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GREGORY A. CHECHKIN, TARAS A. MEL'NYK

Asymptotic Behavior of the Eigenvalues and Eigenfunctions to a Spectral Problem in Thick Cascade Junction with Concentrated Masses

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Asymptotic Behavior of the Eigenvalues and Eigenfunctions to a Spectral Problem in a Thick Cascade Junction with Concentrated Masses

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Abstract

The asymptotic behavior (as $\varepsilon \to 0$) of eigenvalues and eigenfunctions of a boundary-value problem for the Laplace operator in a thick cascade junction with concentrated masses is investigated. This cascade junction consists of the junction's body and great number $5N = \mathcal{O}(\varepsilon^{-1})$ of $\varepsilon-$alternating thin rods belonging to two classes. One class consists of rods of finite length and the second one consists of rods of small length of order $\mathcal{O}(\varepsilon)$. The density of the junction is order $\mathcal{O}(\varepsilon^{-\alpha})$ on the rods from the second class (the concentrated masses if $\alpha > 0$), and $\mathcal{O}(1)$ outside of them. In addition, we study the influence of the concentrated masses on the asymptotic behavior of these magnitudes in the case $\alpha = 1$ and $\alpha \in (0, 1)$.

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1 Introduction

In present paper we continue our investigation of boundary-value problems in a new kind of thick junctions, namely *thick cascade junctions*, which we have begun in [9, 10], see also [11] and [12].

Boundary-value problems in thick one-level junctions (thick junctions) are intensively investigated recently (see for instance [2], [3], [41] and references there).

Here we study a spectral problem in a thick cascade junction. It is known that the asymptotic behavior of the spectrum of a perturbed spectral problem is highly sensitive to the perturbation and it is unexpected; in thick junction it essentially depends on the junction type and on the conditions given on the boundaries of the attached thin domains. This dependence was observed for spectral problems in thick junctions with the Neumann conditions in [25, 26, 28, 29, 32, 33], with the Dirichlet conditions in [30, 36], with the Fourier conditions in [35], with the Steklov ones in [34], and for spectral problems in thick multi-level junctions in [37, 38].

Vibration systems with a concentration of masses on a small set of diameter $\mathcal{O}(\varepsilon)$ have been studied for a long time. It was experimentally established that such concentration leads to the big reduction of the main frequency and to the big localization of vibrations. The new impulse in this research was given by E. Sánchez-Palencia in the paper [47], in which the effect of local vibrations was mathematically described. After this paper, many articles appeared (see for example [5]–[8], [13, 14, 21, 22, 23, 43]) that deal with the asymptotic behavior of vibrations of a body containing a small region (many small regions) where the density is very much higher than elsewhere (see [25, 30, 33, 36] for thick junctions).

1. Statement of the problem.

Let $a,\ b_1,\ b_2,\ h_1,\ h_2$ be positive numbers such that

$$0 < b_1 < b_2 < \frac{1}{2}, \quad 0 < b_1 - \frac{h_1}{2}, \quad b_1 + \frac{h_1}{2} < b_2 - \frac{h_1}{2}, \quad b_2 + \frac{h_1}{2} < \frac{1}{2} - \frac{h_2}{2}.$$

These inequalities mean that the intervals

$$\left(b_1 - \frac{h_1}{2}\,,\, b_1 + \frac{h_1}{2}\right), \quad \left(b_2 - \frac{h_1}{2}\,,\, b_2 + \frac{h_1}{2}\right), \quad \left(\frac{1-h_2}{2}\,,\, \frac{1+h_2}{2}\right),$$
$$\left(1 - b_2 - \frac{h_1}{2}\,,\, 1 - b_2 + \frac{h_1}{2}\right), \quad \left(1 - b_1 - \frac{h_1}{2}\,,\, 1 - b_1 + \frac{h_1}{2}\right)$$

are not intersected and they belong to $(0,1)$. Let us divide the segment $[0,a]$ into N equal segments $[\varepsilon j, \varepsilon(j+1)],\ j = 0, \ldots, N-1$. Here N is a big positive integer, hence the value $\varepsilon = a/N$ is a small discrete parameter.

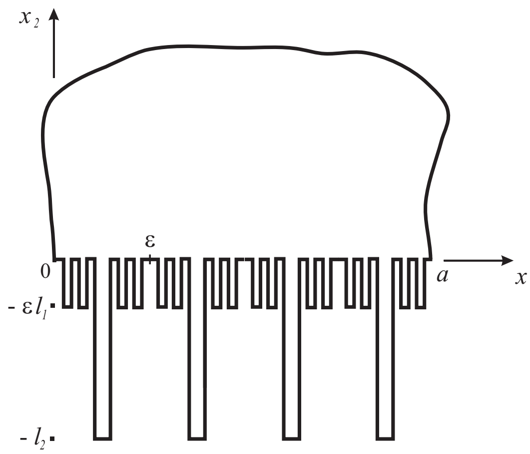

Figure 1: The thick cascade junction Ω_ε.

A model *thick cascade junction* Ω_ε (see Fig. 1) consists of the junction's body

$$\Omega_0 = \{x \in \mathbb{R}^2 :\ 0 < x_1 < a, \quad 0 < x_2 < \gamma(x_1)\},$$

where $\gamma \in C^1([0,a]),\ \ \min_{[0,a]} \gamma > 0,$ and a large number of thin rods

$$G_j^{(1)}(d_k, \varepsilon) = \left\{ x \in \mathbb{R}^2 :\ |x_1 - \varepsilon\,(j + d_k)| < \frac{\varepsilon h_1}{2}, \quad x_2 \in (-\varepsilon\, l_1, 0] \right\}, \quad k = 1, \ldots, 4,$$

$$G_j^{(2)}(\varepsilon) = \left\{ x \in \mathbb{R}^2 : \ |x_1 - \varepsilon\,(j + \frac{1}{2})| < \frac{\varepsilon h_2}{2}, \quad x_2 \in (-l_2, 0] \right\}, \quad j = 0, 1, \ldots, N-1,$$

where $d_1 = b_1,\ d_2 = b_2,\ d_3 = 1 - b_2,\ d_4 = 1 - b_1,$ that is $\Omega_\varepsilon = \Omega_0 \cup G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}$, where

$$G_\varepsilon^{(1)} = \bigcup_{j=0}^{N-1} \Big(\bigcup_{k=1}^{4} G_j^{(1)}(d_k, \varepsilon) \Big), \qquad G_\varepsilon^{(2)} = \bigcup_{j=0}^{N-1} G_j^{(2)}(\varepsilon).$$

Thus the number of the thin rods is equal to $5N$; the thin rods are divided into two classes $G_\varepsilon^{(1)}$ and $G_\varepsilon^{(2)}$ subject to their length and thickness. The length and thickness of the rods from the first class are equal to εl_1 and εh_1 respectively, and these magnitudes are equal to l_2 and εh_2 for the rods from the second class. In addition, the thin rods from each classes are ε-periodically alternated along the segment $I_0 = \{x : x_1 \in [0, a], \quad x_2 = 0\}$.

Such thick cascade junctions are prototypes of widely used engineering, physical and biological systems with very distinct characteristic scales, for instance construction of a bowel with different levels of absorption on various parts of the bowel trunks, construction of an animal's fell consisting of wool and undercoat with different thermal conductivities.

Only vibrations of Ω_ε depending on time by the factor $\exp(-i\sqrt{\lambda}\ t)$ will be considered. Hence we have to investigate the corresponding spectral problem

$$\begin{array}{rcll} -\Delta_x\, u(\varepsilon, x) & = & \lambda(\varepsilon)\ \rho_\varepsilon(x) u(\varepsilon, x), & x \in \Omega_\varepsilon; \\ -\partial_\nu u(\varepsilon, x) & = & 0, & x \in \Upsilon_\varepsilon^{(1)} \cup \Upsilon_\varepsilon^{(2)} \cup \Gamma_\varepsilon; \\ u(\varepsilon, x) & = & 0, & x \in \Gamma_1; \\ [u]_{|x_2=0} & = & [\partial_{x_2} u]_{|x_2=0} = 0, & x_1 \in Q_\varepsilon = \Big(G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)} \Big) \cap \{x_2 = 0\}. \end{array} \tag{1.1}$$

Here $\partial_\nu = \partial/\partial\nu$ is the outward normal derivative; the brackets denote the jump of the enclosed quantities; $\Upsilon_\varepsilon^{(i)}$ is the union of the lateral sides and the lower bases of the rods from the $i-$th class, $i = 1, 2;$ $\Gamma_1 = \{x : \ x_2 = \gamma(x_1),\ x_1 \in [0, a]\};$ $\Gamma_\varepsilon = \partial\Omega_\varepsilon \setminus \big(\Upsilon_\varepsilon^{(1)} \cup \Upsilon_\varepsilon^{(2)} \cup \Gamma_1\big);$ the density

$$\rho_\varepsilon(x) = \left\{ \begin{array}{ll} 1, & x \in \Omega_0 \cup G_\varepsilon^{(2)}, \\ \varepsilon^{-\alpha}, & x \in G_\varepsilon^{(1)}; \end{array} \right. \tag{1.2}$$

the parameter $\alpha \in (-\infty, 2)$.

Thus, the Neumann conditions are imposed on the boundaries of the thin rods and if $\alpha > 0$ then there are concentrated masses on the thin rods from the first class $G_\varepsilon^{(1)}$.

It is known that for each fixed value of ε there is a sequence of eigenvalues of problem (1.1)

$$0 < \lambda_1(\varepsilon) < \lambda_2(\varepsilon) \le \ldots \le \lambda_n(\varepsilon) \le \cdots \to +\infty \quad \text{as} \quad n \to \infty \tag{1.3}$$

and a sequence of the corresponding eigenfunctions $\{u_n(\varepsilon, \cdot) : \ n \in \mathbb{N}\}$, which can be orthonormalized by the following way

$$(u_n, u_m)_{L^2(\Omega_0 \cup G_\varepsilon^{(2)})} + \varepsilon^{-\alpha} (u_n, u_m)_{L^2(G_\varepsilon^{(1)})} = \delta_{n,m}, \quad \{n,\, m\} \in \mathbb{N}. \tag{1.4}$$

Here and below $\delta_{n,m}$ is the Kronecker delta.

Our aim is to study the asymptotic behavior of the eigenvalues $\{\lambda_n(\varepsilon):\ n\in\mathbb{N}\}$ and the eigenfunctions $\{u_n(\varepsilon,\cdot):\ n\in\mathbb{N}\}$ as $\varepsilon\to 0$, i.e., when the number of the attached thin rods from each class infinitely increases and their thickness decreases to zero, to find other limiting points of the spectrum of problem (1.1) and to describe corresponding eigenvibrations.

It should be noted that the limit process is accompanied by the concentrated masses on the rods from the first class. In fact, we have two kinds of perturbations for problem (1.1): the domain perturbation and the density perturbation. We are going to study the influence of both factors on the asymptotic behavior of the eigenvalues and eigenfunctions as well.

We establish five qualitatively different cases in the asymptotic behavior eigenvalues and eigenfunctions of problem (1.1) as $\varepsilon\to 0$, namely $\alpha\in(0,1)$, $\alpha=1$, $\alpha\in(1,2)$, $\alpha=2$, $\alpha>2$. In the present paper we consider two cases $\alpha\in(0,1)$ and $\alpha=1$.

2 The case $\alpha = 1$

2.1 Formal Asymptotics

Combining the algorithm of constructing asymptotics in thin domains with the methods of homogenization theory, we seek the main terms of the asymptotics for the eigenvalue $\lambda_n(\varepsilon)$ and the eigenfunction $u_n(\varepsilon,\cdot)$ in the form (index n is omitted):

$$\lambda(\varepsilon)\approx\lambda_0+\varepsilon\lambda_1+\ldots \tag{2.1}$$

$$u(\varepsilon,x)\approx v_0^+(x)+\varepsilon v_1^+(x)+\ldots,\quad \text{in domain } \Omega_0; \tag{2.2}$$

in the thin rectangle $G_j^{(2)}(\varepsilon)$ $(j=0,\ldots,N-1)$

$$u(\varepsilon,x)\approx v_0^-(x_1,x_2,\eta_1-j)+\varepsilon v_1^-(x_1,x_2,\eta_1-j)+\ldots,\qquad \eta_1=\frac{x_1}{\varepsilon}; \tag{2.3}$$

and in the junction zone of the body and thin rectangles of both classes (which we call internal expansion) the series of the following type:

$$\begin{aligned} u(\varepsilon,x)\approx{}& v_0^+(x_1,0)+\varepsilon\Big(\sum_{i=1}^{2}Z_1^{(i)}(\eta)\partial_{x_i}v_0^+(x_1,0)+Z_1^{(0)}(\eta)v_0^+(x_1,0)\Big)+\\ &+\varepsilon^2\sum_{|\beta|\le 2}Z_2^{(\beta)}(\eta)D^\beta v_0^+(x_1,0)+\ldots,\qquad \eta=\frac{x}{\varepsilon}. \end{aligned} \tag{2.4}$$

We used the following standard notation: $\beta=(\beta_1,\beta_2)$, $|\beta|=\beta_1+\beta_2$, $\beta_i\in\mathbb{N}_0$, $D^\beta=\dfrac{\partial^{|\beta|}}{\partial x_1^{\beta_1}\partial x_2^{\beta_2}}$ and $\partial_{x_i}=\dfrac{\partial}{\partial x_i}$.

Denote $\Gamma_2:=\partial\Omega_0\backslash(\Gamma_1\cup I_0)$. Substituting (2.1) and (2.2) in the problem (1.1) and collecting terms with equal order of ε, we get:

$$\begin{array}{rcll} -\Delta_x\ v_0^+(x) &=& \lambda_0 v_0^+(x), & x\in\Omega_0,\\ \partial_\nu\ v_0^+(x) &=& 0, & x\in\Gamma_2,\\ v_0^+(x) &=& 0, & x\in\Gamma_1. \end{array} \tag{2.5}$$

It remains to ensure the continuity of the asymptotic approximations on the interfaces between the “rectangles” and the “body”. The necessity of the condition

$$v_0^+(x_1,0) = v_0^-(x_1,0), \qquad x \in I_0, \tag{2.6}$$

is evident. Another condition appears when one constructs the junction layer. This condition has the form

$$\partial_{x_2} v_0^+(x_1,0) - h_2 \partial_{x_2} v_0^-(x_1,0) = -4h_1 l_1 \lambda_0 v_0^+(x_1,0), \qquad x \in I_0, \tag{2.7}$$

and will be obtained in the next section.

Collecting terms of order ε, we have

$$\begin{aligned} -\Delta_x\, v_1^+(x) &= \lambda_0 v_1^+(x) + \lambda_1 v_0^+(x), && x \in \Omega_0, \\ \partial_\nu\, v_1^+(x) &= 0, && x \in \Gamma_2, \\ v_1^+(x) &= 0, && x \in \Gamma_1. \end{aligned} \tag{2.8}$$

In the transmission conditions here the following jumps appear

$$v_1^+(x_1,0) - v_1^-(x_1,0) = \mathcal{F}_1(x_1), \qquad x \in I_0, \tag{2.9}$$

and

$$\partial_{x_2} v_1^+(x_1,0) - h_2 \partial_{x_2} v_1^-(x_1,0) = \mathcal{F}_2(x_1), \qquad x \in I_0, \tag{2.10}$$

where $\mathcal{F}_1$, $\mathcal{F}_2$ are given functions on I_0 that will be defined in subsection 2.2.

2.1.1 Formal asymptotics on thin rectangles.

Keeping in mind that in (2.3) v_k^- are smooth functions, using Taylor series for v_k^- and changing variable $x_1 \mapsto \eta_1$ in the neighborhood of the points $x_1 = \varepsilon(j+\frac{1}{2})$, we get

$$u(\varepsilon,x) = \sum_{k=0}^{+\infty} \varepsilon^k W_k^{(j)}(x_2,\eta_1), \quad x \in G_j^{(2)}(\varepsilon), \tag{2.11}$$

where for $k \in \mathbb{N}$ we have

$$W_k^{(j)}(x_2,\eta_1) = v_k^-(\varepsilon(j+\frac{1}{2}), x_2, \eta_1 - j) + \\ + \sum_{m=1}^{k} \frac{1}{m!}\left(\eta_1 - j - \frac{1}{2}\right)^m \frac{\partial^m v_{k-m}^-}{\partial x_1^m}\left(\varepsilon(j+\frac{1}{2}), x_2, \eta_1 - j\right). \tag{2.12}$$

Substituting (2.1) and (2.11) in the problem (1.1) instead of $\lambda_n(\varepsilon)$ and $u_n(\varepsilon,\cdot)$ respectively, collecting terms with equal powers of ε, we obtain the following problems $(k = 0,1,2,3)$:

$$\begin{aligned} -\partial^2_{\eta_1\eta_1} W_k^{(j)}(x_2,\eta_1) &= \partial^2_{x_2x_2} W_{k-2}^{(j)}(x_2,\eta_1) + \sum_{m=0}^{k-2} \lambda_m W_{k-2-m}^{(j)}(x_2,\eta_1), \quad |\eta_1 - \tfrac{1}{2}| < \tfrac{h_2}{2}, \\ \partial_{\eta_1} W_k^{(j)}(x_2, \tfrac{1\pm h_2}{2}) &= 0, \end{aligned} \tag{2.13}$$

where λ_p and the functions $W_p^{(j)}$ with negative p are equal to zero; the variable x_2 is a parameter; $\partial_{\eta_1} = \frac{\partial}{\partial \eta_1}$.

From (2.13) we deduce that $W_0^{(j)}$, $W_1^{(j)}$, $W_2^{(j)}$ and $W_3^{(j)}$ are independent of η_1. Moreover the solvability conditions for the problem (2.13) as $k = 2, 3$, give us the equations

$$h_2\, \partial^2_{x_2x_2} v_0^-(x_1, x_2) + \lambda_0 h_2\, v_0^-(x_1, x_2) = 0, \quad x_2 \in (-l_2, 0),\ x_1 = \varepsilon(j + \frac{1}{2}) \tag{2.14}$$

and

$$h_2\, \partial^2_{x_2x_2} v_1^-(x_1, x_2) + h_2\lambda_0\, v_1^-(x_1, x_2) = -h_2\lambda_1\, v_0^-(x_1, x_2), \quad x_2 \in (-l_2, 0),\ x_1 = \varepsilon(j + \frac{1}{2}). \tag{2.15}$$

Since we seek the smooth functions v_0^- and v_1^- and the points $x_1 = \varepsilon(j + \frac{1}{2})$ form the ε-net in the interval $(0, a)$, then the equations (2.14), (2.15) defined on N segments can be extended to the whole rectangle $D_2 = (0, a) \times (-l_2, 0)$. Bearing in mind the boundary conditions of the original problem, we add

$$\partial_{x_2} v_0^-(x_1, -l_2) = 0, \qquad \partial_{x_2} v_1^-(x_1, -l_2) = 0. \tag{2.16}$$

2.1.2 Junction-layer solutions

Pass to the "fast" variables $\eta = \dfrac{x}{\varepsilon}$ in (1.1). Under this transformation as $\varepsilon \to 0$ the domain Ω_0 transforms to $\{\eta :\ \eta_i > 0,\ i = 1, 2\}$, the thin rectangle $G_0^{(2)}(\varepsilon)$ to the semistrip

$$\Pi^- = \left(\frac{1}{2} - \frac{h_2}{2}, \frac{1}{2} + \frac{h_2}{2}\right) \times (-\infty, 0]$$

and rectangle $G_0^{(1)}(d_k, \varepsilon)$ to the fixed rectangle

$$\Pi_k = \left(d_k - \frac{h_1}{2}, d_k + \frac{h_1}{2}\right) \times (-l_1, 0].$$

Taking into account the periodic structure of Ω_ε in a neighborhood of I_0, we take the following cell of periodicity

$$\Pi = \Pi^- \cup \Pi^+ \cup \Pi_{l_1},$$

in which we will consider boundary value problems. Here $\Pi^+ = (0, 1) \times (0, +\infty)$, $\Pi_{l_1} := \bigcup_{k=1}^{4} \overline{\Pi}_k$ (see Fig.2).

