Oberwolfach Preprints

OWP 2010 - 18

MACIEJ KARPICZ, ANDRZEJ SKOWROŃSKI, KUNIO YAMAGATA

On Selfinjective Artin Algebras Having Generalized Standard Quasitubes

Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

Oberwolfach Preprints (OWP)

Starting in 2007, the MFO publishes a preprint series which mainly contains research results related to a longer stay in Oberwolfach. In particular, this concerns the Research in Pairs-Programme (RiP) and the Oberwolfach-Leibniz-Fellows (OWLF), but this can also include an Oberwolfach Lecture, for example.

A preprint can have a size from 1 - 200 pages, and the MFO will publish it on its website as well as by hard copy. Every RiP group or Oberwolfach-Leibniz-Fellow may receive on request 30 free hard copies (DIN A4, black and white copy) by surface mail.

Of course, the full copy right is left to the authors. The MFO only needs the right to publish it on its website *www.mfo.de* as a documentation of the research work done at the MFO, which you are accepting by sending us your file.

In case of interest, please send a **pdf file** of your preprint by email to *rip@mfo.de* or *owlf@mfo.de*, respectively. The file should be sent to the MFO within 12 months after your stay as RiP or OWLF at the MFO.

There are no requirements for the format of the preprint, except that the introduction should contain a short appreciation and that the paper size (respectively format) should be DIN A4, "letter" or "article".

On the front page of the hard copies, which contains the logo of the MFO, title and authors, we shall add a running number (20XX - XX).

We cordially invite the researchers within the RiP or OWLF programme to make use of this offer and would like to thank you in advance for your cooperation.

Imprint:

Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

On selfinjective artin algebras having generalized standard quasitubes

Maciej Karpicz[a,*], Andrzej Skowroński[a], Kunio Yamagata[b]

[a]*Faculty of Mathematics and Computer Science, Nicolaus Copernicus University, Chopina 12/18, 87-100 Toruń, Poland*
[b]*Department of Mathematics, Tokyo University of Agriculture and Technology, Nakacho 2-24-16, Koganei, Tokyo 184-8588, Japan*

Dedicated to Daniel Simson on the occasion of his seventieth birthday

Abstract

We give a complete description of the Morita equivalence classes of all connected selfinjective artin algebras for which the Auslander-Reiten quiver admits a family of quasitubes having common composition factors, closed under composition factors, and consisting of modules not lying on infinite short cycles.

Keywords: Selfinjective algebra, orbit algebra, quasitilted algebra, quasitube, short cycle
MSC: 16D50, 16G10, 16G70

1. Introduction and the main result

Throughout the paper, by an algebra we mean a basic, connected artin algebra over a commutative artin ring k. For an algebra A we denote by $\operatorname{mod} A$ the category of finitely generated right A-modules. Given a module M in $\operatorname{mod} A$, we denote by $[M]$ the image of M in the Grothendieck group $K_0(A)$ of A. Thus $[M] = [N]$ if and only if the modules M and N have the same composition factors including the multiplicities. An algebra A is called

*Corresponding author
Email addresses: mkarpicz@mat.uni.torun.pl (Maciej Karpicz), skowron@mat.uni.torun.pl (Andrzej Skowroński), yamagata@cc.taut.ac.jp (Kunio Yamagata)

selfinjective if A_A is an injective module, or equivalently, the projective and injective modules in $\operatorname{mod} A$ coincide.

An important combinatorial and homological invariant of the module category $\operatorname{mod} A$ of an algebra A is its Auslander–Reiten quiver Γ_A. The Auslander–Reiten quiver Γ_A describes the structure of the quotient category $\operatorname{mod} A/\operatorname{rad}^{\infty}(\operatorname{mod} A)$, where $\operatorname{rad}^{\infty}(\operatorname{mod} A)$ is the infinite Jacobson radical of $\operatorname{mod} A$. In particular, by a result due to Auslander [6], A is of finite representation type if and only if $\operatorname{rad}^{\infty}(\operatorname{mod} A) = 0$. In general, it is important to study the behavior of the components of Γ_A in the category $\operatorname{mod} A$. Following [45] a component $\mathcal{C}$ of Γ_A is called *generalized standard* if $\operatorname{rad}^{\infty}(X, Y) = 0$ for all modules X and Y in $\mathcal{C}$. It has been proved in [45] that every generalized standard component $\mathcal{C}$ of Γ_A is *almost periodic*, that is, all but finitely many DTr-orbits in $\mathcal{C}$ are periodic. Moreover, by a result of [59], the additive closure $\operatorname{add}(\mathcal{C})$ of a generalized standard component $\mathcal{C}$ of Γ_A is closed under extensions in $\operatorname{mod} A$. We note that, for A selfinjective, every infinite generalized standard component $\mathcal{C}$ of Γ_A is either acyclic with finitely many DTr-orbits or is a *quasitube* (the stable part $\mathcal{C}^s$ of $\mathcal{C}$ is a stable tube).

In the representation theory of selfinjective algebras a prominent role is played by the selfinjective algebras of *quasitilted type*, that is, the orbit algebras $\widehat{B}/G$, where $\widehat{B}$ is the repetitive algebra of a quasitilted algebra B and G is an admissible group of automorphisms of $\widehat{B}$, which is in fact an infinite cyclic group generated by a strictly positive automorphism of $\widehat{B}$. Recall that the quasitilted algebras are those of the form $\operatorname{End}_{\mathcal{H}}(T)$ where T is a tilting object in a hereditary Ext-finite abelian category $\mathcal{H}$, or equivalently, the algebras Λ of global dimension at most two and with every indecomposable module in $\operatorname{mod} \Lambda$ of the projective dimension or the injective dimension at most one [17]. It has been proved in [16] that the class of quasitilted algebras consists of the tilted algebras [18] (endomorphism algebras of tilting modules over hereditary algebras) and the quasitilted algebras of canonical type [31] (endomorphism algebras of tilting objects in hereditary abelian categories whose derived categories are equivalent to the derived categories of canonical algebras in the sense of Ringel [40], [41]). Accordingly the class of selfinjective algebras of quasitilted type consists of the selfinjective algebras of tilted type and the selfinjective algebras of canonical type. We refer to the survey article [58] for the representation theory and the structure of the Auslander–Reiten quivers of selfinjective algebras of quasitilted type. We also mention that a selfinjective algebra A over an algebraically closed field is of polynomial growth if and only if A is a socle and geometric deformation of an orbit algebra $\widehat{B}/G$ with B a quasitilted algebra having nonnegative Euler form (see [44], [51]).

In the paper we are concerned with the structure of selfinjective algebras for which the Auslander–Reiten quiver admits a generalized standard component. A distinguished class of such algebras is formed by the selfinjective algebras of finite type. It is conjectured in [58, Problem 12.4] that these algebras are socle deformations of the orbit algebras $\widehat{B}/G$ of tilted algebras B of Dynkin type. It has been proved in [54], [55] that the selfinjective algebras A having an acyclic generalized standard component in Γ_A are selfinjective algebras of tilted (Euclidean or wild) type. On the other hand, the description of selfinjective algebras whose Auslander–Reiten quiver admits a generalized standard quasitube is an exciting but difficult problem (see [49], [50]). Namely, every algebra Λ over a field k is a factor algebra of a selfinjective algebra A with Γ_A having a generalized standard stable tube (see [50]).

The aim of the paper is to prove the following theorem characterizing a wide class of selfinjective algebras whose Auslander–Reiten quiver admits a family of generalized standard quasitubes satisfying certain conditions.

Theorem 1.1. *Let A be a basic, connected, selfinjective artin algebra. The following statements are equivalent.*

(i) Γ_A *admits a family $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ of quasitubes having common composition factors, closed on composition factors, and consisting of modules which do not lie on infinite short cycles in* $\operatorname{mod} A$.

(ii) *A is isomorphic to an orbit algebra $\widehat{B}/G$, where B is an almost concealed canonical algebra and G is an infinite cyclic group of automorphisms of $\widehat{B}$ of one of the forms*
   (a) *$G = (\varphi\nu_{\widehat{B}}^2)$, for a strictly positive automorphism φ of $\widehat{B}$,*
   (b) *$G = (\varphi\nu_{\widehat{B}}^2)$, for B a tubular algebra and φ a rigid automorphism of $\widehat{B}$,*
   (c) *$G = (\varphi\nu_{\widehat{B}}^2)$, for B of Euclidean or wild type and φ a rigid automorphism of $\widehat{B}$ acting freely on the nonstable tubes of the unique separating family $\mathcal{T}^B$ of ray tubes of Γ_B,*

   where $\nu_{\widehat{B}}$ is the Nakayama automorphism of $\widehat{B}$.

Following [48] a family $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ of components of Γ_A is said to have *common composition factors*, if, for each pair i and j in I, there exist modules $X_i \in \mathcal{C}_i$ and $X_j \in \mathcal{C}_j$ with $[X_i] = [X_j]$. Moreover, $\mathcal{C}$ is *closed under composition factors* if, for every indecomposable modules M and N in $\operatorname{mod} A$ with $[M] = [N]$, $M \in \mathcal{C}$ forces $N \in \mathcal{C}$. Further, by a *short cycle* in $\operatorname{mod} A$ we mean a sequence $M \xrightarrow{f} N \xrightarrow{g} M$ of nonzero nonisomorphisms between indecomposable modules in $\operatorname{mod} A$ [38], and such a cycle is said to be *infinite* if at least one of the homomorphisms f or g belongs to $\operatorname{rad}^\infty(\operatorname{mod} A)$. We also mention that, by a result proved in [38], every indecomposable module M in $\operatorname{mod} A$ which does not lie on a short cycle is uniquely determined by $[M]$ (up to isomorphism).

As a direct consequence of Theorem 1.1 and results on selfinjective algebras of canonical type, established in Section 6 (Proposition 6.4 and 6.5), we obtain the following fact.

Corollary 1.2. *Let A be a basic, connected, selfinjective artin algebra. The following statements are equivalent.*

(i) *Γ_A admits a family $\mathcal{T} = (\mathcal{T}_i)_{i\in I}$ of stable tubes with common composition factors, closed under composition factors and consisting of modules which do not lie on infinite short cycles in* $\operatorname{mod} A$.

(ii) *A is isomorphic to an orbit algebra $\widehat{B}/G$, where B is a concealed canonical algebra and G is an infinite cyclic group of automorphisms of $\widehat{B}$ of the form $(\varphi\nu_{\widehat{B}}^2)$ for a positive automorphism φ of $\widehat{B}$.*

We refer to [11], [22], [27], [28], [29], [30], [41], [42], [48] for contructions and basic properties of concealed canonical algebras.

By general theory (see [33], [61]), an infinite component $\mathcal{C}$ of the Auslander-Reiten quiver Γ_A of a selfinjective algebra A is cyclic (every module in $\mathcal{C}$ lies on an oriented cycle in Γ_A) if and only if $\mathcal{C}$ is a quasitube. Then we obtain the following consequence of Theorem 1.1 and the known structure of the Auslander-Reiten quivers of selfinjective algebras of canonical type (see Theorem 6.3, Proposition 6.4).

Corollary 1.3. *Let A be a basic, connected, selfinjective artin algebra. The following statements are equivalent.*

(i) *Γ_A is cyclic and admits a family $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ of quasitubes having common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles in* $\operatorname{mod} A$.

(ii) Γ_A *is cyclic and admits a family* $\mathcal{T} = (\mathcal{T}_i)_{i \in I}$ *of stable tubes having common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles in* $\operatorname{mod} A$.

(iii) A *is isomorphic to an orbit algebra* $\widehat{B}/G$*, where* B *is a tubular algebra and* G *is an infinite cyclic group of* $\widehat{B}$ *of the form* $(\varphi\nu_{\widehat{B}}^2)$ *for a positive automorphism* φ *of* $\widehat{B}$.

We refer to [23], [24], [25], [26], [27], [40], [41] for constructions and basic properties of tubular algebras.

As an immediate consequence of Theorem 1.1 and the fact that the ordinary valued quivers of quasitilted algebras are acyclic [17] we obtain the following fact.

Corollary 1.4. *Let* A *be a basic, connected, selfinjective artin algebra whose Auslander-Reiten quiver* Γ_A *admits a family* $\mathcal{C}$ *of quasitubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles. Then the center of* A *is a field, and hence* A *is a finite dimensional algebra over a field.*

The paper is organized as follows. In Section 2 we describe basic properties of quasitubes which are fundamental for the proofs of the main results. In Section 3 we recall known characterizations of quasitilted algebras of canonical type, playing a prominent role in the proof of Theorem 1.1. Section 4 is devoted to quasitube enlargements of concealed canonical algebras, essential for further considerations. In Section 5 we recall criteria for selfinjective algebras to be orbit algebras of repetitive algebras established by the second and third named authors, applied in the proof of Theorem 1.1. In Section 6 we describe the module categories of selfinjective algebras of canonical type as well as prove the implication $(ii) \Rightarrow (i)$ of Theorem 1.1. The final Section 7 is devoted to the proof of the implication $(i) \Rightarrow (ii)$ of Theorem 1.1.

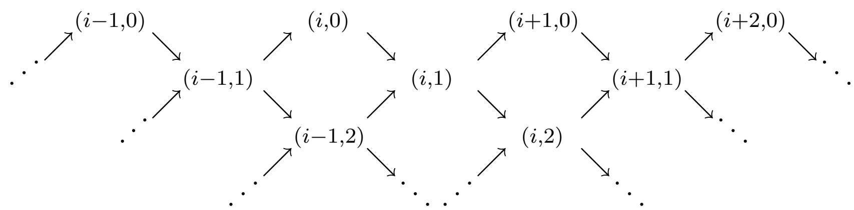
For basic background on the representation theory of algebras applied in the paper we refer to the books [2], [7], [15], [40], [42], [43] and to the survey articles [51], [58], [60].

The main results of the paper have been presented by the first named author during the Fortheenth International Conference on Representations of Algebras (ICRA XIV) held in Tokyo in August 2010.

2. Quasitubes

The purpose of this section is to present results on quasitubes of Auslander-Reiten quivers of algebras, playing a prominent role in the proof of Theorem 1.1.

Recall that if $\mathbb{A}_\infty$ is the quiver $0 \to 1 \to 2 \to \cdots$, then $\mathbb{Z}\mathbb{A}_\infty$ is the translation quiver of the form



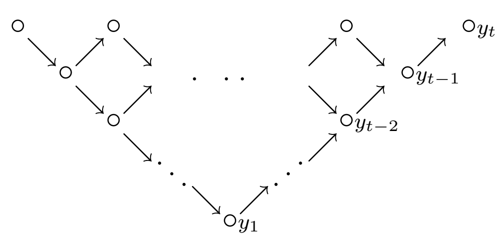
with $\tau(i,j) = (i-1,j)$ for $i \in \mathbb{Z}$, $j \in \mathbb{N}$. For $r \geq 1$, denote by $\mathbb{Z}\mathbb{A}_\infty/(\tau^r)$ the translation quiver obtained from $\mathbb{Z}\mathbb{A}_\infty$ by identifying each vertex (i,j) of $\mathbb{Z}\mathbb{A}_\infty$ with the vertex $\tau^r(i,j)$ and each arrow $x \to y$ in $\mathbb{Z}\mathbb{A}_\infty$ with the arrow $\tau^r x \to \tau^r y$, and call it the *stable tube of rank* r. The τ-orbit of a stable tube Γ formed by all vertices having exactly one immediate predecessor (equivalently, successor) is called the *mouth* of Γ.

Let (Γ, τ) be a translation quiver (with trivial valuations). For some vertices x in Γ, called pivots, we shall define two *admissible operations* [4] modifying (Γ, τ) to a new translation quiver (Γ', τ'), depending on the shape of paths in Γ starting from x.

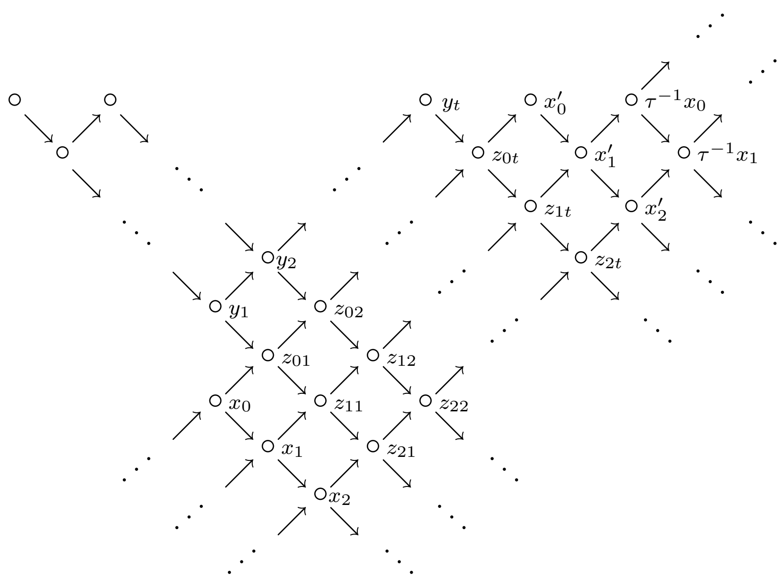
(*ad* 1) Suppose that Γ admits an infinite sectional path

$$x = x_0 \to x_1 \to x_2 \to \cdots$$

starting at x, and assume that every sectional path in Γ starting at x is a subpath of the above path. For $t \geq 1$, let Γ_t be the following translation quiver, isomorphic to the Auslander-Reiten quiver of the full $t \times t$ upper triangular matrix algebra over a field,



We then let Γ' be the translation quiver having as vertices those of Γ, those of Γ_t, additional vertices z_{ij} and x_i' (where $i \geq 0$, $1 \leq j \leq t$) and having arrows as in the figure below

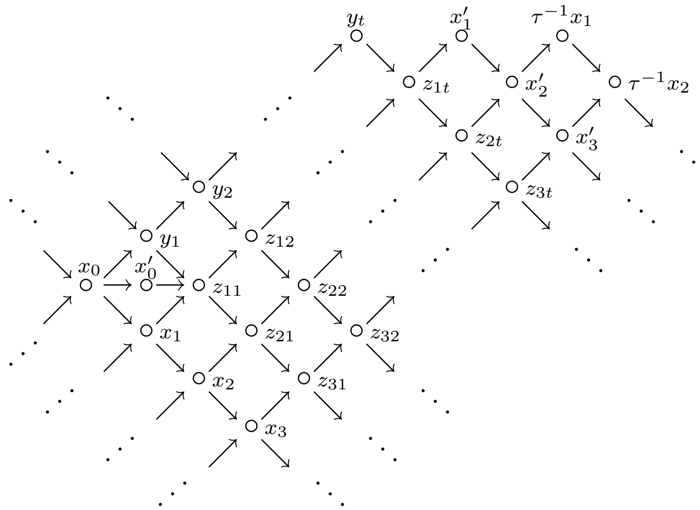


The translation τ' of Γ' is defined as follows: $\tau' z_{ij} = z_{i-1,j-1}$ if $i \geq 1$, $j \geq 2$, $\tau' z_{i1} = x_{i-1}$ if $i \geq 1$, $\tau' z_{0j} = y_{j-1}$ if $j \geq 2$, z_{01} is projective, $\tau' x_0' = y_t, \tau' x_i' = z_{i-1,t}$ if $1 \geq 1$, $\tau'(\tau^{-1}x_i) = x_i'$ provided x_i is not injective in Γ, otherwise x_i' is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation of Γ, or Γ_t, respectively. If $t = 0$, the new translation quiver Γ' is obtained from Γ by inserting only the sectional path consisting of the vertices $x_i', i \geq 0$.

($ad\,2$) Suppose that a vertex x in Γ is injective and Γ admits two sectional paths starting at x, one infinite and the other finite with at least one arrow

$$y_t \leftarrow \cdots \leftarrow y_2 \leftarrow y_1 \leftarrow x = x_0 \rightarrow x_1 \rightarrow x_2 \rightarrow \cdots$$

such that any sectional path starting at x is a subpath of one of these paths. Then Γ' is the translation quiver having as vertices those of Γ, additional vertices denoted by x_0', z_{ij}, x_i' (where $i \geq 1$, $1 \leq j \leq t$), and having arrows as in the figure below



The translation τ' of Γ' is defined as follows: x'_0 is projective-injective, $\tau' z_{ij} = z_{i-1,j-1}$ if $i \geq 2$, $j \geq 2$, $\tau' z_{i1} = x_{i-1}$ if $i \geq 1$, $\tau' z_{1j} = y_{j-1}$ if $j \geq 2$, $\tau' x'_i = z_{i-1,t}$ if $i \geq 2$, $\tau' x'_1 = y_t$, $\tau'(\tau^{-1} x_i) = x'_i$ provided x_i is not injective in Γ, otherwise x'_i is injective in Γ'. For the remaining vertices of Γ', τ' coincides with the translation τ of Γ.

We denote by $(ad\,1^*)$ and $(ad\,2^*)$ the admissible operations dual to the admissible operations $(ad\,1)$ and $(ad\,2)$, respectively.

