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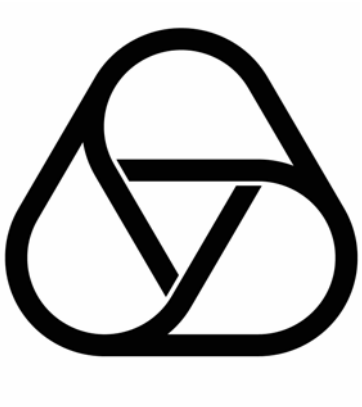

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BERTFRIED FAUSER, PETER D. JARVIS, AND RONALD C. KING

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PLETHYSMS, REPLICATED SCHUR FUNCTIONS AND SERIES, WITH APPLICATIONS TO VERTEX OPERATORS

BERTFRIED FAUSER, PETER D. JARVIS, AND RONALD C. KING

The characters of the orthogonal and symplectic groups have been found by Schur [34] *and Weyl* [35] *respectively. The method used is transcendental, and depends on integration over the group manifold. These characters, however, may be obtained by purely algebraic methods, ... This algebraic method would seem to offer a better prospect of successful application to other restricted groups than the method of group integration.*

D.E. Littlewood, Phil. Trans. Roy. Soc. London, Ser. A, Vol 239, No 809, 1944, p.392

ABSTRACT. Specializations of Schur functions are exploited to define and evaluate the Schur functions $s_\lambda[\alpha X]$ and plethysms $s_\lambda[\alpha s_\nu(X))]$ for any α - integer, real or complex. Plethysms are then used to define pairs of mutually inverse infinite series of Schur functions, M_π and L_π, specified by arbitrary partitions π. These are used in turn to define and provide generating functions for formal characters, $s_\lambda^{(\pi)}$, of certain groups H_π, thereby extending known results for orthogonal and symplectic group characters. Each of these formal characters is then given a vertex operator realization, first in terms of the series $M = M_{(0)}$ and various $L_\sigma^\perp$ dual to L_σ, and then more explicitly in exponential form. Finally the replicated form of such vertex operators are written down.

1. INTRODUCTION

The aim here is to exploit the Hopf algebra structure of the ring $\Lambda(X)$ of symmetric functions of the independent variables $(x_1, x_2, \ldots)$, finite or countably infinite in number, that constitute the alphabet X. An emphasis will be placed on the interconnections between the various products and coproducts that apply to the Schur functions $s_\lambda(X)$ that form an integral basis of $\Lambda(X)$. These allow us to define certain replicated, rational or scaled plethysms that involve an argument α in $\mathbb{N}$, $\mathbb{Q}$ or $\mathbb{R}$, respectively, or even to $\mathbb{C}$ or a sequence of such parameters.

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The first key result is that, for any alphabet $X = (x_1, x_2, \dots)$ and parameter α, and any partitions λ and ν, we have

$$s_\lambda[\alpha s_\nu(X)] = \sum_{\mu,\rho} g^\lambda_{\rho,\mu}\, \dim_\mu(\alpha)\, s_\mu[s_\nu(X)]\,, \tag{1-1}$$

where the coefficients $g^\lambda_{\rho,\mu}$ are Kronecker coefficients associated with products of characters of the symmetric group S_m with $m = |\lambda|$, the weight of the partition λ, while $\dim_\mu(\alpha)$ is the polynomial in α that gives the dimension of the irreducible representation of $GL(n)$ specified by the partition μ evaluated at $n = \alpha$. The map $\dim : \Lambda(X) \longrightarrow R$ is an algebra homomorphism for any target ring R.

Following some notational preliminaries in Section 2, this result is obtained in Section 4 through the use of one of the specializations introduced in Section 3. Section 4 also contains some examples of replicated plethysms and the computer benchmarking of their calculation, showing that the formula (1-1) is very efficient. The relevant algorithm is relegated to Appendix C in the form of appropriate computer pseudo code. In the special case $\nu = (1)$ for which $s_\nu(X) = X$, the above plethysms coincide both with the replicated plethysms of Jarvis and Yung [15] and, setting $\alpha = q$, with the q-analogues of Schur functions introduced by Brenti [4]. Section 4 includes an account of their orthogonality properties as given by Baker [2] and Brenti [4] but obtained here by exploiting the Schur-Hall scalar product for the ring $\Lambda(X)$.

The next result realizes Littlewood's hope that an algebraic treatment of character theory greatly generalizes the scope of the classical approach so as to encompass cases which are very difficult to treat by analytical methods. This same scalar product, in the form of the Cauchy identity, is then exploited in Section 5 to derive the character generating function

$$L_\pi(Z)\, M(XZ) = \sum_\lambda s^{(\pi)}_\lambda(X)\, s_\lambda(Z) \tag{1-2}$$

for formal characters $s^{(\pi)}_\lambda(X)$ of H_π, each specified by a partition λ, where H_π is the subgroup of the general linear group preserving an invariant form of symmetry π, as introduced elsewhere [11]. Here $X = (x_1, x_2, \dots)$ is to be evaluated at the sequence of eigenvalues of group elements of H_π. The notation is such that $M(XZ) = \prod_{i,j\geq 1}(1 - x_i z_j)^{-1}$, while $L_\pi(Z) = L[s_\pi(Z)]$ is an infinite Schur function series plethysm, with $L(Y) = \prod_{k\geq 1}(1 - y_k)$ for all Y, including the case for which the elements y_k of Y are the monomials of $s_\pi(Z)$. In this case, for an alphabet Z of cardinality l, the cardinality of Y is exactly $\dim_\pi(l)$.

By exploiting the same series L, its inverse M and its dual $L^\perp$ (i.e. its adjoint with respect to the Schur-Hall scalar product), together with the Hopf algebra structure of $\Lambda(X)$, a vertex operator realization of the characters $s_{\lambda^{(\pi)}}(X)$ is derived in Section 6. This takes the form of another key result, namely,

$$s^{(\pi)}_\lambda = [Z^\lambda]\, V^\pi(z_1)\, V^\pi(z_2) \cdots V^\pi(z_l) \cdot 1\,, \tag{1-3}$$

where the vertex operators are given by

$$V^\pi(z) = (1 - z^p\, \delta_{\pi,(p)})\, M(z)\, L^\perp(z^{-1}) \prod_{k=1}^{p-1} L^\perp_{\pi/(k)}(z^k)\,. \tag{1-4}$$

Then by means of exponential expressions for both M and $L^{\perp}$, these vertex operators are explicitly constructed in exponential form for all partitions π of weight $|\pi| \leq 3$. The case $\pi = (2,1)$ is also given an alternative normal ordering derivation in Section 6. The explicit evaluation of $L_{\pi}^{\perp}(w)(M(Z))$ is undertaken in Appendix A, with the result expressed, perhaps somewhat surprisingly, in terms of semistandard Young tableaux of shape specified by the partition π.

In Section 7 it is pointed out rather briefly that a wide class of replicated vertex operators may be obtained very easily through the application of the replicated Schur functions of Section 4 to parametrized versions of the vertex operators of Section 6.

Finally, Section 8 consists of a few concluding remarks.

2. NOTATIONAL PRELIMINARIES

2.1. **Partitions and Young diagrams.** Our notation follows in large part that of Macdonald [29]. Partitions are specified by lower case Greek letters. If λ is a partition of n we write $\lambda \vdash n$, and $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ is a sequence of non-negative integers λ_i for $i = 1, 2, \dots, n$ such that $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq 0$, with $\lambda_1 + \lambda_2 + \cdots + \lambda_n = n$. The partition λ is said to be of weight $|\lambda| = n$ and length $\ell(\lambda)$, where $\lambda_i > 0$ for all $i \leq \ell(\lambda)$ and $\lambda_i = 0$ for all $i > \ell(\lambda)$. In specifying λ the trailing zeros, that is those parts $\lambda_i = 0$, are often omitted, while repeated parts are sometimes written in exponent form $\lambda = (\cdots, 2^{m_2}, 1^{m_1})$ where λ contains m_i parts equal to i for $i = 1, 2, \dots$. For each such partition, $n(\lambda) = \sum_{i=1}^{n}(i-1)\lambda_i$ and $z_\lambda = \prod_{i \geq 1} i^{m_i}\, m_i!$.

Each partition λ of weight $|\lambda|$ and length $\ell(\lambda)$ defines a Young or Ferrers diagram, F^λ, consisting of $|\lambda|$ boxes or nodes arranged in $\ell(\lambda)$ left-adjusted rows of lengths from top to bottom $\lambda_1, \lambda_2, \dots, \lambda_{\ell(\lambda)}$ (in the English convention). The partition λ', conjugate to λ, is the partition specifying the column lengths of F^λ read from left to right. The box $(i,j) \in F^\lambda$ in the ith row and jth column is said to have content $c(i,j) = j - i$ and hook length $h(i,j) = \lambda_i + \lambda'_j - i - j + 1$.

By way of illustration, if $\lambda = (4,2,2,1,0,0,0,0,0) = (4,2,2,1) = (4,2^2,1)$ then $|\lambda| = 9$, $\ell(\lambda) = 4$, $\lambda' = (4,3,1^2)$,

$$F^\lambda = F^{(4,2^2,1)} = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & \\ \cline{1-2} & \\ \cline{1-2} \\ \cline{1-1} \end{array} \quad \text{and} \quad F^{\lambda'} = F^{(4,3,1^2)} = \begin{array}{|c|c|c|c|}\hline & & & \\ \hline & & \\ \cline{1-3} \\ \cline{1-1} \\ \cline{1-1} \end{array} \quad . \tag{2-1}$$

The content and hook lengths of F^λ are specified by

$$\begin{array}{|c|c|c|c|}\hline 0 & 1 & 2 & 3 \\ \hline \overline{1} & 0 \\ \cline{1-2} \overline{2} & 1 \\ \cline{1-2} \overline{3} \\ \cline{1-1} \end{array} \quad \text{and} \quad \begin{array}{|c|c|c|c|}\hline 7 & 5 & 2 & 1 \\ \hline 4 & 2 \\ \cline{1-2} 3 & 1 \\ \cline{1-2} 1 \\ \cline{1-1} \end{array} \quad , \tag{2-2}$$

where $\overline{m} = -m$ for all m. In addition, $n(4,2^2,1) = 0 \cdot 4 + 1 \cdot 2 + 2 \cdot 2 + 3 \cdot 1 = 9$ and $z_{(4,2^2,1)} = 4 \cdot 2^2 \cdot 1 \cdot 1! \cdot 2! \cdot 1! = 32$.

2.2. **The ring $\Lambda(X)$ and Schur functions.** There exist various bases of $\Lambda(X)$ as described in [29]: the monomial symmetric functions $\{m_\lambda\}_\lambda$, the complete symmetric functions $\{h_\lambda\}_\lambda$, the elementary symmetric functions $\{e_\lambda\}_\lambda$, the power sum symmetric functions $\{p_\lambda\}_\lambda$ and the Schur symmetric functions $\{s_\lambda\}_\lambda$. Three of these bases are multiplicative, with $h_\lambda = h_{\lambda_1} h_{\lambda_2} \cdots h_{\lambda_n}$, $e_\lambda = e_{\lambda_1} e_{\lambda_2} \cdots e_{\lambda_n}$ and $p_\lambda = p_{\lambda_1} p_{\lambda_2} \cdots p_{\lambda_n}$. Of the relationships between the various bases we just mention at this stage the transitions

$$p_\rho(X) = \sum_{\lambda \vdash n} \chi^\lambda_\rho \, s_\lambda(X) \qquad \text{and} \qquad s_\lambda(X) = \sum_{\rho \vdash n} z_\rho^{-1} \, \chi^\lambda_\rho \, p_\rho(X) \,, \tag{2-3}$$

where χ^λ_ρ is the character of the irreducible representation of the symmetric groups S_n specified by λ in the conjugacy class specified by ρ. These characters satisfy the orthogonality conditions

$$\sum_{\rho \vdash n} z_\rho^{-1} \, \chi^\lambda_\rho \, \chi^\mu_\rho \;=\; \delta_{\lambda,\mu} \qquad \text{and} \qquad \sum_{\lambda \vdash n} z_\rho^{-1} \, \chi^\lambda_\rho \, \chi^\lambda_\sigma \;=\; \delta_{\rho,\sigma} \,. \tag{2-4}$$

The significance of the Schur function basis lies in the fact that with respect to the usual Schur-Hall scalar product $\langle \cdot \,|\, \cdot \rangle_{\Lambda(X)}$ on $\Lambda(X)$ we have

$$\langle s_\mu(X) \,|\, s_\nu(X) \rangle_{\Lambda(X)} = \delta_{\mu,\nu} \,. \tag{2-5}$$

From (2-3) and (2-4) it follows that

$$\langle p_\rho(X) \,|\, p_\sigma(X) \rangle_{\Lambda(X)} = z_\rho \delta_{\rho,\sigma} \,. \tag{2-6}$$

In what follows we shall make considerable use of several infinite series of Schur functions. The most important of these are the mutually inverse pair defined by

$$M(t;X) = \prod_{i \geq 1} (1 - t\, x_i)^{-1} = \sum_{k \geq 0} h_m(X) \, t^m \,; \tag{2-7}$$

$$L(t;X) = \prod_{i \geq 1} (1 - t\, x_i) = \sum_{k \geq 0} (-1)^m \, e_m(X) \, t^m \,, \tag{2-8}$$

where as Schur functions $h_m(X) = s_{(m)}(X)$ and $e_m(X) = s_{(1^m)}(X)$. It might be noted that in Macdonald's notation and λ-ring notation $M(t;X) = H(t) = \sigma_t(X)$ and $L(t;X) = E(-t) = \lambda_{-t}(X)$. For convenience, in the case $t = 1$ we write $M(1;X) = M(X)$ and $L(1;X) = L(X)$.