Substituting the series (2.4) and (2.1) in the problem (1.1) and collecting terms with equal powers of ε, we get problems for $Z_1^{(i)}$, $i = 0, 1, 2$, and $Z_2^{(\beta)}$, $|\beta| \le 2$. Obviously, these solutions have to be 1-periodic in η_1. Therefore we will demand the following periodic conditions

$$\partial^s_{\eta_1} Z(0, \eta_2) = \partial^s_{\eta_1} Z(1, \eta_2), \qquad \eta_2 > 0, \quad s = 0, 1, \tag{2.17}$$

on the vertical sides of semistrip Π^+. In addition, it is easy to see that all these solutions must satisfy the Neumann conditions

$$\partial_{\eta_2} Z(\eta_1, 0) = 0, \quad (\eta_1, 0) \in \partial\Pi, \quad \partial_{\eta_2} Z(\eta_1, -l_1) = 0, \quad (\eta_1, -l_1) \in \partial\Pi, \tag{2.18}$$

on the horizontal parts of the boundary of Π.

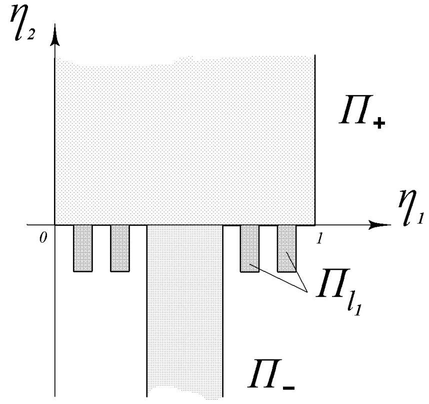

Figure 2: The cell of periodicity.

Denote by $\partial\Pi_{\|}$ the vertical part of $\partial\Pi$ laying in $\{\eta: \ \eta_2 < 0\}$.

Thus for $Z_1^{(i)}, \ i = 0,1,2$, and $Z_2^{(\beta)}, \ |\beta| \le 2$, we have the following problems (to all problems we must add conditions (2.17) and (2.18)):

$$\left\{ \begin{array}{rcl} -\Delta_\eta \ Z_1^{(0)}(\eta) & = & \left\{ \begin{array}{ll} 0, & \eta \in \Pi^+ \cup \Pi^-, \\ \lambda_0, & \eta \in \Pi_{l_1}, \end{array} \right. \\ \partial_{\eta_1} Z_1^{(0)}(\eta) & = & 0, \qquad\qquad \eta \in \partial\Pi_{\|}; \end{array} \right. \tag{2.19}$$

$$\left\{ \begin{array}{rcll} -\Delta_\eta \ Z_1^{(i)}(\eta) & = & 0, & \eta \in \Pi, \\ \partial_{\eta_1} Z_1^{(i)}(\eta) & = & -\delta_{1i}, & \eta \in \partial\Pi_{\|}, \quad i = 1,2; \end{array} \right. \tag{2.20}$$

$$\left\{ \begin{array}{rcl} -\Delta_\eta \ Z_2^{(0,0)}(\eta) & = & \left\{ \begin{array}{ll} \lambda_0, & \eta \in \Pi^+ \cup \Pi^-, \\ \lambda_1 + \lambda_0 Z_1^{(0)}(\eta), & \eta \in \Pi_{l_1}, \end{array} \right. \\ \partial_{\eta_1} Z_2^{(0,0)}(\eta) & = & 0, \qquad\qquad \eta \in \partial\Pi_{\|}; \end{array} \right. \tag{2.21}$$

$$\left\{ \begin{array}{rcl} -\Delta_\eta \ Z_2^{(1,0)}(\eta) & = & \left\{ \begin{array}{ll} 2\partial_{\eta_1} Z_1^{(0)}(\eta), & \eta \in \Pi^+ \cup \Pi^-, \\ 2\partial_{\eta_1} Z_1^{(0)}(\eta) + \lambda_0 Z_1^{(1)}(\eta), & \eta \in \Pi_{l_1}, \end{array} \right. \\ \partial_{\eta_1} Z_2^{(1,0)}(\eta) & = & -Z_1^{(0)}(\eta), \qquad\qquad \eta \in \partial\Pi_{\|}; \end{array} \right. \tag{2.22}$$

$$\left\{ \begin{array}{rcl} -\Delta_\eta \ Z_2^{(0,1)}(\eta) & = & \left\{ \begin{array}{ll} 0, & \eta \in \Pi^+ \cup \Pi^-, \\ \lambda_0 Z_1^{(2)}(\eta), & \eta \in \Pi_{l_1}, \end{array} \right. \\ \partial_{\eta_1} Z_2^{(0,1)}(\eta) & = & 0, \qquad\qquad \eta \in \partial\Pi_{\|}; \end{array} \right. \tag{2.23}$$

$$\left\{ \begin{array}{rcll} -\Delta_\eta \ Z_2^{(0,2)}(\eta) & = & 0, & \eta \in \Pi, \\ \partial_{\eta_1} Z_2^{(0,2)}(\eta) & = & 0, & \eta \in \partial\Pi_{\|}; \end{array} \right. \tag{2.24}$$

$$\left\{ \begin{array}{rcll} -\Delta_\eta \ Z_2^{(1,1)}(\eta) & = & 2\partial_{\eta_1} Z_1^{(2)}(\eta), & \eta \in \Pi, \\ \partial_{\eta_1} Z_2^{(1,1)}(\eta) & = & -Z_1^{(2)}(\eta), & \eta \in \partial\Pi_{\|}; \end{array} \right. \tag{2.25}$$

$$\begin{cases} -\Delta_\eta \, Z_2^{(2,0)}(\eta) = 1 + 2\partial_{\eta_1} Z_1^{(1)}(\eta), & \eta \in \Pi, \\ \partial_{\eta_1} Z_2^{(2,0)}(\eta) = -Z_1^{(1)}(\eta), & \eta \in \partial\Pi_{\|}. \end{cases} \tag{2.26}$$

The existence and the main asymptotic relations for the functions $\{Z_1^{(i)}\}$, $\{Z_2^{(\beta)}\}$ can be obtained from general results about the asymptotic behavior of solutions to elliptic problems in domains with different exits to infinity [17, 19, 20, 44]. The proofs are substantially simplified if the polynomial property of the corresponding quasilinear forms is employed [45]. However, if a domain, where we consider a boundary-value problem, has some symmetry, then we can define more exactly the asymptotic relations and detect other properties of junction-layer solutions (see Lemma 4.1 and Corollary 4.1 from [31]). Using this approach, one can prove the following lemma.

Lemma 2.1. *There exist solutions* $Z_1^{(i)} \in H^1_{loc,\eta_2}(\Pi)$, $i = 0, 1, 2$, *of the problems* (2.19), (2.20) *and* $Z_2^{(\beta)} \in H^1_{loc,\eta_2}(\Pi)$, $|\beta| \le 2$ *of the problems* (2.21), (2.22), (2.23), (2.24), (2.25), (2.26), *which have the following differentiable asymptotics*

$$Z_1^{(0)}(\eta) = \begin{cases} C_1^{(0)} + \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{4h_1 l_1 \lambda_0}{h_2}\,\eta_2 - \dfrac{C_1^{(0)}}{h_2} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.27}$$

$$Z_1^{(1)}(\eta) = \begin{cases} \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \Big(-\eta_1 + \dfrac{1}{2}\Big) + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.28}$$

$$Z_1^{(2)}(\eta) = \begin{cases} \eta_2 + C_1^{(2)} + \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{\eta_2}{h_2} - \dfrac{C_1^{(2)}}{h_2} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.29}$$

$$Z_2^{(0,0)}(\eta) = \begin{cases} -\dfrac{\lambda_0}{2}\eta_2^2 + C_2^{(0,0)} + \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ -\dfrac{\lambda_0}{2}\eta_2^2 + \dfrac{4h_1 l_1 \lambda_1 + \lambda_0 \int_{\Pi_{l_1}} Z_1^{(0)}(\eta)d\eta}{h_2}\,\eta_2 - \dfrac{C_2^{(0,0)}}{h_2} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.30}$$

$$Z_2^{(1,0)}(\eta) = \begin{cases} C_2^{(1,0)} + \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{4h_1 l_1 \lambda_0}{h_2}\,\eta_2\Big(-\eta_1 + \dfrac{1}{2}\Big) - \dfrac{C_2^{(1,0)}}{h_2} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.31}$$

$$Z_2^{(0,1)}(\eta) = \begin{cases} C_2^{(0,1)} + \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{\lambda_0 \int_{\Pi_{l_1}} Z_1^{(2)}(\eta)d\eta}{h_2}\,\eta_2 - \dfrac{C_2^{(0,1)}}{h_2} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.32}$$

$$Z_2^{(0,2)}(\eta) = \begin{cases} \eta_2 + C_1^{(2)} + \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{\eta_2}{h_2} - \dfrac{C_1^{(2)}}{h_2} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.33}$$

$$Z_2^{(1,1)}(\eta) = \begin{cases} C_2^{(1,1)} + \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{\eta_2}{h_2}\Big(-\eta_1 + \dfrac{1}{2}\Big) - \dfrac{C_2^{(1,1)}}{h_2} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.34}$$

$$Z_2^{(2,0)}(\eta) = \begin{cases} -\dfrac{1}{2}\eta_2^2 + C_2^{(2,0)} + \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ \dfrac{\mu_0}{h_2}\eta_2 - \dfrac{C_2^{(2,0)}}{h_2} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty, \end{cases} \tag{2.35}$$

where

$$\mu_0 = 2\int\limits_{\Pi^+} \partial_{\eta_1} Z_1^{(1)}(\eta)\, d\eta + \int\limits_{\Pi_{l_1}\cup\Pi^-} (1 + \partial_{\eta_1} Z_1^{(1)}(\eta))\, d\eta. \tag{2.36}$$

Moreover functions $Z_1^{(1)}$, $Z_2^{(1,0)}$, $Z_2^{(1,1)}$ are odd in η_1 with respect to $\frac{1}{2}$; functions $Z_1^{(0)}$, $Z_1^{(2)}$, $Z_2^{(0,0)}$, $Z_2^{(0,1)}$, $Z_2^{(0,2)}$ and $Z_2^{(2,0)}$ are even in η_1 with respect to $\frac{1}{2}$.

Proof. Recall that a function Ψ belongs to $H^1_{loc,\eta_2}(\Pi)$ if for every $R > 0$ the function $\Psi \in H^1(\Pi) \cap \{\eta : |\eta_2| < R\}$.

We will demonstrate this proof for the junction-layer problem (2.21). In the other cases the proof is similar. We look for the solution $Z_2^{(0,0)}$ to problem (2.21) in the form

$$Z_2^{(0,0)}(\eta) = -\frac{\lambda_0}{2}\eta_2^2 + \mu\, \eta_2 \chi_-(\eta_2) + \widetilde{Z}_2^{(0,0)}(\eta), \qquad \eta \in \Pi,$$

where $\chi_-(\eta_2)$ is a smooth cut-off function such that $0 \le \chi_-(\eta_2) \le 1$; it is equal to 1 if $\eta_2 \le -2$, and to 0 if $\eta_2 \ge -1$. It is easy to see that $\widetilde{Z}_2^{(0,0)}$ must satisfy the problem

$$\begin{cases} -\Delta_\eta\, \widetilde{Z}_2^{(0,0)}(\eta) = \begin{cases} 0, & \eta \in \Pi^+, \\ \mu\big(\eta_2\chi''_-(\eta_2) + 2\chi'_-(\eta_2)\big), & \eta \in \Pi^-, \\ \lambda_1 + \lambda_0\big(Z_1^{(0)}(\eta) - 1\big), & \eta \in \Pi_{l_1}, \end{cases} \\ \partial^s_{\eta_1} \widetilde{Z}_2^{(0,0)}(0, \eta_2) = \partial^s_{\eta_1} \widetilde{Z}_2^{(0,0)}(1, \eta_2), \quad \eta_2 > 0, \quad s = 0, 1, \\ \partial_{\eta_2} \widetilde{Z}_2^{(0,0)}(\eta_1, 0) = 0, \quad (\eta_1, 0) \in \partial\Pi, \\ \partial_{\eta_1} \widetilde{Z}_2^{(0,0)}(\eta) = 0, \quad \eta \in \partial\Pi_{\|}, \quad \eta_2 < 0, \\ \partial_{\eta_2} \widetilde{Z}_2^{(0,0)}(\eta_1, -l_1) = -\lambda_0 l_1, \quad (\eta_1, -l_1) \in \partial\Pi. \end{cases} \tag{2.37}$$

From Lemma 4.1 (see paper [31]) it follows that there exists the energy solution to the problem (3.42) if and only if

$$\mu = \frac{4h_1 l_1 \lambda_1 + \lambda_0 \int_{\Pi_{l_1}} Z_1^{(0)}(\eta) d\eta}{h_2}; \tag{2.38}$$

in addition this solution is defined up to an additive constant. Choosing in an appropriate way this constant (see Remark 4.1 from [31]), we get the asymptotics (2.30).

Since the right-hand sides both in the equation and boundary conditions of problem (3.42) are even in η_1 with respect to $\frac{1}{2}$, the solution $\widetilde{Z}_2^{(0,0)}$ has the same property due to Remark 4.2 from [31]. □

2.2 Homogenized problem and correctors

We have formally constructed the leading terms of the asymptotic expansions (2.2), (2.3), (2.4) in three different parts of the junction Ω_ε. Now we apply the method of matching of asymptotic expansions to complete the constructions. Following this method (see, for instance [16]), the asymptotics of the external expansions (2.2) and (2.3) as $x_2 \to \pm 0$ has to coincide with the corresponding asymptotics of the internal expansion (2.4) as $\eta_2 \to \pm\infty$.

Writing down the Taylor series for v_0^+ and v_1^+ with respect to x_2 in the neighborhood of the point $(x_1, 0)$, where $x_1 \in (0, a)$, and passing to the variables $\eta_2 = \varepsilon^{-1} x_2$, we derive

$$\begin{aligned} u(\varepsilon, x) &= v_0^+(x_1,0) + \varepsilon\Big(\eta_2 \partial_{x_2} v_0^+(x_1,0) + v_1^+(x_1,0)\Big) + \\ &+ \varepsilon^2\Big(\frac{1}{2}\eta_2^2 \partial^2_{x_2x_2} v_0^+(x_1,0) + \eta_2 \partial_{x_2} v_1^+(x_1,0) + v_2^+(x_1,0)\Big) + \mathcal{O}(\varepsilon^3\eta_2^3), \quad x_2 \equiv \varepsilon\eta_2 \to +0. \end{aligned} \tag{2.39}$$

Bearing in mind the asymptotics of the functions $Z_1^{(i)}$ $(i = 0, 1, 2)$, $Z_2^{(\beta)}$ $(|\beta| < 2)$, as $\eta_2 \to +\infty$ (see (2.27)–(2.35)), we write down the asymptotics

$$\begin{aligned} u(\varepsilon, x) &= v_0^+(x_1,0) + \varepsilon\Big(\eta_2 \partial_{x_2} v_0^+(x_1,0) + C_1^{(0)} v_0^+(x_1,0) + C_1^{(2)} \partial_{x_2} v_0^+(x_1,0)\Big) \\ &+ \varepsilon^2\bigg(\Big(-\frac{\lambda_0}{2}\eta_2^2 + C_2^{(0,0)}\Big) v_0^+(x_1,0) + C_2^{(1,0)} \partial_{x_1} v_0^+(x_1,0) + C_2^{(0,1)} \partial_{x_2} v_0^+(x_1,0) \\ &+ \Big(-\frac{1}{2}\eta_2^2 + C_2^{(2,0)}\Big) \partial^2_{x_1x_1} v_0^+(x_1,0) + C_2^{(1,1)} \partial^2_{x_1x_2} v_0^+(x_1,0) \\ &+ \Big(\eta_2 + C_1^{(2)}\Big) \partial^2_{x_2x_2} v_0^+(x_1,0)\bigg) + \mathcal{O}(\varepsilon^3\eta_2^3), \qquad \eta_2 \to +\infty. \end{aligned} \tag{2.40}$$

To match (2.3) and (2.4) we write down (2.3) as $x_2 \to -0$ in fast variables:

$$\begin{aligned} u(\varepsilon, x) &= v_0^-(x_1,0) + \varepsilon\Big(\eta_2 \partial_{x_2} v_0^-(x_1,0) + v_1^-(x_1,0) + Y(\eta_1)\partial_{x_1} v_0^-(x_1,0)\Big) \\ &+ \varepsilon^2\bigg(\frac{1}{2}\eta_2^2 \partial^2_{x_2x_2} v_0^-(x_1,0) + \eta_2 \partial_{x_2} v_1^-(x_1,0) + \eta_2 Y(\eta_1) \partial^2_{x_1x_2} v_0^-(x_1,0) + v_2^-(x_1,0) \\ &+ Y(\eta_1)\partial_{x_1} v_1^-(x_1,0) + \frac{1}{2} Y^2(\eta_1) \partial^2_{x_1x_1} v_0^-(x_1,0)\bigg) + \mathcal{O}(\varepsilon^3\eta_2^3), \quad x_2 \equiv \varepsilon\eta_2 \to -0 \end{aligned} \tag{2.41}$$

and (2.4) as $\eta_2 \to -\infty$:

$$\begin{aligned} u(\varepsilon, x) = v_0^+(x_1,0) &+ \varepsilon\bigg(Y(\eta_1)\partial_{x_1} v_0^+(x_1,0) + \Big(\frac{\eta_2}{h_2} - \frac{C_1^{(2)}}{h_2}\Big) \partial_{x_2} v_0^+(x_1,0) \\ &+ \Big(\frac{4h_1 l_1 \lambda_0}{h_2}\eta_2 - \frac{C_1^{(0)}}{h_2}\Big) v_0^+(x_1,0)\bigg) \\ &+ \varepsilon^2\bigg(\Big(-\frac{\lambda_0}{2}\eta_2^2 + \frac{4h_1 l_1 \lambda_1 + \lambda_0 \int_{\Pi_{l_1}} Z_1^{(0)}(\eta\, d\eta}{h_2}\eta_2 - \frac{C_2^{(0,0)}}{h_2}\Big) v_0^+(x_1,0) \\ + \Big(\frac{4h_1 l_1 \lambda_0}{h_2} Y(\eta_1)\eta_2 - \frac{C_2^{(1,0)}}{h_2}\Big) \partial_{x_1} v_0^+(x_1,0) &+ \Big(\frac{\lambda_0 \int_{\Pi_{l_1}} Z_1^{(2)}\, d\eta}{h_2}\eta_2 - \frac{C_2^{0,1)}}{h_2}\Big) \partial_{x_2} v_0^+(x_1,0) \end{aligned}$$

$$+\Big(\frac{\mu_0}{h_2}\eta_2 - \frac{C_2^{(2,0)}}{h_2}\Big)\partial^2_{x_1x_1}v_0^+(x_1,0) + \Big(\frac{\eta_2}{h_2}Y(\eta_1) - \frac{C_2^{(1,1)}}{h_2}\Big)\partial^2_{x_1x_2}v_0^+(x_1,0)$$

$$+\Big(\frac{\eta_2}{h_2} - \frac{C_1^{(2)}}{h_2}\Big)\partial^2_{x_2x_2}v_0^+(x_1,0)\Big) + \mathcal{O}(\varepsilon^3\eta_2^3), \qquad (2.42)$$

where $Y(\eta_1) = -\eta_1 + \frac{1}{2} + [\eta_1]$, $[\eta_1]$ is the entire part of the number η_1 and μ_0 is defined by (2.36). We convince, that the leading terms of the asymptotic expansions (2.2), (2.3) and (2.4) are matched, if functions $\mathcal{F}_1$ and $\mathcal{F}_2$ from (2.9) and (2.10) are equal respectively

$$\mathcal{F}_1(x_1) = \frac{1+h_2}{h_2}\left(C_1^{(0)}v_0^+(x_1,0) + C_1^{(2)}\partial_{x_2}v_0^+(x_1,0)\right), \quad x_1 \in I_0, \qquad (2.43)$$

and

$$\begin{aligned}\mathcal{F}_2(x_1) = &- \mu_0\partial^2_{x_1x_1}v_0^+(x_1,0) - \lambda_0\int\limits_{\Pi_{l_1}} Z_1^{(2)}(\eta)\, d\eta\partial_{x_2}v_0^+(x_1,0) - \\ &- \Big(4h_1l_1\lambda_1 + \lambda_0\int\limits_{\Pi_{l_1}} Z_1^{(0)}(\eta)\, d\eta\Big)v_0^+(x_1,0), \quad x_1 \in I_0\end{aligned} \qquad (2.44)$$

and the conditions (2.6), (2.7), (2.9) and (2.10) hold true.