A connected translation quiver Γ is said to be a *quasitube* if Γ can be obtained from a stable tube by an iterated application of admissible operations $(ad\,1)$, $(ad\,2)$, $(ad\,1^*)$ or $(ad\,2^*)$. A *tube* (in the sense of [40]) is a quasitube having the property that each admissible operation in the sequence defining it is of the form $(ad\,1)$ or $(ad\,1^*)$. Finally, if we apply only operations of type $(ad\,1)$ (respectively, of type $(ad\,1^*)$) then such a quasitube Γ is called a *ray tube* (respectively, a *coray tube*). Observe that a quasitube without injective (respectively, projective) vertices is a ray tube (respectively, a coray tube). A quasitube Γ whose all nonstable vertices are projective-injective is said to be *smooth*.

The following proposition provides a characterization of quasitubes in the Auslander-Reiten quivers of selfinjective algebras ([34, Theorem A], [33] and [61])

Proposition 2.1. *Let A be a selfinjective algebra and Γ a connected component of Γ_A. The following statements are equivalent.*

(i) Γ *is a quasitube.*
(ii) Γ^s *is a stable tube.*
(iii) Γ *contains an oriented cycle.*

Here, Γ^s denotes the stable part of Γ, obtained from Γ by removing the projective-injective modules and the arrows attached to them.

The following characterization of generalized standard stable tubes of an Auslander-Reiten quiver has been established in [45, Corollary 5.3] (see also [47, Lemma 3.1]).

Proposition 2.2. *Let A be an algebra and Γ a stable tube of Γ_A. The following statements are equivalent:*

(i) Γ *is generalized standard.*
(ii) *The mouth of Γ consists of pairwise orthogonal bricks.*
(iii) $\mathrm{rad}^{\infty}(X, X) = 0$ *for any module X in Γ.*

Recall that an indecomposable A-module X is called a *brick* if its endomorphism algebra $\mathrm{End}_A(X)$ is a division algebra. We note that the division algebras of all modules lying on the mouth of a generalized standard stable tube of Γ are isomorphic.

Let A be an algebra and Γ be a stable tube of Γ_A. Then Γ has two types of arrows: arrows pointing to infinity and arrows pointing to the mouth. Hence, for any module Z lying in Γ, there are a unique sectional path $X_1 \to X_2 \to \ldots \to X_m = Z$ in Γ with X_1 lying on the mouth of Γ (consisting of arrows pointing to infinity) and a unique sectional path $Z = Y_1 \to Y_2 \to \ldots \to Y_m$ with Y_m lying on the mouth of Γ (consisting of arrows pointing to the mouth), and m is called the *quasi-length* of Z in Γ, denoted by $\mathrm{ql}(Z)$. Observe that if Γ is of rank 1 and X its unique module lying on the mouth, then for any module Z in Γ we have $[Z] = \mathrm{ql}(Z)[X]$, and hence Γ consists of modules with pairwise different classes in the Grothendieck group $K_0(A)$.

For stable tubes of ranks bigger that one, we have following theorem (see [47, Theorem 4.3]).

Theorem 2.3. *Let A be an algebra, Γ a generalized standard stable tube of Γ_A of rank $r > 1$, and M, N nonisomorphic modules from Γ. Then $[M] = [N]$ if and only if $\mathrm{ql}(M) = \mathrm{ql}(N) = cr$ for some $c \geq 1$.*

For stable tubes consisting of modules which do not lie on infinite short cycles we have the following results established in [47, Corollaries 4.4 and 4.6].

Theorem 2.4. *Let A be an algebra, Γ a stable tube of rank $r > 1$ in Γ_A consisting of modules which do not lie on infinite short cycles in* $\operatorname{mod} A$*, and M a module in Γ. Then M is uniquely determined (up to isomorphism) by $[M]$ if and only if r does not divide* $\operatorname{ql}(M)$.

Theorem 2.5. *Let A be an algebra, Γ and Γ' different stable tubes in Γ_A consisting of modules which do not lie on infinite short cycles in* $\operatorname{mod} A$*. Let r be the rank of Γ and r' be the rank of Γ'. Assume that $[M] = [N]$ for some modules M in Γ and M' in Γ'. Then r divides* $\operatorname{ql}(M)$*, r' divides* $\operatorname{ql}(M')$*, and the tubes Γ and Γ' are orthogonal.*

Observe that, by Proposition 2.2, every stable tube Γ of an Auslander--Reiten quiver Γ_A consisting of modules which do not lie on infinite short cycles is generalized standard [47, Corollary 3.2]. We shall show that it is also the case for the smooth quasitubes. We need some results on the degrees of irreducible homomorphisms proved by Liu in [33].

By a result of Bautista [8] a homomorphism $f : M \to N$ between indecomposable modules in $\operatorname{mod} A$ is irreducible if and only if $f \in \operatorname{rad}(M, N) \setminus \operatorname{rad}^2(M, N)$. The following more general result has been established by Igusa and Todorov in [21].

Proposition 2.6. *Let A be an algebra and*

$$X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} \cdots \longrightarrow X_{n-1} \xrightarrow{f_n} X_n$$

be a path of irreducible homomorphisms between indecomposable modules in $\operatorname{mod} A$ *corresponding to a sectional path of Γ_A. Then we have $f_n \dots f_2 f_1 \in \operatorname{rad}^n(X_0, X_n) \setminus \operatorname{rad}^{n+1}(X_0, X_n)$.*

Let A be an algebra and $f : X \to Y$ be an irreducible homomorphism in $\operatorname{mod} A$ with X and Y indecomposable modules. Following Liu [33], f is said to be of *infinite left degree* if, for any integer $n \geq 1$ and a homomorphism $g : M \to X$ in $\operatorname{rad}^n(M, X) \setminus \operatorname{rad}^{n+1}(M, Y)$, we have $fg \in \operatorname{rad}^{n+1}(M, Y) \setminus \operatorname{rad}^{n+2}(M, Y)$. Dually, f is said to be of *infinite right degree* if, for any integer $n \geq 1$ and a homomorphism $h : Y \to N$ in $\operatorname{rad}^m(Y, N) \setminus \operatorname{rad}^{n+1}(Y, N)$, we have $hf \in \operatorname{rad}^{n+1}(X, N) \setminus \operatorname{rad}^{n+2}(X, N)$.

The following facts are consequences of [33, Corollary 1.6 and its dual].

Proposition 2.7. *Let A be an algebra. The following statements hold.*

(i) *Assume Γ_A admits a full translation subquiver*

$$\begin{array}{ccccccccc} \cdots \longrightarrow & X_{i+1} & \longrightarrow & X_i & \longrightarrow \cdots \longrightarrow & X_1 & \longrightarrow & X_0 = X \\ & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \cdots \longrightarrow & Y_{i+1} & \longrightarrow & Y_i & \longrightarrow \cdots \longrightarrow & Y_1 & \longrightarrow & Y_0 = Y \end{array}$$

where the upper and lower infinite paths are sectional. Then every irreducible homomorphism $f : X \to Y$ in $\operatorname{mod} A$ is of infinite left degree.

(ii) *Assume Γ_A admits a full translation subquiver*

$$\begin{array}{ccccccccc} M = M_0 & \longrightarrow & M_1 & \longrightarrow \cdots \longrightarrow & M_j & \longrightarrow & M_{j+1} & \longrightarrow \cdots \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & \\ N = N_0 & \longrightarrow & N_1 & \longrightarrow \cdots \longrightarrow & N_j & \longrightarrow & N_{j+1} & \longrightarrow \cdots \end{array}$$

where the upper and lower infinite paths are sectional. Then every irreducible homomorphism $g : M \to N$ in $\operatorname{mod} A$ is of infinite right degree.

Let A be an algebra and $\mathcal{C}$ a smooth quasitube in Γ_A. Then the stable part $\mathcal{C}^s$ of $\mathcal{C}$ is a stable tube, and we may define the *stable quasi-length* $\operatorname{sql}(X)$ of a stable module X in $\mathcal{C}$ as the quasi-length $\operatorname{ql}(X)$ of X in $\mathcal{C}^s$. Moreover, the stable quasi-length of a projective-injective module in $\mathcal{C}$ is defined to be 0.

Lemma 2.8. *Let A be an algebra and $\mathcal{C}$ be a smooth quasitube in Γ_A. Moreover, let r be the rank of $\mathcal{C}^s$ and m be the maximum of stable quasi-length of the radicals of projective-injective modules in $\mathcal{C}$. Then, for all modules X and Y in $\mathcal{C}$ of stable quasi-length bigger than $m + r$, we have* $\operatorname{rad}(X, Y) \neq 0$.

Proof. Let X and Y be modules in $\mathcal{C}$ with $\operatorname{sql}(Y)$ and $\operatorname{sql}(X)$ bigger than $m + r$. Then there are in $\mathcal{C}$ sectional paths

$$X = U_0 \to U_1 \to \ldots \to U_{p-1} \to U_p = Z,$$

consisting of arrows of $\mathcal{C}^s$ pointing to the mouth, and

$$Z = V_0 \to V_1 \to \ldots \to V_{q-1} \to V_q = Y,$$

consisting of arrows of $\mathcal{C}^s$ pointing to infinity. Moreover, $\mathcal{C}$ admits full translation subquivers

$$
\begin{array}{ccccccccc}
\cdots \longrightarrow & W_{s+1}^{(j-1)} & \longrightarrow & W_s^{(j-1)} & \longrightarrow \cdots \longrightarrow & W_1^{j-1} & \longrightarrow & W_0^{(j-1)} & = V_{j-1} \\
& \downarrow & & \downarrow & & \downarrow & & \downarrow & \\
\cdots \longrightarrow & W_{s+1}^{(j)} & \longrightarrow & W_s^{(j)} & \longrightarrow \cdots \longrightarrow & W_1^{j} & \longrightarrow & W_0^{(j)} & = V_j
\end{array}
$$

for $j \in \{1, \ldots, q\}$, formed by parallel infinite sectional paths, consisting of indecomposable modules of stable quasi-length $> m$. Take irreducible homomorphisms in $\operatorname{mod} A$

$$\varphi_i : U_{i-1} \to U_i \text{ and } \psi_j : V_{j-1} \to V_j,$$

for $i \in \{1, \ldots, p\}$ and $j \in \{1, \ldots, q\}$. Then it follows from Proposition 2.6 that $\varphi = \varphi_p \ldots \varphi_1 \in \operatorname{rad}^p(X, Z) \setminus \operatorname{rad}^{p+1}(X, Z)$. On the other hand, by Proposition 2.7, the irreducible homomorphisms $\psi_1, \ldots, \psi_q$ are of infinite left degree. Then, for $\psi = \psi_q \ldots \psi_1 \in \operatorname{Hom}_A(Z, Y)$, we obtain that $\psi\varphi \in \operatorname{rad}^{p+q}(X, Y) \setminus \operatorname{rad}^{p+q+1}(X, Y)$. Therefore, we conclude that $\operatorname{rad}(X, Y) \neq 0$. $\square$

We need also the following lemma (see [46, Lemma 2.1]).

Lemma 2.9. *Let A be an algebra and X, Y be indecomposable modules in* $\operatorname{mod} A$ *with* $\operatorname{rad}^\infty(X, Y) \neq 0$. *Then the following statements hold.*

(i) *There exists an infinite path*

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \longrightarrow X_{i-1} \xrightarrow{f_i} X_i \longrightarrow \cdots$$

of irreducible homomorphisms between indecomposable modules in $\operatorname{mod} A$ *and homomorphisms $g_i \in \operatorname{rad}^\infty(X_i, Y)$, $i \geq 1$, such that $g_i f_i \ldots f_1 \neq 0$ for all $i \geq 1$.*

(ii) *There exists an infinite path*

$$\cdots \longrightarrow Y_j \xrightarrow{h_j} Y_{j-1} \longrightarrow \cdots \longrightarrow Y_2 \xrightarrow{h_2} Y_1 \xrightarrow{h_1} Y_0 = Y$$

of irreducible homomorphisms between indecomposable modules in $\operatorname{mod} A$ *and homomorphisms $u_j \in \operatorname{rad}^\infty(X, Y_j)$, $j \geq 1$, such that $h_1 \ldots h_j u_j \neq 0$ for all $j \geq 1$.*

Proposition 2.10. *Let A be an algebra and $\mathcal{C}$ be a smooth quasitube in Γ_A consisting of modules which do not lie on infinite short cycles in* $\operatorname{mod} A$. *Then $\mathcal{C}$ is a generalized standard component of Γ_A.*

PROOF. Since $\mathcal{C}$ is a smooth quasitube of Γ_A, the stable part $\mathcal{C}^s$ of $\mathcal{C}$ is a stable tube, say of rank r. Denote by m the maximum of stable quasi-lengths $\operatorname{sql}(\operatorname{rad} P)$ of the radicals $\operatorname{rad} P$ of projective-injective modules P in $\mathcal{C}$. Consider the positive integer $n = m + 2r$, and denote by Γ the full translation subquiver of $\mathcal{C}$ consisting of all modules of stable quasi-length $\geq n$. Moreover, let M be the direct sum of all indecomposable modules in $\mathcal{C} \setminus \Gamma$. Clearly M is a module in $\operatorname{mod} A$, and hence $\operatorname{End}_A(M)$ is an artin algebra over k.

Assume that there are modules X and Y in $\mathcal{C}$ such that $\operatorname{rad}^{\infty}(X, Y) \neq 0$. Then it follows from Lemma 2.9(i) that there exist an infinite path

$$X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \longrightarrow X_{s-1} \xrightarrow{f_s} X_s \longrightarrow \cdots$$

of irreducible homomorphisms between indecomposable modules in $\operatorname{mod} A$ and homomorphisms $g_s \in \operatorname{rad}^{\infty}(X_s, Y)$, $s \geq 1$, such that $g_s f_s \dots f_1 \neq 0$ for any $s \geq 1$. Since $\operatorname{rad} \operatorname{End}_A(M)$ is nilpotent and $f_i \in \operatorname{rad}(X_{i-1}, X_i)$ for all $i \geq 1$, we conclude that there is an integer $s_0 \geq 1$ such that all modules X_s, $s \geq s_0$, belong to Γ. Since $\operatorname{rad}^{\infty}(X_{s_0}, Y) \neq 0$, applying Lemma 2.9$(ii)$, we conclude that there exist an infinite path

$$\cdots \longrightarrow Y_t \xrightarrow{h_t} Y_{t-1} \longrightarrow \cdots \longrightarrow Y_2 \xrightarrow{h_2} Y_1 \xrightarrow{h_1} Y_0 = Y$$

of irreducible homomorphisms between indecomposable modules in $\operatorname{mod} A$ and homomorphisms $u_t \in \operatorname{rad}^{\infty}(X_{s_0}, Y_t)$, $t \geq 1$, such that $h_1 \dots h_t u_t \neq 0$ for all $t \geq 1$. Moreover, we conclude as above that, for some integer $t_0 \geq 1$, all modules Y_t, $t \geq t_0$, belong to Γ. Clearly, by our choice of Γ, the modules X_{s_0} and Y_{t_0} have stable quasi-length bigger than $m + r$. Then it follows from Lemma 2.8 that there is a nonzero homomorphism $v \in \operatorname{rad}(Y_{t_0}, X_{t_0})$. Summing up, there is an infinite short cycle in $\operatorname{mod} A$ of the form

$$X_{s_0} \xrightarrow{u} Y_{t_0} \xrightarrow{v} X_{s_0} ,$$

where $u = u_{t_0}$, with X_{s_0} and Y_{t_0} in $\mathcal{C}$, a contradiction. Therefore, $\mathcal{C}$ is a generalized standard component of Γ_A. □

Lemma 2.11. *Let A be an algebra and $\mathcal{C}$ a quasitube in Γ_A. Assume there exist indecomposable modules X, Y, M in* $\operatorname{mod} A$ *such that* $\operatorname{rad}^{\infty}(X, M) \neq 0$, $\operatorname{rad}^{\infty}(M, Y) \neq 0$, *and X, Y lie in $\mathcal{C}$. Then there is an infinite short cycle $N \to M \to N$ in* $\operatorname{mod} A$ *with N in $\mathcal{C}$.*

Proof. Since $\operatorname{rad}^{\infty}(X, M) \neq 0$, then it follows from Lemma 2.9(i) that there exist an infinite path

$$\Theta: \ X = X_0 \xrightarrow{f_1} X_1 \xrightarrow{f_2} X_2 \longrightarrow \cdots \longrightarrow X_{s-1} \xrightarrow{f_s} X_s \longrightarrow \cdots$$

of irreducible homomorphisms between indecomposable modules in $\operatorname{mod} A$ and homomorphisms $g_s \in \operatorname{rad}^{\infty}(X_s, M)$, $s \geq 1$, such that $g_s f_s \dots f_1 \neq 0$ for any $s \geq 1$. Now, suppose that there is a finite family $\{Z_i\}_{i \in I}$ of indecomposable modules in $\mathcal{C}$ which are isomorphic with infinitely many modules from the family $\{X_s\}_{s \geq 0}$. Let Z be the direct sum of all modules from the family $\{Z_i\}_{i \in I}$. Clearly, Z is a module in $\operatorname{mod} A$, and hence $\operatorname{End}_A(Z)$ is an artin algebra over k. Since $f_s \in \operatorname{rad}(X_{s-1}, X_s)$ for all $s \geq 1$, we get then arbitrary large nonzero compositions of homomorphisms from $\operatorname{rad} \operatorname{End}_A(Z)$, and hence, because $\operatorname{rad}_A(Z)$ is nilpotent, a contradiction. Moreover, since $\operatorname{rad}^{\infty}(M, Y) \neq 0$, applying Lemma 2.9($ii$), we conclude that there exist an infinite path

$$\Sigma: \ \cdots \longrightarrow Y_t \xrightarrow{h_t} Y_{t-1} \longrightarrow \cdots \longrightarrow Y_2 \xrightarrow{h_2} Y_1 \xrightarrow{h_1} Y_0 = Y$$

of irreducible homomorphisms between indecomposable modules in $\operatorname{mod} A$ and homomorphisms $u_t \in \operatorname{rad}^{\infty}(M, Y_t)$, $t \geq 1$, such that $h_1 \dots h_t u_t \neq 0$ for all $t \geq 1$. Similarly as above, we conclude that there is no finite family $\{Z_i\}_{i \in I}$ of indecomposable modules from $\mathcal{C}$ which are isomorphic with infinitely many modules from the family $\{Y_t\}_{t \geq 0}$. Therefore, we conclude that the path Θ intersects the path Σ.

Let N be a module in $\Theta \cap \Sigma$. Then there are $s \geq 0$ and $t \geq 0$ such that $X_s = N = Y_t$, and hence we obtain an infinite short cycle $N \xrightarrow{g_s} M \xrightarrow{u_t} N$.

□

Let A be an algebra and $\mathcal{C}$ be a family of components of Γ_A. Then $\mathcal{C}$ is said to be *sincere* if any simple A-module occurs as a composition factor of a module in $\mathcal{C}$, and *faithful* if its annihilator $\operatorname{ann}_A(\mathcal{C})$ in A (the intersection of the annihilators of all modules in $\mathcal{C}$) is zero. Observe that if $\mathcal{C}$ is faithful then $\mathcal{C}$ is sincere. Moreover, in general $\operatorname{ann}_A(\mathcal{C})$ is an ideal of A and $\mathcal{C}$ is a faithful family of components in the Auslander-Reiten quiver $\Gamma_{A/\operatorname{ann}_A(\mathcal{C})}$ of the quotient algebra $A/\operatorname{ann}_A(\mathcal{C})$ of A. Further, by an *external short path* in $\operatorname{mod} A$, with respect to a family $\mathcal{C}$ of components in Γ_A, we mean a sequence $X \to Y \to Z$ of nonzero nonisomorphisms between indecomposable modules with X and Z from $\mathcal{C}$ but Y not in $\mathcal{C}$ [37].

