Oberwolfach Preprints

OWP 2012 - 09

ROBERT BOLTJE; SUSANNE DANZ

Ghost Algebras of Double Burnside Algebras via Schur Functors

Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

Oberwolfach Preprints (OWP)

Starting in 2007, the MFO publishes a preprint series which mainly contains research results related to a longer stay in Oberwolfach. In particular, this concerns the Research in Pairs-Programme (RiP) and the Oberwolfach-Leibniz-Fellows (OWLF), but this can also include an Oberwolfach Lecture, for example.

A preprint can have a size from 1 - 200 pages, and the MFO will publish it on its website as well as by hard copy. Every RiP group or Oberwolfach-Leibniz-Fellow may receive on request 30 free hard copies (DIN A4, black and white copy) by surface mail.

Of course, the full copy right is left to the authors. The MFO only needs the right to publish it on its website *www.mfo.de* as a documentation of the research work done at the MFO, which you are accepting by sending us your file.

In case of interest, please send a **pdf file** of your preprint by email to *rip@mfo.de* or *owlf@mfo.de*, respectively. The file should be sent to the MFO within 12 months after your stay as RiP or OWLF at the MFO.

There are no requirements for the format of the preprint, except that the introduction should contain a short appreciation and that the paper size (respectively format) should be DIN A4, "letter" or "article".

On the front page of the hard copies, which contains the logo of the MFO, title and authors, we shall add a running number (20XX - XX).

We cordially invite the researchers within the RiP or OWLF programme to make use of this offer and would like to thank you in advance for your cooperation.

Imprint:

Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

Ghost Algebras of Double Burnside Algebras via Schur Functors*

Robert Boltje
Department of Mathematics
University of California
Santa Cruz, CA 95064
U.S.A.
boltje@ucsc.edu

Susanne Danz
Department of Mathematics
University of Kaiserslautern
P.O. Box 3049
65653 Kaiserslautern
Germany
danz@mathematik.uni-kl.de

March 6, 2012

Abstract

For a finite group G, we introduce a multiplication on the $\mathbb{Q}$-vector space with basis $\mathscr{S}_{G\times G}$, the set of subgroups of $G\times G$. The resulting $\mathbb{Q}$-algebra $\tilde{A}$ can be considered as a ghost algebra for the double Burnside ring $B(G,G)$ in the sense that the mark homomorphism from $B(G,G)$ to $\tilde{A}$ is a ring homomorphism. Our approach interprets $\mathbb{Q}B(G,G)$ as an algebra eAe, where A is a twisted monoid algebra and e is an idempotent in A. The monoid underlying the algebra A is again equal to $\mathscr{S}_{G\times G}$ with multiplication given by composition of relations (when a subgroup of $G\times G$ is interpreted as a relation between G and G). The algebras A and $\tilde{A}$ are isomorphic via Möbius inversion in the poset $\mathscr{S}_{G\times G}$. As an application we improve results by Bouc on the parametrization of simple modules of $\mathbb{Q}B(G,G)$ and also of simple biset functors, by using results by Linckelmann and Stolorz on the parametrization of simple modules of finite category algebras. Finally, in the case where G is a cyclic group of order n, we give an explicit isomorphism between $\mathbb{Q}B(G,G)$ and a direct product of matrix rings over group algebras of the automorphism groups of cyclic groups of order k, where k divides n.

1 Introduction

The main goal of this paper is to find a ghost ring for the double Burnside ring $B(G,G)$ of a finite group G, as an analogue of the classical ghost ring of the usual Burnside ring $B(G)$, and as a generalization of the results in [BD], where this was achieved for the subring $B^{\triangleleft}(G,G)$ of $B(G,G)$, which is additively generated by the classes of left-free (G,G)-bisets. This goal is achieved over the field $\mathbb{Q}$ of rational numbers.

Recall that the *Burnside ring* $B(G)$, the Grothendieck ring of the category of finite left G-sets with respect to disjoint unions and direct products, is embedded via the *mark homomorphism* $\rho_G : B(G) \to$

*MR Subject Classification: 19A22, 20C20. **Keywords:** Burnside ring, double Burnside ring, mark homomorphism, ghost ring, Schur functor, biset, biset functor, twisted category algebra

$\prod_{U\leqslant G}\mathbb{Z}$, $[X]\mapsto |X^U|$, into a direct product of copies of $\mathbb{Z}$, also called the *ghost ring* of $B(G)$, a ring with a considerably simpler structure than $B(G)$ itself. Here, X denotes a left G-set, $[X]$ denotes its image in $B(G)$, and X^U denotes the set of U-fixed points of X, for a subgroup U of G. Every element in the image of ρ_G is constant on conjugacy classes of subgroups of G. After tensoring with $\mathbb{Q}$ one obtains a $\mathbb{Q}$-algebra isomorphism

$$\rho_G\colon \mathbb{Q}B(G) \xrightarrow{\sim} (\prod_{U\leqslant G}\mathbb{Q})^G \subseteq \prod_{U\leqslant G}\mathbb{Q}$$

onto the subalgebra of G-fixed points under the conjugation action. This isomorphism has become one of the main tools to obtain answers to questions related to the ring structure of $B(G)$: for instance, Dress determined the primitive idempotents of $B(G)$, cf. [D], and Yoshida determined the unit group of $B(G)$, cf. [Y], using the mark homomorphism.

The double Burnside ring $B(G,G)$ is the Grothendieck ring of the category of finite (G,G)-bisets with respect to disjoint unions and the tensor product of bisets over G. In contrast to $B(G)$, it is not commutative when $G\neq 1$. The Burnside ring and the double Burnside ring are linked through a natural embedding $B(G)\to B(G,G)$. The double Burnside ring and related constructions have led to the theory of biset functors, initiated by Bouc in [Bc1], see also the more recent book [Bc3]. This theory was the main tool in the determination of the Dade group, an invariant of a p-group that is important in the modular representation theory of finite groups. It was achieved in a sequence of papers by various authors, including Bouc, Thévenaz and Carlson, cf. [BT1] and [T]. Another application of biset functors was the complete determination of the unit group of $B(P)$ for a p-group P, cf. [Bc2]. Bisets also led to surprising and interesting invariants, called stabilizing bisets, of representations of finite groups over arbitrary commutative base rings, cf. [BT2]. Moreover, there is a connection of the double Burnside ring with algebraic topology, cf. [MP], and fusion systems, cf. [BLO], [RS], although these connections only use the subring $B^{\triangleleft}(G,G)$ of $B(G,G)$.

Given the renewed interest in the double Burnside ring, the potential for applications, and the usefulness of the ghost ring of $B(G)$, it seems desirable to determine a ghost algebra of $B(G,G)$ in the following sense. The double Burnside ring can, as an abelian group, be canonically identified with the Burnside group $B(G\times G)$, and this way one has the usual additive embedding via the mark homomorphism $\rho_{G,G}\colon B(G,G)\to \mathbb{Z}\mathscr{S}_{G\times G}$ into the free $\mathbb{Z}$-module over the set $\mathscr{S}_{G\times G}$ of subgroups of $G\times G$, so that one obtains an isomorphism of $\mathbb{Q}$-vector spaces,

$$\rho_{G,G}\colon \mathbb{Q}B(G,G) \xrightarrow{\sim} (\mathbb{Q}\mathscr{S}_{G\times G})^{G\times G} \subseteq \mathbb{Q}\mathscr{S}_{G\times G}\,,$$

onto the subspace of $G\times G$-fixed points under the conjugation action of $G\times G$ on the set of its subgroups, $\mathscr{S}_{G\times G}$. The multiplication on $\mathbb{Q}B(G,G)$ induces a unique algebra structure on the fixed point set that turns $\rho_{G,G}$ into a $\mathbb{Q}$-algebra isomorphism. It is natural to ask what this multiplication on $(\mathbb{Q}\mathscr{S}_{G\times G})^{G\times G}$ looks like. One can also ask whether there is a 'natural' $\mathbb{Q}$-algebra structure on the bigger vector space $\mathbb{Q}\mathscr{S}_{G\times G}$ extending the multiplication on the fixed points. Of course, such an extension might not be unique. Using the (single) Burnside ring $B(G)$ and its mark homomorphism as a model, one would also hope that such a $\mathbb{Q}$-algebra structure on $\mathbb{Q}\mathscr{S}_{G\times G}$ should be simpler than the one of $\mathbb{Q}B(G,G)$ itself. Also, it is reasonable to expect that this multiplication should involve the construction of 'composition', $L*M$, of two subgroups L and M of $G\times G$, given by

$$L*M := \{(x,y)\in G\times G \mid \exists g\in G\colon (x,g)\in L, (g,y)\in M\}\,,$$

which is used in the explicit description of the multiplication in $\mathbb{Q}B(G,G)$ by a Theorem of Bouc, cf. Proposition 2.4.

The key observation leading to answers to the above questions is that the additive map $\alpha_{G,G}\colon \mathbb{Q}B(G,G) \to \mathbb{Q}\mathscr{S}_{G\times G}$, $[X] \mapsto \sum_{x\in X} \mathrm{stab}_{G\times G}(x)$, is almost multiplicative when $\mathbb{Q}\mathscr{S}_{G\times G}$ is viewed as monoid algebra over the monoid $(\mathscr{S}_{G\times G}, *)$. This map is multiplicative if one introduces a 2-cocycle on the monoid $\mathscr{S}_{G\times G}$ and uses the twisted monoid algebra structure on $\mathbb{Q}\mathscr{S}_{G\times G}$. For the purpose of this introduction, we denote this algebra by A. The map $\alpha_{G,G}$ has the effect to 'straighten out' the Mackey-type formula for the multiplication in $B(G,G)$, by omitting the summation over double cosets. Moreover, $\alpha_{G,G}$ is injective with image eAe for a certain idempotent e of A. Thus, the double Burnside algebra and its representation theory can be viewed as the result of a Schur functor applied to a twisted monoid algebra. The $\mathbb{Q}$-linear isomorphism $\zeta_{G,G}\colon \mathbb{Q}\mathscr{S}_{G\times G} \xrightarrow{\sim} \mathbb{Q}\mathscr{S}_{G\times G}$, $L \mapsto \sum_{L'\leqslant L} L'$, transports the twisted monoid algebra structure A on $\mathbb{Q}\mathscr{S}_{G\times G}$ to another algebra structure, which we denote by $\tilde{A}$ on the same $\mathbb{Q}$-vector space. It turns out that the composition $\zeta_{G,G}\circ\alpha_{G,G}$ is equal to the mark homomorphism $\rho_{G,G}$ and we obtain a commutative diagram (omitting the indices of the maps)

$$\begin{array}{ccccc}
\mathbb{Q}B(G,G) & \xrightarrow{\ \alpha\ } & (\mathbb{Q}\mathscr{S}_{G\times G})^{G\times G} = eAe & \subseteq & \mathbb{Q}\mathscr{S}_{G\times G} = A \\
 & \searrow^{\rho} & \downarrow \zeta & & \downarrow \zeta \\
 & & (\mathbb{Q}\mathscr{S}_{G\times G})^{G\times G} = \tilde{e}\tilde{A}\tilde{e} & \subseteq & \mathbb{Q}\mathscr{S}_{G\times G} = \tilde{A}
\end{array}$$

of algebra isomorphisms, with $\tilde{e} := \zeta(e)$.

The first part of the paper is arranged as follows. Section 2 recalls the basic notions related to bisets and the double Burnside ring, Section 3 establishes the properties related to the above diagram, and Section 4 explicitly determines the multiplication formula for the basis elements L and M in $\tilde{A}$, cf. Theorem 4.1. It turns out that, in general, the product of L and M is an alternating sum of very few subgroups N of $G\times G$; and if the image of the right projection of L to G and the image of the left projection of M to G are different then the product is equal to 0.

The rest of the paper is dedicated to applications of the approach to view $\mathbb{Q}B(G,G)$ as a Schur algebra eAe of the twisted monoid algebra A. The first application is the study of simple $\mathbb{Q}B(G,G)$-modules (and also simple biset functors) by first determining the simple A-modules and then using Green's theory, cf. [Gr2, Section 6.2]. Recall that, by Green's theory, every simple eAe-module is of the form eS, where S is a simple A-module. This way, there is a natural injective map from the set of isomorphism classes of simple eAe-modules into the set of isomorphism classes of simple A-modules. The class of S is contained in the image of this embedding if and only if $e\cdot S \neq \{0\}$, a condition that often cannot be checked easily. Section 5 recalls Green's theory and also results on the parametrization of simple modules of twisted category algebras due to Linckelmann–Stolorz ([LS]), and on the structure of inverse category algebras due to Linckelmann ([L]). This is used in Section 6 to obtain a parametrization of the simple A-modules of A (cf. Theorem 6.5). In Section 7 we give an explicit criterion when a simple A-module S satisfies $e\cdot S\neq\{0\}$, cf. Theorem 7.1. This leads to improvements, but only in characteristic 0, of results by Bouc, as given in [Bc3], on the parametrization of simple $\mathbb{Q}B(G,G)$-modules, cf. Theorem 7.5, and simple biset functors, cf. Theorem 7.8. By Bouc's results, the simple $\mathbb{Q}B(G,G)$-modules are parametrized by pairs (H,W), where H is a subquotient of G and W is a simple $\mathbb{Q}\mathrm{Out}(H)$-module. But not all simple modules W occur in this parametrization. We give a purely representation-theoretic necessary condition for W to occur in this parametrization. Finally, in Section 8, we prove that the algebra $\tilde{A}$ is the category algebra of an inverse category when G is cyclic. In this case we can use the results from Section 5 to determine an explicit isomorphism between $\tilde{A}$ and a product of matrix rings over group algebras, cf. Theorem 8.11.

For simplicity we have, so far, only given an account of the results in the setting where one considers a single finite group G. Throughout the paper we develop a more general theory with respect to double Burnside groups $RB(G,H)$, where R is an appropriate commutative ring and $B(G,H)$ is the double Burnside group of two finite groups G and H. Only this more general setting leads to a description of the category of biset functors as a category of modules over a Schur algebra of a twisted category algebra, cf. Example 5.15(c). This is where one really needs category algebras and not only monoid algebras. Our setup of the underlying category on which biset functors are defined is slightly more general than the one used in [Bc3] by using Condition (16) in Section 6, which avoids to require that all group isomorphisms are present in the category. For results on simple biset functors see also [BST].

Unless specified otherwise, all groups occurring in this paper will be finite, and R will always denote an associative commutative unitary ring.

Acknowledgement. The authors' research on this project was supported through a Marie Curie Intra-European Fellowship (grant PIEF-GA-2008-219543), a 'Rückkehrstipendium' of the German Academic Exchange Service (grant D/11/05654), and a 'Research in Pairs' Grant from the Mathematisches Forschungsinstitut Oberwolfach (MFO) in February 2012. It is a pleasure to thank the Universities of Oxford, Jena and Kaiserslautern as well as the MFO, where part of this paper was written, for their kind hospitality.

2 Preliminaries

Throughout this section, let G, H, and K be finite groups. We begin by fixing some general notation, and summarize some basic results that will be essential for this paper. More details can be found in [Bc3, Part I].

2.1 Subgroups and sections. (a) For every subgroup U of G and every $g \in G$, we set $U^g := g^{-1}Ug$ and ${}^gU := gUg^{-1}$. Moreover, we denote by c_g the automorphism $G \to G : h \mapsto ghg^{-1}$.

The set of all subgroups of G will be denoted by $\mathscr{S}_G$, and $\tilde{\mathscr{S}}_G \subseteq \mathscr{S}_G$ will denote a transversal for the conjugacy classes of subgroups of G. For $U \leqslant G$, we denote its G-conjugacy class by $[U]_G$ and its isomorphism class by $[U]$.

(b) By a *section* of G we understand a pair (U,V) such that $V \trianglelefteq U \leqslant G$. The group G acts on the set of all its section via conjugation: if (U,V) is a section of G and if $g \in G$ then we set ${}^g(U,V) := ({}^gU, {}^gV)$ and $(U,V)^g := (U^g, V^g)$.

