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On Siegel modular forms of level p and their properties mod p

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Abstract

Using theta series we construct Siegel modular forms of level p which behave well modulo p in all cusps. This construction allows us to show (under a mild condition) that all Siegel modular forms of level p and weight 2 are congruent mod p to level one modular forms of weight $p+1$; in particular, this is true for Yoshidal lifts of level p.

In [12] Serre showed (among many other things) that elliptic modular forms of weight 2 for $\Gamma_0(p)$ are congruent mod p to level one modular forms of weight $p+1$. Our principal aim in this paper is to extend this result to Siegel modular forms under some "mild condition" (roughly speaking, the p-denominators of the Fourier expansions in the cusps different from ∞ should not grow too much). This mild condition is always satisfied in Serre's case but also for all linear combinations of degree n theta series attached to quaternary positive definite quadratic forms of determinant p^2 and level p; in other words, *all Yoshida lifts of degree* n *and level* p *are congruent mod* p *to level one Siegel modular forms of weight* $p+1$, at least if p is large compared to n. Results of this type have been quite usefull in the past, see e.g. [2, 11] and we may hope for similar applications in the Siegel case.
To obtain these results we construct Siegel modular forms of level p which are congruent to 1 mod p such that their Fourier expansions at the cusps different from ∞ are as good as possible mod p. This construction is more subtle than the corresponding one in [12] because we have now $n+1$ inequivalent cusps. We use linear combinations of theta series attached to lattices with automorphisms of order p. Sufficiently many such lattices are provided by powers of the ramified prime ideal $\mathfrak{p}$ in the cyclotomic field of p-th roots of

unity. The existence of such modular forms with prescribed behaviour mod p in all cusps may be also of independent interest.
Using these modular forms, we can now generalize Serre's technique of traces to switch form a modular form f of level p and weight k to a level one modular form g of weight $k+p-1$ such that f is congruent to g mod p, at least if f satisfies the "mild condition" mentioned above. The arithmetic of the quaternary quadratic forms involved (they are anisotropic mod p) then implies that this mild condition is automatically satisfied for all Yoshida lifts of level p.
In Section 1 we discuss the existence of sufficiently many lattices with automorphisms of order p. We then proceed to show the existence of modular forms of level p and congruent 1 mod p with prescribed behaviour mod p in the cusps different from ∞. We include here the case of real nebentypus. In section 3 we consider the trace operator and its properties mod p, in particular we discuss the "mild condition". Finally we show in section 4 that Yoshida lifts satisfy the required condions.

Parts of this work were done during a stay within the RESEARCH IN PAIRS programme at the Mathematisches Forschungsinstitut Oberwolfach from december 7 to december 20, 2008. We thank E.Bayer-Fluckiger for advice by email concerning p-special lattices. The first author also thanks Professor T.Oda for financial support in Japan (spring 2009), which allowed us to continue our work.

0 Preliminaries

0.1 Siegel modular forms

For standard facts about Siegel modular forms we refer to [1, 9, 10]. The group $Sp(n,\mathbb{R})$ acts on the upper half space $\mathbb{H}_n$ in the usual way. For an integer k, a function $f:\mathbb{H}_n \longrightarrow \mathbb{C}$ and $M=\begin{pmatrix} A & B \\ C & D\end{pmatrix}$ we define the slash operator by

$$(f\mid_k M)(Z) := det(CZ+D)^{-k} f((AZ+B)(CZ+D)^{-1}).$$

For a congruence subgroup Γ of $Sp(n,\mathbb{Z})$ and a character χ of Γ we denote by $M_n^k(\Gamma,\chi)$ the space of Siegel modular forms for Γ of weight k and character χ. If χ is trivial, we just omit it. We will mainly be concerned with congruence subgroups of type

$$\Gamma_0^n(N) := \{M = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \mid C \equiv 0 \bmod N\}$$

and with groups arising from these by conjugation within $Sp(n,\mathbb{Z})$. If $N=1$ we just write Γ^n instead of $\Gamma_0^n(1)$. The only characters of $\Gamma_0^n(N)$ occuring are those arising from Dirichlet characters mod N in the ususal way (i.e. $\chi(M) = \chi(D)$), the most important one will be the quadratic character

$$\chi_p(*) := \left(\frac{(-1)^{\frac{p-1}{2}} p}{*}\right)$$

for an odd prime p.
If f is an element of $M_k^n(\Gamma)$ for an arbitrary Γ, then f has a Fourier expansion

$$f(Z) = \sum_T a(T) e^{2\pi i tr(TZ)}$$

where T runs over positive semidefinite rational symmetric matrices with bounded denominator. In particular, for $\Gamma = \Gamma_0(N)$, T runs over positive semidefinite matrices in

$$\Lambda_n := \{T = (t_{ij}) \in Sym_n(\mathbb{Q}) \mid t_{ii}, 2t_{ij} \in \mathbb{Z}\}.$$