2.3. **Algebraic properties of $\Lambda(X)$.** The ring, $\Lambda(X)$, of symmetric functions over X has a Hopf algebra structure, and two further algebraic and two coalgebraic operations. For notation and basic properties we refer for example to [10, 11] and references therein. For the moment, in the interest of typographical simplicity, the symbol X for the underlying alphabet is suppressed unless specifically required.

We indicate outer products on Λ either by m, or with infix notation using juxtaposition. Inner products are denoted either by m or as infix by $\star$, while plethysms (compositions) are denoted by $\circ$ or by means of square brackets $[\quad]$. The corresponding coproduct maps are specified by Δ for the outer coproduct, δ for the inner coproduct, and ∇ for the plethysm coproduct. In Sweedler notation the action of these coproducts is distinguished by means of different brackets,

round, square and angular, around the Sweedler indices – the so-called Brouder-Schmitt convention. The coproduct coefficients themselves are obtained from the products by duality using the Schur-Hall scalar product and the self-duality of $\Lambda(X)$. For example, for all $A, B \in \Lambda(X)$:

$$\begin{aligned} m(A \otimes B) &= AB\,; & \Delta(A) &= A_{(1)} \otimes A_{(2)}\,; \\ \mathsf{m}(A \otimes B) &= A \star B\,; & \delta(A) &= A_{[1]} \otimes A_{[2]}\,; \\ A \circ B &= A[B]\,; & \nabla(A) &= A_{<1>} \otimes A_{<2>}\,. \end{aligned}$$

In terms of the Schur function basis $\{s_\lambda\}_{\lambda\vdash n, n\in\mathbb{N}}$ the product and coproduct maps give rise to the particular sets of coefficients specified as follows:

$$\begin{aligned} s_\mu s_\nu &= \sum_\lambda c^\lambda_{\mu,\nu} s_\lambda\,; & \Delta(s_\lambda) &= s_{\lambda_{(1)}} \otimes s_{\lambda_{(2)}} = \sum_{\mu,\nu} c^\lambda_{\mu,\nu} s_\mu \otimes s_\nu\,; \\ s_\mu \star s_\nu &= \sum_\lambda g^\lambda_{\mu,\nu} s_\lambda\,; & \delta(s_\lambda) &= s_{\lambda_{[1]}} \otimes s_{\lambda_{[2]}} = \sum_{\mu,\nu} g^\lambda_{\mu,\nu} s_\mu \otimes s_\nu\,; \\ s_\mu[s_\nu] &= \sum_\lambda p^\lambda_{\mu,\nu} s_\lambda\,; & \nabla(s_\lambda) &= s_{\lambda_{\langle 1\rangle}} \otimes s_{\lambda_{\langle 2\rangle}} = \sum_{\mu,\nu} p^\lambda_{\mu,\nu} s_\mu \otimes s_\nu\,. \end{aligned}$$

Here the $c^\lambda_{\mu,\nu}$ are Littlewood-Richardson coefficients, the $g^\lambda_{\mu,\nu}$ are Kronecker coefficients and the $p^\lambda_{\mu,\nu}$ are plethysm coefficients. All these coefficients are non-negative integers. The Littlewood-Richardson coefficients can be obtained, for example, by means of the Littlewood-Richardson rule [27, 25] or the hive model [5]. The Kronecker coefficients may determined directly from characters of the symmetric group or by exploiting the Jacobi-Trudi identity and the Littlewood-Richardson rule [31], while plethysm coefficients have been the subject of a variety methods of calculation [26, 31, 7]. Note that the above sums are finite, since

$$\begin{aligned} c^\lambda_{\mu,\nu} \geq 0 \qquad &\text{iff} \quad |\lambda| = |\mu| + |\nu|\,; \\ g^\lambda_{\mu,\nu} \geq 0 \qquad &\text{iff} \quad |\lambda| = |\mu| = |\nu|\,; \\ p^\lambda_{\mu,\nu} \geq 0 \qquad &\text{iff} \quad |\lambda| = |\mu|\,|\nu|\,. \end{aligned}$$

The Schur-Hall scalar product may be used to define skew Schur functions $s_{\lambda/\mu}$ through the identities

$$c^\lambda_{\mu,\nu} = \langle\, s_\mu\, s_\nu \,|\, s_\lambda \,\rangle = \langle\, s_\nu \,|\, s^\perp_\mu(s_\lambda) \,\rangle = \langle\, s_\nu \,|\, s_{\lambda/\mu} \,\rangle\,, \tag{2-9}$$

so that

$$s_{\lambda/\mu} = \sum_\nu c^\lambda_{\mu,\nu}\, s_\nu\,. \tag{2-10}$$

Within the outer product Hopf algebra we have a unit Id, a counit ε and an antipode S such that[1]:

$$\mathrm{Id}(1) = s_0\,; \qquad \varepsilon(s_\lambda) = \delta_{\lambda,(0)}\,; \qquad \mathsf{S}(s_\lambda) = (-1)^{|\lambda|} s_{\lambda'}\,. \tag{2-11}$$

[1]Macdonald uses the involution ω which differs from the antipode by a sign factor: $\mathsf{S}(s_\lambda) = (-1)^{\ell(\lambda)}\omega(s_\lambda)$. It is, however, convenient to employ the antipode if Hopf algebra structures are in use.

2.4. **The Cauchy kernel.** It is often convenient to represent an alphabet in an additive manner $X = x_1 + x_2 + \cdots$, as itself an element of the ring $\Lambda(X)$ in the sense that

$$X = x_1 + x_2 + \cdots = h_1(X) = e_1(X) = p_1(X) = s_{(1)}(X)\,.$$

As elements of $\Lambda(X) \otimes \Lambda(Y)$ we have

$$\begin{aligned} X+Y &= x_1 + x_2 + \cdots + y_1 + y_2 + \cdots \\ XY &= (x_1 + x_2 + \cdots)(y_1 + y_2 + \cdots) = (x_1y_1 + x_1y_2 + \cdots + x_2y_1 + x_2y_t + \cdots)\,. \end{aligned}$$

With this notation, the outer coproduct gives

$$\begin{aligned} \Delta(M) &= M_{(1)} \otimes M_{(2)} = M \otimes M & M(X+Y) &= \prod_i \frac{1}{1-x_i} \prod_j \frac{1}{1-y_j}\,; \\ \Delta(L) &= L_{(1)} \otimes L_{(2)} = L \otimes L & L(X+Y) &= \prod_i (1-x_i) \prod_j (1-y_j)\,, \end{aligned}$$

so that

$$M(X+Y) = M(X)\,M(Y) \quad \text{and} \quad L(X+Y) = L(X)\,L(Y)\,. \tag{2-12}$$

For the inner coproduct:

$$\begin{aligned} \delta(M) &= M_{[1]} \otimes M_{[2]} & M(XY) &= \prod_{i,j} \frac{1}{1-x_iy_j}\,; \\ \delta(L) &= L_{[1]} \otimes L_{[2]} & L(XY) &= \prod_{i,j} (1-x_iy_j)\,. \end{aligned}$$

The expansions of the products on the right hand sides of these expressions is effected remarkably easily by evaluating the inner coproducts on the left:

$$\begin{aligned} \delta(M) &= \sum_{k\geq 0} \delta(h_k) = \sum_{k\geq 0} \sum_{\lambda\vdash k} s_\lambda \otimes s_\lambda\,; \\ \delta(L) &= \sum_{k\geq 0} (-1)^k\, \delta(e_k) = \sum_{k\geq 0} (-1)^k \sum_{\lambda\vdash k} s_\lambda \otimes s_{\lambda'}\,. \end{aligned}$$

This gives immediately the well known Cauchy and Cauchy-Binet formulae:

$$M(XY) = \prod_{i,j} \frac{1}{1-x_i\,y_j} = \sum_\lambda s_\lambda(X)\,s_\lambda(Y)\,; \tag{2-13}$$

$$L(XY) = \prod_{i,j} (1-x_i\,y_j) = \sum_\lambda (-1)^{|\lambda|} s_\lambda(X)\,s_{\lambda'}(Y)\,. \tag{2-14}$$

That the Cauchy kernel, $M(XY)$, is a dual version of the Schur-Hall scalar product can be seen by noting that

$$s_\mu(X)\; M(XY) = \sum_\lambda \sum_\nu c^\nu_{\mu,\lambda}\; s_\nu(X)\,s_\lambda(Y)$$

$$= \sum_{\nu} s_\nu(X)\, s_{\nu/\mu}(Y) = s_\mu^\perp(Y)\,(M(XY)). \tag{2-15}$$

More generally, for any $F(X) \in \Lambda(X)$ with dual $F^\perp(X)$, by linearly extending the above result we have

$$F(X)\, M(XY) = F^\perp(Y)\,(M(XY)). \tag{2-16}$$

This is an identity that will be encountered and exploited a number of times in later sections.

2.5. **Plethysms.** Plethysms are defined as compositions whereby for any $A, B \in \Lambda(X)$ the plethysm $A[B]$ is A evaluated over an alphabet Y whose letters are the monomials of $B(X)$, with each letter repeated as many times as the multiplicity of the corresponding monomial. Thus the Schur function plethysm is defined by

$$s_\lambda[s_\mu](X) = s_\lambda(Y) \qquad \text{where} \qquad Y = s_\mu(X)\,. \tag{2-17}$$

For all $A, B, C \in \Lambda(X)$ we have the following rules, due to Littlewood [25], for manipulating plethysms:

$$\begin{aligned} (A+B)[C] &= A[C] + B[C]\,; & A[B+C] &= A_{(1)}[B]A_{(2)}[C]\,; \\ (AB)[C] &= A[C]B[C]\,; & A[BC] &= A_{[1]}[B]A_{[2]}[C]\,; \\ A[B[C]] &= (A[B])[C]\,. \end{aligned} \tag{2-18}$$

To these we can add, see [11]:

$$\begin{aligned} A[-B] &= (\mathsf{S}(A))[B]\,; & A[\mathsf{S}(B)] &= \mathsf{S}(A[B])\,; \\ A[\Delta(B)] &= \Delta(A[B])\,; & A[\delta(B)] &= \delta(A[B])\,, \end{aligned} \tag{2-19}$$

and the plethysm of a tensor product:

$$A[B \otimes C] = A_{[1]}[B] \otimes A_{[2]}[C]\,. \tag{2-20}$$

These rules enable us to evaluate plethysms not only of outer and inner products but also of outer and inner coproducts.

3. Specializations

3.1. **Definition of specializations.** Before dealing with the plethysms of interest here, it is appropriate to define certain specializations. We will do this in some generality so as to be able to use the technique of specialization in a rather broad context, see also [30, Sect.1.12].

Definition 3.1: A **specialization** ϕ is an algebra homomorphism (and thus a 1-cocycle for the outer Hopf algebra) from the Hopf algebra of the ring $\Lambda(X)$ of symmetric functions to another ring R, where R may be any one of $\mathbb{N}, \mathbb{Z}, \mathbb{Q}, \mathbb{R}, \mathbb{C}, \mathbb{Z}[t], \mathbb{Z}[[q]], , \mathbb{Z}[t][[q]], \ldots$. For any $A, B \in \Lambda(X)$ it is required that we have

$$\phi : \Lambda \to R \qquad \text{with} \qquad \phi(AB) = \phi(A)\phi(B)\,. \tag{3-1}$$

■

Specializations ϕ may be defined either through a map, also denoted by ϕ, on the letters x_i of the underlying alphabet $X = (x_1, x_2, \dots) = x_1 + x_2 + \cdots$, or through maps on the generators of $\Lambda(X)$ such as $h_n(X)$, $e_n(X)$ or $p_n(X)$.

3.2. **Fundamental specialization.** We denote by ϵ^1 the map $\epsilon^1 \circ A(X) = A(\epsilon^1(X))$ where $\epsilon^1(X) = (1, 0, \dots) = 1$, that is to say

$$\epsilon^1(x_i) = \begin{cases} 1 & \text{if } i = 1\,; \\ 0 & \text{otherwise}\,. \end{cases} \tag{3-2}$$

This fundamental specialization evaluates on Schur functions as the dimension formula for $GL(1)$ in the sense that

$$\epsilon^1(s_\lambda(X)) = s_\lambda(1, 0, \dots) = \dim\, V^\lambda_{GL(1)} = \dim_\lambda(1)\,, \tag{3-3}$$

where $\dim\, V^\lambda_{GL(1)}$ is the dimension of the irreducible representation $V^\lambda_{GL(1)}$ of $GL(1)$ having highest weight λ. This specialization is such that

$$\epsilon^1(s_\lambda(X)) = \begin{cases} 1 & \text{if } \lambda = (m) \text{ for any } m \geq 0; \\ 0 & \text{otherwise.} \end{cases} \tag{3-4}$$

3.3. t**-specialization.** We generalize the fundamental specialization along the following lines. For all $t \in \mathbb{N}$ we denote by $\epsilon^t \circ A(X) = A(\epsilon^t(X))$ with $\epsilon^t(X) = (1, \dots, 1, 0 \dots) = 1 + 1 + \cdots + 1$ with t occurrences of the 1s. In the sequence notation we thus have $\epsilon^t \circ A(X) = A(1^t)$ while in the additive (ring) notation we have $\epsilon^t \circ A(X) = A[t]$. In both cases we make this precise as

$$\epsilon^t(x_i) = \begin{cases} 1 & \text{if } 1 \leq i \leq t\,; \\ 0 & \text{otherwise}\,. \end{cases} \tag{3-5}$$

For all $t \in \mathbb{N}$ the t-specialization of a Schur function can be interpreted by means of a $GL(t)$ dimension formula, that is $\epsilon^t(s_\lambda(X)) = s_\lambda(1^t) = s_\lambda(1, 1, \dots, 1) = \dim V^\lambda_{GL(t)} = \dim_\lambda(t)$. However, the dimension formula for $GL(t)$ is polynomial in t:

$$\dim_\lambda(t) = \prod_{(i,j)\in F^\lambda} \frac{t + c(i,j)}{h(i,j)}\,, \tag{3-6}$$

and hence can be generalized by analytic continuation to rational, real or even complex t.

