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OWP 2009 - 17

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Approximation of Discrete Functions and Size of Spectrum

Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

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Imprint:

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The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

Approximation of discrete functions and size of spectrum

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Abstract

Let $\Lambda \subset \mathbb{R}$ be a uniformly discrete sequence and $S \subset \mathbb{R}$ a compact set. We prove that if there exists a bounded sequence of functions in Paley-Wiener space PW_S, which approximates $\delta-$functions on Λ with l^2-error d, then measure$(S) \geq 2\pi(1-d^2)D^+(\Lambda)$. This estimate is sharp for every d. Analogous estimate holds when the norms of approximating functions have a moderate growth, and we find a sharp growth restriction.

Keywords: Paley–Wiener space; Bernstein space; Set of interpolation; Approximation of discrete functions

1 Introduction

1.1. Let S be a compact set in $\mathbb{R}$, and let $m(E)$ denote the Lebesgue measure of S. By PW_S we denote the Paley–Wiener space

$$PW_S := \{f \in L^2(\mathbb{R}) : f = \hat{F}, F = 0 \text{ on } \mathbb{R} \setminus S\}$$

endowed with L^2-norm. Here $\hat{F}$ stands for the Fourier transform:

$$\hat{F}(x) := \int_{\mathbb{R}} e^{itx} F(t)\, dt.$$

By B_S we denote the Bernstein space of bounded functions f (with the sup-norm), which are the Fourier transforms of Schwartz distributions supported by S. Clearly, every function $f \in PW_S$ (and every $f \in B_S$) can be extended to an entire function of finite exponential type.

Let Λ be a uniformly discrete set, that is

$$\gamma(\Lambda) := \inf_{\lambda,\lambda' \in \Lambda, \lambda \neq \lambda'} |\lambda - \lambda'| > 0. \tag{1}$$

The restriction operator

$$f \to f|_\Lambda$$

is a bounded linear operator from PW_S into $l^2(\Lambda)$. When this operator is surjective, the set Λ is called a set of interpolation for PW_S. Similarly, if the restriction operator acts surjectively from B_S onto l^∞, then Λ is called a set of interpolation for B_S. The interpolation problem is to determine when Λ is a set of interpolation for PW_S or B_S.

The case $S = [a, b]$ is classical. Beurling and Kahane proved that in this case the answer can be essentially given in terms of the upper uniform density of Λ,

$$D^+(\Lambda) := \lim_{r\to\infty} \max_{a\in\mathbb{R}} \frac{\text{card}(\Lambda \cap (a, a + r))}{r}.$$

Namely, it was shown in [6] that the condition

$$m(S) > 2\pi D^+(\Lambda)$$

is sufficient while the condition

$$m(S) \geq 2\pi D^+(\Lambda) \tag{2}$$

is necessary for Λ to be a set of interpolation for PW_S.

The first condition above is necessary and sufficient for Λ to be a set of interpolation for B_S, see [2].

1.2. The situation becomes more delicate for the disconnected spectra. For the sufficiency part, not only the size but also the arithmetical structure of Λ is important. On the other hand, Landau [6] proved that (2) is still necessary for Λ to be a set of interpolation for PW_S, for every bounded set S.

For compact spectrum S, Landau's result can be stated in a more general form, which requires interpolation of $\delta-$functions only. For each $\xi \in \Lambda$, let δ_ξ denote the corresponding δ–function on Λ:

$$\delta_\xi(\lambda) := \begin{cases} 1 & \lambda = \xi \\ 0 & \lambda \neq \xi \end{cases}, \quad \lambda \in \Lambda.$$

Proposition 1 ([10], Theorem 1) *Let S be a compact. Suppose there exist functions $f_\xi \in PW_S$ satisfying $f_\xi|_\Lambda = \delta_\xi, \xi \in \Lambda$, and*

$$\sup_{\xi \in \Lambda} \|f_\xi\|_{L^2(\mathbb{R})} < \infty. \tag{3}$$

Then inequality (2) holds. The statement is also true for B_S-spaces.