0.2 Traces

We need the explict from of the trace map

$$Tr : \begin{cases} M_n^k(\Gamma_0^n(p)) & \longrightarrow M_n^k(\Gamma^n) \\ f & \longmapsto \sum_\gamma f \mid_k \gamma \end{cases}$$

where γ runs over $\Gamma_0^n(p)\backslash\Gamma^n$, see also [4]. To obtain an explicit set of representatives for these cosets we start from a Bruhat decomposition over the finite field $\mathbb{F}_p$:

$$Sp(n,\mathbb{F}_p) = \cup_{j=0}^n P(\mathbb{F}_p) \cdot \omega_j \cdot P(\mathbb{F}_p),$$

where $P \subset Sp(n,\mathbb{F}_p)$ denotes the Siegel parabolic defined by $C=0$ and for $0 \le j \le n$ we put

$$\omega_j = \begin{pmatrix} 0_j & 0 & -1_j & 0 \\ 0 & 1_{n-j} & 0 & 0_{n-j} \\ 1_j & 0 & 0_j & 0 \\ 0 & 0_{n-j} & 0 & 1_{n-j} \end{pmatrix}.$$

Using the Levi decomposition $P = MN$ with Levi factor

$$M = M_n = \{m(A) = \begin{pmatrix} A & 0 \\ 0 & A^{-t} \end{pmatrix} \mid A \in GL_n(\mathbb{F}_p)\}$$

and unipotent radical

$$N = \{n(B) = \begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix} \mid B \in M_n(\mathbb{F}_p) \text{ symmetric}\}$$

we easily see that

$$\{\omega_j \cdot n(B_j) \cdot m(A) \mid B_j \in M_j(\mathbb{F}_p) \, \text{symmetric}, A \in P_{n,j}(\mathbb{F}_p)\backslash GL_n(\mathbb{F}_p)\}$$

is a complete set of representatives for the cosets $P(\mathbb{F}_p)\backslash P(\mathbb{F}_p) \cdot \omega_j \cdot P(\mathbb{F}_p)$. Here M_j is naturally embedded into M_n by $B_j \longmapsto \begin{pmatrix} B & 0 \\ 0 & 0_{n-j} \end{pmatrix}$ and $P_{n,j}$ is a standard parabolic subgroup of GL_n defined by $0^{(n-j,j)}$ being the lower left corner of g. We tacitly identify the matrices above with corresponding representatives with entries in $\mathbb{Z}$ and obtain sets of representatives for $\Gamma_0^n(n)\backslash\Gamma^n$. We analyse the contribution of fixed j to the trace of a given $f \in M_k^n(\Gamma_0(p))$: The function $f \mid_k \omega_j$ is itself a modular form for the group conjugate to $\Gamma_0^n(p)$ by ω_j, it has a Fourier expansion

$$f \mid_k \omega_j(Z) = \sum_T a_{f,j}(T) e^{2\pi i tr(TZ))}$$

where the t_{it} are integral or semi-integral except for the t_{it} in the upper left block of size j in T, where p may occur in the denominator. Then an elementary calculation using orthogonality of exponential sums shows that

$$\sum_{B_j} (f \mid \omega_j) \mid_k n(B_j)(Z) = p^{\frac{j(j+1)}{2}} \sum_{T \in \Lambda_n} a_{F,j}(T) e^{2\pi i tr(TZ)}.$$

The result of the action of the matrices $m(A)$ is:

$$\sum_{B_j, A} f \mid_k (\omega_j \cdot n(B_j) \cdot m(A)) = p^{\frac{j(j+1)}{2}} \sum_{T \in \Lambda_n} b_{F,j}(T) e^{2\pi i tr(TZ)}$$

with

$$b_{F,j}(T) = \sum_A a_{F,j}(A^{-1} T \, A^{-t}).$$

We can therefore write the contribution of a fixed j to the trace as

$$p^{\frac{j(j+1)}{2}} f \mid \omega_j \mid \tilde{U}(j),$$

where $\tilde{U}(j)$ is an operator which maps a Fourier series to a new Fourier series, where the new coefficients are certain finite sums of of the Fourier coefficients in the series we started from. The exact shape of this operator will not be important for us. We just mention the extreme cases: For $j = 0$ the operator $\tilde{U}(0)$ is just the identity and $\tilde{U}(n)$ is quite similar to the usual U-operator:

$$\tilde{U}(n) : \sum_{T \in \frac{1}{p}\Lambda_n} a(T) e^{2\pi i tr(TZ)} \mapsto \sum_{T \in \Lambda_n} a(T) e^{2\pi i tr(TZ)}.$$

With this terminology we get
Proposition 0.1: For $f \in M_n^k(\Gamma_0(p))$

$$Tr(f) = \sum_{j=0}^{n} p^{\frac{j(j+1)}{2}} (f \mid_k \omega_j) \mid \tilde{U}(j).$$

Remark: It should be clear that this expression for the trace has an analogue for the more general case of taking the trace from $\Gamma_0^n(NR)$ to $\Gamma_0^n(N)$ if N and R are coprime and R is squarefree (see e.g. [4]).

