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On generalizations of Kac-Moody groups

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Abstract

In [7] we define a Curtis-Tits group as a certain generalization of a Kac-Moody group. We distinguish between orientable and non-orientable Curtis-Tits groups and identify all orientable Curtis-Tits groups as Kac-Moody groups associated to twin-buildings. We mention that non-orientable Curtis-Tits groups exist. In the present paper we construct families of orientable and non-orientable Curtis-Tits groups. The resulting groups are quite interesting in their own right. The orientable ones are related to Drinfel'd' s construction of vector bundles over a non-commutative projective line and to the classical groups over cyclic algebras. The non-orientable ones are related to q-CCR algebras in physics and have symplectic, orthogonal and unitary groups as quotients.

1 Introduction

In [25], the author defines Kac-Moody groups to be groups with a twin-root datum, which implies that they are symmetry groups of Moufang twin-buildings. A celebrated theorem of Curtis and Tits on groups with finite BN-pair (later extended by P. Abramenko and B. Mühlherr to Kac-Moody groups [2]) shows that by and large these groups are determined by their local structure, that is by an amalgam of rank 2 algebraic groups.

This motivated the study in [7] of what we call Curtis-Tits groups. These are groups that are the universal completion of an amalgam of rank 2 whose groups are copies of $\mathrm{SL}_2(\mathrm{k})$ and $\mathrm{SL}_3(\mathrm{k})$, where the inclusions of such groups are described by a Dynkin diagram. Examples arise naturally from the Curtis-Tits theorem. However in loc. cit. it was shown that the same diagrams in fact describe a wider class of amalgams, called Curtis-Tits amalgams. In fact the Curtis-Tits amalgams coming from Kac-Moody groups via the Curtis-Tits theorem can be viewed as "orientable" amalgams in the sense that one can coherently pick a "set of positive roots". However, it was shown that there also exist non-orientable Curtis-Tits amalgams. In fact we have the following classification result:

Theorem 1.1 *Let Γ be a simply laced Dynkin diagram with no triangles and* k *a field with at least 4 elements. There is a natural bijection between isomorphism classes of Curtis-Tits amalgams over the field* k *on a graph Γ and elements of the set $\{\Phi\colon \pi(\Gamma, i_0) \to \langle\tau\rangle \times \mathrm{Aut}(\mathrm{k}) \mid \Phi$ is a group homomorphism$\}$, where τ has order 2.*

Here, $\pi(\Gamma, i_0)$ denotes the fundamental group of the graph Γ with base point i_0. The orientable Curtis-Tits amalgams are exactly those for which the image of Φ lies inside $\mathrm{Aut}(\mathrm{k})$.

It is not at all immediate that all the amalgams arising from Theorem 1.1 are non-collapsing, i.e. that their universal completion is non-trivial. We shall call a non-trivial group a Curtis-Tits group if it is the universal completion of a Curtis-Tits amalgam. The purpose of the present paper is to construct orientable and non-orientable Curtis-Tits groups of type $\widetilde{A}_{n-1}$ and to study their properties. More precisely, we prove the following.

Theorem 1 *There is a natural bijection between Curtis-Tits groups of type $\widetilde{A}_{n-1}$ and $\mathrm{Aut}(\mathrm{k}) \times \langle\tau\rangle$, where τ has order 2. Moreover, those corresponding to elements of the torsion subgroup of $\mathrm{Aut}(\mathrm{k}) \times \langle\tau\rangle$ appear as subgroups of a Kac-Moody group of type $\widetilde{A}_{m-1}$ for some positive integer m.*

The resulting groups are quite interesting in their own right. The orientable ones are related to Drinfel'd' s construction of vector bundles over a non-commutative projective line and to the classical groups over cyclic algebras. The non-orientable ones are related to q-CCR algebras in physics and have symplectic, orthogonal and unitary groups as quotients. The reader only interested in applications will find a brief description in Section 2. We note here that some of these groups have been studied in a different context, namely that of abstract involutions of Kac-Moody groups [13]. In that paper, also connectedness, but not simple-connectedness, of geometries such as those defined in Section 6 is proved.

For technical reasons, in this paper we concentrate on describing the groups that correspond to elements of Aut(k) and the element τ respectively. The general mixed case is obtained by combining the two constructions.

The paper is structured as follows. In Section 2 we introduce the Curtis-Tits groups and list some surprising connections to number theory, finite groups and theoretical physics. This chapter is independent of the rest of the paper. In Section 3 we introduce the relevant notions about amalgams and in Section 4 we specialize to the case $\Gamma = \widetilde{A}_{n-1}$ and describe all possible amalgams. Section 5 deals with the description of the universal completion of orientable Curtis-Tits amalgams and Section 6 does the same for the non-orientable amalgam corresponding to τ.

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2 The Curtis-Tits groups and some applications

2.1 The orientable Curtis-Tits groups $\mathrm{SL}_n(A)$

Consider the ring $A = \mathrm{k}\{t, t^{-1}\}$ of skew Laurent polynomials. More precisely if $x \in \mathrm{k}$, then $t^{-1}xt = x^\delta$ for some fixed automorphism δ of k.

In Section 5 we will construct a group $G \leq \mathrm{GL}_n(A)$ and show that it admits a copy of the Curtis-Tits amalgam corresponding to the automorphism δ. Moreover, if the order of δ is finite we show that it can be regarded as $\mathrm{SL}_n(A)$ for a coherent definition of a determinant $\det_R$, and that it is the universal completion of the Curtis-Tits amalgam, hence a Kac-Moody group.

The objects of the twin-building associated to this Kac-Moody group correspond to vector bundles over the non-commutative projective line $\mathbb{P}^1(\delta)$ in the sense of Drinfel'd. More precisely, let $\mathrm{k}\{t\}, \mathrm{k}\{t^{-1}\} \leq \mathrm{k}\{t, t^{-1}\}$ be the corresponding skew polynomial rings and fix $\mathbf{M}$ a free $\mathrm{k}\{t, t^{-1}\}$ module of rank r. Following [14] and [23] one can define a rank r vector bundle over the non-commutative projective line $\mathbb{P}^1(\delta)$ as a collection $(M_+, M_-, \phi_+, \phi_-)$ where M_ε is a free r-dimensional module over $\mathrm{k}\{t^\varepsilon\}$ and $\phi_\varepsilon \colon M_\varepsilon \otimes \mathrm{k}\{t, t^{-1}\} \to \mathbf{M}$ is an isomorphism of $\mathrm{k}\{t, t^{-1}\}$-modules. By analogy to the commutative case (see [18, 19] for example) one can describe the building structure in terms of these vector bundles. We intend to explore these relations to number theory in a future paper.

To give a different perspective on these groups we note that the skew Laurent polynomials are closely related to cyclic algebras as defined by Dickson. More precisely let $\mathrm{k}' \leq \mathrm{k}$ be a cyclic field extension, of degree n, and let δ be the generator of its Galois group. Given any $a \in \mathrm{k}'$, define the k'-algebra $(\mathrm{k}/\mathrm{k}', \delta, a)$ to be generated by the elements of k, viewed as an extension of k', together with some element u subject to the following relations:

$$u^n = a, xu = ux^\delta \text{ for } x \in \mathrm{k}.$$

These algebras are central simple algebras. The celebrated Brauer-Hasse-Noether theorem states that every central division algebra over a number field k' is isomorphic to $(\mathrm{k}/\mathrm{k}', \delta, a)$ for some k, a, δ.

For each $a \in \mathrm{k}'$ one constructs the map $\epsilon_a : \mathrm{k}\{t, t^{-1}\} \to (\mathrm{k}/\mathrm{k}', \delta, a)$ via $t \mapsto u$. This induces a map $\epsilon_a\colon \mathrm{SL}_n(A) \to \mathrm{SL}_n((\mathrm{k}/\mathrm{k}', \delta, a))$, realizing the linear groups over cyclic algebras as completions of the Curtis-Tits amalgams.

2.2 The non-orientable groups G^τ

Let V be a free $\mathrm{k}[t, t^{-1}]$-module of rank $2n$ with basis $\{e_i, f_i \mid i = 1, \ldots, n\}$. In this case $\mathrm{k}[t, t^{-1}]$ denotes the ring of commutative Laurent polynomials in the variable t over a field k. The group G^τ is the isometry group of the unique non-symmetric σ-sesquilinear form β on V with the property that $\beta(e_i, e_j) = \beta(f_i, f_j) = 0, \beta(e_i, f_j) = t\delta_{ij}$ and $\beta(f_i, e_j) = \delta_{ij}$ where $\sigma\colon \mathrm{k}[t, t^{-1}] \to \mathrm{k}[t, t^{-1}]$ is the identity on k and interchanges t and t^{-1}. More precisely

$$G^\tau := \{g \in \mathrm{SL}_{2n}(\mathrm{k}[t, t^{-1}]) | \forall x, y \in V, \beta(gx, gy) = \beta(x, y)\}$$

In Section 6 we prove that G^τ is the Curtis-Tits group corresponding to the element τ from Theorem 1.

It turns out that the group G^τ has some very interesting natural quotients and that its action on certain Clifford-like algebras are related to phenomena in quantum physics.

Let $\overline{\mathrm{k}}$ denote the algebraic closure of k. For any $a \in \overline{\mathrm{k}}^*$ consider the specialization map $\epsilon_a\colon \mathrm{k}[t, t^{-1}] \to \overline{\mathrm{k}}$ given by $\epsilon_a(f) = f(a)$. The map induces a homomorphism $\epsilon_a\colon \mathrm{SL}_{2n}(\mathrm{k}[t, t^{-1}]) \to \mathrm{SL}_{2n}(\mathrm{k}(a))$. In some instances the map commutes with the automorphism σ and so one can define a map $\epsilon_a\colon G^\tau \to \mathrm{SL}_{2n}(\overline{\mathrm{k}})$

The most important specialization maps are those given by evaluating t at $a = \pm 1$ or $a = \zeta$, a $(q^m + 1)$-st root of 1 where q is a power of the characteristic.

Consider first $a = -1$. In this case the automorphism σ becomes trivial. Note that for $g \in G^\tau$ we have $\epsilon_{-1}(g) \in \mathrm{Sp}_{2n}(\mathrm{k})$. In this case, the image of the group G^τ is the group generated by the Curtis-Tits amalgam $\mathcal{A}^\tau$ inside $\mathrm{Sp}_{2n}(\mathrm{k})$. Preliminary studies suggest that we have equality. Similarly, if $a = 1$, the automorphism σ is trivial and the map ϵ_1 takes G^τ into $\mathrm{SO}_{2n}^+(\mathrm{k})$. Preliminary results suggest that in fact the image of this map is $\Omega_{2n}^+(\mathrm{k})$.

Finally assume that $\mathrm{k} = \mathbb{F}_q$ and $a \in \overline{\mathbb{F}}_q$ is a primitive $(q+1)$-st root of 1. The $\mathbb{F}_q$-linear map $\mathbb{F}_q(a) \to \mathbb{F}_q(a)$ induced by σ sends a to a^{-1}. Thus, σ coincides with the Frobenius automorphism of the field $\mathbb{F}_q(a) = \mathbb{F}_{q^2}$. It is easy to verify that a change of coordinates $e_i' = e_i$ and $f_i' = bf_i$ where $b^2 = a$ standardizes the Gram matrix of $\beta \circ (\epsilon_a \times \epsilon_a)$ to a hermitian one, thus idenfying the image of ϵ_a with a subgroup of a conjugate of the unitary group $\mathrm{SU}_{2n}(q)$. Again, preliminary results suggest that in fact the image of this map is isomorphic to $\mathrm{SU}_{2n}(q)$. This easily generalizes to the case where a is a $(q^m + 1)$-st root of unity and indeed to other cases where a is Galois-conjugate to a^{-1}.

An intriguing connection comes from mathematical physics, where the form β has been considered in the context of q-CCR algebras (see for example [12, 3]). The related infinite dimensional Clifford algebra is a higher GK-dimensional version of Manin's quantum plane. This algebra is related to both the Clifford algebra of the orthogonal groups and the Heisenberg algebra for the symplectic groups in a similar fashion.

These application will be discussed in more detail in an upcoming paper.

3 CT-groups

In this section we introduce the notion of a Curtis-Tits group over a commutative field and define their category. Throughout the paper k will be a commutative field.