(c) We denote the canonical projections $G \times H \to G$ and $G \times H \to H$ by p_1 and p_2, respectively, and for every $L \leqslant G \times H$ we further set

$$k_1(L) := \{g \in G \mid (g,1) \in L\} \quad \text{and} \quad k_2(L) := \{h \in H \mid (1,h) \in L\}\,.$$

Note that, for $i = 1,2$, we have $k_i(L) \trianglelefteq p_i(L)$ and p_i induces a group isomorphism $\bar{p}_i : L/(k_1(L) \times k_2(L)) \to p_i(L)/k_i(L)$. Thus one has a group isomorphism

$$\eta_L := \bar{p}_1 \circ \bar{p}_2^{-1} : p_2(L)/k_2(L) \xrightarrow{\sim} p_1(L)/k_1(L)$$

with $\eta_L(hk_2(L)) = gk_1(L)$, for $(g,h) \in L$. In this way one obtains a bijection between the set $\mathscr{S}_{G\times H}$ and the set of quintuples (B,A,η,D,C), where (B,A) is a section of G, (D,C) is a section of H, and $\eta : D/C \xrightarrow{\sim} B/A$ is a group isomorphism; the inverse of this bijection maps a quintuple (B,A,η,D,C) to the group $\{(g,h) \in B \times D \mid \eta(hC) = gA\}$. Since $L/(k_1(L) \times k_2(L)) \cong p_i(L)/k_i(L)$, for $i = 1, 2$, one has

$$|L| = |p_1(L)| \cdot |k_2(L)| = |k_1(L)| \cdot |p_2(L)|\,. \tag{1}$$

In the sequel we will freely identify every subgroup $L \leqslant G \times H$ with its corresponding quintuple $(p_1(L), k_1(L), \eta_L, p_2(L), k_2(L))$. The common isomorphism class of the groups $p_1(L)/k_1(L)$ and $p_2(L)/k_2(L)$ will be denoted by $q(L)$. If α is an automorphism of G then we abbreviate the group $(G, 1, \alpha, G, 1) \leqslant G \times G$ by $\Delta_\alpha(G)$; if $\alpha = c_g$ for some $g \in G$ then we just set $\Delta_g(G) := \Delta_\alpha(G)$, and if $g = 1$ then we set $\Delta(G) := \Delta_1(G)$.

(d) For subgroups $L \leqslant G \times H$ and $M \leqslant H \times K$, we set

$$L * M := \{(g,k) \in G \times K \mid \exists\, h \in H : (g,h) \in L,\ (h,k) \in M\},$$

which is a subgroup of $G \times K$.

2.2 Bisets and tensor products. (a) By a (G,H)-*biset* we will always understand a finite set, equipped with a left G-action and a right H-action that commute with each other. The (G,H)-bisets form a category whose morphisms are the (G,H)-equivariant set maps. As usual, we will freely identify (G,H)-bisets with left $G \times H$-sets, by setting

$$(g,h)x := gxh^{-1} \quad \text{and} \quad gyh := (g, h^{-1})y,$$

for every (G,H)-biset X, every left $G \times H$-set Y, $g \in G$, $h \in H$, $x \in X$, $y \in Y$. In particular, for every (G,H)-biset X and every $L \leqslant G \times H$, one can define the L-*fixed points* X^L of X.

(b) For every (G,H)-biset X, we denote by X° its *opposite biset*, which an (H,G)-biset: as a set X° equals X, and the biset structure on X° is given by $hx^\circ g := (g^{-1}xh^{-1})$, where $g \in G$, $h \in H$, $x \in X$, and x° is x viewed as an element in X°. Note that $X = (X^\circ)^\circ$ as (G,H)-bisets.

On the other hand, one defines, for every subgroup $L \leqslant G \times H$, its *opposite group*

$$L^\circ := \{(h,g) \in H \times G \mid (g,h) \in L\}\,,$$

which is clearly a subgroup of $H \times G$. If $M \leqslant H \times K$ then $(L * M)^\circ = M^\circ * L^\circ$. Moreover, one has an isomorphism of (H,G)-bisets

$$(H \times G)/L^\circ \to (G \times H/L)^\circ, \quad (h,g)L^\circ \mapsto ((g,h)L)^\circ.$$

(c) Suppose that X is a (G,H)-biset and that Y is an (H,K)-biset. Then their cartesian product $X \times Y$ becomes a (G,K)-biset in the obvious way. Moreover, $X \times Y$ is also a left H-set via $h(x,y) := (xh^{-1}, hy)$, for $h \in H$, $x \in X$, $y \in Y$. This H-action commutes with the $G \times K$-action, and the set $X \times_H Y$ of H-orbits on $X \times Y$ inherits the (G,K)-biset structure. We call $X \times_H Y$ the *tensor product* of X and Y and denote the H-orbit of an element $(x,y) \in X \times Y$ by $x \times_H y \in X \times_H Y$.

2.3 (Double) Burnside rings. (a) The *Burnside ring* $B(G)$ of a finite group G is the Grothendieck ring of the category of finite left G-sets with respect to disjoint unions and direct products. The element in $B(G)$ associated to a finite left G-set X will be denoted by $[X]$. Thus, if $\tilde{\mathscr{S}}_G \subseteq \mathscr{S}_G$ is a set of representatives of the conjugacy classes of subgroups of G then the elements $[G/U]$, $U \in \tilde{\mathscr{S}}_G$, form a $\mathbb{Z}$-basis of $B(G)$, the *standard basis* of $B(G)$.

(b) The Grothendieck group of the category of (G,H)-bisets will be denoted by $B(G,H)$, and is called the *double Burnside group* of G and H. Identifying (G,H)-bisets with left $G\times H$-sets as in 2.2(a), we may also identify $B(G,H)$ with the classical Burnside group $B(G\times H)$. As in Part (a), for every (G,H)-biset X, the image of its isomorphism class in $B(G,H)$ will be denoted by $[X]$, and if $\tilde{\mathscr{S}}_{G\times H} \subseteq \mathscr{S}_{G\times H}$ denotes a transversal for the conjugacy classes of subgroup of $G\times H$ then the elements $[G\times H/L]$, $L \in \tilde{\mathscr{S}}_{G\times H}$, form a $\mathbb{Z}$-basis of $B(G,H)$.

(c) The tensor product of bisets gives rise to a $\mathbb{Z}$-bilinear map

$$- \cdot_H - : B(G,H) \times B(H,K) \to B(G,K),\ ([X],[Y]) \mapsto [X \times_H Y],$$

where X is a (G,H)-biset and Y is an (H,K)-biset. In the case where $G = H = K$ this defines a multiplication turning $B(G,G)$ into a ring with identity element $[G] = [G\times G/\Delta(G)]$; we call $B(G,G)$ the *double Burnside ring of G*.

(d) Sending each (G,H)-biset to its opposite biset induces a group isomorphism

$$-^\circ : B(G,H) \to B(H,G),\ [X] \mapsto [X^\circ]$$

with

$$([X] \cdot_H [Y])^\circ = [Y]^\circ \cdot_H [X]^\circ \in B(K,G) \quad \text{and} \quad ([X]^\circ)^\circ = [X] \in B(G,H),$$

where X is a (G,H)-biset and Y is an (H,K)-biset. Thus, in the case where $G = H = K$, this defines an anti-involution of the ring $B(G,G)$.

The following Mackey-type formula shows how to express the tensor product of a standard basis element of $B(G,H)$ and a standard basis element of $B(H,K)$ as a sum of standard basis elements of $B(G,K)$.

2.4 Proposition ([Bc3], 2.3.34) *For $L \leqslant G\times H$ and $M \leqslant H\times K$, one has*

$$[G\times H/L] \cdot_H [H\times K/M] = \sum_{h\in[p_2(L)\backslash H/p_1(M)]} [G\times K/(L * {}^{(h,1)}M)] \in B(G,K),$$

where $[p_2(L)\backslash H/p_1(M)] \subseteq H$ denotes a set of double coset representatives.

In order to analyze the map $- \cdot_H - : B(G,H)\times B(H,K) \to B(G,K)$ in more detail, it will, therefore, be important to examine the $*$-product of subgroups of $G\times H$ with subgroups of $H\times K$.

The first of the following two lemmas is a classical result due to Zassenhaus. The second one is most likely well known to experts; however, we have not been able to find a suitable reference, and therefore include a proof for the reader's convenience.

2.5 Lemma (Butterfly Lemma, [Hup] Hilfssatz I.11.3) *Let (B,A) and (D,C) be two sections of G. Then there exists a canonical isomorphism*

$$\beta(B',A';D',C')\colon D'/C \to B'/A'\,,$$

where $A \leqslant A' \trianglelefteq B' \leqslant B$ and $C \leqslant C' \trianglelefteq D' \leqslant D$ are defined as

$$B' := (B \cap D)A\,, \quad A' := (B \cap C)A\,, \quad D' := (D \cap B)C\,, \quad \text{and} \quad C' := (D \cap A)C\,.$$

The isomorphism $\beta(B', A'; D', C')$ is uniquely determined by the property that it maps xC' to xA' for every $x \in B \cap D$.

The following definition is due to Bouc, cf. [Bc3, Definition 4.3.12].

2.6 Definition Sections (B, A) and (D, C) of G are called *linked* if

$$D \cap A = C \cap B, \quad (D \cap B)C = D, \quad \text{and} \quad (D \cap B)A = B;$$

in this case we write $(D, C) \,=\!\!=\, (B, A)$.

2.7 Lemma *Let $L = (P_1, K_1, \phi, P_2, K_2) \leqslant G \times H$ and $M = (P_3, K_3, \psi, P_4, K_4) \leqslant H \times K$. Then*

$$L * M = (P_1', K_1', \bar\phi \circ \beta(P_2', K_2'; P_3', K_3') \circ \bar\psi, P_4', K_4')\,,$$

where

- *$K_2 \leqslant K_2' \leqslant P_2' \leqslant P_2$ and $K_3 \leqslant K_3' \leqslant P_3' \leqslant P_3$ are determined by the Butterfly Lemma applied to the sections (P_2, K_2) and (P_3, K_3) of H;*
- *$K_1 \leqslant K_1' \leqslant P_1' \leqslant P_1$ and $K_4 \leqslant K_4' \leqslant P_4' \leqslant P_4$ are determined by*

$$P_1'/K_1 = \phi(P_2'/K_2)\,, \quad K_1'/K_1 = \phi(K_2'/K_2)\,,$$
$$P_4'/K_4 = \psi^{-1}(P_3'/K_3)\,, \quad K_4'/K_4 = \psi^{-1}(K_3'/K_3)\,;$$

- *the isomorphisms $\bar\phi\colon P_2'/K_2' \to P_1'/K_1'$ and $\bar\psi\colon P_4'/K_4' \to P_3'/K_3'$ are induced by the isomorphisms ϕ and ψ.*

In particular,

(i) *if $P_2 = P_3$ then $(P_2', K_2') = (P_2, K_2K_3) = (P_3', K_3')$ and*

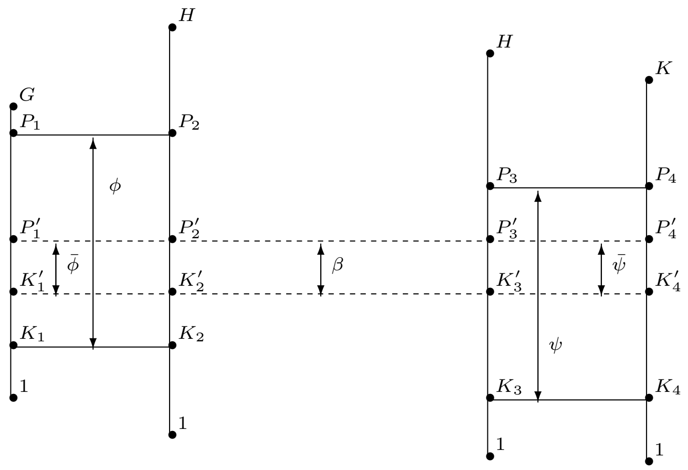
$$L * M = (P_1, K_1', \bar\phi \circ \bar\psi, P_4, K_4'),$$

where K_1' and K_4' are such that $K_1'/K_1 = \phi(K_2K_3/K_2)$ and $K_4'/K_4 = \psi^{-1}(K_2K_3/K_3)$;

(ii) *if $(P_2, K_2) = (P_3, K_3)$ then*

$$L * M = (P_1, K_1, \phi \circ \psi, P_4, K_4).$$

The following diagram illustrates the result of the lemma:



Proof For $g \in G$ one has

$$\begin{aligned} g \in p_1(L*M) &\iff \exists h \in H, k \in K : (g,h) \in L, (h,k) \in M \\ &\iff \exists h \in P_2 \cap P_3 : (g,h) \in L \\ &\iff gK_1 \in \phi((P_2 \cap P_3)K_2/K_2)\,. \end{aligned}$$

This implies that $p_1(L*M) = P_1'$. Similarly one shows that $p_2(L*M) = P_4'$. Next, for $g \in G$ one has

$$\begin{aligned} g \in k_1(L*M) &\iff \exists h \in H\colon (g,h) \in L, (h,1) \in M \\ &\iff \exists h \in k_1(M) : (g,h) \in L \\ &\iff gK_1 \in \phi((K_3 \cap P_2)K_2/K_2)\,. \end{aligned}$$

This implies that $k_1(L*M) = K_1'$. Similarly one shows that $k_2(L*M) = K_4'$. Finally, for $(g,k) \in G \times K$ one has

$$\begin{aligned} (g,k) \in L*M &\iff \exists h \in H : (g,h) \in L, (h,k) \in M \\ &\iff \exists h \in P_2 \cap P_3 : gK_1 = \phi(hK_2), hK_3 = \psi(kK_4) \\ &\Longrightarrow \exists h \in P_2 \cap P_3 : gK_1' = \bar{\phi}(hK_2'), hK_3' = \bar{\psi}(kK_4') \\ &\Longrightarrow gK_1' = \big(\bar{\phi} \circ \beta(P_2', K_2'; P_3', K_3') \circ \bar{\psi}\big)(kK_4')\,. \end{aligned}$$

This implies that the isomorphism $\eta_{L*M}\colon p_2(L*M)/k_2(L*M) \to p_1(L*M)/k_1(L*M)$ is equal to $\bar{\phi} \circ \beta(P_2', K_2'; P_3', K_3') \circ \bar{\psi}$. ∎

3 Natural embeddings of the (double) Burnside ring

Recall from 2.1 that, for every finite group G, we denote by $\mathscr{S}_G$ the set of subgroups of G, and by $\tilde{\mathscr{S}}_G \subseteq \mathscr{S}_G$ a transversal for the conjugacy classes of $\mathscr{S}_G$. Moreover, for $U \leqslant G$, we denote by $[U]_G$ the G-conjugacy class of U. If R is a commutative ring then we denote by $[U]_G^+ \in R\mathscr{S}_G$ the sum of the elements in $[U]_G$; here, $R\mathscr{S}_G$ denotes the free R-module with basis $\mathscr{S}_G$. Since G acts on $\mathscr{S}_G$ by conjugation, we can view $R\mathscr{S}_G$ as a permutation RG-module. Its fixed points are denoted by $(R\mathscr{S}_G)^G$. We use the same notation X^G for the set of G-fixed points on any G-set X.

3.1 Definition We define additive maps

$$\begin{aligned}
\alpha_G\colon B(G) \to \mathbb{Z}\mathscr{S}_G\,, &\quad [X] \mapsto \sum_{x\in X} \mathrm{stab}_G(x)\,,\\
\rho_G\colon B(G) \to \mathbb{Z}\mathscr{S}_G\,, &\quad [X] \mapsto \sum_{U\leqslant G} |X^U|\cdot U\,,\\
\zeta_G\colon \mathbb{Z}\mathscr{S}_G \to \mathbb{Z}\mathscr{S}_G\,, &\quad \quad U \mapsto \sum_{U'\leqslant U} U'\,.
\end{aligned}$$

The map ρ_G is the well-studied classical *mark homomorphism*. Clearly, ζ_G is an isomorphism with inverse μ_G given by $\mu_G(U) = \sum_{U'\leqslant U} \mu_{U',U}\cdot U'$, where $\mu_{U',U}$ denotes the Möbius function with respect to the poset $\mathscr{S}_G$. Extending scalars from $\mathbb{Z}$ to R one obtains R-module homomorphisms $\alpha_G\colon RB(G)\to R\mathscr{S}_G$, $\zeta_G\colon R\mathscr{S}_G \to R\mathscr{S}_G$ and $\rho_G\colon RB(G)\to R\mathscr{S}_G$. Then $\zeta_G\colon R\mathscr{S}_G\to R\mathscr{S}_G$ is an isomorphism of RG-modules and induces an isomorphism

$$\zeta_G\colon (R\mathscr{S}_G)^G \to (R\mathscr{S}_G)^G\,.$$

3.2 Proposition *Let G be a finite group and let R be a commutative ring.*

(a) *For $U\leqslant G$, one has $\alpha_G([G/U]) = [N_G(U):U]\cdot[U]_G^+$. In particular, $\alpha_G\colon RB(G)\to R\mathscr{S}_G$ is injective.*

(b) *One has $\zeta_G\circ\alpha_G = \rho_G$.*

(c) *The images of α_G and ρ_G are contained in $(R\mathscr{S}_G)^G$. In particular, one obtains a commutative diagram*

$$\begin{array}{ccccc}
RB(G) & \xrightarrow{\ \alpha_G\ } & (R\mathscr{S}_G)^G & \subseteq & R\mathscr{S}_G\\
 & \searrow^{\rho_G} & \downarrow{\scriptstyle\zeta_G} & & \downarrow{\scriptstyle\zeta_G}\\
 & & (R\mathscr{S}_G)^G & \subseteq & R\mathscr{S}_G
\end{array}$$

of injective R-module homomorphisms in which both vertical maps ζ_G are isomorphisms. If $|G|$ is invertible in R then also α_G and ρ_G in the above diagram are isomorphisms. For $R=\mathbb{Z}$ one has

$$[(\mathbb{Z}\mathscr{S}_G)^G : \alpha_G(B(G))] = \prod_{U\in\tilde{\mathscr{S}}_G} [N_G(U):U] = [(\mathbb{Z}\mathscr{S}_G)^G : \rho_G(B(G))].$$

Proof (a) The stabilizer subgroup of gU is equal to ${}^gU$ and it occurs as the stabilizer of precisely $[N_G(U):U]$ elements in G/U, namely the elements gnU for $n\in N_G(U)$.