0.3 Congruences

For a prime number p we denote by ν_p the normalized additive valuation on $\mathbb{Q}$ (i.e. $\nu_p(p) = 1$). For a Siegel modular form $f \in M_n^k(\Gamma)$ with Fourier expansion $f(Z) = \sum_T a_F(T) e^{2\pi i tr(TZ)}$ we define

$$\nu_p(f) := min\{\nu_p(A_f(T) \mid T \geq 0\}.$$

We remark that this makes sense not only for modular forms with rational Fourier coefficients but also for the arbitrary case by tacitly extending the valuation to the field generated by all Fourier coefficients. For two modular forms f and g we define

$$f \equiv g \bmod p :\Longleftrightarrow \nu_p(f - g) \geq 1 + \nu_p(f).$$

We finally remark that in this setting, $\nu_p(f \mid_k \gamma)$ also makes sense for arbitrary $\gamma \in \Gamma^n$. In particular, for $f \in M_n^k(\Gamma_0(p), \chi)$ we may consider $\nu_p(f \mid_k \omega_j)$;

the Fourier expansions of $f\mid_k \omega_j$ may be viewed as "expansion of f in the cusp ω_j". (Strictly speaking, we should consider the double coset $\Gamma_0^n(p)\cdot\omega_j\cdot P(\mathbb{Z})$ as a cusp for $\Gamma_0^n(p)$; by abuse of language we will call the ω_j "the cusps for $\Gamma_0^n(p)$").

0.4 Lattices and theta series

For an even integral positive definite matrix S of size $m = 2k$ we define the degree n theta series in the usual way:

$$\theta_S^n(Z) := \sum_{R\in\mathbb{Z}^{(m,n)}} e^{\pi i tr({}^tXSXZ)} \qquad (Z \in \mathbb{H}_n).$$

We will freely switch between the languages of matrices S and corresponding lattices L and we write sometimes $\theta^n(L)$ instead of θ_S^n. For the transformation properties of such theta series see e.g. [1].
Following [6] a lattice L will be called *p-special*, if it has an isometry of order p with no fixed point in $L \setminus \{0\}$. The theta series of such a lattices automatically satisfies

$$\theta^n(L) \equiv 1 \bmod p.$$

In this paper, we consider the theta series of such p-special lattice as *principal source for congruences among modular forms.*

1 On the construction of many p-special lattices

For later applications, we need p-special lattices with many different determinants.
Proposition 1.1:[1] *Let p be an odd prime, then there are p-special (positive definite, even) lattices of rank $p-1$, level p and determinant p^t for all $1 \le t \le p-2$.*

Proof: Let ξ be a primitive p-th root of unity and consider the cyclotomic field $\mathbf{K} := \mathbb{Q}(\xi)$ together with its ring of integers $\mathcal{O}_{\mathbf{K}}$. Then the discriminant of this field is $\pm p^{p-2}$ and the dual $\mathcal{O}_K^*$ of $\mathcal{O}_{\mathbf{K}}$ (the inverse of the different)

[1]this result is based on an email communication by E.Bayer-Fluckiger

is $\mathfrak{p}^{-p+2}$ where $\mathfrak{p}$ is the unique ramified prime ideal in $\mathcal{O}_{\mathbf{K}}$. For $i \in \mathbb{Z}$ we consider $\mathfrak{p}^i$; this is a $\mathbb{Z}$- lattice of rank p-1 , equipped with the $\mathbb{Q}$ - bilinear form

$$q_i(x,y) := tr_{\mathbf{K}/\mathbb{Q}}(x \cdot \overline{y}).$$

This form is is positive definite, it is $\mathbb{Q}$- valued in general, its values are in $\mathbb{Z}$ iff

$$\mathfrak{p}^{2i} \subseteq \mathcal{O}_{\mathbf{K}}^*, \quad \text{i.e.} \quad 2i \geq 2-p.$$

The discriminant of the (fractional) ideal $\mathfrak{p}^i$ is

$$N_{\mathbf{K}/\mathbb{Q}}(\mathfrak{p}^i)^2 \cdot disc_K = p^{2i} \cdot p^{p-2}.$$

Let $2i \geq 2-p$.
Then lattice $\mathfrak{p}^i$ has level p $\iff$ the quadratic form given by $tr_{\mathbf{K}/\mathbb{Q}}$ is $\mathbb{Z}$-valued on $p \cdot (\mathfrak{p}^i)^{dual}$, i.e.