Lemma 2.12. *Let A be an algebra, $\mathcal{C}$ and $\mathcal{C}'$ two different ray tubes in Γ_A having infinitely many modules with common composition factors and consisting of modules which do not lie on infinite short cycles. Then there are no external short paths in* $\operatorname{mod} A$ *with respect to the components $\mathcal{C}$ and $\mathcal{C}'$.*

Proof. Assume that there is an external short path $M \to L \to M'$, where M is in $\mathcal{C}$, M' is in $\mathcal{C}'$ and L is neither in $\mathcal{C}$ nor in $\mathcal{C}'$. First we will show that then there is an external short path $M \to L \to N$ with N in $\mathcal{C}$. It follows from Lemma 2.9(i) that there exist an infinite path

$$\Theta: \cdots \longrightarrow X_s \xrightarrow{h_s} X_{s-1} \longrightarrow \cdots \longrightarrow X_2 \xrightarrow{h_2} X_1 \xrightarrow{h_1} X_0 = M'$$

of irreducible homomorphism between indecomposable modules from $\mathcal{C}'$ and homomorphisms $u_s \in \operatorname{rad}^\infty(L, X_s)$, $s \geq 1$, such that $h_1 \dots h_s u_s \neq 0$ for all $s \geq 1$. Now, assume that there is a finite family $\{Z_i\}_{i \in I}$ of indecomposable modules in $\mathcal{C}'$ which are isomorphic with infinitely many modules from the family $X = \{X_s\}_{s\geq 1}$. Let Z be the direct sum of all modules from the family $\{Z_i\}_{i\in I}$. Clearly, Z is a module in $\operatorname{mod} A$, and hence $\operatorname{End}_A(Z)$ is an artin algebra over k. Since $h_s \in \operatorname{rad}(X_s, X_{s-1})$ for all $s \geq 1$, we get then arbitrary large nonzero compositions of homomorphisms from $\operatorname{rad}\operatorname{End}_A(Z)$, and hence, because $\operatorname{rad}\operatorname{End}_A(Z)$ is nilpotent, a contradiction. Therefore, we conclude that the path Θ intersects each ray in $\mathcal{C}'$ at least once. Moreover, it follows from our assumption that there is a ray in $\mathcal{C}'$ with infinitely many modules N' such that $[N] = [N']$ for a module N in $\mathcal{C}$. Using the fact that the irreducible homomorphisms lying on rays of the ray tube $\mathcal{C}'$ are monomorphisms, we conclude that there are an external short path $M \to L \to M'$, with M in $\mathcal{C}$, M' in $\mathcal{C}'$, L neither in $\mathcal{C}$ nor in $\mathcal{C}'$, and a module N in $\mathcal{C}$ such that $[M'] = [N]$. Because $[M'] = [N]$, applying [47, Proposition 4.1], we obtain the equality

$$|\operatorname{Hom}_A(L, N)| - |\operatorname{Hom}_A(N, \tau L)| = |\operatorname{Hom}_A(L, M')| - |\operatorname{Hom}_A(M', \tau L)|,$$

where $|V|$ denotes the length of a k-module $|V|$. If $\mathrm{Hom}(M', \tau L) \neq 0$, then, by [38, Theorem 1.6], M' is the middle of a short chain, so is on a short cycle $M' \to E \to M'$, with E an indecomposable direct summand of the middle term of an almost split sequence with the left term L, and so E does not belong to $\mathcal{C}'$. Hence this cycle is infinite what contradicts our assumption. Thus $\mathrm{Hom}_A(M', \tau L) = 0$ and so $\mathrm{Hom}_A(L, N) \neq 0$. Therefore, we get an external short path $M \to L \to N$, with M and N in $\mathcal{C}$. Obviously, then we have $\mathrm{rad}^\infty(M, L) \neq 0$ and $\mathrm{rad}^\infty(L, N) \neq 0$, and hence, applying Lemma 2.11, we conclude that there exist an infinite short cycle $X \to Z \to X$ in $\mathrm{mod}\, A$ with X in $\mathcal{C}$. □

Lemma 2.13. *Let A be an algebra, $B = A/I$ a quotient algebra of A, and $\mathcal{T}$ a stable tube of Γ_B. Assume that the modules of $\mathcal{T}$ belong to a stable tube $\mathcal{C}$ of Γ_A. Then $\mathcal{C} = \mathcal{T}$.*

Proof. In order to prove that $\mathcal{C} = \mathcal{T}$ it suffices to show that every module M in $\mathcal{C}$ is a B-module. Because $\mathcal{T} \subseteq \mathcal{C}$ and the stable tube $\mathcal{T}$ consists of infinitely many B-modules, then for every A-module M in $\mathcal{C}$ there are an A-module monomorphism $f : M \to N$, where N is a module lying on a ray in $\mathcal{C}$ containing M, and an A-module epimorphism $g : Z \to N$, where Z is a B-module from $\mathcal{T}$ lying on a coray in $\mathcal{C}$ containing N. Therefore, $NI = g(Z)I = g(ZI) = g(0) = 0$. Hence, $f(MI) = f(M)I = 0$, and so $MI = 0$, because f is a monomorphism. Therefore, M is a B-module. □

Lemma 2.14. *Let A be an algebra, Λ a quotient algebra of A, and $\mathcal{T}$ a stable tube of Γ_Λ. Assume that the modules of $\mathcal{T}$ belong to a family $\mathcal{C}$ of smooth quasitubes of Γ_A consisting of modules which do not lie on infinite short cycles. Then the modules of $\mathcal{T}$ belong to one quasitube of $\mathcal{C}$.*

Proof. Assume that there are two different quasitubes $\mathcal{C}_x$ and $\mathcal{C}_y$ in $\mathcal{C}$ and modules $M, N \in \mathcal{T}$ such that $M \in \mathcal{C}_x$ and $N \in \mathcal{C}_y$. Let Θ be the infinite sectional path in $\mathcal{T}$ starting at M and pointing to infinity and Σ be the infinite sectional path in $\mathcal{T}$ from infinity to N. Let Z be a module in $\Theta \cap \Sigma$ and $f : M \to Z$ the composition of irreducible monomorphism corresponding to arrows of the subpath of Θ from M to Z and g the composition of irreducible epimorphisms corresponding to arrows of the subpath of Σ from Z to N. Then $f \in \mathrm{rad}_A^\infty(M, Z)$ or $g \in \mathrm{rad}_A^\infty(Z, N)$, because $N \in \mathcal{C}_y$ or $N \in \mathcal{C}_x$ or $N \in \mathcal{C}_z$, where $z \neq x, y$. Assume, without lost of generality, that Z is not in $\mathcal{C}_x$, and hence $f \in \operatorname{rad}_A^\infty(M,Z)$. Let L be a module in $\mathcal{T}$, lying on Θ, with $\operatorname{ql}(Z) < \operatorname{ql}(L)$, and such that there is a sectional path in $\mathcal{T}$ from L to M. Then the composed monomorphism $h : M \to L$ belongs to $\operatorname{rad}_A^\infty(M,L)$. Hence, we have the infinite short cycle $M \xrightarrow{h} L \xrightarrow{v} M$ in $\operatorname{mod} A$, where v is the composition of irreducible epimorphism corresponding to the arrows of the sectional path from L to M, which contradicts our assumption on $\mathcal{C}$. □

Lemma 2.15. *Let A be an algebra, Λ a quotient algebra of A, and $\mathcal{T}$ and $\mathcal{T}'$ be orthogonal stable tubes of Γ_Λ. Assume that there exist smooth quasitubes $\mathcal{C}$ and $\mathcal{C}'$ of Γ_A such that $\mathcal{C}$ contains all modules of $\mathcal{T}$ and $\mathcal{C}'$ contains all modules of $\mathcal{T}'$. Then $\mathcal{C}$ is different from $\mathcal{C}'$.*

Proof. Suppose $\mathcal{C} = \mathcal{C}'$. Let r be the rank of $\mathcal{C}^s$ and m the maximum of stable quasi-lengths of the radicals of projective-injective modules in $\mathcal{C}$. Since $\mathcal{T}$ and $\mathcal{T}'$ have infinitely many modules, there exist $X \in \mathcal{T}$ and $X' \in \mathcal{T}'$ of stable quasi-length $> m+r$ in $\mathcal{C}^s$. Then, by Lemma 2.8, we have $\operatorname{Hom}_A(X, X') \neq 0$, which contradicts the orthogonality of $\mathcal{T}$ and $\mathcal{T}'$ in $\operatorname{mod} \Lambda$. □

Lemma 2.16. *Let A be an algebra, Λ a quotient algebra of A, and $\mathcal{T} = (\mathcal{T}_x)_{x\in\mathbb{X}}$ a generalized standard family of stable tubes of Γ_Λ with common composition factors and consisting of modules which do not lie on infinite short cycles. Assume that there exists a stable tube $\mathcal{T}_x$ in $\mathcal{T}$ such that the modules of $\mathcal{T}_x$ belong to a family $\mathcal{C}$ of smooth quasitubes of Γ_A closed under composition factors and consisting of modules which do not lie on infinite short cycles. Then all modules of $\mathcal{T}$ belong to the family $\mathcal{C}$.*

Proof. For each $x \in \mathbb{X}$, we denote by r_x the rank of $\mathcal{T}_x$. It follows from Lemma 2.14 that the modules of $\mathcal{T}_x$ belong to one quasitube $\mathcal{C}_x$ of $\mathcal{C}$. Because $\mathcal{T}$ is a family of stable tubes with common composition factors, then, for any $y \in \mathbb{X}$, there are modules M_y in $\mathcal{T}_x$ and N_y in $\mathcal{T}_y$ such that $[M_y] = [N_y]$. Now, using the fact that the modules from $\mathcal{T}$ do not lie on infinite short cycles in $\operatorname{mod} A$, we conclude, by Theorem 2.5, that r_y divides $\operatorname{ql}(N_y)$ and r_x divides $\operatorname{ql}(M_y)$. It follows from Theorem 2.3 that for any two modules N, N' in a stable tube $\mathcal{T}_y$ with $\operatorname{ql}(N) = \operatorname{ql}(N') = cr_y$, for some $c \geq 1$, we have $[N] = [N']$. Moreover, for any N in $\mathcal{T}_y$ and M in $\mathcal{T}_x$ with $\operatorname{ql}(N) = c\,\operatorname{ql}(N_y)$ and $\operatorname{ql}(M) = c\,\operatorname{ql}(M_y)$, we get $[N] = [M]$. Now, because the family $\mathcal{C}$ is closed on composition factors, we conclude that, for all $y \in \mathbb{X}$ and for any N in $\mathcal{T}_y$ with $ql(N) = c\,\operatorname{ql}(N_y)$, for some $c \geq 1$, N is in the family $\mathcal{C}$.

Let N be any module in $\mathcal{T}_y$ with $y \in \mathbb{X}$. We will show that N belongs to the family $\mathcal{C}$. Suppose N is not in $\mathcal{C}$. We may choose a module N_y in $\mathcal{T}_y$ such that $[N_y] = [M_y]$ for a module $M_y \in \mathcal{T}_x$ and with $\mathrm{ql}(N) < n = \mathrm{ql}(N_y)$. Then there are sectional paths in $\mathcal{T}_y$

$$N''_y = M_0 \to M_1 \to \ldots \to M_m \to N$$

and

$$N \to N_m \to \ldots \to N_1 \to N_0 = N'_y,$$

where $\mathrm{ql}(N'_y) = n = \mathrm{ql}(N''_y)$. Since r_y divides n, we conclude that the modules N'_y and N''_y belong to the family $\mathcal{C}$. Hence we obtain a nonzero homomorfizm from $\mathrm{rad}^{\infty}(N''_y, N'_y)$, what contradicts Proposition 2.10. □

For convenience of the reader, we give a proof of the following well known fact, which we need in further considerations.

Lemma 2.17. *Let A be an algebra and $\mathcal{T}$ a faithful stable tube of Γ_A. Then all but finitely many indecomposable modules in $\mathcal{T}$ are faithful A-modules.*

Proof. First notice that for any ray Σ in $\mathcal{T}$ and a module M lying on Σ, we have $\mathrm{ann}(M) \subseteq \mathrm{ann}(M')$ for every module M' lying on Σ with $\mathrm{ql}(M') > \mathrm{ql}(M)$. Now, because $\mathcal{T}$ is a faithful stable tube of Γ_A, then there are indecomposable modules $M_1, \ldots, M_s$ in $\mathcal{T}$ such that $\mathrm{ann}(\bigoplus_{i=1}^{s} M_i) = 0$. Let n be the maximum of quasi-lengths of modules $M_1, \ldots, M_s$. Then for every module Z with quasi-length bigger than $n+r$, where r is the rank of $\mathcal{T}$, the unique sectional path in $\mathcal{T}$ starting at Z and pointing to the mouth, intersects every ray containing modules from $\{M_1, \ldots, M_s\}$ and consists of empimorphisms. Note, that for an epimorphism $f : X \to Y$ we have $\mathrm{ann}(X) \subseteq \mathrm{ann}(Y)$, because $Y \,\mathrm{ann}(X) = f(X) \,\mathrm{ann}(X) = f(X \,\mathrm{ann}(X)) = f(0) = 0$. Therefore, $\mathrm{ann}(Z) \subseteq \mathrm{ann}(\bigoplus_{i=1}^{s} M_i) = 0$, and hence Z is a faithful module. □

3. Quasitilted algebras of canonical type

The purpose of this section is to present characterizations of quasitilted algebras of canonical type.

Let A be an algebra. Then a family $\mathcal{C}$ of components of Γ_A is said to be *separating* in $\mathrm{mod}\, A$ if the indecomposable modules in $\mathrm{mod}\, A$ split into three disjoint classes $\mathcal{P}^A$, $\mathcal{C}^A = \mathcal{C}$ and $\mathcal{Q}^A$ such that:

(S1) $\mathcal{C}^A$ is a sincere generalized standard family of components;
(S2) $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{P}^A) = 0$, $\mathrm{Hom}_A(\mathcal{Q}^A, \mathcal{C}^A) = 0$, $\mathrm{Hom}_A(\mathcal{C}^A, \mathcal{P}^A) = 0$;
(S3) any homomorphism from $\mathcal{P}^A$ to $\mathcal{Q}^A$ factors through the additive category $\mathrm{add}\,\mathcal{C}^A$ of $\mathcal{C}^A$.

Algebras with a separating family of stable tubes have attracted much attention. A prominent class of algebras with this property is formed by the canonical algebras, introduced by Ringel (see [40], [41]). Hence, for a canonical algebra Λ, Γ_Λ admits a decomposition $\Gamma_\Lambda = \mathcal{P}^\Lambda \vee \mathcal{T}^\Lambda \vee \mathcal{Q}^\Lambda$, where $\mathcal{T}^\Lambda$ is a (canonical) family of stable tubes separating $\mathcal{P}^\Lambda$ from $\mathcal{Q}^\Lambda$. Following [28], an algebra C is called *concealed canonical* of type Λ if C is the endomorphism algebra $\mathrm{End}_\Lambda(T)$ for a tilting module from $\mathrm{add}\,\mathcal{P}^\Lambda$. Then the images of all modules from $\mathcal{T}^\Lambda$ via the functor $\mathrm{Hom}_\Lambda(T, -)$ form a separating family $\mathcal{T}^C$ of stable tubes of Γ_C. In particular, we have a decomposition $\Gamma_C = \mathcal{P}^C \vee \mathcal{T}^C \vee \mathcal{Q}^C$. We note that $\mathcal{T}^C$ is a family of stable tubes $\mathcal{T}^C_x$, $x \in \mathbb{X}$, where the index set $\mathbb{X}$ is in a natural bijection with the set of stable tubes of a tame hereditary algebra $\begin{bmatrix} F & M \\ 0 & G \end{bmatrix}$, where F and G are finite central skew field extensions of a field k and the F-G-bimodule ${}_FM_G$ satisfies $(\dim {}_FM)(\dim M_G) = 4$ (see [10], [39], [41]). Moreover, if k is an algebraically closed field, then $\mathbb{X}$ is in a natural bijection with the projective line $\mathbb{P}_1(k)$ [40], and is equipped with the structure of a weighted projective line [14]. It has been proved in [30, Theorem 1.1] that an algebra C is a concealed canonical algebra if and only if Γ_C admits a separating family of stable tubes.

An algebra A is said to be a *quasitilted algebra of canonical type* if $A = \mathrm{End}_{\mathcal{H}}(T)$, where T is a tilting object in an abelian hereditary category $\mathcal{H}$ whose derived category $D^b(\mathcal{H})$ is equivalent (as a triangulated category) to the derived category $D^b(\mathrm{mod}\,\Lambda)$ of the module category $\mathrm{mod}\,\Lambda$ of a canonical algebra Λ.

An algebra A is said to be an *almost concealed canonical algebra* if A is the endomorphism algebra $\mathrm{End}_\Lambda(T)$ of a tilting module T from the additive category $\mathrm{add}(\mathcal{P}^\Lambda \vee \mathcal{T}^\Lambda)$, for the canonical decomposition $\Gamma_\Lambda = \mathcal{P}^\Lambda \vee \mathcal{T}^\Lambda \vee \mathcal{Q}^\Lambda$ with $\mathcal{T}^\Lambda$ the canonical family of stable tubes separating $\mathcal{P}^\Lambda$ from $\mathcal{Q}^\Lambda$ over a canonical algebra Λ. It has been proved in [31, Theorem 3.4] that A is quasitilted if and only if Γ_A admits a separating family of semiregular (ray or coray) tubes. Moreover, the class of almost concealed algebras coincides with the class of tubular extensions of concealed canonical algebras (using modules from the canonical family of stable tubes), and with the class of algebras having a separating family of ray tubes (see [28, Theorem 3.1] and [31, Theorem 3.4]).

We will need the following deeper characterization of almost concealed canonical algebras (see [47, Proposition 3.5] and [49, Theorem 1.6]).

Theorem 3.1. *An algebra A is an almost concealed canonical algebra if and only if Γ_A has a sincere generalized standard family of ray tubes without external short paths.*

We have also the following characterization of concealed canonical algebras (see [37, Theorem 3.1], [48, Theorem C] and [49, Theorem 1.6]).

Theorem 3.2. *An algebra A is a concealed canonical algebra if and only if Γ_A has a sincere family of pairwise orthogonal stable tubes without external short paths.*

We will need the following consequence of the above theorem (see [49, Corollary 1.7]).

Corollary 3.3. *Let A be an algebra and $\mathcal{T}$ a sincere stable tube in Γ_A without external short cycles. Then $\mathcal{T}$ is a faithful generalized standard stable tube and A is a concealed canonical algebra.*

4. Quasitube enlargements of algebras

In this section we introduce quasitube enlargements of algebras, essential for our further considerations.

Let A be an algebra, F a division algebra and ${}_FM_A$ an F-A-bimodule such that M_A is in $\operatorname{mod} A$ and k acts centrally on ${}_FM_A$. Then the *one-point extension* of A by M is the matrix algebra of the form

$$A[M] = \begin{bmatrix} F & {}_FM_A \\ 0 & A \end{bmatrix} = \left\{ \begin{pmatrix} f & m \\ 0 & a \end{pmatrix} \ : \ f \in F, a \in A, m \in M \right\}$$

with the usual addition and multiplication. Dually, one defines also the *one-point coextension* of A by ${}_FM_A$ as the matrix algebra

$$[M]A = \begin{bmatrix} A & D({}_FM_A) \\ 0 & F \end{bmatrix}.$$

Let A be an algebra and Γ a generalized standard component of Γ_A. For each indecomposable module X in Γ which is a pivot of an admissible operation of type $(ad\,1)$, $(ad\,2)$, $(ad\,1^*)$, or $(ad\,2^*)$, we shall define the corresponding admissible operation on A in such a way that the modified translation quiver Γ' is a component of the Auslander-Reiten quiver $\Gamma_{A'}$ of the modified algebra A' (see [4], [5]). Since Γ is generalized standard, such a pivot X is necessarily a brick, and we denote by F the division algebra $\operatorname{End}_A(X)$. Clearly, X is F-A-bimodule. Suppose X is the pivot of an admissible operation of type $(ad\,1)$ and $t \geq 1$. Denote $D = D_t$ the full $t \times t$ upper triangular matrix algebra over the division algebra F and by Y the unique indecomposable projective-injective D-module, which we consider as an F-D-bimodule. Then $A' = (A \times D)[X \oplus Y]$ is the required modified algebra. If X is the pivot of an admissible operation of type $(ad\,2)$ then the modified algebra A' is defined to be $A' = A[X]$. Dually, invoking the one-point coextensions, one defines the modified algebra A', if X is a pivot of an admissible operation of type $(ad\,1^*)$ or $(ad\,2^*)$. Then the following fact mentioned above holds (see [4, Section 2]).

Lemma 4.1. *The modified translation quiver Γ' of Γ is a component of $\Gamma_{A'}$.*

Let C be an algebra and $\mathcal{T}$ a generalized standard family of stable tubes in Γ_C. Following [5] an algebra B is said to be a *quasitube enlargement* of C using modules from $\mathcal{T}$ if there is a finite sequence of algebras $A_0 = C, A_1, \ldots, A_m = B$ such that, for each $0 \leq j < m$, A_{j+1} is obtained from A_j by an admissible operation of type $(ad\,1)$, $(ad\,2)$, $(ad\,1^*)$, or $(ad\,2^*)$, with pivot either in a stable tube of $\mathcal{T}$ or in a quasitube of Γ_{A_j} obtained from a stable tube of $\mathcal{T}$ by means of the sequence of admissible operations (of types $(ad\,1)$, $(ad\,2)$, $(ad\,1^*)$, $(ad\,2^*)$) done so far. We note that a *tubular extension* (respectively, *tubular coextension*) of C (in the sense of [40]), using modules from $\mathcal{T}$, is just an enlargement of C invoking only admissible operations of type $(ad\,1)$ (respectively, of type $(ad\,1^*)$).

We have the following proposition (see [4, Lemmas 2.2 and 2.3] and [35, Theorem C]).