(b) For any finite G-set X one has

$$\zeta_G(\alpha_G([X])) = \sum_{x\in X}\sum_{K\leqslant \mathrm{stab}_G(x)} K = \sum_{\substack{(K,x)\in\mathscr{S}_G\times X\\ x\in X^K}} K = \sum_{K\leqslant G} |X^K|\cdot K = \rho_G([X])\,.$$

(c) By Part (a), the image of α_G is contained in $(R\mathscr{S}_G)^G$ and, since ζ_G is an isomorphism of RG-modules, Part (b) implies that $\rho_G(RB(G)) \subseteq (R\mathscr{S}_G)^G$. The last two statements follow from Part (a) and the fact that the elements $[U]_G^+$, $U \in \tilde{\mathscr{S}}_G$, form an R-basis of $(R\mathscr{S}_G)^G$. ∎

3.3 Remark Let G and H be finite groups and let R be a commutative ring.

(a) Although we will not make use of the following fact in this paper it seems worth to point it out. One can endow $\mathbb{Z}\mathscr{S}_G$ with two ring structures. The first multiplication is given by $(U, V) \mapsto U \cap V$ and the second one by $(U, V) \mapsto \delta_{U,V} U$, for $U, V \in \mathscr{S}_G$. Extending scalars to R we obtain two R-algebra structures on $R\mathscr{S}_G$. It is easy to verify that the G-fixed points $(R\mathscr{S}_G)^G$ form a subalgebra for both R-algebra structures. Moreover, it is easy to verify that all maps in the commutative diagram in Proposition 3.2(c) are R-algebra homomorphisms if one endows $RB(G)$ with the usual multiplication coming from the ring structure of the Burnside ring $B(G)$, and if one endows $R\mathscr{S}_G$ in the top and bottom row with the first and second multiplication, respectively.

(b) From Proposition 3.2(c) we obtain a commutative diagram

$$\begin{array}{ccccc} RB(G,H) & \xrightarrow{\alpha_{G,H}} & (R\mathscr{S}_{G\times H})^{G\times H} & \subseteq & R\mathscr{S}_{G\times H} \\ & \searrow^{\rho_{G,H}} & \downarrow{\scriptstyle \zeta_{G,H}} & & \downarrow{\scriptstyle \zeta_{G,H}} \\ & & (R\mathscr{S}_{G\times H})^{G\times H} & \subseteq & R\mathscr{S}_{G\times H} \end{array}$$

by replacing G with $G \times H$ and slightly renaming the maps. If also K is a finite group we will next define a multiplication map $R\mathscr{S}_{G\times H} \times R\mathscr{S}_{H\times K} \to R\mathscr{S}_{G\times K}$ that is compatible with the tensor product construction $RB(G,H) \times RB(H,K) \to RB(G,K)$ via the embedding α.

3.4 Definition Let G, H, and K be finite groups, and let R be a commutative ring such that $|H|$ is invertible in R. For $L \leqslant G \times H$ and $M \leqslant H \times K$ we set

$$\kappa(L, M) := \frac{|k_2(L) \cap k_1(M)|}{|H|}$$

and we define the R-bilinear map

$$- *_H^\kappa - \colon R\mathscr{S}_{G\times H} \times R\mathscr{S}_{H\times K} \to R\mathscr{S}_{G\times K}$$

by

$$L *_H^\kappa M := \kappa(L, M) \cdot (L * M)\,.$$

3.5 Proposition *Let G, H, K, and I be finite groups, and let R be a commutative ring such that $|H|$ and $|K|$ are invertible in R. For any $L \leqslant G \times H$, $M \leqslant H \times K$, and $N \leqslant K \times I$, the 2-cocycle relation*

$$\kappa(L, M)\,\kappa(L * M, N) = \kappa(L, M * N)\,\kappa(M, N)$$

holds in $R^\times$. In particular, the bilinear maps defined in Definition 3.4 behave associatively:

$$(L *_H^\kappa M) *_K^\kappa N = L *_H^\kappa (M *_K^\kappa N)\,.$$

Moreover, if $|G|$ is invertible in R then the R-module $R\mathscr{S}_{G\times G}$ together with the multiplication $*_G^\kappa$ is an R-algebra with identity element $|G|\cdot\Delta(G)$.

Proof It suffices to show that

$$|k_2(L)\cap k_1(M)|\cdot|k_2(L*M)\cap k_1(N)| = |k_2(L)\cap k_1(M*N)|\cdot|k_2(M)\cap k_1(N)|$$

holds in $\mathbb{Z}$. To show this it suffices to show that one has a group isomorphism

$$\frac{k_2(L)\cap k_1(M*N)}{k_2(L)\cap k_1(M)} \cong \frac{k_2(L*M)\cap k_1(N)}{k_2(M)\cap k_1(N)}\,.$$

In order to construct a homomorphism between these two groups, let $h\in k_2(L)\cap k_1(M*N)$. Then $(1,h)\in L$ and $(h,1)\in M*N$. This implies that there exists $k\in K$ such that $(h,k)\in M$ and $(k,1)\in N$. Thus, $k\in k_2(L*M)\cap k_1(N)$. If also $k'\in k_1(N)$ is such that $(h,k')\in M$ then $k^{-1}k'\in k_2(M)\cap k_1(N)$. Therefore we have a well-defined function that maps the class of h to the class of k, with $(h,k)\in M$ and $(k,1)\in N$. Symmetrically, one has a well-defined function in the other direction that maps the class of an element $k\in k_2(L*M)\cap k_1(N)$ to the class of h, where $(h,k)\in M$ and $(1,h)\in L$. Clearly these two functions are inverses of each other. It is also easy to see that they are group homomorphisms.

The last two statements are immediate consequences of the 2-cocycle relation. □

3.6 Proposition *Let G, H, and K be finite groups, and let R be a commutative ring such that $|H|$ is invertible in R. Moreover let $a\in RB(G,H)$ and $b\in RB(H,K)$. Then*

$$\alpha_{G,K}(a\cdot_H b) = \alpha_{G,H}(a) *_H^\kappa \alpha_{H,K}(b)\,.$$

Proof We may assume that $a=[X]$ and $b=[Y]$ for a finite (G,H)-biset X and a finite (H,K)-biset Y. Then we have

$$\begin{aligned}\alpha_{G,K}([X]\cdot_H[Y]) = \alpha_{G,K}([X\times_H Y]) &= \sum_{x\times_H y\in X\times_H Y} \mathrm{stab}_{G\times K}(x\times_H y)\\ &= \sum_{x\times_H y\in X\times_H Y} \mathrm{stab}_{G\times H}(x)*\mathrm{stab}_{H\times K}(y)\\ &= \sum_{(x,y)\in X\times Y}\frac{|k_2(\mathrm{stab}_{G\times H}(x))\cap k_1(\mathrm{stab}_{H\times K}(y))|}{|H|}\cdot \mathrm{stab}_{G\times H}(x)*\mathrm{stab}_{H\times K}(y)\\ &= \alpha_{G,H}([X]) *_H^\kappa \alpha_{H,K}([Y])\,.\end{aligned}$$

□

For any finite group G and any commutative ring R we set

$$e_G := \sum_{g\in G}\Delta_g(G) = |Z(G)|\cdot\sum_{c\in\mathrm{Inn}(G)}\Delta_c(G)\ \in R\mathscr{S}_{G\times G}\,.$$

In the case where $|G|$ is invertible in R it follows immediately from the definition of the bilinear map $- *_H^\kappa -$ in Definition 3.4 that e_G is an idempotent in the R-algebra $(R\mathscr{S}_{G\times G}, *_G^\kappa)$.

3.7 Proposition *Let R be a commutative ring, and let G and H be finite groups whose orders are invertible in R.*

(a) *One has*

$$\alpha_{G,H}(RB(G,H)) = (R\mathscr{S}_{G\times H})^{G\times H} = e_G *_G^\kappa (R\mathscr{S}_{G\times H}) *_H^\kappa e_H\,.$$

(b) *The double Burnside algebra $RB(G,G)$ is isomorphic to $e_G *_G^\kappa (R\mathscr{S}_{G\times G}) *_G^\kappa e_G$.*

Proof (a) The first equation was already proved in Proposition 3.2(c). For the proof of the second equation note that, for $g \in G$, $h \in H$ and $L \leqslant G \times H$, one has

$$\Delta_g(G) * L * \Delta_h(H) = {}^{(g,h^{-1})}L\,,$$

by Lemma 2.7. This implies that

$$e_G *_G^\kappa L *_H^\kappa e_H = \frac{|N_{G\times H}(L)|}{|G\times H|} \cdot [L]^+_{G\times H}\,.$$

Now the second equation is immediate.

(b) This is an immediate consequence of Proposition 3.6 and Part (a), since $\alpha_{G,G}$ is injective. ∎

Finally, we will show in the next proposition that the bilinear map $- *_H^\kappa -$ from Definition 3.4 is translated via the isomorphism ζ into the following bilinear map.

3.8 Definition Let G, H, and K be finite groups, and let R be a commutative ring such that $|H|$ is invertible in R. We define the R-bilinear map

$$-\tilde{*}_H^\kappa-\colon R\mathscr{S}_{G\times H} \times R\mathscr{S}_{H\times K} \to R\mathscr{S}_{G\times K}\,, \quad (L,M) \mapsto \sum_{N\leqslant G\times K} a_{L,M}^N \cdot N\,,$$

where

$$a_{L,M}^N := \frac{1}{|H|} \sum_{\substack{(L',M')\leqslant(L,M)\\ N\leqslant L'*M'}} \mu^{\mathscr{S}_{G\times H}\times\mathscr{S}_{H\times K}}_{(L',M'),(L,M)} \cdot |k_2(L')\cap k_1(M')|$$

$$= \sum_{\substack{(L',M')\leqslant(L,M)\\ N\leqslant L'*M'}} \mu^{\mathscr{S}_{G\times H}\times\mathscr{S}_{H\times K}}_{(L',M'),(L,M)} \cdot \kappa(L',M')\,.$$

Here, $\mu^{\mathscr{S}_{G\times H}\times\mathscr{S}_{H\times K}}_{(L',M'),(L,M)}$ denotes the Möbius function on the poset $\mathscr{S}_{G\times H}\times\mathscr{S}_{H\times K}$ equipped with the direct product partial order on $\mathscr{S}_{G\times H}\times\mathscr{S}_{H\times K}$, i.e., $(L',M') \leqslant (L,M)$ if and only if $L' \leqslant L$ and $M' \leqslant M$. Note that $a^{N^\circ}_{M^\circ,L^\circ} = a^N_{L,M}$ and $(a\tilde{*}_H^\kappa b)^\circ = b^\circ\tilde{*}_H^\kappa a^\circ$ for $a \in R\mathscr{S}_{G\times H}$ and $b \in R\mathscr{S}_{H\times K}$. Here, $-^\circ\colon R\mathscr{S}_{G\times H} \to R\mathscr{S}_{H\times G}$ is defined as the R-linear extension of the map $\mathscr{S}_{G\times H} \to \mathscr{S}_{H\times G}$, $L \mapsto L^\circ$.

3.9 Proposition *Let G, H, and K be finite groups, and let R be a commutative ring such that $|H|$ is invertible in R.*

(a) *For all $a \in R\mathscr{S}_{G\times H}$ and $b \in R\mathscr{S}_{H\times K}$ one has $\zeta_{G,K}(a *_H^\kappa b) = \zeta_{G,H}(a)\ \tilde{*}_H^\kappa\ \zeta_{H,K}(b)$.*

(b) *If $|G|$ is invertible in R then the map $\rho_{G,G}$ defines an isomorphism between the double Burnside R-algebra $RB(G,G)$ and the subring*

$$(R\mathscr{S}_{G\times G})^{G\times G} = \tilde{e}_G \ \tilde{*}_G^{\kappa} \ R\mathscr{S}_{G\times G} \ \tilde{*}_G^{\kappa} \ \tilde{e}_G$$

of $(R\mathscr{S}_{G\times G}, \tilde{*}_G^{\kappa})$, where $\tilde{e}_G := \zeta_G(e_G)$ is an idempotent in $(R\mathscr{S}_{G\times G}, \tilde{*}_G^{\kappa})$.

Proof (a) Recalling from Definition 3.1 that the maps ζ are isomorphisms with inverse μ, it suffices to prove the equation for $a = \mu_{G,H}(L)$ and $b = \mu_{H,K}(M)$ with $L \leqslant G \times H$ and $M \leqslant H \times K$. In this case we have

$$\begin{aligned}
\zeta_{G,K}(a *_H^{\kappa} b) &= \sum_{L'\leqslant L} \sum_{M'\leqslant M} \mu_{L',L} \cdot \mu_{M',M} \cdot \zeta_{G,K}(L' *_H^{\kappa} M') \\
&= \sum_{(L',M')\leqslant(L,M)} \mu^{\mathscr{S}_{G\times H}\times\mathscr{S}_{H\times K}}_{(L',M'),(L,M)} \cdot \kappa(L',M') \cdot \sum_{N\leqslant L'*M'} N \\
&= \sum_{N\leqslant G\times K} \Big(\sum_{\substack{(L',M')\leqslant(L,M) \\ N\leqslant L'*M'}} \mu^{\mathscr{S}_{G\times H}\times\mathscr{S}_{H\times K}}_{(L',M'),(L,M)} \cdot \kappa(L',M') \Big) \cdot N \\
&= \sum_{N\leqslant G\times K} a^N_{L,M} \cdot N = L \tilde{*}_H^{\kappa} M = \zeta_{G\times H}(a) \ \tilde{*}_H^{\kappa} \ \zeta_{H\times K}(b)\,.
\end{aligned}$$

(b) This follows immediately from Part (a) and Proposition 3.7. □

4 Simplifying the coefficient $a^N_{L,M}$

Throughout this section, G, H, and K denote finite groups, and R denotes a commutative ring such that $|H|$ is invertible in R. Moreover, we fix subgroups $L \leqslant G \times H$, $M \leqslant H \times K$, and $N \leqslant G \times K$. The goal of this section is a simplification of the formula for $a^N_{L,M}$ in Definition 3.8. For this we will need to consider chains

$$\sigma = \big((L_0, M_0) < \cdots < (L_n, M_n)\big) \tag{2}$$

in the partially ordered set $\mathscr{S}_{G\times H} \times \mathscr{S}_{H\times K}$ endowed with the direct product poset structure of $\mathscr{S}_{G\times H}$ and $\mathscr{S}_{H\times K}$, i.e., $(L', M') \leqslant (L, M)$ if and only if $L' \leqslant L$ and $M' \leqslant M$, for $L', L \in \mathscr{S}_{G\times H}$ and $M', M \in \mathscr{S}_{H\times K}$. We define the set

$$\mathcal{X} = \mathcal{X}^N_{L,M} := \{(L', M') \in \mathscr{S}_{G\times H} \times \mathscr{S}_{H\times K} \mid (L', M') \leqslant (L, M) \text{ and } N \leqslant L' * M'\}\,.$$

Note that for every $(L', M') \in \mathcal{X}$ we have

$$p_1(N) \leqslant p_1(L' * M') \leqslant p_1(L') \leqslant p_1(L) \quad \text{and} \quad p_2(N) \leqslant p_2(L' * M') \leqslant p_2(M') \leqslant p_2(M)\,. \tag{3}$$

Next, we define the set

$$\mathcal{C} = \mathcal{C}^N_{L,M}$$

of chains σ as in (2) with $(L_i, M_i) \in \mathcal{X}$, for $i = 0 \ldots, n$, satisfying the extra condition that $(L_n, M_n) = (L, M)$. In particular, $\mathcal{C} = \emptyset$ if $(L, M) \notin \mathcal{X}$. For a chain σ as in (2), its *length* $|\sigma|$ is defined by $|\sigma| = n$, and we set

$$\kappa(\sigma) := \kappa(L_0, M_0) = \frac{|k_2(L_0) \cap k_1(M_0)|}{|H|} \in R\,.$$

With this notation we can rewrite $a_{L,M}^N$ from Definition 3.8 as

$$a_{L,M}^N = \sum_{\sigma \in \mathcal{C}} (-1)^{|\sigma|} \kappa(\sigma)\,. \tag{4}$$

We aim to show that the above expression for $a_{L,M}^N$ can be simplified as follows: we define the subset $\mathcal{Z} = \mathcal{Z}_{L,M}^N$ of $\mathcal{X}_{L,M}^N$ as the set of pairs $(L', M') \in \mathcal{X}_{L,M}^N$ satisfying

$$p_1(N) = p_1(L'), \quad p_2(L') = p_1(M'), \quad \text{and} \quad p_2(M') = p_2(N)\,. \tag{5}$$

Note that $\mathcal{Z} = \emptyset$ unless $N \leqslant L * M$, since $L' * M' \leqslant L * M$ for $(L', M') \leqslant (L, M)$. We further define the set

$$\mathcal{E} = \mathcal{E}_{L,M}^N$$

as the set of all chains σ as in (2) with $(L_i, M_i) \in \mathcal{Z}$, for $i = 0, \ldots, n$, satisfying the extra condition $(L_n, M_n) = (L, M)$. Thus, $\mathcal{Z} \subseteq \mathcal{X}$ and $\mathcal{E} \subseteq \mathcal{C}$. Moreover, $\mathcal{Z} = \emptyset$ and $\mathcal{E} = \emptyset$ unless (L, M) itself satisfies the conditions in (5) and $N \leqslant L * M$.