1.3. The present paper is a direct continuation of [10]. We prove that the possibility of *approximation* of $\delta-$functions on Λ with a given l^2-error already implies an estimate from below on the measure of spectrum:

Theorem 1 *Let $0 < d < 1$, S be a compact set, and Λ be a uniformly discrete set. Suppose there exist functions $f_\xi \in PW_S$ satisfying (3) and such that*

$$\|f_\xi|_\Lambda - \delta_\xi\|_{l^2(\Lambda)} \leq d, \quad \textit{for every } \xi \in \Lambda. \tag{4}$$

Then

$$m(S) \geq 2\pi(1 - d^2)D^+(\Lambda). \tag{5}$$

Inequality (5) is sharp for every d.

In sec. 4 we show that a similar result holds when the norms of f_ξ have a moderate growth. No estimate on the measure of spectrum is possible when the norms of approximating functions have a fast growth, and we present a sharp growth restriction (see Theorem 2 below).

Clearly, by letting $d \to 0$, Theorem 1 implies the necessary condition (2) for interpolation in PW_S.

However, the possibility of $l^\infty-$approximation does not imply any estimate on the measure of S:

Proposition 2 *Given a number $0 < d < 1$ and a uniformly discrete set Λ, there exist a compact set S of measure zero and a bounded sequence of functions $f_\xi \in B_S$ satisfying*

$$\|f_\xi|_\Lambda - \delta_\xi\|_{l^\infty(\Lambda)} \leq d, \quad \textit{for every } \xi \in \Lambda.$$

The set S can be chosen depending only on d and the separation constant in (1).

Versions of the theorems above for approximation in l^p are discussed in sec. 6.

Some of results of this paper were announced in [9].

2 Lemmas

Our approach to proof of Theorem 1 includes Landau's method (see [8] and sec. 2 in [10]) and some arguments from Kolmogorov's width theory.

2.1. **Concentration.**

Definition: Given a number $c, 0 < c < 1$, we say that a linear subspace X of $L^2(\mathbb{R})$ is *c-concentrated* on a set Q if

$$\int_Q |f(x)|^2\, dx \geq c\|f\|^2_{L^2(\mathbb{R})}, \quad \text{for every} \quad f \in X.$$

Lemma 1 *Given sets $S, Q \subset \mathbb{R}$ of positive measure and a number $0 < c < 1$, let X be a linear subspace of PW_S which is c-concentrated on Q. Then*

$$dim\, X \leq \frac{m(Q)\, m(S)}{2\pi c}.$$

This lemma is contained in [8] (see statements (iii) and (iv) in Lemma 1).

2.2. **A remark on Kolmogorov's width estimate.**

Lemma 2 *Let $0 < d < 1$, and $\{\mathbf{u}_j\}, 1 \leq j \leq n$, be an orthonormal basis in an n-dimensional complex Euclidean space U. Suppose that $\{\mathbf{v}_j\}, 1 \leq j \leq n$, is a family of vectors in U satisfying*

$$\|\mathbf{v}_j - \mathbf{u}_j\| \leq d, \;\; j = 1, ..., n. \tag{6}$$

Then for every α, $1 < \alpha < 1/d$, there is a linear subspace X in $\mathbb{C}^n$ such that

(i) $dim\, X > (1 - \alpha^2 d^2)n - 1$;

(ii) the estimate

$$Q(\mathbf{c}) := \| \sum_{j=1}^{n} c_j \mathbf{v}_j \|^2 \geq (1 - \frac{1}{\alpha})^2 \sum_{j=1}^{n} |c_j|^2,$$

holds for every vector $\mathbf{c} = (c_1, ..., c_n) \in X$.

The classical equality for Kolmogorov's width of "octahedron" (see [7]) implies that the dimension of the linear span of $\mathbf{v}_j$ is at least $(1 - d^2)n$. This means that there exists a linear space $X \subset \mathbb{C}^n$, $\dim X \geq (1 - d^2)n$, such that the quadratic form $Q(\mathbf{c})$ is positive on the unite sphere of X. Lemma 2 shows that by a small relative reduction of the dimension, one can get an estimate of this form from below by a positive constant independent of n.

We are indebted to E.Gluskin for the following simple proof of this lemma.

Proof. Given an $n \times n$ matrix $T = (t_{k,l}), k, l = 1, ..., n$, denote by $s_1(T) \geq ... \geq s_n(T)$ the singular values of this matrix (=the positive square roots of the eigenvalues of TT^*).