Proposition 4.2. *Let B be a quasitube enlargement of an algebra C using modules from a generalized standard family $\mathcal{T}$ of stable tubes of Γ_C, and $\mathcal{C}$ the family of components of Γ_B obtained from $\mathcal{T}$ by means of admissible operations leading from C to B. Then $\mathcal{C}$ is a generalized standard family of quasitubes of Γ_B.*

Moreover, we have the following theorem (see [5, Theorem 3.5] and [35, Theorem C]).

Theorem 4.3. *Let C be a concealed canonical algebra and $\mathcal{T}$ a separating family of stable tubes of Γ_C. Let B be a quasitube enlargement of C, using modules from $\mathcal{T}$, and $\mathcal{C}$ the associated generalized standard family of quasitubes of Γ_B. Then the following statements hold:*

(i) *There is a unique maximal tubular coextension B_l of C inside B and a generalized standard family $\mathcal{C}^l$ of coray tubes of Γ_{B_l} such that B is obtained from B_l (respectively, $\mathcal{C}$ is obtained from $\mathcal{C}^l$) by a sequence of admissible operations of types $(ad\,1)$ and $(ad\,2)$, using modules from $\mathcal{C}^l$.*

(ii) *There is a unique maximal tubular extension B_r of C inside B and a generalized standard family $\mathcal{C}^r$ of ray tubes of Γ_{B_r} such that B is obtained from B_r (respectively, $\mathcal{C}$ is obtained from $\mathcal{C}^r$) by a sequence of admissible operations of types $(ad\,1^*)$ and $(ad\,2^*)$, using modules from $\mathcal{C}^r$.*

For a quasitube enlargement B of a concealed canonical algebra C, the maximal tubular extension B_r of C inside B is an almost concealed canonical algebra, called the *right quasitilted part* of B. Similarly, the maximal tubular coextension B_l of C inside B is the opposite algebra of an almost concealed algebra, called the *left quasitilted part* of B.

We note that a quasitube of an Auslander-Reiten quiver is an almost cyclic coherent component in the sense of [34]. The following theorem is then a special case of a characterization of algebras with separating families of almost cyclic coherent Auslander-Reiten components established in [35, Theorem A].

Theorem 4.4. *Let A be a basic, connected, artin algebra. The following statements are equivalent.*

(i) *Γ_A admits a separating family of quasitubes.*

(ii) *Γ_A admits a sincere generalized standard family of quasitubes without external short paths.*

(iii) *A is a quasitube enlargement of a concealed canonical algebra C.*

5. Selfinjective orbit algebras

For an algebra Λ, we denote by D the standard duality $\mathrm{Hom}_k(-, E)$ on $\mathrm{mod}\,\Lambda$, where E is a minimal injective cogenerator in $\mathrm{mod}\,k$. Then an algebra

Λ is *selfinjective* if and only if $\Lambda \cong \mathrm{D}(\Lambda)$ in $\operatorname{mod} \Lambda$. If Λ is selfinjective, then the left and the right socle of Λ coincide, and we denote them by $\operatorname{soc} \Lambda$. Two selfinjective algebras A and Λ are said to be *socle equivalent* if the factor algebras $A/\operatorname{soc} A$ and $\Lambda/\operatorname{soc} \Lambda$ are isomorphic.

Let A be a selfinjective algebra and $\{e_i \mid 1 \leq i \leq s\}$ a complete set of orthogonal primitive idempotents of A. We denote by $\nu = \nu_A$ the *Nakayama automorphism* of A inducing an A-bimodule isomorphism $A \cong \mathrm{D}(A)_\nu$, where $\mathrm{D}(A)_\nu$ denotes the right A-module obtained from $\mathrm{D}(A)$ by changing the right operation of A as follows: $f \cdot a = f\nu(a)$ for each $a \in A$ and $f \in \mathrm{D}(A)$. Hence we have $\operatorname{soc}(\nu(e_i)A) \cong \operatorname{top}(e_iA)$ $(= e_iA/\operatorname{rad}(e_iA))$ as right A-modules for all $i \in \{1, \ldots, s\}$. Since $\{\nu(e_i)A \mid 1 \leq i \leq s\}$ is a complete set of representatives of indecomposable projective right A-modules, there is a *(Nakayama) permutation* of $\{1, \ldots, s\}$, denoted again by ν, such that $\nu(e_i)A \cong e_{\nu(i)}A$ for all $i \in \{1, \ldots, s\}$. Invoking the Krull-Schmidt theorem, we may assume that $\nu(e_iA) = \nu(e_i)A = e_{\nu(i)}A$ for all $i \in \{1, \ldots, s\}$.

Let B be an algebra. The *repetitive algebra* $\widehat{B}$ of B [20] is an algebra (without identity) whose k-module structure is that of

$$\bigoplus_{m \in \mathbb{Z}} (B_m \oplus \mathrm{D}(B)_m)$$

where $B_m = B$ and $\mathrm{D}(B)_m = \mathrm{D}(B)$ for all $m \in \mathbb{Z}$, and the multiplication is defined by

$$(a_m, f_m)_m \cdot (b_m, g_m)_m = (a_m b_m, a_m g_m + f_m b_{m-1})_m$$

for $a_m, b_m \in B_m$, $f_m, g_m \in \mathrm{D}(B)_m$. For a fixed set $\mathcal{E} = \{e_i \mid 1 \leq i \leq n\}$ of orthogonal primitive idempotents of B with $1_B = e_1 + \cdots + e_n$, consider the canonical set $\widehat{\mathcal{E}} = \{e_{m,i} \mid m \in \mathbb{Z},\ 1 \leq i \leq n\}$ of orthogonal primitive idempotents of $\widehat{B}$ such that $e_{m,i}\widehat{B} = (e_iB)_m \oplus (e_i \mathrm{D}(B))_m$ for $m \in \mathbb{Z}$ and $1 \leq i \leq n$. By an automorphism of $\widehat{B}$ we mean a k-algebra automorphism of $\widehat{B}$ which fixes the chosen set $\widehat{\mathcal{E}}$ of orthogonal primitive idempotents of $\widehat{B}$. A group G of automorphisms of $\widehat{B}$ is said to be *admissible* if the induced action of G on $\widehat{\mathcal{E}}$ is free and has finitely many orbits. Then the *orbit algebra* $\widehat{B}/G$ is a finite dimensional selfinjective algebra and the G-orbits in $\widehat{\mathcal{E}}$ form a canonical set of orthogonal primitive idempotents of $\widehat{B}/G$ whose sum is the identity of $\widehat{B}/G$. We denote by $\nu_{\widehat{B}}$ the *Nakayama automorphism* of $\widehat{B}$ such that $\nu_{\widehat{B}}(e_{m,i}) = e_{m+1,i}$ for all $m \in \mathbb{Z}$, $1 \leq i \leq n$. Then the infinite cyclic group $(\nu_{\widehat{B}})$ generated by $\nu_{\widehat{B}}$ is admissible and $\widehat{B}/(\nu_{\widehat{B}})$ is the trivial extension $B \ltimes \mathrm{D}(B)$ of B by $\mathrm{D}(B)$. An automorphism φ of $\widehat{B}$ is said to be *positive* (respectively, *rigid*) if $\varphi(B_m) \subseteq \sum_{j\geq m} B_j$ (respectively, $\varphi(B_m) = B_m$) for any $m \in \mathbb{Z}$. Finally, φ is said to be *strictly positive* if φ is positive but not rigid.

Let A be a selfinjective algebra, I an ideal of A, $B = A/I$, and e an idempotent of A such that $e + I$ is the identity of B. We may assume that $e = e_1 + \cdots + e_n$, where $\{e_i \mid 1 \leq i \leq n\}$ is a complete set of orthogonal primitive idempotents of A which are not in I. Then such an idempotent e is uniquely determined by I, up to an inner automorphism of A, and we call it a *residual identity* of B [52]. Note that $B \cong eAe/eIe$ and $1 - e \in I$. We denote by $l_A(I)$ and $r_A(I)$ the left and right annihilator of I in A, respectively. Following [52, (2.1)] the ideal I is said to be *deforming* if $eIe = l_{eAe}(I) = r_{eAe}(I)$ and A/I is triangular (the ordinary quiver of A/I has no oriented cycles). The following lemma has been proved in [57, Lemma 4.1].

Lemma 5.1. *Let A be a selfinjective algebra, e an idempotent of A, and assume that $l_A(I) = Ie$ or $r_A(I) = eI$. Then e is a residual identity of the factor algebra A/I.*

Moreover, the following proposition has been proved in [52, Proposition 2.3].

Proposition 5.2. *Let A be a selfinjective algebra, I an ideal of A, $B = A/I$, e a residual identity of B, and assume that $IeI = 0$. Then the following conditions are equivalent.*

(i) *Ie is an injective cogenerator in* $\operatorname{mod} B$.
(ii) *eI is an injective cogenerator in* $\operatorname{mod} B^{op}$.
(iii) $l_A(I) = Ie$.
(iv) $r_A(I) = eI$.

Moreover, under these equivalent conditions, we have $eIe = l_{eAe}(I) = r_{eAe}(I)$.

Let A be a selfinjective algebra, I a deforming ideal of A and e a residual identity of A/I. Then I can be considered as a (not necessarily unitary) (eAe/eIe)-bimodule. Denote by $A[I]$ the direct sum of k-modules $(eAe/eIe) \oplus I$ with the multiplication

$$(b, x) \cdot (b', x') = (bb', bx' + xb' + xx')$$

for $b, b' \in eAe/eIe$ and $x, x' \in I$. Then $A[I]$ is an algebra with the identity $(e, 1-e)$ and, by identifying $x \in I$ with $(0, x) \in A[I]$, we may consider I as an ideal of $A[I]$.

The following combination of results proved in [52, Theorem 4.1], [53, Theorem 3] and [56, Proposition 3.2] establishes the relationship between A and $A[I]$.

Theorem 5.3. *Let A be a selfinjective algebra, I a deforming ideal of A, and e a residual identity of A/I. Then the following statements hold.*

(i) *$A[I]$ is a selfinjective algebra, I is a deforming ideal of $A[I]$, and the Nakayama permutations of A and $A[I]$ are the same.*
(ii) *A and $A[I]$ are socle equivalent.*
(iii) *Assume $IeI = 0$ and $e_i \neq e_{\nu(i)}$ for any primitive summand e_i of e. Then A and $A[I]$ are isomorphic.*

The following criterion is a direct consequence of [54, Theorems 3.8 and 4.1] and Proposition 5.2.

Theorem 5.4. *Let A be a selfinjective algebra, I an ideal of A, $B = A/I$ and e a residual identity of B. Assume that B is triangular and $l_A(I) = Ie$. Then $A[I]$ is isomorphic to an algebra $\widehat{B}/(\psi\nu_{\widehat{B}})$, for some positive automorphism ψ of $\widehat{B}$.*

6. Selfinjective algebras of canonical type

A selfinjective algebra A is said to be a *selfinjective algebra of canonical type* if A is isomorphic to an orbit algebra $\widehat{B}/G$, where B is a quasitilted algebra of canonical type and G is an admissible torsion-free automorphism group of $\widehat{B}$.

The following general result is a consequence of results proved in [1], [12], [13], [32], [36], [44].

Theorem 6.1. *Let B be a quasitilted algebra of canonical type, G an admissible torsion-free group of automorphisms of $\widehat{B}$, and $A = \widehat{B}/G$ the associated orbit algebra. Then the following statements hold.*

(i) *G is an infinite cyclic group generated by a strictly positive automorphism ψ of $\widehat{B}$.*

(ii) *The push-down functor* $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} A$ *associated to the Galois covering* $F : \widehat{B} \to \widehat{B}/G = A$ *with Galois group* G *is dense.*

(iii) *The Auslander-Reiten quiver* Γ_A *of* A *is isomorphic to the orbit quiver* $\Gamma_{\widehat{B}}/G$ *of the Auslander-Reiten quiver* $\Gamma_{\widehat{B}}$ *of* $\widehat{B}$ *with respect to the induced action of* G *on* $\Gamma_{\widehat{B}}$*.*

The following proposition (see [1], [32], [36], [44]) relates the selfinjective algebras of canonical type with almost concealed canonical algebras.

Proposition 6.2. *Let* B *be a quasitilted algebra of canonical type. Then there exists an almost concealed canonical algebra* B^* *such that* $\widehat{B} = \widehat{B^*}$*.*

We note that in general we may have several almost concealed canonical algebras whose repetitive algebras are isomorphic.

The class of selfinjective algebras of canonical type may be divided into three disjoint classes, according to the natural division of almost concealed canonical algebras into three disjoint classes.

Let B be an almost concealed canonical algebra, G an admissible infinite cyclic automorphism group of $\widehat{B}$, and $A = \widehat{B}/G$. Then A is said to be

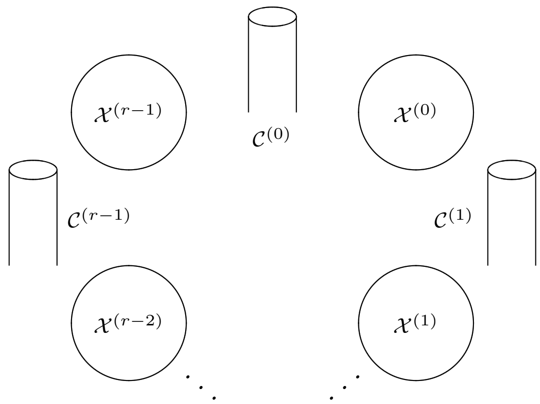
- a *selfinjective algebra of Euclidean type*, if B is a tilted algebra of Euclidean type;
- a *selfinjective algebra of tubular type*, if B is a tubular algebra;
- a *selfinjective algebra of wild canonical type*, if B is of wild canonical type

(see [58, Section 7]).

The following theorem gives more precise information on the structure of the Auslander-Reiten quiver of a selfinjective algebra of canonical type (see [1], [32], [36], [44]).

Theorem 6.3. *Let* A *be a selfinjective algebra of canonical type. Then the*

Auslander-Reiten quiver Γ_A *of* A *has the form*



for some integer $r \geq 1$, *where each* $\mathcal{C}^{(i)}$, $i \in \{0, \ldots, r-1\}$, *is an infinite family of quasitubes, and*

(1) *If* A *is of Euclidean type, then every* $\mathcal{X}^{(i)}$, $i \in \{0, \ldots, r-1\}$, *is an acyclic component of Euclidean type (the stable part is of the form* $\mathbb{Z}\Delta$ *for an Euclidean quiver* Δ*);*
(2) *If* A *is of tubular type, then every* $\mathcal{X}^{(i)}$, $i \in \{0, \ldots, r-1\}$, *is a disjoint union* $\bigvee_{q \in \mathbb{Q}^i_{i+1}} \mathcal{C}^{(i)}_q$, *where* $\mathcal{C}^{(i)}_q$ *is an infinite family of stable tubes for each* $q \in \mathbb{Q}^i_{i+1} = \mathbb{Q} \cap (i, i+1)$*;*
(3) *If* A *is of wild canonical type, then every* $\mathcal{X}^{(i)}$, $i \in \{0, \ldots, r-1\}$, *is an infinite family of components whose stable parts are of the form* $\mathbb{Z}\mathbb{A}_\infty$.

We call the above decomposition of Γ_A a *canonical decomposition* of Γ_A.

The main aim of the remaining part of this section is to prove two propositions which show the implication $(ii) \Rightarrow (i)$ of Theorem 1.1.

Proposition 6.4. *Let* B *be a tubular algebra,* G *an infinite cyclic admissible group of automorphisms of* $\widehat{B}$, *and* $A = \widehat{B}/G$. *Then the following statements are equivalent:*

(i) Γ_A *admits a family of quasitubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles.*
(ii) Γ_A *admits a family of stable tubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles.*

(iii) $G = (\varphi \nu_{\widehat{B}}^2)$ *for a positive automorphism* φ *of* $\widehat{B}$.

Proof. It follows from the results established in [14], [19], [36], [44] (see also [9], [24]) that the Auslander-Reiten quiver $\Gamma_{\widehat{B}}$ of $\widehat{B}$ has a decomposition

$$\Gamma_{\widehat{B}} = \bigvee_{q\in\mathbb{Q}} \mathcal{C}_q^{\widehat{B}} = \bigvee_{q\in\mathbb{Q}} \bigvee_{x\in\mathbb{X}_q} \mathcal{C}_{q,x}^{\widehat{B}}$$

such that

(1) For each $q \in \mathbb{Z}$, $\mathcal{C}_q^{\widehat{B}}$ is an infinite family $\mathcal{C}_{q,x}^{\widehat{B}}$, $x \in \mathbb{X}_q$, of quasitubes containing at least one projective module.
(2) For each $q \in \mathbb{Q} \setminus \mathbb{Z}$, $\mathcal{C}_q^{\widehat{B}}$ is an infinite family $\mathcal{C}_{q,x}^{\widehat{B}}$, $x \in \mathbb{X}_q$, of stable tubes.
(3) For each $q \in \mathbb{Q}$, $\mathcal{C}_q^{\widehat{B}}$ is a family of pairwise orthogonal generalized standard quasitubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles in $\operatorname{mod} \widehat{B}$.
(4) There is a positive integer m such that $3 \le m \le \operatorname{rk} K_0(B)$ and $\nu_{\widehat{B}}(\mathcal{C}_q^{\widehat{B}}) = \mathcal{C}_{q+m}^{\widehat{B}}$ for any $q \in \mathbb{Q}$.
(5) $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_q^{\widehat{B}}, \mathcal{C}_r^{\widehat{B}}) = 0$ for all $q > r$ in $\mathbb{Q}$.
(6) $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_q^{\widehat{B}}, \mathcal{C}_r^{\widehat{B}}) = 0$ for all $r > q + m$ in $\mathbb{Q}$.
(7) For $q \in \mathbb{Q}$, we have $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_q^{\widehat{B}}, \mathcal{C}_{q+m}^{\widehat{B}}) \neq 0$ if and only if $q \in \mathbb{Z}$.
(8) For $p < q$ in $\mathbb{Q}$ with $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_p^{\widehat{B}}, \mathcal{C}_q^{\widehat{B}}) \neq 0$, we have $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_p^{\widehat{B}}, \mathcal{C}_r^{\widehat{B}}) \neq 0$ and $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_r^{\widehat{B}}, \mathcal{C}_q^{\widehat{B}}) \neq 0$ for any $r \in \mathbb{Q}$ with $p \le r \le q$.
(9) For all $p \in \mathbb{Q} \setminus \mathbb{Z}$ and all $q \in \mathbb{Q}$ with $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_p^{\widehat{B}}, \mathcal{C}_q^{\widehat{B}}) \neq 0$, we have $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_{p,x}^{\widehat{B}}, \mathcal{C}_{q,y}^{\widehat{B}}) \neq 0$ for all $x \in \mathbb{X}_p$ and $y \in \mathbb{X}_q$.
(10) For all $p \in \mathbb{Q}$ and all $q \in \mathbb{Q} \setminus \mathbb{Z}$ with $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_p^{\widehat{B}}, \mathcal{C}_q^{\widehat{B}}) \neq 0$, we have $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_{p,x}^{\widehat{B}}, \mathcal{C}_{q,y}^{\widehat{B}}) \neq 0$ for all $x \in \mathbb{X}_p$ and $y \in \mathbb{X}_q$.

We know also from [44] (see Theorem 6.1(i)) that G is generated by strictly positive automorphism g of $\widehat{B}$. Consider the canonical Galois covering $F : \widehat{B} \to \widehat{B}/G = A$ and the associated push-down functor $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} A$. Since F_λ is dense, we obtain natural isomorphisms of k-modules

$$\bigoplus_{i\in\mathbb{Z}} \operatorname{Hom}_{\widehat{B}}(X, {}^{g^i}Y) \xrightarrow{\sim} \operatorname{Hom}_A(F_\lambda(X), F_\lambda(Y)),$$

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\widehat{B}}({}^{g^i}X, Y) \xrightarrow{\sim} \mathrm{Hom}_A(F_\lambda(X), F_\lambda(Y)),$$

for all indecomposable modules X and Y in $\mathrm{mod}\,\widehat{B}$.