3.4. **Principal** $(q;n)$**-specialization.** A further important specialization is given by the map $\epsilon^1_{q;n} \circ A(X) = A(1, q, q^2, \dots, q^{n-1}, 0, \dots) = A[\frac{1-q^n}{1-q}]$ or

$$\epsilon^1_{q;n}(x_i) = \begin{cases} q^{i-1} & \text{for } 1 \leq i \leq n; \\ 0 & \text{otherwise.} \end{cases} \tag{3-7}$$

In the case of Schur functions, with the notation described earlier, we have [29, p.44]

$$\epsilon^1_{q;n} \circ s_\lambda(X) = s_\lambda(1, q, \dots, q^{n-1}) = q^{n(\lambda)} \prod_{(i,j)\in F^\lambda} \frac{1 - q^{n+c(i,j)}}{1 - q^{h(i,j)}}\,. \tag{3-8}$$

In the special case of $\lambda = (1^m)$ this takes the form

$$\epsilon^1_{q;n} \circ s_{(1^m)}(X) = \epsilon^1_{q;n} \circ e_m(X) = q^{m(m-1)/2} \begin{bmatrix} n \\ m \end{bmatrix}_q ,$$

where the q-binomial coefficient is given by

$$\begin{bmatrix} n \\ m \end{bmatrix}_q = \frac{(1-q^n)(1-q^{n-1})\cdots(1-q^{n-m+1})}{(1-q)(1-q^2)\cdots(1-q^m)} .$$

3.5. **Three parameter specialization.** Note that for $q \to 1$ we recover the t-specialization from the principal $(q;t)$-specialization. However, we keep these two specializations apart so as to have the opportunity to employ a combination of both. This will be denoted by

$$\epsilon^t_{q;n} \circ A(X) = A(1,\dots,1,q,\dots,q,\dots,q^{n-1},\dots,q^{n-1},0,\dots) = A\left[t\,\frac{1-q^n}{1-q}\right] , \qquad (3\text{-}9)$$

with t repetitions of each distinct power of q, while $t = 1+1+\cdots+1$ with t repetitions of 1.

4. PARAMETERIZED PLETHYSMS

The idea now is to exploit the above findings to see if we can derive a general formula for the plethysm $s_\lambda[\alpha\, s_\nu]$ for any α: integer, rational or complex.

4.1. **Replicated Schur functions as plethysms.** First we deal with the case $\alpha = t \in \mathbb{N}$. In this case we may use the iterated outer coproduct identity

$$\begin{aligned} \Delta^{(t-1)} s_\lambda &= (\mathrm{Id} \otimes \Delta^{(t-2)})\Delta s_\lambda = (\mathrm{Id} \otimes \Delta^{(t-2)}) s_{\lambda_{(1)}} \otimes s_{\lambda_{(2)}} \\ &= s_{\lambda_{(1)}} \otimes (\Delta^{(t-2)} s_{\lambda_{(2)}}) = \cdots = s_{\lambda_{(1)}} \otimes s_{\lambda_{(2)}} \otimes \cdots \otimes s_{\lambda_{(t)}} , \end{aligned} \qquad (4\text{-}1)$$

with $\Delta^{(2)} = \Delta$, $\Delta^{(1)} = \mathrm{Id}$, $\Delta^{(0)} = \epsilon^0$ and some relabelling has been applied to the iterated outer product Sweedler indices. Then using outer product multiplication $t-1$ times one finds

$$\begin{aligned} s_\lambda[t\, s_\nu(X)] &= s_\lambda[s_\nu(X) + s_\nu(X) + \cdots + s_\nu(X)] \\ &= s_{\lambda_{(1)}}[s_\nu(X)]\; s_{\lambda_{(2)}}[s_\nu(X)]\; \cdots s_{\lambda_{(t)}}[s_\nu(X)] . \end{aligned} \qquad (4\text{-}2)$$

Example 4.2:

$$\begin{aligned} s_{(2)}[2s_{(2)}] &= s_{(2)}[s_{(2)} + s_{(2)}] = s_{(2)}[s_{(2)}] + s_{(1)}[s_{(2)}]\; s_{(1)}[s_{(2)}] + s_{(2)}[s_{(2)}] \\ &= (s_{(4)} + s_{(2,2)}) + (s_{(4)} + s_{(3,1)} + s_{(2,2)}) + (s_{(4)} + s_{(2,2)}) \\ &= 3s_{(4)} + s_{(3,1)} + 3s_{(2,2)} ; \\ s_{(1,1)}[2s_{(2)}] &= s_{(1,1)}[s_{(2)} + s_{(2)}] = s_{(1,1)}[s_{(2)}] + s_{(1)}[s_{(2)}]\; s_{(1)}[s_{(2)}] + s_{(1,1)}[s_{(2)}] \\ &= s_{(3,1)} + (s_{(4)} + s_{(3,1)} + s_{(2,2)}) + s_{(3,1)} \\ &= s_{(4)} + 3s_{(3,1)} + s_{(2,2)} . \end{aligned}$$

■

Alternatively, we may use the inner coproduct identity $\delta s_\lambda = s_{\lambda_{[1]}} \otimes s_{\lambda_{[2]}}$ to obtain

$$\begin{aligned} s_\lambda[t\, s_\nu(X)] &= s_{\lambda_{[1]}}[t]\; s_{\lambda_{[2]}}[s_\nu(X)] = \sum_{\rho,\mu} g^\lambda_{\rho,\mu}\; s_\rho[t]\; s_\mu[s_\nu(X)] \\ &= \sum_{\rho,\mu} g^\lambda_{\rho,\mu}\; \dim_\rho(t)\; s_\mu[s_\nu](X) = \sum_\mu\; b^\lambda_\mu(t)\; s_\mu[s_\nu](X)\,, \end{aligned} \tag{4-3}$$

where

$$b^\lambda_\mu(t) = \sum_\rho\; g^\lambda_{\rho,\mu}\; \dim_\rho(t)\,. \tag{4-4}$$

Example 4.3:

$$\begin{aligned} s_{(2)}[2s_{(2)}] &= \dim_{(2)}(2)\; s_{(2)}[s_{(2)}] + \dim_{(1,1)}(2)\; s_{(1,1)}[s_{(2)}] \\ &= 3\,(s_{(4)} + s_{(2,2)}) + (s_{(4)} + 1\, s_{(3,1)} \\ &= 3s_{(4)} + s_{(3,1)} + 3s_{(2,2)}\,; \\ s_{(1,1)}[2s_{(2)}] &= \dim_{(1,1)}(2)\; s_{(2)}[s_{(2)}] + \dim_{(2)}(2)\; s_{(1,1)}[s_{(2)}]\,; \\ &= 1, (s_{(4)} + s_{(2,2)}) + 3\, s_{(3,1)} \\ &= s_{(4)} + 3s_{(3,1)} + s_{(2,2)}\,, \end{aligned}$$

as before. ∎

In the special case $\nu = (1)$, for which $s_\nu(X) = X$, (4-3) gives

$$s_\lambda[t\, X] = \sum_{\rho,\mu} g^\lambda_{\rho,\mu}\; \dim_\rho(t)\; s_\mu(X) = \sum_\mu\; b^\lambda_\mu(t)\; s_\mu(X)\,. \tag{4-5}$$

Example 4.4:

$$\begin{aligned} s_{(2)}[2\, X] &= \dim_{(2)}(2)\; s_{(2)}(X) + \dim_{(1,1)}(2)\; s_{(1,1)}(X) \\ &= 3\, s_{(2)}(X) + s_{(1,1)}(X)\,; \end{aligned}$$

$$\begin{aligned} s_{(1,1)}[2\, X] &= \dim_{(1,1)}(2)\; s_{(2)}(X) + \dim_{(2)}(2)\; s_{(1,1)}(X) \\ &= s_{(2)}(X) + 3s_{(1,1)}(X)\,. \end{aligned}$$

∎

4.2. **Benchmarking replicated plethysm calculations.** The above shows that we may use either iterated outer coproducts, or a single inner coproduct augmented by a dimensionality formula, to evaluate replicated plethysms. Although the above examples might suggest that these two methods are comparable in complexity, this is far from being the case. The iteration may be very tedious, with the second method much more efficient, at least for sufficiently large n. Symbolic computations show a dramatic increase of speed for even modestly large n (greater than 10). The relevant algorithm is given in the form of computer pseudo code in Appendix C.

We have investigated this process via the use of both Maple using the SchurFkt package [1] and the open source software SCHUR [36]. In arbitrary time units we can compare the computation of the plethysms as shown in Table 1. Both algorithms make use of Maple remember tables,

TABLE 1. Timing of the iterated and directly evaluated plethysms $s_{(3)}[n \cdot s_{(1,1)}]$ using SchurFkt. (Note that figures are obscured by Maple's garbage collection and not as accurate as shown)

multiplicity	recursive	direct
n=1	0.01	0.02
n=10	0.08	0.02
n=100	0.89	0.01
n=1000	7.32	0.01
n=10000	—	0.01

so a plethysm is never computed twice. It is clear that the second method is $O(1)$ with respect to n, while the first one increases rapidly. Maple fails to do the iteration for $n = 10000$. Very similar results can be obtained by using SCHUR, but the inner coproduct case has to be carried out in two stages in order to insert the dimensionality factors appropriately.

4.3. **α-plethysms and α-Schur functions.** Since the coefficients $b^\lambda_\mu(t)$ are polynomials in t, the formulae (4-3) and (4-5) may be extended so as to define α-plethysms and α-Schur functions by means of the formulae

$$s_\lambda[\alpha\, s_\nu(X)] = \sum_\mu b^\lambda_\mu(\alpha)\; s_\mu[s_\nu](X) \tag{4-6}$$

and

$$s_\lambda[\alpha\, X] = \sum_\mu b^\lambda_\mu(\alpha)\; s_\mu(X)\,, \tag{4-7}$$

where

$$b^\lambda_\mu(\alpha) = \sum_\rho g^\lambda_{\rho,\mu}\; \dim_\rho(\alpha)\,. \tag{4-8}$$

The symmetric functions $s_\lambda[\alpha X]$ are precisely those that were introduced and studied by Baker [2] as replicated Schur functions, and independently by Brenti [4] as q-analogues of Schur functions. Our notation is such that $s_\lambda[\alpha\, X]$ is identical to Baker's $s_\lambda(x^{(\alpha)})$ and Brenti's $s_\lambda[x]_q$ under the identifications $x = X$ and $q = \alpha$.

The case of replicated and α-power sum functions is even easier. Since $p_n(X) = x_1^n + x_2^n + \cdots$, it follows immediately that for any $t, n \in \mathbb{N}$ we have

$$p_n(t\,X) = p_n(X + X + \cdots + X) = t\,(x_1^n + x_2^n + \cdots) = t\,p_n(X)\,, \tag{4-9}$$

so that, replacing X by $Y = p_\mu(X)$, we have

$$p_n(t\,p_\mu(X)) = t\,p_n[p_\mu(X)]\,. \tag{4-10}$$

The multiplicative nature of $p_\lambda = p_{\lambda_1}\,p_{\lambda_2}\,\cdots\,p_{\lambda_{\ell(\lambda)}}$, where $\ell(\lambda)$ is the number of non-zero parts of λ, is then such that:

$$p_\lambda[t\,X] = t^{\ell(\lambda)}\,p_\lambda(X)\,; \tag{4-11}$$

$$p_\lambda[t\,p_\mu(X)] = t^{\ell(\lambda)}\,p_\lambda[p_\mu(X)]\,. \tag{4-12}$$

Once again we are at liberty to extend the domain of t to give, as a matter of definition:

$$p_\lambda[\alpha\,X] = \alpha^{\ell(\lambda)}\,p_\lambda(X)\,; \tag{4-13}$$

$$p_\lambda[\alpha\,p_\mu(X)] = \alpha^{\ell(\lambda)}\,p_\lambda[p_\mu(X)]\,. \tag{4-14}$$

The first of these is really the starting point in Brenti's development of q-analogues of symmetric functions, and both Baker [2] and Brenti [4] have pointed out that the Jack symmetric functions $J_{(n)}(X;\alpha^{-1})$ can be expressed in the form

$$J_{(n)}(X;\alpha^{-1}) = \frac{n!}{\alpha^n}\,s_{(n)}[\alpha\,X]\,, \tag{4-15}$$

which specialize to zonal symmetric functions for $\alpha = 2$.