Definition 3.1 *Let V be a vector space of dimension* 3 *over* k. *We call (S_1, S_2) a* standard pair *for $S = \mathrm{SL}(V)$ if there are decompositions $V = U_i \oplus V_i$, $i = 1, 2$, with $\dim(U_i) = 1$ and $\dim(V_i) = 2$ such that $U_1 \subseteq V_2$ and $U_2 \subseteq V_1$ and S_i centralizes U_i and preserves V_i.*

One also calls S_1 a standard complement *of S_2 and vice-versa. We set $D_1 = N_{S_1}(S_2)$ and $D_2 = N_{S_2}(S_1)$. A simple calculation shows that D_i is a maximal torus in S_i, for $i = 1, 2$. In general if $G \cong \mathrm{SL}_3(\mathrm{k})$, then (G_1, G_2) is a standard pair for G if there is an isomorphism $\psi: G \to S$ such that $\psi(G_i) = S_i$ for $i = 1, 2$.*

Definition 3.2 *A* simply laced Dynkin diagram *over the set I is a simple graph $\Gamma = (I, E)$. That is, Γ has vertex set I, and an edge set E that contains no loops or double edges.*

Definition 3.3 *An* amalgam *over a set I is a collection $\mathcal{A} = \{G_i, G_{i,j} \mid i, j \in I\}$ of groups, together with a collection $\varphi = \{\varphi_{i,j} \mid i, j \in I\}$ of monomorphisms $\varphi_{i,j}: G_i \hookrightarrow G_{i,j}$, called* inclusion maps. *A* completion *of $\mathcal{A}$ is a group G together with a collection $\phi = \{\phi_i, \phi_{i,j} \mid i, j \in I\}$ of homomorphisms $\phi_i: G_i \to G$ and $\phi_{i,j}: G_{i,j} \to G$, such that for any i, j we have $\phi_{i,j} \circ \varphi_{i,j} = \phi_i$. For simplicity we denote by $\overline{G}_i = \varphi_{i,j}(G_i) \leq G_{i,j}$. The amalgam $\mathcal{A}$ is* non-collapsing *if it has a non-trivial completion. A completion $(\widehat{G}, \widehat{\phi})$ is called* universal *if for any completion (G, ϕ) there is a unique surjective group homomorphism $\pi: \widehat{G} \to G$ such that $\phi = \pi \circ \widehat{\phi}$.*

Definition 3.4 *Let $\Gamma = (I, E)$ be a simply laced Dynkin diagram. A* Curtis-Tits amalgam over Γ *is a non-collapsing amalgam $\mathcal{A}(\Gamma) = \{G_i, G_{i\,j} \mid i, j \in I\}$, with connecting maps $\varphi = \{\varphi_{i,j} \mid i, j \in I\}$, such that*

(CT1) for any vertex i, the group $G_i = \mathrm{SL}_2(\mathrm{k})$ and for each pair $i, j \in I$,

$$G_{i,j} \cong \begin{cases} \mathrm{SL}(V_{i,j}) & \text{if } \{i,j\} \in E \\ G_i \times G_j & \text{if } \{i,j\} \notin E \end{cases},$$

where $V_{i,j}$ is a 3-dimensional vector space over k*;*

(CT2) if $\{i, j\} \in E$ then $(\overline{G}_i, \overline{G}_j)$ is a standard pair in $G_{i,j}$.

Definition 3.5 *A Dynkin diagram is* admissible *if it is connected and has no circuits of length ≤ 3.*

From now on $\Gamma = (I, E)$ will be an admissible Dynkin diagram and $\mathcal{A} = \mathcal{A}(\Gamma) = \{G_i, G_{i,j} \mid i, j \in I\}$ will be a non-collapsing Curtis-Tits amalgam over Γ with connecting maps $\varphi = \{\varphi_{i,j} \mid i, j \in I\}$.

It is proved in [7] that if the Dynkin diagram is admissible then the following is well-defined.

Definition 3.6 *For $i, j \in I$, we let $\overline{D}_i = N_{G_{i,j}}(\overline{G}_j) \cap \overline{G}_i$, where $\{i, j\} \in E$. Note that this defines $\overline{D}_i$ for all i since Γ is connected. We also denote $D_i = \varphi_{i,j}^{-1}(\overline{D}_i)$.*

We then have the following.

Lemma 3.7 [7, Section 2] *If $\{i, j\} \in E$, then $\overline{D}_i$ and $\overline{D}_j$ are contained in a unique common maximal torus $D_{i,j}$ of $G_{i,j}$.*

Definition 3.8 *Note that a torus in* $\mathrm{SL}_2(\mathrm{k})$ *uniquely determines a pair of opposite root groups X_+ and X_-. We now choose one root group X_i normalized by the torus D_i of G_i for each i. An orientable Curtis-Tits (OCT) amalgam (respectively orientable Curtis-Tits (OCT) group) is a Curtis-Tits amalgam that admits a system $\{X_i \mid i \in I\}$ of root groups as above such that for any $i, j \in I$, the groups $\varphi_{i,j}(X_i)$ and $\varphi_{j,i}(X_j)$ are contained in a common Borel subgroup $B_{i,j}$ of $G_{i,j}$.*

3.1 Morphisms

In this subsection, for $k = 1, 2$, let $\Gamma^k = (I^k, E^k)$ be a Dynkin diagram.
Now, for $k = 1, 2$, let $\mathcal{A}^k = \{G_i^k, G_{i,j}^k \mid i, j \in I^k\}$ be a Curtis-Tits amalgam with admissible Dynkin diagram Γ^k.

Definition 3.9 *A* homomorphism *between the amalgams $\mathcal{A}^1(\Gamma)$ and $\mathcal{A}^2(\Gamma)$ is a collection $\phi = \{\phi_i, \phi_{i,j} \mid i, j \in I^1\}$ of group homomorphisms $\phi\colon G_i^1 \to G_i^2$ and $\phi_{i,j}\colon G_{i,j}^1 \to G_{i,j}^2$ such that*

$$\phi_{i,j} \circ \varphi_{i,j}^1 = \varphi_{i,j}^2 \circ \phi_i.$$

We call ϕ an isomorphism *of amalgams if ϕ_i and $\phi_{i,j}$ are bijective for all $i, j \in I$, and ϕ^{-1} is a homomorphism of amalgams.*

4 Classification of Curtis-Tits groups of type $\widetilde{A}_{n-1}$

Theorem 1.1 classifies all Curtis-Tits amalgams. For the rest of this paper however we will only consider Curtis-Tits amalgams with Dynkin diagram Γ of type $\widetilde{A}_{n-1}$ where $n \geq 4$. Therefore we can assume that the set of indices is $I = \{0, \ldots, n-1\}$.

4.1 The role of $\mathrm{Aut}(\mathrm{k}) \times \langle \tau \rangle$

In this subsection we describe all amalgams of type $\widetilde{A}_{n-1}$ using Theorem 1.1.

To this end we first discuss certain automorphisms of the Curtis-Tits amalgam with diagram A_2. Let W be a (left) vector space of dimension n over k. Let $G = \mathrm{SL}(W)$ act on W as the matrix group $\mathrm{SL}_n(\mathrm{k})$ with respect to some fixed basis $\mathsf{E} = \{e_i \mid i = 1, 2, \ldots, n\}$. Let $\tau \in \mathrm{Aut}(\mathrm{SL}_n(\mathrm{k}))$ be the automorphism given by

$$A \mapsto {}^tA^{-1}$$

where ${}^tA$ denotes the transpose of A.

Let $\Phi = \{(i,j) \mid 1 \leq i \neq j \leq n\}$. For any $(i,j) \in \Phi$ and $\lambda \in k$, we define the *root group* $X_{i,j} = \{X_{i,j}(\lambda) \mid \lambda \in k\}$, where $X_{i,j}(\lambda)$ acts as

$$\begin{array}{ll} e_j \mapsto e_j + \lambda e_i & \text{and} \\ e_k \mapsto e_k & \text{for all } k \neq j. \end{array}$$

Let $\Phi_+ = \{(i,j) \in \Phi \mid i < j\}$ and $\Phi_- = \{(i,j) \in \Phi \mid j < i\}$. We call $X_{i,j}$ *positive* if $(i,j) \in \Phi_+$ and *negative* otherwise. Let H be the torus of diagonal matrices in $\mathrm{SL}_n(\mathrm{k})$ and for $\varepsilon \in \{+,-\}$, let $X_\varepsilon = \langle X_{i,j} \mid (i,j) \in \Phi_\varepsilon \rangle$ and $B_\varepsilon = H \ltimes X_\varepsilon$. The following lemma describes the action of τ on these root groups.

Lemma 4.1 *$X_{i,j}^\tau = X_{j,i}$ for all $(i,j) \in \Phi$ and $B_\varepsilon^\tau = B_{-\varepsilon}$, for $\varepsilon \in \{+,-\}$.*

Let $\Gamma\mathrm{L}_n(\mathrm{k})$ be the group of all semilinear automorphisms of the vector space W and let $\mathrm{P}\Gamma\mathrm{L}_n(\mathrm{k}) = \Gamma\mathrm{L}_n(\mathrm{k})/Z(\Gamma\mathrm{L}_n(\mathrm{k}))$. Then $\Gamma\mathrm{L}_n(\mathrm{k}) \cong \mathrm{GL}_n(\mathrm{k}) \rtimes \mathrm{Aut}(\mathrm{k})$, where we view $t \in \mathrm{Aut}(\mathrm{k})$ as an element of $\Gamma\mathrm{L}_n(\mathrm{k})$ by setting $((a_{i,j})_{i,j=1}^n)^t = (a_{i,j}^t)_{i,j=1}^n$. The automorphism group of $\mathrm{SL}_n(\mathrm{k})$ can be expressed using $\mathrm{P}\Gamma\mathrm{L}_n(\mathrm{k})$ and τ as follows [20].

Lemma 4.2

$$\mathrm{Aut}(\mathrm{SL}_n(\mathrm{k})) = \begin{cases} \mathrm{P}\Gamma\mathrm{L}_n(\mathrm{k}) & \text{if } n = 2; \\ \mathrm{P}\Gamma\mathrm{L}_n(\mathrm{k}) \rtimes \langle \tau \rangle & \text{if } n \geq 3. \end{cases}$$

Definition 4.3 *Given an element $\delta \in \mathrm{Aut}(\mathrm{k}) \times \langle \tau \rangle \leq \mathrm{Aut}(\mathrm{SL}_2(\mathrm{k}))$ we shall now construct a Curtis-Tits amalgam $\mathcal{A}^\delta$ of type $\widetilde{A}_{n-1}$. For each $i \in \{0, 1, \ldots, n-1\}$ we let $G_i = \mathrm{SL}_2(\mathrm{k})$ and $\mathcal{A}^\delta = \{G_i, G_{i\,j} \mid i, j \in I\}$ with connecting maps $\psi = \{\psi_{i,j} \mid i, j \in I\}$, where*

(SCT1) for any vertex i, the group $G_i = \mathrm{SL}_2(\mathrm{k})$ and for each pair $i, j \in I$,

$$G_{i,j} \cong \left\{ \begin{array}{ll} \mathrm{SL}_3(\mathrm{k}) & \text{if } \{i,j\} = \{i, i+1\} \\ G_i \times G_j & \text{if } \{i,j\} \neq \{i, i+1\} \end{array} \right. ;$$

(SCT2) For $i = 0, 1, \ldots, n-2$ we have

$$\begin{array}{llll} \psi_{i,i+1} \colon G_i & \to G_{i,i+1} & \qquad \psi_{i+1,i} \colon G_{i+1} & \to G_{i,i+1} \\ A & \mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix} & \qquad A & \mapsto \begin{pmatrix} 1 & 0 \\ 0 & A \end{pmatrix} \end{array},$$

and we have

$$\begin{aligned}\psi_{n-1,0}\colon G_{n-1} &\to G_{n-1,0} \\ A &\mapsto \begin{pmatrix} A & 0 \\ 0 & 1 \end{pmatrix}\end{aligned} \qquad \begin{aligned}\psi_{0,n-1}\colon G_0 &\to G_{0,n-1} \\ A &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & A^\delta \end{pmatrix}\end{aligned} \quad ,$$

whereas for all other pairs (i,j), $\psi_{i,j}$ *is the natural inclusion of* G_i *in* $G_i \times G_j$.

Theorem 1.1 reduces to the following particular case:

Corollary 4.4 *Every Curtis-Tits amalgam with diagram* $\widetilde{A}_{n-1}$ *is isomorphic to* $\mathcal{A}^\delta$ *for some unique* $\delta \in \mathrm{Aut}(\mathrm{k}) \times \langle \tau \rangle$.