The following theorem states that we may replace $\mathcal{C}$ with $\mathcal{E}$ in the formula (4).

4.1 Theorem *One has*

$$a_{L,M}^N = \sum_{\sigma \in \mathcal{E}} (-1)^{|\sigma|} \kappa(\sigma)\,.$$

In particular, $L \,\tilde{*}_H^\kappa\, M$ is a linear combination of subgroups $N \leqslant G \times K$ satisfying $p_1(N) = p_1(L)$, $p_2(N) = p_2(M)$, and $N \leqslant L * M$. Moreover, if $p_2(L) \neq p_1(M)$ then $L \,\tilde{*}_H^\kappa\, M = 0$.

Proof We prove the theorem in two steps.

Step 1. Recall from (3) that, for every $(L', M') \in \mathcal{X}$, we have $p_1(N) \leqslant p_1(L')$ and $p_2(N) \leqslant p_2(M')$. We define $\mathcal{Y} = \mathcal{Y}_{L,M}^N$ as the set of all pairs $(L', M') \in \mathcal{X}$ satisfying $p_1(L') = p_1(N)$ and $p_2(M') = p_2(N)$. Moreover, we define $\mathcal{D}$ as the set of chains σ as in (2) with $(L_i, M_i) \in \mathcal{Y}$, for $i = 0, \ldots, n$, satisfying the extra condition that $(L_n, M_n) = (L, M)$. Then $\mathcal{E} \subseteq \mathcal{D} \subseteq \mathcal{C}$. Next we define a function

$$\phi\colon \mathcal{C} \smallsetminus \mathcal{D} \to \mathcal{C} \smallsetminus \mathcal{D}$$

as follows. Let $\sigma \in \mathcal{C} \smallsetminus \mathcal{D}$ be as in (2). Since $\sigma \notin \mathcal{D}$, there exists a minimal index $i \in \{0, \ldots, n\}$ such that $p_1(N) < p_1(L_i)$ or $p_2(N) < p_2(M_i)$. We set

$$L_i' := (p_1(N) \times p_2(L_i)) \cap L_i \quad \text{and} \quad M_i' := (p_1(M_i) \times p_2(N)) \cap M_i\,.$$

Then it is easy to verify that

$$p_1(L_i') = p_1(N)\,, \quad p_2(M_i') = p_2(N)\,, \quad k_2(L_i') = k_2(L_i)\,, \quad k_1(M_i') = k_1(M_i)\,,$$
$$(L_i', M_i') < (L_i, M_i)\,, \quad \text{and} \quad N \leqslant L_i' * M_i'\,.$$

Moreover, if $i \geqslant 1$ then one also has

$$(L_{i-1}, M_{i-1}) \leqslant (L'_i, M'_i)\,.$$

Now, if $i \geqslant 1$ and $(L_{i-1}, M_{i-1}) = (L'_i, M'_i)$ then we define $\phi(\sigma)$ as the chain obtained from σ by removing (L_{i-1}, M_{i-1}), and if $i = 0$ or if $i \geqslant 1$ and $(L_{i-1}, M_{i-1}) < (L'_i, M'_i)$ then we define $\phi(\sigma)$ as the chain obtained from σ by inserting (L'_i, M'_i). Using the above properties of L'_i and M'_i it is easy to see that $\phi(\sigma) \in \mathcal{C} \smallsetminus \mathcal{D}$ and that

$$\phi \circ \phi = \mathrm{id}\,, \quad |\phi(\sigma)| - |\sigma| \in \{\pm 1\}\,, \quad \text{and} \quad \kappa(\phi(\sigma)) = \kappa(\sigma)\,. \tag{6}$$

This implies immediately that

$$\sum_{\sigma \in \mathcal{C}} (-1)^{|\sigma|} \kappa(\sigma) = \sum_{\sigma \in \mathcal{D}} (-1)^{|\sigma|} \kappa(\sigma)\,. \tag{7}$$

Step 2. Similarly we will construct a function

$$\phi\colon \mathcal{D} \smallsetminus \mathcal{E} \to \mathcal{D} \smallsetminus \mathcal{E}$$

such that the three conditions in (6) hold. This immediately implies that

$$\sum_{\sigma \in \mathcal{D}} (-1)^{|\sigma|} \kappa(\sigma) = \sum_{\sigma \in \mathcal{E}} (-1)^{|\sigma|} \kappa(\sigma)\,, \tag{8}$$

and the combination of Equations (7) and (8) yields the desired result.

In order to define ϕ, let $\sigma \in \mathcal{D} \smallsetminus \mathcal{E}$ be as in (2). Since $\sigma \notin \mathcal{E}$, there exists a minimal index $i \in \{0, \ldots, n\}$ such that $p_2(L_i) \neq p_1(M_i)$. We set

$$L'_i := L_i \cap \big(p_1(L_i) \times (p_2(L_i) \cap p_1(M_i))\big) \quad \text{and} \quad M'_i := M_i \cap \big((p_2(L_i) \cap p_1(M_i)) \times p_2(M_i)\big)\,,$$

and claim that

$$\begin{gathered}
\text{(i) } p_2(L'_i) = p_2(L_i) \cap p_1(M_i) = p_1(M'_i)\,, \quad \text{(ii) } (L'_i, M'_i) < (L_i, M_i)\,, \\
\text{(iii) } \kappa(L'_i, M'_i) = \kappa(L_i, M_i)\,, \quad \text{(iv) } N \leqslant L'_i * M'_i\,, \quad \text{(v) } p_1(L'_i) = p_1(N)\,, \\
\text{(vi) } p_2(M'_i) = p_2(N)\,, \quad \text{(vii) } (L_{i-1}, M_{i-1}) \leqslant (L'_i, M'_i) \text{ if } i \geqslant 1\,,
\end{gathered}$$

For the first claim, note that $p_2(L'_i) \leqslant p_2(L_i) \cap p_1(M_i)$ by construction, and if $h \in p_2(L_i) \cap p_1(M_i)$ then there exists $g \in p_1(L_i)$ such that $(g, h) \in L_i$. This implies that $(g, h) \in L'_i$. Similarly, one shows that $p_1(M'_i) = p_2(L_i) \cap p_1(M_i)$.

For the second claim, note that $(L'_i, M'_i) \leqslant (L_i, M_i)$. Moreover, $p_2(L'_i) = p_2(L_i) \cap p_1(M_i) = p_1(M'_i)$ is a proper subgroup of $p_2(L)$ or of $p_1(M)$. This implies $L'_i < L_i$ or $M'_i < M_i$ and, therefore, $(L'_i, M'_i) < (L_i, M_i)$.

Concerning the third claim, note first that $k_2(L'_i) \cap k_1(M'_i) \leqslant k_2(M_i) \cap k_1(L_i)$, since $L'_i \leqslant L_i$ and $M'_i \leqslant M_i$. For the reverse inclusion let $h \in k_2(M_i) \cap k_1(L_i)$. Then clearly $(1, h) \in L'_i$ and $(h, 1) \in M'_i$, so that $h \in k_2(L'_i) \cap k_1(M'_i)$.

To prove the fourth claim, let $(g, k) \in N$. Then there exists $h \in H$ such that $(g, h) \in L_i$ and $(h, k) \in M_i$. This implies $h \in p_2(L) \cap p_1(M)$ and, therefore, $(g, h) \in L'_i$ and $(h, k) \in M'_i$.

In order to prove the fifth claim, note that $p_1(N) \leqslant p_1(L'_i * M'_i) \leqslant p_1(L'_i) \leqslant p_1(L_i) = p_1(N)$ which implies equality everywhere. Similarly the sixth claim follows.

Finally, for the seventh claim, assume that $i \geqslant 1$. If $(g,h) \in L_{i-1}$ then $(g,h) \in L_i$, $g \in p_1(L_i)$, and $h \in p_2(L_{i-1}) = p_1(M_{i-1}) \leqslant p_2(L_i) \cap p_1(M_i)$. Similarly, we have $M_{i-1} \leqslant M'_i$.

Now, if $i \geqslant 1$ and $(L_{i-1}, M_{i-1}) = (L'_i, M'_i)$ then we define $\phi(\sigma)$ as the chain obtained from σ by removing (L_{i-1}, M_{i-1}), and if $i = 0$ or if $i \geqslant 1$ and $(L_{i-1}, M_{i-1}) < (L'_i, M'_i)$ then we define $\phi(\sigma)$ as the chain obtained from σ by inserting the pair (L'_i, M'_i) in front of (L_i, M_i). The seven claims imply immediately that $\phi(\sigma) \in \mathcal{D} \smallsetminus \mathcal{E}$ and that ϕ satisfies the three conditions in (6). This completes the proof of the theorem. ☐

4.2 Proposition *Assume that $k_1(L) = 1$ and $k_1(M) = 1$ (or that $k_2(L) = 1$ and $k_2(M) = 1$). Then*

$$L \tilde{*}_H^{\kappa} M = \begin{cases} \frac{1}{|H|} \cdot L * M & \text{if } p_2(L) = p_1(M), \\ 0 & \text{if } p_2(L) \neq p_1(M). \end{cases}$$

Proof We only prove the statement under the assumption $k_1(L) = 1$ and $k_1(M) = 1$. In the other case the result can be derived from the first one by applying $-^{\circ}$. By Theorem 4.1 we may assume that $p_2(L) = p_1(M)$, since otherwise the product is 0. It suffices to show for arbitrary $N \leqslant G \times K$ that

$$\mathcal{E}_{L,M}^N \neq \emptyset \text{ implies } N = L * M\,, \quad \text{and that } \mathcal{Z}_{L,M}^{L*M} = \{(L,M)\}\,. \tag{9}$$

Assume that $\mathcal{E}_{L,M}^N \neq \emptyset$. Then $(L,M) \in \mathcal{Z}_{L,M}^N$ and therefore $p_2(N) = p_2(M)$. Note that Lemma 2.7(i) implies that $p_2(M) = p_2(L * M)$ and $1 = k_1(L) = k_1(L * M)$. Thus, $p_2(N) = p_2(L * M)$, and $N \leqslant L * M$ implies $k_1(N) = 1$. Equation (1) yields

$$|N| = |k_1(N)| \cdot |p_2(N)| = |k_1(L * M)| \cdot |p_2(L * M)| = |L * M|,$$

and with $N \leqslant L * M$ we obtain $N = L * M$. This shows the first part of (9).

Assume now that $(L', M') \in \mathcal{Z}_{L,M}^{L*M}$. Then $L * M \leqslant L' * M' \leqslant L * M$ and we obtain $L * M = L' * M'$. Moreover, since $p_2(L') = p_1(M')$ and $p_2(L) = p_1(M)$, Lemma 2.7(i) implies $p_2(M) = p_2(L*M) = p_2(L' * M') = p_2(M')$. Further, $k_1(M') \leqslant k_1(M) = 1$ implies $k_1(M') = 1$. Now Equation (1) yields $M' = M$, and we obtain $p_2(L') = p_1(M') = p_1(M) = p_2(L)$. Finally, Lemma 2.7(i) and $k_1(M) = k_1(M') = 1$ imply $k_1(L * M) = k_1(L)$ and $k_1(L' * M') = k_1(L')$. Thus $k_1(L) = k_1(L')$, and Equation (1) implies $L' = L$. ☐

4.3 Remark (a) Proposition 4.2 allows us to recover Theorem 4.7 in [BD], where a ghost ring for the left-free double Burnside ring $B^{\lhd}(G,G)$ was introduced. This is the $\mathbb{Z}$-span of the standard basis elements $[(G \times G)/L]$ of $B(G,G)$, with $L \leqslant G \times G$ satisfying $k_1(L) = 1$. Warning: in [BD] the mark homomorphism $\rho_{G,G}$ was defined differently, with an additional scaling factor.

(b) Note that in [BD] it was possible to define an integral version for the ghost ring of $B^{\lhd}(G,G)$, cf. [BD, Lemma 4.5(c)]. In order to generalize this result to our situation one would have to find non-zero rational numbers b_L, for $L \in \mathscr{S}_{G \times G}$, such that the product

$$\left(b_L \cdot [L]^{+}_{G\times G}\right) \tilde{*}_G^{\kappa} \left(b_M \cdot [M]^{+}_{G \times G}\right)$$

is contained in the $\mathbb{Z}$-span of the elements $b_N \cdot [N]^{+}_{G \times G}$, $N \in \mathscr{S}_{G \times G}$. We do not know whether this is possible.

5 Twisted category algebras

Throughout this section, R denotes a commutative ring and C denotes a category whose objects form a set, denoted by $\mathrm{Ob}(\mathsf{C})$. We denote by $\mathrm{Mor}(\mathsf{C})$ the set of morphisms of C.

The main purpose of this section is to recall basic notions associated with Green's theory of idempotent condensation (also called *Schur functor*) and with twisted category algebras. We will mostly be concerned with the case where C is a finite category, that is, where $\mathrm{Mor}(\mathsf{C})$ (and consequently $\mathrm{Ob}(\mathsf{C})$) is a finite set. Category algebras generalize the concept of monoid algebras: see [Gr1], [St], [L], [LS] for the development of these topics. In Sections 6, 7 and 8 we will apply Green's theory, Theorem 5.8 (by Linckelmann and Stolorz, cf. [LS]) and Theorem 5.13 (by Linckelmann, cf. [L]) to algebras related to the category of biset functors, which in turn are related to the double Burnside algebra by specializing to categories C with just one object.

Two morphisms $s\colon X \to Y$ and $t\colon Y' \to Z$ are called *composable* if $Y = Y'$. In this case we also say that $t \circ s$ *exists*.

5.1 Definition (a) The *category algebra* $R\mathsf{C}$ is the R-algebra defined as follows: the underlying R-module is free with basis $\mathrm{Mor}(\mathsf{C})$. The product ts of two morphisms s and t is defined as $t \circ s$ if $t \circ s$ exists, and it is defined to be 0 if $t \circ s$ does not exist.

(b) A 2-*cocycle* of C with values in the unit group $R^\times$ is a function α that assigns to any two morphisms s and t such that $t \circ s$ exists an element $\alpha(t,s) \in R^\times$ with the following property: for any three morphisms s, t, u of C such that $t \circ s$ and $u \circ t$ exist, one has $\alpha(u \circ t, s)\alpha(u,t) = \alpha(u, t \circ s)\alpha(t,s)$. The *twisted category algebra* $R_\alpha\mathsf{C}$ is the free R-module with R-basis $\mathrm{Mor}(\mathsf{C})$, and with multiplication defined by

$$t \cdot s := \begin{cases} \alpha(t,s) \cdot t \circ s & \text{if } t \circ s \text{ exists,} \\ 0 & \text{otherwise.} \end{cases}$$

5.2 Remark If C has finitely many objects then the sum of the identity morphisms is an identity element of $R\mathsf{C}$, and also $R_\alpha\mathsf{C}$ has an identity element as we will see. If C has infinitely many objects then neither of the R-algebras $R\mathsf{C}$ and $R_\alpha\mathsf{C}$ has an identity element.

5.3 Idempotent morphisms in categories. We recall several notions from [LS].

(a) The set of idempotent endomorphisms in $\mathrm{Mor}(\mathsf{C})$ carries a poset structure: if $e, f \in \mathrm{End}_\mathsf{C}(X)$ are idempotent morphisms of the same object X of C then we set

$$e \leqslant f :\Longleftrightarrow e = e \circ f = f \circ e\,.$$

If e and f are idempotent morphisms of different objects then they are not comparable.

(b) For any object X of C and any idempotent $e \in \mathrm{End}_\mathsf{C}(X)$, the group of invertible elements in $e \circ \mathrm{End}_\mathsf{C}(X) \circ e$ is denoted by Γ_e, and we set $J_e := e \circ \mathrm{End}_\mathsf{C}(X) \circ e \smallsetminus \Gamma_e$.

