $\mathrm{Fib}(M, B)$

This section is devoted to the proof of Theorem 3.

Proof . — Let $\pi_0 \in \mathrm{Fib}(M, B)$ and σ be a global section of π_0, i.e., $\pi_0 \circ \sigma = \mathrm{Id}_B$. We denote by $\mathrm{Fib}_0(M, B)$ the connected component of π_0 in $\mathrm{Fib}(M, B)$ and consider the set $\mathcal{S}$ of all fibrations for which σ is a section:

$$\mathcal{S} = \{\pi \in \mathrm{Fib}_0(M, B) \,|\, \pi \circ \sigma = \mathrm{Id}_B\}.$$

As a first step, let us prove that $\mathcal{S}$ is a Fréchet submanifold of $\mathrm{Fib}_0(M, B)$. To see this, first remark that since $\mathrm{Fib}_0(M, B)$ is a homogeneous space (Theorem 1), we can work in the neighbourhood of π_0 without lack of generality. Then, remember that there exist an open set $\mathcal{U}$ in $\mathrm{Fib}_0(M, B)$ containing π_0 and a Fréchet diffeomorphism



$\Psi : \mathcal{U} \to \mathcal{V}$ onto a neighbourhood $\mathcal{V}$ of the zero section in the Fréchet linear space of sections $\Gamma(\pi_0^* TB)$ of the vector bundle $\pi_0^* TB$ over M. Moreover, this chart Ψ can be chosen so that for all $x \in M$ and all $\pi \in \mathcal{U}$, $\pi(x) = \pi_0(x)$ if and only if $\Psi(\pi)(x) = 0$.

Now, if we denote by Σ the submanifold of M which is the image of the section σ, we see that for $\pi \in \mathcal{U}$,

$$\pi \in \mathcal{S} \quad \Longleftrightarrow \quad \pi|_\Sigma = \pi_0|_\Sigma \quad \Longleftrightarrow \quad \Psi(\pi)|_\Sigma = 0.$$

Let us consider the space $\Gamma_\Sigma(\pi_0^* TB)$ of sections of $\pi_0^* TB$ which vanish on Σ and let us show that it is a closed subspace of $\Gamma(\pi_0^* TB)$ with a closed complement.

First, $\Gamma_\Sigma(\pi_0^* TB)$ is obviously closed, so that it only remains to construct a closed complement. To any vector field χ over B, we associate the section $\widetilde{\chi}$ of $\pi_0^* TB$ defined by

$$\widetilde{\chi} = \chi \circ \pi_0.$$

Now, we consider $\mathcal{F} = \{\widetilde{\chi} \,|\, \chi \in \Gamma(TB)\}$ the set of sections of $\pi_0^* TB$ which lift vector fields on B. The set $\mathcal{F}$ is a closed linear subspace of $\Gamma(\pi_0^* TB)$. Moreover, for any $s \in \Gamma(\pi_0^* TB)$, we can consider the vector field $s \circ \sigma$ over B. In the decomposition

$$s = \big(s - \widetilde{s \circ \sigma}\big) + \widetilde{s \circ \sigma},$$

the first term lies in $\Gamma_\Sigma(\pi_0^* TB)$ while the second is in $\mathcal{F}$, hence

$$\Gamma(\pi_0^* TB) = \Gamma_\Sigma(\pi_0^* TB) + \mathcal{F}.$$

Since moreover $\Gamma_\Sigma(\pi_0^* TB)$ and $\mathcal{F}$ have trivial intersection, the sum is direct and $\mathcal{F}$ is a closed complement of $\Gamma_\Sigma(\pi_0^* TB)$. This proves our claim that $\Gamma_\Sigma(\pi_0^*TB)$ is a closed subspace of $\Gamma(\pi_0^*TB)$, with closed complement.

Consequently, the image of $\mathcal{S} \cap \mathcal{U}$ by Ψ coincides with the intersection of $\mathcal{V}$ with a closed complemented linear subspace of $\Gamma(\pi_0^*TB)$. This shows that $\mathcal{S}$ is a Fréchet submanifold of $\mathrm{Fib}_0(M, B)$.

Let us now consider $\mathcal{W} = \{\pi \in \mathrm{Fib}_0(M, B) \,|\, \pi \circ \sigma \in \mathrm{Diff}_0(B)\}$. Since the right composition by σ is a continuous map, $\mathcal{W}$ is an open subset of $\mathrm{Fib}_0(M, B)$. It is moreover invariant under the action of $\mathrm{Diff}_0(B)$ by composition on the left. We are going to show that there is an equivariant Fréchet diffeomorphism

$$\mathcal{W} \simeq \mathrm{Diff}_0(B) \times \mathcal{S},$$

where the action of $\mathrm{Diff}_0(B)$ on the right hand side is given by composition on the left on the first factor. Because of Lemma 5, this will achieve the proof of Theorem 3.

Let $\pi \in \mathcal{W}$ and set

$$\Phi(\pi) = (\pi \circ \sigma, (\pi \circ \sigma)^{-1} \circ \pi).$$

The first factor is obviously in $\mathrm{Diff}_0(B)$ and the second in $\mathcal{W} \cap \mathcal{S}$. Since composition and inversion are smooth maps Φ is also a smooth map. Moreover, the fact that Φ is equivariant is easily checked: for any $\phi \in \mathrm{Diff}_0(B)$, $\pi \in \mathcal{W}$, one has

$$\begin{aligned} \Phi(\phi \circ \pi) &= (\phi \circ \pi \circ \sigma, (\phi \circ \pi \circ \sigma)^{-1} \circ \phi \circ \pi) \\ &= (\phi \circ (\pi \circ \sigma), (\pi \circ \sigma)^{-1} \circ \pi). \end{aligned}$$

Finally, Φ has a smooth inverse given by the map $(\phi, \pi_S) \mapsto \phi \circ \pi_S$. □

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