4.4. **Orthogonality properties of α-Schur functions.** We may use the Schur-Hall scalar product to extract from (4-7) the formula

$$\begin{aligned} b_\mu^\lambda(\alpha) &= \langle\, s_\mu(X)\,,\, s_\lambda[\alpha\,X]\,\rangle_{\Lambda(X)} \\ &= \sum_{\sigma,\tau}\,\chi_\sigma^\mu\,\chi_\tau^\lambda\,\langle\, p_\sigma(X)\,,\, p_\tau(\alpha\,X)\,\rangle_{\Lambda(X)} \\ &= \sum_{\sigma,\tau}\,\chi_\sigma^\mu\,\chi_\tau^\lambda\,\alpha^{\ell(\tau)}\langle\, p_\sigma(X)\,,\, p_\tau(X)\,\rangle_{\Lambda(X)} \\ &= \sum_{\sigma,\tau}\,\chi_\sigma^\mu\,\chi_\tau^\lambda\,\alpha^{\ell(\tau)}\,z_\tau^{-1}\,\delta_{\sigma,\tau} = \sum_\sigma\,z_\sigma^{-1}\,\chi_\sigma^\mu\,\chi_\sigma^\lambda\,\alpha^{\ell(\sigma)}\,, \end{aligned} \tag{4-16}$$

where use has been made of (2-3) and (4-13).

With this determination of the coefficients $b_\mu^\lambda(\alpha)$ we can establish the following result due to Baker [2] and Brenti [4]:

Theorem 4.5: For all non-zero α

$$\langle\, s_\mu(\alpha\,X)\,,\, s_\lambda(\alpha^{-1}\,X)\,\rangle_{\Lambda(X)} \;=\; \delta_{\mu,\lambda}\,. \tag{4-17}$$

■

Proof:

$$\begin{aligned}
&\langle\, s_\mu(\alpha\, X)\,,\, s_\lambda(\beta\, X)\,\rangle_{\Lambda(X)} \\
&= \sum_{\sigma,\tau}\, \chi^\mu_\sigma\, \chi^\lambda_\tau\, \langle\, p_\sigma(\alpha\, X)\,,\, p_\tau(\beta\, X)\,\rangle_{\Lambda(X)} \\
&= \sum_{\sigma,\tau}\, \chi^\mu_\sigma\, \chi^\lambda_\tau\, \alpha^{\ell(\sigma)}\beta^{\ell(\tau)}\langle\, p_\sigma(X)\,,\, p_\tau(X)\,\rangle \\
&= \sum_{\sigma,\tau}\, \chi^\mu_\sigma\, \chi^\lambda_\tau\, \alpha^{\ell(\sigma)}\beta^{\ell(\tau)}\, z^{-1}_\tau\, \delta_{\sigma,\tau} = \sum_{\sigma}\, z^{-1}_\sigma\, \chi^\mu_\sigma\, \chi^\lambda_\sigma\, (\alpha\beta)^{\ell(\sigma)}\,.
\end{aligned}$$

Hence, taking $\beta = 1/\alpha$ we have the Baker-Brenti orthogonality condition

$$\langle\, s_\mu(\alpha\, X)\,,\, s_\lambda(\alpha^{-1}\, X)\,\rangle_{\Lambda(X)} = \sum_{\sigma}\, z^{-1}_\sigma\, \chi^\mu_\sigma\, \chi^\lambda_\sigma\, = \delta_{\mu,\lambda}\,.$$

□

Now consider the following technical result.

Lemma 4.6: For any positive integer n and any partitions ν and ρ of the same weight

$$n^{\ell(\rho)}\, \chi^\nu_\rho = \sum_{\mu,\sigma,\tau,\ldots,\zeta}\, \chi^\mu_\rho\, c^\mu_{\sigma,\tau,\ldots,\zeta}\, c^\nu_{\sigma,\tau,\ldots,\zeta}\,, \tag{4-18}$$

where the sum is over $n+1$ partitions $\mu, \sigma, \tau, \ldots, \zeta$. ■

Proof: Consider

$$\begin{aligned}
p_\rho(X, Y, \ldots, Z) &= \sum_{\mu}\, \chi^\mu_\rho\, s_\mu(X, Y, \ldots, Z) \\
&= \sum_{\mu,\sigma,\tau,\ldots,\zeta}\, \chi^\mu_\rho\, c^\mu_{\sigma,\tau\ldots,\zeta}\, s_\sigma(X)\, s_\tau(Y)\cdots s_\zeta(Z)
\end{aligned}$$

where the coproduct Δ has been applied $n-1$ times. If we now apply the multiplication operator $n-1$ times, that is we set $X = Y = \cdots = Z$ and take outer products, we obtain

$$\begin{aligned}
p_\rho(X, X, \ldots, X) = p_\rho(n\, X)\, &= n^{\ell(\rho)}\, p_\rho(X) = n^{\ell(\rho)}\, \sum_{\nu}\, \chi^\nu_\rho\, s_\nu(X) \\
&= \sum_{\mu,\sigma,\tau,\ldots,\zeta}\, \chi^\mu_\rho\, c^\mu_{\sigma,\tau\ldots,\zeta}\, c^\nu_{\sigma,\tau\ldots,\zeta}\, s_\nu(X)\,.
\end{aligned}$$

Comparing the coefficients of $s_\nu(X)$ and using (4-18) proves the Lemma. □

This Lemma allows us to prove the following more general orthogonality theorem:

Theorem 4.7: [2] For n alphabets $X, Y, \ldots, Z$, let

$$\tilde{s}_\lambda(X, Y, \ldots, Z) = \sum_\mu b_{\lambda,\mu}(1/n)\, s_\mu(X, Y, \ldots, Z)\,. \tag{4-19}$$

Then

$$\langle\, \tilde{s}_\lambda(X, Y, \ldots, Z)\,,\; s_\nu(X, Y, \ldots, Z)\,\rangle_{\Lambda(X)\otimes\Lambda(Y)\otimes\cdots\otimes\Lambda(Z)} = \delta_{\lambda,\nu}\,. \tag{4-20}$$

■

Proof:

$$\begin{aligned}
&\langle\, \tilde{s}_\lambda(X,Y,\ldots,Z)\,,\; s_\nu(X,Y,\ldots,Z)\,\rangle_{\Lambda(X)\otimes\Lambda(Y)\otimes\cdots\otimes\Lambda(Z)} \\
&\quad= \sum_\mu b_{\lambda,\mu}(1/n)\,\langle\, s_\mu(X,Y,\ldots,Z)\,,\; s_\nu(X,Y,\ldots,Z)\,\rangle_{\Lambda(X)\otimes\Lambda(Y)\otimes\cdots\otimes\Lambda(Z)} \\
&\quad= \sum_\mu b_{\lambda,\mu}(1/n) \sum_{\sigma,\tau,\ldots,\zeta,\eta,\theta\ldots,\phi} c^\mu_{\sigma,\tau,\ldots,\zeta}\, c^\nu_{\eta,\theta\ldots,\phi}\, \delta_{\sigma,\eta}\, \delta_{\tau,\theta}\, \cdots\, \delta_{\zeta,\phi} \\
&\quad= \sum_{\mu,\rho} z_\rho^{-1}\, \chi^\lambda_\rho\, \chi^\mu_\rho\, n^{-\ell(\rho)} \sum_{\sigma,\tau,\ldots,\zeta} c^\mu_{\sigma,\tau,\ldots,\zeta}\, c^\nu_{\sigma,\tau,\ldots,\zeta} \\
&\quad= \sum_\rho z_\rho^{-1}\, \chi^\lambda_\rho\, \chi^\nu_\rho \\
&\quad= \delta_{\lambda,\nu}\,.
\end{aligned}$$

□

5. Series plethysms and character generating functions

5.1. **Series defined by plethysms.** Given

$$M(t; X) = \prod_{i\geq 1}(1 - t\, x_i)^{-1} = \sum_{k=0}^\infty t^k\, s_{(k)}(X)\,, \tag{5-1}$$

new Schur function series may be generated from $M(t; X)$ by means of the plethysm operation, as explained elsewhere [11]. For each fixed partition π one merely replaces X by $Y = s_\pi(X)$ to give

$$\begin{aligned}
M_\pi(t; X) &= M(t; [s_\pi])(X) = M(t; s_\pi(X)) \\
&= M(t; Y) = \prod_{j\geq 1} (1 - t\, y_j)^{-1} = \sum_{k=0}^\infty t^k\, s_{(k)}[s_\pi](X)\,,
\end{aligned} \tag{5-2}$$

where the product is taken over all monomials y_j of $s_\pi(X)$. Similarly, from

$$L(t; X) = \prod_{i\geq 1}(1 - t\, x_i) = \sum_{k=0}^\infty (-1)^k t^k\, s_{(1^k)}(X)\,, \tag{5-3}$$

one obtains

$$L_\pi(t;X) = L(t;[s_\pi])(X) = L(t;s_\pi(X))$$
$$= L(t;Y) = \prod_{j\geq 1}(1-t\,y_j) = \sum_{k=0}^{\infty}(-1)^k t^k\, s_{(1^k)}[s_\pi](X)\,. \tag{5-4}$$

5.2. **Character generating functions and the M_π and L_π series.** The Cauchy kernel $M(XZ)$ serves as a generating function for characters of $GL(n)$ in the sense that

$$M(XZ) = \prod_{i,j}(1-x_i z_j)^{-1} = \sum_\lambda\, s_\lambda(X)\, s_\lambda(Z)\,, \tag{5-5}$$

where $s_\lambda(X)$ is the character of the irreducible representation $V^\lambda_{GL(n)}$ of highest weight λ evaluated at group elements whose eigenvalues are the elements of X. As pointed out earlier this implies, and is implied by

$$s_\lambda(X) = [s_\lambda(Z)](\,M(XZ)\,) = \langle\, s_\lambda(Z)\,|\,M(XZ)\,\rangle_{\Lambda(Z)}\,, \tag{5-6}$$

where $[s_\lambda](\cdots)$ denotes the coefficient of $s_\lambda(Z)$ in $(\cdots)$.

Now we are in a position to determine the analogous generating functions for certain formal characters, $s^{(\pi)}_\lambda(X)$, of subgroups $H_\pi(n)$ of $GL(n)$ introduced elsewhere [11] by exploiting the mutually inverse series $M_\pi = M[s_\pi]$ and $L_\pi = L[s_\pi]$. To be more precise, we let

$$s^{(\pi)}_\lambda(X) = L^\perp_\pi(X)\,(s_\lambda(X)). \tag{5-7}$$

The generating function for these characters may then be found as follows:

$$s^{(\pi)}_\lambda(X) = L^\perp_\pi(X)\,(s_\lambda(X)) = L^\perp_\pi(X)\,([s_\lambda(Z)]\,M(XZ))$$
$$= [s_\lambda(Z)]\;L^\perp_\pi(X)\,(M(XZ))\,. \tag{5-8}$$

It then follows from (2-16) that

$$s^{(\pi)}_\lambda(X) = [s_\lambda(Z)]\;L_\pi(Z)\,M(XZ)\,, \tag{5-9}$$

and hence

$$L_\pi(Z)\,M(XZ) = \sum_\lambda\, s^{(\pi)}_\lambda(X)\, s_\lambda(Z)\,. \tag{5-10}$$

Example 5.8:

$$\prod_{i\leq j}(1-z_i z_j)\;\prod_{i,j}(1-x_i z_j)^{-1} = \sum_\lambda\, s^{(2)}_\lambda(X)\, s_\lambda(Z)\,;$$
$$\prod_{i<j}(1-z_i z_j)\;\prod_{i,j}(1-x_i z_j)^{-1} = \sum_\lambda\, s^{(1^2)}_\lambda(X)\, s_\lambda(Z)\,;$$

$$\prod_{i\le j\le k}(1-z_iz_jz_k)\prod_{i,j}(1-x_iz_j)^{-1}=\sum_\lambda s_\lambda^{(3)}(X)\,s_\lambda(Z)\,;$$
$$\prod_{i\neq j}(1-z_i^2z_j)\prod_{i<j<k}(1-z_iz_jz_k)^2\prod_{i,j}(1-x_iz_j)^{-1}=\sum_\lambda s_\lambda^{(21)}(X)\,s_\lambda(Z)\,;$$
$$\prod_{i<j<k}(1-z_iz_jz_k)\prod_{i,j}(1-x_iz_j)^{-1}=\sum_\lambda s_\lambda^{(1^3)}(X)\,s_\lambda(Z)\,.$$

■

In the first two cases, $s_\lambda^{(2)}(X)$ and $s_\lambda^{(1^2)}(X)$, with the appropriate specification of X, are nothing other than the irreducible orthogonal and symplectic group characters, variously denoted by $o_\lambda(X)=[\lambda](X)$ and $sp_\lambda(X)=\langle\lambda\rangle(X)$, respectively [25, 3]. In the remaining cases, $s_\lambda^{(3)}(X)$, $s_\lambda^{(21)}(X)$ and $s_\lambda^{(1^3)}(X)$, again with appropriate specifications of X, are formal, not necessarily irreducible, characters of the subgroups $H_3(n)$, $H_{21}(n)$ and $H_{1^3}(n)$ of $GL(n)$ that leave invariant cubic forms of symmetry specified by the partitions (3), (21) and (1^3), respectively [11].