(c) Let X and Y be objects of C, and let $e \in \mathrm{End}_\mathsf{C}(X)$ and $f \in \mathrm{End}_\mathsf{C}(Y)$ be idempotents. We call e and f *equivalent* if there exist morphisms $s \in f \circ \mathrm{Hom}_\mathsf{C}(X,Y) \circ e$ and $t \in e \circ \mathrm{Hom}_\mathsf{C}(Y,X) \circ f$ such that $t \circ s = e$ and $s \circ t = f$; in this case, the morphisms s and t induce mutually inverse group isomorphisms

$$\Gamma_e \to \Gamma_f,\ u \mapsto s \circ u \circ t; \quad \Gamma_f \to \Gamma_e,\ v \mapsto t \circ v \circ s\,.$$

We remark that in [L], for instance, the idempotents e and f are called isomorphic rather than equivalent. We will later consider categories whose morphisms are finite groups, and it will then be important to not confuse isomorphisms between groups with equivalence of morphisms in the relevant category, whence the different terminology.

(d) Assume now that C is finite. Let X and Y be objects of C and let $e \in \mathrm{End}_{\mathsf{C}}(X)$ and $f \in \mathrm{End}_{\mathsf{C}}(Y)$ be idempotents. The 2-cocycle α restricts to 2-cocycles on the groups Γ_e and Γ_f, and we may consider the twisted group algebras $R_\alpha\Gamma_e$ and $R_\alpha\Gamma_f$ as subrings of $R_\alpha\mathsf{C}$. Let T be a simple $R_\alpha\Gamma_e$-module and let T' be a simple $R_\alpha\Gamma_f$-module. The pairs (e, T) and (f, T') are called *isomorphic* if there exist s and t as in (c) such that $t \circ s = e$ and $s \circ t = f$, and such that the isomorphism classes of T and T' correspond to each other under the R-algebra isomorphism $R_\alpha\Gamma_e \cong R_\alpha\Gamma_f$ induced by s and t, cf. [LS, Proposition 5.2] for a precise definition of this isomorphism. For different choices of s and t the different isomorphisms differ only by inner automorphisms, cf. [LS, Proposition 5.4]. Thus, the definition of (e, T) and (f, T') being isomorphic does not depend on the choice of s and t.

5.4 Condensation functors. We recall Green's theory of idempotent condensation, cf. [Gr2, Section 6.2].

Suppose that A is any ring with identity and that $e \in A$ is an idempotent. Then eAe is a ring with identity e and one has an exact functor

$$A\text{-}\mathbf{Mod} \to eAe\text{-}\mathbf{Mod},\ V \mapsto eV\,, \tag{10}$$

which is often called the *Schur functor* or *condensation functor* with respect to e.

One also has a functor

$$eAe\text{-}\mathbf{Mod} \to A\text{-}\mathbf{Mod},\ W \mapsto Ae \otimes_{eAe} W\,; \tag{11}$$

this functor is in general not exact.

Whenever S is a simple A-module, the eAe-module eS is either 0 or again simple, and every simple eAe-module occurs in this way. The construction $S \mapsto eS$ induces a bijection between the isomorphism classes of simple A-modules that are not annihilated by e and the isomorphism classes of simple eAe-modules. Its inverse can be described as follows: for every simple eAe-module T, the A-module $M := Ae \otimes_{eAe} T$ has a unique simple quotient S, and $eS \cong T$ as eAe-modules. Thus, $S = \mathrm{Hd}(M) := M/\mathrm{Rad}(M)$, the *head* of M, where $\mathrm{Rad}(M)$ denotes the radical of the A-module M, i.e., the intersection of all its maximal submodules.

With this notation, one also has the following basic statements.

5.5 Proposition *Let A be a unitary ring, let $e \in A$ be an idempotent, and let T be a simple eAe-module. Moreover, set $M := Ae \otimes_{eAe} T$ and $S := M/\mathrm{Rad}(M) = \mathrm{Hd}(M)$. Then:*

(a) *$eM \cong T$ as eAe-modules;*

(b) *$e\mathrm{Rad}(M) = \{0\}$;*

(c) *$AeM = M$.*

Proof Since $eS \cong T$ as eAe-modules, Parts (a) and (b) are equivalent. Moreover, Part (a) is proved in [Gr2, Section 6.2]. As for (c), recall that either $AeM = M$ or $AeM \subseteq \mathrm{Rad}(M)$, since AeM is an A-submodule of M. If $AeM \subseteq \mathrm{Rad}(M)$ then $\{0\} = eAeM = (eAe) \cdot eM$, by (b), thus $\{0\} = (eAe) \cdot T = T$, by (a), which is impossible, whence (c). □

In Section 7 we will have to compare the annihilator of the simple A-module S with the annihilator of the simple eAe-module T. The assertion of the following lemma is probably well known; we do, however, include a proof for the reader's convenience.

5.6 Lemma *Let A be a unitary ring, let $e \in A$ be an idempotent, let T be a simple eAe-module, and let S be the simple head of the A-module $Ae \otimes_{eAe} T$. Then, for $x \in A$, one has*

$$xS = \{0\} \quad \textit{if and only if} \quad (eAxAe)T = \{0\}\,.$$

Proof Again we set $M := Ae \otimes_{eAe} T$. Since $S = M/\mathrm{Rad}(M)$, we infer that $aS = \{0\}$ if and only if $xM \subseteq \mathrm{Rad}(M)$, which in turn is equivalent to $AxM \subseteq \mathrm{Rad}(M)$, since $\mathrm{Rad}(M)$ is an A-submodule of M. Since AxM is an A-submodule of M, it is either equal to M or is contained in $\mathrm{Rad}(M)$. Thus, by Proposition 5.5 (b), $AxM \subseteq \mathrm{Rad}(M)$ if and only if $eAxM = \{0\}$. By Proposition 5.5 (c), we have $eAxM = eAxAeM = (eAxAe)eM$, and by Proposition 5.5 (a), $eM \cong T$ as eAe-modules. This now implies that $eAxM = \{0\}$ if and only if $(eAxAe) \cdot T = \{0\}$, and the proof of the lemma is complete. □

5.7 Condensation functors and twisted category algebras. Now consider the case where $A = R_\alpha\mathsf{C}$, for a finite category C and a 2-cocycle α of C with coefficients in $R^\times$. Theorem 5.8 below is due to Linckelmann–Stolorz and describes how the simple A-modules can be constructed via the functor in (11). In order to state the theorem, we fix some further notation: let $e \in \mathrm{End}_\mathsf{C}(X)$ be an idempotent endomorphism in C. Then

$$e' := \alpha(e,e)^{-1}e$$

is an idempotent in $R_\alpha\mathsf{C}$, and we also have $e' \cdot R_\alpha\mathsf{C} \cdot e' = R_\alpha(e \circ \mathrm{End}_\mathsf{C}(X) \circ e)$. One has an R-module decomposition

$$R_\alpha(e \circ \mathrm{End}_\mathsf{C}(X) \circ e) = R_\alpha\Gamma_e \oplus RJ_e\,, \tag{12}$$

where the second summand denotes the R-span of J_e. Note that RJ_e is an ideal and that $R_\alpha\Gamma_e$ is a unitary R-subalgebra of $e' \cdot R_\alpha\mathsf{C} \cdot e'$. For every $R_\alpha\mathsf{C}$-module V, the condensed module $e'V$ becomes an $R_\alpha\Gamma_e$-module by restriction from $R_\alpha(e \circ \mathrm{End}_\mathsf{C}(X) \circ e)$ to $R_\alpha\Gamma_e$. Conversely, given an $R_\alpha\Gamma_e$-module W, we obtain the $R_\alpha\mathsf{C}$-module $R_\alpha\mathsf{C}e' \otimes_{e'R_\alpha\mathsf{C}e'} \tilde{W}$, where $\tilde{W}$ denotes the inflation of W from $R_\alpha\Gamma_e$ to $e'R_\alpha\mathsf{C}e'$ with respect to the decomposition (12).

With this notation one has the following theorem, due to Linckelmann and Stolorz.

5.8 Theorem ([LS], Theorem 1.2) *Let C be a finite category, and let α be a 2-cocycle of C with coefficients in $R^\times$. There is a bijection between the set of isomorphism classes of simple $R_\alpha\mathsf{C}$-modules and the set of isomorphism classes of pairs (e,T), where e is an idempotent endomorphism in C and T is a simple $R_\alpha\Gamma_e$-module; the isomorphism class of a simple $R_\alpha\mathsf{C}$-module S is sent to the isomorphism class of the pair $(e, e'S)$, where e is an idempotent endomorphism that is minimal with the property $e'S \neq \{0\}$. Conversely, the isomorphism class of a pair (e,T) is sent to the isomorphism class of the $R_\alpha\mathsf{C}$-module* $\mathrm{Hd}(R_\alpha\mathsf{C}e' \otimes_{e'R_\alpha\mathsf{C}e'} \tilde{T})$.

5.9 Definition ([Ka]) The category C is called an *inverse category* if, for every $X, Y \in \mathrm{Ob}(\mathsf{C})$ and every $s \in \mathrm{Hom}_\mathsf{C}(X,Y)$, there is a unique $\hat{s} \in \mathrm{Hom}_\mathsf{C}(Y,X)$ such that

$$s \circ \hat{s} \circ s = s \quad \text{and} \quad \hat{s} \circ s \circ \hat{s} = \hat{s}\,.$$

5.10 Category algebras of inverse categories. From now on, let C be a finite inverse category. Note that if s is a morphism in C then $\hat{\hat{s}} = s$ and the morphisms $\hat{s} \circ s$ and $s \circ \hat{s}$ are idempotent morphisms. We recall several constructions from [L], which are based on constructions in [St] for inverse monoids.

(a) One can extend the poset structure on the set of idempotent morphisms of $\mathrm{Mor}(\mathsf{C})$ from 5.3(a) to the set $\mathrm{Mor}(\mathsf{C})$ of all morphisms: for $X, Y \in \mathrm{Ob}(\mathsf{C})$ and $s, t \in \mathrm{Hom}_{\mathsf{C}}(X, Y)$, one defines

$$s \leqslant t :\Longleftrightarrow s = t \circ e, \text{ for some idempotent } e \in \mathrm{End}_{\mathsf{C}}(X)\,.$$

If s and t are morphisms with different domain or codomain then they are incomparable. For $s \in \mathrm{Mor}(\mathsf{C})$ we consider the element

$$\underline{s} := \sum_{t \leqslant s} \mu_{t,s} t \in R\mathsf{C}$$

where $\mu_{t,s}$ denotes the Möbius function with respect to the partial order on $\mathrm{Mor}(\mathsf{C})$. Note that the elements in $\{\underline{s} \mid s \in \mathrm{Mor}(\mathsf{C})\}$ form an R-basis of $R\mathsf{C}$.

(b) Let X and Y be objects of C and let $e \in \mathrm{End}_{\mathsf{C}}(X)$ and $f \in \mathrm{End}_{\mathsf{C}}(Y)$ be idempotent endomorphisms. Then 5.3(c) and Definition 5.9 show that e and f are equivalent if and only if there exists some $s \in f \circ \mathrm{Hom}_{\mathsf{C}}(X, Y) \circ e$ such that $\hat{s} \circ s = e$ and $s \circ \hat{s} = f$. Moreover, the group Γ_e then consists of those endomorphisms $u \in \mathrm{End}_{\mathsf{C}}(X)$ with $\hat{u} \circ u = e = u \circ \hat{u}$.

(c) From the inverse category C one can construct another finite category $G(\mathsf{C})$ as follows:

(i) the objects in $G(\mathsf{C})$ are the pairs (X, e), where $X \in \mathrm{Ob}(\mathsf{C})$ and e is an idempotent in $\mathrm{End}_{\mathsf{C}}(X)$;

(ii) the morphisms in $G(\mathsf{C})$ are triples $(f, s, e) : (X, e) \to (Y, f)$, where $s \in f \circ \mathrm{Hom}_{\mathsf{C}}(X, Y) \circ e$ is such that $\hat{s} \circ s = e$ and $s \circ \hat{s} = f$; in particular, e and f are equivalent endomorphisms in C;

(iii) if $(f, s, e) : (X, e) \to (Y, f)$ and $(g, t, f) : (Y, f) \to (Z, g)$ are morphisms in $G(\mathsf{C})$ then their composition $(X, e) \to (Z, g)$ is defined as $(g, t, f) \circ (f, s, e) := (g, t \circ f \circ s, e)$.

Note that $G(\mathsf{C})$ is a *groupoid*, i.e., a category in which every morphism is an isomorphism.

5.11 Lemma ([L], Theorem 4.1; [St], Theorem 4.2) *Let C be a finite inverse category. Then the category algebras $R\mathsf{C}$ and $RG(\mathsf{C})$ are isomorphic. More precisely, the maps*

$$RG(\mathsf{C}) \to R\mathsf{C}\,,\ (f, s, e) \mapsto \underline{s}\,, \quad \textit{and} \quad R\mathsf{C} \to RG(\mathsf{C})\,,\ s \mapsto \sum_{t \leqslant s} (t \circ \hat{t}, t, \hat{t} \circ t)\,,$$

define mutually inverse R-algebra isomorphisms.

5.12 Remark It is well known that the category algebra $R\mathsf{D}$ of a finite groupoid D is isomorphic to a direct product of matrix algebras over group algebras. More precisely, let E be a set of representatives of the isomorphism classes of objects e of D, and for each $e \in E$ let $n(e)$ denote the number of objects isomorphic to e. Since D is a groupoid, every endomorphism of $e \in E$ is an automorphism of e, and id_e is the unique idempotent in $\mathrm{End}_{\mathsf{D}}(e)$; in particular, we have $\Gamma_{\mathrm{id}_e} = \mathrm{End}_{\mathsf{D}}(e)$. For convenience we may thus denote $\mathrm{End}_{\mathsf{D}}(e)$ by Γ_e.

Note that, in the case where $\mathsf{D} = G(\mathsf{C})$ for some finite inverse category C, this is consistent with 5.3(b): if $(X, e) \in G(\mathsf{C})$ then $\mathrm{End}_{G(\mathsf{C})}((X, e)) = \{(e, s, e) \mid s \in e \circ \mathrm{End}_{\mathsf{C}}(X) \circ e,\ \hat{s} \circ s = e = s \circ \hat{s}\}$, and $\{s \in e \circ \mathrm{End}_{\mathsf{C}}(X) \circ e \mid \hat{s} \circ s = e = s \circ \hat{s}\}$ is the set of invertible elements in $e \circ \mathrm{End}_{C}(X) \circ e$, thus equal to Γ_e as defined in 5.3(b).

Suppose again that D is an arbitrary finite groupoid. Then there exists an R-algebra isomorphism

$$\epsilon\colon R\mathsf{D} \to \bigtimes_{e\in E} \mathrm{Mat}_{n(e)}(R\Gamma_e), \tag{13}$$

which can be defined as follows: we fix $e \in E$, denote by $e = e_1, \ldots, e_{n(e)}$ the objects that lie in the isomorphism class of e, and choose an isomorphism $s_i\colon e \to e_i$ for every $i \in \{1, \ldots, n(e)\}$. Let s be a morphism in D. If the domain of s is not isomorphic to e then the matrix in the e-component of $\epsilon(s)$ is defined to be 0. If the domain of s is isomorphic to e then there exist unique elements $i, j \in \{1, \ldots, n(e)\}$ such that $s\colon e_j \to e_i$. In this case the e-component of $\epsilon(s)$ is the matrix all of whose entries are equal to 0 except the (i, j)-entry, which is equal to $s_i^{-1} \circ s \circ s_j$. The isomorphism ϵ depends on the ordering of the elements in an isomorphism class and on the choices of the isomorphisms s_i. However, any two isomorphisms defined as above differ only by an inner automorphism.

Combining the two isomorphisms in Lemma 5.11 and Remark 5.12 one obtains the following theorem, due to Linckelmann, cf. [L, Corollary 4.2].

5.13 Theorem *Let C be a finite inverse category. Let E be a set of representatives of the equivalence classes of idempotents in* $\mathrm{Mor}(\mathsf{C})$ *and, for $e \in E$, let $n(e)$ denote the cardinality of the equivalence class of e. Then there exists an R-algebra isomorphism*

$$\omega\colon R\mathsf{C} \to \bigtimes_{e\in E} \mathrm{Mat}_{n(e)}(R\Gamma_e)\,. \tag{14}$$

5.14 Remark (a) Using the explicit description of the isomorphism ϵ in Remark 5.12 we obtain the following description of the e-component of the isomorphism ω in Theorem 5.13: let $e = e_1, \ldots, e_{n(e)}$ denote the idempotent morphisms of C that are equivalent to e and for every $i \in \{1, \ldots, n(e)\}$ let s_i be a morphism such that $\hat{s}_i \circ s_i = e$ and $s_i \circ \hat{s}_i = e_i$. For every morphism s of C, the e-component of $\omega(\underline{s})$ is defined to be 0 if the idempotent $\hat{s} \circ s$ is not equivalent to e, and if it is equivalent to e then there exist unique elements $i, j \in \{1, \ldots, n(e)\}$ such that $\hat{s} \circ s = e_j$ and $s \circ \hat{s} = e_i$. In this case the e-component of $\omega(\underline{s})$ is the matrix all of whose entries are equal to 0 except the (i, j)-entry, which is equal to $s_i \circ s \circ \hat{s}_j$.