Finally, for

$$v_0(x) = \begin{cases} v_0^+(x), & x \in \Omega, \\ v_0^-(x), & x \in D_2 = (0,a)\times(-l_2,0), \end{cases}$$

and the number λ_0 we have the problem

$$\begin{cases} -\Delta_x\, v_0^+(x) = \lambda_0 v_0^+(x), & x \in \Omega_0, \\ -\partial^2_{x_2x_2}\, v_0^-(x) = \lambda_0\, v_0^-(x), & x \in D_2, \\ \partial_\nu\, v_0^+(x) = 0, & x \in \Gamma_2, \\ v_0^+(x) = 0, & x \in \Gamma_1, \\ v_0^+(x_1,0) = v_0^-(x_1,0), & x_1 \in (0,a), \\ \partial_{x_2}v_0^+(x_1,0) - h_2\partial_{x_2}v_0^-(x_1,0) = -4h_1l_1\lambda_0\, v_0^+(x_1,0), & x_1 \in (0,a), \\ \partial_{x_2}v_0^-(x_1,-l_2) = 0, & x_1 \in (0,a), \end{cases} \qquad (2.45)$$

which called *homogenized spectral problem* for problem (1.1). The spectrum of this problem is studied in § 2.3. Let λ_0 be an eigenvalue of problem (2.45) and v_0 is the corresponding eigenfunction that we normalize as follows

$$\int\limits_{\Omega_0}\left(v_0^+\right)^2 dx + h_2\int\limits_{D_2}\left(v_0^-\right)^2 dx + 4h_1l_1\int\limits_{I_0}\left(v_0^+(x_1,0)\right)^2 dx_1 = 1. \qquad (2.46)$$

Then for

$$v_1(x) = \begin{cases} v_1^+(x), & x \in \Omega, \\ v_1^-(x), & x \in D_2, \end{cases}$$

and λ_1 we get the following boundary-value problem

$$
\begin{cases}
-\Delta_x \, v_1^+(x) = \lambda_0 v_1^+(x) + \lambda_1 v_0^+(x), \qquad x \in \Omega_0, \\
\partial_\nu \, v_1^+(x) = 0, \quad x \in \Gamma_2; \qquad v_1^+(x) = 0, \quad x \in \Gamma_1, \\
-h_2 \, \partial^2_{x_2 x_2} v_1^-(x_1, x_2) = h_2 \lambda_0 \, v_1^-(x_1, x_2) + h_2 \lambda_1 \, v_0^-(x_1, x_2), \qquad x \in D_2, \\
\partial_{x_2} v_1^-(x_1, -l_2) = 0, \qquad x_1 \in (0, a), \\
v_1^+(x_1, 0) - v_1^-(x_1, 0) = \dfrac{1 + h_2}{h_2} \Big(C_1^{(0)} v_0^+(x_1, 0) + C_1^{(2)} \partial_{x_2} v_0^+(x_1, 0) \Big), \quad x \in I_0, \\
\partial_{x_2} v_1^+(x_1, 0) - h_2 \partial_{x_2} v_1^-(x_1, 0) = -\mu_0 \partial^2_{x_1 x_1} v_0^+(x_1, 0) - \lambda_0 \displaystyle\int_{\Pi_{l_1}} Z_1^{(2)}(\eta) \, d\eta \, \partial_{x_2} v_0^+(x_1, 0) \\
\qquad - \Big(4 h_1 l_1 \lambda_1 + \lambda_0 \displaystyle\int_{\Pi_{l_1}} Z_1^{(0)}(\eta) \, d\eta \Big) \, v_0^+(x_1, 0), \qquad x \in I_0.
\end{cases}
\tag{2.47}
$$

We see that the corresponding homogeneous problem has nontrivial solution since λ_0 is the eigenvalue of problem (2.45). Therefore, we should choose λ_1 such that the solvability condition for problem (2.47) is satisfied. Obviously, in this case the solution to problem (2.47) is not uniquely defined. For the uniqueness we demand the following orthogonality condition:

$$
\int_{I_0} v_1^+(x_1, 0) \, v_0^+(x_1, 0) \, dx_1 = 0. \tag{2.48}
$$

Multiplying the equation in Ω_0 by v_0^+, integrating it over the domain and using twice the Green's formula and repeating these procedures for the domain D_2 (only difference is that we multiply the equation by v_0^-) and then summarizing these identities, we obtain

$$
\begin{aligned}
&\int_{I_0} \partial_{x_2} v_1^+(x_1, 0) \, v_0^+(x_1, 0) \, dx_1 - \int_{I_0} v_1^+(x_1, 0) \, \partial_{x_2} v_0^+(x_1, 0) \, dx_1 - \\
&- h_2 \int_{I_0} \partial_{x_2} v_1^-(x_1, 0) \, v_0^-(x_1, 0) \, dx_1 + h_2 \int_{I_0} v_1^-(x_1, 0) \, \partial_{x_2} v_0^-(x_1, 0) \, dx_1 = \\
&= \lambda_1 \int_{\Omega_0} \Big(v_0^+ \Big)^2 \, dx + \lambda_1 h_2 \int_{D_2} \Big(v_0^- \Big)^2 \, dx
\end{aligned}
\tag{2.49}
$$

or, keeping in mind the transmission conditions in problem (2.45) on I_0,

$$
\begin{aligned}
&\int_{I_0} \big(\partial_{x_2} v_1^+(x_1, 0) - h_2 \partial_{x_2} v_1^-(x_1, 0) \big) \, v_0^+ \, dx_1 - \int_{I_0} \big(v_1^+(x_1, 0) - v_1^-(x_1, 0) \big) \, \partial_{x_2} v_0^+ \, dx_1 \\
&+ 4 h_1 l_1 \int_{I_0} v_1^-(x_1, 0) \, v_0^+(x_1, 0) \, dx_1 = \lambda_1 \int_{\Omega_0} \big(v_0^+(x_1, 0) \big)^2 \, dx + \lambda_1 h_2 \int_{D_2} \big(v_0^-(x_1, 0) \big)^2 \, dx.
\end{aligned}
\tag{2.50}
$$

Taking into account the transmission conditions in problem (2.47), the normalized condition (2.46) and the orthogonality condition (2.48), we get from (2.50)

$$\begin{aligned}\lambda_1 =&\mu_0 \int\limits_{I_0} \left(\partial_{x_1} v_0^+\right)^2 dx_1 - \left(\frac{1+h_2}{h_2} C_1^{(0)} + \lambda_0 \int_{\Pi_{l_1}} Z_1^{(2)}(\eta)\, d\eta\right) \int\limits_{I_0} v_0^+\, \partial_{x_2} v_0^+\, dx_1 \\ &- \frac{1+h_2}{h_2}\, 4h_1 l_1 \lambda_0 \int\limits_{I_0} \left(C_1^{(0)} v_0^+ + C_1^{(2)} \partial_{x_2} v_0^+\right) v_0^+\, dx_1 \\ &- \frac{1+h_2}{h_2}\, C_1^{(2)} \int\limits_{I_0} \left(\partial_{x_2} v_0^+\right)^2 dx_1 - \lambda_0 \int\limits_{\Pi_{l_1}} Z_1^{(0)}(\eta)\, d\eta \int\limits_{I_0} \left(v_0^+\right)^2 dx_1,\end{aligned} \tag{2.51}$$

where μ_0 is defined by (2.36).

2.3 The spectrum of the homogenized spectral problem (2.45)

It is obvious that any eigenvalue of problem (2.45) is real and positive. By solving the ordinary equation of problem (2.45) in the rectangle D_2 with regard to the boundary condition on $\Gamma_{-l_2} = \{x : \ x_1 \in (0,a), \ x_2 = -l_2\}$ and the first transmission condition on I_0, we find

$$v_0^-(x) = \frac{v_0^+(x_1,0)}{\cos(\sqrt{\lambda_0}\, l_2)} \cos(\sqrt{\lambda_0}\,(x_2 + l_2))\,. \tag{2.52}$$

Now, according to the second transmission condition in problem (2.45), we obtain the following spectral problem

$$\left\{\begin{array}{rcll} -\Delta\, v_0^+(x) &=& \lambda_0\, v_0^+(x)\,, & x \in \Omega_0, \\ \partial_\nu\, v^+(x) &=& 0\,, & x \in \Gamma_2, \\ v^+(x) &=& 0\,, & x \in \Gamma_1, \\ \partial_{x_2} v_0^+(x_1,0) &=& -\left(h_2 \sqrt{\lambda_0}\, \tan(\sqrt{\lambda_0}\, l_2) \ + \ 4h_1 l_1 \lambda_0\right) v_0^+(x_1,0), & x \in I_0, \end{array}\right. \tag{2.53}$$

with the spectral parameter λ_0 occurring both in the differential equation and in the boundary condition on I_0, where it enters in a nonlinear way.

Multiplying the differential equation of problem (2.53) with an arbitrary function $\psi \in H^1(\Omega_0;\Gamma_1)$ and integrating by parts in Ω_0, we can reduce the spectral problem (2.53) to a spectral problem for the following operator-function

$$\mathbf{L}(\lambda_0) = \lambda_0\, \mathbf{A}_1 + \Big(h_2 \sqrt{\lambda_0}\, \tan(\sqrt{\lambda_0}\, l_2) \ + \ 4h_1 l_1 \lambda_0\Big)\, \mathbf{A}_2 - \mathbf{I},$$

where $H^1(\Omega_0;\Gamma_1) = \{u \in H^1(\Omega_0; \ u|_{\Gamma_1} = 0\}$ and the scalar product is defined as follows $(u,v)_{H^1(\Omega_0;\Gamma_1)} := \int_{\Omega_0} \nabla u \cdot \nabla v\, dx,$ $\mathbf{I}$ is the identity operator in $H^1(\Omega_0;\Gamma_1)$, $\mathbf{A}_1$, $\mathbf{A}_2$ are self-adjoint compact operators in $H^1(\Omega_0;\Gamma_1)$ and

$$(\mathbf{A}_1 \varphi, \psi)_{H^1(\Omega_0;\Gamma_1)} = \int_{\Omega_0} \varphi(x)\, \psi(x)\, dx\,,$$

$$(\mathbf{A}_2 \varphi, \psi)_{H^1(\Omega_0;\Gamma_1)} = \int_{I_0} \varphi(x_1,0)\, \psi(x_1,0)\, dx_1 \quad \text{for any} \quad \varphi, \psi \in H^1(\Omega_0;\Gamma_1).$$

Theorems on existence and concentration of the spectrum for such self-adjoint operator-functions and mini-max principles for the eigenvalues were proved in [27, 15]. From these results we have the following theorem.

Theorem 2.1. *The spectrum of operator-function* $\mathbf{L}$ *and problem* (2.45) *contains normal eigenvalues (they have finite multiplicity and the corresponding eigenvectors have no Jordan chain) and also the left accumulation points*

$$P_m = \left(\frac{\pi + 2\pi(m-1)}{2l_2}\right)^2, \qquad m \in \mathbb{N},$$

which divide the eigenvalues into the sequences

$$0 < \lambda_0^{(1,1)} \le \ldots \le \lambda_0^{(1,n)} \le \ldots \to P_1 \quad \textit{as} \quad n \to \infty\,, \tag{2.54}$$

$$P_{m-1} < \lambda_0^{(m,1)} \le \ldots \le \lambda_0^{(m,n)} \le \ldots \to P_m \quad \textit{as} \quad n \to \infty, \quad m = 2, 3, \ldots\,. \tag{2.55}$$

2.4 Asymptotic approximations

Let λ_0 be an eigenvalue of problem (2.45), v_0 is the corresponding eigenfunction, i.e., $v_0 = v_0^+$ in Ω_0, where v_0^+ is the corresponding eigenfunction to problem (2.53), and $v_0 = v_0^-$ in D_2, where v_0^- is defined by (2.52). Then we can define λ_1 with the help of (2.51) and the unique solution $v_1^\pm$ to problem (2.47).

Using the method of matched asymptotic expansions for the leading terms of (2.2), (2.3) and (2.4), we construct the approximation $R_\varepsilon \in H^1(\Omega_0;\Gamma_1)$:

$$\begin{aligned} R_\varepsilon(x) &= v_0^+(x) + \varepsilon v_1^+(x) + \\ &+ \varepsilon\chi_0(x_2)\Big(\sum_{i=1}^{2}\big(Z_1^{(i)}(\eta) - \delta_{i,2}(\eta_2 + C_1^{(2)})\big)\partial_{x_i}v_0^+(x_1,0) + (Z_1^{(0)}(\eta) - C_1^{(0)})v_0^+(x_1,0)\Big) \\ &+ \varepsilon^2\chi_0\Big(\Big(Z_2^{(0,0)}(\eta) + \frac{\lambda_0\eta_2^2}{2}\Big)v_0^+(x_1,0) + \sum_{|\beta|=1} Z_2^{(\beta)}(\eta)D^\beta v_0^+(x_1,0) + \Big(Z_2^{(2,0)}(\eta) + \frac{\eta_2^2}{2}\Big)\partial^2_{x_1x_1}v_0^+(x_1,0) \\ &+ Z_2^{(1,1)}(\eta)\,\partial^2_{x_1x_2}v_0^+(x_1,0) + \big(Z_2^{(0,2)}(\eta) - \eta_2\big)\,\partial^2_{x_2x_2}v_0^+(x_1,0)\Big), \quad \eta = \frac{x}{\varepsilon}, \quad x \in \Omega_0; \end{aligned} \tag{2.56}$$

$$\begin{aligned} R_\varepsilon(x) &= v_0^-(x) + \varepsilon\big(v_1^-(x) + Y(\eta_1)\partial_{x_1}v_0^-(x)\big) \\ &+ \varepsilon\chi_0(x_2)\Big(\big(Z_1^{(1)}(\eta) - Y(\eta_1)\big)\partial_{x_1}v_0^+(x_1,0) + \Big(Z_1^{(2)}(\eta) - \frac{\eta_2}{h_2} + \frac{C_1^{(2)}}{h_2}\Big)\partial_{x_2}v_0^+(x_1,0) \\ &+ \Big(Z_1^{(0)}(\eta) - \frac{4h_1l_1\lambda_0}{h_2}\,\eta_2 + \frac{C_1^{(0)}}{h_2}\Big)v_0^+(x_1,0)\Big) \\ &+ \varepsilon^2\chi_0(x_2)\Big(\Big(Z_2^{(0,0)}(\eta) + \frac{\lambda_0}{2}\eta_2^2 - \frac{4h_1l_1\lambda_1 + \lambda_0\int_{\Pi_{l_1}} Z_1^{(0)}(\eta)d\eta}{h_2}\,\eta_2\Big)v_0^+(x_1,0) \\ &+ \Big(Z_2^{(1,0)}(\eta) - \frac{4h_1l_1\lambda_0}{h_2}\,\eta_2Y(\eta_1)\Big)\partial_{x_1}v_0^+(x_1,0) + \Big(Z_2^{(0,1)}(\eta) - \frac{\lambda_0\int_{\Pi_{l_1}} Z_1^{(2)}(\eta)d\eta}{h_2}\,\eta_2\Big)\partial_{x_2}v_0^+(x_1,0) \\ &+ \big(Z_2^{(2,0)}(\eta) - \frac{\mu_0}{h_2}\,\eta_2\big)\partial^2_{x_1x_1}v_0^+(x_1,0) + \big(Z_2^{(1,1)}(\eta) - \frac{\eta_2Y(\eta_1)}{h_2}\big)\partial^2_{x_1x_2}v_0^+(x_1,0) \\ &+ \big(Z_2^{(0,2)}(\eta) - \frac{\eta_2}{h_2}\big)\,\partial^2_{x_2x_2}v_0^+(x_1,0)\Big), \quad \eta = \frac{x}{\varepsilon}, \quad x \in G_\varepsilon^{(1)} \cup G_\varepsilon^{(2)}. \end{aligned} \tag{2.57}$$

Here χ_0 is a smooth cut-off function that equals 1 in a neighborhood of zero.

Substituting R_ε and $\lambda_0 + \varepsilon\lambda_1$ into problem (1.1) instead of u and $\lambda(\varepsilon)$ respectively, and finding residuals, we get

$$\| R_\varepsilon - (\lambda_0 + \varepsilon\lambda_1) A_\varepsilon R_\varepsilon \|_{\mathcal{H}_\varepsilon} \le c(\delta)\, \varepsilon^{2-\delta} \quad (\delta > 0). \tag{2.58}$$

Here operator $A_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_\varepsilon$ is defined by the following equality

$$(A_\varepsilon u, v)_{\mathcal{H}_\varepsilon} = (u, v)_{\mathcal{V}_\varepsilon} \quad \forall\, u, v \in \mathcal{H}_\varepsilon, \tag{2.59}$$

where by $\mathcal{H}_\varepsilon$ we denote the space $\{u \in H^1(\Omega_\varepsilon) : \ \ u|_{\Gamma_1} = 0\}$ with the scalar product

$$(u, v)_{\mathcal{H}_\varepsilon} := \int_{\Omega_\varepsilon} \nabla u \cdot \nabla v \, dx,$$

and by $\mathcal{V}_\varepsilon$ we denote the space $L^2(\Omega_\varepsilon)$ with the scalar product

$$(u, v)_{\mathcal{V}_\varepsilon} := \int_{\Omega_\varepsilon} \rho_\varepsilon \, u\, v \, dx.$$

Obviously, operator A_ε is self-adjoint, positive, and compact. In addition, problem (1.1) is equivalent to the spectral problem $A_\varepsilon u = \lambda^{-1}(\varepsilon)\, u$ in $\mathcal{H}_\varepsilon$.