$$\left(p \cdot (\mathfrak{p}^i)^{dual}\right)^2 \subseteq \mathcal{O}_{\mathbf{K}}^*$$

Using

$$(\mathfrak{p}^i)^{dual} = \mathfrak{p}^{-i} \cdot \mathcal{O}_{\mathbf{K}}^* = \mathfrak{p}^{-i+2-p}$$

and $(p) = \mathfrak{p}^{p-1}$ we obtain

$$\mathfrak{p}^i \text{ has level } p \iff 2-p \leq 2i \leq p$$

Clearly, these lattices $\mathfrak{p}^i$ are p-special, because the multiplication with ξ defines an automorphism of order p with no fixed points (except zero).
It remains to show that the lattices $\mathfrak{p}^i$ are even for $2i \geq 2-p$:
Let $\mathbf{K}^+$ be the maximal real subfield of $\mathbf{K}$. Then for $x \in \mathfrak{p}^i$ we have $x \cdot \overline{x} \in \mathbf{K}^+$ and

$$tr_{\mathbf{K}/\mathbb{Q}}(x \cdot \overline{x}) = 2tr_{\mathbf{K}^+/\mathbb{Q}}(x \cdot \overline{x})$$

We only need to observe that

$$\mathcal{O}_{\mathbf{K}}^* \cap \mathbf{K}^+ = \mathcal{O}_{\mathbf{K}^+}^*.$$

By the definition of the dual and writing $tr_{\mathbf{K}/\mathbb{Q}} = tr_{\mathbf{K}/\mathbf{K}^+} \circ tr_{\mathbf{K}^+/\mathbb{Q}}$ we first obtain $\mathcal{O}_{\mathbf{K}^+}^* \subseteq \mathcal{O}_{\mathbf{K}}^* \cap \mathbf{K}^+ \subseteq \frac{1}{[\mathbf{K}:\mathbf{K}^+]} \cdot \mathcal{O}_{\mathbf{K}^+}$; the prime 2 however is unramified in these extensions.

The corollaries below are formulated with later applications in mind:

By taking two (orthogonal) copies of the lattices $\mathfrak{p}^i$ in the proposition above, we obtain (taking into account results from [6] and also [3, 8]):
Corollary 1.2: *Let p be an odd prime. There exist positive definite p-special even lattices of rank $2p-2$, level p and determinant p^{2t} $(1 \leq t \leq p-2)$. In particular, the number of such lattices with pairwise different determinants p^{2t} is larger or equal to the number of cusps for $\Gamma_0^n(p)$ if $n+1 \leq p-1$. In the case $p \equiv 1 \bmod 4$ we have in addition a p-special even unimodular lattice L of rank $2p-2$ together with the lattice $p \cdot L$. For $p \equiv 3 \bmod 4$ we have as a substitute for L (or better: as a substitute for its theta series) a modular form f of level 1 with $f \equiv 1 \bmod p$ together with $f(p \cdot Z)$, if $p \geq n+3$, see [6].*

The case of odd powers of p yields a slightly different result: We request p to be larger, but at the same time the rank of the lattices is $p-1$ (not $2p-2$ as in Corollary 1.2):
Corollary 1.3: *Let p be an odd prime. There exist positive definite p-special lattices of rank $p-1$, level p and determinant p^{1+2t} with $0 \leq t \leq \frac{p-3}{2}$. The number of such lattices with pairwise different determinants is larger or equal to the number of cusps for $\Gamma_0^n(p)$ if $n+1 \leq \frac{p-1}{2}$. The corresponding theta series are elements of $M_n^{\frac{p-1}{2}}(\Gamma_0^n(p), \chi_p)$ with $\chi_p = \left(\frac{(-1)^{\frac{p-1}{2}} \cdot p}{\cdot}\right)$.*

2 Existence of good modular forms of level p

We start with two examples, whose behaviour with respect to p-integrality of Fourier coefficients in the cusps is typical in some sense:
Example 1: It is easy to construct modular forms of level p such that

$$\nu_p(f\mid_k \omega_i) < \nu_p(f\mid_k \omega_j) \quad \forall i > j,$$

namely we can take $f = \theta^n(S, Z)$ with a quadratic form S in $m = 2k$ variables of level p and $\det(S) = p^r$; the transformation properties of theta series then imply $\nu_p(\theta^n(S)\mid_k \omega_i) = -\frac{ri}{2}$.