The following properties are well–known (see [3], ch. 3):

(a) (Hilbert–Schmidt norm of T via singular values)

$$\sum_{j=1}^{n} s_j^2(T) = \sum_{k,l=1}^{n} |t_{k,l}|^2.$$

(b) (Minimax–principle for singular values)

$$s_k(T) = \max_{L_k} \min_{x \in L_k, \|x\|=1} \|Tx\|,$$

where $\| \cdot \|$ is the norm in $\mathbb{C}^n$, and the maximum is taken over all linear subspaces $L_k \subseteq \mathbb{C}^n$ of dimension k.

(c) $s_{k+j-1}(T_1+T_2) \leq s_k(T_1)+s_j(T_2)$, for all $k, j \geq 1, k+j-1 \leq n$.

Denote by T_1 the matrix, whose columns are the coordinates of $\mathbf{v}_l$ in the basis $\mathbf{u}_k$, and set $T_2 := I - T_1$, where I is the identity matrix. Then property (a) and (6) imply:

$$\sum_{j=1}^{n} s_j^2(T_2) < d^2 n,$$

and hence:

$$s_j^2(T_2) \le d^2 \frac{n}{j},\ 1 \le j \le n.$$

This and (c) give:

$$s_k(T_1) \ge s_n(I) - s_{n-k+1}(T_2) \ge 1 - d\sqrt{\frac{n}{n-k+1}}.$$

Since $s_n(I) = 1$, by setting $k = n - [\alpha^2 d^2 n]$, where $[\cdot]$ means the integer part, we obtain:

$$s_k(T_1) \ge 1 - \frac{1}{\alpha},\ k = n - [\alpha^2 d^2 n].$$

Now, one can obtain from (b) that there exists X satisfying the conclusions of the lemma.

3 Proof of Theorem 1

3.1. Observe that condition (3) implies the uniform boundedness of interpolating functions f_ξ:

$$|f_\xi(x)| = \left| \int_S F_\xi(t) e^{ixt} dt \right| \le \sqrt{m(S)} \|F_\xi\|_{L^2(\mathbb{R})} < C_1. \tag{7}$$

We shall also use the following well–known fact: given a bounded spectrum S and a uniformly discrete set S, there exists $C(S, \Lambda)$ such that

$$\sum_{\lambda \in \Lambda} |f(\lambda)|^2 \le C(S, \Lambda) \int_{\mathbb{R}} |f(x)|^2\, dx,\ \text{for every } f \in PW_S. \tag{8}$$

3.2. Fix a small number $\delta > 0$. Set $S(\delta) := S + [-\delta, \delta]$ and

$$g_\xi(x) := f_\xi(x)\varphi(x - \xi),\ \xi \in \Lambda,\ \varphi(x) := \left(\frac{\sin(\delta x/2)}{\delta x/2} \right)^2. \tag{9}$$

Clearly, $\varphi \in PW_{[-\delta,\delta]}$, so that $g_\xi \in PW_{S(\delta)}$. Also, since $\varphi(0) = 0$ and $|\varphi(x)| \le 1, x \in \mathbb{R}$, it follows from (4) that each $g_\xi|_\Lambda$ approximates δ_ξ with an l^2-error $\le d$:

$$\|g_\xi|_\Lambda - \delta_\xi\|_{l^2(\Lambda)} \le d,\ \xi \in \Lambda. \tag{10}$$

3.3. Fix numbers $a \in \mathbb{R}$ and $r > 0$, and set

$$I := (a - r, a + r), \;\; \nu = \nu(I) := \mathrm{card}(\Lambda \cap I).$$

From (1) we have:

$$\nu < C|I|. \tag{11}$$

Here and below in this proof we denote by C constants which do not depend on I.

Denote by $\lambda_1 < ... < \lambda_\nu$ the elements of $\Lambda \cap I$. It follows from (10) that the vectors

$$\mathbf{v}_j := (g_{\lambda_j}(\lambda_1), ..., g_{\lambda_j}(\lambda_\nu)) \in \mathbb{C}^\nu, \;\; j = 1, ..., \nu,$$

satisfy (6) where $\{\mathbf{u}_j, j = 1, ..., \nu\}$ is the standard orthonormal basis in $\mathbb{C}^\nu$.