Our next goal is to construct universal completions of each one of the amalgams $\mathcal{A}^\delta$. More precisely we shall construct such completions for the special cases $\delta \in \mathrm{Aut}(\mathrm{k})$ and $\delta = \tau$. All other completions arise by taking a suitable composition of these constructions.

5 Orientable Curtis-Tits groups

Let $\mathrm{k}[T, T^{-1}]$ be the ring of Laurent polynomials over the field k and let $\delta \in \mathrm{Aut}(\mathrm{k})$.

Theorem 2 *If* δ *has order* s *then the universal completion* G^δ *of* $\mathcal{A}^\delta$ *is a simply connected Kac-Moody group of type* $\widetilde{A}_{n-1}$. *It is a subgroup of finite index* $\aleph$ *inside* $\mathrm{SL}_{sn}(\mathrm{k}[T, T^{-1}])$. *Moreover if the norm* $\mathrm{k} \to \mathrm{k}^\delta$ *is surjective then* $\aleph = ns[(\mathrm{k}^\delta)^* : ((\mathrm{k}^\delta)^*)^{sn}]$.

5.1 Linear groups over twisted Laurent polynomials

Let k be a commutative field and $\delta \in \mathrm{Aut}(\mathrm{k})$. The ring of *twisted Laurent polynomials* is the non-commutative ring

$$R = \mathrm{k}\{t, t^{-1}\}$$

where $t^{-1}xt = x^\delta$ for all $x \in \mathrm{k}$. For some given $n \geq 1$, let $I = \{1, 2, \ldots, n\}$ and let M be an n-dimensional free left R-module with ordered basis $\mathcal{E} = \{e_1, \ldots, e_n\}$. The group of all R-linear invertible transformations of M is denoted $\mathrm{GL}_R(\mathsf{M})$. Representation of transformations as matrices w.r.t. the basis $\mathcal{E}$ acting from the left yields the usual identification:

$$\begin{aligned}\mathrm{End}_R(\mathsf{M}) &\to M_n(R) \\ g &\mapsto (g_{i,j})_{i,j\in I}, \quad \text{where, for all } j \in I, ge_j = \textstyle\sum_i g_{i,j} e_i\end{aligned}$$

Note that since R is in general not commutative, for $a, b, c \in \mathrm{End}_R(\mathsf{M})$ with $ab = c$, we have

$$c_{ik} = \sum_{j \in I} b_{jk} a_{i,j}.$$

At the very end of [25] it is claimed that a realization of the Kac-Moody group G^δ can be obtained as a subgroup of index n inside $\mathrm{PGL}_n(\mathrm{k}\{t, t^{-1}\})$. We shall now proceed to give an explicit description of this realization.

Consider the following collection $\mathcal{L}^\delta = \{L_i, L_{i,j} \mid i,j = 0,1,\ldots,n-1\}$ of subgroups of $\mathrm{SL}_n(\mathrm{k}\{t,t^{-1}\})$. For $i = 0,1,\ldots,n-2$, let

$$L_i = \left\{ \begin{pmatrix} I_i & & \\ & A & \\ & & I_{n-i-2} \end{pmatrix} \mid A \in \mathrm{SL}_2(\mathrm{k}) \right\}$$

and

$$L_{n-1} = \left\{ \begin{pmatrix} d^{\delta^{-1}} & & ct^{-1} \\ & I_{n-2} & \\ tb & & a \end{pmatrix} \mid \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathrm{k}) \right\}$$

Moreover, for each $i,j \in I$ we let

$$L_{i,j} = \langle L_i, L_j \rangle.$$

Finally we let the inclusion maps $\varphi_{i,j}$ be given by natural inclusion of subgroups of $\mathrm{GL}_R(\mathsf{M})$.

Proposition 5.1 *We have an isomorphism of CT amalgams $\mathcal{L}^\delta \cong \mathcal{A}^\delta$.*

Proof Consider the following matrix:

$$F = \left(\begin{array}{c|c} 0 & I_{n-1} \\ \hline t & 0 \end{array}\right).$$

We now define the automorphism Φ of $\mathrm{PGL}_n(\mathrm{k}\{t,t^{-1}\})$ given by $X \mapsto F^{-1}XF$. We first note that we have isomorphisms $\phi_i\colon \mathrm{SL}_2(\mathrm{k}) \to L_i$. For $i = 0,1,\ldots,n-2$ we take

$$\phi_i\colon A \mapsto \begin{pmatrix} I_i & & \\ & A & \\ & & I_{n-i-2} \end{pmatrix}.$$

Moreover, we define

$$\begin{aligned} \phi_{n-1}\colon \mathrm{SL}_2(\mathrm{k}) &\to L_{n-1} \\ \begin{pmatrix} a & b \\ c & d \end{pmatrix} &\mapsto \begin{pmatrix} d^{\delta^{-1}} & & ct^{-1} \\ & I_{n-2} & \\ tb & & a \end{pmatrix}. \end{aligned}$$

One verifies that, for $i = 0,1,\ldots,n-1$ we have $\phi_i = \Phi^i \circ \phi_0$. In particular ϕ_{n-1} is an isomorphism. We now turn to the rank 2 groups. For distinct $i,j \in \{0,1,\ldots,n-1\}$, let $\phi_{i,j}$ be the canonical isomorphism between $G_{i,j} = \langle G_i, G_j \rangle$ and $L_{i,j} = \langle L_i, L_j \rangle$ induced by ϕ_i and ϕ_j. Note that this implies that $\phi_{i,i+1} = \Phi^i \circ \phi_{0,1}$.

We claim that the collection $\phi = \{\phi_i, \phi_{i,j} \mid i,j \in I\}$ is the required isomorphism between $\mathcal{A}^\delta$ and $\mathcal{L}^\delta$. This is completely straightforward except for the maps ϕ_0, $\phi_{n-1,0}$. Note that

$$\phi_{n,0}\colon \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \mapsto \left(\begin{array}{cc|c|c} tet^{-1} & tft^{-1} & & dt^{-1} \\ tht^{-1} & tit^{-1} & & gt^{-1} \\ \hline & & I_{n-3} & \\ \hline tb & tc & & a \end{array}\right)$$

Thus we have

$$\phi_{i,j} \circ \psi_{i,j} = \varphi_{i,j} \circ \phi_i,$$

for all $i, j \in I$. □

5.2 A presentation over the ring $\mathrm{k}[T, T^{-1}]$

In the case when the order of the automorphism $\delta \in \mathrm{Aut}(\mathrm{k})$ is finite we give another interpretation of the group G^δ. To do so, let $s = |\delta|$ and consider the rings

$$\begin{aligned} R &= \mathrm{k}\{t, t^{-1}\} \\ A &= \mathrm{k}[T, T^{-1}] \end{aligned}$$

where $T = t^s$. Note that T commutes with k so that A is the usual ring of Laurent polynomials in T over k.

Now let M be the free left R-module of dimension n with basis $e_1, \cdots, e_n$. Then M is also a free A-module of dimension sn with basis $\mathcal{B} = \{t^i e_j \mid i = 0, \cdots s-1 \text{ and } j = 1, \cdots n\}$, ordered lexicographically (that is, $t^i e_j < t^k e_l$ whenever $i < k$ and j, l are arbitrary, or $i = k$ and $j < l$). Using the basis $\mathcal{B}$ we have an embedding

$$\rho \colon \mathrm{End}_R(\mathsf{M}) \hookrightarrow \mathrm{End}_A(\mathsf{M}) \cong M_{sn}(A).$$

Scalar multiplication on M by the element $t \in R$ is a δ^{-1}-semi A-linear transformation on M and so we can interpret this as an element from $\Gamma\mathrm{L}_{sn}(A)$, acting on the basis $\mathcal{B}$ as x^n, where x is given by

$$x = \left(\begin{array}{c|c} & T \\ \hline I_{ns-1} & \end{array}\right).$$

Now an A-linear map g represents an R-linear transformation precisely if it satisfies $tg = gt$. We phrase this in a lemma.

Lemma 5.2 $\mathrm{End}_R(\mathsf{M})^\rho = C_{M_{sn}(A)}(t)$.

In matrix notation this means that $x^n g^{\delta^{-1}} = g x^n$. More explicitly, if we represent g with respect to $\mathcal{B}$ as a block-matrix $g = (g_{i,j})_{i,j=1}^s$, where $g_{i,j} \in M_n(A)$, then the condition $tg = gt$ is equivalent to choosing $g_{1,j}$ randomly, and setting

$$\begin{aligned} g_{i+1,j+1} &= g_{i,j}^{\delta^{-1}} & 1 \le i, j \le s-1 \\ g_{i+1,1} &= g_{i,s-1}^{\delta^{-1}} T^{-1} & i = 1, \ldots, s-1. \end{aligned} \tag{5.1}$$

Definition 5.3 *For any $g \in \mathrm{End}_R(\mathsf{M})$, let $\det_R(g) = \det_A(g^\rho)$, where the latter denotes the determinant in the matrix ring $M_{sn}(A)$.*

Lemma 5.4 *We have $\mathrm{GL}_R(\mathsf{M}) = \{g \in \mathrm{End}_R(\mathsf{M}) \mid \det_R(g) \in A^*\}$.*

Proof Let $g \in \mathrm{End}_R(\mathsf{M})$. Clearly if $g \in \mathrm{GL}_R(\mathsf{M})$, then g^ρ is invertible in $M_{sn}(A)$ so that $\det_R(g) \in A^*$, the ring of units of A. Conversely, suppose that $\det_R(g) \in A^*$, and let g^{-1} be its inverse in $M_{sn}(A)$. Since $g \in C_{M_{sn}(A)}(t)$, so is g^{-1} and the result follows from Lemma 5.2. □

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Lemma 5.5 *Consider the map* $\det_R\colon \mathrm{End}_R(\mathsf{M}) \to A$ *and assume that the norm* $N_\delta\colon \mathrm{k} \to \mathrm{k}^\delta$ *is surjective. Then, we have the following.*

(a) *The image of* $\mathrm{GL}_R(\mathsf{M})$ *under* $\det_R$ *is equal to* $\{\lambda T^l \mid \lambda \in \mathrm{k}^\delta, l \in \mathbb{Z}\}$.

(b) *The image of* $Z_n(R) = Z(\mathrm{GL}_R(\mathsf{M}))$ *under* $\det_R$ *is equal to* $\{\lambda^{sn} T^{lsn} \mid \lambda \in \mathrm{k}^\delta, l \in \mathbb{Z}\}$.

Proof The relation $x^n g^\delta = g x^n$ implies that $\det(g^\delta) = \det(g)$, that is, $\det(g) \in \mathrm{k}^\delta[T, T^{-1}]^* = \{aT^l \mid a \in \mathrm{k}^\delta, l \in \mathbb{Z}\}$. This shows $\subseteq$. Moreover, note that the element $x \in \mathrm{GL}_R(\mathsf{M})^\rho$ has determinant T and the diagonal matrix corresponding to the transformation $e_1 \to a e_1$ with $a \in \mathrm{k}$ and $e_i \to e_i$ for all $i \geq 2$ has determinant $N_\delta(a)$. This shows the inclusion $\supseteq$ and we have proved part (a).

(b) As in commutative matrix algebra it is clear that any element of $Z_n(R)$ must be of the form $z\,\mathrm{id}$, for some $z \in R$. Moreover, since such an element must commute with all other scalar matrices, z must belong to $Z(R)^* = (A^\delta)^* = \{aT^l \mid a \in (\mathrm{k}^\delta)^*, l \in \mathbb{Z}\}$. The image of $z\,\mathrm{id}$ under ρ is a matrix of the form zI_{sn} and therefore has determinant z^{sn}. □

From now on we shall make the following assumption:

(S) The norm $N_\delta\colon \mathrm{k} \to \mathrm{k}^\delta$ is surjective.

Corollary 5.6 *The index* $[\mathrm{PGL}_n(R)\colon \mathrm{PSL}_n(R)] = sn[\mathrm{k}^\delta : (\mathrm{k}^\delta)^{sn}]$.