6. VERTEX OPERATORS

6.1. **Vertex operators associated with formal characters.** Let X be the underlying alphabet of all our Schur functions and Schur function series unless otherwise indicated, with X itself suppressed unless it is necessary to exhibit it. In addition let $Z=(z_1,z_2,\dots,z_l)$, and for any partition λ of length $\ell(\lambda)\le l$ let $Z^\lambda=z_1^{\lambda_1}z_1^{\lambda_2}\cdots z_l^{\lambda_l}$ and let $[Z^\lambda](\cdots)$ be the coefficient of Z^λ in $(\cdots)$. For any non-zero z let $\overline{z}=1/z$. Then the vertex operator $V(z)$ is defined by

$$V(z)=M(z)\,L^\perp(\overline{z})\,. \tag{6-1}$$

If phrased in the language of symmetric functions, see for example [29, Ex.29 p.95] and [6], vertex operators are also sometimes called Bernstein vertex operators, or simply Bernstein operators, a name coined by Zelevinsky [37, p.69]. The following formula is well-known [29, 3]:

Proposition 6.9: Let $\lambda=(\lambda_1,\lambda_2,\dots,\lambda_l)$ be a partition of length $\ell(\lambda)\le l$. Then

$$s_\lambda=[Z^\lambda]\,V(z_1)\,V(z_2)\cdots V(z_l)\cdot 1\,. \tag{6-2}$$

■

Proof: One way to see this is as follows:

$$[Z^\lambda]\, V(z_1)\, V(z_2)\, \cdots\, V(z_l)\cdot 1$$
$$= [Z^\lambda]\, M(z_1)\, L^\perp(\overline{z_1})\, M(z_2)\, L^\perp(\overline{z_2})\, \cdots\, M(z_l)\, L^\perp(\overline{z_l})\, \cdot 1\,. \tag{6-3}$$

However, since the outer coproduct of L is just $\Delta L = L\otimes L$, we have

$$L^\perp(\overline{z})\,(M(w)\,G) = (M(w)/L(\overline{z}))\;\;(G/L(\overline{z})) \tag{6-4}$$

for any G, w and non-zero z, while

$$M(w)/L(\overline{z}) = M(w)/(s_0 - \overline{z}s_1) = M(w) - \overline{z}\,w\,M(w). \tag{6-5}$$

Noting that $L^\perp(\overline{z})\cdot 1 = 1$ for all non-zero z, this implies that

$$[Z^\lambda]\, V(z_1)\, V(z_2)\, \cdots\, V(z_l)\cdot 1$$
$$= [Z^\lambda] \prod_{1\le i<j\le l} (1-\overline{z_i}\,z_j) M(z_1)\, M(z_2)\, \cdots\, M(z_l)$$
$$= [Z^{\lambda+\delta}] \prod_{1\le i<j\le l} (z_i - z_j)\; M(Z) = [s_\lambda(Z)]\, M(Z)\,, \tag{6-6}$$

where $\delta = (n-1,\ldots,1,0)$ and use has been made of the fact that

$$s_\lambda = (Z^{\lambda+\delta} + \cdots)/\prod_{i<j}(z_i - z_j),$$

while $M(z_1)\, M(z_2)\, \cdots\, M(z_l) = M(Z)$. Restoring the X dependence for the moment,

$$M(XZ) = \prod_{i=1}^{n}\prod_{j=1}^{l} (1-x_i z_j)^{-1} = \sum_\lambda\; s_\lambda(X)\; s_\lambda(Z)\,, \tag{6-7}$$

so that $[s_\lambda(Z)]\, M(XZ) = s_\lambda(X)$. That is to say, without the explicit X-dependence, we have $[s_\lambda(Z)]\, M(Z) = s_\lambda$, as required to complete the proof of (6-2). □

In order to generalise Proposition 6.9 to the characters $s_\lambda^{(\pi)}$ for arbitrary partitions π, it is helpful to first establish

Lemma 6.10: For all w, z and all partitions π of weight $|\pi| = p \ge 1$,

$$L^\perp_\pi(w)\, M(z) = (1 - w\, z^p\, \delta_{\pi,(p)})\, M(z) \prod_{k=1}^{p-1} L^\perp_{\pi/(k)}(wz^k)\;\, L^\perp_\pi(w)\,, \tag{6-8}$$

where the product over k is absent if $p = 1$, that is to say if $\pi = (1)$. ■

Proof: For arbitrary G,

$$\begin{aligned} L_\pi^\perp(w)\,(M(z)\,G) &= (M(z)\,G)/L_\pi(w) \\ &= M(z)/(L_\pi(w))_{(1)} \quad G/(L_\pi(w))_{(2)}\,, \end{aligned} \tag{6-9}$$

where, in Sweedler notation,

$$\Delta L_\pi(w) = (L_\pi(w))_{(1)} \otimes (L_\pi(w))_{(2)}\,. \tag{6-10}$$

More explicitly, in terms of Littlewood-Richardson coefficients,

$$\begin{aligned} \Delta L_\pi(w) &= (\, L_\pi(w) \otimes L_\pi(w)\,) \prod_{0\neq\xi,\eta\neq\pi} \prod_{c=1}^{c^\pi_{\xi\eta}} \sum_{\rho(\xi,\eta,c)} (-w)^{|\rho(\xi,\eta,c)|}\; s_{\rho(\xi,\eta,c)}[s_\xi] \otimes s_{\rho(\xi,\eta,c)'}[s_\eta] \\ &= (L_\pi(w))_{(1)} \otimes (L_\pi(w))_{(2)}\,. \end{aligned} \tag{6-11}$$

However, since $M(z) = \sum_{m\geq 0}\, s_{(m)}\, z^m$, we have

$$M(z)/s_\lambda = \begin{cases} z^k\, M(z) & \text{if } \lambda = (k) \text{ with } k \geq 0; \\ 0 & \text{otherwise.} \end{cases} \tag{6-12}$$

Moreover, for any partitions ρ and ξ, the plethysm $s_\rho[s_\xi]$ is such that

$$M(z)/(\, s_\rho[s_\xi]\,) = \begin{cases} z^{rk}\, M(z) & \text{if } \rho = (r) \text{ and } \xi = (k) \text{ with } r, k \geq 0; \\ 0 & \text{otherwise.} \end{cases} \tag{6-13}$$

It follows first from this that

$$M(z)/L_\pi(w) = \sum_{r\geq 0} (-w)^r M(z)/(\, s_{1^r}[s_\pi]\,) = (1 - w\, z^p\, \delta_{\pi,(p)})\, M(z)\,. \tag{6-14}$$

Then in evaluating all other contributions of the form $M(z)/(L_\pi(w))_{(1)})$, with $(L_\pi(w))_{(1)}$ identified as in (6-11), the domain of ξ may be restricted to one-part partitions (k) with $0 < k < p$, for which all non-zero $c^\pi_{(k),\eta}$ are equal to 1, so that $c = 1$. In addition all $\rho(\xi,\eta,c)$ may be restricted to partitions of the form (r), with $\rho(\xi,\eta,c)' = (1^r)$. If for fixed π we let $m(k)$ be such that $s_{\pi/(k)} = \sum_{m=1}^{m(k)}\, s_{\eta(k,m)}$ for each $k = 1, 2, \ldots, p-1$, then

$$\begin{aligned} L_\pi^\perp(w)\,(\, M(z)\,G\,) = \sum_{r(k,m)} &(\, M(z)/(L_\pi(w) \prod_{k=1}^{p-1} \prod_{m=1}^{m(k)}\, s_{(r(k,m))}[s_{(k)}]\,)\,) \\ &(\, G/(L_\pi(w) \prod_{k=1}^{p-1} \prod_{m=1}^{m(k)}\, (-w)^{r(k,m)}\; s_{(1^{r(k,m)})}[s_{\eta(k,m)}]\,)\,)\,. \end{aligned}$$

Then from (6-13) it follows that

$$L_\pi^\perp(w)\,(\,M(z)\,G\,) = \sum_{r(k,m)} (\prod_{k=1}^{p-1}\prod_{m=1}^{m(k)} z^{k\,r(k,m)}\; M(z)/L_\pi(w)\,)$$
$$(\,G/(L_\pi(w)\prod_{k=1}^{p-1}\prod_{m=1}^{m(k)}(-w)^{r(k,m)}\; s_{(1^{r(k,m)})}[s_{\eta(k,m)}]\,)\,)$$
$$= \sum_{r(k,m)} (\,M(z)/L_\pi(w)\,)$$
$$(\,G/(L_\pi(w)\prod_{k=1}^{p-1}\prod_{m=1}^{m(k)}(-w)^{r(k,m)}\; z^{k\,r(k,m)}\; s_{(1^{r(k,m)})}[s_{\eta(k,m)}]\,)\,). \tag{6-15}$$

Hence

$$L_\pi^\perp(w)(\,M(z)\,G\,) = (\,M(z)/L_\pi(w)\,)\;\;(\,G/(L_\pi(w)\prod_{k=1}^{p-1}\prod_{m=1}^{m(k)} L_{\eta(k,m)}(wz^k)\,)\,)$$
$$= (\,M(z)/L_\pi(w)\,)\;\;(\,G/(L_\pi(w)\prod_{k=1}^{p-1} L_{\pi/(k)}(wz^k)\,)\,). \tag{6-16}$$

From (6-14) it follows that for all G

$$L_\pi^\perp(w)\,(\,M(z)\,G\,) = (1-w\,z^p\,\delta_{\pi,(p)})\,M(z)\,(\,G/(L_\pi(w)\prod_{k=1}^{p-1} L_{\pi/(k)}(wz^k)\,)\,), \tag{6-17}$$

which implies the validity of (6-8). □

This Lemma leads immediately to the following generalization of Proposition 6.9

Proposition 6.11: Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l)$ be a partition of length $\ell(\lambda) \le l$, and let π be a partition of weight $|\pi| = p \ge 1$. Then

$$s_\lambda^{(\pi)} = [Z^\lambda]\; V^\pi(z_1)\,V^\pi(z_2)\,\cdots\,V^\pi(z_l)\cdot 1\,, \tag{6-18}$$

where

$$V^\pi(z) = (1-z^p\,\delta_{\pi,(p)})\,M(z)\;L^\perp(z^{-1})\prod_{k=1}^{p-1} L_{\pi/(k)}^\perp(z^k)\,. \tag{6-19}$$

■

Proof: Proceeding as in the proof of Proposition 6.9, one can make use of the fact that

$$L^\perp(\overline{z})\,((1-w^p\,\delta_{\pi,(p)})\,M(w)\,G) = L^\perp(\overline{z})\,(\,(M(w)/L_\pi)\,G)$$
$$= M(w)/(L_\pi\,L(\overline{z})\;\;(G/L(\overline{z})) = (1-w\,\overline{z})\,(M(w)/L_\pi)\;(G/L(\overline{z}))$$

$$= (1 - w\,\overline{z})\,(1 - w^p\,\delta_{\pi,(p)})\,M(w)\,(G/L(\overline{z}))\,, \tag{6-20}$$

for any G, w and non-zero z, to show that

$$\begin{aligned}
&[Z^\lambda]\,V^\pi(z_1)\,V^\pi(z_2)\,\cdots\,V^\pi(z_l)\cdot 1\\
&\quad= [Z^\lambda]\prod_{1\le i<j\le l}(1-\overline{z_i}\,z_j)\prod_{k=1}^{l}(1-z_k^p\,\delta_{\pi,(p)})\\
&\qquad M(z_1)\prod_{k=1}^{p-1}L^\perp_{\pi/(k)}(z_1^k)\,\cdots M(z_l)\prod_{k=1}^{p-1}L^\perp_{\pi/(k)}(z_l^k)\cdot 1\\
&\quad= [Z^{\lambda+\delta}]\prod_{1\le i<j\le l}(z_i-z_j)\prod_{k=1}^{l}(1-z_k^p\,\delta_{\pi,(p)})\\
&\qquad M(z_1)\prod_{k=1}^{p-1}L^\perp_{\pi/(k)}(z_1^k)\,\cdots M(z_l)\prod_{k=1}^{p-1}L^\perp_{\pi/(k)}(z_l^k)\cdot 1\\
&\quad= [s_\lambda]\prod_{k=1}^{l}(1-z_k^p\,\delta_{\pi,(p)})\\
&\qquad M(z_1)\prod_{k=1}^{p-1}L^\perp_{\pi/(k)}(z_1^k)\,\cdots M(z_l)\prod_{k=1}^{p-1}L^\perp_{\pi/(k)}(z_l^k)\cdot 1\\
&\quad= [s_\lambda]\;L^\perp_\pi\,M(z_1)\,\cdots\,M(z_l)\cdot 1\;=[s_\lambda]\,L^\perp_\pi\,(M(Z))\;=\;s^\pi_\lambda\,,
\end{aligned} \tag{6-21}$$

as required, where the first step in the last line involves the use of (6-8) extended iteratively, and the final step is a consequence of (5-8). □

6.2. **Vertex operators in exponential form.** Given

$$M(z;X)=\prod_{i\ge 1}(1-z\,x_i)^{-1}\,, \tag{6-22}$$

it follows that

$$\begin{aligned}
\ln M(z;X) &= -\sum_{i\ge1}\ln(1-z\,x_i)=\sum_{i\ge1}\Big(z\,x_i+\frac{(z\,x_i)^2}{2}+\frac{(z\,x_i)^3}{3}+\cdots\Big)\\
&= z\,p_1(X)+\frac{z^2}{2}\,p_2(X)+\frac{z^3}{3}\,p_3(X)+\cdots=\sum_{k\ge1}\frac{z^k}{k}\,p_k(X)\,.
\end{aligned} \tag{6-23}$$

Hence

$$M(z;X)=\exp\left(\sum_{k\ge1}\frac{z^k}{k}\,p_k(X)\right)\,. \tag{6-24}$$

It follows that for any partition π

$$\ln M_\pi(z;X)=\sum_{k\ge1}\frac{z^k}{k}\,p_k(Y)=\sum_{k\ge1}\frac{z^k}{k}\,p_k[s_\pi(X)]=\sum_{k\ge1}\frac{z^k}{k}\,s_\pi[p_k(X)]$$

$$= \sum_{k\geq 1} \frac{z^k}{k} \left(\sum_\rho \frac{1}{z_\rho} \chi^\pi_\rho \, p_\rho[p_k(X)] \right) = \sum_{k\geq 1} \frac{z^k}{k} \left(\sum_\rho \frac{1}{z_\rho} \chi^\pi_\rho \, p_{k\rho}(X) \right) , \qquad (6\text{-}25)$$

where we have used the identities $p_k[s_\pi(X)] = s_\pi[p_k(X)]$ and $p_r[p_k(X)] = p_{rk}(X)$ that apply for all π, k, r and X, and the notation $k\rho = (k\rho_1, k\rho_2, \ldots)$ if $\rho = (\rho_1, \rho_2, \ldots)$.