By virtue of the minimax principle for eigenvalues, we have that for each $n \in \mathbb{N}$ $\lambda_n(\varepsilon) \le C_n$ and then due to (1.4) we get

$$\|u_n(\varepsilon, \cdot)\|_{\mathcal{H}_\varepsilon} = \lambda_n(\varepsilon) \le C_n. \tag{2.60}$$

3 The case $0 < \alpha < 1$

3.1 Formal Asymptotics

In this case we seek the main terms of the asymptotics for the eigenvalue $\lambda_n(\varepsilon)$ and the eigenfunction $u_n(\varepsilon, \cdot)$ of problem (1.1) in the form (index n is omitted):

$$\lambda(\varepsilon) \approx \lambda_0 + \varepsilon^{1-\alpha}\lambda_{1-\alpha} + \varepsilon\lambda_1 + \varepsilon^{2-\alpha}\lambda_{2-\alpha} + \ldots \tag{3.1}$$

$$u(\varepsilon, x) \approx v_0^+(x) + \varepsilon^{1-\alpha} v_{1-\alpha}^+(x) + \varepsilon v_1^+(x) + \varepsilon^{2-\alpha} v_{2-\alpha}^+(x) + \ldots, \quad \text{in domain } \Omega_0; \tag{3.2}$$

in the thin rectangle $G_j^{(2)}(\varepsilon)$ $(j = 0, \ldots, N-1)$

$$\begin{aligned} u(\varepsilon, x) \approx\ & v_0^-(x_1, x_2, \eta_1 - j) + \varepsilon^{1-\alpha} v_{1-\alpha}^-(x_1, x_2, \eta_1 - j) + \varepsilon v_1^-(x_1, x_2, \eta_1 - j) + \\ & + \varepsilon^{2-\alpha} v_{2-\alpha}^-(x_1, x_2, \eta_1 - j) + \ldots, \qquad \eta_1 = \frac{x_1}{\varepsilon}; \end{aligned} \tag{3.3}$$

and in the junction zone of the body and thin rectangles of both classes (which we call internal expansion) the series of the following type:

$$\begin{aligned} u(\varepsilon, x) \approx\ & v_0^+(x_1, 0) + \varepsilon^{1-\alpha} v_{1-\alpha}^+(x_1, 0) + \varepsilon \sum_{i=1}^{2} Z_1^{(i)}(\eta) \partial_{x_i} v_0^+(x_1, 0) + \\ & + \varepsilon^{2-\alpha} \Big(Z_{2-\alpha}^{(0)}(\eta) v_0^+(x_1, 0) + \sum_{i=1}^{2} Z_{2-\alpha}^{(i)}(\eta) \partial_{x_i} v_{1-\alpha}^+(x_1, 0) \Big) + \\ & + \varepsilon^2 \sum_{|\beta| \le 2} Z_2^{(\beta)}(\eta) D^\beta v_0^+(x_1, 0) + \ldots, \qquad \eta = \frac{x}{\varepsilon}. \end{aligned} \tag{3.4}$$

Substituting (3.1) and (3.2) in the problem (1.1) and collecting terms with equal order of ε, we get:

$$\begin{array}{rcll} -\Delta_x\ v_0^+(x) &=& \lambda_0 v_0^+(x), & x\in\Omega_0,\\ \partial_\nu\ v_0^+(x) &=& 0, & x\in\Gamma_2,\\ v_0^+(x) &=& 0, & x\in\Gamma_1. \end{array} \tag{3.5}$$

It remains to ensure the continuity of the asymptotic approximations and their gradients on the interfaces between the “rectangles” and the “body”. As in the previous section the necessity of the condition

$$v_0^+(x_1,0)=v_0^-(x_1,0),\qquad x\in I_0, \tag{3.6}$$

is evident. Another condition appears when one constructs the junction layer. This condition has the form

$$\partial_{x_2}v_0^+(x_1,0)=h_2\partial_{x_2}v_0^-(x_1,0),\qquad x\in I_0, \tag{3.7}$$

and will be obtained in the next section.

Collecting terms of order $\varepsilon^{1-\alpha}$, we obtain

$$\begin{array}{rcll} -\Delta_x\ v_{1-\alpha}^+(x) &=& \lambda_0 v_{1-\alpha}^+(x)+\lambda_{1-\alpha}v_0^+(x), & x\in\Omega_0,\\ \partial_\nu\ v_{1-\alpha}^+(x) &=& 0, & x\in\Gamma_2,\\ v_{1-\alpha}^+(x) &=& 0, & x\in\Gamma_1. \end{array} \tag{3.8}$$

Using the same arguments, we conclude that

$$v_{1-\alpha}^+(x_1,0)=v_{1-\alpha}^-(x_1,0),\qquad x\in I_0, \tag{3.9}$$

The second condition also appears when one constructs the junction layer. This condition is the following:

$$\partial_{x_2}v_{1-\alpha}^+(x_1,0)-h_2\partial_{x_2}v_{1-\alpha}^-(x_1,0)=-4h_1l_1\lambda_0v_0^+(x_1,0),\qquad x\in I_0 \tag{3.10}$$

and will be obtained in the next section.

Collecting terms of order ε, we have

$$\begin{array}{rcll} -\Delta_x\ v_1^+(x) &=& \lambda_0 v_1^+(x)+\lambda_1 v_0^+(x), & x\in\Omega_0,\\ \partial_\nu\ v_1^+(x) &=& 0, & x\in\Gamma_2,\\ v_1^+(x) &=& 0, & x\in\Gamma_1. \end{array} \tag{3.11}$$

In the transmission conditions here the following jumps appear

$$v_1^+(x_1,0)-v_1^-(x_1,0)=\mathcal{F}_3(x_1),\qquad x\in I_0, \tag{3.12}$$

and

$$\partial_{x_2}v_1^+(x_1,0)-h_2\partial_{x_2}v_1^-(x_1,0)=\mathcal{F}_4(x_1),\qquad x\in I_0, \tag{3.13}$$

where $\mathcal{F}_3$ and $\mathcal{F}_4$ are given functions on I_0 that will be defined in subsection 3.2.

Finally, collecting terms of order $\varepsilon^{2-\alpha}$, we obtain

$$\begin{array}{rcll} -\Delta_x\ v_{2-\alpha}^+(x) &=& \lambda_0 v_{2-\alpha}^+(x)+\lambda_1 v_{1-\alpha}^+(x)+\lambda_{1-\alpha}v_1^+(x)+\lambda_{2-\alpha}v_0^+(x), & x\in\Omega_0,\\ \partial_\nu\ v_{2-\alpha}^+(x) &=& 0, & x\in\Gamma_2,\\ v_{2-\alpha}^+(x) &=& 0, & x\in\Gamma_1. \end{array} \tag{3.14}$$

Similarly we obtain

$$v^+_{2-\alpha}(x_1,0) - v^-_{2-\alpha}(x_1,0) = \mathcal{F}_5(x_1), \qquad x \in I_0, \tag{3.15}$$

and to simplify the constructions we set

$$\partial_{x_2} v^+_{2-\alpha}(x_1,0) = h_2 \partial_{x_2} v^-_{2-\alpha}(x_1,0), \qquad x \in I_0. \tag{3.16}$$

The function $\mathcal{F}_5(x_1)$ also is given (see subsection 3.2).

3.1.1 Formal asymptotics on thin rectangles.

Let us enumerate the set $\{p-1+(2-\alpha)q\}_{p,q=0}^{\infty} \setminus \{-1\}$ for fixed α in increasing order $0 = \varsigma_1 < \varsigma_2 \le \ldots$. Obviously, $\varsigma_2 = 1-\alpha$, $\varsigma_3 = 1$ as $0 < \alpha \le 1$. Keeping in mind that in (3.3) $v^-_{\varsigma_k}$ are smooth functions, using Taylor series for $v^-_{\varsigma_k}$ and changing variable $x_1 \mapsto \eta_1$ in the neighborhood of the points $x_1 = \varepsilon(j+\frac{1}{2})$, we get

$$u(\varepsilon, x) = \sum_{k=0}^{+\infty} \varepsilon^{\varsigma_k} W^{(j)}_{\varsigma_k}(x_2, \eta_1), \quad x \in G^{(2)}_j(\varepsilon), \tag{3.17}$$

where, for instance for $k \in \mathbb{N}$ we also have (as in the previous section)

$$W^{(j)}_k(x_2,\eta_1) = v^-_k(\varepsilon(j+\frac{1}{2}), x_2, \eta_1 - j) + \\ + \sum_{m=1}^{k} \frac{1}{m!}\left(\eta_1 - j - \frac{1}{2}\right)^m \frac{\partial^m v^-_{k-m}}{\partial x_1^m}\left(\varepsilon(j+\frac{1}{2}), x_2, \eta_1 - j\right) \tag{3.18}$$

and, in particular,

$$\begin{aligned} W^{(j)}_0(x_2,\eta_1) &= v^-_0(\varepsilon(j+\frac{1}{2}), x_2, \eta_1 - j), \\ W^{(j)}_{1-\alpha}(x_2,\eta_1) &= v^-_{1-\alpha}(\varepsilon(j+\frac{1}{2}), x_2, \eta_1 - j), \\ W^{(j)}_1(x_2,\eta_1) &= v^-_1(\varepsilon(j+\frac{1}{2}), x_2, \eta_1 - j) + \left(\eta_1 - j - \frac{1}{2}\right)\frac{\partial v^-_0}{\partial x_1}(\varepsilon(j+\frac{1}{2}), x_2, \eta_1 - j), \\ W^{(j)}_{2-\alpha}(x_2,\eta_1) &= v^-_{2-\alpha}(\varepsilon(j+\frac{1}{2}), x_2, \eta_1 - j) + \left(\eta_1 - j - \frac{1}{2}\right)\frac{\partial v^-_{1-\alpha}}{\partial x_1}(\varepsilon(j+\frac{1}{2}), x_2, \eta_1 - j). \end{aligned} \tag{3.19}$$

Substituting (3.1) and (3.17) in the problem (1.1) instead of $\lambda_n(\varepsilon)$ and $u_n(\varepsilon,\cdot)$ respectively, collecting terms with equal powers of ε, we obtain the following problems $(k = 0,1,2,3):$

$$\begin{aligned} -\partial^2_{\eta_1\eta_1} W^{(j)}_k(x_2,\eta_1) &= \partial^2_{x_2x_2} W^{(j)}_{k-2}(x_2,\eta_1) + \sum_{m=0}^{k-2} \lambda_m W^{(j)}_{k-2-m}(x_2,\eta_1), \quad |\eta_1 - \tfrac{1}{2}| < \tfrac{h_2}{2}, \\ \partial_{\eta_1} W^{(j)}_k(x_2, \tfrac{1\pm h_2}{2}) &= 0 \end{aligned} \tag{3.20}$$

and (k=1, 2, 3, 4)

$$\begin{aligned} -\partial^2_{\eta_1\eta_1} W^{(j)}_{k-\alpha}(x_2,\eta_1) &= \partial^2_{x_2x_2} W^{(j)}_{k-2-\alpha}(x_2,\eta_1) + \sum_{m=0}^{k-3} \lambda_m W^{(j)}_{k-2-m-\alpha}(x_2,\eta_1) + \\ &+ \sum_{m=0}^{k-3} \lambda_{k-2-m-\alpha} W^{(j)}_m(x_2,\eta_1), \qquad |\eta_1 - \tfrac{1}{2}| < \tfrac{h_2}{2}, \\ \partial_{\eta_1} W^{(j)}_{k-\alpha}(x_2, \tfrac{1\pm h_2}{2}) &= 0, \end{aligned} \tag{3.21}$$

where λ_{ς_p} and the functions $W^{(j)}_{\varsigma_p}$ with negative ς_p are equal to zero; the variable x_2 is a parameter; $\partial_{\eta_1} = \frac{\partial}{\partial \eta_1}$.

From (3.20) and (3.21) we deduce that $W^{(j)}_0$, $W^{(j)}_1$, $W^{(j)}_2$, $W^{(j)}_3$, $W^{(j)}_{1-\alpha}$, $W^{(j)}_{2-\alpha}$, $W^{(j)}_{3-\alpha}$ and $W^{(j)}_{4-\alpha}$ are independent of η_1. Moreover the solvability conditions for the problem (3.20) as $k = 2, 3$ and (3.21) as $k = 3, 4$, give us the equations

$$h_2\, \partial^2_{x_2x_2} v^-_0(x_1, x_2) + \lambda_0 h_2\, v^-_0(x_1, x_2) = 0, \quad x_2 \in (-l_2, 0),\ x_1 = \varepsilon(j + \frac{1}{2}) \tag{3.22}$$

$$\begin{aligned} h_2\, \partial^2_{x_2x_2} v^-_{1-\alpha}(x_1, x_2) + h_2\lambda_0\, v^-_{1-\alpha}(x_1, x_2) = \\ = -h_2\lambda_{1-\alpha} v^-_0(x_1, x_2), \quad x_2 \in (-l_2, 0),\ x_1 = \varepsilon(j + \frac{1}{2}). \end{aligned} \tag{3.23}$$

$$h_2\, \partial^2_{x_2x_2} v^-_1(x_1, x_2) + h_2\lambda_0\, v^-_1(x_1, x_2) = -h_2\lambda_1\, v^-_0(x_1, x_2), \quad x_2 \in (-l_2, 0),\ x_1 = \varepsilon(j + \frac{1}{2}) \tag{3.24}$$

and

$$\begin{aligned} h_2\, \partial^2_{x_2x_2} v^-_{2-\alpha}(x_1, x_2) + \lambda_0 h_2\, v^-_{2-\alpha}(x_1, x_2) = -h_2\lambda_{2-\alpha} v^-_0(x_1, x_2) - h_2\lambda_{1-\alpha} v^-_1(x_1, x_2) \\ - h_2\lambda_1 v^-_{1-\alpha}(x_1, x_2), \quad x_2 \in (-l_2, 0),\ x_1 = \varepsilon(j + \frac{1}{2}). \end{aligned} \tag{3.25}$$

Since we seek the smooth functions v^-_0, $v^-_{1-\alpha}$, v^-_1 and $v^-_{2-\alpha}$ and the points $x_1 = \varepsilon(j + \frac{1}{2})$ form the ε-net in the interval $(0, a)$, then the equations (3.22), (3.23), (3.24), (3.25) defined on N segments can be extended to the whole rectangle $D_2 = (0, a) \times (-l_2, 0)$. Bearing in mind the boundary conditions of the original problem, we add

$$\begin{aligned} \partial_{x_2} v^-_0(x_1, -l_2) = 0, \qquad \partial_{x_2} v^-_{1-\alpha}(x_1, -l_2) &= 0, \\ \partial_{x_2} v^-_1(x_1, -l_2) = 0, \qquad \partial_{x_2} v^-_{2-\alpha}(x_1, -l_2) &= 0. \end{aligned} \tag{3.26}$$

3.1.2 Junction-layer solutions

Similarly as in subsection 2.1.2 we substitute series (3.4) and (3.1) in problem (1.1) and collect terms with equal powers of ε to obtain boundary-value problems in Π for $Z^{(i)}_1$, $i = 1, 2$, $Z^{(\imath)}_{2-\alpha}, \imath = 0, 1, 2$, and $Z^{(\beta)}_2$, $|\beta| \le 2$. Obviously, these solutions have to be 1-periodic in η_1, i.e., they must satisfy conditions (2.17). In addition, they must satisfy the Neumann conditions (2.18) as well. We discover that

- function $Z^{(i)}_1$, $(i = 1, 2)$ is the solution to problem (2.20) and it has the asymptotics (2.28) for $i = 1$ ((2.29) for $i = 2$);
- function $Z^{(0)}_{2-\alpha}$ coincides with function $Z^{(0)}_1$ from subsection 2.1.2, i.e., it satisfies problem (2.19) and has the asymptotics (2.27);
- function $Z^{(1)}_{2-\alpha} \equiv Z^{(1)}_1$, i.e., it satisfies problem (2.20) and has the asymptotics (2.28);
- $Z^{(2)}_{2-\alpha} \equiv 0$; $Z^{(1,0)}_2 \equiv 0$;
- $Z^{(0,1)}_2 \equiv Z^{(2)}_1$, i.e., it satisfies the problem (2.20) and has the asymptotics (2.29));

- $Z_2^{(2,0)}$ is identically equal to $Z_2^{(2,0)}$ from subsection 2.1.2, i.e., it satisfies problem (2.26) and has the asymptotics (2.35);
- $Z_2^{(1,1)}$ is identically equal to $Z_2^{(1,1)}$ from subsection 2.1.2, i.e., it satisfies the problem (2.25) and has the asymptotics (2.34);
- $Z_2^{(0,2)}$ is identically equal to $Z_1^{(2)}$, i.e., it satisfies problem (2.20) and has the asymptotics (2.29);
- for function $Z_2^{(0,0)}$ we obtain

$$\begin{cases} -\Delta_\eta \, Z_2^{(0,0)}(\eta) & = & \begin{cases} \lambda_0, & \eta \in \Pi^+ \cup \Pi^-, \\ 0, & \eta \in \Pi_{l_1}, \end{cases} \\ \partial_{\eta_1} Z_2^{(0,0)}(\eta) & = & 0, \qquad \eta \in \partial\Pi_{\|}. \end{cases} \tag{3.27}$$

Similarly to the proof of Lemma 2.1 we deduce the following statement.

Lemma 3.1. *Problem* (3.27) *has a solution from space* $H^1_{loc,\eta_2}(\Pi)$ *and this solution has the differentiable asymptotics*

$$Z_2^{(0,0)}(\eta) = \begin{cases} -\dfrac{\lambda_0}{2}\eta_2^2 + C_2^{(0,0)} + \mathcal{O}(\exp(-2\pi\eta_2)), & \eta_2 \to +\infty, \\ -\dfrac{\lambda_0}{2}\eta_2^2 - \dfrac{C_2^{(0,0)}}{h_2} + \mathcal{O}(\exp(\pi h_2^{-1}\eta_2)), & \eta_2 \to -\infty. \end{cases} \tag{3.28}$$

Moreover, $Z_2^{(0,0)}$ *is even in* η_1 *with respect to* $\frac{1}{2}$.

3.2 Homogenized problem and correctors

As in subsection 2.2, here we should match the leading terms of the asymptotic expansions (3.2), (3.3) and (3.4). Following the method of matching of asymptotic expansions (see [16]), the asymptotics of the external expansions (3.2) and (3.3) as $x_2 \to \pm 0$ has to coincide respectively with the corresponding asymptotics of the internal expansion (3.4) as $\eta_2 \to \pm\infty$.