Proof We have $[\mathrm{PGL}_n(R) : \mathrm{PSL}_n(R)] = [\mathrm{GL}_R(\mathsf{M}) : \mathrm{SL}_n(R) \cdot Z_n(R)] = [(A^\delta)^* : ((A^\delta)^*)^{sn}]$, so the result follows from Lemma 5.5. □

5.3 Proof of Theorem 2

Let $\Delta = (\Delta_+, \Delta_-, \delta_*)$ be the affine twin-building of type $\widetilde{A}_{sn-1}$ afforded by $V = \mathsf{M} \otimes_A \mathrm{k}(T)$. Consider the standard twin-apartment $\Sigma = (\Sigma_+, \Sigma_-)$ corresponding to the A-basis $\mathcal{B} = \{t^i e_j \mid i = 0, \ldots, s-1; j = 1, \ldots, n\}$. For $\varepsilon = +, -$, let v_ε be the discrete valuation on k such that $v_\varepsilon(T^\varepsilon) = 1$ and let $\mathcal{O}_\varepsilon \leq \mathrm{k}$ be its valuation ring. Then, let $c_\varepsilon = c_\varepsilon(\mathcal{B})$, where $\mathcal{B}$ is considered as an ordered basis. Moreover, let Θ_ε be the flag-complex of Δ_ε.

Lemma 5.7 *Let* $\varepsilon = +, -$. *Then,* t *acts as a type-permuting automorphism on* Θ_ε *fixing* c_ε. *Moreover,* $\mathrm{typ}(t)$ *acts as a deck-transformation group on the diagram* Γ *of* Δ *in the sense of Mühlherr [15].*

Proof That t acts as an automorphism follows from the fact that it sends free $\mathcal{O}_\varepsilon$ lattices to free $\mathcal{O}_\varepsilon$ lattices while preserving their rank and inclusion among such lattices. Since it is δ^{-1}-semilinear over $\mathrm{k}(T)$, it preserves the A-module M, thereby preserving the opposition relation of Δ.

That t preserves c_ε is an easy exercise. In fact t permutes the objects of c_ε by sending the object of type i to the object of type $i + n$ modulo ns. Since t is an automorphism of Δ the graph automorphism $\mathrm{typ}(t)$ acts accordingly. Thus the cyclic group of order s generated by $\mathrm{typ}(t)$ is a deck transformation group of M in the sense of [15]. □

Theorem 5.8 *The universal completion of $\mathcal{A}^\delta$ is* $\mathrm{SL}_n(R)$.

Proof In view of Lemma 5.7 we can apply Theorem B. of loc. cit. which says that the subcomplex of Δ fixed by t contains a Moufang twin-building $\tilde{\Delta}$. Since Γ has finite rank, in fact, this fixed subcomplex is equal to $\tilde{\Delta}$.

Note that $\mathrm{SL}_n(R)$, the centralizer in $G = \mathrm{SL}_{sn}(A)$ of t, is a flag-transitive group of automorphisms of $\tilde{\Delta}$. Namely, identify Δ with $(G/B_+, G/B_-)$ via the Birkhoff decomposition associated to the twin-BN pair B_+, B_-, N for G. Here B_+ and B_- are the stabilizers of the fundamental chambers c_+ and c_-, which are fixed by t. Then, t preserves B_+ and B_-, so that the action of t on Δ is given entirely by its action on G. Therefore the fixed complex $\tilde{\Delta}$ consists of those chambers gB_+, gB_-, where $g \in C_G(t)$. Clearly now the group $C_G(t)$ is flag-transitive on $\tilde{\Delta}$, acting by left-multiplication on these cosets.

Since $\widetilde{A}_{n-1}$ is simply-laced, Δ satisfies condition (co) of [16]. Then, by the twin-building version of the Curtis-Tits' theorem [2] the automorphism group $C_G(t)$ of $\tilde{\Delta}$ is the universal completion of its Levi-components of rank 2 and 3. One verifies that the amalgam of Levi-components of rank 2 and 3 in $C_G(t)$ is exactly $\mathcal{L}^\delta$.

The result follows from the fact that $C_G(t) = \mathrm{SL}_n(R)$ by Lemmas 5.2 and 5.4. □

6 The non-orientable Curtis-Tits group G^τ

In this section $\mathrm{k}[t, t^{-1}]$ denotes the ring of commuting Laurent polynomials with coefficients in the field k. Consider the group $G = \mathrm{SL}_{2n}(\mathrm{k}[t, t^{-1}])$ and let

$$s = \left(\begin{array}{c|c} 0_n & t^{-1}I_n \\ \hline I_n & 0_n \end{array}\right).$$

Let σ be the involutory automorphism of $\mathrm{k}(t)$ that fixes all of k and interchanges t and t^{-1}. We define the automorphism $\tau\colon G \mapsto G$ by $A \mapsto s^{-1}\, {}^tA^{-\sigma} s$. As before let V be a $\mathrm{k}(t)$-vector space of dimension $2n$ with basis $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$. Let M be the free $\mathrm{k}[t, t^{-1}]$-lattice spanned by this basis.

Define a σ-sesquilinear form β on V such that $\beta(e_i, e_j) = \beta(f_i, f_j) = 0, \beta(e_i, f_j) = t\delta_{ij}$ and $\beta(f_i, e_j) = \delta_{ij}$ and in addition, for $u, v \in V$ and $\lambda, \mu \in \mathrm{k}(t)$, we have

$$\beta(\lambda u, \mu v) = \lambda\beta(u, v)\mu^\sigma.$$

Theorem 3 *Let* k *be a field of size at least* 5*. The universal completion G^τ of $\mathcal{A}^\tau$ is the group of symmetries in* $\mathrm{SL}_{2n}(\mathrm{k}[t, t^{-1}])$ *of the σ-sesquilinear form β.*

In Subsection 6.2 we will prove that the geometry Δ^τ is connected and simply connected which by Tits' Lemma implies that the group G^τ is the universal completion of the amalgam of maximal parabolics. We then observe that $\mathcal{L}^\tau$ is the amalgam of parabolics of rank 2 and 3. Moreover, we note that the maximal parabolics are all linear groups over k. Theorem 3 will then follow by applying the Curtis-Tits theorem for linear groups to the maximal parabolics.

We will first construct the amalgam $\mathcal{A}^\tau$ from Corollary 4.4 inside $\mathrm{SL}_{2n}(\mathrm{k}[t,t^{-1}])$. Consider the following matrix:

$$F = \left(\begin{array}{c|c} 0 & I_{2n-1} \\ \hline 1 & 0 \end{array}\right).$$

We now define the automorphism Φ of $\mathrm{SL}_{2n}(\mathrm{k}[t,t^{-1}])$ given by $X \mapsto F^{-1}XF$. Also define the map $i\colon \mathrm{SL}_2(\mathrm{k}) \to \mathrm{SL}_{2n}(\mathrm{k}[t,t^{-1}])$ by

$$A \quad \mapsto \left(\begin{array}{c|c} A & \\ \hline & I_{2n-2} \end{array}\right).$$

Next, for $k = 0, \ldots, n-1$, let $\phi_k\colon \mathrm{SL}_2(\mathrm{k}) \to \mathrm{SL}_{2n}(\mathrm{k}[t,t^{-1}])$ by

$$\phi_k(A) = F^k(i(A)) \cdot \tau(F^k(i(A)))$$

and let L_k be the image of ϕ_k. Note that for each $k = 0, \ldots n-2$ we have

$$L_k = \left\{ \left(\begin{array}{ccc|ccc} I_k & & & & & \\ & A & & & & \\ & & I_{n-k-2} & & & \\ \hline & & & I_k & & \\ & & & & {}^tA^{-1} & \\ & & & & & I_{n-k-2} \end{array}\right) | A \in \mathrm{SL}_2(\mathrm{k}) \right\}$$

and

$$L_{n-1} = \left\{ \left(\begin{array}{ccc|ccc} a & & & & & -bt^{-1} \\ & I_{n-2} & & & & \\ & & a & b & & \\ \hline & & c & d & & \\ & & & & I_{n-2} & \\ -ct & & & & & d \end{array}\right) | \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathrm{k}) \right\}.$$

For distinct $i, j \in \{0, 1, \ldots, n-1\}$, let $\phi_{i,j}$ be the canonical isomorphism between $G_{i,j} = \langle G_i, G_j \rangle$ and $L_{i,j} = \langle L_i, L_j \rangle_G$ induced by ϕ_i and ϕ_j. It follows that $L_{ij} \cong \mathrm{SL}_3(\mathrm{k})$ if $i - j \equiv \pm 1 (\operatorname{mod} n)$ and $G_{ij} \cong L_i \times L_j$ otherwise.

Now let $\mathcal{L}^\tau = \{L_i, L_{i,j}, \varphi_{i,j} \mid i, j \in \{0, 1, \ldots, n-1\}\}$ be the amalgam of the L_i, L_{ij} where the maps $\varphi_{i,j}$ are the natural inclusion maps.

Proposition 6.1 *We have an isomorphism of amalgams $\mathcal{L}^\tau \cong \mathcal{A}^\tau$.*

Proof We claim that the collection $\phi = \{\phi_i, \phi_{i,j} \mid i, j \in I\}$ is the required isomorphism between $\mathcal{A}^\tau$ and $\mathcal{L}^\tau$. This is completely straightforward after noting that if we define ϕ_n in the same manner as ϕ_k for $k = 0, 1, \ldots, n-1$, then we have $\phi_0 \circ \phi_n^{-1} = \tau$. □

6.1 The geometry Δ^τ for G^τ

We now describe a group G^τ acting flag-transitively on a geometry Δ^τ so that $\mathcal{L}^\tau$ is the amalgam of parabolic subgroups of rank 2 and 3.

Lemma 6.2 *β is a non-degenerate trace-valued (σ, t)-sesquilinear form, that is for all $u, v \in V$ we have $\beta(v, u) = t\beta(u, v)^\sigma$ and there exists $x \in \mathrm{k}(t)$ such that $\beta(u, u) = x + x^\sigma t$.*

Proof Let $u = \sum_{i=1}^n \lambda_i e_i + \mu_i f_i$ and let $u' = \sum_{i=1}^n \lambda'_i e_i + \mu'_i f_i$. Then

$$\beta(u, u') = \sum_{i=1}^n \lambda_i \mu_i'^\sigma t + \mu_i \lambda_i'^\sigma = t\beta(u', u)^\sigma$$

In particular, setting $u = u'$ we get $x = \sum_{i=1}^n \mu_i \lambda_i^\sigma$. □

Given a $\mathrm{k}(t)$-basis $\{a_1, \ldots, a_{2n}\}$ for V, the *right dual basis* for V with respect to β is the unique basis $\{a_1^*, \ldots, a_{2n}^*\}$ such that $\beta(a_i, a_j^*) = \delta_{ij}$ (note the order within β). The *right adjoint* of a transformation $g \in \Gamma\mathrm{L}(V)$, is the transformation $g^* \in \Gamma\mathrm{L}(V)$ such that $\beta(gu, g^*v) = \beta(u, v)$ for all $u, v \in V$.

One easily verifies the following two lemmas.

Lemma 6.3 *If $g \in \mathrm{GL}(V)$ is represented by a matrix (g_{ij}) with respect to $\{a_1, \ldots, a_{2n}\}$, then $g^* = ^t(g_{ij}^\sigma)^{-1}$ with respect to $\{a_1^*, \ldots, a_{2n}^*\}$*

Lemma 6.4 *The right dual basis for $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ is $\{tf_1, \ldots, tf_n, e_1, \ldots, e_n\}$. As a consequence, $g^* = g^\tau$*

Proof Let $u, v \in V$. Then $\beta(gu, g^\tau v) = {}^t u {}^t g {}^t s^{-1}(s^{-1t}g^{-\sigma} s v)^\sigma = {}^t u {}^t g {}^t s^{-1} s^{-\sigma t} g^{-1} s^\sigma v^\sigma = {}^t u s^\sigma v^\sigma = \beta(u, v)$ and since this holds for all $u, v \in V$ and β is non-degenerate, we are done. □

Let Δ be the twin-building for the group $G = \mathrm{SL}_{2n}(\mathrm{k}[t, t^{-1}])$ with twinning determined by M. Let (W, S) be the Coxeter system with diagram Γ of type $\widetilde{A}_{2n-1}$. Call $S = \{s_i \mid i = 0, \ldots, 2n-1\}$.

Lemma 6.5 *The map induced by τ on Δ, is given by*

$$\Lambda_\varepsilon^\tau = \{v \in V \mid \beta(u, v) \in \mathcal{O}_\varepsilon \text{ for all } u \in \Lambda_\varepsilon\}$$

for all $\mathcal{O}_\varepsilon$-lattices Λ_ε.