Example 2: This is more interesting: We denote by f_n any modular form of level 1, weight $p-1$ such that $f_n \equiv 1 \bmod p$. For $0 \leq i \leq n$ we put

$$h^{(i)}(Z) := f_n - p^{(p-1)\cdot i} f_n(p \cdot Z)$$

Then we easily see that

$$\nu_p(h^{(i)} \mid_{p-1} \omega_j) \begin{cases} = & 0 & \text{if} \quad i > j \\ \geq & 1 & \text{if} \quad j = i \\ = & -(j-i)(p-1) & \text{if} \quad i < j \end{cases} .$$

The case $n = i = 1$ is considered in [12]. The case $i = 0$ is extreme in some sense, because $\nu_p(h^{(0)}) \geq 1$ and for $j \geq 1$

$$\nu_p(h^{(0)} \mid \omega_j) = -j(p-1) \leq -j(p-1) - 1 + \nu_p(h^{(0)}),$$

in particular, the p-denominator of $h^{(0)} \mid \omega_n$ is very large.
These examples suggest that $\nu_p(f \mid \omega_i) > \nu_p(f \mid \omega_j)$ for $i > j$ is very rare. Therefore we call $f \in M_k^n(\Gamma_0(p))$ *regular* if the sequence $(\nu_p(f \mid \omega_i))_{(0 \leq i \leq n)}$ is non-increasing and irregular otherwise.
In view of the applications we have in mind it is desirable to construct irregular modular forms H with $H \equiv 1 \bmod p$ and $\nu_p(H \mid \omega_j)$ as large as possible for $j > 0$ in the sense that we want to maximize the $\nu_p(H \mid \omega_j)$ successively. We now present such a construction using the existence of sufficiently many p-special lattices :
Theorem 2.1: *Assume that $p \geq n$ if $p \equiv 1 \bmod 4$ or $p \geq n+3$ if $p \equiv 3 \bmod 4$. Then there exists a modular form h of level p, weight $p-1$ with the following properties:*

(A)

$$h \equiv 1 \mod p$$

(B) For $1 \leq j \leq n$

$$\nu_p(h \mid \omega_j) \geq -\frac{j(j-1)}{2} + 1$$

The Fourier expansion of $h \mid \omega_j$ has coefficients in $\mathbb{Z}$ for $j = 0$ and in $\mathbb{Z}[\frac{1}{p}]$ for $j \geq 1$.

A similar statement is true for case of real nebentypus χ_p:
Theorem 2.2: *Assume that $p \geq 2n+3$. Then there exists a modular form of weight $\frac{p-1}{2}$, level p and nebentypus χ_p such that*

(A)

$$h \equiv 1 \bmod p$$

(B) For $1 \le j \le n$

$$\nu_p(h \mid \omega_j) \ge -\frac{j(j+1)}{4} + 1$$

The Fourier expansion of $h \mid \omega_j$ *has coefficients in* $\mathcal{O}_K$ *for* $K = \mathbb{Q}(\sqrt{p})$ *for* $j = 0$ *and in* $\mathcal{O}_K[\frac{1}{\sqrt{p}}])$ *for* $j \ge 1$.

We start with a simple fact from elementary number theory:
Lemma 2.3: *Suppose that nonnegative integers*

$$\alpha(0) < \alpha(1) < \cdots < \alpha(n)$$

are given. We put

$$m(j) := \alpha(0) + \cdots + \alpha(j-1)$$

Then there are integers $A_0, \ldots, A_n$ *such that*

$$\begin{aligned} A_0 &\equiv 1 \bmod p \\ \min\{\nu_p(A_j p^{-i\alpha(j)}) \mid 0 \le j \le n\} &= -m(i) \qquad (1 \le i \le n) \\ \nu_p(\sum_{j=0}^{n} A_j p^{-i\alpha(j)}) &\ge -m(i) + 1 \qquad (1 \le i \le n) \end{aligned}$$

Proof (of the lemma): We put

$$A_j := (-1)^j p^{j\cdot\alpha(j) - m(j)}.$$

These are integers because $j \cdot \alpha(j) \ge m(j)$.
Then for $j \ge i$:

$$\begin{aligned} \nu_p(A_j \cdot p^{-i\alpha(j)}) &= -m(j) + j\alpha(j) - i\alpha(j) \\ &= -\alpha(0) - \cdots - \alpha(j-1) + (j-i)\alpha(j) \\ &\ge -\alpha(0) - \cdots - \alpha(i-1) \end{aligned}$$

with equality only for $j = i$.
Furthermorefor $j < i$:

$$\begin{aligned} \nu_p(A_j \cdot p^{-i\alpha(j)}) &= -m(j) + j\alpha(j) - i\alpha(j) \\ &= -m(i) + \alpha(j) + \ldots \alpha(i-1) - (i-j)\alpha(j) \\ &\ge -m(i) \end{aligned}$$

with equality only for $j = i-1$. Therefore in $\nu_p(\sum_{j=0}^{n} A_j p^{-i\alpha(j)})$ we only need to consider the summands with $j = i-1$ and $j = i$. The desired property follows from

$$A_{i-1}p^{-i\alpha(i-1)} + A_i p^{-i\alpha(i)} = \big((-1)^{i-1} + (-1)^i\big) \cdot p^{-m(i)}.$$