(b) It follows immediately from (a) that the central idempotent of $R\mathsf{C}$ corresponding to the identity matrix in the e-component, for $e \in E$ and with the notation introduced in (a), is equal to

$$\tilde{e} := \sum_{i=1}^{n(e)} \underline{e_i} \in R\mathsf{C}\,.$$

This element is independent of the choices made when constructing the isomorphism ω. Moreover, the e-th component $\mathrm{Mat}_{n(e)}(R\Gamma_e)$ in (14) corresponds to a two-sided ideal $I_e = R\mathsf{C} \cdot \tilde{e}$, which is independent of the choices involved in the definition of ω, and one has $R\mathsf{C} = \bigoplus_{e\in E} I_e$. By (a), the elements $\underline{s} \in R\mathsf{C}$, where s runs through all morphisms in C such that the idempotent $\hat{s} \circ s$ is equivalent to e, form an R-basis of I_e.

We will use Theorems 5.8 and 5.13 on category algebras in two situations. The following example describes the first one; it will be used in Sections 6 and 7.

5.15 Example (a) We denote by B a category with the following properties: its objects form a set of finite groups such that every isomorphism type of finite group is represented in $\mathrm{Ob}(\mathsf{B})$. The morphisms in B are defined by

$$\mathrm{Hom}_{\mathsf{B}}(H,G) := \mathscr{S}_{G\times H}\,,$$

for $G,H \in \mathrm{Ob}(\mathsf{B})$, and the composition is defined by

$$L \circ M := L * M\,,$$

for $L \in \mathscr{S}_{G\times H}$, $M \in \mathscr{S}_{H\times K}$, with $G,H,K \in \mathrm{Ob}(\mathsf{B})$. The identity element of the object G of B is equal to $\Delta(G)$.

(b) In the sequel we assume that C is a subcategory of B. For $G,H \in \mathrm{Ob}(\mathsf{C})$, we set $\mathsf{C}_{G,H} := \mathrm{Hom}_{\mathsf{C}}(H,G)$. For completeness we also set $\mathsf{C}_{G,H} := \emptyset$ if $G \in \mathrm{Ob}(\mathsf{B}) \smallsetminus \mathrm{Ob}(\mathsf{C})$ or $H \in \mathrm{Ob}(\mathsf{B}) \smallsetminus \mathrm{Ob}(\mathsf{C})$. Let R be a commutative ring such that $|G| \in R^\times$ for all $G \in \mathrm{Ob}(\mathsf{C})$. Then, by Proposition 3.5, the function κ defined by

$$\kappa(L,M) := \frac{|k_2(L)\cap k_1(M)|}{|H|} \in R^\times\,,$$

with $L \in \mathsf{C}_{G,H}$, $M \in \mathsf{C}_{H,K}$, for $G,H,K \in \mathrm{Ob}(\mathsf{C})$, defines a 2-cocycle on C. We obtain a twisted category algebra $R_\kappa\mathsf{C}$, which we will denote by $A^\kappa_{\mathsf{C},R}$.

(c) Recall from [W, Page 105] that biset functors over R, for an arbitrary commutative ring R, can be considered as modules for the algebra $\bigoplus_{G,H\in\mathrm{Ob}(\mathsf{B})} RB(G,H)$, where the multiplication on two components $RB(G,H)$ and $RB(H',K)$ is induced by the tensor product of bisets over H if $H = H'$, and is defined to be 0 if $H \neq H'$. Note that this algebra has no identity element, since $\mathrm{Ob}(\mathsf{B})$ is infinite. If C is a subcategory of B and if $\mathsf{C}_{G,H} \subseteq \mathscr{S}_{G\times H}$ is closed under $G\times H$-conjugation then we define

$$RB^{\mathsf{C}}(G,H) := \langle [(G\times H)/L] \mid L \in \mathsf{C}_{G,H}\rangle_R \subseteq RB(G,H)\,,$$

the R-span of the elements $[G\times H/L]$ with $L \in \mathsf{C}_{G,H}$. If C is finite then the corresponding R-subalgebra $\bigoplus_{G,H\in\mathrm{Ob}(\mathsf{C})} RB^{\mathsf{C}}(G,H)$ of $\bigoplus_{G,H\in\mathrm{Ob}(\mathsf{B})} RB(G,H)$ has an identity element. If, moreover, $|G| \in R^\times$ for every $G \in \mathrm{Ob}(\mathsf{C})$ then the maps $\alpha_{G,H}$, $G,H \in \mathrm{Ob}(\mathsf{C})$, yield an R-algebra isomorphism

$$\bigoplus_{G,H\in\mathrm{Ob}(\mathsf{C})} RB^{\mathsf{C}}(G,H) \cong e_{\mathsf{C}} A^\kappa_{\mathsf{C},R} e_{\mathsf{C}}\,, \tag{15}$$

where

$$e_{\mathsf{C}} := \sum_{G\in\mathrm{Ob}(\mathsf{C})} e_G$$

is an idempotent in $A^\kappa_{\mathsf{C},R}$, cf. Proposition 3.7 and the paragraph preceding it. Recall that $e_G := \sum_{g\in G} \Delta_g(G)$, for $G \in \mathrm{Ob}(\mathsf{C})$. Many considerations about biset functors on the category of all finite groups can be reduced to the case of finitely many groups. Thus, the algebra $A^\kappa_{\mathsf{C},R}$ through its condensed algebra (15) can be used to study biset functors over R.

We will study the simple modules of $A^\kappa_{\mathsf{C},R}$ and $e_{\mathsf{C}} A^\kappa_{\mathsf{C},R} e_{\mathsf{C}}$ in Sections 6 and 7, respectively, for appropriate choices of C and R.

We will see in Section 8 that, in special cases, the multiplication $\tilde{*}^\kappa_H$ from Definition 3.8 can be related to the following construction.

For the remainder of this section we fix a set $\mathscr{D}$ of finite groups. We will introduce an inverse category $\tilde{\mathsf{C}} = \tilde{\mathsf{C}}(\mathscr{D})$ related to this set $\mathscr{D}$. Later, in Section 8, we will see that in the case where the groups in $\mathscr{D}$ are cyclic we obtain an R-algebra isomorphism between the category algebra $R\tilde{\mathsf{C}}$ and $\bigoplus_{G,H\in\mathscr{D}} RB(G,H)$. There, R will denote a commutative ring such that $|G|$ and $|\mathrm{Aut}(G)|$ are units in R, for all $G \in \mathscr{D}$.

5.16 Definition For finite groups G and H and a subgroup $L \leqslant G \times H$, we set

$$\mathscr{P}(L) := \{L' \leqslant L \mid p_1(L') = p_1(L) \text{ and } p_2(L') = p_2(L)\}\,.$$

Given the set $\mathscr{D}$ of finite groups, we define a category $\tilde{\mathsf{C}} = \tilde{\mathsf{C}}(\mathscr{D})$ as follows:
(i) $\mathrm{Ob}(\tilde{\mathsf{C}}) = \{(G, G') \mid G \in \mathscr{D}, G' \leqslant G\}$;
(ii) for $(G, G'), (H, H') \in \mathrm{Ob}(\tilde{\mathsf{C}})$, we set $\mathrm{Hom}_{\tilde{\mathsf{C}}}((H, H'), (G, G')) := \mathscr{P}(G' \times H')$;
(iii) for $(G, G'), (H, H'), (K, K') \in \mathrm{Ob}(\tilde{\mathsf{C}})$, ${}_GL_H \in \mathrm{Hom}_{\tilde{\mathsf{C}}}((H, H'), (G, G'))$, and ${}_HM_K \in \mathrm{Hom}_{\tilde{\mathsf{C}}}((K, K'), (H, H'))$, we define the composition ${}_GL_H \circ {}_HM_K := {}_GL_H * {}_HM_K$.
Here, we use the notation ${}_GL_H$, since every morphism needs to determine its source and target objects. While $G' = p_1(L)$ and $H' = p_2(L)$ are determined by the notation L, G and H are not.

With this notation, we have the following proposition.

5.17 Proposition (a) *Let $(G, G'), (H, H') \in \mathrm{Ob}(\tilde{\mathsf{C}})$, and let ${}_GL_H \in \mathrm{Hom}_{\tilde{\mathsf{C}}}((H, H'), (G, G'))$. Then $({}_GL_H)^\circ$ is the unique element ${}_HM_G \in \mathrm{Hom}_{\tilde{\mathsf{C}}}((G, G'), (H, H'))$ such that*

$${}_GL_H * {}_HM_G * {}_GL_H = {}_GL_H \quad \text{and} \quad {}_HM_G * {}_GL_H * {}_HM_G = {}_HM_G\,;$$

in particular, $\tilde{\mathsf{C}}$ is an inverse category.

(b) *The idempotent endomorphisms in $\tilde{\mathsf{C}}$ are precisely the endomorphisms of the form*

$${}_G(P, K, \mathrm{id}, P, K)_G \qquad (K \trianglelefteqslant P \leqslant G \in \mathscr{D})\,.$$

Moreover, two idempotent endomorphisms $e := {}_G(P, K, \mathrm{id}, P, K)_G$ and $f := {}_H(P', K', \mathrm{id}, P', K')_H$ in $\tilde{\mathsf{C}}$ are equivalent if and only if $P/K \cong P'/K'$.

(c) *If $e := {}_G(P, K, \mathrm{id}, P, K)_G$ is an idempotent endomorphism in $\tilde{\mathsf{C}}$ then one has $\Gamma_e = \{{}_G(P, K, \alpha, P, K)_G \mid \alpha \in \mathrm{Aut}(P/K)\}$; in particular, $\Gamma_e \cong \mathrm{Aut}(P/K)$, by Lemma 2.7(ii).*

Proof (a) It follows immediately from Lemma 2.7 that ${}_HM_G := ({}_GL_H)^\circ$ has the desired property. Conversely, suppose that ${}_HM_G \in \mathrm{Hom}_{\tilde{\mathsf{C}}}((G, G'), (H, H'))$ satisfies $L * M * L = L$ and $M * L * M = M$. By Lemma 2.7(i), this forces $k_2(L) = k_1(M)$ as well as $k_1(L) = k_2(M)$. Note that $p_2(L) = H' = p_1(M)$ and $p_1(L) = G' = p_2(M)$ by definition. Now, $L * M * L = L$ and Lemma 2.7(i) again force that $\eta_L \circ \eta_M \circ \eta_L = \eta_L$. This implies $\eta_L = \eta_M^{-1}$ and $M = L^\circ$.

Assertions (b) and (c) are now easy consequences of (a) and 5.10. □

5.18 Remark Proposition 5.17 thus shows, in particular, that the equivalence classes of idempotent endomorphisms in $\tilde{\mathsf{C}}$ are in bijection with the isomorphism classes of groups occurring as subquotients of groups in $\mathscr{D}$.

6 The simple $A^{\kappa}_{\mathsf{C},R}$-modules

Throughout this section, let B be a category as in Example 5.15(a) and let C be a finite subcategory of B. Moreover, let R be a commutative ring such that $|G|$ is invertible in R for each $G \in \mathrm{Ob}(\mathsf{C})$. In this section we will use Theorem 5.8 to construct the simple modules of the twisted category R-algebra $A^{\kappa}_{\mathsf{C},R}$ introduced in 5.15(b). This is in preparation for Section 7, where we determine the simple modules of the R-algebra $e_{\mathsf{C}} A^{\kappa}_{\mathsf{C},R} e_{\mathsf{C}}$, which is related to the category of biset functors. Note that all the results apply, in particular, to the case where the category C has only one object G and $\mathsf{C}_{G,G} = \mathscr{S}_{G\times G}$.

We say that a finite group Q is *realized* (*by a morphism*) in C if there exist $G, H \in \mathrm{Ob}(\mathsf{C})$ and $L \in \mathsf{C}_{G,H} = \mathrm{Hom}_{\mathsf{C}}(H,G)$ such that $q(L) = [Q]$, the isomorphism class of Q; recall that $q(L)$ denotes the isomorphism class of $p_1(L)/k_1(L)$. Note that, in particular, each object G of C is realized in C by the identity morphism $\Delta(G) \in \mathrm{Hom}_{\mathsf{C}}(G,G)$.

Throughout this section, we will assume that C satisfies the following property:

> If Q is a finite group that is realized in C, if G and H are objects in C, and if (P_1, K_1) and (P_2, K_2) are sections of G and H, respectively, such that $P_1/K_1 \cong Q \cong P_2/K_2$, then there exists an isomorphism $\alpha\colon P_2/K_2 \xrightarrow{\sim} P_1/K_1$ such that $(P_1, K_1, \alpha, P_2, K_2) \in \mathsf{C}_{G,H}$. (16)

We first derive two consequences from Condition (16). For a section (P, K) of a group G we set

$$e_{(P,K)} := (P, K, \mathrm{id}_{P/K}, P, K) \in \mathscr{S}_{G\times G}\,;$$

note that, by Lemma 2.7(ii), the element $e_{P,K}$ is an idempotent in the monoid $(\mathscr{S}_{G\times G}, *)$.

6.1 Lemma *Let Q be a finite group that is realized by a morphism in C. Let G be an object of C and let (P, K) be a section of G such that $P/K \cong Q$. Then the idempotent $e_{(P,K)}$ belongs to $\mathsf{C}_{G,G}$.*

Proof Since C satisfies (16), there exists an automorphism $\alpha \in \mathrm{Aut}(P/K)$ such that $(P, K, \alpha, P, K) \in C_{G,G}$. By Lemma 2.7(ii), the $|\mathrm{Aut}(P/K)|$-th power of (P, K, α, P, K) is equal to $e_{(P,K)}$, showing that $e_{(P,K)}$ belongs to $\mathsf{C}_{G,G}$. □

For a section (P, K) of an object G of C with $e_{(P,K)} \in \mathsf{C}_{G,G}$, we set

$$\mathrm{Aut}_{\mathsf{C}}(P/K) := \{\alpha \in \mathrm{Aut}(P/K) \mid (P, K, \alpha, P, K) \in \mathsf{C}_{G,G}\}\,.$$

Clearly, this is a subgroup of $\mathrm{Aut}(P/K)$. The following lemma shows that the isomorphism type of $\mathrm{Aut}_{\mathsf{C}}(P/K)$ does in fact only depend on the isomorphism type of the group P/K (not on the actual section (P, K)). We omit the straightforward proof.

6.2 Lemma *Assume that G and G' are objects of C and that (P, K) and (P', K') are sections of G and G', respectively, such that $e_{(P,K)} \in \mathsf{C}_{G,G}$, $e_{(P',K')} \in \mathsf{C}_{G',G'}$ and $P/K \cong P'/K'$. Let $\gamma\colon P'/K' \xrightarrow{\sim} P/K$ be an isomorphism such that $(P, K, \gamma, P', K') \in \mathsf{C}_{G,G'}$ (which exists by (16)). Then the map*

$$c_\gamma\colon \mathrm{Aut}_{\mathsf{C}}(P'/K') \to \mathrm{Aut}_{\mathsf{C}}(P/K)\,, \quad \alpha \mapsto \gamma\alpha\gamma^{-1}\,,$$

is a group isomorphism, and if also $\delta\colon P'/K' \xrightarrow{\sim} P/K$ is an isomorphism with $(P, K, \delta, P', K') \in \mathsf{C}_{G,G'}$ then $c_\delta = c_\alpha \circ c_\gamma$, where $\alpha := \delta \circ \gamma^{-1} \in \mathrm{Aut}_{\mathsf{C}}(P/K)$.

Recall from 5.3(c) that two idempotents $e \in \mathsf{C}_{G,G}$ and $f \in \mathsf{C}_{H,H}$ are called equivalent if there exist elements $s \in f * \mathsf{C}_{H,G} * e$ and $t \in e * \mathsf{C}_{G,H} * f$ such that $t * s = e$ and $s * t = f$.

6.3 Proposition *Let G and H be objects in C.*

(a) *A morphism $L = (P_1, K_1, \alpha, P_2, K_2) \in \mathsf{C}_{G,G}$ is an idempotent if and only if (P_1, K_1) and (P_2, K_2) are linked and $\alpha = \beta(P_1, K_1; P_2, K_2)$.*

(b) *If $L = (P_1, K_1, \alpha, P_2, K_2) \in \mathsf{C}_{G,G}$ is an idempotent morphism then L is equivalent to $e_{(P_i,K_i)}$, for $i = 1, 2$.*

(c) *Two idempotent morphisms $L \in \mathsf{C}_{G,G}$ and $M \in \mathsf{C}_{H,H}$ are equivalent if and only if $q(L) = q(M)$.*

Proof (a) By Lemma 2.7, we have $L * L = L$ if and only if (P_2, K_2) and (P_1, K_1) are linked and $\alpha = \alpha\circ\beta(P_2, K_2; P_1, K_1)\circ\alpha$. But the last equality is equivalent to the equality $\beta(P_2, K_2; P_1, K_1)\circ\alpha = \mathrm{id}_{P_2/K_2}$. Since $\beta(P_2, K_2; P_1, K_1)^{-1} = \beta(P_1, K_1; P_2, K_2)$, the result follows.