Proof This follows from the fact that $g^* = g^\tau$. □

Lemma 6.6 (a) *If $\{a_1, \ldots, a_{2n}\}$ is a basis for V with right dual $\{a_1^*, \ldots, a_{2n}^*\}$ with respect to β, then $\Lambda_\varepsilon^\tau(a_1, \ldots, a_{2n}) = \Lambda_{-\varepsilon}(a_1^*, \ldots, a_{2n}^*)$.*

(b) *For all i, j we have $(t^j a_i)^* = t^j a_i^*$ so $\Lambda_\varepsilon^\tau(t^{j_1}a_1, \ldots, t^{j_{2n}}a_{2n}) = \Lambda_{-\varepsilon}(t^{j_1}a_1^*, \ldots, t^{j_{2n}}a_{2n}^*)$.*

(c) *The right dual of an A-basis for M is an A-basis for M.*

Proof (a) and (b) are straightforward consequences of the fact that β is σ-sesquilinear.
(c) This follows from Lemma 6.4 and 6.3. □

The *standard ordered t-hyperbolic basis* for M is $\{e_1, \ldots, e_n, f_1, \ldots, f_n\}$ whose Gram matrix is given by s^σ. The standard chamber in Δ_ε is $c_\varepsilon(e_1, \ldots, e_n, f_1, \ldots, f_n)$.

Proposition 6.7 *The map τ induces isomorphisms $\tau\colon \Delta_\varepsilon \to \Delta_{-\varepsilon}$ where $\mathrm{typ}(\tau)\colon I \to I$ is the graph isomorphism defined by $i \to i - n \bmod (2n)$. Moreover the standard chambers c_+ and c_- are interchanged by τ.*

Proof Let $X_{i,\varepsilon}$ be the object of type i on c_ε. We show that $X_{i,\varepsilon}^\tau = X_{n+i,-\varepsilon}$. This follows immediately from Lemmas 6.6 and 6.4. In particular c_+ and c_- are interchanged.

We now consider an arbitrary lattice $\Lambda_\varepsilon = \langle a_1, \ldots, a_{2n}\rangle_{\mathcal{O}_\varepsilon}$, where $\{a_1, \ldots, a_{2n}\}$ is some $\mathrm{k}(t)$-basis for M (note that this is always possible as the Kac Moody group acts flag transitively on the twin building).

Let g be the transformation sending e_i to a_i and f_i to a_{n+i} for $i = 1, 2, \ldots, n$. It follows that $\det g = at^l$ for some $a \in k, l \in \mathbb{Z}$. Now $s^{-1t}g^{-\sigma} = g^\tau s^{-1}$ is the transformation sending $e_1, \ldots, f_n$ to $a_1^*, \ldots, a_{2n}^*$. Taking determinants we see that the type of Λ_ε^τ is $\varepsilon v_\varepsilon(\det(g)^{-\sigma} t^{-n}) = \varepsilon v_\varepsilon(\det(g)) - n \bmod (2n)$. □

Definition 6.8 *Let*

$$\Delta^\tau = \{(d_+, d_+^\tau) \mid d_+ \operatorname{opp} d_+^\tau\}$$

Adjacency is induced by adjacency in Δ so that

$$(d_+, d_+^\tau) \sim_i (e_+, e_+^\tau) \iff d_+ \sim_i e_+ (\text{ and } d_- \sim_{\tau(i)} e_-)$$

Lemma 6.9 *$(d_+, d_-) \in \Delta^\tau$ if and only if there is an A-basis $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ for M whose Gram matrix is s^σ and such that $d_\varepsilon = c_\varepsilon(a_1, \ldots, a_n, b_1, \ldots, b_n)$.*

Proof As in the proof of Proposition 6.7, one verifies that any such basis gives rise to a pair of chambers in Δ^τ. Conversely, let $(d_+, d_-) \in \Delta^\tau$. That means that $d_- = d_+^\tau$. Let $\Sigma = \Sigma(d_+, d_-)$ be the twin-apartment containing d_+ and d_-. Then $\Sigma^\tau = \Sigma$. Let $\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ be an A-basis for M such that $\Sigma = \Sigma\{a_1, \ldots, a_n, b_1, \ldots, b_n\}$ and $d_\varepsilon = c_\varepsilon(a_1, \ldots, a_n, b_1, \ldots, b_n)$, where $X_0 = \langle a_1, \ldots, a_n, b_1, \ldots, b_n\rangle_{\mathcal{O}_\varepsilon}$ has type 0. Let $\{a_1^*, \ldots, a_n^*, b_1^*, \ldots, b_n^*\}$ be the right dual basis with respect to β. Then,

$$\begin{aligned} \Sigma &= \Sigma\{a_1^*, \ldots, a_n^*, b_1^*, \ldots, b_n^*\} \\ d_\varepsilon &= c_\varepsilon(a_1^*, \ldots, a_n^*, b_1^*, \ldots, b_n^*) \end{aligned}$$

Note that by Lemma 6.6 both bases are A-bases for M. Note that the type of the lattice $\langle a_1^*, \dots, a_n^*, b_1^*, \dots, b_n^* \rangle_{\mathcal{O}_\varepsilon} = \langle a_1, \dots, a_n, b_1, \dots, b_n \rangle^\tau_{\mathcal{O}_{-\varepsilon}}$ is n. Now consider the $\mathrm{k}(t)$-linear map

$$\begin{aligned} \phi \colon V &\to V \\ b_i &\mapsto a_i^* \\ ta_i &\mapsto b_i^* \end{aligned}$$

for all $i = 1, 2, \dots, n$. It is easy to check that ϕ is a type-preserving automorphism of Δ_ε such that $d_\varepsilon^\phi = d_\varepsilon$ since it is a $\mathrm{k}(t)$-linear map that sends the object of type i on d_ε to the object of type i on d_ε. This implies that $\phi \in H = N \cap B_+ \cap B_-$ and it follows (see e.g. [1]) that

$$\begin{aligned} b_i &= \lambda_i a_i^* \\ ta_i &= \mu_i b_i^* \end{aligned}$$

where $\lambda_i, \mu_i \in \mathrm{k}^*$ and in fact since $(a_i^*)^* = ta_i$ we have $\mu_i = \lambda_i^{-1}$. Without modifying the chambers d_ε, we may scale so that $\lambda_i = 1$ for all i, that is

$$\begin{aligned} b_i &= a_i^* \\ ta_i &= b_i^* \end{aligned}$$

so the Gram matrix of $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ is s^σ. □

Recall the definition of G^τ.

$$G^\tau := \{g \in \mathrm{SL}_{2n}(\mathrm{k}[t, t^{-1}]) | \forall x, y \in V, \beta(gx, gy) = \beta(x, y)\}$$

Theorem 6.10 *The group G^τ acts flag-transitively on Δ^τ.*

Proof Let $(d_+, d_-) \in \Delta^\tau$. By Lemma 6.9 there exists an A-basis $\{a_1, \dots, a_n, b_1, \dots, b_n\}$ for M whose Gram matrix is s^σ. The A-linear map

$$\begin{aligned} \phi \colon V &\to V \\ e_i &\mapsto a_i \\ f_i &\mapsto b_i \end{aligned}$$

for all $i = 1, 2, \dots, n$ belongs to G^τ and sends (c_+, c_-) to (d_+, d_-). □

6.2 Simple connectedness

We will use the techniques developed in [8] to show that Δ^τ is simply connected. In the terminology of loc. cit. a collection $\{\mathcal{C}_m\}_{m \in \mathbb{N}}$ of subsets of Δ_+ is a filtration if the following are satisfied:

(F1) For any $m \in \mathbb{N}$ $\mathcal{C}_m \subseteq \mathcal{C}_{m+1}$,

(F2) $\bigcup_{m \in \mathbb{N}} \mathcal{C}_m = \Delta_+$, (F3) For any $m \in \mathbb{N}_{>0}$, if $\mathcal{C}_{m-1} \neq \emptyset$, there exists an $i \in I$ such that for any $c \in \mathcal{C}_m$, there is a $d \in \mathcal{C}_{m-1}$ that is i-adjacent to c.

It is called a residual filtration if the intersections of $\mathcal{C}$ with any given residue is a filtration of that residue.

For any $c \in \Delta$, let $|c| = \min\{\lambda \mid c \in \mathcal{C}_\lambda\}$. For a subset $X \subseteq \Delta$ we accordingly define $|X| = \min\{|c| \mid c \in X\}$ and $\operatorname{aff}(X) = \{c \in X \mid |c| = |X|\}$. We shall make use of the following result from loc. cit..

Theorem 6.11 *Suppose $\mathcal{C}$ is a residual filtration such that for any rank 2 residue R, $\operatorname{aff}(R)$ is connected and any rank 3 residue R, $\operatorname{aff}(R)$ is simply connected, then the following are equivalent.*

(a) *Δ is simply connected,*

(b) *$\mathcal{C}_n$ is simply connected for all $n \in \mathbb{N}$.*

We shall define a residual filtration $\mathcal{C}$ with the property that $\mathcal{C}_0 = \Delta^\tau$. Then, since we know that Δ is simply connected, it suffices to show that $\mathcal{C}$ satisfies the conditions of the theorem.

6.3 The filtration $\mathcal{C}$

In order to define the filtration $\mathcal{C}$ we first let

$$W^\tau = \{w \in W \mid \exists d_\varepsilon \in \Delta_\varepsilon \colon w = \delta_*(d_\varepsilon, d_\varepsilon^\tau)\}.$$

We also fix an injective map $|\cdot| \colon W^\tau \to \mathbb{N}$ such that whenever $l(w) > l(w')$, we have $|w| > |w'|$ and for any $m \in \mathbb{N}$. We then define a filtration on Δ_+ using $|\cdot|$ as follows: Let

$$\mathcal{C}_m = \{c_+ \in \Delta_+ \mid |\delta_*(c_+, c_+^\tau)| \leq m\}.$$

In the remainder of this section we prove that $\mathcal{C}$ is a residual filtration. First however, we will need some technical lemmas about W^τ. Let

$$W(\tau) = \{u \in W \mid u^\tau = u^{-1}\}.$$

These elements are called twisted involutions in [22] and [13]. Some of the results bellow have somewhat weaker forms in the most general case of a quasi-twist. See [13] for details on both twisted involutions and of the corresponding geometries.

We now characterize $W(\tau)$ as follows:

Lemma 6.12

$$W(\tau) = \{w(w^{-1})^\tau \mid w \in W\}.$$

More precisely, given any $u \in W(\tau)$ there exists a word $w \in W$ such that $w(w^{-1})^\tau$ is a reduced expression for u.

Proof It is obvious that we have $\supseteq$. We now proceed to prove the reverse inclusion. Let $u \in W^\tau$. We prove that u can be written as a reduced expression of the form $w(w^{-1})^\tau$ by induction on $l = l(u)$. If $l = 0$, then $1 = u = 1 \cdot (1^{-1})^\tau$. Now let $l \geq 1$ and write $u = s_{i_1} \cdots s_{i_l}$. By assumption we can also write $u = s_{\tau(i_l)} \cdots s_{\tau(i_1)}$. Consider $u' = s_{i_1} u s_{\tau(i_1)}$. Note that $u' \in W(\tau)$. We note the following: $l(s_{i_1} u) < l(u)$ and so writing $s_{i_1} u = s_{i_1} s_{\tau(i_l)} \cdots s_{\tau(i_1)}$ it follows from the exchange property that there is some j such that $s_{i_1} u = s_{\tau(i_l)} \cdots \widehat{s_{\tau(i_j)}} \cdots s_{\tau(i_1)}$. There are two cases:

(i) $j > 1$

(ii) $j = 1$

In case (i) it follows that $l(s_{i_1} u s_{\tau i_1}) = l(u) - 2$. By induction we have a word w' of length $(l(u) - 2)/2$ such that $u = s_{i_1} w'(w'^{-1})^\tau s_{\tau i_1}$ and since this expression has length $l(u)$ it is reduced and we are done.

In case (ii) it follows that $s_{i_1} u s_{\tau i_1} = u$. This means that u can also be written in the form $u = s_{i_2} \cdots s_{i_l} s_{\tau(i_1)}$. Repeating this process we either decrease the length as in case (i), or u has the property that it can be written such that any of the s_{i_j} come first. By Theorem 2.16 of [17] this means that if $J = \{i_1, \ldots, i_l, \tau(i_l), \ldots, \tau(i_1)\}$, then J is finite and u is the longest word in W_J. In particular $J \neq I$, then since $\text{typ}(\tau)$ acts on $\widetilde{A}_{2n-1}$ by interchanging opposite nodes, there is a subset $K \subseteq J$ such that J is the disjoint union of K and K^τ. As a consequence, $u = w_K(w_K)^\tau$. □

The following lemma characterizes W^τ.