We first show that (iii) implies (ii). Assume that $g = \varphi\nu_{\widehat{B}}^2$ for some positive automorphism φ of $\widehat{B}$. Then it follows from (4) that there is a positive integer $l \geq 2m$ such that $g(\mathcal{C}_q^{\widehat{B}}) = \mathcal{C}_{q+l}^{\widehat{B}}$ for any $q \in \mathbb{Q}$. Since $g = \varphi\nu_{\widehat{B}}^2 = (\varphi\nu_{\widehat{B}})\nu_{\widehat{B}}$ with $\varphi\nu_{\widehat{B}}$ a strictly positive automorphism of $\widehat{B}$, invoking the knowledge of the supports of indecomposable modules in $\mathrm{mod}\,\widehat{B}$ (see [36, Section 3]), we conclude that the images $F_\lambda(S)$ and $F_\lambda(T)$ of any nonisomorphic simple $\widehat{B}$-modules S and T which occur as composition factors of modules in a fixed family $\mathcal{C}_q^{\widehat{B}}$ are nonisomorphic simple A-modules. Therefore, it follows from Theorem 6.1 and properties (1)-(4), that, for each $q \in \mathbb{Q}$, $\mathcal{C}_q^A = F_\lambda(\mathcal{C}_q^{\widehat{B}})$ is an infinite family $\mathcal{C}_{q,x}^A = F_\lambda(\mathcal{C}_{q,x}^{\widehat{B}})$, $x \in \mathbb{X}_q$, of quasitubes of Γ_A with common composition factors and closed under composition factors. Take now $p \in \mathbb{Q} \setminus \mathbb{Z}$. Then, by (2), $\mathcal{C}_p^A = (\mathcal{C}_{p,x}^A)_{x\in\mathbb{X}_p}$ is a family of stable tubes of Γ_A. We claim that $\mathcal{C}_p^A$ consists of indecomposable A-modules which do not lie on infinite short cycles in $\mathrm{mod}\,A$. Observe first that, for two indecomposable modules M and N in $\mathcal{C}_p^A$, we have $M = F_\lambda(X)$ and $N = F_\lambda(Y)$, for some indecomposable modules X and Y in $\mathcal{C}_p^{\widehat{B}}$, and F_λ induces an isomorphism of k-modules $\mathrm{Hom}_A(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\widehat{B}}(X, Y)$, by (5) and (6), and $q+l \geq q+2m > q+m$. In particular, by Theorem 2.5 and (3), $\mathcal{C}_p^A$ is a family of pairwise orthogonal generalized standard stable tubes of Γ_A. Suppose now that there is an infinite short cycle $M \to L \to M$ in $\mathrm{mod}\,A$ with M in $\mathcal{C}_{p,x}^A$ for some $x \in \mathbb{X}_p$. Since $\mathcal{C}_p^A$ is a family of pairwise orthogonal generalized standard stable tubes of Γ_A, we conclude that L does not belong to $\mathcal{C}_p^A$. Then $M = F_\lambda(X)$ for some X in $\mathcal{C}_{p,x}^{\widehat{B}}$ and $L = F_\lambda(Z)$ for some Z in $\mathcal{C}_r^{\widehat{B}}$ with $r > p$. We have an isomorphism of k-modules, induced by F_λ,

$$\mathrm{Hom}_A(M, L) \xrightarrow{\sim} \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\widehat{B}}(X, {}^{g^i}Z).$$

Since $\mathrm{Hom}_A(M, L) \neq 0$, we may choose, invoking (5), a minimal $r > p$ and $Z \in \mathcal{C}_r^{\widehat{B}}$ such that $L = F_\lambda(Z)$ and $\mathrm{Hom}_{\widehat{B}}(X, Z) \neq 0$. Since $p \in \mathbb{Q} \setminus \mathbb{Z}$ and X lies in $\mathcal{C}_p^{\widehat{B}}$, applying (6) and (7), we infer that $p < r < p + m$. Further, we have also an isomorphism of k-modules, induced by F_λ,

$$\mathrm{Hom}_A(L, M) \xrightarrow{\sim} \bigoplus_{i\in\mathbb{Z}} \mathrm{Hom}_{\widehat{B}}(Z, {}^{g^i}X).$$

Observe that, for each $i \in \mathbb{Z}$, ${}^{g^i}X$ is an indecomposable module from $\mathcal{C}_{p+li}^{\widehat{B}}$, and clearly with $F_\lambda({}^{g^i}X) = F_\lambda(X) = M$. Since $\mathrm{Hom}_A(L, M) \neq 0$, $L = F_\lambda(Z)$ for $Z \in \mathcal{C}_r^{\widehat{B}}$ with $r > p$ and $X \in \mathcal{C}_p^{\widehat{B}}$, applying (5), we conclude that $\mathrm{Hom}_{\widehat{B}}(Z, {}^{g^i}X) \neq 0$, for some $i \geq 1$. But then $p+li \geq p+l \geq p+2m > r+m$, because $r < p+m$, and we obtain a contradiction with (6).

Summing up, we have proved that $\mathcal{C}_p^A = F_\lambda(\mathcal{C}_p^{\widehat{B}})$ is a family of stable tubes of Γ_A with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles in $\mathrm{mod}\, A$. Therefore, (iii) implies (ii).

Since clearly (ii) implies (i), it remains to show that (i) implies (iii). Assume that Γ_A admits a family $\mathcal{C} = (\mathcal{C}_x)_{x \in \mathbb{X}}$ of quasitubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles in $\mathrm{mod}\, A$. We know from the property (3) that, for each $q \in \mathbb{Q}$, $\mathcal{C}_q^A = F_\lambda(\mathcal{C}_q^{\widehat{B}})$ is a family $\mathcal{C}_{q,x}^A = F_\lambda(\mathcal{C}_{q,x}^{\widehat{B}})$, $x \in \mathbb{X}_q$, of quasitubes with common composition factors. Moreover, the push-down functor F_λ induces an isomorphism of translation quivers $\Gamma_{\widehat{B}}/G \xrightarrow{\sim} \Gamma_A$ (see Theorem 6.1), and hence every component of Γ_A is a quasi-tube of the form $\mathcal{C}_{q,x}^A = F_\lambda(\mathcal{C}_{q,x}^{\widehat{B}})$ for some $q \in \mathbb{Z}$ and $x \in \mathbb{X}_q$. Then, since the family $\mathcal{C}$ is closed under composition factors, we conclude that there is $r \in \mathbb{Q}$ such that $\mathcal{C}$ contains all quasitubes $\mathcal{C}_{r,x}^A$, $x \in \mathbb{X}_r$, of $\mathcal{C}_r^A$. In particular, we conclude that the family $\mathcal{C}_r^A = (\mathcal{C}_{r,x}^A)_{x \in \mathbb{X}_r}$ consists of modules which do not lie on infinite short cycles in $\mathrm{mod}\, A$. We claim that this forces g to be of the form $g = \varphi \nu_{\widehat{B}}^2$ for some positive automorphism φ of $\widehat{B}$. Suppose it is not the case. Since g is a strictly positive automorphism of $\widehat{B}$ and all projective $\widehat{B}$-modules lie in $\bigvee_{p \in \mathbb{Z}} \mathcal{C}_p^{\widehat{B}}$, invoking (4), we conclude that there exists a positive integer $s < 2m$ such that $g(\mathcal{C}_q^{\widehat{B}}) = \mathcal{C}_{q+s}^{\widehat{B}}$ for any $q \in \mathbb{Q}$. Let p be the natural number such that $r \in [p, p+1) \cap \mathbb{Q}$. We have two cases to consider.

Assume $s < m$. Take $q \in (\mathbb{Q} \setminus \mathbb{Z}) \cap (p+1, p+s)$. Since $m \geq 3$, we have the inequalities

$$p \leq r < p+1 < q < p+s \leq r+s < p+m$$

which, together with (7) and (8), implies that $\mathrm{Hom}_{\widehat{B}}(\mathcal{C}_r^{\widehat{B}}, \mathcal{C}_q^{\widehat{B}}) \neq 0$ and $\mathrm{Hom}_{\widehat{B}}(\mathcal{C}_q^{\widehat{B}}, \mathcal{C}_{r+s}^{\widehat{B}}) \neq 0$. Moreover, from the properties (9) and (10), we obtain that

$$\mathrm{Hom}_{\widehat{B}}(\mathcal{C}_{r,x}^{\widehat{B}}, \mathcal{C}_{q,y}^{\widehat{B}}) \neq 0 \text{ for any } x \in \mathbb{X}_r \text{ and } y \in \mathbb{X}_q,$$
$$\mathrm{Hom}_{\widehat{B}}(\mathcal{C}_{q,y'}^{\widehat{B}}, \mathcal{C}_{r+s,x'}^{\widehat{B}}) \neq 0 \text{ for any } x' \in \mathbb{X}_{r+s} \text{ and } y' \in \mathbb{X}_q,$$

because $q \in \mathbb{Q} \setminus \mathbb{Z}$. Since $\mathcal{C}^{\widehat{B}}_{r+s} = g(\mathcal{C}^{\widehat{B}}_r)$, there exist $x \in \mathbb{X}_r$, $y \in \mathbb{X}_q$ and modules $X \in \mathcal{C}^{\widehat{B}}_{r,x}$, $Y \in \mathcal{C}^{\widehat{B}}_{q,y}$ and $X' \in \mathcal{C}^{\widehat{B}}_{r+s,x}$ such that $\operatorname{Hom}_{\widehat{B}}(X,Y) \neq 0$, $\operatorname{Hom}_{\widehat{B}}(Y,X') \neq 0$ and $F_\lambda(X) = F_\lambda(X')$. Hence we have an infinite short cycle $F_\lambda(X) \to F_\lambda(Y) \to F_\lambda(X') = F_\lambda(X)$ in $\operatorname{mod} A$ with $F_\lambda(X)$ in $\mathcal{C}^A_r$, what contradicts our assumption.

Finally assume that $m \leq s < 2m$. Take $q = p + m$. We have the inequalities

$$p \leq r < p+1 < q \leq p+s \leq r+s < p+2m.$$

Because $p+m-1 \in \mathbb{Z}$, then $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}^{\widehat{B}}_{p+m-1}, \mathcal{C}^{\widehat{B}}_{p+2m-1}) \neq 0$, and hence from the property (8) we get $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}^{\widehat{B}}_q, \mathcal{C}^{\widehat{B}}_{p+2m-1}) \neq 0$, and so $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}^{\widehat{B}}_q, \mathcal{C}^{\widehat{B}}_{r+s}) \neq 0$. Using the properties (9) and (10) we obtain

$$\operatorname{Hom}_{\widehat{B}}(\mathcal{C}^{\widehat{B}}_{r,x}, \mathcal{C}^{\widehat{B}}_{q,y}) \neq 0 \text{ for any } x \in \mathbb{X}_r \text{ and } y \in \mathbb{X}_q,$$
$$\operatorname{Hom}_{\widehat{B}}(\mathcal{C}^{\widehat{B}}_{q,y'}, \mathcal{C}^{\widehat{B}}_{r+s,x'}) \neq 0 \text{ for any } x' \in \mathbb{X}_{r+s} \text{ and } y' \in \mathbb{X}_q,$$

because $q \in \mathbb{Q} \setminus \mathbb{Z}$. Similarly as above we conclude that there is an infinite short cycle $F_\lambda(X) \to F_\lambda(Y) \to F_\lambda(X') = F_\lambda(X)$ in $\operatorname{mod} A$ with $F_\lambda(X)$ in $\mathcal{C}^A_r$, a contradiction with our assumption. □

Proposition 6.5. *Let B be an almost concealed canonical algebra of Euclidean or wild type, G an infinite cyclic admissible group of automorphisms of $\widehat{B}$, and $A = \widehat{B}/G$. Then the following statements are equivalent:*

- (i) *Γ_A admits a family of quasitubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles.*
- (ii) *G is one of the forms*
  - (a) *$G = (\varphi\nu^2_{\widehat{B}})$, for a strictly positive automorphism φ of $\widehat{B}$,*
  - (b) *$G = (\varphi\nu^2_{\widehat{B}})$, for a rigid automorphism φ of $\widehat{B}$ whose restriction to B does not fix any nonstable ray tube of the unique separating family $\mathcal{T}^{\widehat{B}}$ of ray tubes of $\Gamma_{\widehat{B}}$.*

Proof. It follows from [1], [3], [32] that the Auslander-Reiten quiver $\Gamma_{\widehat{B}}$ of $\widehat{B}$ has a decomposition

$$\Gamma_{\widehat{B}} = \bigvee_{q\in\mathbb{Z}} (\mathcal{C}^{\widehat{B}}_q \vee \mathcal{X}^{\widehat{B}}_q)$$

such that

(1) For each $q \in \mathbb{Z}$, $\mathcal{C}_q^{\widehat{B}}$ is an infinite family $\mathcal{C}_{q,x}^{\widehat{B}}$, $x \in \mathbb{X}_q$, of quasitubes.
(2) For each $q \in \mathbb{Z}$, $\mathcal{X}_q^{\widehat{B}}$ is either an acyclic component of Euclidean type, if B is of Euclidean type, or an infinite family of components whose stable parts are of the form $\mathbb{Z}\mathbb{A}_\infty$, if B is of wild type.
(3) For each $q \in \mathbb{Z}$, $\mathcal{C}_q^{\widehat{B}}$ is a family $\mathcal{C}_{q,x}^{\widehat{B}}$, $x \in \mathbb{X}_q$, of pairwise orthogonal generalized standard quasitubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles in $\operatorname{mod} \widehat{B}$.
(4) For each $q \in \mathbb{Z}$, we have $\nu_{\widehat{B}}(\mathcal{C}_q^{\widehat{B}}) = \mathcal{C}_{q+2}^{\widehat{B}}$ and $\nu_{\widehat{B}}(\mathcal{X}_q^{\widehat{B}}) = \mathcal{X}_{q+2}^{\widehat{B}}$.
(5) For each $q \in \mathbb{Z}$, we have $\operatorname{Hom}_{\widehat{B}}(\mathcal{X}_q^{\widehat{B}}, \mathcal{C}_q^{\widehat{B}} \vee \bigvee_{r<q}(\mathcal{C}_r^{\widehat{B}} \vee \mathcal{X}_r^{\widehat{B}})) = 0$ and $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_q^{\widehat{B}}, \bigvee_{r<q}(\mathcal{C}_r^{\widehat{B}} \vee \mathcal{X}_r^{\widehat{B}})) = 0$.
(6) For each $q \in \mathbb{Z}$, we have $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_q^{\widehat{B}}, \mathcal{X}_{q+2}^{\widehat{B}} \vee \bigvee_{r>q+2}(\mathcal{C}_r^{\widehat{B}} \vee \mathcal{X}_r^{\widehat{B}})) = 0$ and $\operatorname{Hom}_{\widehat{B}}(\mathcal{X}_q^{\widehat{B}}, \bigvee_{r>q+2}(\mathcal{C}_r^{\widehat{B}} \vee \mathcal{X}_r^{\widehat{B}})) = 0$.
(7) For $q \in \mathbb{Z}$, $x \in \mathbb{X}_q$ and $y \in \mathbb{X}_{q+2}$, we have $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_{q,x}^{\widehat{B}}, \mathcal{C}_{q+2,y}^{\widehat{B}}) \neq 0$ if and only if the quasitube $\mathcal{C}_{q,x}^{\widehat{B}}$ is nonstable and $\nu_{\widehat{B}}(\mathcal{C}_{q,x}^{\widehat{B}}) = \mathcal{C}_{q+2,y}^{\widehat{B}}$.
(8) For all $q \in \mathbb{Z}$, $x \in \mathbb{X}_q$ and $y \in \mathbb{X}_{q+1}$, we have $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_{q,x}^{\widehat{B}}, \mathcal{C}_{q+1,y}^{\widehat{B}}) \neq 0$.
(9) For each $q \in \mathbb{Z}$ and any stable tubes $\mathcal{C}_{q,x}^{\widehat{B}}$ in $\mathcal{C}_q^{\widehat{B}}$ and $\mathcal{C}_{q+3,y}^{\widehat{B}}$ in $\mathcal{C}_{q+3}^{\widehat{B}}$, there is an indecomposable projective $\widehat{B}$-module P in $\mathcal{X}_{q+1}^{\widehat{B}}$ such that $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_{q,x}^{\widehat{B}}, P) \neq 0$ and $\operatorname{Hom}_{\widehat{B}}(P, \mathcal{C}_{q+3,y}^{\widehat{B}}) \neq 0$.

We know also from [1], [3], [32], [44] that G is generated by a strictly positive automorphism g of $\widehat{B}$. Hence there exists a positive integer l such that $g(\mathcal{C}_q^{\widehat{B}}) = \mathcal{C}_{q+l}^{\widehat{B}}$ and $g(\mathcal{X}_q^{\widehat{B}}) = \mathcal{X}_{q+l}^{\widehat{B}}$ for any $q \in \mathbb{Z}$. Consider the canonical Galois covering $F : \widehat{B} \to \widehat{B}/G = A$ and the associated push-down functor $F_\lambda : \operatorname{mod} \widehat{B} \to \operatorname{mod} A$. Since F_λ is dense, we obtain natural isomorphisms of k-modules

$$\bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\widehat{B}}(X, {}^{g^i}Y) \xrightarrow{\sim} \operatorname{Hom}_A(F_\lambda(X), F_\lambda(Y)),$$

$$\bigoplus_{i \in \mathbb{Z}} \operatorname{Hom}_{\widehat{B}}({}^{g^i}X, Y) \xrightarrow{\sim} \operatorname{Hom}_A(F_\lambda(X), F_\lambda(Y)),$$

for all indecomposable modules X and Y in $\operatorname{mod} \widehat{B}$.

We show first that $(i) \Rightarrow (ii)$. Assume that Γ_A admits a family $\mathcal{C} = (\mathcal{C}_x)_{x \in \mathbb{X}}$ of quasitubes with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles in $\operatorname{mod} A$. It follows from Proposition 2.10 that the quasitubes $\mathcal{C}_x$, $x \in \mathbb{X}$, are generalized standard. In fact they are also pairwise orthogonal. Indeed, because $A = \widehat{B}/G$, where B is an almost concealed canonical algebra with a separating family of ray tubes $\mathcal{T} = (\mathcal{T}_i)_{i\in I}$, we infer that the quasitubes $\mathcal{C}_x$, $x \in \mathbb{X}$, are obtained from the ray tubes $\mathcal{T}_i$ by admissible operations of types $(ad\,1^*)$ and $(ad\,2^*)$. Therefore, using Lemma 2.9 and arguments as in the proof of Lemma 2.11, we obtain that the quasitubes $\mathcal{C}_x$, $x \in \mathbb{X}$, are pairwise orthogonal, because the ray tubes $\mathcal{T}_i$ are pairwise orthogonal.

We know from the property (3) that, for each $q \in \mathbb{Z}$, $\mathcal{C}_q^A = F_\lambda(\mathcal{C}_q^{\widehat{B}})$ is an infinite family $\mathcal{C}_{q,x}^A = F_\lambda(\mathcal{C}_{q,x}^{\widehat{B}})$, $x \in \mathbb{X}_q$, of quasitubes with common composition factors. Moreover,

$$\Gamma_A = \mathcal{C}_0^A \vee \mathcal{X}_0^A \vee \mathcal{C}_1^A \vee \mathcal{X}_1^A \vee \ldots \vee \mathcal{C}_{l-1}^A \vee \mathcal{X}_{l-1}^A,$$

with $\mathcal{X}_q^A = F_\lambda(\mathcal{X}_q^{\widehat{B}})$ for $q \in \{0, 1, \ldots, l-1\}$, since F_λ induces an isomorphism of translation quivers $\Gamma_{\widehat{B}}/G \xrightarrow{\sim} \Gamma_A$, $G = (g)$, and $g(\mathcal{C}_q^{\widehat{B}}) = \mathcal{C}_{q+l}^{\widehat{B}}$, $g(\mathcal{X}_q^{\widehat{B}}) = \mathcal{X}_{q+l}^{\widehat{B}}$, for any $q \in \mathbb{Z}$. Then, since $\mathcal{C}$ is the family of quasitubes in Γ_A closed under composition factors, we conclude that there is $r \in \{0, 1, \ldots, l-1\}$ such that $\mathcal{C}$ contains all quasitubes $\mathcal{C}_{r,x}^A$, $x \in \mathbb{X}_r$, of $\mathcal{C}_r^A$. In particular, we conclude that $\mathcal{C}_r^A = (\mathcal{C}_{r,x}^A)_{x\in\mathbb{X}_r}$ is a family of pairwise orthogonal generalized standard quasitubes consisting of modules which do not lie on infinite short cycles in $\operatorname{mod} A$. We may assume, without loss of generality, that $r = 0$.

We claim that this forces G to be one of two forms (a) and (b) required in (ii). We show first that $g = \varphi\nu_{\widehat{B}}^2$ for a positive automorphism φ of $\widehat{B}$. Suppose it is not the case. Then, by the property (4), we conclude that $l \in \{1, 2, 3\}$. We have three cases to consider.

Assume $l = 1$. Then we have $F_\lambda(\mathcal{C}_0^{\widehat{B}}) = \mathcal{C}_0^A = F_\lambda(\mathcal{C}_1^{\widehat{B}})$. Applying then the property (8), we conclude that, for any $x \in \mathbb{X}_0$, the quasitube $\mathcal{C}_{0,x}^A$ is not generalized standard, a contradiction.