Example 6.12: Suppressing the X dependence

$$\begin{aligned}
\ln M_{(1)}(z) &= \sum_{k\geq 1} \left(p_k \right) z^k/k \,; \\
\ln M_{(2)}(z) &= \sum_{k\geq 1} \left(p_{(k,k)} + p_{(2k)} \right) z^k/2k \,; \\
\ln M_{(1,1)}(z) &= \sum_{k\geq 1} \left(p_{(k,k)} - p_{(2k)} \right) z^k/2k \,; \\
\ln M_{(3)}(z) &= \sum_{k\geq 1} \left(p_{(k,k,k)} + 3p_{(2k,k)} + 2p_{(3k)} \right) z^k/6k \,; \\
\ln M_{(2,1)}(z) &= \sum_{k\geq 1} \left(p_{(k,k,k)} - p_{(3k)} \right) z^k/3k \,; \\
\ln M_{(1,1,1)}(z) &= \sum_{k\geq 1} \left(p_{(k,k,k)} - 3p_{(2k,k)} + 2p_{(3k)} \right) z^k/6k \,. \qquad (6\text{-}26)
\end{aligned}$$

■

Since

$$L(z;X) = \prod_{i\geq 1} (1 - z\, x_i) = 1/M(z;X) \,, \qquad (6\text{-}27)$$

it follows that

$$\ln L(z;X) = -\ln M(x;Z) = -\sum_{k\geq 1} \frac{z^k}{k} \, p_k(X) \,, \qquad (6\text{-}28)$$

and more generally

$$\ln L_\pi(z;X) = -\ln M_\pi(x;Z) = -\sum_{k\geq 1} \frac{z^k}{k} \left(\sum_\rho \frac{1}{z_\rho} \chi^\pi_\rho \, p_{k\rho}(X) \right) . \qquad (6\text{-}29)$$

Then, if we recall that for all positive integers k [29, p.76],

$$p_k^\perp(X) = k \frac{\partial}{\partial p_k(X)} \,, \qquad (6\text{-}30)$$

we are in a position to see that

$$L^\perp(z;X) = L^\perp_{(1)}(z) = \exp\left(-\sum_{k\geq 1} z^k \, \frac{\partial}{\partial p_k(X)} \right) , \qquad (6\text{-}31)$$

while, exploiting the data of Example 6.12, we have:

$$L^{\perp}_{(2)}(z;X) = \exp\left(-\sum_{k\geq 1} z^k \left(\frac{k}{2}\frac{\partial^2}{\partial p_k(X)^2} + \frac{\partial}{\partial p_{2k}(X)}\right)\right);$$
$$L^{\perp}_{(1^2)}(z;X) = \exp\left(-\sum_{k\geq 1} z^k \left(\frac{k}{2}\frac{\partial^2}{\partial p_k(X)^2} - \frac{\partial}{\partial p_{2k}(X)}\right)\right). \qquad (6\text{-}32)$$

Proposition 6.11 implies:

$$\begin{aligned}
V^{(1)}(z) &= (1-z)\, M(z)\, L^{\perp}(z^{-1});\\
V^{(2)}(z) &= (1-z^2)\, M(z)\, L^{\perp}(z^{-1})\, L^{\perp}_{(1)}(z);\\
V^{(1^2)}(z) &= M(z)\, L^{\perp}(z^{-1})\, L^{\perp}_{(1)}(z);\\
V^{(3)}(z) &= (1-z^3)\, M(z)\, L^{\perp}(z^{-1})\, L^{\perp}_{(2)}(z)\, L^{\perp}_{(1)}(z^2);\\
V^{(21)}(z) &= M(z)\, L^{\perp}(z^{-1})\, L^{\perp}_{(2)}(z)\, L^{\perp}_{(1^2)}(z)\, L^{\perp}_{(1)}(z^2);\\
V^{(1^3)}(z) &= M(z)\, L^{\perp}(z^{-1})\, L^{\perp}_{(1^2)}(z). \qquad (6\text{-}33)
\end{aligned}$$

The results (6-31) and (6-32) are then sufficient for us to express the vertex operators of (6-33) in exponential form as follows:

$$\begin{aligned}
V^{(1)}(z) &= (1-z)\ \exp\left(\sum_{k\geq 1}\frac{z^k}{k}\, p_k\right)\exp\left(-\sum_{k\geq 1} z^{-k}\frac{\partial}{\partial p_k}\right);\\
V^{(2)}(z) &= (1-z^2)\ \exp\left(\sum_{k\geq 1}\frac{z^k}{k}\, p_k\right)\exp\left(-\sum_{k\geq 1}(z^{-k}+z^k)\frac{\partial}{\partial p_k}\right);\\
V^{(1^2)}(z) &= \exp\left(\sum_{k\geq 1}\frac{z^k}{k}\, p_k\right)\exp\left(-\sum_{k\geq 1}(z^{-k}+z^k)\frac{\partial}{\partial p_k}\right);\\
V^{(3)}(z) &= (1-z^3)\ \exp\left(\sum_{k\geq 1}\frac{z^k}{k}\, p_k\right)\\
&\qquad \exp\left(-\sum_{k\geq 1}\left((z^{-k}+z^{2k})\frac{\partial}{\partial p_k} + \frac{k\, z^k}{2}\frac{\partial^2}{\partial p_k^2} + z^k\frac{\partial}{\partial p_{2k}}\right)\right);\\
V^{(21)}(z) &= \ \exp\left(\sum_{k\geq 1}\frac{z^k}{k}\, p_k\right)\ \exp\left(-\sum_{k\geq 1}\left((z^{-k}+z^{2k})\frac{\partial}{\partial p_k} + k\, z^k\frac{\partial^2}{\partial p_k^2}\right)\right);\\
V^{(1^3)}(z) &= \ \exp\left(\sum_{k\geq 1}\frac{z^k}{k}\, p_k\right)\exp\left(-\sum_{k\geq 1}\left(z^{-k}\frac{\partial}{\partial p_k} + \frac{k\, z^k}{2}\frac{\partial^2}{\partial p_k^2} - z^k\frac{\partial}{\partial p_{2k}}\right)\right), \qquad (6\text{-}34)
\end{aligned}$$

where once again the explicit dependence on X has been omitted.

The first result expresses the fact that $V^{(1)}(z) = (1-z)\,V(z)$. At first sight this appears rather surprising, but it should be noted that it yields

$$\begin{aligned} s_\lambda^{(1)} &= [Z^\lambda]\, V^{(1)}(z_1)\, V^{(1)}(z_2)\, \cdots\, V^{(1)}(z_l)\cdot 1 \\ &= [Z^\lambda]\, \prod_{i=1}^{l}(1-z_i)\; V(z_1)\, V(z_2)\, \cdots\, V(z_l)\cdot 1 \\ &= [s_\lambda(Z)]\, \prod_{i=1}^{l}(1-z_i)\; M(Z)\,. \end{aligned} \tag{6-35}$$

Implicit in this is the dependence on an arbitrary alphabet $X = (x_1, x_2, \dots)$. Making this explicit gives

$$\begin{aligned} s_\lambda^{(1)}(X) &= [s_\lambda(Z)]\, \prod_{i=1}^{l}(1-z_i)\; M(XZ) = [s_\lambda(Z)]\; L(Z)\; M(XZ) \\ &= [s_\lambda(Z)]\; L^\perp(X)\,(M(XZ)) = [s_\lambda(Z)]\; L^\perp(X)\left(\sum_\mu s_\mu(X)\, s_\mu(Z)\right) \\ &= L^\perp(X)\,(s_\lambda(X)) = L^\perp_{(1)}\,(s_\lambda(X))\,, \end{aligned} \tag{6-36}$$

as required by the definition (5-7) of such a character.

The next two results in (6-34) have been derived by Baker [3] using different techniques involving rather more traditional operator reordering methods. In what follows next, this operator ordering approach is outlined and is used, by way of example, to recover the formula for $V^{(21)}$ as given in (6-34). However, it is clear that our Proposition 6.11 allows further vertex operators V^π specified by partitions π of weight higher than 3 to be written down rather easily.

6.3. **Vertex operators via normal ordering.** The expressions $L^\perp_\pi(w;X)(M(XZ))$ can be evaluated rather easily using (2-16), or more explicitly for any given π as in Appendix A. However, from the point of view of operator ordering the more general expressions $L^\perp_\pi(w;X)M(XZ)$ are not in so-called normal-ordered form since they involve exponentials of various partial derivatives with respect to power sum symmetric functions standing to the left of other exponentials of power sum symmetric functions.

Algebraically, if we introduce operators K and P such that $e^K = L^\perp_\pi(w;X)$ and $e^P = M(XZ)$, the normal ordering problem can be tackled using the following formula:

$$e^K e^P = e^P\big(e^{-P} e^K e^P\big) = e^P\, e^{\,K\,+\,[K,P]\,+\,\frac{1}{2}[[K,P],P]\,+\,\frac{1}{6}[[[K,P],P],P]\,+\,\cdots}\,. \tag{6-37}$$

Even though we suspect the formula used in making the second step may be well known as an adjoint action result in the theory of Lie groups and their algebras, we have been unable to locate a statement or proof of this result. We therefore attach a strictly combinatorial proof in Appendix B.

In our case, K symbolises the partial derivative operator of degree p that is defined by $L^{\perp}_{\pi}(w;X) = \exp K$ with $p = |\pi|$, and P is the usual infinite sum of power sum symmetric functions appearing as the exponent in the formula $M(XZ) = \exp P$. One must retain terms up to those involving $1/p!$ in the expansion of the exponent that appears in (6-37) in order to extract all scalar and differential contributions arising from the reordering. Note that none of the surviving terms in the exponent of the final term will contain symmetric functions in the alphabet X, but only scalars and partial derivatives that all mutually commute. This enables this term to be written as a product of exponentials, each with a single multi-commutator argument.

As an illustration of this method, we deal with the case $\pi = (21)$ for which $p = 3$. For ease of writing, we suppress the alphabet X and abbreviate $\partial/\partial p_k(X)$ as ∂_k for all positive integers k. In this case

$$M_{(21)}(w) = \exp\left(\sum_{k\geq 1}\frac{w^k}{3k}(-p_{3k}+p_k^3)\right); \qquad L_{(21)}(w) = \exp\left(\sum_{k\geq 1}\frac{w^k}{3k}(p_{3k}-p_k^3)\right);$$
$$L^{\perp}_{(21)}(w) = \exp\left(\sum_{k\geq 1} w^k\left(\partial_{3k}-\frac{1}{3}k^2\partial_k^3\right)\right); \qquad M(Z) = \exp\left(\sum_{m\geq 1}\frac{p_m\cdot p_m(Z)}{m}\right). \quad \text{(6-38)}$$

This gives $K = \sum_{k\geq 1} w^k\left(\partial_{3k} - \frac{1}{3}k^2\partial_k^3\right)$ and $P = \sum_{m\geq 1} p_m\cdot p_m(Z)/m$. We calculate directly

$$\begin{aligned}
[\partial_{3k}, P] &= \frac{1}{3k}\, p_{3k}(Z)\,;\\
[k^2\partial_k^3, P] &= 3k\, p_k(Z)\,\partial_k^2;\\
[[k^2\partial_k^3, P], P] &= 6\, p_k(Z)^2\,\partial_k;\\
[[[k^2\partial_k^3, P], P], P] &= \frac{6}{k}\, p_k(Z)^3\,.
\end{aligned} \quad \text{(6-39)}$$

Hence

$$[K,P] = \sum_{k\geq 1} w^k\left(\frac{1}{3k}p_{3k}(Z) - k\, p_k(Z)\,\partial_k^2\right), \quad \text{(6-40)}$$

which clearly commutes with K as claimed. Moreover,

$$\begin{aligned}
&K + [K,P] + \frac{1}{2}[[K,P],P] + \frac{1}{6}[[[K,P],P],P]\\
&\quad = \sum_{k\geq 1} w^k\left(\frac{1}{3k}\big(p_{3k}(Z) - p_k(Z)^3\big) + \partial_{3k} - p_k^2(Z)\,\partial_k - k\, p_k(Z)\,\partial_k^2 - \frac{1}{3}k^2\partial_k^3\right).
\end{aligned} \quad \text{(6-41)}$$

It follows that

$$L^{\perp}_{(21)}(w)\, M(Z) = M(Z)\, L_{(21)}(w;Z)\, \exp\left(-\sum_{k\geq 1} w^k\big(p_k^2(Z)\,\partial_k + k\, p_k(Z)\,\partial_k^2\big)\right) L^{\perp}_{(21)}(w)\,. \quad \text{(6-42)}$$

There are a number of special cases of this normal ordered formula that are of interest. First, acting on 1, or any other scalar, this gives

$$L^{\perp}_{(21)}(w)\, M(Z)\cdot 1 = \; M(Z)\, L_{(21)}(w;Z)\cdot 1 \tag{6-43}$$

in agreement with the identity (2-16).