Lemma 6.13 $W^\tau = W(\tau)$.

Proof Let $c_\varepsilon \in \Delta_\varepsilon$. Then $u = \delta_*(c_\varepsilon, c_\varepsilon^\tau)$ satisfies $u^\tau = u^{-1}$. Therefore the inclusion $\subseteq$ follows by definition. Conversely, consider a chamber c_ε such that c_ε opp c_ε^τ. Then the apartment $\Sigma(c_\varepsilon, c_\varepsilon^\tau)$ is preserved by τ and identifying it with the Coxeter group we see that τ acts on Σ as it acts on W. Let $u \in W(\tau)$. Then, by Lemma 6.12 it is of the form $w(w^{-1})^\tau$ for some $w \in W$. Let d_ε be the chamber such that $\delta_\varepsilon(c_\varepsilon, d_\varepsilon) = w$, then $\delta_*(d_\varepsilon, d_\varepsilon^\tau) = w(w^{-1})^\tau = u$ as desired. □

In the sequel we shall use the following notation for projections. Given a residue R of Δ_ε, we denote projection from Δ_ε onto R by proj_R and denote (co-) projection from $\Delta_{-\varepsilon}$ onto R by proj^*_R.

Lemma 6.14 *Suppose that $c_\varepsilon \in \Delta$ satisfies $\delta_*(c_\varepsilon, c_\varepsilon^\tau) = w$, let $i \in I$ and suppose that π is the i-panel on c_ε. Then,*

(a) *If $l(s_i w) > l(w)$, then all chambers $d_\varepsilon \in \pi - \{c_\varepsilon\}$ except one satisfy $\delta_*(d_\varepsilon, d_\varepsilon^\tau) = w$. The remaining chamber $\check{c}_\varepsilon$ satisfies $\delta_*(\check{c}_\varepsilon, (\check{c}_\varepsilon)^\tau) = s_i w s_{\tau(i)}$.*

(b) *If $l(s_i w) < l(w)$, then all chambers $d_\varepsilon \in \pi - \{c_\varepsilon\}$ satisfy $\delta_*(d_\varepsilon, d_\varepsilon^\tau) = s_i w s_{\tau(i)}$.*

In particular, if $w = 1$, then all chambers $d_\varepsilon \in \pi - \{c_\varepsilon\}$ except one satisfy $\delta_*(d_\varepsilon, d_\varepsilon^\tau) = 1$.

Proof (a) In this case, by the twin-building axioms, there is a unique chamber, called $\check{c} = \mathrm{proj}_\pi^*(c_\varepsilon^\tau)$ such that $\delta_*(\check{c}, c_\varepsilon^\tau) = s_i w$. Let d_ε be any other chamber in π. Then, again by the twin-building axioms we have $\delta_*(d_\varepsilon, c_\varepsilon^\tau) = w$. By applying τ we see that $\delta_*(d_\varepsilon^\tau, c_\varepsilon) = w^\tau = w^{-1}$. It follows that for any other chamber $d' \in \pi$ we either have $\delta_*(d_\varepsilon^\tau, d') = w^\tau s_{\tau(i)}$ or w^τ. Note here that $l(w^\tau s_{\tau(i)}) = l(w^\tau) + 1$. However, $\delta_*(d_\varepsilon, d_\varepsilon^\tau) \in W^\tau$, where all lengths are even. Since $w^\tau \in W^\tau$, $w^\tau s_{\tau(i)} \notin W^\tau$ and so we must have $\delta_*(d_\varepsilon, d_\varepsilon^\tau) = w$. By the same token, the distance $\delta_*(\check{c}, \check{c}^\tau) = s_i w s_{\tau(i)}$.

(b) In this case, by the twin-building axioms, every chamber $d_\varepsilon \in \pi - \{c_\varepsilon\}$ satisfies $\delta_*(d_\varepsilon, c_\varepsilon^\tau) = s_i w$, since now $c_\varepsilon = \mathrm{proj}_\pi^*(c_\varepsilon^\tau)$, which is unique. Applying τ we see that $\delta_*(d_\varepsilon^\tau, c_\varepsilon) = s_{\tau(i)} w^\tau$. It follows that for any other chamber $d' \in \pi$ we either have $\delta_*(d_\varepsilon^\tau, d') = s_{\tau(i)} w^\tau s_i$ or $s_{\tau(i)} w^\tau$. However, since $w^\tau \in W^\tau$, by looking at the lengths, $w^\tau s_i \notin W^\tau$, and so we must have $\delta_*(d_\varepsilon^\tau, d_\varepsilon) = s_{\tau(i)} w^\tau s_i$ and we are done. □

Lemma 6.15 *τ does not commute with any reflection.*

Proof Let r be any reflection such that $r^\tau = r$. Then in fact $r \in W^\tau$. However, all elements of W^τ have even length and r being a conjugate of a fundamental reflection does not. □

Lemma 6.16 *For $u \in W^\tau$ and $i \in I$, we have $l(s_i u s_{\tau(i)}) = l(u) \pm 2$.*

Proof By Lemma 6.12 u has a reduced expression of the form $ww^{-\tau}$. First note that by Lemma 6.15 we cannot have $s_i u s_{\tau(i)} = u$ because that would imply that the reflection $w^{-1} s_i w$ is fixed by τ. There are two cases to consider, namely,

(a) $l(s_i u) > l(u)$,

(b) $l(s_i u) < l(u)$.

In case (a) note that $l(s_i u) = l(u s_{\tau(i)}) > l(u)$, so that by Proposition 4.1(b) of [8] we have $l(s_i u s_{\tau(i)}) = l(u) + 2$ or $s_i u s_{\tau(i)} = u$. The latter is impossible by the preceding argument.

In case (b) consider $u' = s_i u$ and assume that $l(s_i u s_{\tau(i)}) = l(u)$. We now have $l(u' s_{\tau(i)}) = l(s_i u s_{\tau(i)}) = l(u) > l(u')$ and $l(s_i u') = l(u) > l(s_i u) = l(u')$. Applying the aforementioned Proposition again, we find that either $l(s_i u' s_{\tau(i)}) = l(u') + 2$ or $s_i u' s_{\tau(i)} = u'$. In the first case we find that $l(s_i u) = l(u s_{\tau(i)}) - 2$, which contradicts the equality $l(s_i u) = l((s_i u)^\tau) = l(s_{\tau(i)} u^{-1}) = l(u s_{\tau(i)})$. The second case is ruled out as in (a). □

We define the following subset of a given residue R:

$$A_\tau(R) = \{c \in R \mid l(\delta_*(c, c^\tau)) \text{ is minimal among all such distances}\}.$$

Lemma 6.17 *Let R be a J-residue. Let $c \in A_\tau(R)$ and let $w = \delta_*(c, c^\tau)$. Then, $d \in A_\tau(R)$ if and only if $w = \delta_*(d, d^\tau)$. Moreover, w is determined by the fact that for any $j \in J$ we have $l(s_j w) = l(w) + 1$.*

Proof First note that by Lemma 6.14, $\{\delta_*(x, x^\tau) \mid x \in R\} = \{uwu^\tau \mid u \in W_J\}$. Moreover, the coset $W_J w W_{\tau(J)}$ has a minimal element m that is characterized by the fact that $l(s_j m) = l(m)+1$ and $l(m s_{\tau(j)}) = l(m)+1$ for all $j \in J$. We claim that w has that property as well. Namely, let $j \in J$ have the property that $l(w s_{\tau(j)}) = l(s_j w) < l(w)$. Then, by Lemma 6.14 (b) any element d in the j-panel on c has the property that $\delta_*(d, d^\tau) = s_j w s_{\tau(j)}$ and by Lemma 6.16 this must have length $l(w) - 2$, a contradiction to the fact that $c \in A_\tau(R)$. Thus, w satisfies the conditions on m and it follows that $w = m$. □

Proposition 6.18 *Let $c \in R$ and let $w = \delta_*(c, c^\tau)$. The following are equivalent:*

(i) *$c \in A_\tau(R)$,*

(ii) *$w = w_R$, the unique element of minimal length in $W_J w W_{\tau(J)}$,*

(iii) *$c \in \mathcal{C}_k$, where $k = \min\{l \mid \mathcal{C}_l \cap R \neq \emptyset\}$.*

In particular, we have $A_\tau(R) = \mathrm{aff}(R)$.

Proof By Lemma 6.17 (i) and (ii) are equivalent. Since $|\cdot|$ is strictly increasing, also (ii) and (iii) are equivalent. □

Proposition 6.19 *$\mathcal{C}$ is a residual filtration.*

Proof Part (F1) and (F2) are immediate. Now let R be a J-residue, suppose that $R \cap \mathcal{C}_{n-1} \neq \emptyset$ and let $c \in R \cap \mathcal{C}_n - \mathcal{C}_{n-1}$. Let $w = \delta_*(c, c^\tau)$. By Proposition 6.18, $c \notin A_\tau(R)$ and so, by Lemma 6.17, there exists a $j \in J$ with $l(s_j w) < l(w)$. Therefore by Lemma 6.16, any j-neighbor d of of c has $l(\delta(d, d^\tau)) = l(w) - 2$ and therefore belongs to $\mathcal{C}_{n-1}$. □

Proposition 6.19 allows us to apply Theorem 6.11 and, by Proposition 6.18, in order to show simple connectedness of Δ^τ, it suffices to show that $\mathrm{aff}(R) = A_\tau(R)$ is connected when R has rank 2 and is simply connected when R has rank 3. We shall first obtain some general properties of $A_\tau(R)$ and then verify the connectedness properties using concrete models of $A_\tau(R)$.

Proposition 6.20 (See Corollary 7.4 of [6]) *For $\varepsilon = \pm$, let $S_\varepsilon \subsetneq R_\varepsilon$ be residues of Δ_ε such that $S_\varepsilon = \mathrm{proj}^*_{R_\varepsilon}(R_{-\varepsilon})$ and let $x_\varepsilon \in R_\varepsilon$ be an arbitrary chamber and assume in addition that $R_{-\varepsilon} = R_\varepsilon^\tau$ and $x_{-\varepsilon} = x_\varepsilon^\tau$, for $\varepsilon = \pm$. Then, $x_\varepsilon \in A_\tau(R_\varepsilon)$ if and only if*

(i) *x_ε belongs to a residue opposite to S_ε in R_ε whose type is also opposite to the type of S_ε in R_ε and*

(ii) $\mathrm{proj}_{S_\varepsilon}(x_\varepsilon) \in A_\tau(S_\varepsilon)$.

Proof This is exactly the same as the proof in loc. cit. noting that it suffices for τ to be an isomorphism between Δ_+ and Δ_- that preserves lengths of codistances. □

Lemma 6.21 *With the notation of Proposition 6.20,* $\mathrm{proj}^*_{S_\varepsilon}$, $\mathrm{proj}^*_{S_{-\varepsilon}}$ *define adjacency preserving bijections between* $S_{-\varepsilon}$ *and* S_ε *and* $(\mathrm{proj}^*_{S_\varepsilon})^{-1} = \mathrm{proj}^*_{S_{-\varepsilon}}$. *Let* $l = \max\{l(\delta_*(c_\varepsilon, d_{-\varepsilon})) \mid c_\varepsilon \in S_\varepsilon, d_{-\varepsilon} \in S_{-\varepsilon}\}$. *Then,* $d_{-\varepsilon} = \mathrm{proj}^*_{S_{-\varepsilon}}(c_\varepsilon)$ *if and only if* $l(\delta_*(c_\varepsilon, d_{-\varepsilon})) = l$.

Proof This is the twin-building version of the main result of [9]. □

In view of Proposition 6.20, in order to study $A_\tau(R)$ entirely inside R we need to know what $A_\tau(S)$ looks like if $\mathrm{proj}^*_S \circ \tau$ is a bijection on S. From now on we shall write $\tau_S = \mathrm{proj}^*_S \circ \tau$.

Corollary 6.22 *In the notation of Proposition 6.20,* τ_{S_ε} *has order* 2.