Fix a number $\alpha, 1 < \alpha < 1/d$. By Lemma 2 there exists a subspace $X = X(a, r, \alpha) \subset \mathbb{C}^\nu$ such that:

(i) $\dim X > (1 - \alpha^2 d^2)\nu - 1$,

(ii) for every vector $\mathbf{c} = (c_1, c_2, ..., c_\nu) \in X$ the inequality holds:

$$\|\sum_{j=1}^{\nu} c_j \mathbf{v}_j\|^2 = \sum_{k=1}^{\nu} \left| \sum_{j=1}^{\nu} c_j g_{\lambda_j}(\lambda_k) \right|^2 \geq (1 - \frac{1}{\alpha})^2 \sum_{j=1}^{\nu} |c_j|^2.$$

Hence, we have from (8) that

$$\int_{\mathbb{R}} \left| \sum_{j=1}^{\nu} c_j g_{\lambda_j}(x) \right|^2 dx \geq C \sum_{j=1}^{\nu} |c_j|^2, \;\; (c_1, ..., c_\nu) \in X. \tag{12}$$

3.4. Set $I' := (a - r(1 + \delta), a + r(1 + \delta))$. Then, due to (7), (9) and (11), every function

$$g(x) := \sum_{j=1}^{\nu} c_j g_{\lambda_j}(x)$$

satisfies:

$$\int_{\mathbb{R}\setminus I'} |g(x)|^2 \, dx = \int_{\mathbb{R}\setminus I'} \left| \sum_{j=1}^{\nu} c_j f_{\lambda_j}(x) \left(\frac{\sin \delta(x - \lambda_j)/2}{\delta(x - \lambda_j)/2} \right)^2 \right|^2 dx$$

$$\leq C\left(\sum_{j=1}^{\nu}|c_j|^2\right)\int_{\mathbb{R}\setminus I'}\sum_{j=1}^{\nu}\frac{1}{\delta^4(x-\lambda_j)^4}\,dx$$

$$\leq C|I|\left(\sum_{j=1}^{\nu}|c_j|^2\right)\frac{1}{\delta^4}\int_{|y|>\delta r}\frac{dy}{y^4}\leq\frac{C}{\delta^7 r^2}\sum_{j=1}^{\nu}|c_j|^2. \qquad (13)$$

Fix $\epsilon>0$. Inequalities (12) and (13) show that there is a number $r_0=r(\delta,\epsilon)$ (not depending on a and $\mathbf{c}$) such that $r>r_0$ implies:

$$\int_{I'}|g(x)|^2\,dx\geq(1-\epsilon)\int_{\mathbb{R}}|g(x)|^2\,dx.$$

This means that the subspace

$$G:=\{g(x)=\sum_{j=1}^{\nu}c_j g_{\lambda_j}(x);\ (c_1,...,c_\nu)\in X\}\subset L^2(\mathbb{R})$$

is $(1-\epsilon)$-concentrated on I', provided $r>r_0$.

3.5. Clearly, $\dim G\geq\dim X$, so Lemma 1 now implies:

$$\dim X\leq\frac{m(S_\delta)|I'|}{2\pi(1-\epsilon)}.$$

Using inequality (i) for $\dim X$, we obtain:

$$(1-\alpha^2d^2)\nu-1\leq 2r(1+\delta)\frac{m(S_\delta)}{2\pi(1-\epsilon)},$$

and so

$$\frac{\operatorname{card}(\Lambda\cap(a-r,a+r))}{2r}\leq\frac{(1+\delta)m(S_\delta)}{2\pi(1-\epsilon)(1-\alpha^2d^2)}+\frac{1}{2r(1-\alpha^2d^2)}.$$

Now, for each fixed number r we choose a so that the left part is maximal, and then take limit as $r\to\infty$:

$$D^+(\Lambda)\leq\frac{(1+\delta)\,m(S_\delta)}{2\pi(1-\epsilon)(1-\alpha^2d^2)}.$$

Since this inequality is true for all positive ϵ,δ and every $\alpha\in(1,1/d)$, we conclude that estimate (5) is true.