(b) Suppose that $L * L = L$. By Part (a), the sections (P_1, K_1) and (P_2, K_2) are linked and $\alpha = \beta(P_1, K_1; P_2, K_2)$. By Lemma 6.1, the morphisms $M_i := e_{(P_i,K_i)}$ belong to $\mathsf{C}_{G,G}$, for $i = 1, 2$. Moreover, Lemma 2.7 implies that

$$L = M_1 * L\,, \quad M_1 = L * M_1\,, \quad L = L * M_2\,, \quad \text{and} \quad M_2 = M_2 * L\,.$$

Thus, L and M_i are equivalent for $i = 1, 2$.

(c) If L and M are equivalent then there exist $S \in \mathsf{C}_{H,G}$ and $T \in \mathsf{C}_{G,H}$ such that $L = T * S$ and $M = S*T$. This implies $L = L*L = T*S*T*S = T*M*S$ and $M = M*M = S*T*S*T = S*L*T$. Therefore, $p_1(L)/k_1(L)$ is isomorphic to a subquotient of $p_1(M)/k_1(M)$ and also $p_1(M)/k_1(M)$ is isomorphic to a subquotient of $p_1(L)/k_1(L)$. This implies $q(L) = q(M)$. Conversely, assume that $q(L) = q(M)$. By Lemma 6.1 and Part (b), we may assume that $L = e_{(P_1,K_1)} \in \mathsf{C}_{G,G}$ and $M = e_{(P_2,K_2)} \in \mathsf{C}_{H,H}$ for a section (P_1, K_1) of G and a section (P_2, K_2) of H, with $P_1/K_1 \cong P_2/K_2$. By (16), there exist isomorphisms $\alpha\colon P_1/K_1 \xrightarrow{\sim} P_2/K_2$ and $\beta\colon P_2/K_2 \xrightarrow{\sim} P_1/K_1$ such that $S := (P_2, K_2, \alpha, P_1, K_1) \in \mathsf{C}_{H,G}$ and $T := (P_1, K_1, \beta, P_2, K_2) \in \mathsf{C}_{G,H}$. For $T' := T * (S * T)^{|\mathrm{Aut}(P_2/K_2)|-1} \in \mathsf{C}_{G,H}$ we then obtain $S * T' = e_{(P_2,K_2)}$ and $T' * S = e_{(P_1,K_1)}$, and the proof is complete. ∎

Recall from Section 5 that, for an idempotent morphism e in $\mathsf{C}_{G,G}$, we denote by Γ_e the group of invertible elements of the monoid $e * \mathsf{C}_{G,G} * e$. The identity element of Γ_e is equal to e.

6.4 Proposition *Let G be an object of C, and let (P, K) be a section of G such that $e := e_{(P,K)}$ belongs to $\mathsf{C}_{G,G}$.*

(a) *One has*

$$\Gamma_e = \{(P, K, \alpha, P, K) \mid \alpha \in \mathrm{Aut}_{\mathsf{C}}(P/K)\},$$

and the map $\alpha \mapsto (P, K, \alpha, P, K)$ defines a group isomorphism $\mathrm{Aut}_{\mathsf{C}}(P/K) \to \Gamma_e$.

(b) *The 2-cocycle κ on the monoid $(\mathsf{C}_{G,G}, *)$ restricted to the subgroup Γ_e is the constant function with value $[G : K]^{-1}$, and the map $\alpha \mapsto [G : K] \cdot (P, K, \alpha, P, K)$ defines an R-algebra isomorphism between the group algebra $R\mathrm{Aut}_{\mathsf{C}}(P/K)$ and the twisted group algebra $R_\kappa\Gamma_e$.*

Proof (a) Let $\alpha, \beta \in \mathrm{Aut}_{\mathsf{C}}(P/K)$. Then $L_\alpha := (P, K, \alpha, P, K) \in \mathsf{C}_{G,G}$ satisfies $L_\alpha = e * L_\alpha * e$ and $L_\alpha * L_\beta = L_{\alpha\circ\beta}$, by Lemma 2.7. Thus, $L_\alpha \in \Gamma_e$ for all $\alpha \in \mathrm{Aut}_{\mathsf{C}}(P/K)$ and we obtain an injective group homomorphism $\mathrm{Aut}_{\mathsf{C}}(P/K) \to \Gamma_e$. Conversely, for $L \in \Gamma_e$ we have $L = e * L * e$ and there exists

$M = e * M * e \in \mathsf{C}_{G,G}$ with $M * L = L * M = e$. Since $L = e * L * e$, we have $p_i(L) \leqslant p_i(e) = P$ and $k_i(L) \geqslant k_i(e) = K$ for $i = 1, 2$. Since $L * M = e$, we also have $P = p_1(e) \leqslant p_1(L)$ and $K = k_1(e) \geqslant k_1(L)$. Similarly, $M * L = e$ implies $P \leqslant p_2(L)$ and $K \geqslant k_2(L)$. Altogether we obtain $p_i(L) = P$ and $k_i(L) = K$ for $i = 1, 2$. This implies that $L = (P, K, \alpha, P, K)$ for some $\alpha \in \mathrm{Aut}_{\mathsf{C}}(P/K)$.

(b) Writing again $L_\alpha := (P, K, \alpha, P, K)$ for $\alpha \in \mathrm{Aut}_{\mathsf{C}}(P/K)$, we have $\kappa(L_\alpha, L_\beta) = |K \cap K|/|G| = [G : K]^{-1}$, for $\alpha, \beta \in \mathrm{Aut}_{\mathsf{C}}(P/K)$. The remaining statement is an easy verification. □

We call a section (P_1, K_1) of a group G and a section (P_2, K_2) of a group G' *isomorphic* if $P_1/K_1 \cong P_2/K_2$. In this case we write $(P_1, K_1) \cong (P_2, K_2)$. For an object G of C and a section (P, K) of G with $e_{(P,K)} \in \mathsf{C}_{G,G}$ we set $e'_{(P,K)} := [G : K] \cdot e_{(P,K)}$, which is the corresponding idempotent in $R_\kappa \mathsf{C}_{G,G}$, cf. 5.7. It is also the identity in $R_\kappa \Gamma_{e_{(P,K)}}$. As in 5.3(b) we write $J_{e_{(P,K)}} = (e_{(P,K)} * \mathsf{C}_{G,G} * e_{(P,K)}) \smallsetminus \Gamma_{e_{(P,K)}}$. Let $A := A^\kappa_{\mathsf{C},R}$. Recall from 5.7 that the R-span $RJ_{e_{(P,K)}}$ is an ideal of the R-algebra $e'_{(P,K)} A e'_{(P,K)} = e'_{(P,K)} R_\kappa \mathsf{C}_{G,G} e'_{(P,K)}$, and that we have

$$e'_{(P,K)} A e'_{(P,K)} = R_\kappa \Gamma_{e_{(P,K)}} \oplus RJ_{e_{(P,K)}} \,. \tag{17}$$

The following theorem is now an immediate consequence of Theorem 5.8 and Propositions 6.3 and 6.4.

6.5 Theorem *Let $\mathcal{S}$ be a set of representatives of the isomorphism classes of sections (P, K) of groups $G \in \mathrm{Ob}(\mathsf{C})$ with $e_{P,K} \in \mathsf{C}_{G,G}$, and for each $(P, K) \in \mathcal{S}$ let $\mathcal{T}_{(P,K)}$ denote a set of representatives of the isomorphism classes of simple left $R[\mathrm{Aut}_{\mathsf{C}}(P/K)]$-modules. Moreover, set $A := A^\kappa_{\mathsf{C},R}$. Then the map*

$$((P, K), T) \mapsto \mathrm{Hd}\big(Ae'_{(P,K)} \otimes_{e'_{(P,K)} A e'_{(P,K)}} \mathrm{Inf}^{e'_{(P,K)} A e'_{(P,K)}}_{R_\kappa \Gamma_{e_{(P,K)}}}(T)\big)$$

induces a bijection between the set of pairs $((P, K), T)$, where $(P, K) \in \mathcal{S}$ and $T \in \mathcal{T}_{(P,K)}$, and the set of isomorphism classes of simple left A-modules. Here, the simple $R[\mathrm{Aut}_{\mathsf{C}}(P/K)]$-module T is viewed as an $R_\kappa \Gamma_{e_{(P,K)}}$-module via the isomorphism from Proposition 6.4(b) and is inflated via the decomposition in (17).

7 Simple $e_{\mathsf{C}} A^\kappa_{\mathsf{C},R} e_{\mathsf{C}}$-modules in characteristic 0

Throughout this section we assume that R is a field of characteristic 0, that B is a category as in Example 5.15(a) and that C is a subcategory of B satisfying the condition in (16). We also assume throughout this section that, for any two objects G and H of C, the set $\mathsf{C}_{G,H}$ is closed under $G \times H$-conjugation. However, we do not assume at this point that C is finite, but will do so in some parts of this section. Using Theorem 6.5 and the condensation results of Green, cf. 5.4, we will first determine the simple $e_{\mathsf{C}} A^\kappa_{\mathsf{C},R} e_{\mathsf{C}}$-modules when C is finite, cf. Theorem 7.1. The remainder of this secton is devoted to several consequences of Theorem 7.1.

One consequence, presented at the end of the section, will be on simple biset functors: recall from Proposition 3.7 and Example 5.15(c) that, when C is finite, the maps $\alpha_{G,H}$, $G, H \in \mathrm{Ob}(\mathsf{C})$, induce an isomorphism

$$\alpha_{\mathsf{C}} \colon \bigoplus_{G,H \in \mathrm{Ob}(\mathsf{C})} RB^{\mathsf{C}}(G, H) \xrightarrow{\sim} e_{\mathsf{C}} A^\kappa_{\mathsf{C},R} e_{\mathsf{C}}$$

of R-algebras. Thus, the simple biset functors for C over R are parametrized by the simple $A^{\kappa}_{\mathsf{C},R}$-modules S with the property that $e_{\mathsf{C}} \cdot S \neq \{0\}$, cf. 5.4. In Theorem 7.8 we use the Schur functor approach to determine the simple biset functors on B, or on any subcategory C of B that satisfies the general assumptions in this section and is closed under taking subquotients in a sense defined in Corollary 7.6.

Recall from Theorem 6.5 that, in the case where C is finite, the simple $A^{\kappa}_{\mathsf{C},R}$-modules are described as

$$S_{(P,K),T} := \mathrm{Hd}(Ae'_{(P,K)} \otimes_{e'_{(P,K)}Ae'_{(P,K)}} \tilde{T})\,,$$

where $A = A^{\kappa}_{\mathsf{C},R}$, (P,K) is a section of some $G \in \mathrm{Ob}(\mathsf{C})$, T is a simple $R\mathrm{Aut}_{\mathsf{C}}(P/K)$-module and $\tilde{T}$ is the same R-module as T but viewed as an $e'_{(P,K)}Ae'_{(P,K)}$-module via the canonical isomorphism $R_{\kappa}\Gamma_{e_{(P,K)}} \cong R\mathrm{Aut}_{\mathsf{C}}(P/K)$ and the inflation from $R_{\kappa}\Gamma_{e_{(P,K)}}$ to $e'_{(P,K)}Ae'_{(P,K)}$.

In the case where C has only one object G we will obtain results on simple modules of $RB^{\mathsf{C}}(G,G)$ and, in particular, on simple modules of $RB(G,G)$. In [Bc3, Section 6.1], Bouc shows that the isomorphism classes of simple $RB(G,G)$-modules can be parametrized by pairs consisting of an isomorphism class of a section (P,K) of G and an isomorphism class of a simple $R\mathrm{Out}(P/K)$-module T, where $\mathrm{Out}(P/K)$ denotes the outer automorphism group of P/K. More precisely, there exists an injective map from the set of isomorphism classes of simple $RB(G,G)$-modules S to the set of such pairs. In this section we give a necessary condition for such a pair to belong to the image of the parametrization, i.e., we bound the image of this map from above, cf. Theorem 7.5.

7.1 Theorem *Assume that the category C is finite, that it satisfies Condition (16), and assume that for any two objects G and H of C, the morphism set $\mathsf{C}_{G,H}$ is closed under $G \times H$-conjugation. Let G be an object of C, let (P,K) be a section of G such that $e_{(P,K)} \in \mathsf{C}_{G,G}$, let T be a simple $R\mathrm{Aut}_{\mathsf{C}}(P/K)$-module, let $S_{(P,K),T}$ be the corresponding simple $A^{\kappa}_{\mathsf{C},R}$-module from Theorem 6.5, and let χ denote the character of T. Then, $e_{\mathsf{C}} \cdot S_{(P,K),T} \neq \{0\}$ if and only if there exist an object H of C and a morphism $L = (P'',K'',\tau,P',K') \in C_{H,H}$ with $q(L) = [P/K]$ such that*

$$\sum_{\substack{h\in H\\ (P',K')={}^{h}(P'',K'')}} |K' \cap {}^{h}K''| \cdot \chi''(\tau \circ \beta(P',K';\, {}^{h}P'',\, {}^{h}K'') \circ \bar{c}_h) \neq 0 \tag{18}$$

in the field R. Here χ'' is the character of the simple $R\mathrm{Aut}_{\mathsf{C}}(P''/K'')$-module T'' corresponding to T (cf. Lemma 6.2) and $\bar{c}_h \colon P''/K'' \xrightarrow{\sim} {}^{h}P''/{}^{h}K''$ is the isomorphism induced by conjugation with h.

Before we prove the theorem, note that the argument $\tau \circ \beta(P',K';\, {}^{h}P'',\, {}^{h}K'') \circ \bar{c}_h$ of χ'' is actually an element of $\mathrm{Aut}_{\mathsf{C}}(P''/K'')$, since it is the isomorphism η_M for the group $M := L * {}^{(h,1)}e_{(P'',K'')} \in \mathsf{C}_{H,H}$.

Proof We set $e := e_{(P,K)}$ and $e' := [G:K]e$. By Lemma 5.6, we have $e_{\mathsf{C}} \cdot S \neq \{0\}$ if and only if $(e'Ae_{\mathsf{C}}Ae') \cdot \tilde{T} \neq \{0\}$, and the latter is equivalent to the existence of an object H of C and morphisms $L \in \mathsf{C}_{G,H}$ and $M \in \mathsf{C}_{H,G}$ such that

$$e'Le_HMe' \cdot \tilde{T} \neq \{0\}\,. \tag{19}$$

Let $\pi\colon e'Ae' = R_{\kappa}\Gamma_e \oplus RJ_e \to R_{\kappa}\Gamma_e$ denote the canonical projection. Since RJ_e annihilates $\tilde{T}$, the condition in (19) is equivalent to the condition $\pi(e'Le_HMe') \cdot T \neq \{0\}$. This in turn is equivalent to

$$e_{\chi} \cdot \pi(e'Le_HMe') \neq 0$$

in $R\Gamma_e$, where

$$e_\chi := \sum_{\sigma \in \mathrm{Aut}_{\mathsf{C}}(P/K)} \chi(\sigma^{-1}) \cdot L_\sigma \,, \quad \text{with } L_\sigma = (P, K, \sigma, P, K),$$

which is (up to a unit in R) the primitive central idempotent of $R_\kappa\Gamma_e$ corresponding to T. Replacing, if necessary, L by $e * L$ and M by $M * e$, we may assume that

$$K \leqslant k_1(L) \leqslant p_1(L) \leqslant P \quad \text{and} \quad K \leqslant k_2(M) \leqslant p_2(M) \leqslant P\,.$$

Further, using Lemma 2.7, we may assume that $K = k_1(L) = k_2(M)$ and $P = p_1(L) = p_2(M)$, since otherwise $\pi(e'Le_HMe') = 0$. Altogether, since $e' = [G : K]e$, we obtain that $e_{\mathsf{C}} \cdot S \neq \{0\}$ if and only if there exist an object $H \in \mathrm{Ob}(\mathsf{C})$ and morphisms $L \in \mathsf{C}_{G,H}$ and $M \in \mathsf{C}_{H,G}$ satisfying

$$k_1(L) = K = k_2(M)\,, \quad p_1(L) = P = p_2(M)\,, \tag{20}$$

and

$$e_\chi \cdot \pi(e'Le_HMe') \neq 0 \quad \text{in } R_\kappa\Gamma_e. \tag{21}$$