Second, restricting Z to the one letter alphabet z gives

$$p_k(z) = z^k, \quad p_{2k}(z) = z^{2k} \qquad \text{and} \qquad p_{3k}(z) - p_k(z)^3 = z^{3k} - (z^k)^3 = 0\,, \tag{6-44}$$

for all $k \geq 1$, so that $L_{(21)}(z) = 1$, and (6-42) reduces to

$$L^{\perp}_{(21)}(w)\, M(z) = M(z)\, \exp\left(-\sum_{k\geq 1} w^k \big(z^{2k}\,\partial_k + k z^k\,\partial_k^2\big)\right)\, L^{\perp}_{(21)}(w)\,. \tag{6-45}$$

This is nothing other than an illustrative example of Lemma 6.10 since the identity

$$z^{2k}\,\partial_k + k z^k\,\partial_k^2 = \big(\frac{1}{2} k z^k\,\partial_k^2 + z^k \partial_{2k}\big) + \big(\frac{1}{2} k z^k\,\partial_k^2 - z^k\partial_{2k}\big) + z^{2k}\,\partial_k \tag{6-46}$$

enables (6-45) to be rewritten in the form

$$\begin{aligned} L^{\perp}_{(21)}(w)\, M(z) &= M(z)\, L^{\perp}_{(2)}(wz)\, L^{\perp}_{(1^2)}(wz)\, L^{\perp}_{(1)}(wz^2)\, L^{\perp}_{(21)}(w) \\ &= M(z)\, \sum_{k=1}^{2}\, L^{\perp}_{(21/k)}(wz^k)\, L^{\perp}_{(21)}(w)\,, \end{aligned} \tag{6-47}$$

where use has been made of (6-31) and (6-32).

In addition, it follows from (6-45) that if we now set $w = 1$ and $Z = (z_1, z_2, \ldots, z_l)$ once again and reverse the sequence of steps used in (6-21) we obtain

$$s^{(21)}_{\lambda} = [s_\lambda]\, L^{\perp}_{(21)}\, M(Z)\cdot 1 = [Z^\lambda]\, V^{(21)}(z_1)\, V^{(21)}(z_2)\, \cdots\, V^{(21)}(z_l)\cdot 1\,, \tag{6-48}$$

with

$$\begin{aligned} V^{(21)}(z) &= M(z)\, \exp\left(-\sum_{k\geq 1} \big(z^{2k}\,\partial_k + k z^k\,\partial_k^2\big)\right) \\ &= \exp\left(\sum_{k\geq 1} \frac{z^k}{k}\, p_k\right)\, \exp\left(-\sum_{k\geq 1} \big((-z^{-k} + z^{2k})\,\partial_k + k z^k\,\partial_k^2\big)\right), \end{aligned} \tag{6-49}$$

precisely as in (6-34).

The other results of (6-34) may be obtained in the same way.

7. Replicated vertex operators

Since their introduction in string theory, vertex operators have played a fruitful role in mathematical constructions of group representations as well as combinatorial objects. We cite for example applications to affine Lie algebras [24, 13], quantum affine algebras [12] and sporadic discrete groups [14], see also [22, Chapter 14]. Variations on the theme of symmetric functions [29] are applications to Q-functions [19, 32], Hall-Littlewood functions [20], Macdonald functions [28, 29, 21, 9], Jack functions [6] and Kerov symmetric functions [2, Chapter 6].

As a modest approach to generalising the vertex operators of Section 6, the observations made in Section 4 allow us to write down immediately expressions for replicated or parameterized vertex operators. In the simplest case, this is exemplified by

$$\begin{aligned} V_\alpha(z) &= M(\alpha z)\, L^\perp(\alpha z^{-1}) \\ &= \exp\left(\alpha \sum_{k\geq 1} \frac{z^k}{k}\, p_k\right) \exp\left(-\alpha \sum_{k\geq 1} z^{-k}\, \frac{\partial}{\partial p_k}\right) . \end{aligned} \tag{7-1}$$

for any α, integer, rational, real or complex. Here, making the usual dependence on X quite explicit,

$$\begin{aligned} M(\alpha z; X) = M(z; X)^\alpha &= \prod_{i\geq 1} (1 - z\, x_i)^{-\alpha} \\ &= \sum_\sigma s_\sigma(\alpha z)\, s_\sigma(X) \\ &= \sum_\sigma z^{|\sigma|} \dim_\sigma(\alpha)\, s_\sigma(X) , \end{aligned} \tag{7-2}$$

while

$$\begin{aligned} L(\alpha z^{-1}; X) = L(z^{-1}; X)^\alpha &= \prod_{i\geq 1} (1 - z^{-1}\, x_i)^{\alpha} \\ &= \sum_\tau (-1)^{|\tau|}\, s_\tau(\alpha z^{-1})\, s_{\tau'}(X) \\ &= \sum_\tau (-z)^{-|\tau|} \dim_\tau(\alpha)\, s_{\tau'}(X) , \end{aligned} \tag{7-3}$$

as given first in [16].

More generally, we can define in a similar way:

$$V^\pi_\alpha(z) = (1 - z^p\, \delta_{\pi,(p)})^\alpha\, M(\alpha z)\, L^\perp(\alpha z^{-1}) \prod_{k=1}^{p-1} L^\perp_{\pi/(k)}(\alpha z^k) , \tag{7-4}$$

where each term on the right has an expansion of the type shown above involving sums over partitions $\sigma, \tau, \ldots$. Still more variations may be constructed in which the α's on the right are not all identical. It should be stressed that the normal ordering relations for products of such vertex operators are very much more complicated than those encountered in Section 6.

8. CONCLUSION

This work allows us to conclude that Littlewood's attempt to develop character theory algebraically, instead of using a group manifold integration approach, leads much further than expected. In particular, it allows us to obtain generating functions for formal characters of a range of subgroups H_π of the general linear group, well beyond the classical orthogonal and symplectic subgroups. Furthermore, we see that the algebraic approach does not suffer from the infinities encountered using analytic methods. Instead, the algebraic infinities are just those associated with readily manipulated infinite series of Schur functions. A further advantage of this approach is that the results of these manipulations take very compact forms if carried out in Hopf algebraic terms, as illustrated in the proof of Proposition 6.11, by way of Lemma 6.10, that was based on a knowledge of the coproduct of an infinite Schur function series. While the explicit exponential form of vertex operators cannot easily yield such general results, it can be used to express the algebraic results in a physically more desirable form, as exemplified in (6-34). Moreover, thanks to the plethystic approach to replication and parameterization, these exponential forms can be readily generalized, while still remaining susceptible to actual (machine) calculations.

Moreover we have shown in Table 1 that the use of inner coproducts and the dimension map can dramatically speed up the computation of plethysms. This approach has two major benefits. It allows us i) to compute plethysms with large multiplicities, for example $s_\mu[n\, s_\nu]$ for large integers n, and ii) to extend plethysms to those involving an argument which need not be integral, but can be in a ring extension, and evaluate them. This opens the way for dealing with q-deformations, as introduced by Jarvis and Yung [16], Baker [2] and Brenti [4], who considered plethysms of the form $s_\mu[q\, s_{(1)}]$, which correspond to scalings. In Appendix C we provide pseudo code to implement an algorithm for their evaluation.

An extension of algebraic and operator methods in combinatorial settings, which we have not pursued in the present work, invokes the fermion-boson correspondence (see for example [33, 8, 18]). In the present case, our explicit vertex operator constructions for the formal H_π characters $s_\lambda^{(\pi)}$ can be expected to have their equivalents in terms of free fermions, and hence via Wick's theorem, to be amenable to determinantal evaluations [17]. The resulting determinantal expressions can be expected to be helpful in an attack on the notorious problem of determining the modification rules for H_π characters involving a finite alphabet [11]. We leave further developments along these lines to future work.

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APPENDIX A. AN EXPLICIT EXPRESSION FOR $L_\pi^\perp(w)\,(M(Z))$

Proposition A.13: For any partition π, any w and any alphabet $Z = (z_1, z_2, \ldots, z_l)$ with l a positive integer,

$$L_\pi^\perp(w)\,(M(Z)) = \prod_{T\in\mathcal{T}^\pi[l]} (1 - w\, Z^{\mathrm{wgt}(T)})\; M(Z)\,, \tag{A-1}$$

where the product is taken over all tableaux, T, in the set, $\mathcal{T}^\pi[l]$, of all semistandard or column-strict tableaux [29] with entries taken from the set $\{1, 2, \ldots, l\}$. For each tableau T its weight is defined to be $\mathrm{wgt}(T) = (\#1, \#2, \ldots, \#l))$, with $\#k$ the number of entries k in T for $k = 1, 2, \ldots, l$. ∎

Proof: The proof is by induction with respect to p, the weight of π.

For $p = 1$ we have $\pi = (1)$ and $L_\pi = L_{(1)} = L$ so that

$$\begin{aligned} L_{(1)}^\perp(w)\,(M(z)) &= L^\perp(w)\,(\,M(z_1)\,M(z_2)\,\cdots\,M(z_l)\,) \\ &= M(z_1)/L(w)\;M(z_1)/L(w)\;\cdots\;M(z_l)/L(w) \\ &= (1 - w\,z_1)(1 - w\,z_2)\,\cdots\,(1 - w\,z_l)\;M(z_1)\,M(z_2)\,\cdots\,M(z_l) \\ &= \prod_{1\le i\le l} (1 - w\,z_i)\;M(Z) \;=\; \prod_{T\in\mathcal{T}^{(1)}[l]} (1 - w\,Z^{\mathrm{wgt}(T)})\;M(Z)\,, \end{aligned} \tag{A-2}$$

since each semistandard tableau of shape $\pi = (1)$ consists of a single box whose entry is to be taken from $\{1, 2, \ldots, l\}$. This proves the required result in the case $p = 1$.

Now we assume the result to be true for all partitions η with weight $|\eta| < p$. Then by means of the coproduct argument used in the proof of Lemma 6.10 we have

$$L_\pi^\perp(w)\,(M(Z)) = L_\pi^\perp(w)\,(\,M(z_l)\cdots M(z_2)\,M(z_1)\,)$$
$$= (\,M(z_l)/L_\pi(w)\,)\,(\,M(Z')/(L_\pi(w)\prod_{k=1}^{p-1}\prod_{m=1}^{m(k)} L_{\eta(k,m)}(w\,z_l^k)\,)\,)\,, \qquad \text{(A-3)}$$

where $Z' = (z_1, z_2, \ldots, z_{l-1})$.

Consider the first factor $M(z_l)/L_\pi(w)$. The use of (6-14) implies that $M(z_l)/L_\pi(w) = M(z_l)$ unless $\pi = (p)$ in which case one obtains

$$M(z_l)/L_{(p)}(w) = (1 - w\,z^p)\,M(z_l) = (1 - w\,z_l^{\mathrm{wgt}(T)})\,M(z_l)\,, \qquad \text{(A-4)}$$

where T is the single semistandard tableau of one-rowed shape (p) whose entries are all l.

Turning to the second factor involving $M(Z')$, by the induction hypothesis

$$M(Z')/L_{\eta(k,m)}(w\,z_l^k) = \prod_{T'\in\mathcal{T}^{\eta(k,m)}[l-1]} (1 - w\,z_l^k\,Z'^{\,\mathrm{wgt}(T')})\,M(Z')\,. \qquad \text{(A-5)}$$

It remains to take the product over all $k = 1, 2, \ldots, p-1$ and $m = 1, 2, \ldots, m(k)$, but this is a product over all shapes $\eta(k,m)$ that are obtained by the removal of a horizontal strip [29, p.5] of k-boxes from the shape of π. For each semistandard tableau T' of shape $\eta(k,m)$ with entries from $\{1, 2, \ldots, l-1\}$, if we then fill the k boxes of the horizontal strip with k entries l we obtain a semistandard tableau T of shape π with entries from $\{1, 2, \ldots, l\}$. All semistandard tableaux T of shape π containing at least one l and no more than $p-1$ entries l can be obtained in this way.