3.6. Let us show that estimate (5) is sharp for every d. Pick up a number $a \in (0, \pi)$, and set $S := [-a, a]$, $\Lambda := \mathbb{Z}$ and

$$f_j(x) := \frac{\sin a(x-j)}{\pi(x-j)} \in PW_S, \; j \in \mathbb{Z}.$$

We have for every $j \in \mathbb{Z}$ that

$$\|f_j|_{\mathbb{Z}} - \delta_j\|^2_{l^2(\mathbb{Z})} = \|f_0|_{\mathbb{Z}} - \delta_0\|^2_{l^2(\mathbb{Z})} = \sum_{k \neq 0} \left(\frac{\sin ak}{\pi k}\right)^2 + \left(\frac{a}{\pi} - 1\right)^2 =$$

$$\frac{a}{\pi} - \frac{a^2}{\pi^2} + \left(\frac{a}{\pi} - 1\right)^2 = 1 - \frac{a}{\pi}.$$

Hence, the assumptions of Theorem 1 hold with $d^2 = 1 - a/\pi$. On the other hand, since $D^+(\mathbb{Z}) = 1$, we see that $m(S) = 2\pi(1-d^2)D^+(\mathbb{Z})$, so that estimate (5) is sharp.

4 Interpolation with moderate growth of norms

Assume that the norms of functions satisfying (4) satisfy

$$\|f_\xi\|_{L^2(\mathbb{R})} \le Ce^{|\xi|^\gamma}, \; \xi \in \Lambda, \tag{14}$$

where C and γ are some positive constants. In this section we show that the statement of Theorem 1 remains true, provided $\gamma < 1$ and the density $D^+(\Lambda)$ is replaced by the upper density $D^*(\Lambda)$,

$$D^*(\Lambda) := \limsup_{a \to \infty} \frac{\text{card}\,(\Lambda \cap (-a, a))}{2a}.$$

Restriction $\gamma < 1$ is sharp: we show also that no estimate on the measure of spectrum is possible, when the norms of f_ξ grow exponentially.

Observe that $D^*(\Lambda) \le D^+(\Lambda)$, for each Λ. However, one has $D^*(\Lambda) = D^+(\Lambda)$ whenever Λ is regularly distributed (in particular, when Λ is a bounded perturbation of integers).

Theorem 2 *Let* $0 < d < 1$.

(i) Suppose S *is a compact set and* Λ *is a uniformly discrete set. If there exist functions* $f_\xi \in PW_S$ *satisfying (4) and (14) with some* $0 < \gamma < 1$, *then*

$$m(S) \ge 2\pi(1 - d^2)D^*(\Lambda). \tag{15}$$

(ii) For every $\epsilon > 0$ and $d > 0$ there is a compact $S, m(S) < \epsilon$, a sequence $\Lambda = \{n + o(1)\}$ and functions $f_\xi \in PW_S$ that satisfy (4) and (14) with $\gamma = 1$.

Remarks. 1. The upper density in part (i) of this theorem cannot be replaced with the upper uniform density, see Theorem 2.3 in [10].

2. Similarly to [10], Theorem 2.4, one can check that the assumption $\gamma < 1$ in part (i) can be weakened by replacing it with any 'non–quasianalytic' growth. It looks likely that the assumption $\gamma = 1$ in part (ii) can be replaced with any 'quasianalytic' growth. We leave this question open.

3. The compact S in part (ii) must be disconnected. Indeed, every sequence Λ in (ii) is a sampling set for $PW_{[a,b]}$, whenever $0 < b - a < 2\pi$. Hence, the boundedness of the norms $\|f_\xi\|_{l^2(\Lambda)}$ implies (3). Therefore, due to Theorem 1, condition (4) yields: $m(S) \geq 2\pi(1 - d^2)$.

On the other hand we shall see that S can be chosen a union of two intervals.

Proof of Theorem 2.

The proof of part (i) is quite similar to the proof of Theorem 1.

4.1. Fix numbers $\delta > 0$ and β, $\gamma < \beta < 1$. There exists a function $\psi \in PW_{(-\delta,\delta)}$ with the properties:

$$\psi(0) = 1, \ |\psi(x)| \leq 1, \ |\psi(x)| \leq Ce^{-|x|^\beta}, \ x \in \mathbb{R}, \tag{16}$$

where $C > 0$ is some constant. It is well-known that such a function can be constructed as a product of $\sin(\delta_j x)/(\delta_j x)$ for a certain sequence $\delta_j \to 0$.