Writing down the Taylor series for functions v_0^+, v_1^+ and $v_{2-\alpha}^+$ with respect to x_2 in the neighborhood of the point $(x_1, 0)$, where $x_1 \in (0, a)$, and passing to the variables $\eta_2 = \varepsilon^{-1}x_2$, we derive

$$\begin{aligned} u(\varepsilon, x) = {} & v_0^+(x_1,0) + \varepsilon^{1-\alpha} v_{1-\alpha}^+(x_1,0) + \varepsilon\Big(\eta_2 \partial_{x_2} v_0^+(x_1,0) + v_1^+(x_1,0)\Big) + \\ & + \varepsilon^{2-\alpha}\Big(\eta_2 \partial_{x_2} v_{1-\alpha}^+(x_1,0) + v_{2-\alpha}^+(x_1,0)\Big) + \\ & + \varepsilon^2\Big(\frac{1}{2}\eta_2^2 \partial^2_{x_2x_2} v_0^+(x_1,0) + \eta_2 \partial_{x_2} v_1^+(x_1,0) + v_2^+(x_1,0)\Big) + \vartheta_{up}^+(\varepsilon, \eta_2), \end{aligned} \tag{3.29}$$

where $\vartheta_{up}^+(\varepsilon,\eta_2) = \mathcal{O}(\max(\varepsilon^3\eta_2^3, \varepsilon^{3-\alpha}\eta_2))$ as $x_2 \equiv \varepsilon\eta_2 \to +0$. Bearing in mind the asymptotics of the functions $Z_1^{(i)}$ $(i = 1,2)$, $Z_{2-\alpha}^{(j)}$ $(j = 0,1,2)$, as $\eta_2 \to +\infty$, we convince, that the leading terms of the asymptotic expansions (3.2) and (3.4) are matched.

In fact, keeping in mind the asymptotics of the functions $Z_j^{(i)}$, we rewrite (3.4) as $\eta_2 \to 0$

$$\begin{aligned} u(\varepsilon, x) &= v_0^+(x_1,0) + \varepsilon^{1-\alpha} v_{1-\alpha}^+(x_1,0) + \varepsilon\Big(\eta_2 \partial_{x_2} v_0^+(x_1,0) + C_1^{(2)} \partial_{x_2} v_0^+(x_1,0)\Big) + \\ &+ \varepsilon^{2-\alpha}\Big(C_{2-\alpha}^{(0)} v_0^+(x_1,0) + \eta_2 v_0^+(x_1,0)\Big) + \\ &+ \varepsilon^2\Big(\big(-\frac{\lambda_0}{2}\eta_2^2 + C_2^{(0,0)}\big) v_0^+(x_1,0) + \big(\eta_2 + C_1^{(2)}\big)\partial_{x_2} v_0^+(x_1,0) + \\ &+ \big(-\frac{\eta_2^2}{2} + C_2^{(2,0)}\big)\partial^2_{x_1x_1} v_0^+(x_1,0) + C_2^{(1,1)} \partial^2_{x_1x_2} v_0^+(x_1,0) + \\ &+ \big(\eta_2 + C_1^{(2)}\big)\partial^2_{x_2x_2} v_0^+(x_1,0)\Big) + \vartheta^+_{down}(\varepsilon, \eta_2), \end{aligned} \tag{3.30}$$

where $\vartheta^+_{down}(\varepsilon,\eta_2) = \mathcal{O}(\max(\varepsilon^3\eta_2^3, \varepsilon^{3-\alpha}\eta_2))$ as $\eta_2 \to +\infty$.

To match the asymptotics (3.3) and (3.4) we write down the asymptotics (3.3) as $x_2 \to -0$ in fast variables:

$$\begin{aligned} u(\varepsilon, x) &= v_0^-(x_1,0) + \varepsilon^{1-\alpha} v_{1-\alpha}^-(x_1,0) + \varepsilon\left(\eta_2 \partial_{x_2} v_0^-(x_1,0) + v_1^-(x_1,0) + Y(\eta_1)\partial_{x_1} v_0^-(x_1,0)\right) + \\ &+ \varepsilon^{2-\alpha}\left(\eta_2 \partial_{x_2} v_{1-\alpha}^-(x_1,0) + v_{2-\alpha}^-(x_1,0) + Y(\eta_1)\partial_{x_1} v_{1-\alpha}^-(x_1,0)\right) + \\ &+ \varepsilon^2\left(\frac{1}{2}\eta_2^2 \partial^2_{x_2x_2} v_0^-(x_1,0) + \eta_2 \partial_{x_2} v_1^-(x_1,0) + \eta_2 Y(\eta_1)\partial^2_{x_1x_2} v_0^-(x_1,0) + v_2^-(x_1,0) + \right. \\ &\left. + Y(\eta_1)\partial_{x_1} v_1^-(x_1,0) + \frac{1}{2} Y^2(\eta_1)\partial^2_{x_1x_1} v_0^-(x_1,0)\right) + \vartheta^-_{down}(\varepsilon,\eta_2), \end{aligned} \tag{3.31}$$

where $\vartheta^-_{down}(\varepsilon,\eta_2) = \mathcal{O}(\max(\varepsilon^3\eta_2^3, \varepsilon^{3-\alpha}\eta_2))$ as $x_2 \equiv \varepsilon\eta_2 \to -0$ and (3.4) as $\eta_2 \to -\infty$:

$$\begin{aligned} u(\varepsilon, x) &= v_0^+(x_1,0) + \varepsilon^{1-\alpha} v_{1-\alpha}^+(x_1,0) + \varepsilon\Big(Y(\eta_1)\partial_{x_1} v_0^+(x_1,0) + \frac{\eta_2}{h_2}\partial_{x_2} v_0^+(x_1,0) - \\ &- \frac{C_1^{(2)}}{h_2}\partial_{x_2} v_0^+(x_1,0)\Big) + \varepsilon^{2-\alpha}\Big(\frac{4h_1 l_1 \lambda_0 + 1}{h_2}\eta_2 v_0^+(x_1,0) - \frac{C_{2-\alpha}^{(0)}}{h_2} v_0^+(x_1,0) + \\ &+ Y(\eta_1)\partial_{x_1} v_{1-\alpha}^+(x_1,0)\Big) + \varepsilon^2\Big(\big(-\frac{\lambda_0}{2}\eta_2^2 - \frac{C_2^{(0,0)}}{h_2}\big) v_0^+(x_1,0) + \\ &+ \big(\frac{\eta_2}{h_2} - \frac{C_1^{(2)}}{h_2}\big)\partial_{x_2} v_0^+(x_1,0) + \big(\frac{\mu_0}{h_2}\eta_2 - \frac{C_2^{(2,0)}}{h_2}\big)\partial^2_{x_1x_1} v_0^+(x_1,0) + \\ &+ \big(\frac{\eta_2}{h_2} Y(\eta_1) - \frac{C_2^{(1,1)}}{h_2}\big)\partial^2_{x_1x_2} v_0^+(x_1,0) + \big(\frac{\eta_2}{h_2} - \frac{C_1^{(2)}}{h_2}\big)\partial^2_{x_2x_2} v_0^+(x_1,0)\Big) + \vartheta^+_{up}(\varepsilon,\eta_2), \end{aligned} \tag{3.32}$$

where $\vartheta^+_{down}(\varepsilon,\eta_2) = \mathcal{O}(\max(\varepsilon^3\eta_2^3, \varepsilon^{3-\alpha}\eta_2))$ as $\eta_2 \to -\infty$ and μ_0 is defined by (2.36).

We convince that the leading terms of the asymptotic expansions (2.2), (3.3) and (3.4) are matched, if

$$\mathcal{F}_3(x_1) = \frac{1 + h_2}{h_2} C_1^{(2)} \partial_{x_2} v_0^+(x_1, 0), \quad x_1 \in (0, a), \tag{3.33}$$

$$\mathcal{F}_4(x_1) = -\mu_0 \partial^2_{x_1x_1} v_0^+(x_1, 0), \quad x_1 \in (0, a), \tag{3.34}$$

and

$$\mathcal{F}_5(x_1) = \frac{1+h_2}{h_2} C^{(0)}_{2-\alpha} v_0^+(x_1,0), \quad x_1 \in (0,a), \tag{3.35}$$

and conditions (3.6), (3.7), (3.9), (3.10), (3.12), (3.13), (3.15) and (3.16) hold true.

Finally, for

$$v_0(x) = \begin{cases} v_0^+(x), & x \in \ \Omega, \\ v_0^-(x), & x \in \ D_2, \end{cases}$$

and the number λ_0 we have the problem

$$\left\{ \begin{array}{rcll} -\Delta_x \ v_0^+(x) & = & \lambda_0 v_0^+(x), & x \in \Omega_0, \\ -\partial^2_{x_2x_2} \ v_0^-(x) & = & \lambda_0 \ v_0^-(x), & x \in D_2, \\ \partial_\nu \ v_0^+(x) & = & 0, & x \in \Gamma_2, \\ v_0^+(x) & = & 0, & x \in \Gamma_1, \\ v_0^+(x_1,0) & = & v_0^-(x_1,0), & x_1 \in (0,a), \\ \partial_{x_2} v_0^+(x_1,0) & = & h_2 \partial_{x_2} v_0^-(x_1,0), & x_1 \in (0,a), \\ \partial_{x_2} v_0^-(x_1,-l_2) & = & 0, & x_1 \in (0,a), \end{array} \right. \tag{3.36}$$

that called *homogenized spectral problem* for problem (1.1) in the case $\alpha \in (0,1)$. This problem coincides with the homogenized spectral problem for a spectral problem in a thick one-level junction (see [28]). This means that there is no any influence of the concentrated masses in the first terms of the asymptotics both for the eigenvalues and for eigenfunctions of problem (1.1) if $\alpha \in (0,1)$. From [28, Theorem 2.1] (see also subsection 2.3) it follows the following theorem.

Theorem 3.1. *The spectrum of problem* (3.36) *contains normal eigenvalues and the left accumulation points*

$$P_m = \left(\frac{\pi + 2\pi(m-1)}{2l_2} \right)^2, \qquad m \in \mathbb{N},$$

which divide the eigenvalues into the sequences

$$0 < \lambda_0^{(1,1)} \leq \ldots \leq \lambda_0^{(1,n)} \leq \ldots \to \ P_1 \quad as \quad n \to \infty\,, \tag{3.37}$$

$$P_{m-1} < \lambda_0^{(m,1)} \leq \ldots \leq \lambda_0^{(m,n)} \leq \ldots \to \ P_m \quad as \quad n \to \infty, \quad m = 2,3,\ldots\ . \tag{3.38}$$

Let λ_0 be an eigenvalue of problem (3.36). We normalize the corresponding eigenfunction as follows

$$\int\limits_{\Omega_0} \left(v_0^+\right)^2 dx + h_2 \int\limits_{D_2} \left(v_0^-\right)^2 dx = 1. \tag{3.39}$$

Then for

$$v_{1-\alpha}(x) = \begin{cases} v_{1-\alpha}^+(x), & x \in \ \Omega, \\ v_{1-\alpha}^-(x), & x \in \ D_2, \end{cases}$$

and the number $\lambda_{1-\alpha}$ we get the following boundary-value problem

$$\begin{array}{rcll} -\Delta_x \ v_{1-\alpha}^+(x) & = & \lambda_0 \ v_{1-\alpha}^+(x) + \lambda_{1-\alpha} \ v_0^+(x), & x \in \Omega_0, \\ -h_2 \ \partial^2_{x_2x_2} \ v_{1-\alpha}^-(x) & = & h_2\lambda_0 \ v_{1-\alpha}^-(x) + h_2\lambda_{1-\alpha} \ v_0^-(x), & x \in D_2, \\ \partial_\nu \ v_{1-\alpha}^+(x) & = & 0, & x \in \Gamma_2, \\ v_{1-\alpha}^+(x) & = & 0, & x \in \Gamma_1, \\ v_{1-\alpha}^+(x_1,0) & = & v_{1-\alpha}^-(x_1,0), & x_1 \in (0,a), \\ \partial_{x_2} v_{1-\alpha}^+(x_1,0) - h_2 \partial_{x_2} v_{1-\alpha}^-(x_1,0) & = & -4h_1 l_1 \lambda_0 v_0^+(x_1,0), & x_1 \in (0,a), \\ \partial_{x_2} v_{1-\alpha}^-(x_1,-l_2) & = & 0, & x_1 \in (0,a). \end{array} \tag{3.40}$$

Since λ_0 is the eigenvalue of the corresponding uniform problem for problem (3.40), we should choose $\lambda_{1-\alpha}$ such that the solvability condition for problem (3.40) is satisfied. Obviously, in this case the solution to problem (3.40) is not uniquely defined. For the uniqueness we demand the following orthogonality condition:

$$\int\limits_{\Omega_0} v_{1-\alpha}^+ v_0^+ \, dx + h_2 \int\limits_{D_2} v_{1-\alpha}^- v_0^- \, dx = 0. \tag{3.41}$$

From the solvability condition of the problem (3.40) we derive the formula for $\lambda_{1-\alpha}$. Multiplying the equation in Ω_0 by v_0^+, integrating it over the domain and using twice the Green's formula and repeating these procedures for the domain D_2 (only difference is that we multiply the equation by v_0^-) and then summarizing these identities, we obtain

$$\begin{gathered}
-\int\limits_{\partial\Omega_0} \frac{\partial v_{1-\alpha}^+}{\partial \nu} v_0^+ \, ds + \int\limits_{\partial\Omega_0} \frac{\partial v_0^+}{\partial \nu} v_{1-\alpha}^+ \, ds - \int\limits_0^a h_2 \frac{\partial v_{1-\alpha}^-}{\partial x_2} v_0^- \Big|_{-l_2}^0 \, dx_1 + \\
+ \int\limits_0^a h_2 \frac{\partial v_0^-}{\partial x_2} v_{1-\alpha}^- \Big|_{-l_2}^0 \, dx_1 = \lambda_{1-\alpha} \int\limits_{\Omega_0} \left(v_0^+\right)^2 dx + \lambda_{1-\alpha} h_2 \int\limits_{D_2} \left(v_0^-\right)^2 dx
\end{gathered} \tag{3.42}$$

or, keeping in mind the normalization condition (3.39) and the boundary conditions of the problems (3.36) and (3.40), we get

$$\int\limits_{I_0} \left(\frac{\partial v_{1-\alpha}^+}{\partial x_2} - h_2 \frac{\partial v_{1-\alpha}^-}{\partial x_2} \right) v_0^+ \, dx_1 - \int\limits_{I_0} \left(v_{1-\alpha}^+ - v_{1-\alpha}^- \right) \frac{\partial v_0^+}{\partial x_2} \, dx_1 = \lambda_{1-\alpha} \tag{3.43}$$

and finally

$$\lambda_{1-\alpha} = -4 h_1 l_1 \lambda_0 \int\limits_{I_0} \left(v_0^+\right)^2 dx_1. \tag{3.44}$$

For $v_1(x) = \begin{cases} v_1^+(x), & x \in \Omega, \\ v_1^-(x), & x \in D_2, \end{cases}$ and λ_1 we have

$$\begin{cases}
-\Delta_x \, v_1^+(x) = \lambda_0 v_1^+(x) + \lambda_1 v_0^+(x), & x \in \Omega_0, \\
\partial_\nu \, v_1^+(x) = 0, & x \in \Gamma_2, \\
v_1^+(x) = 0, & x \in \Gamma_1, \\
-h_2 \, \partial^2_{x_2 x_2} v_1^-(x) = h_2 \lambda_0 \, v_1^-(x) + h_2 \lambda_1 \, v_0^-(x), & x \in D_2, \\
\partial_{x_2} v_1^-(x_1, -l_2) = 0, & x_1 \in (0, a), \\
v_1^+(x_1, 0) - v_1^-(x_1, 0) = \dfrac{1 + h_2}{h_2} C_1^{(2)} \partial_{x_2} v_0^+(x_1, 0), & x \in (0, a), \\
\partial_{x_2} v_1^+(x_1, 0) - h_2 \partial_{x_2} v_1^-(x_1, 0) = -\mu_0 \partial^2_{x_1 x_1} v_0^+(x_1, 0), & x_1 \in (0, a).
\end{cases} \tag{3.45}$$

For the uniqueness of the solution to problem (3.45) we demand the following orthogonality condition:

$$\int\limits_{\Omega_0} v_1^+ v_0^+ \, dx + h_2 \int\limits_{D_2} v_1^- v_0^- \, dx = 0. \tag{3.46}$$

From the solvability condition of problem (3.45), similarly as before, we derive the formula

$$
\begin{aligned}
&-\int\limits_{\partial\Omega_0} \frac{\partial v_1^+}{\partial\nu} v_0^+ \, ds + \int\limits_{\partial\Omega_0} \frac{\partial v_0^+}{\partial\nu} v_1^+ \, ds - \int\limits_0^a h_2 \frac{\partial v_1^-}{\partial x_2} v_0^- \Big|_{-l_2}^0 \, dx_1 + \int\limits_0^a h_2 \frac{\partial v_0^-}{\partial x_2} v_1^- \Big|_{-l_2}^0 \, dx_1 = \\
&= \lambda_1 \int\limits_{\Omega_0} \left(v_0^+\right)^2 \, dx + \lambda_1 h_2 \int\limits_{D_2} \left(v_0^-\right)^2 \, dx
\end{aligned}
\tag{3.47}
$$

or, keeping in mind the normalization condition (3.39) and the boundary conditions of the problems (3.36) and (3.45), we get

$$
\int\limits_{I_0} \left(\frac{\partial v_1^+}{\partial x_2} - h_2 \frac{\partial v_1^-}{\partial x_2} \right) v_0^+ \, dx_1 - \int\limits_{I_0} \left(v_1^+ - v_1^- \right) \frac{\partial v_0^+}{\partial x_2} \, dx_1 = \lambda_1 \tag{3.48}
$$

and finally

$$
\lambda_1 = \mu_0 \int\limits_{I_0} \left(\partial_{x_1} v_0^+\right)^2 dx_1 - \frac{1+h_2}{h_2} C_1^{(2)} \int\limits_{I_0} \left(\partial_{x_2} v_0^+\right)^2 \, dx_1. \tag{3.49}
$$

Here μ_0 is defined by (2.36).