Proof Pick any $c \in S_\varepsilon$. Then $l(\delta_*(c^\tau, (\mathrm{proj}^*_{S_{-\varepsilon}}(c))^\tau)) = l(\delta_*(c, (\mathrm{proj}^*_{S_{-\varepsilon}}(c))))$. Therefore, by Lemma 6.21, $\mathrm{proj}^*_{S_\varepsilon}(c^\tau) = (\mathrm{proj}^*_{S_{-\varepsilon}}(c))^\tau$. The claim of the lemma follows. □

Lemma 6.23 *Let* R *be a residue of type* $M_J \cong A_m$ *for some* m *and assume that* $\mathrm{proj}^*_{R^\tau}$ *defines a bijection between* R *and* R^τ. *Then,* τ_R *is a type preserving automorphism of* R.

Proof Note first that both τ and $\mathrm{proj}^*_{R^\tau}$ define a bijection between the type set of R and the type set of $\tau(R)$. Both maps can either be equal or differ by opposition. We now prove that they cannot differ by opposition.

Let $x \in A_\tau(R)$ and consider an arbitrary twin-apartment Σ on x and x^τ. Note that $\mathrm{proj}^*_{R^\tau}(x) \in \Sigma$ and $\mathrm{proj}^*_R(x^\tau) \in \Sigma$. Moreover, since $x \in A_\tau(R)$, the chambers $\mathrm{proj}^*_{R^\tau}(x)$ and x^τ are opposite in $R^\tau \cap \Sigma$.

Let $y = \mathrm{proj}^*_\pi(x^\tau)$, where π is the j-panel on x in R. Then $y \in \Sigma \cap R$ and $l(\delta_*(y, y^\tau)) = l(\delta_*(x, x^\tau)) + 2$ by Lemma 6.14. More precisely, that lemma says that $y^\tau = \mathrm{proj}^*_{\pi^\tau}(y)$. In particular $y^\tau \in \Sigma$.

Note that $l(\delta_*(x, \mathrm{proj}^*_{R^\tau}(x))) = l(\delta_*(y, \mathrm{proj}^*_{R^\tau}\, y))$, but $l(\delta_*(x, x^\tau)) \neq l(\delta_*(y, y^\tau))$. Therefore, by definition of projection $\delta_{-\varepsilon}(\mathrm{proj}^*_{R^\tau}(y), y^\tau) \neq \delta_{-\varepsilon}(\mathrm{proj}^*_{R^\tau}(x), x^\tau) = w_{\tau(J)}$. Therefore if $\mathrm{proj}^*_{R^\tau}(y)$ and $\mathrm{proj}^*_{R^\tau}(x)$ are j' adjacent, then j' and $\tau(j)$ are not opposite. □

Proposition 6.24 *Assume the terminology of Proposition 6.20. Then, we have the following.*

(i) τ_{S_ε} *cannot preserve a panel.*

(ii) S_ε *cannot be of type* A_1;

(iii) S_ε *cannot be of type* A_2;

(iv) *if* S_ε *has type* $A_1 \times A_1$, *then either* $A_\tau(S_\varepsilon) = S_\varepsilon$ *or* τ_{S_ε} *interchanges the types;*

Proof Suppose π is an i-panel that is preserved by τ_{S_ε}. Thus the bijection $\mathrm{proj}^*_{S_\varepsilon}: S^\tau_\varepsilon \to S_\varepsilon$ restricts to a bijection between π^τ and π. Note that this bijection is proj^*_π.

However, by Lemma 6.16 we see that there is a chamber $c_\varepsilon \in \pi$ and a $w \in W^\tau$ with the property that $\delta_*(c_\varepsilon, c^\tau_\varepsilon) = s_i w s_{\tau(i)}$ and $\delta_*(d_\varepsilon, d^\tau_\varepsilon) = w$, for all $d_\varepsilon \in \pi - \{c_\varepsilon\}$ and $l(s_i w s_{\tau(i)}) = l(w)+2$. From the twin-building axioms it now follows that $c_\varepsilon = \mathrm{proj}^*_\pi(d^\tau_\varepsilon)$ for all $d_\varepsilon \in \pi$. Thus, proj^*_π is not bijective on π^τ, hence neither is $\mathrm{proj}^*_{S_\varepsilon}$ on S^τ_ε, a contradiction.

Part (ii) follows immediately from (i). To see (iii) note that in this case S_ε is a projective plane and any automorphism of order 2 necessarily has a fixed point. This fixed point is a panel that is preserved by $\mathrm{proj}^*_{S_\varepsilon} \circ \tau$, contradicting (i).

(iv) Suppose S_ε has type $A_1 \times A_1$. Then, by (i) τ_{S_ε} cannot preserve a panel. Therefore if it fixes type, then, τ_{S_ε} has no fixed points so that $A_\tau(S_\varepsilon) = S_\varepsilon$. □

Lemma 6.25 *If $R \neq S$ and $S = A_\tau(S)$, then $A_\tau(R)$ is connected in rank 2 and simply connected in rank 3.*

Proof By Proposition 6.20, $A_\tau(R)$ is the geometry opposite S. Connectedness is proved in [5, 4, 1]. Now let R have rank 3. If the diagram of R is disconnected, $A_\tau(R)$ is the product of connected residues, hence it is simply connected. Finally suppose R has type A_3. If S is a chamber then we are done by [1]. In view of Proposition 6.24 this leaves the case where S has type $A_1 \times A_1$. Now $A_\tau(R)$ is the geometry of all points, lines and planes of a projective 3-space that are opposite a fixed line l. That is the points and planes are those not incident to l and the lines are those not intersecting l. Consider any gallery γ in $A_\tau(R)$. It corresponds to a path of points and lines that all belong to $A_\tau(R)$. One easily verifies the following: Any two points are on some plane. Hence the collinearity graph Ξ on the point set of $A_\tau(R)$ has diameter 2. Any triangle in Ξ lies on a plane. Given any line m and two points p_1 and p_2 off that line, there is a point q on m that is collinear to p_1 and p_2 since lines have at least three points. It follows that quadrangles and pentagons in Ξ can be decomposed into triangles. Since triangles are geometric, γ is null-homotopic. □

Lemma 6.26 *If R has rank 2, then $A_\tau(R)$ is connected.*

Proof There are two cases: R has type A_2 or $A_1 \times A_1$. If R has type A_2, then by Proposition 6.24, S is a chamber and so by Lemma 6.25 we are done. Now let R have type $A_1 \times A_1$, then S is a chamber, in which case we are done again, or it is R. By Proposition 6.24, either $A_\tau(R) = R$, which is connected, or τ_R switches types and $A_\tau(R)$ is a complete bipartite graph with a perfect matching removed. This is connected since panels have at least three elements. □

Lemma 6.27 *Assume the notation of Proposition 6.20. Suppose that $R \cong R_1 \times R_2$ and $S \cong S_1 \times S_2$, where $\mathrm{typ}(S_i) \subseteq \mathrm{typ}(R_i)$ for $i = 1, 2$. Suppose moreover, that τ_S preserves the type sets I_i of the residue S_i (not necessarily point-wise). Then,*

(i) $\tau_R = \tau_{R_1} \times \tau_{R_2}$.

(ii) $A_\tau(R) \cong A_\tau(R_1) \times A_\tau(R_2)$.

Proof For $i = 1, 2$, let $J_i = \text{typ}(R_i)$ and let $I_i = \text{typ}(S_i)$. (i) Note that if, for $i = 1, 2$, R'_i is a residue of type J'_i in R then $R'_i \cap R'_2 = \{c\}$ for some chamber c and, for any $x \in R'_i$, $\text{proj}_{R'_2}(x) = c$. By assumption on S the same is true for residues S'_i of type I_i. Note further that the same applies to the residues R^τ and S^τ. Recall now that the isomorphism $R \cong R_1 \times R_2$ is given by $x \mapsto (x_1, x_2)$, where $x_i = \text{proj}_{R_i}(x)$. Thus in order to prove (i) it suffices to show that

$$\text{proj}_{R_i} \circ \text{proj}^*_R \circ \tau = \text{proj}^*_{R_i} \circ \tau \circ \text{proj}_{R_i}.$$

However, note that in fact

$$\tau_R \quad = \text{proj}^*_R \circ \tau = \text{proj}^*_S \circ \tau,$$

By Lemma 7.3 of [6] we have $\text{proj}^*_S = \text{proj}^*_S \circ \text{proj}_{S^\tau}$ so that

$$\tau_R \quad = \text{proj}^*_S \circ \tau = \text{proj}^*_S \circ \text{proj}_{S^\tau} \circ \tau,$$

and the same holds for R_i and S_i. Since τ is an isomorphism we also have $\text{proj}_{S^\tau} \circ \tau = \tau \circ \text{proj}_S$, so that

$$\begin{aligned} \tau_R \quad &= \text{proj}^*_S \circ \text{proj}_{S^\tau} \circ \tau = \text{proj}^*_S \circ \tau \circ \text{proj}_S, \\ \tau_{R_i} \quad &= \text{proj}^*_{S_i} \circ \text{proj}_{S_i{}^\tau} \circ \tau = \text{proj}^*_{S_i} \circ \tau \circ \text{proj}_{S_i}, \text{ for } i = 1, 2. \end{aligned}$$

Note at this point that $\text{proj}_S(x) = \text{proj}_S((x_1, x_2)) = (\text{proj}_{S_1} \circ \text{proj}_{R_1}(x)), \text{proj}_{S_2} \circ \text{proj}_{R_2}(x))$. In other words: $\text{proj}_S = \text{proj}_{S_1} \times \text{proj}_{S_2} = (\text{proj}_{S_1} \circ \text{proj}_{R_1}, \text{proj}_{S_2} \circ \text{proj}_{R_2})$. Thus in order to prove (i) it suffices to show that

$$\text{proj}_{S_i} \circ \text{proj}^*_S \circ \tau \circ \text{proj}_S = \text{proj}^*_{S_i} \circ \tau \circ \text{proj}_{S_i}, \text{ for } i = 1, 2.$$

This is equivalent to showing that on S we have

$$\text{proj}_{S_i} \circ \text{proj}^*_S \circ \tau = \text{proj}^*_{S_i} \circ \tau \circ \text{proj}_{S_i}, \text{ for } i = 1, 2.$$

To see this, first pick some $x \in S$ and note that if x lies on the I_2-residue S'_2, then $x, \text{proj}_{S_1}(x) \in S'_2$, thus $\tau(x), \tau \circ \text{proj}_{S_1}(x) \in S'^\tau_2$. But since τ_S is type-preserving, we have $\text{proj}^*_S \circ \tau(x), \text{proj}^*_S \circ \tau \circ \text{proj}_{S_1}(x) \in \text{proj}^*_S(S'_2) = S''_2$, and S''_2 is again of type I_2. Therefore, the projection on S_1 of these two chambers is the same, namely $S_1 \cap S''_2$. Namely, $\text{proj}_{S_1} \circ \text{proj}^*_S \circ \tau(x) = \text{proj}_{S_1} \circ \text{proj}^*_S \circ \tau \circ \text{proj}_{S_1}(x) = S_1 \cap S''_2$. Noting that $\text{proj}_{S_1} \circ \text{proj}^*_S(y) = \text{proj}^*_{S_1}(y)$ for any $y \in S^\tau$, we have $\text{proj}_{S_1} \circ \text{proj}^*_S \circ \tau(x) = (\text{proj}_{S_1} \circ \text{proj}^*_S) \circ \tau \circ \text{proj}_{S_1}(x) = \text{proj}^*_{S_1} \circ \tau \circ \text{proj}_{S_1}(x)$, that is, $\text{proj}_{S_1} \circ \tau_S = \tau_{S_1} \circ \text{proj}_{S_1}$,which proves the claim.

(ii) Let $x = (x_1, x_2) \in R_1 \times R_2$, and suppose $R \subseteq \Delta_\varepsilon$. Then, by (i),

$$\begin{aligned} \delta_\varepsilon(x, x^\tau) = &\quad \delta((x_1, x_2), \tau_R(x_1, x_2)) \\ = &\quad \delta((x_1, x_2), (\tau_{R_1}(x_1), \tau_{R_2}(x_2))) \\ = &\quad \delta_1(x_1, \tau_{R_1}(x_1)) \cdot \delta_2(x_2, \tau_{R_2}(x_2)). \end{aligned}$$

Since $A_\tau(R_1) \times A_\tau(R_2) \subseteq R_1 \times R_2$, we see that $\delta(x, \tau_R(x))$ is maximal if and only if $\delta(x_i, \tau_{R_i}(x_i))$ is maximal for $i = 1, 2$. Thus $A_\tau(R) \cong A_\tau(R_1) \times A_\tau(R_2)$. $\square$

Theorem 6.28 *Suppose that* $|\mathrm{k}| \geq 5$. *If* R *has rank* 3, *then* $A_\tau(R)$ *is connected and simply* 2*-connected.*

Proof The residue R has one of three possible types: A_3, $A_2 \times A_1$, or $A_1 \times A_1 \times A_1$. In view of Lemma 6.25 we will ignore the cases where $S = A_\tau(S)$ is a proper residue of R.