Combining this with our earlier result on the first factor, implies that

$$M(Z)/L_\pi(w) = (\,M(z_l)/L_{(p)}(w)\,)\,(\prod_{k=1}^{p-1}\prod_{m=1}^{m(k)} L_{\eta(k,m)}(w\,z_l^k)\,)\,)$$
$$= M(z_l)\prod_{T\in\mathcal{T}^\pi[l]_l} (1 - w\,Z^{\,\mathrm{wgt}(T)})\,(\,M(Z')/L_\pi(w)\,)\,, \qquad \text{(A-6)}$$

where the subscript l on $[l]$ is intended to indicate that the product is taken over all those semistandard tableaux T of shape π containing at least one entry l.

By applying the same process to $M(Z')/L_\pi(w)$, one obtains factors corresponding to all semistandard T of shape π containing no entry l but at least one entry $l-1$. Continuing with this iteration procedure one obtains the result

$$M(Z)/L_\pi(w) = \prod_{T\in\mathcal{T}^\pi[l]} (1 - w\,Z^{\,\mathrm{wgt}(T)})\,M(z_l)\,M(z_{l-1})\cdots M(z_1)\,, \qquad \text{(A-7)}$$

thereby completing the proof of (A-1). □

It should be noted that, as one possible definition of Schur functions,

$$s_\pi(Z) = \sum_{T\in\mathcal{T}^\pi[l]} Z^{\,\mathrm{wgt}(T)}\,, \tag{A-8}$$

since the monomials in the expansion of $s_\pi(Z)$ are precisely the various $Z^{\,\mathrm{wgt}(T)}$ specified by all the semistandard tableaux T appearing in $\mathcal{T}^\pi[l]$. Then, thanks to the plethystic definition of $L_\pi(w;Z)$, (A-1) immediately implies the validity of:

Corollary A.14:

$$L_\pi^\perp(w)\,(M(Z)) = L_\pi(w;Z)M(Z)\,. \tag{A-9}$$

■

Once it is recalled that the dependence on X has been omitted, this can be seen to be nothing other than an exemplification of the more general result (2-16) associated with the Cauchy kernel.

APPENDIX B. PROOF OF ADJOINT ACTION IDENTITY

Theorem B.15: Let x and y be arbitrary elements of a ring R with identity 1 but which is in general non-commutative. Then

$$\exp(x)\exp(y)\exp(-x) = \exp\left(\sum_{n=0}^{\infty}\frac{1}{n!}[x,\cdots,[x,[x,y]]\cdots]\right)\,, \tag{B-1}$$

where the displayed commutator $[x,\cdots,[x,[x,y]]\cdots]$ is of degree n in x. ■

Proof: For all $x\in R$ the mutully inverse functions $\exp$ and $\ln$ are defined by

$$\exp(x) = \sum_{k=0}^{\infty}\frac{1}{k!}\,x^k \qquad \text{and} \qquad \ln(x) = \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\,(x-1)^m\,. \tag{B-2}$$

Now let $\exp(x)\exp(y)\exp(-x) = \exp(z)$ so that $z = \ln(\exp(x)\exp(y)\exp(-x))$. It follows that

$$\begin{aligned} z &= \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\left(\sum_{p,q,r\geq 0}(-1)^r\frac{x^p y^q x^r}{p!q!r!}\;-\;1\right)^m \\ &= \sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\sum_{\substack{p_i,\,q_i,\,r_i\,\geq\,0\\ p_i+q_i+r_i\,>\,0}}(-1)^{r_1}\frac{x^{p_1}y^{q_1}x^{r_1}}{p_1!q_1!r_1!}(-1)^{r_2}\frac{x^{p_2}y^{q_2}x^{r_2}}{p_2!q_2!r_2!}\cdots(-1)^{r_m}\frac{x^{p_m}y^{q_m}x^{r_m}}{p_m!q_m!r_m!}\,. \end{aligned} \tag{B-3}$$

In this expansion as a signed sum of products of triples consider those contributions for which a triple contains no y that is to say a triple of the form $(x^p y^0 x^r)$ with $p+r=n>0$. For each such n, if we collect together all those terms that differ only in the values of p and r their sum contains the factor

$$\sum_{r=0}^{n}(-1)^r\frac{x^{n-r}x^r}{(n-r)!r!}=\frac{1}{n!}(x-x)^n=0. \tag{B-4}$$

It follows that in (B-3) we need retain only those terms for which $q_k>0$ for all $k=1,2,\dots,m$.

Now consider those terms for which there are two neighbouring triples $(\cdots yx^r)(x^p y\cdots)$ with $r+p=n>0$. Then as before, for each such n, if we collect together all those terms that differ only in the values of p and r their sum contains once again the factor

$$\sum_{r=0}^{n}(-1)^r\frac{x^{n-r}x^r}{(n-r)!r!}=\frac{1}{n!}(x-x)^n=0. \tag{B-5}$$

It follows that in (B-3) we need retain only those terms for which no two y's are separated by any x's.

This leaves only terms of the form $x^p y^q x^r$, with $q>0$ and $p,r\geq 0$. As far as the constituent triples are concerned the x^p and x^r must be attached to at least one y on their right and left, respectively, since all triples consisting of just x's have been eliminated. Thus the contribution of the x's is a fixed common factor, namely $(-1)^r x^{p+r}/(p!r!)$. Apart from this common factor the contribution of all terms $x^p y^q x^r$ to z in (B-3) is given by

$$\begin{aligned}
&\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\sum_{q_i>0;\ q_1+q_2+\cdots+q_m=q}\frac{y^{q_1+q_2+\cdots+q_m}}{q_1!q_2!\cdots q_m!}\\
&=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\frac{y^q}{q!}\sum_{q_i>0;\ q_1+q_2+\cdots+q_m=q}\begin{pmatrix} q\\ q_1\,q_2\,\dots\,q_m\end{pmatrix}\\
&=\sum_{m=1}^{\infty}\frac{(-1)^{m-1}}{m}\frac{y^q}{q!}\,m!\,S(q,m)=\frac{y^q}{q!}\sum_{m=1}^{\infty}s(m,1)\,S(q,m)=\frac{y^q}{q!}\,\delta_{q,1}\,.
\end{aligned} \tag{B-6}$$

where $s(m,1)$ and $S(q,m)$ are Stirling numbers of the first and second kind, respectively.

It follows that the only surviving terms in (B-3) are those of the form $x^p y x^r$ with $p,r\geq 0$, and each of these terms must constitute a single triple, with $m=1$. Thus

$$z=\sum_{p,r\geq 0}(-1)^r\frac{x^p y x^r}{p!r!}=\sum_{n=0}^{\infty}\frac{1}{n!}\sum_{r=0}^{n}(-1)^r\begin{pmatrix} n\\ r\end{pmatrix}x^{n-r}y\,x^r\,. \tag{B-7}$$

To complete the proof of Theorem B.15 it only remain to prove the following:

Lemma B.16: For all x and y and all non-negative integers n

$$[x,\cdots,[x,[x,y]]\cdots]=\sum_{r=0}^{n}(-1)^r\begin{pmatrix} n\\ r\end{pmatrix}x^{n-r}\,y\,x^r\,, \tag{B-8}$$

where the commutator on the left is of degree n in x. ∎

Proof: We offer a proof by induction with respect to n. For $n = 0$ the right hand side of (B-8) is just y, and this is how the left hand side must be interpreted in this $n = 0$ case. Perhaps more significantly, for $n = 1$ the right hand side of (B-8) reduces to $xy - yx = [x, y]$, as required.

Now, for convenience, let $[x^{(k)}, y]$ denote the commutator $[x, \cdots, [x, [x, y]] \cdots]$ of degree k for any positive integer k. Then assuming the validity of (B-8) in the case $n = k$ we have

$$\begin{aligned}
[x^{(k+1)}, y] &= [x, [x^{(k)}, y]] \\
&= \sum_{r=0}^{k} (-1)^r \begin{pmatrix} k \\ r \end{pmatrix} x^{k-r+1}\, y\, x^r - \sum_{r=0}^{k} (-1)^r \begin{pmatrix} k \\ r \end{pmatrix} x^{k-r}\, y\, x^{r+1} \\
&= x^{k+1}\, y + \sum_{r=1}^{k} \left(\begin{pmatrix} k \\ r \end{pmatrix} + \begin{pmatrix} k \\ r-1 \end{pmatrix} \right) x^{k+1-r}\, y\, x^r + (-1)^{k+1} y\, x^{k+1} \\
&= \sum_{r=0}^{k+1} (-1)^r \begin{pmatrix} k+1 \\ r \end{pmatrix} x^{k+1-r}\, y\, x^r \,. \qquad \text{(B-9)}
\end{aligned}$$

This proves the required result for $n = k + 1$ and completes the induction argument, thereby proving Lemma B.16 and hence also Theorem B.15. □

APPENDIX C. A GENERAL ROUTINE TO COMPUTE SCALED PLETHYSMS

In this appendix we want to give pseudo code for an algorithm to compute plethysms with scaled arguments. Such an algorithm was implemented in the Maple package `SchurFkt` [1]. To the best knowledge of the authors no other computer algebra system uses this fast algorithm, so it seems appropriate to present this method here.

We assume that we have a basis $\{u_\lambda\}$ of the ring of symmetric functions $\Lambda(X)$ in countably many variables. We distinguish basis monomials `SymB`, terms `SymT` and polynomials `SymFkt`. We need also types for the tensor product and call this `SymBxB` for tensor basis monomials and `SymFktBxB` for general tensor polynomials. We also assume that we can compute the following functions for this basis:

- $\dim : \Lambda(X) \times R \longrightarrow R$ the dimension function for vector spaces V^λ having an $GL(\alpha)$ action for $\alpha \in R$. Such vector spaces need not be irreducible. We call this map
  `dim :: SymB, Ring -> Ring`
- $\Delta : \Lambda(X) \longrightarrow \Lambda(X) \otimes \Lambda(X)$ the outer coproduct. Due to self duality this is equivalent to computing skew products. This function is called
  `Delta :: SymB -> SymFktBxB`.
  The fast evaluation of outer coproducts is done using, for example, the Lascoux-Schützenberger algorithm for skew Schur functions, see [23].

- $\delta : \Lambda(X) \longrightarrow \Lambda(X) \otimes \Lambda(X)$ the inner coproduct. Due to self duality this is equivalent to computing an inner product. This function is called
  `delta :: SymB -> SymFktBxB`.
  The inner coproduct is computed from the Kronecker coefficients of inner products evaluated, for example in the Schur basis, by the method of Robinson [31].
- We also assume that we can compute plethysms for basis monomials $\{u_\lambda\}$, choosing our favorite method. This map is called
  `plethB :: SymB, SymB -> SymFkt`.
  Good algorithms for plethysms in standard bases are available [7, 23].

Let us further assume, that a symmetric function (tensor) polynomial is stored so that we can access terms by a function `listOfTerms` and that a term is a pair (triple) consisting of a coefficient in R and a basis monomial in $\{u_\lambda\}$ (a pair of basis monomials) which we can access by functions `first` for the coefficient and `second` (and `third`) for the basis monomial(s).

We know from the properties of plethysms displayed in (2-18), that the plethysm is linear in the first argument but not linear in the second argument. Our task is hence to provide a procedure for expanding with respect to a general symmetric function in the second argument. This reads as follows:

LISTING 1

```
1  // Declarations of predefined functions
2  SymFkt          plethysmB(SymB, SymB);
3  SymFktBxB       delta(SymB), Delta(SymB);
4  Ring            dim(SymB,Ring);
5
6  // Declarations
7  SymFkt          plethysmRight(SymB,SymFkt);
8  SymFkt          plethysm(SymFkt, SymFkt);
9
10 // Procedures
11 // right non-linear expansion
12 plethysmRight(sMon,sPoly){
13         SymB            ty1; // local variables
14         SymT            term,head;
15         SymFktBxB       coProd;
16         SymFkt          res;
17         List[SymFkt]    tail;
18         List[SymFktBxB] lstTerms;
19         if zero=second(sMon) { return(sMon); };
20         lstTerms := listOfTerms(sPoly);
21         if #lstTerms=1 {
22           coProd := delta(sMon); //inner coproduct
```

```
23            lstTerms := listOfTerms(coProd);
24            res := 0;
25            for term in lstTerms do{
26               res := res +
27               dim(second(term),first(sPoly))
28                  *plethysmB(third(term),sPoly);
29            }
30            return(res);
31          } else {
32            head := first(lstTerms);
33            tail := rest(lstTerms);
34            coProd := Delta(sMon); //outer coproduct
35            lstTerms := listOfTerms(coProd);
36            res := 0;
37            for term in lstTerms do{
38              res := res +
39                     first(term)*plethysmRight(second(term),head)
40                                *plethysmRight(third(term),tail);
41            }
42            return(res);
43          }
44
45  }
46
47  // left linearity
48  plethysm(sPloly1, sPoly2){
49          SymT    term;
50          SymFkt  res :=0;
51          if sPoly1=0 or sPoly2=0 then { return 0 };
52          for term in listOfTerms(sPoly1) do {
53            res := res +
54                   first(term)*plethysmRight(second(term),sPoly2);
55          }
56          return(res);
57  }
```

We end this Appendix by noting that many standard maps have a plethystic interpretation and hence are available via the above algorithm. Among them are the identity map Id seen as plethysm with $s_{(1)}$ and the antipode map S seen as plethysms with $((-1)s_{(1)})$.

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