For $v_{2-\alpha}(x) = \begin{cases} v_{2-\alpha}^+(x), & x \in \ \Omega, \\ v_{2-\alpha}^-(x), & x \in \ D_2, \end{cases}$ and $\lambda_{2-\alpha}$ we have the problem

$$
\begin{cases}
-\Delta_x \ v_{2-\alpha}^+(x) = \lambda_0 v_{2-\alpha}^+(x) + \lambda_1 v_{1-\alpha}^+(x) + \lambda_{1-\alpha} v_1^+(x) + \lambda_{2-\alpha} v_0^+(x), & x \in \Omega_0, \\
-\partial_{x_2x_2}^2 \ v_{2-\alpha}^-(x) = \lambda_0 \ v_{2-\alpha}^-(x) + \lambda_1 \ v_{1-\alpha}^-(x) + \lambda_{1-\alpha} \ v_1^-(x) + \lambda_{2-\alpha} \ v_0^-(x), & x \in D_2, \\
\partial_\nu \ v_{2-\alpha}^+(x) = 0, & x \in \Gamma_2, \\
v_{2-\alpha}^+(x) = 0, & x \in \Gamma_1, \\
v_{2-\alpha}^+(x_1,0) - v_{2-\alpha}^-(x_1,0) = \frac{1+h_2}{h_2} C_1^{(0)} v_0^+(x_1,0), & x \in I_0, \\
\partial_{x_2} v_{2-\alpha}^+(x_1,0) = h_2 \partial_{x_2} v_{2-\alpha}^-(x_1,0), & x \in I_0, \\
\partial_{x_2} v_{2-\alpha}^-(x_1,-l_2) = 0, & x \in I_0,
\end{cases}
\tag{3.50}
$$

For the uniqueness we demand the following orthogonality condition:

$$
\int\limits_{\Omega_0} v_{2-\alpha}^+ v_0^+ \, dx + h_2 \int\limits_{D_2} v_{2-\alpha}^- v_0^- \, dx = 0. \tag{3.51}
$$

From the solvability condition of the problem (3.50) we derive the formula for $\lambda_{2-\alpha}$. Similarly as before, we obtain

$$
\begin{aligned}
&-\int\limits_{\partial\Omega_0} \frac{\partial v_{2-\alpha}^+}{\partial\nu} v_0^+ \, ds + \int\limits_{\partial\Omega_0} \frac{\partial v_0^+}{\partial\nu} v_{2-\alpha}^+ \, ds - \int\limits_0^a h_2 \frac{\partial v_{2-\alpha}^-}{\partial x_2} v_0^- \Big|_{-l_2}^0 \, dx_1 + \int\limits_0^a h_2 \frac{\partial v_0^-}{\partial x_2} v_{2-\alpha}^- \Big|_{-l_2}^0 \, dx_1 = \\
&= \lambda_1 \int\limits_{\Omega_0} v_{1-\alpha}^+ v_0^+ \, dx + \lambda_1 h_2 \int\limits_{D_2} v_{1-\alpha}^- v_0^- \, dx + \lambda_{1-\alpha} \int\limits_{\Omega_0} v_1^+ v_0^+ \, dx + \lambda_{1-\alpha} h_2 \int\limits_{D_2} v_1^- v_0^- \, dx + \\
&+ \lambda_{2-\alpha} \int\limits_{\Omega_0} \left(v_0^+\right)^2 \, dx + \lambda_{2-\alpha} h_2 \int\limits_{D_2} \left(v_0^-\right)^2 \, dx
\end{aligned}
\tag{3.52}
$$

or, keeping in mind the normalization condition (3.39) and the boundary conditions of the problems (3.36) and (3.45), we get

$$\int\limits_{I_0}\left(\frac{\partial v^+_{2-\alpha}}{\partial x_2}-h_2\frac{\partial v^-_{2-\alpha}}{\partial x_2}\right)v^+_0\,dx_1-\int\limits_{I_0}\left(v^+_{2-\alpha}-v^-_{2-\alpha}\right)\frac{\partial v^+_0}{\partial x_2}\,dx_1-$$

$$-\lambda_1\int\limits_{\Omega_0}v^+_{1-\alpha}v^+_0\,dx-\lambda_1h_2\int\limits_{D_2}v^-_{1-\alpha}v^-_0\,dx-\lambda_{1-\alpha}\int\limits_{\Omega_0}v^+_1v^+_0\,dx-\lambda_{1-\alpha}h_2\int\limits_{D_2}v^-_1v^-_0\,dx=\lambda_{2-\alpha} \tag{3.53}$$

and finally, using (3.46) and (3.51), we derive

$$\lambda_{2-\alpha}=-\frac{1+h_2}{h_2}C^{(0)}_1\int\limits_{I_0}v^+_0\partial_{x_2}v^+_0\,dx_1. \tag{3.54}$$

3.3 Asymptotic approximations

Let λ_0 be an eigenvalue of problem (3.36), v_0 is the corresponding eigenfunction normalized with (3.39). Then we can define $\lambda_{1-\alpha}$ with the help of (3.44), λ_1 with the help of (3.49), $\lambda_{2-\alpha}$ with the help of (3.54), the unique solution $v^\pm_{1-\alpha}$ to problem (3.40), the unique solution $v^\pm_1$ to problem (3.45) and the unique solution $v^\pm_{2-\alpha}$ to problem (3.50).

Using the method of matched asymptotic expansions for the leading terms of (3.2), (3.3) and (3.4), we construct the approximation $R_\varepsilon\in H^1(\Omega_0;\Gamma_1)$:

$$\begin{aligned}R_\varepsilon(x)&=v^+_0(x)+\varepsilon^{1-\alpha}v^+_{1-\alpha}(x)+\varepsilon v^+_1(x)+\varepsilon\chi_0(x_2)\Bigg(\sum_{i=1}^{2}\big(Z^{(i)}_1(\eta)-\delta_{i,2}(\eta_2+C^{(2)}_1)\big)\partial_{x_i}v^+_0(x_1,0)\Bigg)\\&+\varepsilon^{2-\alpha}v^+_{2-\alpha}(x)+\varepsilon^{2-\alpha}\chi_0(x_2)\Big((Z^{(0)}_{2-\alpha}(\eta)-C^{(0)}_1)v^+_0(x_1,0)+Z^{(1)}_{2-\alpha}(\eta)\partial_{x_1}v^+_{1-\alpha}(x_1,0)\Big)\\&+\varepsilon^2\chi_0\Bigg(\Big(Z^{(0,0)}_2(\eta)+\frac{\lambda_0\eta_2^2}{2}\Big)v^+_0(x_1,0)+\Big(Z^{(0,1)}_2(\eta)-\eta_2\Big)\partial_{x_2}v^+_0(x_1,0)+\Big(Z^{(2,0)}_2(\eta)+\frac{\eta_2^2}{2}\Big)\partial^2_{x_1x_1}v^+_0(x_1,0)\\&+Z^{(1,1)}_2(\eta)\,\partial^2_{x_1x_2}v^+_0(x_1,0)+\big(Z^{(0,2)}_2(\eta)-\eta_2\big)\,\partial^2_{x_2x_2}v^+_0(x_1,0)\Bigg),\quad\eta=\frac{x}{\varepsilon},\quad x\in\Omega_0;\end{aligned} \tag{3.55}$$

$$\begin{aligned}R_\varepsilon(x)&=v^-_0(x)+\varepsilon^{1-\alpha}v^-_{1-\alpha}(x)+\varepsilon\big(v^-_1(x)+Y(\eta_1)\partial_{x_1}v^-_0(x)\big)\\&+\varepsilon\chi_0(x_2)\Big(\big(Z^{(1)}_1(\eta)-Y(\eta_1)\big)\partial_{x_1}v^+_0(x_1,0)+\Big(Z^{(2)}_1(\eta)-\frac{\eta_2}{h_2}+\frac{C^{(2)}_1}{h_2}\Big)\partial_{x_2}v^+_0(x_1,0)\Big)\\&+\varepsilon^{2-\alpha}\big(v^-_{2-\alpha}(x)+Y(\eta_1)\partial_{x_1}v^-_{1-\alpha}(x)\big)\\&+\varepsilon^{2-\alpha}\chi_0(x_2)\Big(Z^{(0)}_{2-\alpha}(\eta)-\frac{4h_1l_1\lambda_0}{h_2}\,\eta_2+\frac{C^{(0)}_1}{h_2}\Big)v^+_0(x_1,0)+\big(Z^{(1)}_{2-\alpha}(\eta)-Y(\eta_1)\big)\partial_{x_1}v^+_0(x_1,0)\Big)\\&+\varepsilon^2\chi_0(x_2)\Bigg(\Big(Z^{(0,0)}_2(\eta)+\frac{\lambda_0}{2}\eta_2^2\Big)v^+_0(x_1,0)+\Big(Z^{(0,1)}_2(\eta)-\frac{\eta_2}{h_2}\Big)\partial_{x_2}v^+_0(x_1,0)\\&+\big(Z^{(2,0)}_2(\eta)-\frac{\mu_0}{h_2}\,\eta_2\big)\partial^2_{x_1x_1}v^+_0(x_1,0)+\Big(Z^{(1,1)}_2(\eta)-\frac{\eta_2Y(\eta_1)}{h_2}\Big)\partial^2_{x_1x_2}v^+_0(x_1,0)\\&+\big(Z^{(0,2)}_2(\eta)-\frac{\eta_2}{h_2}\big)\,\partial^2_{x_2x_2}v^+_0(x_1,0)\Bigg),\quad\eta=\frac{x}{\varepsilon},\quad x\in G^{(1)}_\varepsilon\cup G^{(2)}_\varepsilon.\end{aligned} \tag{3.56}$$

Here χ_0 is a smooth cut-off function that equals 1 in a neighborhood of zero.

Substituting R_ε and $\lambda_0+\varepsilon^{1-\alpha}\lambda_{1-\alpha}+\varepsilon\lambda_1+\varepsilon^{2-\alpha}\lambda_{2-\alpha}$ into problem (1.1) instead of u and $\lambda(\varepsilon)$ respectively, and finding residuals, we get that for arbitrary $\delta>0$

$$\| R_\varepsilon - (\lambda_0+\varepsilon^{1-\alpha}\lambda_{1-\alpha}+\varepsilon\lambda_1+\varepsilon^{2-\alpha}\lambda_{2-\alpha})A_\varepsilon R_\varepsilon \|_{\mathcal{H}_\varepsilon} \le c(\delta)\,\varepsilon^{2-\delta}, \qquad (3.57)$$

where operator $A_\varepsilon:\mathcal{H}_\varepsilon\mapsto\mathcal{H}_\varepsilon$ is defined by (2.59).

4 The extension operator

For domains of the type under consideration there exist no extension operators that would be bounded uniformly in ε in the Sobolev space H^1 (see [28, 32]). But as was shown in [28, 32], for eigenfunctions of spectral problems in thick junctions it was possible to construct special extensions that are bounded on each eigenfunction. Here we prove the similar result for the eigenfunctions of problem (1.1) in the case when the parameter $\alpha\le 1$.

Theorem 4.1 ($\alpha\le 1$)**.** *There exists an extension operator $\mathbf{P}_\varepsilon:\mathcal{H}_\varepsilon\mapsto H^1(\Omega,\Gamma_1)$ which is asymptotically bounded in ε on each eigenfunctions $\{u_n(\varepsilon,\cdot)\}$ of problem* (1.1)*, i.e., for any $n\in\mathbb{N}$ there exist positive constants C_n and ε_n that for all values of the parameter ε from $(0,\varepsilon_n)$ the following estimate holds:*

$$\| \mathbf{P}_\varepsilon u_n(\varepsilon,\cdot) \|_{H^1(\Omega,\Gamma_1)} \le C_n \| u_n(\varepsilon,\cdot) \|_{\mathcal{H}_\varepsilon} \le C_n, \qquad (4.1)$$

where Ω is the interior of the union $\overline{\Omega}_0\cup\overline{D}_2$.

Proof. Let χ_0 be a smooth cut-off function such that $\chi_0(x_2)=0$ for $x_2\ge\gamma_0$, and $\chi_0(x_2)=1$ for $x_2\le\frac{\gamma_0}{2}$, where $\gamma_0=\min\{\gamma(x_1):\ x_1\in[0,a]\}$.

If u_n is an eigenfunction of problem (1.1) normalized by condition (1.4), then the function $v_n=\chi_0 u_n$ is the solution to the following problem

$$\begin{array}{ll} -\Delta_x v_n(x) = f_n(x)+\lambda_n(\varepsilon)\ v_n(x), & x\in\Omega_{0,\gamma_0}, \\ -\Delta_x v_n(x) = \lambda_n(\varepsilon)\ v_n(x), & x\in G_\varepsilon^{(2)}, \\ -\Delta_x v_n(x) = \varepsilon^{-\alpha}\lambda_n(\varepsilon)\ v_n(x), & x\in G_\varepsilon^{(1)}, \\ v_n(x_1,\gamma_0)=0, & (x_1,\gamma_0)\in\Gamma_{\gamma_0}, \\ \partial_\nu v_n(x)=0, & x\in\partial\Omega_{\varepsilon,\gamma_0}\setminus\Gamma_{\gamma_0}\ . \end{array} \qquad (4.2)$$

Here $\Omega_{\varepsilon,\gamma_0}$ is the interior of the union $\overline{\Omega}_{0,\gamma_0}\cup\overline{G_\varepsilon^{(1)}\cup G_\varepsilon^{(2)}}$, $f_n(x)=2\chi_0'\,\partial_{x_2}u_n+\chi_0''\,u_n$, $\mathrm{supp}(\chi_0')\subset[0,a]\times(\frac{\gamma_0}{2},\gamma_0)$, $\Omega_{0,\gamma_0}=(0,a)\times(0,\gamma_0)$, $\Gamma_{\gamma_0}=\{x:x_1\in[0,a],\quad x_2=\gamma_0\}$.

In the sequel we interpret $\widehat{Y}$ as follows : if Y is a set, then $\widehat{Y}$ is the union of Y and of its image symmetric with respect to the ordinate axis $\{x:x_1=0\}$; if Y is a function, then $\widehat{Y}$ is its even extension into the relevant domain with respect to the axis $\{x:x_1=0\}$.

We extend this problem to the left into the domain $\Omega_{\varepsilon,\gamma_0}$ in the even way and require $2a-$periodicity conditions on the corresponding side of the rectangle $\widehat{\Omega}_{0,\gamma_0}$.

Since the extended problem is invariant with respect to shifts by ε along the axis Ox_1, the function (the index n is omitted)

$$V_\varepsilon(x)=\varepsilon^{-1}(\,\widehat{v}(x+\varepsilon\bar{e}_1)-\widehat{v}(x)\,),\quad (\,\bar{e}_1=(1,0)\,), \qquad (4.3)$$

that is $2a-$periodic in x_1, satisfies the following relations

$$\begin{aligned}
-\Delta_x V_\varepsilon(x) &= \varepsilon^{-1}(\widehat{f}(x+\varepsilon\bar{e}_1)-\widehat{f}(x)) + \lambda_n(\varepsilon)\ V_\varepsilon(x), && x\in\widehat{\Omega}_{0,\gamma_0},\\
-\Delta_x V_\varepsilon(x) &= \lambda_n(\varepsilon)\ V_\varepsilon(x), && x\in\widehat{G}^{(2)}_\varepsilon,\\
-\Delta_x V_\varepsilon(x) &= \varepsilon^{-\alpha}\lambda_n(\varepsilon)\ V_\varepsilon(x), && x\in\widehat{G}^{(1)}_\varepsilon,\\
V_\varepsilon(x_1,\gamma_0) &= 0, && (x_1,\gamma_0)\in\widehat{\Gamma}_{\gamma_0},\\
\partial_\nu V_\varepsilon(x) &= 0, && x\in\partial\widehat{\Omega}_{\varepsilon,\gamma_0}\cap\{x:\ x_2\le 0\},
\end{aligned}$$

whence we get the integral equality

$$\|\nabla V_\varepsilon\|^2_{L_2(\widehat{\Omega}_{\varepsilon,\gamma_0})} = \lambda_n(\varepsilon)\|V_\varepsilon\|^2_{L_2(\widehat{\Omega}_{0,\gamma_0})} + \lambda_n(\varepsilon)\|V_\varepsilon\|^2_{L_2(\widehat{G}^{(2)}_\varepsilon)} + \varepsilon^{-\alpha}\lambda_n(\varepsilon)\|V_\varepsilon\|^2_{L_2(\widehat{G}^{(1)}_\varepsilon)} +$$

$$+\varepsilon^{-1}\int_{\widehat{\Omega}_{0,\gamma_0}}(\widehat{f}(x+\varepsilon\bar{e}_1)-\widehat{f}(x))V_\varepsilon\,dx =: I_1(\varepsilon)+I_2(\varepsilon)+I_3(\varepsilon)+I_4(\varepsilon). \tag{4.4}$$

Let us estimate the right-hand side of (4.4). Since

$$\int_{\widehat{\Omega}_{0,\gamma_0}}(V_\varepsilon)^2\ dx = \int_0^{\gamma_0}dx_2\int_{-a}^{a}dx_1\ \varepsilon^{-2}\left|\int_{x_1}^{x_1+\varepsilon}\partial_t\widehat{v}^{(1)}(t,x_2)\ dt\right|^2 \le$$

$$\le \int_0^{\gamma_0}dx_2\int_{-a}^{a}(\partial_t\widehat{v}(t,x_2))^2\ dt \le 2\|\partial_{x_1}u_n\|^2_{L_2(\Omega_0)},$$

we have

$$|I_4(\varepsilon)| \le \|\varepsilon^{-1}(\widehat{f}(x+\varepsilon\bar{e}_1)-\widehat{f}(x))\|_{L_2(\widehat{\Omega}_{0,\gamma_0})}\cdot\|V_\varepsilon\|_{L_2(\widehat{\Omega}_{0,\gamma_0})} \le$$

$$\le c\|\partial_{x_1}f\|_{L_2(\Omega_{0,\gamma_0})}\ \|\partial_{x_1}u\|_{L_2(\Omega_0)} \le$$

$$\le c\left(\|u\|_{H^1(\Omega_0)} + \|(\chi_0)'\partial^2_{x_1x_2}u\|_{L_2(\Omega_{0,\gamma_0})\|}\right)\ \|\partial_{x_1}u\|_{L_2(\Omega_0)} \le c\|u(\varepsilon,\cdot)\|^2_{H^1(\Omega_0)}.$$

Here, in order to estimate the mixed second-order derivative, we have used so-called the second energy inequality for elliptic operators in the domain $(0,a)\times(\frac{\gamma_0}{2},\gamma_0)$, i.e., the a-priori estimate $\|u\|^2_{H^2(\Omega)} \le c(\|\Delta u\|^2_{L_2(\Omega)} + \|u\|^2_{L_2(\Omega)})$ (see [18]) with a suitable cut-off function.

In order to estimate I_2 and I_3 we use the approach of Theorem 4.1 ([28]). Since the singularity is greater on the rods $G^{(1)}_\varepsilon$, we estimate I_3. Let us represent V_ε on the rod $G^{(1)}_j(d_k,\varepsilon)$ in the following form :

$$V_\varepsilon(x) = \varphi_j(x_2) + U_j(x), \quad x\in G^{(1)}_j(d_k,\varepsilon), \tag{4.5}$$

$$\int_{\varkappa_j(d_k,\varepsilon)} U_j(x)\,dx_1 = 0 \quad \forall x_2\in[-\varepsilon l_1, 0],$$

where $\varkappa_j(d_k,\varepsilon)$ is the cross-section of the rod $G^{(1)}_j(d_k,\varepsilon)$.