Proof of theorem 2.1:
First we consider a more general situation with arbitrary $\alpha(i)$ as in the lemma; assume that L_i are p -special lattices of level p, rank $2p-2$ and determinant $p^{2\alpha(i)}$ and we put $g_i = \theta^n(L_i)$. We remark that (see e.g. [5])

$$g_i \mid \omega_j = (-1)^{(p-1)j} p^{-j\alpha(i)}(1 + p \cdot X)$$

where X denotes a Fourier series with integral Fourier coefficients (the p in front of X is a consequence of the special automorphism of L_i).
Then we put

$$g := \sum_{j=0}^{n} (-1)^{(p-1)j} A_j \cdot g_j.$$

The lemma yields

$$g \equiv 1 \bmod p$$

and for all $1 \leq i \leq n$ we have

$$\begin{aligned} \nu_p(g \mid \omega_i) &\geq \min\{\nu_p(A_j g_j \mid \omega_i) \mid 0 \leq j \leq n\} \\ &\geq \min\{\nu_p(A_j p^{-\alpha(i)j} \mid 0 \leq j \leq n\} = -m(i) \end{aligned}$$

Moreover the constant term of $p^{m(i)} g \mid \omega_i$ is $p^{m(i)} \sum_{j=0}^{n} A_j p^{\alpha(i)j}$, hence divisible by p and therefore

$$\nu_p(g \mid \omega_i) \geq -m(i) + 1.$$

The assumptions of the theorem guarantee the existence of p-special lattices L_i with $\alpha(i) = i$ and therefore we get in this case $m(i) = \frac{(i-1)i}{2}$. If necessary (in the case $p \equiv 3 \bmod 4$) we could choose f_0^n as a substitute for g_0 and also $f_0^n(p \cdot Z)$ as substitute for g_n.

The proof of Theorem 2.2 goes along the same lines, but we must allow half-integers in the lemma. We may choose $2\alpha(i) = i+1$ and we get $m(j) = \frac{j(j+1)}{4}$ in this case.

A refinement of theorem 2.1 is possible by varying the number i as in example 2:

Theorem 2.4: *For* $1 \leq i \leq n$ *and* $p \geq n-i+1$ *if* $p \equiv 1 \bmod 4$ *and* $p \geq (n-i+1)+3$ *if* $p \equiv 3 \bmod 4$ *there is* $H^{(i)} \in M_n^{p-1}(\Gamma_0(p))$ *such that*

$$\begin{aligned} H^{(i)} \mid \omega_j &\equiv 1 \bmod p \quad (0 \leq j < i) \\ H^{(i)} \mid \omega_i &\equiv 0 \bmod p \\ \nu_p(H^{(i)} \mid \omega_j) &\geq -\frac{(j-i)(j-i-1)}{2} + 1 \quad (j > i) \end{aligned}$$

Proof: The case $i = 1$ is just Theorem 2.1.For the general case we apply Theorem 2.1 for degree n-i+1: With the notion of the proof of that theorem we have

$$g := \sum_{t=0}^{n-i+1} A_t \cdot \theta^{n-i+1}(L_i)$$

Then indeed $g \equiv 1 \bmod p$ and

$$\nu_p(g \mid \omega_j) \geq \frac{j(j-1)}{2} + 1 \qquad (j \geq 1)$$

It is easy to see that

$$H^{(i)} := \sum_{t=0}^{n-i+1} (-1)^{(i-1)(p-1)} p^{t(i-1)} A_t \cdot \theta^n(L_t)$$

has the requested properties.

Remark 1: An inspection of the proof of the lemma shows the results above are the best possible if we allow as a resource for congruences only the action of order p- automorphisms of our lattices (we omit details here).

Remark 2: Some further variants of theorem 3 are possible: For given numbers $0 \leq i' < i$ we may construct modular forms $H^{i',i}$ of weight $p-1$ and level p such that $H \mid \omega_j$ is divisible by p for $j < i'$, $H \mid \omega_j \equiv 1 \bmod p$ for $i' \leq j < i$, $H \mid \omega_i \equiv 0 \bmod p$ and maximized values of $\nu_p(H \mid \omega_j)$ for $j > i$.

Remark 3: It seems very difficult to construct (for $n \geq 2$) an irregular modular form f such that $f \equiv 1 \bmod p$ and $f \mid \omega_j \equiv 0 \bmod p$ for all $j > 0$. Indeed, it seems that for an irregular modular form f the p-denominators grow "rapidly after the occurence of the irregularity".