Assume $l = 2$. Then we have $F_\lambda(\mathcal{C}_0^{\widehat{B}}) = \mathcal{C}_0^A = F_\lambda(\mathcal{C}_2^{\widehat{B}})$. We know from (1) that $\mathcal{C}_0^{\widehat{B}} = (\mathcal{C}_{0,x}^A)_{x\in\mathbb{X}_0}$ and $\mathcal{C}_1^A = (\mathcal{C}_{1,x}^A)_{x\in\mathbb{X}_1}$ are infinite families of quasitubes. Since Γ_A has only finitely many projective modules, we may choose $x_0 \in \mathbb{X}_0$ and $x_1 \in \mathbb{X}_1$ such that $\mathcal{C}_{0,x_0}^A$ and $\mathcal{C}_{1,x_1}^A$ are stable tubes. Observe that $\mathcal{C}_{0,x_0}^A = F_\lambda(\mathcal{C}_{0,x_0}^{\widehat{B}})$, $\mathcal{C}_{1,x_1}^A = F_\lambda(\mathcal{C}_{1,x_1}^{\widehat{B}})$, and $\mathcal{C}_{0,x_0}^A = F_\lambda(\mathcal{C}_{2,x_2}^{\widehat{B}})$ for $x_2 \in \mathbb{X}_2$ such that $\nu_{\widehat{B}}(\mathcal{C}_{0,x_0}^{\widehat{B}}) = \mathcal{C}_{2,x_2}^{\widehat{B}}$, by the property (4). Applying the property (8), we conclude that $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_{0,x_0}^{\widehat{B}}, \mathcal{C}_{1,x_1}^{\widehat{B}}) \neq 0$ and $\operatorname{Hom}_{\widehat{B}}(\mathcal{C}_{1,x_1}^{\widehat{B}}, \mathcal{C}_{2,x_2}^{\widehat{B}}) \neq 0$, and hence

$\mathrm{Hom}_A(\mathcal{C}^A_{0,x_0}, \mathcal{C}^A_{1,x_1}) \neq 0$ and $\mathrm{Hom}_A(\mathcal{C}^A_{1,x_1}, \mathcal{C}^A_{0,x_0}) \neq 0$. Then it follows from Lemma 2.11 that there is in $\mathrm{mod}\, A$ an infinite short cycle $M \to N \to M$ with M in $\mathcal{C}^A_{0,x_0}$ and $N \in \mathcal{C}^A_{1,x_1}$, a contradiction because $\mathcal{C}^A_{0,x_0}$ is a quasitube of the family $\mathcal{C} = (\mathcal{C}_x)_{x\in\mathbb{X}}$.

Assume $l = 3$. Then we have $F_\lambda(\mathcal{C}^{\widehat{B}}_0) = \mathcal{C}^A_0 = F_\lambda(\mathcal{C}^{\widehat{B}}_3)$. Since $\mathcal{C}^A_0 = (\mathcal{C}^A_{0,x})_{x\in\mathbb{X}_0}$ is an infinite family of quasitubes and the number of projective modules in Γ_A is finite, we may choose $x_0 \in \mathbb{X}_0$ such that $\mathcal{C}^A_{0,x_0}$ is a stable tube of Γ_A. Observe that then $\mathcal{C}^A_{0,x_0} = F_\lambda(\mathcal{C}^{\widehat{B}}_{0,x_0})$, $\mathcal{C}^{\widehat{B}}_{0,x_0}$ is a stable tube of $\Gamma_{\widehat{B}}$, and hence $g(\mathcal{C}^{\widehat{B}}_{0,x_0})$ is a stable tube $\mathcal{C}^{\widehat{B}}_{3,x_3}$, for some $x_3 \in \mathbb{X}_3$, of $\Gamma_{\widehat{B}}$. Applying now the property (9), we conclude that there is an indecomposable projective module P in $\mathcal{X}^{\widehat{B}}_1$ such that $\mathrm{Hom}_{\widehat{B}}(\mathcal{C}^{\widehat{B}}_{0,x_0}, P) \neq 0$ and $\mathrm{Hom}_{\widehat{B}}(P, \mathcal{C}^{\widehat{B}}_{3,x_3}) \neq 0$. Then we have $F_\lambda(\mathcal{C}^{\widehat{B}}_{0,x_0}) = \mathcal{C}^A_{0,x_0} = F_\lambda(\mathcal{C}^{\widehat{B}}_{3,x_3})$ and $F_\lambda(P)$ is an indecomposable projective A-module in $F_\lambda(\mathcal{X}^{\widehat{B}}_1)$ such that $\mathrm{Hom}_A(\mathcal{C}^A_{0,x_0}, F_\lambda(P)) \neq 0$ and $\mathrm{Hom}_A(F_\lambda(P), \mathcal{C}^A_{0,x_0}) \neq 0$. Then it follows from Lemma 2.11 that there is in $\mathrm{mod}\, A$ an infinite short cycle $M \to F_\lambda(P) \to M$ with M in $\mathcal{C}^A_{0,x_0}$, again a contradiction since $\mathcal{C}^A_{0,x_0}$ is a quasitube of the family $\mathcal{C} = (\mathcal{C}_x)_{x\in\mathbb{X}}$.

Summing up, we proved that indeed $g = \varphi\nu^2_{\widehat{B}}$ for a positive automorphism φ of $\widehat{B}$.

Assume now that φ is a rigid automorphism of $\widehat{B}$ and B is an almost concealed canonical algebra (of Euclidean or wild type) whose unique separating family $\mathcal{T}^B$ of ray tubes contains at least one projective module, or equivalently (see [30], [32]), B is not a concealed canonical algebra. Then the family $\mathcal{C}^{\widehat{B}}_0$ of quasitubes of $\Gamma_{\widehat{B}}$, and hence the family $\mathcal{C}^A_0 = F_\lambda(\mathcal{C}^{\widehat{B}}_0)$ of quasitubes in Γ_A, contains at least one projective module. We also note that, since φ is a rigid automorphism of $\widehat{B}$, its restriction φ_B to $B = B_0$ is a k-algebra automorphism of B and φ_B acts on the unique separating family $\mathcal{T}^B$ of ray tubes of Γ_B. Suppose φ_B fixes a nonstable tube (a ray tube containing projective module) of $\mathcal{T}^B$. Then there is $x_0 \in \mathbb{X}_0$ such that $\mathcal{C}^{\widehat{B}}_{0,x_0}$ is a quasitube containing at least one projective module such that $\varphi(\mathcal{C}^{\widehat{B}}_{0,x_0}) = \mathcal{C}^{\widehat{B}}_{0,x_0}$. Since $g = \varphi\nu^2_{\widehat{B}}$, applying the property (4), we then obtain that $g(\mathcal{C}^{\widehat{B}}_{0,x_0}) = \mathcal{C}^{\widehat{B}}_{4,x_0}$. Take now an indecomposable projective $\widehat{B}$-module P in $\mathcal{C}^{\widehat{B}}_{0,x_0}$. Then, by (4), we conclude that $\nu_{\widehat{B}}(P) \in \mathcal{C}^{\widehat{B}}_{2,x_0}$ and $\nu^2_{\widehat{B}}(P) \in \mathcal{C}^{\widehat{B}}_{4,x_0}$. Clearly, we have $\mathrm{Hom}_{\widehat{B}}(P, \nu_{\widehat{B}}(P)) \neq 0$ and $\mathrm{Hom}_{\widehat{B}}(\nu_{\widehat{B}}(P), \nu^2_{\widehat{B}}(P)) \neq 0$. Moreover, $g(P)$ and $\nu^2_{\widehat{B}}(P)$ belong to the same quasitube $\mathcal{C}^{\widehat{B}}_{4,x_0}$. Therefore, we conclude that there are indecomposable projective A-modules $F_\lambda(P)$ and $F_\lambda(\nu^2_{\widehat{B}}(P))$ in

$\mathcal{C}^A_{0,x_0}$ and an indecomposable projective A-module $F_\lambda(\nu_{\widehat{B}}(P))$ in $\mathcal{C}^A_{2,x_0}$ such that $\mathrm{Hom}_A(F_\lambda(P), F_\lambda(\nu_{\widehat{B}}(P))) \neq 0$ and $\mathrm{Hom}_A(F_\lambda(\nu_{\widehat{B}}(P)), F_\lambda(\nu^2_{\widehat{B}}(P))) \neq 0$. Applying now Lemma 2.11, we conclude that there is in $\mathrm{mod}\, A$ an infinite short cycle $M \to F_\lambda(\nu_{\widehat{B}}(P)) \to M$ with M in $\mathcal{C}^A_{0,x_0}$, a contradiction since $\mathcal{C}^A_{0,x_0}$ is in $\mathcal{C} = (\mathcal{C}_x)_{x\in\mathbb{X}}$.
This finishes the proof that (i) implies (ii).

Assume now that (ii) holds. In particular, we have $g = \varphi\nu^2_{\widehat{B}}$ for a positive automorphism of $\widehat{B}$. Then it follows from (4) that there is a positive integer $l \geq 4$ such that $g(\mathcal{C}^{\widehat{B}}_q) = \mathcal{C}^{\widehat{B}}_{q+l}$ for any $q \in \mathbb{Z}$. Let $\mathcal{C} = (\mathcal{C}_x)_{x\in\mathbb{X}}$ with $\mathbb{X} = \mathbb{X}_0$ and $\mathcal{C}_x = \mathcal{C}^A_{0,x}$ for any $x \in \mathbb{X}$. Since $g = \varphi\nu^2_{\widehat{B}} = (\varphi\nu_{\widehat{B}})\nu_{\widehat{B}}$ with $\varphi\nu_{\widehat{B}}$ a strictly positive automorphism of $\widehat{B}$, invoking the knowledge of the supports of indecomposable modules in $\mathrm{mod}\, \widehat{B}$ (see [1], [32]) we conclude that the images $F_\lambda(S)$ and $F_\lambda(T)$ of any nonisomorphic simple $\widehat{B}$-modules S and T which occur as composition factors of modules in a fixed family $\mathcal{C}^{\widehat{B}}_q$ are nonisomorphic simple A-modules. Therefore it follows from Theorem 6.1 and the properties (1)-(4) that $\mathcal{C}$ is an infinite family of quasitubes with common composition factors and closed under composition factors. We show now that $\mathcal{C}$ consists of indecomposable A-modules which do not lie on infinite short cycles in $\mathrm{mod}\, A$. Observe that for two indecomposable modules M and N in $\mathcal{C}$, we have $M = F_\lambda(X)$ and $N = F_\lambda(Y)$, for some indecomposable $\widehat{B}$-modules X and Y in $\mathcal{C}^{\widehat{B}}_0$, and F_λ induces an isomorphism of k-modules $\mathrm{Hom}_A(M, N) \xrightarrow{\sim} \mathrm{Hom}_{\widehat{B}}(X, Y)$, by the properties (5), (6), and $l \geq 4 > 2$. In particular, by (2) and (3), $\mathcal{C} = (\mathcal{C}_x)_{x\in\mathbb{X}}$ is a family of pairwise orthogonal generalized standard quasitubes of Γ_A. Suppose there is an infinite short cycle $M \to L \to M$ in $\mathrm{mod}\, A$ with M in $\mathcal{C}_{x_0} = \mathcal{C}^A_{0,x_0}$ for some $x_0 \in \mathbb{X} = \mathbb{X}_0$. Clearly, then L does not belong to $\mathcal{C}_{x_0}$. Then $M = F_\lambda(X)$ for some X in $\mathcal{C}^{\widehat{B}}_{0,x_0}$ and $L = F_\lambda(Z)$ for some indecomposable module Z in $\mathrm{mod}\, \widehat{B}$ such that $\mathrm{Hom}_{\widehat{B}}(X, Z) \neq 0$. Applying (5) and (6), we conclude that $Z \in \mathcal{X}^{\widehat{B}}_0 \vee \mathcal{C}^{\widehat{B}}_1 \vee \mathcal{X}^{\widehat{B}}_1 \vee \mathcal{C}^{\widehat{B}}_2$. Since $\mathrm{Hom}_A(L, M) \neq 0$, applying (5) and (6) again, we infer that $\mathrm{Hom}_{\widehat{B}}(Z, {}^gX) \neq 0$. Observe that ${}^gX \in g(\mathcal{C}^{\widehat{B}}_0) = \mathcal{C}^{\widehat{B}}_l$ with $l \geq 4$. Hence, invoking (5) and (6), we obtain that Z belongs to $\mathcal{C}^{\widehat{B}}_2$ and $l = 4$. But then the property (7) forces $Z \in \nu_{\widehat{B}}(\mathcal{C}^{\widehat{B}}_{0,x_0})$ and ${}^gX \in \nu^2_{\widehat{B}}(\mathcal{C}^{\widehat{B}}_{0,x_0})$. In particular we obtain that

$$(\nu^2_{\widehat{B}}\varphi)(\mathcal{C}^{\widehat{B}}_{0,x_0}) = (\varphi\nu^2_{\widehat{B}})(\mathcal{C}^{\widehat{B}}_{0,x_0}) = g(\mathcal{C}^{\widehat{B}}_{0,x_0}) = \nu^2_{\widehat{B}}(\mathcal{C}^{\widehat{B}}_{0,x_0}),$$

and hence $\varphi(\mathcal{C}^{\widehat{B}}_{0,x_0}) = \mathcal{C}^{\widehat{B}}_{0,x_0}$. Therefore, φ is a rigid automorphism of $\widehat{B}$ which fixes the nonstable quasitube $\mathcal{C}^{\widehat{B}}_{0,x_0}$ of $\Gamma_{\widehat{B}}$. Then the restriction φ_B of φ to B is a k-algebra automorphism of B which fixes the nonstable tube $\mathcal{T}^B_{x_0}$ of the unique separating family $\mathcal{T}^B$ of ray tubes of Γ_B whose all modules belong to the quasitube $\mathcal{C}^{\widehat{B}}_{0,x_0}$ of $\Gamma_{\widehat{B}}$. This contradicts the assumption (ii). Therefore, the family $\mathcal{C} = \mathcal{C}^A_0$ of quasitubes $\mathcal{C}_x = \mathcal{C}^A_{0,x}$, $x \in \mathbb{X} = X_0$, consists of the indecomposable A-modules which do not lie on infinite short cycles in $\operatorname{mod} A$. This completes the proof that (ii) implies (i). □

7. Proof of the Theorem 1.1

The aim of this section is to complete the proof of Theorem 1.1, by showing the implication $(i) \Rightarrow (ii)$.

Assume that A is a basic, connected, selfinjective algebra and $\mathcal{C} = (\mathcal{C}_i)_{i\in I}$ a family of quasitubes in Γ_A with common composition factors, closed under composition factors, and consisting of modules which do not lie on infinite short cycles. Then it follows from Proposition 2.10 that all quasitubes $\mathcal{C}_i$ in $\mathcal{C}$ are generalized standard components of Γ_A.

We will show first that $\mathcal{C}$ is a family of quasitubes of a quasitube enlargement Λ of a concealed canonical algebra C.

Fix $i \in I$, and consider the quotient algebra $A_i = A/\operatorname{ann}_A(\mathcal{C}_i)$. Then the quasitube $\mathcal{C}_i$ is a generalized standard faithful, hence sincere, component of Γ_{A_i}. Moreover, it follows from Lemma 2.12 that $\mathcal{C}_i$ is a quasitube without external short paths. Applying Theorem 4.4, we conclude that A_i is a quasitube enlargement of a concealed canonical algebra C_i, there is a separating family $\mathcal{T}^{C_i} = (\mathcal{T}^{C_i}_x)_{x\in\mathbb{X}_i}$ of stable tubes of Γ_{C_i} and a stable tube $\mathcal{T}^{C_i}_{x_i}$, for some $x_i \in \mathbb{X}_i$, such that $\mathcal{C}_i$ is obtained from $\mathcal{T}^{C_i}_{x_i}$ by a sequence of admissible operations of types $(ad\,1)$, $(ad\,2)$, $(ad\,1^*)$ and $(ad\,2^*)$, corresponding to those admissible operations leading from C_i to A_i. We recall that the index set $\mathbb{X}_i$ is infinite. Hence $\mathcal{T}^{C_i}$ is an infinite family of pairwise orthogonal stable tubes consisting of modules which do not lie on infinite short cycles in $\operatorname{mod} C_i$, because $\mathcal{T}^{C_i}$ is a separating family of stable tubes of Γ_{C_i}. Observe also that C_i is a quotient algebra of A, say $C_i = A/J_i$ for an ideal J_i of A, since C_i is a quotient algebra of A_i. We note that $\mathcal{T}^{C_i} = (\mathcal{T}^{C_i}_x)_{x\in\mathbb{X}_i}$ is a family of stable tubes of Γ_{C_i} with common composition factors (see [30], [48]). Since the quasitube $\mathcal{C}_i$, containing all modules of $\mathcal{T}^{C_i}_{x_i}$, belongs to $\mathcal{C}$ and $\mathcal{C}$ is closed under composition factors, we conclude that all modules of the family $\mathcal{T}^{C_i}$ belong to $\mathcal{C}$. Applying Lemma 2.14, we conclude that, for each $x \in \mathbb{X}_i$, there exists a quasitube $\mathcal{C}^{(i)}_x$ in $\mathcal{C}$ containing all modules of the stable tube $\mathcal{T}^{C_i}_x$ of

Γ_{C_i}. Moreover, by Lemma 2.12, we have $\mathcal{C}_x^{(i)} \neq \mathcal{C}_y^{(i)}$ for all $x \neq y$ in $\mathbb{X}_i$, since the tubes $\mathcal{T}_x^{(i)}$ and $\mathcal{T}_y^{(i)}$ are orthogonal. In fact, it follows from Lemma 2.13 that $\mathcal{T}_x^{C_i} = \mathcal{C}_x^{(i)}$ for all but finitely many indices x in $\mathbb{X}_i$, namely those $x \in \mathbb{X}_i$ for which $\mathcal{C}_x^{(i)}$ is a stable tube. We also note that $\mathcal{C}_x^{(i)}$, $x \in \mathbb{X}_i$, is a family of quasitubes with common composition factors. Further, for each $x \in \mathbb{X}_i$ with $\mathcal{T}_x^{C_i} = \mathcal{C}_x^{(i)}$, we have $J_i = \operatorname{ann}_A(\mathcal{C}_x^{(i)})$, because $\mathcal{T}_x^{C_i}$ is a faithful component of Γ_{C_i}.

We claim now that all concealed canonical algebras C_i, $i \in I$, coincide. Take $i \neq j$ in I. Since the sets $\mathbb{X}_i$ and $\mathbb{X}_j$ are infinite, we may take $x \in \mathbb{X}_i$ and $y \in \mathbb{X}_j$ such that $\mathcal{T}_x^{C_i} = \mathcal{C}_x^{(i)}$ and $\mathcal{T}_y^{C_j} = \mathcal{C}_y^{(j)}$. In particular, we have $J_i = \operatorname{ann}_A(\mathcal{T}_x^{C_i})$ and $J_j = \operatorname{ann}_A(\mathcal{T}_y^{C_j})$. We may assume that $\mathcal{T}_x^{C_i}$ and $\mathcal{T}_y^{C_j}$ are different, because $\mathcal{T}_x^{C_i} = \mathcal{T}_y^{C_j}$ forces $J_i = J_j$ and then $C_i = A/J_i = A/J_j = C_j$. Observe that $\mathcal{T}_x^{C_i}$ and $\mathcal{T}_y^{C_j}$ are stable tubes of Γ_A with common composition factors and consist of modules which do not lie on infinite short cycles in $\operatorname{mod} A$, because $\mathcal{T}_x^{C_i}$ and $\mathcal{T}_y^{C_j}$ belong to the family $\mathcal{C} = (\mathcal{C}_i)_{i \in I}$. Applying Theorem 2.5 and Lemma 2.17, we conclude that there exist indecomposable A-modules $M_i \in \mathcal{T}_x^{C_i}$ and $M_j \in \mathcal{T}_y^{C_j}$ such that $[M_i] = [M_j]$ in $K_0(A)$ and $J_i = \operatorname{ann}_A(M_i)$, $J_j = \operatorname{ann}_A(M_j)$. Since $[M_i] = [M_j]$, there is a quotient algebra $\mathrm{D} = A/L$, for an ideal $L = AgA$ of A given by an idempotent g of A, such that M_i and M_j are sincere indecomposable D-modules. Clearly, then $J_i \subseteq L$ and $J_j \subseteq L$, so $\mathcal{T}_x^{C_i} = \mathcal{C}_x^{(i)}$ and $\mathcal{T}_y^{C_j} = \mathcal{C}_y^{(j)}$ are sincere stable tubes of Γ_D consisting of indecomposable D-modules which do not lie on infinite short cycles in $\operatorname{mod} D$. Moreover, we have $\operatorname{ann}_D(\mathcal{T}_x^{C_i}) = L/J_i$ and $\operatorname{ann}_D(\mathcal{T}_y^{(j)}) = L/J_j$. Applying Corollary 3.3, we conclude that D is a concealed canonical algebra, $\mathcal{T}_x^{C_i}$ and $\mathcal{T}_y^{C_j}$ are faithful stable tubes of Γ_D, and consequently $J_i = L = J_j$. Therefore, indeed we have $C_i = A/J_i = A/J_j = C_j$ for all $i, j \in I$.