Integrating the equation for V_ε in $G^{(1)}_j(d_k,\varepsilon)$ over the cross-section $\varkappa_j(d_k,\varepsilon)$, we get

$$\partial^2_{x_2x_2}\varphi_j(x_2) + \varepsilon^{-\alpha}\lambda_n(\varepsilon)\varphi_j(x_2) = 0\,, \quad x_2\in(-\varepsilon l_1, 0)\,; \qquad \partial_{x_2}\varphi_j(-\varepsilon l_1) = 0\ ,$$

which implies

$$\varphi_j(x_2) = A_j\,\frac{\cos\left(\varepsilon^{-\frac{\alpha}{2}}\lambda_n^{\frac{1}{2}}(\varepsilon)(x_2+\varepsilon l_1)\right)}{\cos\left(\varepsilon^{1-\frac{\alpha}{2}}\lambda_n^{\frac{1}{2}}(\varepsilon)\,l_1\right)}, \quad x_2\in[-\varepsilon l_1, 0],$$

$$A_j = \frac{1}{\varepsilon h_1}\int_{\varkappa_j(d_k,\varepsilon)} V_\varepsilon(x_1,0)\,dx_1.$$

It is easy to calculate that

$$\|\varphi_j\|^2_{L^2(G_j^{(1)}(d_k,\varepsilon))} = \frac{\varepsilon h_1 A_j^2}{2\left[\cos\left(\varepsilon^{1-\frac{\alpha}{2}}\lambda_n^{\frac{1}{2}}(\varepsilon)\,l_1\right)\right]^2}\left(\varepsilon l_1 + \frac{\sin\left(2\varepsilon^{1-\frac{\alpha}{2}}\lambda_n^{\frac{1}{2}}(\varepsilon)l_1\right)}{2\varepsilon^{-\frac{\alpha}{2}}\lambda_n^{\frac{1}{2}}(\varepsilon)}\right).$$

Because of $\alpha<2$ and $\lambda_n(\varepsilon)=\mathcal{O}(1)$ as $\varepsilon\to 0$,

$$\|\varphi_j\|^2_{L^2(G_j^{(1)}(d_k,\varepsilon))} \le c_1\varepsilon^2 A_j^2 \le c_2\varepsilon\int_{\varkappa_j(d_k,\varepsilon)} V_\varepsilon^2(x_1,0)\,dx_1.$$

Now using the Poincare inequality for U_j, we get

$$|I_3(\varepsilon)| \le 4\varepsilon^{-\alpha}\lambda_n(\varepsilon)\sum_{j=0}^{N-1}\sum_{k=1}^{4}\left(\|\varphi_j\|^2_{L^2(G_j^{(1)}(d_k,\varepsilon))} + \|U_j\|^2_{L_2(G_j^{(1)}(d_k,\varepsilon))}\right) \le$$

$$\le c_1\varepsilon^{-\alpha}\sum_{j=0}^{N-1}\sum_{k=1}^{4}\left(\varepsilon\int_{\varkappa_j(d_k,\varepsilon)} V_\varepsilon^2(x_1,0)\,dx_1 + \varepsilon^2\|\partial_{x_1}V_\varepsilon\|^2_{L_2(G_j^{(1)}(d_k,\varepsilon))}\right) \le$$

$$\le c_1\varepsilon^{-\alpha}\left(\varepsilon\int_0^a V_\varepsilon^2(x_1,0)\,dx_1 + \varepsilon^2\|\partial_{x_1}V_\varepsilon\|^2_{L_2(G_\varepsilon^{(1)})}\right) \le$$

$$\le c_2\varepsilon^{1-\alpha}\left(\delta_3\|\nabla V_\varepsilon\|^2_{L_2(\Omega_{0,\gamma_0})} + \frac{2}{\delta_3}\|V_\varepsilon\|^2_{L_2(\Omega_{0,\gamma_0})} + \varepsilon^2\|\partial_{x_1}V_\varepsilon\|^2_{L_2(G_\varepsilon^{(1)})}\right).$$

By the same arguments we obtain

$$|I_2(\varepsilon)| \le c_3\left(\delta_2\|\nabla V_\varepsilon\|^2_{L_2(\Omega_{0,\gamma_0})} + \frac{2}{\delta_2}\|V_\varepsilon\|^2_{L_2(\Omega_{0,\gamma_0})} + \varepsilon^2\|\partial_{x_1}V_\varepsilon\|^2_{L_2(G_\varepsilon^{(2)})}\right).$$

Choosing δ_2, δ_3 and ε such that $c_2\delta_3 + c_3\delta_2 + 2\varepsilon^2 < 1/2$, we obtain from (4.4) that for ε small enough

$$\|V_\varepsilon\|_{H^1(\widehat{\Omega}_{\varepsilon,\gamma_0})} \le c(n)\|u_n(\varepsilon,\cdot)\|_{H^1(\Omega_\varepsilon)}. \tag{4.6}$$

This inequality shows that the eigenfunctions have no strong variation of values on neighboring rods.

Now we can conduct the construction of the extension operator $\mathbf{P}_\varepsilon : \mathcal{H}_\varepsilon \mapsto H^1(\Omega,\Gamma_1)$. Since the construction closely follows that of Theorem 4.1 in ([28]), where such an extension operator was constructed for eigenfunctions of the Neumann spectral problem in a thick plane junction and without any concentrated masses, we omit the proof. From (2.60) it follows the second inequality in (4.1). □

5 Justification of the asymptotics

To justify the asymptotic approximations constructed above, we use the scheme proposed in [40] for investigation of the asymptotic behavior of the eigenvalues and eigenvectors of an abstract operator $A_\epsilon : H_\epsilon \mapsto H_\epsilon$ losing the compactness in the limit passage as $\epsilon \to 0$. This scheme generalizes procedure of the justification of the asymptotic behavior of eigenvalues and eigenvectors of boundary value problems in perturbed domains that was proposed in [39]. To prove Theorem 5.4 – Theorem 5.7, we additionally use the same arguments as in [24, Theorem 3.1] (see items 2 – 4 of the proof).

5.1 Condition $\mathbf{D_1} - \mathbf{D_6}$

In our case this is the family of the operators $\{A_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{H}_\varepsilon\}_{\varepsilon>0}$ defined in (2.59). Recall that A_ε corresponds to problem (1.1).

Let us define an operator that corresponds to the homogenized problem (2.45) in the case $\alpha = 1$ and to the homogenized problem (3.36) if $\alpha \in (0,1)$. In the case $\alpha = 1$ we denote by $\mathcal{V}_0$ the space $L^2(\Omega_0) \times L^2(D_2) \times L^2(I_0)$ with the scalar product

$$(\mathbf{u}, \mathbf{v})_{\mathcal{V}_0} := \int\limits_{\Omega_0} u^+ v^+ \, dx + h_2 \int\limits_{D_2} u^- v^- \, dx + 4 h_1 l_1 \int\limits_{I_0} u^0 v^0 \, dx_1,$$

where $\mathbf{u} = \big(u^+, u^-, u^0\big)$, $\mathbf{v} = \big(v^+, v^-, v^0\big)$. If $\alpha \in (0,1)$, then $\mathcal{V}_0 = L^2(\Omega_0) \times L^2(D_2)$ and in the scalar product the integral over I_0 is absent.

It is easy to see that the anisotropic Sobolev space

$$\mathcal{H}_0 := \{u \in L^2(\Omega) : \quad u^+ \in H^1(\Omega_0, \Gamma_1), \quad \exists\, \partial_{x_2} u^- \in L^2(D_2), \quad u^+|_{I_0} = u^-|_{I_0}\}, \tag{5.1}$$

where $u^+ = u|_{\Omega_0},\ u^- = u|_{D_2}$ and the last equality in (5.1) is understood in the sense of traces, with the scalar product

$$\big(u, v\big)_{\mathcal{H}_0} = \int\limits_{\Omega_0} \nabla u^+ \cdot \nabla v^+ \, dx \; + \; h_2 \int\limits_{D_2} \partial_{x_2} u^- \, \partial_{x_2} v^- \, dx$$

is densely and only continuously embedded into $\mathcal{V}_0$.

Problem (2.45) ((3.36)) is equivalent to the spectral problem $A_0 v = \lambda_0^{-1}\, v$ in $\mathcal{H}_0$, where the operator $A_0 : \mathcal{H}_0 \mapsto \mathcal{H}_0$ is defined by the equality

$$\big(A_0 u, v\big)_{\mathcal{H}_0} = \big(\mathbf{u}, \mathbf{v}\big)_{\mathcal{V}_0} \quad \forall\, u, v \in \mathcal{H}_0. \tag{5.2}$$

Here $\mathbf{u} = (u|_{\Omega_0}, u|_{D_2}, u|_{I_0})$. Obviously, A_0 is self-adjoint, positive, continuous, but non-compact.

Also denote by $\mathcal{Z}_0 := H^1(\Omega, \Gamma_1)$. Obviously, that $\mathcal{Z}_0$ is densely and compactly embedded into $\mathcal{V}$.

Now let us verify conditions $\mathbf{D_1} - \mathbf{D_6}$ of the scheme from [40].

The operator $S_\varepsilon : \mathcal{Z}_0 \mapsto \mathcal{H}_\varepsilon$ assigns to each function $v \in \mathcal{Z}_0$ its restriction on Ω_ε. Clearly, S_ε is uniformly bounded with respect to ε. Thus condition $\mathbf{D_1}$ is satisfied.

The operator $P_\varepsilon : \mathcal{H}_\varepsilon \mapsto \mathcal{Z}_0$ from condition $\mathbf{D_2}$ is associated with the extension operator $\mathbf{P}_\varepsilon$ from Theorem 4.1.

Let us verify condition $\mathbf{D_3}$. Consider the sequence $\{u_n(\varepsilon,\cdot)\}_{\varepsilon>0}$ for any fixed index $n\in\mathbb{N}$. Due to Theorem 4.1 there exists some subsequence $\{\varepsilon'\}\subset\{\varepsilon\}$ (again denoted by $\{\varepsilon\}$) such that $\mathbf{P}_\varepsilon u(\varepsilon,\cdot)\to v$ weakly in $\mathcal{Z}_0$ (index n is omitted) as $\varepsilon\to 0$.

Since

$$\int\limits_{D_2}\chi_{h_2}(\tfrac{x_1}{\varepsilon})\,\partial_{x_2}\mathbf{P}_\varepsilon(u(\varepsilon,x))\,\phi(x)\,dx=-\int\limits_{D_2}\chi_{h_2}\tfrac{x_1}{\varepsilon})\,\mathbf{P}_\varepsilon(u(\varepsilon,x))\,\partial_{x_2}\phi\,dx\quad\forall\,\phi\in C_0^\infty(D_2),$$

we get

$$\chi_{h_2}(\frac{x_1}{\varepsilon})\partial_{x_2}\mathbf{P}_\varepsilon(u(\varepsilon,x))\to h_2\,\partial_{x_2}v(x)\quad\text{weakly in}\quad L^2(D_2)\quad\text{as}\quad\varepsilon\to 0.\tag{5.3}$$

Here $\chi_{h_2}(\eta_1),\ \eta_1\in\mathbb{R}$, is a 1-periodic function that equals 1 on the interval $\left(\frac{1-h_2}{2},\ \frac{1+h_2}{2}\right)$ and vanishing on the rest of the segment $[0,1]$.

Consider the corresponding integral identity for problem (1.1) with the following test function

$$\psi(x)=\begin{cases}0, & x\in\Omega_0\cup G_\varepsilon^{(1)},\\ \varepsilon Y\left(\frac{x_1}{\varepsilon}\right)\phi(x), & x\in G_\varepsilon^{(2)},\end{cases}\qquad \phi\in C_0^\infty(D_2),$$

where Y is defined in (2.42). As a result, we have

$$\int\limits_{D_2}\chi_{h_2}(x_1/\varepsilon)\partial_{x_1}\mathbf{P}_\varepsilon(u(\varepsilon,x))\,\phi\,dx=O(\varepsilon),\quad\varepsilon\to 0.\tag{5.4}$$

Due to the second inequality in (4.1), it is easy to verify that

$$\int\limits_{G_\varepsilon^{(1)}}\nabla u(\varepsilon,x)\cdot\nabla\varphi(x)\,dx\to 0\quad\text{as}\quad\varepsilon\to 0\quad\forall\,\varphi\in\mathcal{Z}_0.\tag{5.5}$$

Taking into account limits (5.3)-(5.5) , we ascertain that

$$\lim_{\varepsilon\to 0}\big(u(\varepsilon,\cdot),S_\varepsilon\varphi\big)_{\mathcal{H}_\varepsilon}=\big(v,\varphi\big)_{\mathcal{H}_\varepsilon}\quad\forall\,\varphi\in\mathcal{Z}_0,$$

i.e., condition $\mathbf{D_3}$ is satisfied.

Let for certain functions $u^\varepsilon,v^\varepsilon\in\mathcal{H}_\varepsilon$ one has $\mathbf{P}_\varepsilon u^\varepsilon\to u^0$ and $\mathbf{P}_\varepsilon v^\varepsilon\to v^0$ weakly in $\mathcal{Z}_0$ as $\varepsilon\to 0$. Then

$$\lim_{\varepsilon\to 0}\big(u^\varepsilon,v^\varepsilon\big)_{\mathcal{V}_\varepsilon}=\int\limits_{\Omega_0}u^+v^+\,dx+h_2\int\limits_{D_2}u^-v^-\,dx+\lim_{\varepsilon\to 0}\varepsilon^{-\alpha}\int\limits_{G_\varepsilon^{(1)}}u^\varepsilon v^\varepsilon\,dx,\tag{5.6}$$

where $u^\pm$ and $v^\pm$ are the restrictions of u^0 and v^0 on Ω_0 and D_2 respectively.

To find the limit in the right-hand side of (5.6) for $\alpha<1$ we use the following inequality

$$\varepsilon^{-\alpha}\int\limits_{G_\varepsilon^{(1)}}\varphi^2\,dx\le C_1\varepsilon^{2-\alpha}\int\limits_{G_\varepsilon^{(1)}}\big(\partial_{x_2}\varphi\big)^2\,dx+C_2\varepsilon^{1-\alpha}\int\limits_{I_0}\varphi^2(x_1,0)\,dx_1.\tag{5.7}$$

Thus $\lim_{\varepsilon\to 0}\varepsilon^{-\alpha}\int_{G_\varepsilon^{(1)}}u^\varepsilon v^\varepsilon\,dx=0$ for $\alpha\in(0,1)$.

If $\alpha = 1$, then with the help of the inequality

$$\varepsilon^{-1} \int\limits_{G_\varepsilon^{(1)}} \big(\varphi(x) - \varphi(x_1,0)\big)^2\, dx \le \varepsilon\, l_1 \int\limits_{G_\varepsilon^{(1)}} \big(\partial_{x_2}\varphi(x)\big)^2\, dx \quad \forall\, \varphi \in H^1(G_\varepsilon^{(1)}),$$

we deduce that $\lim_{\varepsilon\to 0} \varepsilon^{-1} \int_{G_\varepsilon^{(1)}} u^\varepsilon v^\varepsilon\, dx = 4h_1 l_1 \int_{I_0} u^0(x_1,0)\, v^0(x_1,0)\, dx_1$.

Therefore, $\lim_{\varepsilon\to 0} \big(u^\varepsilon, v^\varepsilon\big)_{\mathcal{V}_\varepsilon} = \big(u^0, v^0\big)_{\mathcal{V}_0}$ for $\alpha \in (0,1]$. This means that the first part of condition $\mathbf{D_4}$ holds.

We put by definition that for each function $v \in \mathcal{Z}_0$ $P_\varepsilon \mathrm{S}_\varepsilon v = v$. Then the second condition $\mathbf{D_4}$ is satisfied.

Condition $\mathbf{D_5}$, in fact, has been verified in subsection 3.3 and in subsection 2.4: the result of the action of the operator R_ε in $\mathbf{D_5}$ is the construction of the approximating function on the basis of an eigenfunction of the homogenized problem. Furthermore, the approximating function satisfies inequality (2.58) for $\alpha = 1$ and (3.57) for $\alpha \in (0,1)$ that are analog of the corresponding inequality in condition $\mathbf{D_5}$.

5.1.1 Condition $\mathbf{D_6}$. Pseudovibrations

To verify this condition, we choose the approximating function W_ε in the case when λ_0 coincides with one of the numbers $P_m = \left(\frac{\pi + 2\pi(m-1)}{2l_2}\right)^2$, $m \in \mathbb{N}$ (points of the essential spectrum of the homogenized problems (2.45) and (3.36)) as follows:

$$W_\varepsilon(x) = \begin{cases} \sqrt{\frac{2}{\varepsilon h_2 l_2 P_m}} \cos \sqrt{P_m}(x_2 + l_2), & x \in G_{j_0}^{(2)}(\varepsilon), \\ 0, & x \in \Omega_\varepsilon \setminus G_{j_0}^{(2)}(\varepsilon), \end{cases} \tag{5.8}$$

where $G_{j_0}^{(2)}(\varepsilon)$ is certain fixed rod from the second class.

It is easy to verify that W_ε satisfies the boundary conditions of problem (1.1), $\|W_\varepsilon\|_{\mathcal{H}_\varepsilon} = 1$,

$$-\Delta W_\varepsilon(x) = P_m \rho_\varepsilon(x) W_\varepsilon(x), \quad x \in \Omega_\varepsilon,$$

$$\partial_{x_2} W_\varepsilon(x_1, 0+0) - \partial_{x_2} W_\varepsilon(x_1, 0-0) = b_m(\varepsilon), \quad x_1 \in I_{h_2}^\varepsilon(j_0),$$

where $b_m(\varepsilon) = \varepsilon^{-\frac{1}{2}}(-1)^m \sqrt{\frac{2}{h_2\, l_2}}$, $\;I_{h_2}^\varepsilon(j_0) = \big(\varepsilon(j_0 + \frac{1-h_2}{2}), \varepsilon(j_0 + \frac{1+h_2}{2})\big)$.

From these relations and the definition of operator A_ε (see (2.59) it follows the following integral identity

$$\big(W_\varepsilon - P_m A_\varepsilon W_\varepsilon, \psi\big)_{\mathcal{H}_\varepsilon} = -b_m(\varepsilon) \int_{I_{h_2}^\varepsilon(j_0)} \psi(x_1, 0)\, dx_1 \quad \forall\, \psi \in \mathcal{H}_\varepsilon. \tag{5.9}$$

Using Lemma 1.5 [46, Sec.1] and inequality

$$v^2(x_1,0) \le 2\varepsilon^{-1/2} \int_0^{\gamma_0} v^2(x_1,x_2)\, dx_2 \; + \; 2\varepsilon^{1/2} \int_0^{\gamma_0} (\partial_{x_2} v(x_1,x_2))^2\, dx_2, \tag{5.10}$$

(see Lemma 6 [42, p. 412]), we get

$$\Big| - b_m(\varepsilon) \int_{I_h^\varepsilon(j_0)} \psi(x_1,0)\, dx_1 \Big| \;\le\; \sqrt{2 l_2^{-1}} \|\psi\|_{L_2(I_{h_2}^\varepsilon(j_0))} \;\le\; c\, \varepsilon^{\frac{1}{4}} \|\psi\|_{H^1(\Omega_0)}. \tag{5.11}$$

Then we deduce from (5.9) and (5.11) the following estimate

$$\|W_\varepsilon - P_m A_\varepsilon W_\varepsilon\|_{\mathcal{H}_\varepsilon} \;\le\; c_0\, \varepsilon^{\frac{1}{4}}, \tag{5.12}$$

which shows that condition $\mathbf{D_6}$ holds. Here the constant c_0 is independent of m.

Eigenvibrations with eigenfrequencies near to the discrete spectrum of the homogenized problems are vibrations of the junction Ω_ε like an entire system. From (5.12) it will follow that there are eigenvibrations that have structure of function W_ε (obviously we could take function W_ε that is not equal to zero on several different rods from the second class). This means that different rods of the junction can vibrate and the other stay immobile. Such vibrations were discovered in paper [36] and called *pseudovibrations*. It turn out that there are pseudovibrations in which each rod can have its own frequency and can have quickly oscillating character (see [36, Sec. 5]). In should be noted that energy of a pseudovibration is concentrated on the thin rods.

5.2 The main results

Thus, all conditions $\mathbf{D_1}$–$\mathbf{D_6}$ of the scheme from [40] are satisfied both for problem (1.1) and the corresponding homogenized problem (2.45) for $\alpha = 1$ and the homogenized problem (3.36) for $\alpha \in (0,1)$. Applying this scheme, we get the following theorems.