3 Changing the weight by $p-1$ or by $\frac{p-1}{2}$

We put

$$M_n^k(\Gamma_0^n(p))^0 := \{F \in M_n^k(\Gamma_0^n(p) \mid \nu_p(F \mid \omega_j) > -j-1+\nu_p(F)\}$$

and for $\chi_p(-1)^n = (-1)^{nk}$

$$M_n^k(\Gamma_0^n(p),\chi_p)^0 := \{F \in M_n^k(\Gamma_0(p),\chi_p) \mid \nu_p(F \mid \omega_j) > -\frac{j(j+1)}{4}-1+\nu_p(F)\}.$$

We remark that these sets are in general not vector spaces ! The aim in this part is

Theorem 3.1: *Assume that* $p \geq n$ *if* $p \equiv 1 \bmod 4$ *or* $p \geq n+3$ *if* $p \equiv 3 \bmod 4$.
Then for all $F \in M_n^k(\Gamma_0^n(p))^0$ *there is* $G \in M_n^{k+p-1}(\Gamma^n)$ *with*

$$F \equiv G \bmod p.$$

Proof: Let $h \in M_n^{p-1}(\Gamma_0^n(p)$ be as in Theorem 1 of the previous section.
We consider the trace

$$tr(F \cdot h) = F \cdot h + \sum_{j=1}^{n} p^{\frac{j(j+1}{2}}(F \cdot h) \mid \omega_j \mid \tilde{U}(j)$$

Then the contributions for $j \geq 1$ are all congruent zero mod$p$.

Theorem 3.2: *Assume that* $p \geq 2n+3$.
Then for all $F \in M_n^k(\Gamma_0^n(p),\chi_p)^0$ *there is* $G \in M_n^{k+\frac{p-1}{2}}(\Gamma^n)$ *with*

$$F \equiv G \bmod p$$

The proof works along the same lines of reasoning as in Theorem 3.1, using the h from Theorem 2.2.

Remarks:

- The set of modular forms satisfying the conditions of the Theorems is *not* a vector space in general.

- Clearly certain theta series do satisfy the conditions above, namely if $\det(T) = p^2$ in the first case and $\det(T) = p$ in the second case. For slightly different statements about theta series we refer to Skoruppa [13].

- A statement similar to theorem 2 can be found in [11] for the special case of degree 1.

- In [2] we treated a similar situation for degree 1. There is was easy to apply the theorem also for situations where the condition on $\nu_p(F \mid_k \omega_1)$ was not satisfied: We just enlarged the weight of the function h by taking an appropriate power of h. This does no longer work in our case because $\nu_p(h \mid \omega_j)$ is negative for $j \geq 2$. The same is true in the case of nebentypus.

- To get results like Theorem 3.1 for congruences modulo p^l arbitrary $l \geq 1$ one should use a somewhat different method, which does in general not give exact information about the weight of the level one modular form, see [7].

It is remarkable that the full space generated by quaternary theta series of determinant p^2 satisfies the condition above:
Definition: For a prime p we put

$$Y^n(p) := \mathbb{C} < \theta^n(S) >,$$

where S runs over all positive definite quaternary quadratic forms of level p and determinant p^2. This is precisely the space of "Yoshida liftings" , see [14, 5]. Indeed, we will show in the next section that

$$Y^n(p) \subseteq M_n^2(\Gamma_0^n(p))^0.$$

From this we obtain
Corollary 3.3: *Assume that* $p \geq n$ *if* $p \equiv 1 \bmod 4$ *or* $p \geq n+3$ *if* $p \equiv 3 \bmod 4$.
Then all elements of the space $Y^n(p)$ *of Yoshida liftings are congruent* $\bmod\, p$ *to modular forms of level one of weight* $p+1$.

Remark: Assume that n is greater or equal to 5. Then the Yoshida lifts are singular modular forms [9]. The Corollary asserts that we have found degree n modular forms of level one, weight $p+1$ such that all their Fourier coefficients $a(T)$ with T of rank greater than 4 are congruent zero mod p.

Remark: The theory of singular modular forms [9] assures that there are no modular forms of level 1, weight $p+1$ and degree $n > \frac{p+1}{2}$ if $p+1 \equiv 2 \bmod 4$. Therefore it is natural that there is some condition on n and p in the Corollar We do not know whether the bound $p \geq n$ is best possible for $p \equiv 1 \bmod 4$.