Summing up, we have proved that there exists a concealed canonical algebra C such that C is a quotient algebra of A and, for each $i \in I$, $A_i = A/\operatorname{ann}_A(\mathcal{C}_i)$ is a quasitube enlargement of C, and $\mathcal{C}_i$ is obtained from a stable tube $\mathcal{T}_{x_i}^{C}$, $x_i \in \mathbb{X}_i$, by the corresponding iterated application of admissible operations of types $(ad\,1)$, $(ad\,2)$, $(ad\,1^*)$ and $(ad\,2^*)$, where $\mathbb{X}_i$ is the index set of a separating family of stable tubes of Γ_C. Since the family $\mathcal{C} = (\mathcal{C}_i)_{i \in I}$ consists of quasitubes with common composition factors and is closed under composition factors, we conclude that Γ_C has a canonical decomposition

$$\Gamma_C = \mathcal{P}^C \vee \mathcal{T}^C \vee \mathcal{Q}^C$$

where $\mathcal{T}^C = (\mathcal{T}^C_x)_{x\in\mathbb{X}}$ is a separating family of stable tubes such that, for any $i \in I$, $\mathcal{T}^C_{x_i}$ is a stable tube of $\mathcal{T}^C$. In particular, we have $\mathbb{X}_i = \mathbb{X}$ for any $i \in I$. Moreover, we proved that, for a fixed $x \in \mathbb{X}$, all modules of the stable tube $\mathcal{T}^C_x$ are contained in a quasitube $\mathcal{C}_x$ from the family $\mathcal{C}$. Therefore, we conclude that $\Lambda = A/\operatorname{ann}_A(\mathcal{C})$ is a quasitube enlargement of the concealed canonical algebra C, using modules from the separating family $\mathcal{T}^C = (\mathcal{T}^C_x)_{x\in\mathbb{X}}$ of stable tubes of Γ_C, and $\mathcal{C}$ is the separating family of quasitubes of Γ_Λ, obtained from the family $\mathcal{T}^C$ by the corresponding iterated application of admissible operations of types $(ad\,1)$, $(ad\,2)$, $(ad\,1^*)$ and $(ad\,2^*)$. Applying Theorem 4.3, we conclude also that there is a unique almost concealed canonical quotient algebra $B = \Lambda_r$ of Λ (the right quasitilted part of Λ), which is a tubular extension of C and Γ_B admits the separating family $\mathcal{C}^r = (\mathcal{C}^r_x)_{x\in\mathbb{X}}$ obtained from the family $\mathcal{T}^C = (\mathcal{T}^C_x)_{x\in\mathbb{X}}$ of Γ_C by the corresponding iterated application of admissible operations of type $(ad\,1)$. Moreover, the family $\mathcal{C} = (\mathcal{C}_x)_{x\in\mathbb{X}}$ of quasitubes of Γ_Λ (and Γ_A) is obtained from the family $\mathcal{C}^r$ by a sequence of admissible operations of types $(ad\,1^*)$ and $(ad\,2^*)$.

Let I be the annihilator $\operatorname{ann}_A(\mathcal{C}^r)$ of the family $\mathcal{C}^r$ (of modules) in A. Since $\mathcal{C}^r$ is a faithful family of ray tubes of Γ_B, we conclude that $B = A/I$. We may assume that there exists a complete set of pairwise orthogonal idempotents $e_1, \dots, e_n$ of A such that $1_A = e_1 + \dots + e_n$ and $e = e_1 + \dots + e_m$, for some $m \leq n$, is a residual identity of $B = A/I$. We will show that I is a deforming ideal of A with $l_A(I) = Ie$ and $r_A(I) = eI$.

In order to prove that I is a deforming ideal of A we need several technical results.

Denote by J the trace ideal of the family $\mathcal{C}^r$ in A, that is, the sum of the images of all homomorphisms from modules in $\mathcal{C}^r$ to the right A-module A. Similarly, by J' we denote the trace ideal of the dual family $\mathrm{D}(\mathcal{C}^r)$ of left A-modules in A.

Proposition 7.1. *$J \cup J' \subseteq I$.*

Proof. Observe first that the annihilator $I = \operatorname{ann}_A(\mathcal{C}^r)$ of $\mathcal{C}^r$ is the annihilator $\operatorname{ann}_A(M)$ of a module from the additive closure $\operatorname{add}(\mathcal{C}^r)$ of $\mathcal{C}^r$. Indeed, since A is of finite length over k, we have

$$I = \operatorname{ann}_A(\mathcal{C}^r) = \bigcap_{X\in\mathcal{C}^r} \operatorname{ann}_A(X) = \bigcap_{i=1}^{r} \operatorname{ann}_A(M_i) = \operatorname{ann}_A(\bigoplus_{i=1}^{r} M_i)$$

for a finite family $M_1, \dots, M_r$ of indecomposable modules from $\mathcal{C}^r$, so we may take $M = M_1 \oplus \dots \oplus M_r$. We also note that I is the annihilator of the left A-module $D(M)$, from $\operatorname{add} \mathrm{D}(\mathcal{C}^r)$. In particular, we obtain that M is a faithful right B-module and $\mathrm{D}(M)$ is a faithful left B-module.

Invoking again the fact that A_A is of finite length over k, we obtain also that

$$J = \sum_{h \in \operatorname{Hom}_A(Y, A_A),\, Y \in \mathcal{C}^r} \operatorname{Im} h = \sum_{i=1}^{s} \operatorname{Im} h_i$$

for some homomorphisms $h_i \in \operatorname{Hom}_A(Y_i, A)$ with Y_i in $\mathcal{C}^r$, for $i \in \{1, \dots, s\}$, and hence an epimorphism of right A-modules

$$[h_1 \dots h_s] : Y = \bigoplus_{i=1}^{s} Y_i \to J.$$

Then $N = M \oplus Y$ is a module from $\operatorname{add} \mathcal{C}^r$ with $\operatorname{ann}_A(N) = \operatorname{ann}_A(M) = I$, hence N is a faithful right B-module, and there exists an epimorphism of right A-modules $g : N \to J$. Clearly, then J is a right B-module, because $JI = g(N)I = g(NI) = g(0) = 0$.

We will show now the inclusion $J \subseteq I$. Suppose we have $J \nsubseteq I$. Since $I = \operatorname{ann}_A(N)$ is the intersection of the kernels of all homomorphisms from $\operatorname{Hom}_A(A_A, N)$, we conclude that there is a homomorphism $f : A \to N$ in $\operatorname{mod} A$ such that $f(J) \neq 0$. Then there are indecomposable direct summands U and V of N and P of A_A such that $f(g(U) \cap P) \cap V \neq 0$, and consequently we obtain a short path in $\operatorname{mod} A$

$$U \xrightarrow{u} P \xrightarrow{v} V \ ,$$

with U and V in $\mathcal{C}^r$, P an indecomposable projective right A-module, and $vu \neq 0$. Moreover, $\operatorname{Im} u$ contains $\operatorname{soc} P$, and so $\operatorname{soc} P$ is a simple right B-module, because $\operatorname{Im} u$ is a right B-module. On the other hand, the family of quasitubes $\mathcal{C}$ is obtained from the family of ray tubes $\mathcal{C}^r$ by a sequence of admissible operations of types $(ad\,1^*)$ and $(ad\,2^*)$, we then infer that $P \notin \mathcal{C}^r$. Hence u and v belong to $\operatorname{rad}^\infty(\operatorname{mod} A)$, and so $0 \neq vu \in \operatorname{rad}^\infty(U, V)$, a contradiction since $\mathcal{C}^r$ is a generalized standard family of modules in $\operatorname{mod} B$, and hence in $\operatorname{mod} A$. Therefore, we have indeed $J \subseteq I$.

Further, since ${}_A A$ is of finite length over k, we obtain that

$$J' = \sum_{h' \in \operatorname{Hom}_{A^{op}}(\mathrm{D}(Y'), {}_A A),\, Y' \in \mathcal{C}^r} \operatorname{Im} h' = \sum_{j=1}^{t} \operatorname{Im} h'_j$$

for some homomorphisms $h'_j \in \mathrm{Hom}_{A^{op}}(\mathrm{D}(Y'_j), {}_AA)$ with $Y'_j \in \mathcal{C}^r$, for $j \in \{1, \dots, t\}$, and hence an epimorphism of left A-modules

$$[h'_1 \dots h'_t] : \mathrm{D}(Y') = \bigoplus_{j=1}^{t} \mathrm{D}(Y'_j) \to {}_AA.$$

Then $N' = M \oplus Y'$ is a module from $\mathcal{C}^r$, $\mathrm{D}(N')$ is a module in $\operatorname{add} \mathrm{D}(\mathcal{C}^r)$, and $\operatorname{ann}_A \mathrm{D}(N') = \operatorname{ann}_A \mathrm{D}(M) = I$. Hence $\mathrm{D}(N')$ is a faithful left B-module and there exists an epimorphism $g' : \mathrm{D}(N') \to J'$ of left A-modules. Obviously, then J' is a left B-module, because $IJ' = Ig'(\mathrm{D}(N')) = g'(I\,\mathrm{D}(N')) = g'(0) = 0$.

We claim now that $J' \subseteq I$. Suppose $J' \not\subseteq I$. Since $I = \operatorname{ann}_A \mathrm{D}(N')$ is the intersection of the kernels of all homomorphisms from $\mathrm{Hom}_{A^{op}}({}_AA, \mathrm{D}(N'))$, there exists a homomorphism $f' : {}_AA \to \mathrm{D}(N')$ of left A-modules such that $f'(J') \neq 0$. Then we have the sequence of homomorphisms of left A-modules

$$\mathrm{D}(N') \xrightarrow{g'} J' \xrightarrow{w'} {}_AA \xrightarrow{f'} \mathrm{D}(N'),$$

where w' is the canonical embedding, with $f'w'g' \neq 0$. Applying the duality, we obtain homomorphisms in $\operatorname{mod} A$

$$N' \xrightarrow{\mathrm{D}(f')} \mathrm{D}({}_AA) \xrightarrow{\mathrm{D}(w'g')} N'$$

with $\mathrm{D}(w'g')(D(f')(U') \cap P') \cap V' \neq 0$, and consequently a short path in $\operatorname{mod} A$

$$U' \xrightarrow{u'} P' \xrightarrow{v'} V',$$

with U' and V' in $\mathcal{C}^r$, P' an indecomposable projective right A-module, and $v'u' \neq 0$. Since $\operatorname{Im} u'$ is a nonzero right B-module, $\operatorname{soc} P'$ is a simple right B-module, and so we infer as above that $P' \notin \mathcal{C}^r$. Hence u' and v' belong to $\operatorname{rad}^\infty(\operatorname{mod} A)$, and then $0 \neq v'u' \in \operatorname{rad}^\infty(U', V')$, a contradiction since $\mathcal{C}^r$ is a generalized standard family of modules in $\operatorname{mod} A$. □

Lemma 7.2. *We have $l_A(I) = J$, $r_A(I) = J'$ and $I = r_A(J) = l_A(J')$.*

Proof. Because J is a right B-module then $I \subseteq r_A(J)$. Let N be a module from $\operatorname{add} \mathcal{C}^r$ such that $I = r_A(N)$. Let $\rho : N \to A^t$ be an embedding of N into a finite dimensional free right A-module. Denote by $\rho_i : N \to A$, for

$i \in \{1, \ldots, t\}$, the composite of ρ with the projection on the i-th component of A^t. Then there is an embedding of N into the direct sum $\bigoplus_{i=1}^t \rho_i(N)$, which is contained in $\bigoplus_{i=1}^t J$. Hence we have

$$I = r_A(N) \supseteq r_A(\bigoplus_{i=1}^{t} \rho_i(N)) \supseteq r_A(\bigoplus_{i=1}^{t} J) = r_A(J).$$

Consequently, we obtain $I = r_A(J)$. Applying now a theorem by Nakayama [60, Theorem 2.3.3], we get $J = l_A r_A(J) = l_A(I)$.
We will show now that $J' = r_A(I)$. First notice that, because J' is a left B-module, $I \subseteq l_A(J')$. Let N' be a module from $\operatorname{add} \mathcal{C}^r$ such that $I = l_A(\mathrm{D}(N'))$. Let $\rho' : \mathrm{D}(N') \to A^s$ be an embedding of $\mathrm{D}(N')$ into a finite dimensional free left A-module. Denote by $\rho'_i : \mathrm{D}(N') \to A$, for $i \in \{1, \ldots, s\}$, the composite of ρ' with the projection on the i-th component of A^s. Then there is an embedding of $\mathrm{D}(N')$ into the direct sum $\bigoplus_{i=1}^s \rho'_i(\mathrm{D}(N'))$, which is contained in $\bigoplus_{i=1}^s J'$. Hence we have

$$I = l_A(\mathrm{D}(N')) \supseteq l_A(\bigoplus_{i=1}^{s} \rho'_i(\mathrm{D}(N'))) \supseteq l_A(\bigoplus_{i=1}^{s} J') = l_A(J').$$

Thus we obtain $I = l_A(J')$. Applying now the theorem by Nakayama mentioned above, we get $J' = r_A l_A(J') = r_A(I)$. □

Lemma 7.3. *We have $eIe = eJe = eJ'e$. In particular $(eIe)^2 = 0$.*

Proof. Since e is a residual identity of $B = A/I$, we have $B \cong eAe/eIe$. Thus $\mathcal{C}^r$ is a faithful generalized standard family of ray tubes in $\Gamma_{eAe/eIe}$. Further, J is a right B-module, $1 - e \in I$, and so $J = Je + J(1 - e) = Je$, because $J(1 - e) \subseteq JI = 0$. Then eJ is an ideal of eAe with $eJ \subseteq eIe$, by Proposition 7.1.

Consider the algebra $B' = eAe/eJ$. Then $\mathcal{C}^r$ is a sincere generalized standard family of ray tubes in $\Gamma_{B'}$. Because the family $\mathcal{C}^r$ in Γ_A consists of B-modules which do not lie on infinite short cycles in $\operatorname{mod} A$, the modules from the family $\mathcal{C}^r$ in $\Gamma_{B'}$ do not lie on infinite short cycles in $\operatorname{mod} B'$. Moreover, for any $x \neq y$ in $\mathbb{X}$, the ray tubes $\mathcal{C}_x^r$ and $\mathcal{C}_y^r$ have infinitely many modules with common composition factors, since $\mathcal{C}_x^r$ contains all modules of $\mathcal{T}_x^C$ and $\mathcal{C}_y^C$ contains all modules of $\mathcal{T}_y^C$. Therefore, by Lemma 2.12, the family $\mathcal{C}^r$ consists of modules which do not lie on external short paths in $\operatorname{mod} B'$.

Hence, applying Theorem 3.1, we conclude that B' is an almost concealed canonical algebra and $\mathcal{C}^r$ a separating family of ray tubes of $\Gamma_{B'}$. But then the sincere generalized standard family $\mathcal{C}^r$ of ray tubes of $\Gamma_{B'}$ is faithful in $\operatorname{mod} B'$. This implies that $eIe/eJ = \operatorname{ann}_{B'}(\mathcal{C}^r) = 0$, and hence $eIe = eJ$. In a similar way we show that $eIe = J'e$. Applying Lemma 7.2, we obtain the equalities $(eIe)^2 = eJeeIe = eJeIe = (eJe)Ie = eJIe = 0$. □

We shall use also the following general lemma on almost split sequences over triangular algebras (see [52, Lemma 5.6]).

Lemma 7.4. *Let R and S be algebras and N an S-R-bimodule. Let $\Gamma = \begin{pmatrix} S & N \\ 0 & R \end{pmatrix}$ be the triangular matrix algebra defined by the bimodule ${}_SN_R$. Then an almost split sequence $0 \to X \to Y \to Z \to 0$ in $\operatorname{mod} R$ is an almost split sequence in $\operatorname{mod} \Gamma$ if and only if $\operatorname{Hom}_R(N, X) = 0$.*

Lemma 7.5. *Let f be a primitive idempotent in I such that $fJ \neq fAe$. Then $K = fAeAf + fJ + fAeAfAe + eAf + eIe$ is an ideal of the algebra $F = (e+f)A(e+f)$, and $N = fAe/fKe$ is a right B-module such that $\operatorname{Hom}_B(\mathcal{C}^r, N) \neq 0$ and $\operatorname{Hom}_B(N, \mathcal{C}^r) = 0$.*

Proof. It follows from Lemma 7.3 that $eIe = eJe$. Since $eJe \subseteq J$, we obtain the inclusions $fAeIe \subseteq f(eIe) \subseteq fJ$. Therefore K is an ideal of F. Observe also that $fKe = fJ + fAeAfAe$, $fKf \subseteq \operatorname{rad}(fAf)$, because $(fKf)^2 = (fAeAf)(fAeAf) \subseteq IeIeI \subseteq IeJI = 0$, $eKe = eIe$ and $eKf = eAf$. Moreover $N \neq 0$. Indeed, if $fAe = fKe$, then, since $eAfAe \subseteq eIe \subseteq \operatorname{rad}(eAe)$, we have from Lemma 7.3 that $fAe = fJ + fAe(\operatorname{rad}(eAe))$, and so $fAe = fJ$, a contradiction with our assumption. Further, $B = eAe/eIe$ and $(fAe)(eIe) = fAeJ \subseteq fJ \subseteq fKe$, and hence N is a right B-module. Finally, N is also a left module over $S = fAf/fKf$ and $\Gamma = F/K$ is isomorphic to the triangular matrix algebra $\Gamma = \begin{pmatrix} S & N \\ 0 & B \end{pmatrix}$. Invoking now the structure of the family $\mathcal{C} = (\mathcal{C}_x)_{x \in I}$ of quasitubes of Γ_A, we conclude that the family $\mathcal{C}^r = (\mathcal{C}_x^r)_{x \in \mathbb{X}}$ of ray rubes of Γ_B is the image of the family $\mathcal{C}$ via the restriction functor $(-)(e+f) : \operatorname{mod} A \to \operatorname{mod} F$, and consequently $\mathcal{C}^r$ is a family of ray tubes of Γ_F. We note also that the ray tubes $\mathcal{C}_x^r$, $x \in I$, do not contain injective modules, and hence for any module X in $\mathcal{C}^r$ there exists an almost split sequence $0 \to X \to Y \to Z \to 0$ in $\operatorname{mod} F$ consisting entirely of B-modules. Therefore, applying Lemma 7.4, we obtain $\operatorname{Hom}_B(N, X) = 0$ for any module X in $\mathcal{C}^r$, and so $\mathrm{Hom}_B(N, \mathcal{C}^r) = 0$. Further, $\mathcal{C}^r$ is a separating family of ray tubes of Γ_B, and hence every indecomposable module in $\mathrm{mod}\, B$ is either generated or cogenerated by $\mathcal{C}^r$. This implies that $\mathrm{Hom}_B(\mathcal{C}^r, N) \neq 0$. □

Denote by ν the Nakayama automorphism of A and by ν^- its inverse. Then for any primitive idempotent f of A we have $\mathrm{soc}(\nu(f)A) \cong \mathrm{top}(fA) = fA/\mathrm{rad}(fA)$. We have then the following two lemmas proved in [52, Lemmas 1.1 and 5.11].

Lemma 7.6. *The right ideal $\nu(e)l_A(I)$ is a minimal injective cogenerator in $\mathrm{mod}\, B$, and the left ideal $r_A(I)\nu^-(e)$ is a minimal injective cogenerator in $\mathrm{mod}\, B^{op}$.*

Lemma 7.7. *We have $\nu(e)J = l_{\nu(e)Ae}(eIe)$ and $J'\nu^-(e) = r_{eA\nu^-(e)}(eIe)$.*

Lemma 7.8. *We have $\nu(e)Ie = \nu(e)J$ and $eI\nu^-(e) = J'\nu^-(e)$.*

Proof. Let e_i be a primitive direct summand of e and put $f = \nu(e_i)$. We shall show that $fIe = fJ$. It is enough to prove that $fIeI = 0$, because then Lemma 7.7 implies $fIe \subseteq l_{fAe}(eIe) = fJ$, and $fJ \subseteq fIe$ follows from Proposition 7.1. Suppose that $fIeI \neq 0$. Then $f \in I$, because $\mathrm{soc}(fIeI_A) \subseteq \mathrm{top}(e_iA)$, and so $fIeIe_i \neq 0$ but $(eIe)^2 = 0$, by Lemma 7.3. Moreover, if $fAe = fJ$ then, since $f \in I$, it follows that $(fIe)I \subseteq (fAe)I = fJI = 0$, which contradicts our assumption. Therefore, we get $fAe \neq fJ$. Now consider K and N as in Lemma 7.5. Then we have $\mathrm{Hom}_B(\mathcal{C}^r, N) \neq 0$ and $\mathrm{Hom}_B(N, \mathcal{C}^r) = 0$. Take a module M from $\mathcal{C}^r$ such that $\mathrm{Hom}_B(M, N) \neq 0$.