Theorem 5.1 (the Hausdorff convergence)**.** *Only points of the spectrum of the homogenized problem* (2.45) *if* $\alpha = 1$ ((3.36) *if* $\alpha \in (0,1)$) *are accumulation points for the spectrum of problem* (1.1) *as* $\varepsilon \to 0$.

The eigenvalues $\{\lambda_n(\varepsilon)\}$ at fixed indices n, are usually called *low eigenvalues* (see [36]); the corresponding eigenfunctions are called *low frequency oscillations*.

Definition 5.1. ([36]) *The value* $\mathcal{T} := \sup_{n\in\mathbf{N}} \limsup_{\varepsilon\to 0} \lambda_n(\varepsilon)$ *is called threshold of the low eigenvalues of problem* (1.1).

This value indicate the frequency range where pseudovibrations can appeared.

Recall that $\{\lambda_n(\varepsilon) : n \in \mathbb{N}\}$ is the ordered sequence (1.3) of eigenvalues of problem (1.1), $\{u_n(\varepsilon,\cdot) : n \in \mathbb{N}\}$ is the corresponding sequence of eigenfunctions that are orthonormalized with relations (1.4), and $\{\lambda_0^{(1,n)} : n \in \mathbb{N}\}$ is the first series of eigenvalues of the homogenized problem (2.45) if $\alpha = 1$ (see Th. 2.1) and (3.36) if $\alpha \in (0,1)$ (see Th. 3.1)).

Theorem 5.2 (Low-frequency convergence; $\alpha \in (0,1)$ and $\alpha = 1$)**.** *For any* $n \in \mathbb{N}$

$$\lambda_n(\varepsilon) \to \lambda_0^{(1,n)} \quad \textit{as} \;\; \varepsilon \to 0,$$

and the threshold of the low eigenvalues of problem (1.1) *is equal to* P_1.

There exists a subsequence of the sequence $\{\varepsilon\}$ *(again denoted by* $\{\varepsilon\}$*) such that*

$$\forall\; n \in \mathbb{N} \quad \mathbf{P}_\varepsilon u_n(\varepsilon,\cdot) \to v_0^{(1,n)} \quad \textit{weakly in } H^1(\Omega,\Gamma_1) \quad \textit{as} \;\; \varepsilon \to 0,$$

where $\{v_0^{(1,n)} : n \in \mathbb{N}\}$ *are the corresponding eigenfunctions of the homogenized problem* (2.45) ((3.36)) *that satisfy the following orthonormalized condition*

$$\begin{aligned}\left(v_0^{(1,n)}, v_0^{(1,k)}\right)_{\mathcal{V}_0} &= \int\limits_{\Omega_0} v_0^{(1,n)}\, v_0^{(1,k)}\, dx + h_2 \int\limits_{D_2} v_0^{(1,n)}\, v_0^{(1,k)}\, dx \\ &+ \delta_{\alpha,1} 4 h_1 l_1 \int\limits_{I_0} v_0^{(1,n)}(x_1,0)\, v_0^{(1,k)}(x_1,0)\, dx_1 = \delta_{n,k}.\end{aligned}$$

Next using condition $\mathbf{D_6}$ we get the following theorem.

Theorem 5.3 (Asymptotic behavior near the essential spectrum. Pseudovibrations)**.** *Let* λ_0 *coincides with one of the points* $\left\{P_m = \left(\frac{\pi+2\pi(m-1)}{2l_2}\right)^2,\ m \in \mathbb{N}\right\}$ *of the essential spectrum of the homogenized problem* (2.45) (or (3.36))*.*

Then there exist $c_0 > 0$ *and* $\varepsilon_0 > 0$ *such that for all values of the parameter* $\varepsilon \in (0, \varepsilon_0)$ *the interval*

$$\left(\frac{1}{\lambda_0} - c_0\, \varepsilon^{\frac{1}{4}}\ ,\ \frac{1}{\lambda_0} + c_0\, \varepsilon^{\frac{1}{4}}\right)$$

contains finitely many eigenvalues of the operator A_ε*.*

In addition, there exists a finite linear combination $\widetilde{U}_\varepsilon$ ($\|\widetilde{U}_\varepsilon\|_{\mathcal{H}_\varepsilon} = 1$) *of the eigenfunctions* $\{u_{k(\varepsilon)+i}(\varepsilon,\cdot) : i = \overline{1,\ p(\varepsilon)}\}$ *that correspond, respectively, to the eigenvalues* $\left\{\left(\lambda_{k(\varepsilon)+i}(\varepsilon)\right)^{-1} : i = \overline{1,\ p(\varepsilon)}\right\}$ *of operator* A_ε *from the segment* $\left[\frac{1}{\lambda_0} - c_0\, \varepsilon^{\frac{1}{8}}\ ,\ \frac{1}{\lambda_0} + c_0\, \varepsilon^{\frac{1}{8}}\right]$ *such that*

$$\left\| W_\varepsilon - \widetilde{U}_\varepsilon \right\|_{\mathcal{H}_\varepsilon} \le 2\varepsilon^{\frac{1}{8}},$$

where W_ε *is defined by* (5.8)*.*

For next theorems, where asymptotic estimates are established, we have to consider two cases $\alpha \in (0,1)$ and $\alpha = 1$ separately.

5.2.1 The case $\alpha = 1$

Let $\lambda_0^{(1,n+1)} = \ldots = \lambda_0^{(1,n+r)}$ be an $r-$multiple eigenvalue of the homogenized problem (2.45) from the first series and the corresponding eigenfunctions $v_0^{(1,n+1)}, \ldots, v_0^{(1,n+r)}$ are orthonormalized in $\mathcal{V}_0$. Using formula (2.51), we can construct next term $\varepsilon\, \lambda_1^{(1,n+i)}$ of the asymptotic expansion (2.1) ($i = 1, \ldots, r$) and then define the unique solution $v_1^{(1,n+i)}$ to problem (2.47), which satisfies condition (2.48). Denote by

$$\Lambda_i^{(1,n)}(\varepsilon) := \lambda_0^{(1,n+i)} + \varepsilon\, \lambda_1^{(1,n+i)}$$

the partial sum of (2.1). Assume that $\{\Lambda_i^{(1,n)}(\varepsilon) : i = 1, \ldots, r\}$ split into k groups

$$\Lambda_1^{(1,n)}(\varepsilon) = \ldots = \Lambda_{r_1}^{(1,n)}(\varepsilon) < \ldots < \Lambda_{r_1+\ldots+r_{k-1}+1}^{(1,n)}(\varepsilon) = \ldots = \Lambda_r^{(1,n)}(\varepsilon),$$

where $r_1 + \ldots + r_k = r$.

Theorem 5.4 (Asymptotic estimates for the low eigenvalues; $\alpha = 1$)**.** *For any $\delta > 0$ and $s \in \{1, \dots, k\}$ and for sufficiently small ε, we have*

$$\left| \lambda_{n+r_1+\ldots+r_{s-1}+t}(\varepsilon) - \Lambda^{(1,n)}_{r_1+\ldots+r_s}(\varepsilon) \right| \leq C_1(n, \delta)\, \varepsilon^{2-\delta} \qquad \forall\, t = 1, \dots, r_s \quad (r_0 = 0).$$

In addition, for any $t \in \{1, \dots, r_s\}$ there exist $\{a^{(t,s)}_p(\varepsilon),\ p = 1, \dots, r_s\} \subset \mathbb{R}$ such that $0 < c_1 \leq \sum_{p=1}^{r_s} \big(a^{(t,s)}_p(\varepsilon)\big)^2 \leq c_2$ and

$$\Big\| \sum_{p=1}^{r_s} a^{(t,s)}_p(\varepsilon)\, u_{n+r_1+\ldots+r_{s-1}+p}(\varepsilon, \cdot) - R^{(n+r_1+\ldots+r_{s-1}+t)}_\varepsilon \Big\|_{H^1(\Omega_\varepsilon)} \leq C_2(n, \delta)\ \varepsilon^{2-\delta},$$

where $R^{(n+r_1+\ldots+r_{s-1}+t)}_\varepsilon$ is the approximation function constructed with the help of solutions $v^{(1,n+r_1+\ldots+r_{s-1}+t)}_0$ and $v^{(1,n+r_1+\ldots+r_{s-1}+t)}_1$ by formulas (2.56) *and* (2.57)*.*

It follows from Theorems 5.1 and 5.2 that there exist other converging sequences of eigenvalues $\lambda_{n(\varepsilon)}(\varepsilon)$ so-called *high frequency convergences*; the corresponding eigenfunctions are called *high frequency oscillations.* Obviously, in this case the index n depends on ε and $n(\varepsilon) \to +\infty$ as $\varepsilon \to 0$.

Let $\lambda^{(m,n+1)}_0 = \ldots = \lambda^{(m,n+r)}_0$ be an $r-$multiple eigenvalue of the homogenized problem (2.45) from the m-th series ($m > 1$) and the corresponding eigenfunctions $v^{(m,n+1)}_0, \dots, v^{(m,n+r)}_0$ are orthonormalized in $\mathcal{V}_0$. Using formula (2.51), we construct next term $\varepsilon\, \lambda^{(m,n+i)}_1$ of the asymptotic expansion (2.1) ($i = 1, \dots, r$) and then define the unique solution $v^{(m,n+i)}_1$ to problem (2.47), which satisfies condition (2.48). Denote by

$$\Lambda^{(m,n)}_i(\varepsilon) := \lambda^{(m,n+i)}_0 + \varepsilon\, \lambda^{(m,n+i)}_1$$

the partial sum of (2.1). Assume that $\{\Lambda^{(m,n)}_i(\varepsilon) :\ i = 1, \dots, r\}$ split into k groups

$$\Lambda^{(m,n)}_1(\varepsilon) = \ldots = \Lambda^{(m,n)}_{r_1}(\varepsilon) < \ldots < \Lambda^{(m,n)}_{r_1+\ldots+r_{k-1}+1}(\varepsilon) = \ldots = \Lambda^{(m,n)}_r(\varepsilon),$$

where $r_1 + \ldots + r_k = r$.

Theorem 5.5 (High frequency convergences and estimates; $\alpha = 1$)**.** *For any $\delta > 0$ and $s \in \{1, \dots, k\}$ there exist $\varepsilon_0 > 0$ and $c > 0$ such that for all value of the parameter $\varepsilon \in (0, \varepsilon_0)$ the interval*

$$I^{(m,n)}_s(\varepsilon) := \Big(\Lambda^{(m,n)}_{r_1+\ldots+r_s}(\varepsilon) - c\, \varepsilon^{2-\delta}\ ,\ \Lambda^{(m,n)}_{r_1+\ldots+r_s}(\varepsilon) + c\, \varepsilon^{2-\delta} \Big)$$

contains exactly r_s eigenvalues of problem (1.1).

In addition, for the approximation function $R^{(m,\, n+r_1+\ldots+r_{s-1}+t)}_\varepsilon$ ($t = 1, \dots, r_s$), which constructed with the help of solutions $v^{(m,n+r_1+\ldots+r_{s-1}+t)}_0$ and $v^{(m,n+r_1+\ldots+r_{s-1}+t)}_1$ by formulas (2.56) *and* (2.57)*, the following asymptotic estimate*

$$\Big\| R^{(m,\, n+r_1+\ldots+r_{s-1}+t)}_\varepsilon - \widetilde{U}_i(\varepsilon, \cdot) \Big\|_{H^1(\Omega_\varepsilon)} \leq C\, \varepsilon^{2-\delta}$$

holds, where $\widetilde{U}_i(\varepsilon, \cdot)$ is a linear combination of eigenfunctions of problem (1.1) *that correspond to the eigenvalues from the interval $I^{(m,n)}_s(\varepsilon)$.*

5.2.2 The case $\alpha \in (0,1)$

Let $\lambda_0^{(1,n+1)} = \ldots = \lambda_0^{(1,n+r)}$ be an $r-$multiple eigenvalue of the homogenized problem (3.36) from the first series and the corresponding eigenfunctions $v_0^{(1,n+1)}, \ldots, v_0^{(1,n+r)}$ are orthonormalized in $\mathcal{V}_0$. Using formulas (3.44), (3.49) and (3.54), we successively construct next terms $\varepsilon^{1-\alpha}\,\lambda_{1-\alpha}^{(1,n+i)}$, $\varepsilon\,\lambda_1^{(1,n+i)}$, $\varepsilon^{2-\alpha}\,\lambda_{2-\alpha}^{(1,n+i)}$ of the asymptotic expansion (3.1) ($i = 1, \ldots, r$) and define the unique solutions $v_{1-\alpha}^{(1,n+i)}$, $v_1^{(1,n+i)}$, $v_{2-\alpha}^{(1,n+i)}$ to problems (3.40), (3.45) and (3.50) respectively. Denote by

$$\Lambda_i^{(1,n)}(\varepsilon) := \lambda_0^{(1,n+i)} + \varepsilon^{1-\alpha}\lambda_{1-\alpha}^{(1,n+i)} + \varepsilon\lambda_1^{(1,n+i)} + \varepsilon^{2-\alpha}\lambda_{2-\alpha}^{(1,n+i)}$$

the partial sum of (3.1). Assume that $\{\Lambda_i^{(1,n)}(\varepsilon) :\ i = 1, \ldots, r\}$ split into k groups

$$\Lambda_1^{(1,n)}(\varepsilon) = \ldots = \Lambda_{r_1}^{(1,n)}(\varepsilon) < \ldots < \Lambda_{r_1+\ldots+r_{k-1}+1}^{(1,n)}(\varepsilon) = \ldots = \Lambda_r^{(1,n)}(\varepsilon),$$

where $r_1 + \ldots + r_k = r$.

Theorem 5.6 (Asymptotic estimates for the low eigenvalues; $\alpha \in (0,1)$)**.** *For any $\delta > 0$ and $s \in \{1, \ldots, k\}$ and for sufficiently small ε, we have*

$$\left|\lambda_{n+r_1+\ldots+r_{s-1}+t}(\varepsilon) - \Lambda_{r_1+\ldots+r_s}^{(1,n)}(\varepsilon)\right| \ \le\ C_1(n,\delta)\,\varepsilon^{2-\delta} \qquad \forall\ t = 1, \ldots, r_s \quad (r_0 = 0).$$

In addition, for any $t \in \{1, \ldots, r_s\}$ there exist $\{a_p^{(t,s)}(\varepsilon),\ p = 1, \ldots, r_s\} \subset \mathbb{R}$ such that $0 < c_1 \le \sum_{p=1}^{r_s}\left(a_p^{(t,s)}(\varepsilon)\right)^2 \le c_2$ and

$$\left\|\sum\nolimits_{p=1}^{r_s} a_p^{(t,s)}(\varepsilon)\, u_{n+r_1+\ldots+r_{s-1}+p}(\varepsilon,\cdot) - R_\varepsilon^{(n+r_1+\ldots+r_{s-1}+t)}\right\|_{H^1(\Omega_\varepsilon)} \le C_2(n,\delta)\ \varepsilon^{2-\delta},$$

where $R_\varepsilon^{(n+r_1+\ldots+r_{s-1}+t)}$ is the approximation function constructed with the help of solutions $v_0^{(1,n+r_1+\ldots+r_{s-1}+t)}$, $v_{1-\alpha}^{(1,n+r_1+\ldots+r_{s-1}+t)}$, $v_1^{(1,n+r_1+\ldots+r_{s-1}+t)}$ and $v_{2-\alpha}^{(1,n+r_1+\ldots+r_{s-1}+t)}$ by formulas (3.55) *and* (3.56)*.*

Let $\lambda_0^{(m,n+1)} = \ldots = \lambda_0^{(m,n+r)}$ be an $r-$multiple eigenvalue of the homogenized problem (3.36) from the m-th series ($m > 1$) and the corresponding eigenfunctions $v_0^{(m,n+1)}, \ldots, v_0^{(m,n+r)}$ are orthonormalized in $\mathcal{V}_0$. Using formulas (3.44), (3.49) and (3.54), we successively construct next terms $\varepsilon^{1-\alpha}\,\lambda_{1-\alpha}^{(m,n+i)}$, $\varepsilon\,\lambda_1^{(m,n+i)}$, $\varepsilon^{2-\alpha}\,\lambda_{2-\alpha}^{(m,n+i)}$ of the asymptotic expansion (3.1) ($i = 1, \ldots, r$) and define the unique solutions $v_{1-\alpha}^{(m,n+i)}$, $v_1^{(m,n+i)}$, $v_{2-\alpha}^{(m,n+i)}$ to problems (3.40), (3.45) and (3.50) respectively. Denote by

$$\Lambda_i^{(m,n)}(\varepsilon) := \lambda_0^{(m,n+i)} + \varepsilon^{1-\alpha}\lambda_{1-\alpha}^{(m,n+i)} + \varepsilon\lambda_1^{(m,n+i)} + \varepsilon^{2-\alpha}\lambda_{2-\alpha}^{(m,n+i)}$$

the partial sum of (3.1). Assume that $\{\Lambda_i^{(m,n)}(\varepsilon) :\ i = 1, \ldots, r\}$ split into k groups

$$\Lambda_1^{(m,n)}(\varepsilon) = \ldots = \Lambda_{r_1}^{(m,n)}(\varepsilon) < \ldots < \Lambda_{r_1+\ldots+r_{k-1}+1}^{(m,n)}(\varepsilon) = \ldots = \Lambda_r^{(m,n)}(\varepsilon),$$

where $r_1 + \ldots + r_k = r$.

Theorem 5.7 (High frequency convergences and estimates; $\alpha \in (0,1)$)**.** *For any* $\delta > 0$ *and* $s \in \{1, \dots, k\}$ *there exist* $\varepsilon_0 > 0$ *and* $c > 0$ *such that for all value of the parameter* $\varepsilon \in (0, \varepsilon_0)$ *the interval*

$$I_s^{(m,n)}(\varepsilon) := \left(\Lambda^{(m,n)}_{r_1+\ldots+r_s}(\varepsilon) - c\,\varepsilon^{2-\delta} \ , \ \Lambda^{(m,n)}_{r_1+\ldots+r_s}(\varepsilon) + c\,\varepsilon^{2-\delta} \right)$$

contains exactly r_s *eigenvalues of problem* (1.1)*.*

In addition, for the approximation function $R_\varepsilon^{(m,\, n+r_1+\ldots+r_{s-1}+\, t)}$ $(t = 1, \dots, r_s)$*, which constructed with the help of solutions* $v_0^{(m,\, n+r_1+\ldots+r_{s-1}+\, t)}$*,* $v_{1-\alpha}^{(m,\, n+r_1+\ldots+r_{s-1}+\, t)}$*,* $v_1^{(m,\, n+r_1+\ldots+r_{s-1}+\, t)}$ *and* $v_{2-\alpha}^{(m,\, n+r_1+\ldots+r_{s-1}+\, t)}$ *by formulas* (3.55) *and* (3.56)*, the following asymptotic estimate*

$$\left\| R_\varepsilon^{(m,\, n+r_1+\ldots+r_{s-1}+\, t)} - \widetilde{U}_i(\varepsilon, \cdot) \right\|_{H^1(\Omega_\varepsilon)} \le C\,\varepsilon^{2-\delta}$$

holds, where $\widetilde{U}_i(\varepsilon, \cdot)$ *is a linear combination of eigenfunctions of problem* (1.1) *that correspond to the eigenvalues from the interval* $I_s^{(m,n)}(\varepsilon)$*.*

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