Set

$$h_\xi(x) := f_\xi(x)\psi(x - \xi), \ \xi \in \Lambda.$$

Then each h_ξ belongs to $PW_{S(\delta)}$ and the restriction $h_\xi|_\Lambda$ approximates δ_ξ with an l^2-error $\leq d$.

4.2. Set

$$\Lambda_r := \Lambda \cap (-r, r),$$

and denote by C different positive constants independent on r.

The argument in step 3.3 of the previous proof shows that there exists a linear space $X = X(r)$ of dimension $> (1-\alpha^2 d^2)\text{card}(\Lambda_r)-1$

such that

$$\| \sum_{\xi\in\Lambda_r} c_\xi h_\xi(x)\|^2_{L^2(\mathbb{R})} \geq C \sum_{\xi\in\Lambda_r} |c_\xi|^2,$$

for every vector $(c_\xi) \in X$.

4.3. Since Λ is uniformly discrete, we have $\mathrm{card}(\Lambda_r) \leq Cr$. Further, using (14), similarly to (7), we show that

$$|f_\xi(x)|^2 \leq Cm(S)\|f_\xi\|^2_{L^2(\mathbb{R})} \leq Ce^{C|\xi|^\gamma} \leq Ce^{Cr^\gamma},\ \xi \in \Lambda_r.$$

These estimates and (16) imply:

$$\int_{|x|\geq r+\delta r} \left| \sum_{\xi\in\Lambda_r} c_\xi h_\xi(x) \right|^2 dx =$$

$$\int_{|x|\geq r+\delta r} \left| \sum_{\xi\in\Lambda_r} c_\xi f_\xi(x)\psi(x-\xi) \right|^2 dx \leq$$

$$\left(\sum_{\xi\in\Lambda_r} |c_\xi|^2 \right) \left(Cre^{Cr^\gamma} \int_{|x|>\delta r} e^{-2|x|^\beta} dx \right).$$

Since $\beta > \gamma$, the last factor tends to zero as $r \to \infty$. This and the estimate in step 4.2 show that for every $\epsilon > 0$ there exists $r_0 = r(\delta, \epsilon)$ such that the linear space of functions

$$\{h(x) = \sum_{\xi\in\Lambda_r} c_\xi h_\xi(x);\ (c_\xi) \in X\}$$

is $(1-\epsilon)-$concentrated on $(-r-\delta r, r+\delta r)$, for all $r \geq r_0$. Moreover, the dimension of this space is at least $(1-\alpha^2 d^2)\mathrm{card}(\Lambda_r) - 1$.

4.4. By Lemma 1, we obtain:

$$m(S(\delta)) \geq \frac{2\pi(1-\epsilon)}{1+\delta} \frac{(1-\alpha^2 d^2)(\mathrm{card}(\Lambda \bigcap (-r,r)) - 1)}{2r}.$$

Take now the upper limit as $r \to \infty$:

$$m(S(\delta)) \geq \frac{2\pi(1-\epsilon)}{1+\delta}(1-\alpha^2 d^2)D^*(\Lambda).$$

Since this inequality holds for all $\epsilon > 0, \delta > 0$ and $\alpha \in (1, 1/d)$, we conclude that (15) is true.

4.5. We shall now prove part (ii) of Theorem 2. We choose S a union of two intervals and Λ a small perturbation of integers, as follows:

$$S := [-\pi - \epsilon, \pi + \epsilon] \cup [\pi - \epsilon, \pi + \epsilon], \ \Lambda := \{n + R^{-|n|-1}, n \in \mathbb{Z}\}.$$

Here $\epsilon > 0$ is a given small number and $R > 1$.