Since S is a residue, but not a chamber, a panel, or a residue of type A_2, and $S \neq A_\tau(S)$, it follows from Proposition 6.24 that either $R = S$ or S has type $A_1 \times A_1$ and τ_S switches types on S. The theorem will now follow from Lemmas 6.29 and 6.30. □

Lemma 6.29 *If* R *has disconnected diagram of rank* 3, *then* $A_\tau(R)$ *is connected and simply connected.*

Proof We show that in all cases Lemma 6.27 applies. If R has type $A_1 \times A_1 \times A_1$, then let τ act on the types of R. It either fixes all types or it has two orbits I_1 and I_2, where we may assume $|I_2| = 2$. Moreover, if S has type $A_1 \times A_1$, then we can write $S \cong S_1 \times S_2$, where $S_1 = \{c\} \subseteq R_1$, $S_2 = R_2$ and R_i has type I_i, for $i = 1, 2$. If $R = S$, then we can take $S_i = R_i$, where R_i as above. One verifies that Lemma 6.27 applies.

We now turn to the case, where R has type $A_2 \times A_1$. Let J_i be the underlying type set of type A_i. Since τ is an adjacency preserving permutation of R of order 2, it must preserve the type sets J_1 and J_2. In particular if S has type $A_1 \times A_1$, τ_S must be type preserving. Take R_i to be a residue on $c \in R$ of type J_i. Let $S_1 = R_1$ and let $S_2 = S \cap R_2$. Now again Lemma 6.27 applies.

By Lemma 6.27, $A_\tau(R) \cong A_\tau(R_1) \times A_\tau(R_2)$. By Lemma 6.26, $A_\tau(R_i)$ is connected, hence $A_\tau(R)$ is connected and simply connected. □

Lemma 6.30 *If* R *is of type* A_3 *and* $|\mathrm{k}| \geq 5$ *then the geometry* $A_\tau(R)$ *is connected and simply connected.*

Proof

Case 1: $S = R$**.** By Lemma 6.23, τ_R is given by an involutory semilinear map ϕ on a 4-dimensional vector space. Since $S = R$, we also know that ϕ has no fixed points. We now define the objects of the geometry $A_\tau(R)$. All points and all planes of $\mathrm{PG}(V)$ belong to $A_\tau(R)$. The only lines in the geometry are those 2-dimensional spaces of V that are not fixed by ϕ. These will be called *good lines*. Points will be denoted by lowercase letters, good lines will be denoted by uppercase letters and planes will be denoted by greek letters.

We now describe incidence. We shall use containment relations only for containment in $\mathrm{PG}(V)$, not to be confused with incidence in $A_\tau(R)$. Any point contained in a good line will be incident to it and any plane containing a good line will be incident to it. A point p will be incident to a plane π if and only if $p \subseteq \pi$ and $p \not\subseteq \pi^\phi$.

We now gather some basic properties of $A_\tau(R)$. Any plane π is incident to any point p that is not contained in the only bad line $\pi \cap \pi^\phi$ of π. It follows that any two points incident to a plane will be collinear. and any point p is incident to all planes π so that

$p \subseteq \pi$ but π does not contain the only bad line $\langle p, p^\phi \rangle$ containing p. If a line L is incident to a plane π, then all but one point incident to L is incident to π.

Connectivity is quite immediate since any two points p_1, p_2 that are not collinear will be collinear to any other point not in the unique bad line $\langle p_1, p_2 \rangle$ on p_1 (and p_2).

In order to prove simple connectivity we first reduce any path to a path in the collinearity graph. Indeed any path $p_1 \pi p_2$ will be homotopically equivalent to the path $p_1 L p_2$ where $L = \langle p_1, p_2 \rangle$. Any path $p \pi L$ will be homotopically equivalent to the path $pL'p'L$ where p' is a point on L that is also incident to π and $L' = \langle p, p' \rangle$. Note that since p' is incident to π, L' is a good line. Finally a path $L_1 \pi L_2$ is homotopically equivalent to the path $L_1 p_1 L' p_2 L_2$ where p_i are points on L_i that are incident to π and $L' = \langle p_1, p_2 \rangle$.

Therefore, to show simple connectedness we can restrict to paths in the collinearity graph. Note also the fact that if p is a point and L is a good line not incident to p then p will be collinear to all but at most one point on L (namely the intersection of the unique bad line on p and L if this intersection exists). This enables the decomposition of any path in the collinearity graph to triangles. Indeed, the diameter of the collinearity graph is two and so any path can be decomposed into triangles, quadrangles and pentagons. Moreover, if p_1, p_2, p_3, p_4 is a quadrangle then, since $|k| \geq 4$, the line $\langle p_2, p_3 \rangle$ will admit a point collinear to both p_1 and p_4 decomposing the quadrangle into triangles. Similarly, if p_1, p_2, p_3, p_4, p_5 is a pentagon, then there will be a point on the good line $\langle p_3, p_4 \rangle$ that is collinear to p_1. Thus, the pentagon decomposes into quadrangles. Therefore it suffices to decompose triangles into geometric triangles.

Assume that p_1, p_2, p_3 is a triangle. The plane $\pi = \langle p_1, p_2, p_3 \rangle$ is incident to all three (good) lines in the triangle and so, either the triangle is geometric and then we are done, or one of the points is not incident to π. Let us assume that p_1 is not incident to π.

Consider a plane π' that contains the line $\langle p_2, p_3 \rangle$ and so that p_2 and p_3 are incident to π'. This is certainly possible since $|\mathrm{k}| \geq 4$ and one only need to stay clear of the planes $\langle p_2, p_3, p_3^\phi \rangle$ and $\langle p_2, p_3, p_2^\phi \rangle$. Note that by choice of π', any line L with $p_i \subseteq L \subseteq \pi'$ is good.

Let now for each $i = 2, 3$

$$\mathcal{L}_i = \{L \subseteq \pi' | L \neq \langle p_2, p_3 \rangle, p_i \subseteq L, \text{ and } p_i \text{ is incident to the plane } \langle p_1, p_i, L \rangle\}.$$

We have $\mathcal{L}_i = |\mathrm{k}| - 1$, the only lines of π' on p_i not in $\mathcal{L}_i$ are $\langle p_2, p_3 \rangle$ and $\langle p_1, p_i, p_i^\phi \rangle \cap \pi'$. Note that if $L \in \mathcal{L}_i$ then L only admits one point not incident to π'. Pick lines distinct lines $L_{i,j} \in \mathcal{L}_i$ with $j = 1, 2, 3$. Of the 9 intersection points at most 6 are not incident to one of the three planes that they define. Pick any one of the remaining 3 points and use it as the point p above.

Case 2: S **of type** $A_1 \times A_1$**.** The geometry is rather similar to the previous one. There is a line $\mathbf{L}$ so that S is the residue corresponding to $\mathbf{L}$ and the map τ_S induces a pairing between points of $\mathbf{L}$ and planes on $\mathbf{L}$. The geometry $A_\tau(R)$ is described as follows. The points of the geometry are all the points of V not in $\mathbf{L}$, the lines of the geometry are all the lines of V not intersecting $\mathbf{L}$ and the planes are all planes of V not containing $\mathbf{L}$.

We now describe incidence. Any line included in a plane is incident to it and any point included in a line is incident to it. A point p is incident to a plane π if and only if the plane $\pi' = \langle p, \mathbf{L} \rangle$ is not paired to the point $p' = \mathbf{L} \cap \pi$.

We now gather a few useful properties of this geometry. Note a number of similarities with the previous geometry. Any plane π is incident to all the points $p \subseteq \pi$ so that p is not contained in the bad line $\pi' \cap \pi$ where π' is the plane paired to the point $\pi \cap \mathbf{L}$. Similarly any point p is incident to any plane π if $p \subseteq \pi$ and $\langle p, p' \rangle \nsubseteq \pi$ where p' is the point paired to the plane $\langle p, \mathbf{L} \rangle$. If p is a point and L is a good line not incident to p then p will be collinear to all but one point on L; namely the non-collinear point on L is the intersection of L with the bad plane $\langle p, \mathbf{L} \rangle$.

Any two points p_1, p_2 that are not collinear have the property that $\langle p_1, p_2 \rangle$ intersects $\mathbf{L}$ and so any point not in $\langle p_1, p_2, \mathbf{L} \rangle$ will be collinear to both p_1 and p_2. In particular, the geometry $A_\tau(R)$ is connected and the diameter of the collinearity graph is 2.

The reduction to the collinearity graph is a little more involved because not every two points on a good plane will be collinear. However any two non-collinear points incident to a good plane π are collinear to any other point incident to π since $\mathbf{L}$ intersects π in exactly one point.

The previous remark immediately shows that a path of type $p_1 \pi p_2$ can be replaced by a path p_1, L_1, p', L_2, p_2, where all elements are incident to π. Suppose we have a path of type $p\pi L$. Since π is incident to all but one point on the line L and p is collinear to all but one point on the line L, we can replace this path by one of type $p_1 L_1 p_2 L$, where all objects are incident to π. Suppose we have a path of type $L_1 \pi L_2$. This reduces to the previous case since all but one point of L_1 are incident to π.

As before, given any line L and two points p_1 and p_2 not on L, there are only two points on L that are not collinear to at least one of p_1 and p_2. The proof that all paths in the collinearity graph decompose into triangles is identical. Therefore it suffices to show that any triangle decomposes into geometric triangles.

Finally we need to modify the argument above to decompose triangles. The only difference is once more the fact that not every two points incident to a good plane are collinear. As a consequence the sets $\mathcal{L}_i$ only have $|\mathrm{k}| - 2$ lines because one needs to exclude the space $\langle p_i, \pi' \cap \mathbf{L} \rangle$. Moreover, each line of $\mathcal{L}_i$ has three forbidden points. Namely, in addition to the two as in the previous case, it has one point that is not collinear to p_1 since it lies on the plane $\langle p_1, \mathbf{L} \rangle$. If $|\mathrm{k}| \geq 5$, then we can choose four lines from $\mathcal{L}_2$ and $\mathcal{L}_3$ and see that out of the 16 intersection points at most 9 are bad. Pick any one of those remaining points p and notice that it is collinear to all of the p_i and pp_ip_j is a geometric triangle for all $i \neq j$. This decomposes the initial triangle into geometric triangles. $\square$

Proof (of Theorem 3) By Lemma 6.26 and Theorem 6.28 the residual filtration $\mathcal{C}$ satisfies the conditions of Theorem 6.11. It follows that Δ^τ is connected and simply connected and so by Tits' Lemma [24, Corollaire 1], G^τ is the universal completion of the amalgam of maximal parabolics $\{P_i\}_{i \in I}$ with respect to the action on Δ^τ. From the diagram Γ of type $\widetilde{A}_{n-1}$ we see that each of the maximal parabolics of G^τ is a quotient of $\mathrm{SL}_n(\mathrm{k})$ and the intersection $\mathcal{L}_i = \{L_j, L_{j,k} \mid j, k \neq i\}$ of $\mathcal{L}^\tau$ with the maximal parabolic P_i is exactly the Curtis-Tits amalgam for that linear group. Now let $\tilde{G}$ be the universal completion of $\mathcal{L}^\tau$. Since $\mathcal{L}^\tau$ generates G^τ, there is a unique surjective homomorphism $\tilde{\phi}: \tilde{G} \to G^\tau$ that restricts to the inclusions on the groups in $\mathcal{L}^\tau$. The classical Curtis-Tits theorem ensures that each maximal parabolic P_i is the universal completion of the subamalgam $\mathcal{L}_i$. In particular there exists a unique homomorphism $\phi_i: P_i \to \tilde{G}$ that maps surjectively to the subgroup of $\tilde{G}$ generated by $\mathcal{L}_i$. This makes $\tilde{G}$ a completion of the amalgam of maximal parabolics. It follows that there exists a unique surjective homomorphism $\phi^\tau: G^\tau \to \tilde{G}$. The standard universality argument applied to $\tilde{\phi}$ and ϕ^τ now ensures that $G^\tau \cong \tilde{G}$. □

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