(1) Let $L = fKe/fJ$. Observe that L is a right B-module, because $B \cong eAe/eIe$ and $eIe = eJ$ from Lemma 7.3. We claim that $\mathrm{Hom}_B(L, M) = 0$. It is enough to show that L is generated by N, because $\mathrm{Hom}_B(N, M) = 0$. In fact,

$$L \cong (fAeAf)fAe/(fJ \cap fAeAfAe)$$

as B-modules and the module on the right-hand side is generated by $N = fAe/(fJ + fAeAfAe)$, where we note that

$$(fAeAf)fJ \subseteq fJ \cap fAeAfAe,$$
$$(fAeAf)(fAeAfAe) = (fAe)(eAfAe)(eAfAe) \subseteq (fAe)(eIe)^2,$$

and $(eIe)^2 = 0$ by Lemma 7.3. Since $\tau_B M = 0$ or $\tau_B M$ belongs to $\mathcal{C}^r$, we have also $\mathrm{Hom}_B(N, \tau_B M) = 0$, and so $\mathrm{Hom}_B(L, \tau_B M) = 0$.

(2) We show that $\mathrm{Hom}_{eAe}(fKe, \tau_{eAe}M) = 0$. Applying now the functor $\mathrm{Hom}_{eAe}(-, \tau_{eAe}M)$ to the exact sequence $0 \to fJ \to fKe \to L \to 0$, we obtain the exact sequence

$$0 \longrightarrow \mathrm{Hom}_{eAe}(L, \tau_{eAe}M) \xrightarrow{\alpha} \mathrm{Hom}_{eAe}(fKe, \tau_{eAe}M) \longrightarrow$$
$$\longrightarrow \mathrm{Hom}_{eAe}(fJ, \tau_{eAe}M)$$

where $\mathrm{Hom}_{eAe}(fJ, \tau_{eAe}M) = \mathrm{Hom}_B(fJ, \tau_B M)$. Since, by Lemmas 7.2 and 7.6, fJ is an indecomposable injective B-module, it is generated by $\mathcal{C}^r$ but is not in $\mathcal{C}^r$. Invoking now the fact that $\mathcal{C}^r$ is a separating family of ray tubes of Γ_B, we obtain $\mathrm{Hom}_{eAe}(fJ, \tau_{eAe}M) = 0$, and consequently α is an isomorphism. Hence, by (i), we obtain $\mathrm{Hom}_{eAe}(fKe, \tau_{eAe}M) = 0$.

(3) Finally, applying $\mathrm{Hom}_{eAe}(M, -)$ to the canonical exact sequence $0 \to fKe \to fAe \to N \to 0$, we have the exact sequence

$$\mathrm{Hom}_{eAe}(M, fKe) \xrightarrow{\beta} \mathrm{Hom}_{eAe}(M, fAe) \longrightarrow$$
$$\longrightarrow \mathrm{Hom}_{eAe}(M, N) \longrightarrow \mathrm{Ext}^1_{eAe}(M, fKe),$$

where β is an isomorphism because $l_{fAe}(eIe) = fJ \subseteq fKe$ by Lemma 7.7. Further, $\mathrm{Ext}^1_{eAe}(M, fKe) \cong \mathrm{D}\,\overline{\mathrm{Hom}}_{eAe}(fKe, \tau_{eAe}M) = 0$ by (2). This implies that $\mathrm{Hom}_B(M, N) = \mathrm{Hom}_{eAe}(M, N) = 0$, contradicting the choice of M. Therefore, we have proved that $\nu(e)J = l_{\nu(e)Ae}(eIe)$. The proof of the second equality is dual. □

Lemma 7.9. *We have* $IeIe = 0$.

PROOF. Suppose that $IeIe \neq 0$. Then $\nu(e)IeIe \neq 0$ because $\mathrm{soc}({}_A IeIe) \cong \mathrm{top}(A\nu(e))$. But, by Lemma 7.8, we have $\nu(e)IeIe = \nu(e)JIe = 0$, a contradiction. Hence $IeIe = 0$. □

Lemma 7.10. *Let f be a primitive idempotent in I with $fAe \neq fJe$. Then* $\mathrm{Hom}_B(\mathcal{C}^r, fAe/fJe) \neq 0$ *and* $\mathrm{Hom}_B(eAf/eJ', \mathrm{D}(\mathcal{C}^r)) = 0$.

PROOF. Consider K and N as in Lemma 7.5. Observe that $fAeAfAe = (fAe)(eAfAe) \subseteq IeIe$. Since $IeIe = 0$, by Lemma 7.9, we then have $N = fAe/fKe = fAe/fJ$. The claim follows from Lemma 7.5, and from the left-right dual argument. □

Lemma 7.11. *Let f be a primitive idempotent in I such that $\nu^{-}(f) \in I$. We have* $\operatorname{Hom}_B(\mathcal{C}^r, fAe) = 0$.

Proof. We note that fAe is a right B-module, because $B \cong eAe/eIe$ and $(fAe)(eIe) \subseteq IeIe$ and $IeIe = 0$, by Lemma 7.9. As a restriction of the isomorphism $\mathrm{D}(A) \cong A_{\nu^-}$ of A-A-bimodules, we obtain the isomorphism $\mathrm{D}(fAe) \cong eA\nu^{-}(f)$ of left (eAe/eIe)-modules. Further, since $\operatorname{top}(A\nu^{-}(f)) \cong \operatorname{soc}(Af)$ as left A-modules and $f \in I$, we obtain $eJ'\nu^{-}(f) = 0$. Thus we have the isomorphism of left (eAe/eIe)-modules $eA\nu^{-}(f)/eJ'\nu^{-}(f) = eA\nu^{-}(f) \cong \mathrm{D}(fAe)$, where we note that $\nu^{-}(f) \in I$ and $eIe = eJe$, by Lemma 7.3. Consequently, it follows from Lemma 7.10 that $\operatorname{Hom}_B(\mathrm{D}(fAe), \mathrm{D}(\mathcal{C}^r)) = 0$, which implies $\operatorname{Hom}_B(\mathcal{C}^r, fAe) = 0$. □

Lemma 7.12. *Let f be a primitive idempotent from I. Then we have $fAe = fJe$ and $eAf = eJ'f$.*

Proof. It is enough to show the first equality. We assume $fAe \neq 0$, since the assertion is obvious in the case when $fAe = 0$. Suppose that $fAe \neq fJe$. Take K and N as in Lemma 7.5. Observe that, as in the proof of Lemma 7.10, we have $N = fAe/fKe = fAe/fJ$. Applying Lemma 7.5 we obtain $\operatorname{Hom}_B(\mathcal{C}^r, N) \neq 0$. Note that $\nu^{-}(f) \in I$. Indeed, if $\nu^{-}(f) \notin I$ then $fIe = fJ$, by Lemma 7.8, and hence $fJe = fAe$, a contradiction. But $\nu^{-}(f) \in I$ implies $fJ = 0$, because fJ is a right ideal of A, $JI = 0$ and $\operatorname{soc}(fJ) \cong \operatorname{top}(\nu^{-}(f)A)$ if $fJ \neq 0$. Therefore, $N = fAe/fJ = fAe$ and, applying Lemma 7.11, we get $\operatorname{Hom}_B(\mathcal{C}^r, fAe) = 0$, a contradiction to the fact established above. □

Now we are in position to prove the following crucial result.

Proposition 7.13. *We have $Ie = J$, $eI = J'$ and $eIe = J \cap J'$.*

Proof. Observe that $Ie = eIe \oplus (1-e)Ie$. From Lemma 7.3 we have $eIe = eJe = eJ$. Further, by Lemma 7.12, we obtain that $(1-e)Ie = (1-e)Ae = (1-e)Je = (1-e)J$, because $1-e \in I$ by the definition of e. Hence $IeI = 0$. Invoking Lemma 7.2, we then get $Ie \subseteq l_A(I) = J$, and so $Ie = J$. The equality $eI = eJ'$ follows in a similar way. Finally, observe that $J \cap J' = e(J \cap J')e = eJ \cap J'e = eIe$. □

Theorem 7.14. *I is a deforming ideal of A with $l_A(I) = Ie$ and $r_A(I) = eI$.*

Proof. From Lemma 7.2 and Proposition 7.13 we know that $l_A(I) = J = Ie$ and $r_A(I) = J' = eI$. In particular we have $IeI = 0$. Therefore from Proposition 5.2 we get $eIe = l_{eAe}(I) = r_{eAe}(I)$. Finally, $B = A/I$ is an almost concealed canonical algebra, and hence a quasitilted algebra. Then the ordinary quiver Q_B of B is acyclic, by [17, Proposition III.1.1]. This shows that I is a deforming ideal of A. □

We complete now the proof of the implication $(i) \Rightarrow (ii)$ of Theorem 1.1. We know that $I = \text{ann}_A(\mathcal{C}^r)$ is a deforming ideal of A, with $l_A(I) = Ie$, and $B = A/I$ is an almost concealed canonical algebra. Then it follows from Theorem 5.4 that the deformed selfinjective algebra $A[I]$ is isomorphic to the orbit algebra $\widehat{B}/(\psi\nu_{\widehat{B}})$ for some positive automorphism ψ of $\widehat{B}$. Moreover, by Theorem 5.3(ii), the algebras A and $A[I]$ are socle equivalent, and consequently the module categories $\text{mod}(A/\operatorname{soc} A)$ and $\text{mod}(A[I]/\operatorname{soc} A[I])$ coincide. We note also that the Auslander-Reiten quivers Γ_A and $\Gamma_{A[I]}$ are isomorphic. Then our assumption (i) on A forces that $\Gamma_{A[I]}$ admits a family $\mathcal{C}' = (\mathcal{C}'_i)_{i \in I}$ of quasitubes with common composition factors, closed under composition factors, and consisting of indecomposable $A[I]$-modules which do not lie on infinite short cycles in $\text{mod}\, A[I]$. Namely, for each $i \in I$, the quasitube $\mathcal{C}'_i$ is obtained from the quasitube $\mathcal{C}_i$ by replacing any indecomposable projective A-module P by the corresponding indecomposable projective $A[I]$-module P', and keeping the remaining indecomposable A-modules in $\mathcal{C}_i$. Then it follows from Propositions 6.4 and 6.5 that $G = (\psi\nu_{\widehat{B}})$ satisfies the conditions (ii) of Theorem 1.1. In particular, we conclude that $e_i \neq e_{\nu(i)}$ for any primitive summand e_i of the residual identity e. Applying Theorem 5.3(iii), we conclude that A and $A[I]$ are isomorphic k-algebras. Therefore, A is isomorphic to the orbit algebra $\widehat{B}/G$ with G satisfying the conditions (ii) of Theorem 1.1. This finishes the proof of the implication $(i) \Rightarrow (ii)$ of Theorem 1.1.

Acknowledgements

The authors gratefully acknowledge support from the Reserch Grant No. N N201 263 135 of Polish Ministry of Science and Higher Education and the Grant-in-Aid for Scientific Research (B) 21340003 of Japan. The research of the second and third named authors was supported through the programme "Research in Pairs" by the Mathematisches Forschungsinstitut Oberwolfach in 2009. Moreover the first named author gratefully acknowledges support from the Warmińsko-Mazurskie Przedsiębiorstwo Drogowe Sp. z o.o.

References

[1] I. Assem, J. Nehring, A. Skowroński, Domestic trivial extensions of simply connected algebras, Tsukuba J. Math. 13 (1989) 31–72.

[2] I. Assem, D. Simson, A. Skowroński, Elements of the Representation Theory of Associative Algebras 1: Techniques of Representation Theory, London Mathematical Society Student Texts 65, Cambridge University Press, Cambridge, 2006.

[3] I. Assem, A. Skowroński, Iterated tilted algebras of type $\tilde{\mathbb{A}}_n$, Math. Z. 195 (1987) 269–290.

[4] I. Assem, A. Skowroński, Multicoil algebras, in: Representations of Algebras, Canad. Math. Soc. Conference Proc. 14, Amer. Math. Soc., 1993, pp. 29–68.

[5] I. Assem, A. Skowroński, B. Tomé, Coil enlargements of algebras, Tsukuba J. Math. 19 (1995) 453–478.

[6] M. Auslander, Representation theory of artin algebras II, Comm. Algebra 1 (1974) 269–310.

[7] M. Auslander, I. Reiten, S.O. Smalø, Representation Theory of Artin Algebras, Cambridge Studies in Advenced Mathematics 36, Cambridge University Press, Cambridge, 1995.

[8] R. Bautista, On irreducible maps, Bull. Amer. Math. Soc. 2 (1980) 177–180.

[9] J. Białkowski, On the trivial extensions of tubular algebras, Colloq. Math. 101 (2004) 259–269.

[10] W. Crawley-Boevey, Regular modules for tame hereditary algebras, Proc. Lond. Math. Soc. 63 (1991) 241–265.

[11] V. Dlab, C.M. Ringel, Indecomposable representations of graphs and algebras, Memories Amer. Math. Soc. 6 (1976) no. 173.

[12] P. Dowbor, A. Skowroński, Galois coverings of representation-infinite algebras, Comment. Math. Helv. 62 (1987) 311–337.

[13] P. Gabriel, The universal cover of a representation-finite algebra, in: Representations of Algebras, Lecture Notes in Math. 903, Springer-Verlag, Berlin-Heidelberg, 1981, pp. 68–105.

[14] W. Geigle, H. Lenzing, A class of weighted projective curves arising in representation theory of finite dimensional algebras, in: Singularities, Representation of Algebras and Vector Bundles, Lecture Notes in Mathematics 1273, Springer-Verlag, Berlin-Heidelberg, 1987, pp. 265–297.

[15] D. Happel, Triangulated Categories in the Representation Theory of Finite Dimensional Algebras, London Mathematical Society Lecture Note Series 119, Cambridge Univerisity Press, Cambridge, 1988.

[16] D. Happel, I. Reiten, Hereditary abelian categories with tilting object over arbitrary basic fields, J. Algebra 256 (2002) 414–432.

[17] D. Happel, I. Reiten, S.O. Smalø, Tilting in abelian categories and quasitilted algebras, Memoires Amer. Math. Soc. 120 (1996) no. 575.

[18] D. Happel, C.M. Ringel, Tilted algebras, Trans. Amer. Math. Soc. 274 (1982) 399–443.

[19] D. Happel, C.M. Ringel, The derived category of a tubular algebra, in: Representation Theory I. Finite Dimensional Algebras, Lecture Notes in Mathematics 1177, Springer-Verlag, Berlin-Heidelberg, 1986, pp. 156–180.

[20] D. Hughes, J. Waschbüsch, Trivial extensions of tilted algebras, Proc. London Math. Soc. 46 (1983) 347–364.

[21] K. Igusa, G. Todorov, A characterization of finite Auslander-Reiten quivers, J. Algebra 89 (1984) 148–177.

[22] O. Kerner, A. Skowroński, Quasitilted one-point extensions of wild hereditary algebras, J. Algebra 244 (2001) 785–827.

[23] D. Kussin, Non-isomorphic derived-equivalent tubular curves and their associated tubular algebras, J. Algebra 226 (2000) 436–450.

[24] D. Kussin, On the K-theory of tubular algebras, Colloq. Math. 86 (2000) 137–152.

[25] D. Kussin, Noncommutative curves of genus zero related to finite dimensional algebras, Memoirs Amer. Math. Soc., 201 (2009) no. 942.

[26] H. Lenzing, A K-theoretic study of canonical algebras, in: Representation Theory of Algebras, Canad. Math. Soc. Conf. Proc. 18, Amer. Math. Soc., Providence, RI, 1996, pp. 433–454.

[27] H. Lenzing, H. Meltzer, Sheaves on a weighted projective line of genus one and representations of a tubular algebra, in: Representation Theory of Algebras, Canad. Math. Soc. Conf. Proc. 18, Amer. Math. Soc., Providence, RI, 1993, pp. 313–337.

[28] H. Lenzing, H. Meltzer, Tilting sheaves and concealed-canonical algebras, in: Representation Theory of Algebras, Canad. Math. Soc. Conf. Proc. 18, Amer. Math. Soc., Providence, RI, 1996, pp. 455–473.

[29] H. Lenzing, J.A. de la Peña, Wild canonical algebras, Math. Z. 224 (1997) 403–425.

[30] H. Lenzing, J.A. de la Peña, Concealed-canonical algebras and separating tubular families, Proc. Lond. Math. Soc. 78 (1999) 513–540.

[31] H. Lenzing, A. Skowroński, Quasi-tilted algebras of canonical type, Colloq. Math. 71 (1996) 161–181.

[32] H. Lenzing, A. Skowroński, Selfinjective algebras of wild canonical type, Colloq. Math. 96 (2003) 245–275.

[33] S. Liu, The degrees of irreducible maps and the shapes of the Auslander-Reiten quivers, J. London Math. Soc. 45 (1992) 32–54.

[34] P. Malicki, A. Skowroński, Almost cyclic coherent components of an Auslander-Reiten quiver, J. Algebra 229 (2000) 695–749.

[35] P. Malicki, A. Skowroński, Algebras with separating almost cyclic coherent Auslander-Reiten components, J. Algebra 291 (2005) 208–237.

[36] J. Nehring, A. Skowroński, Polynomial growth trivial extensions of simply connected algebras, Fund. Math. 132 (1989) 117–134.

[37] I. Reiten, A. Skowroński, Sincere stable tubes, J. Algebra 232 (2000) 64–75.

[38] I. Reiten, A. Skowroński, S.O. Smalø, Short chains and short cycles of modules, Proc. Amer. Math. Soc. 117 (1993) 343–354.

[39] C.M. Ringel, Representations of K-species and bimodules, J. Algebra 41 (1976) 269–302.

[40] C.M. Ringel, Tame Algebras and Integral Quadratic Forms, Lecture Notes in Math. 1099, Springer-Verlag, Berlin-Heidelberg, 1984.

[41] C.M. Ringel, The canonical algebras, with an appendix by W. Crawley-Boevey, Topics in Algebra, Banach Center Publications 26, Part 1, PWN, Warsaw (1990) 407–432.

[42] D. Simson, A. Skowroński, Elements of the Representation Theory of Associative Algebras 2: Tubes and Concealed Algebras of Euclidean Type, London Mathematical Society Student Texts 71, Cambridge University Press, Cambridge, 2007.

[43] D. Simson, A. Skowroński, Elements of the Representation Theory of Associative Algebras 3: Representation-Infinite Tilted Algebras, London Mathematical Society Student Texts 72, Cambridge University Press, Cambridge, 2007.

[44] A. Skowroński, Selfinjective algebras of polynomial growth, Math. Ann. 285 (1989) 177–199.

[45] A. Skowroński, Generalized standard Auslander-Reiten components, J. Math. Soc. Japan 46 (1994) 517–543.

[46] A. Skowroński, Minimal representation-infinite artin algebras, Math. Proc. Cambridge Phil.. Soc. 116 (1994) 229–243.

[47] A. Skowroński, On the composition factors of periodic modules, J. London Math. Soc. 49 (1994) 477–492.

[48] A. Skowroński, On omnipresent tubular families of modules, in: Representation Theory of Algebras, Canad. Math. Soc. Conf. Proc. 18, Amer. Math. Soc., Providence, RI, 1996, pp. 641–657.

[49] A. Skowroński, Generalized canonical algebras and standard stable tubes, Colloq. Math. 90 (2001) 77–93.

[50] A. Skowroński, A construction of complex syzygy periodic modules over symmetric algebras, Colloq. Math. 103 (2005) 61–69.

[51] A. Skowroński, Selfinjective algebras: finite and tame type, in: Trends in Representation Theory of Algebras and Related Topics, Contemporary Mathematics 406, Amer. Math. Soc., Providence, RI, 2006, pp. 169–238.

[52] A. Skowroński, K. Yamagata, Socle deformations of self-injective algebras, Proc. London Math. Soc. 72 (1996) 545–566.

[53] A. Skowroński, K. Yamagata, Stable equivalence of selfinjective algebras of tilted type, Arch. Math. (Basel) 70 (1998) 341–350.

[54] A. Skowroński, K. Yamagata, Galois coverings of selfinjective algebras by repetitive algebras, Trans. Amer. Math. Soc. 351 (1999) 715–734.

[55] A. Skowroński, K. Yamagata, On selfinjective artin algebras having nonperiodic generalized standard Auslander-Reiten components, Colloq. Math. 96 (2003) 235–244.

[56] A. Skowroński, K. Yamagata, On invariability of selfinjective algebras of tilted type under stable equivalences, Proc. Amer. Math. Soc. 132 (2004) 659–667.

[57] A. Skowroński, K. Yamagata, Positive Galois coverings of selfinjective algebras, Adv. Math. 194 (2005) 398–436.

[58] A. Skowroński, K. Yamagata, Selfinjective algebras of quasitilted type, in: Trends in Representation Theory of Algebras and Related Topics, European Math. Soc., Series of Congress Reports, European Math. Soc. Publ. House, Zurich, 2008, pp. 639–708.

[59] A. Skowroński, G. Zwara, Degeneration-like orders on the additive categories of generalized standard Auslander-Reiten components, Arch. Math. 74 (2000) 11–21.

[60] K. Yamagata, Frobenius algebras, Handbook of Algebra 1, Elsevier, North-Holland, Amsterdam, 1996.

[61] Y. Zhang, The structure of stable components, Canad. J. Math. 43 (1991) 652–672.