Denote by $\lambda_n := n + R^{-|n|-1}$ the elements of Λ, and set

$$f_{\lambda_0}(x) := \frac{\sin \pi x}{\sin \pi \lambda_0} \cdot \frac{\sin \epsilon (x - \lambda_0)}{\epsilon (x - \lambda_0)},$$

and

$$f_{\lambda_n}(x) := \frac{\sin \pi x}{\sin \pi \lambda_n} \cdot \frac{\sin \nu(n)(x - \lambda_n)}{\nu(n)(x - \lambda_n)} \cdot \prod_{|j| \le 2|n|, j \ne n} \frac{\sin \nu(j)(x - \lambda_j)}{\sin \nu(j)(\lambda_n - \lambda_j)}, n \ne 0,$$

where $\nu(n) := \epsilon/(4|n|+1)$. Observe that $m(S) = 4\epsilon$, so to prove part (ii) it suffices to show that the functions f_{λ_n} satisfy (4), provided R is sufficiently large.

It is clear that $f_{\lambda_n} \in PW_S$, and that we have

$$f_{\lambda_n}(\lambda_n) = 1, \ n \in \mathbb{Z}, \ f_{\lambda_n}(\lambda_k) = 0, |k| \le 2n, k \ne n, n \ne 0. \qquad (17)$$

Further, we assume that R is large enough so that the following three estimates hold for every $n \ne 0$ and every $|k| > 2|n|$:

$$\left| \frac{\sin \pi \lambda_k}{\sin \pi \lambda_n} \right| \le \frac{2\pi R^{-|k|-1}}{\pi R^{-|n|-1}} = 2R^{-|k|+|n|};$$

$$\left| \frac{\sin \nu(n)(\lambda_k - \lambda_n)}{\nu(n)(\lambda_k - \lambda_n)} \right| \le \frac{2}{\nu(n)(|k| - |n|)} \le \frac{8}{\epsilon},$$

and

$$\left| \prod_{|j| \le 2|n|, j \ne n} \frac{\sin \nu(j)(\lambda_k - \lambda_j)}{\sin \nu(j)(\lambda_n - \lambda_j)} \right| \le \prod_{|j| \le 2|n|, j \ne n} \frac{2}{\nu(j)|j - n|} \le$$

$$\left(\frac{2}{\nu(2n)} \right)^{4|n|} \frac{1}{|n|!(3|n|)!} \le \left(\frac{C}{\epsilon} \right)^{4|n|},$$

where $C > 1$ is an absolute constant. These estimates yield:

$$|f_{\lambda_n}(\lambda_k)| \le 16 \left(\frac{C}{\epsilon} \right)^{4|n|+1} R^{-|k|+|n|}, \ |k| > 2|n|, \ n \ne 0.$$

A similar estimate holds for $f_{\lambda_0}(\lambda_k)$ for each $k \ne 0$. Clearly, these estimates and (17) prove (4), provided R is large enough.

5 $l^\infty-$approximation.

5.1. In a sharp contrast to Theorem 1, the possibility of l^∞–approximation of δ–functions on Λ does not imply any restrictions on the measure of spectrum.

For approximation by $PW-$functions this follows from Lemma 3.1 in [8]: *For every $N \geq 2$ there exists a set $S(N) \subset (-N, N)$, $\operatorname{mes} S(N) = \frac{2}{N}$, such that*

$$\left| \frac{N}{2} \int_{S(N)} e^{itx}\, dt - \frac{\sin Nx}{Nx} \right| \leq \frac{C}{N},\ x \in \mathbb{R},$$

where $C > 0$ is an absolute constant.

The function $\sin Nx/Nx$ is essentially localized in a small neighborhood of the origin. Its Fourier transform is the unite mass uniformly distributed over the interval $[-N, N]$. It follows that one can re-distribute this mass over a set of small measure so that the 'uniform error' in the Fourier transform is $O(1/N)$.

For the B_S-functions, the result can be stated even in a stronger form, with the spectrum S of measure zero:

Lemma 3 *For every $\epsilon > 0$ there is a compact set $S \subset \mathbb{R}$ of Lebesgue measure zero and a function $f \in B_S$, such that*

$$f(0) = \|f\|_{L^\infty(\mathbb{R})} = 1, \quad \textit{and} \quad |f(t)| < \epsilon, \quad |t| > \epsilon.$$

This lemma is a consequence of an important Menshov's example (see [1] ch.14, sec.12, and remark in sec.18): *There is a singular probability measure ν with compact support, such that*

$$\hat{\nu}(x) \to 0, \quad |x| \to \infty.$$

Indeed, it suffices to set $f(x) = \hat{\nu}(cx)$, where c is sufficiently large.