4 On certain theta series and their behaviour in the cusps

It follows from the standard transformation properties of theta series that

$$\theta_S^n \mid_k \omega_j = (-1)^{jk} det(S)^{-\frac{j}{2}} \sum_{(X,Y)\in S^{-1}\mathbb{Z}^{(m,j)}\times\mathbb{Z}^{(m,n-j)}} e^{\pi i tr((X,Y)^t S(X,Y)\cdot Z)}.$$

This is true for S positiv definite, of even rank $m = 2k$. In particular, if S has determinant p^ν

$$\nu_p(\theta^n \mid_k \omega_j) = -\frac{j\nu}{2}.$$

This property may get lost, if we consider linear combinations of theta series. Certain theta series however behave well in this respect:
We consider positive definite quadratic forms of even rank $m = 2k$, level p and determinant p^ν with the additional property

$$\{\mathfrak{x} \in \mathbb{Z}^m \mid S[\mathfrak{x}] \equiv 0(p)\} = (p \cdot S^{-1}) \cdot \mathbb{Z}^m \qquad (*)$$

We denote by $\Theta_k^n(p)$ the space of linear combinations of such theta series.

Proposition 4.1 :
For $1 \leq j \leq n$ we decompose $Z \in \mathbb{H}_n$ as

$$Z = \begin{pmatrix} \tau & z \\ z^t & w \end{pmatrix} \qquad (\tau \in \mathbb{H}_j, w \in \mathbb{H}_{n-j})$$

Then we have for all $f \in M_k^n(\Gamma_0(p))^0$ and all j

$$f \mid_k \omega_j(\begin{pmatrix} p\tau_1 & z \\ z^t & \frac{1}{p}w \end{pmatrix}) = (-1)^{jk} p^{-\frac{j\nu}{2}} f(\frac{1}{p}Z) \mid \tilde{U}^j(p).$$

In particular, all $f \in M_k(\Gamma_0(p))^0$ *satisfy*

$$\nu_p(f \mid \omega_j) \geq -\frac{j\nu}{2} + \nu_p(f) \qquad\qquad (0 \leq j \leq n).$$

Here $\tilde{U}^j(p)$ acts on periodic functions defined on $\mathbb{H}_n$ which are periodic for $p \cdot Sym_n(\mathbb{Z})$ by

$$f = \sum_T a(T) exptr(2\pi i tr(\frac{1}{p}TZ) \longmapsto f \mid \tilde{U}^j(p) = \sum_{T, t_1 \equiv 0(p)} a(T) exptr(\frac{1}{p}TZ)$$

and t_1 denotes the symmetric matrix of size j in the upper left corner of T.

Proof: It suffices to prove this proposition for theta series. We compute both sides separately for a theta series θ_S^n:

$$\theta_S \mid_k \omega_j = (-1)^{jk} p^{-\frac{j\nu}{2}} \sum_{(\mathfrak{x},\mathfrak{y}) \in \mathbb{Z}^{m,j} \times \mathbb{Z}^{m,n-j}} exptr(\pi i \begin{pmatrix} S^{-1}[\mathfrak{x}] & \mathfrak{x}^t \cdot \mathfrak{y} \\ \mathfrak{y}^t \cdot \mathfrak{x} & S[\mathfrak{y}] \end{pmatrix} \cdot Z)$$

and on the other hand

$$\theta_S^n(\frac{1}{p} \cdot Z) \mid \tilde{U}^j(p) =$$

$$\sum_{\mathfrak{x},\mathfrak{y}, S[\mathfrak{x}] \equiv 0(p)} exptr(\pi i \frac{1}{p} \begin{pmatrix} S[\mathfrak{x}] & \mathfrak{x}^t S \mathfrak{y} \\ \mathfrak{y}^t S \mathfrak{x} & S[\mathfrak{y}] \end{pmatrix} \cdot Z).$$

By the property (*) mentioned before we can write such $\mathfrak{x}$ as

$$\mathfrak{x} = pS^{-1} \cdot \mathfrak{z} \qquad (\mathfrak{z} \in \mathbb{Z}^{m,j}).$$

Then

$$S[\mathfrak{x}] = S[pS^{-1}\mathfrak{z}] = p^2 S^{-1}[\mathfrak{z}]$$
$$\mathfrak{x}^t S \mathfrak{y} = p \mathfrak{z}^t \mathfrak{y}$$

and hence

$$\theta_S^n(\frac{1}{p} \cdot Z) \mid \tilde{U}^j(p) = \sum_{\mathfrak{z},\mathfrak{y}} exptr(\pi i \begin{pmatrix} (pS^{-1}[\mathfrak{z}] & \mathfrak{z}^t \cdot \mathfrak{y} \\ \mathfrak{y}^t \cdot \mathfrak{z} & \frac{1}{p} S[\mathfrak{y}] \end{pmatrix} \cdot Z).$$

Remark: *The condition (*) is essentially a local condition. It is satisfied if S is anisotropic mod p, in particular, it is satisfied for positive definite quaternary quadratic forms of level p determinant p^2 and we have the inclusion already mentioned in section 3:*

$$Y^n(p) = \Theta_2^n(p) \subseteq M_n^2(\Gamma_0^n(p))^0.$$

Remark: There is also a version of Proposition 4.1 for theta series with harmonic polynomials.

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