Proposition 2 follows from Lemma 3: take a positive number $\epsilon < \min\{d, \gamma(\Lambda)\}$, where $\gamma(\Lambda)$ is defined in (1). Let f be a function from Lemma 3. Then the functions $f_\xi(x) := f(x - \xi), \xi \in \Lambda$, satisfy the assumptions of Proposition 2.

5.2. Notice that the Bernstein space B_S can be defined in a similar way for every unbounded closed spectrum S of finite measure. In [10] (see Theorem 3.1) we constructed unbounded spectra S of arbitrarily small measure such that every uniformly discrete set Λ is a set of interpolation for B_S. This was done by a certain iteration argument, using Lemma 3.1 from that paper. Using instead Lemma 3 above, one obtains the following slightly stronger result:

Proposition 3 *There is a closed set S of measure zero such that every uniformly discrete set Λ is a set of interpolation for B_S.*

Remark Assumption $m(S) = 0$ in Propositions 2 and 3 can be replaced by a stronger metrical 'thinness' condition: S may have measure zero with respect to any given Hausdorff scaling function. For such an improvement one needs to use measures ν constructed in [4].

6 B^p_S-spaces and l^p-approximation

One can include spaces PW_S and B_S into a continuous chain of Banach spaces: Given a compact set S and a number $p, 1 \leq p \leq \infty$, denote by B^p_S the space of all entire functions $f \in L^p(\mathbb{R})$ which can be represented as the Fourier transform of a distribution F supported by S. Clearly, $B^2_S = PW_S$ and $B^\infty_S = B_S$.

Observe that for $p < p'$, one has the embedding

$$B^p_S \subset B^{p'}_S \tag{18}$$

with the corresponding inequality for norms.

Let Λ be a uniformly discrete set. It is well-known that the restriction operator $f \to f|_\Lambda$ acts boundedly from B^p_S into $l^p(\Lambda)$ (see, for example, [11], p.82). Λ is called a set of interpolation for B^p_S if this operator is surjective.

Proposition 1 implies:

Proposition 4 *Let S be a compact and $p \geq 1$. If there exist functions $f_\xi \in B^p_S$ satisfying $f_\xi|_\Lambda = \delta_\xi, \xi \in \Lambda$, and*

$$sup_{\xi \in \Lambda} \|f_\xi\|_{L^p(\mathbb{R})} < \infty, \tag{19}$$

then condition (2) holds.

In particular, this shows that if Λ is a set of interpolation for B^p_S, then estimate (2) is true.

However, when considering $l^p(\Lambda)-$approximation by functions from B^p_S, one should distinguish between the following two cases:

$1 \leq p \leq 2$ and $2 < p \leq \infty$. In the first case, the measure of spectrum admits an estimate from below as in Theorem 1, while in the second case it does not as in Proposition 2:

Theorem 3 *Let $0 < d < 1$, S be a compact set and Λ be a uniformly discrete set.*

(i) Suppose $1 \leq p \leq 2$. If every $\delta_\xi, \xi \in \Lambda$, admits approximation

$$\|f_\xi|_\Lambda - \delta_\xi\|_{l^p(\Lambda)} \leq d, \;\; \xi \in \Lambda, \tag{20}$$

by functions $f_\xi \in B^p_S$ satisfying (19), then condition (5) holds true.

(ii) Suppose $p > 2$. There exist a compact set $S \subset \mathbb{R}$ of measure zero and functions $f_\xi \in B^p_S$ satisfying (19) and (20).

Part (i) is a consequence of Theorem 1, embedding (18) and the standard inequality between l^p norms.

Part (ii) follows form the refinement of Menshov's example (see [5]): *There is a singular measure ν with compact support satisfying*

$$\hat{\nu}(x) = O(|x|^{-1/2}), \quad |x| \to \infty.$$

Acknowledgment: A part of the present work was done at the Mathematisches Forschungsinstitut Oberwolfach during a stay within the Research in Pairs Programme from April 5 - 25, 2009. The authors appreciate the hospitality of the Institute.

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