Setting $(P', K') := (p_2(L), k_2(L))$, $(P'', K'') := (p_1(M), k_1(M))$, $\phi := \eta_L \colon P'/K' \xrightarrow{\sim} P/K$, and $\psi := \eta_M \colon P/K \xrightarrow{\sim} P''/K''$, and assuming the conditions in (20), we obtain, by Lemma 2.7, that

$$\begin{aligned} \pi(e'Le_HMe') &= \sum_{h \in H} \pi(e'L\Delta_h(H)Me') \\ &= \sum_{\substack{h \in H \\ (P',K') =_h (P'',K'')}} \frac{|K' \cap {}^hK''|}{|H|^2} \cdot (P, K, \phi \circ \beta(P', K';\, {}^hP'',\, {}^hK'') \circ \bar{c}_h \circ \psi, P, K), \end{aligned}$$

where $\bar{c}_h \colon P''/K'' \xrightarrow{\sim} {}^hP''/{}^gK''$ is induced by the conjugation map c_h. Thus the condition in (21) is equivalent to

$$\sum_{\sigma \in \mathrm{Aut}_{\mathsf{C}}(P/K)} \; \sum_{\substack{h \in H \\ (P',K') =_h (P'',K'')}} a_h \cdot \chi(\sigma^{-1}) \;\cdot\; \big(\sigma \circ \phi \circ \beta(P', K';\, {}^hP'',\, {}^hK'') \circ \bar{c}_h \circ \psi\big) \neq 0$$

in $R\mathrm{Aut}_{\mathsf{C}}(P/K)$, where

$$a_h = |K' \cap {}^hK''| \cdot \frac{|K|}{|H|^2 \cdot |G|} \quad (\text{for } h \in H \text{ and } (P', K') =_h (P'', K''))\,.$$

Applying the isomorphism $c_\psi \colon \mathrm{Aut}_{\mathsf{C}}(P/K) \xrightarrow{\sim} \mathrm{Aut}_{\mathsf{C}}(P''/K'')$, $\sigma \mapsto \psi \circ \sigma \circ \psi^{-1} := \sigma''$ and switching the summation translates the last condition into

$$\sum_{\substack{h \in H \\ (P',K') =_h (P'',K'')}} \; \sum_{\sigma'' \in \mathrm{Aut}_{\mathsf{C}}(P''/K'')} |K' \cap {}^hK''| \cdot \chi''((\sigma'')^{-1}) \;\cdot\; \big(\sigma'' \circ \psi \circ \phi \circ \beta(P', K';\, {}^hP'',\, {}^hK'') \circ \bar{c}_h\big) \neq 0$$

in $R\mathrm{Aut}_{\mathsf{C}}(P''/K'')$, where χ'' is the character of the simple $R\mathrm{Aut}_{\mathsf{C}}(P''/K'')$-module T'' arising from T via c_ψ. Substituting $\sigma'' \circ \psi \circ \phi \circ \beta(P', K'';\, {}^hP'',\, {}^hK'') \circ \bar{c}_h$ with θ'' and then switching the summation again, one obtains the equivalent condition

$$\sum_{\theta'' \in \mathrm{Aut}_{\mathsf{C}}(P''/K'')} \; \sum_{\substack{h \in H \\ (P',K') =_h (P'',K'')}} |K' \cap {}^hK''| \cdot \chi''\big((\theta'')^{-1} \circ \psi \circ \phi \circ \beta(P', K';\, {}^hP'',\, {}^hK'') \circ \bar{c}_h\big) \cdot \theta'' \;\neq\; 0$$

in $R\mathrm{Aut}_{\mathsf{C}}(P''/K'')$; note that here we used the fact that χ'' is a class function. Thus, $e_{\mathsf{C}} \cdot S \neq \{0\}$ if and only if there exist an object H of C, sections (P', K') and (P'', K'') of H that are isomorphic to (P, K), and an isomorphism $\tau\colon P'/K' \to P''/K''$ with $(P'', K'', \tau, P', K') \in \mathsf{C}_{H,H}$ such that

$$\sum_{\substack{h \in H \\ (P',K') = {}^h(P'',K'')}} |K' \cap {}^hK''| \cdot \chi''\big(\tau \circ \beta(P', K''; {}^hP'', {}^hK'') \circ \bar{c}_h\big) \neq 0$$

in R. This completes the proof of the theorem. ∎

The derive an immediate corollary of the previous theorem.

7.2 Corollary *Assume that* C *satisfies the same hypotheses as in Theorem 7.1, that* G *is an object of* C*,* (P, K) *is a section of* G *such that* $e_{P,K} \in \mathsf{C}_{G,G}$*, that* T *is a simple* $R\mathrm{Aut}_{\mathsf{C}}(P/K)$*-module, and that* $S := S_{(P,K),T}$ *is the corresponding simple* $A^{\kappa}_{\mathsf{C},R}$*-module.*

(a) *If there exists an object* H *of* C *such that* H *and* P/K *are isomorphic as groups and if* $\mathrm{Inn}(P/K)$ *acts trivially on* T *then* $e_{\mathsf{C}} \cdot S \neq \{0\}$*.*

(b) *If* P/K *is abelian then* $e_{\mathsf{C}} \cdot S \neq \{0\}$*.*

Proof (a) The condition in (18) is satisfied for H and $L = \Delta(H) \in \mathsf{C}_{H,H}$: in fact, the left-hand side of (18) is equal to $\sum_{c \in \mathrm{Inn}(H)} \chi''(c)$. This element is non-zero if and only if $\mathrm{Inn}(H)$ acts trivially on T''. But this is equivalent to $\mathrm{Inn}(P/K)$ acting trivially on T. Now the result follows.

(b) The condition in (18) is satisfied for $H = G$, $(P', K') = (P'', K'') = (P, K)$, and $\tau = \mathrm{id}_{P/K}$. In fact, the left-hand side is equal to $|G| \cdot |K| \cdot \chi''(\mathrm{id}_{P/K})$, which is clearly non-zero. ∎

Before we can derive further consequences from Theorem 7.1 we need first an auxiliary result. Let G be a finite group, let χ be the character of a finite-dimensional RG-module, and let $X \subseteq G$ be a subset. We set $X^+ := \sum_{x \in X} x \in RG$ and $\chi(X^+) := \sum_{x \in X} \chi(x)$. Note that if H is a subgroup of G then $\chi(H^+)$ is a non-zero multiple of the Schur inner product of the trivial character of H and the restriction of χ to H. With this notation the following lemma holds. Its proof will appear in [BK]. We repeat it here for the convenience of the reader.

7.3 Lemma *Let* χ *be the character of a finite-dimensional* RG*-module and assume that* $g \in G$ *and* $H \leqslant G$ *satisfy* $\chi((gH)^+) \neq 0$*. Then also* $\chi(H^+) \neq 0$ *and, in particular, the restriction of* χ *to* H *contains the trivial character of* H *as a constituent.*

Proof Let $\Delta\colon RG \to \mathrm{Mat}_n(R)$ be a representation affording the character χ and assume that $\chi(H^+) = 0$. Then also $\chi(e_H) = 0$, where $e_H = \frac{1}{|H|}H^+$ is an idempotent in RG. Since e_H is an idempotent, the matrix $\Delta(e_H)$ is diagonalizable and its only eigenvalues are 0 and 1. Since $0 = \chi(e_H) = \mathrm{tr}(\Delta(e_H))$, we obtain $\Delta(e_H) = 0$. Thus, $\Delta(RG \cdot e_H) = 0$. This finally implies that $\chi((gH)^+) = |H| \cdot \chi(ge_H) = 0$, and the proof of the lemma is complete. ∎

For any section (P, K) of a finite group G we set $N_G(P, K) := N_G(P) \cap N_G(K)$ and define

$$\mathrm{Aut}_G(P/K) := \{\bar{c}_g \mid g \in N_G(P, K)\}\,.$$

Note that $\mathrm{Inn}(P/K) \leqslant \mathrm{Aut}_G(P/K) \leqslant \mathrm{Aut}(P/K)$. For each $G \in \mathrm{Ob}(\mathsf{C})$ and each section (P,K) of G with $e_{(P,K)} \in \mathsf{C}_{G,G}$, one has $\mathrm{Aut}_G(P/K) \leqslant \mathrm{Aut}_{\mathsf{C}}(P/K)$. In fact, for each $g \in N_G(P,K)$ one has ${}^{(g,1)}e_{(P,K)} = (P,K,\bar{c}_g,P,K) \in \mathsf{C}_{G,G}$. Thus, we obtain

$$\mathrm{Inn}(P/K) \leqslant \mathrm{Aut}_G(P/K) \leqslant \mathrm{Aut}_{\mathsf{C}}(P/K) \leqslant \mathrm{Aut}(P/K)\,.$$

If also $H \in \mathrm{Ob}(\mathsf{C})$ and if (P',K') is a section of H with $P'/K' \cong P/K$ then there exists an isomorphism $\tau\colon P/K \xrightarrow{\sim} P'/K'$ such that $(P,K,\tau,P',K') \in \mathsf{C}_{G,H}$, since C satisfies (16). Conjugation by τ induces an isomorphism $\mathrm{Aut}_{\mathsf{C}}(P'/K') \to \mathrm{Aut}_{\mathsf{C}}(P/K)$ that maps $\mathrm{Aut}_H(P'/K')$ to a subgroup $\tau\mathrm{Aut}_H(P'/K')\tau^{-1}$ of $\mathrm{Aut}_{\mathsf{C}}(P/K)$ that also contains $\mathrm{Inn}(P/K)$. Different choices of τ yield conjugate subgroups of $\mathrm{Aut}_{\mathsf{C}}(P/K)$. We denote by $\mathscr{A}_{\mathsf{C}}(P/K)$ the set of subgroups of $\mathrm{Aut}_{\mathsf{C}}(P/K)$ that arise this way. Thus, $\mathscr{A}_{\mathsf{C}}(P/K)$ is a collection of subgroups of $\mathrm{Aut}_{\mathsf{C}}(P/K)$ that is closed under $\mathrm{Aut}_{\mathsf{C}}(P/K)$-conjugation and all of whose elements contain $\mathrm{Inn}(P/K)$.

7.4 Corollary *Assume that C is a finite subcategory of B that satisfies Condition (16), and assume that for any two objects G and H of C, the morphism set $\mathsf{C}_{G,H}$ is closed under $G \times H$-conjugation. Furthermore, let G be an object of C, let (P,K) be a section of G such that $e_{(P,K)} \in \mathsf{C}_{G,G}$, let T be a simple $R\mathrm{Aut}_{\mathsf{C}}(P/K)$-module, and let $S = S_{(P,K),T}$ be the corresponding simple $A^{\kappa}_{\mathsf{C},R}$-module, cf. Theorem 6.5. If $e_{\mathsf{C}} \cdot S \neq \{0\}$ then there exists a subgroup B of $\mathrm{Aut}_{\mathsf{C}}(P/K)$ that belongs to $\mathscr{A}_{\mathsf{C}}(P/K)$ such that the restriction of T to B has the trivial module as constituent. In particular, $\mathrm{Inn}(P/K)$ acts trivially on T.*

Proof By Theorem 7.1, there exist $H \in \mathrm{Ob}(\mathsf{C})$ and sections (P',K') and (P'',K'') of H, both isomorphic to (P,K), and an isomorphism $\tau\colon P'/K' \xrightarrow{\sim} P''/K''$ satisfying (18). Note that if $h \in H$ satisfies $(P',K') = {}^{h}(P'',K'')$ then every element in the coset $hN_H(P'',K'')$ satisfies this condition. Thus, the condition in (18) implies

$$\chi''\big((\sigma'' \circ \mathrm{Aut}_H(P''/K''))^+\big) \neq 0$$

for some $\sigma'' \in \mathrm{Aut}_{\mathsf{C}}(P''/K'')$. Lemma 7.3 implies that $\chi''(\mathrm{Aut}_H(P''/K'')^+) \neq 0$. Here, χ denotes the character of T and χ'' denotes the corresponding character of the group $\mathrm{Aut}_{\mathsf{C}}(P''/K'')$. The last condition implies that $\chi(B^+) \neq 0$ for some $B \in \mathscr{A}_{\mathsf{C}}(P/K)$. Thus, the restriction of χ to B contains the trivial character as a constituent. Since $\mathrm{Inn}(P/K) \leqslant B$, this also holds for χ restricted to $\mathrm{Inn}(P/K)$. Finally, since χ is the character of a simple $R\mathrm{Aut}(P/K)$-module and since $\mathrm{Inn}(P/K)$ is normal in $\mathrm{Aut}(P/K)$, this implies that $\mathrm{Inn}(P/K)$ acts trivially on T. □

If C has only one object G then we obtain results on the simple $RB^{\mathsf{C}}(G,G)$-modules. Let $\mathcal{S}$ denote a set of representatives for the isomorphism classes of sections (P,K) of G satisfying $e_{(P,K)} \in \mathsf{C}$, and for each $(P,K) \in \mathcal{S}$, let $\mathcal{T}_{(P,K)}$ denote a set of representatives for the isomorphism classes of simple $R\mathrm{Aut}_{\mathsf{C}}(P/K)$-modules. The simple $RB^{\mathsf{C}}(G,G)$-modules are parametrized by pairs $((P,K),T)$ with $(P,K) \in \mathcal{S}$ and $T \in \mathcal{T}_{(P,K)}$ such that there exist sections (P',K') and (P'',K'') of G and an isomorphism $\tau\colon P''/K'' \xrightarrow{\sim} P'/K'$ with (P',K',τ,P'',K'') such that the condition (18) is satisfied. In this case we say that $((P,K),T)$ *parametrizes a simple $RB^{\mathsf{C}}(G,G)$-module.*

The following theorem improves the results in [Bc3, Chapter 6] on the parametrization of simple $RB(G,G)$-modules in the case that R is a field of characteristic 0. It follows immediately from Corollary 7.4 and Corollary 7.2(b).

7.5 Theorem *Assume that G is a finite group and that C is a submonoid of $(\mathscr{S}_{G\times G}, *)$ satisfying the condition in (16) and that C is closed under $G \times G$-conjugation, and assume the notation from above. Let $(P, K) \in \mathcal{S}$ and $T \in \mathcal{T}_{P,K}$.*

(a) *If $((P, K), T)$ parametrizes a simple $RB^{\mathsf{C}}(G, G)$-module then the restriction of T to a subgroup B of $\mathrm{Aut}_{\mathsf{C}}(P/K)$ contains the trivial RB-module as a constituent; in particular, $\mathrm{Inn}(P/K)$ acts trivially on T.*

(b) *If P/K is abelian then $((P, K), T)$ parametrizes a simple $RB^{\mathsf{C}}(G, G)$-module.*

The following corollary is also a consequence of Corollary 7.2(a) and Corollary 7.4.

7.6 Corollary *Assume that C satisfies Condition (16) and that for any two objects G and H of C the set $\mathsf{C}_{G,H}$ is closed under $G \times H$-conjugation. Assume further that C is finite and that the objects of C are closed under taking subquotients in the following sense: for any object G of C and any section (P, K) of G with $e_{(P,K)} \in \mathsf{C}_{G,G}$ there exists an object H of C with $H \cong P/K$.*

Then the isomorphism classes of simple $e_{\mathsf{C}} A^{\kappa}_{\mathsf{C},R} e_{\mathsf{C}}$-modules are parametrized by pairs (G, T), where G runs through a set $\mathcal{S}$ of representatives of the isomorphism classes of groups G occurring as objects of C and, for each $G \in \mathcal{S}$, T runs through a set of representatives for the isomorphism classes of simple $R\mathrm{Out}_{\mathsf{C}}(G)$-modules. Here, $\mathrm{Out}_{\mathsf{C}}(G)$ is defined as $\mathrm{Aut}_{\mathsf{C}}(G)/\mathrm{Inn}(G)$.

Proof First note that, since C is closed under taking subquotients, the set $\mathcal{S}$ (in Theorem 6.5) of representatives for the isomorphism classes of sections (P, K) of objects G of C satisfying $e_{(P,K)} \in \mathsf{C}_{G,G}$ can be chosen such that $P = G$ and $K = 1$. Next assume that if T is a simple $R\mathrm{Aut}_{\mathsf{C}}(G)$-module. If $e_{\mathsf{C}} \cdot S = \{0\}$ for the corresponding simple $A^{\kappa}_{\mathsf{C},R}$-module S then Corollary 7.4 implies that $\mathrm{Inn}(G)$ acts trivially on T. Conversely, if $\mathrm{Inn}(G)$ acts trivially on T then Corollary 7.2(a) implies that $e_{\mathsf{C}} \cdot S \neq \{0\}$. This completes the proof of the corollary. □

7.7 Remark Assume that C satisfies Condition (16), that for any two objects G and H of C the set $\mathsf{C}_{G,H}$ is closed under $G \times H$-conjugation, and that the objects of C are closed under taking subquotients in the sense of Corollary 7.6.

Corollary 7.6 can be considered as a parametrization result for simple biset functors on C over R, when C is *finite* (cf. Example 5.15(c)). To make the transition from C being finite to C being arbitrary, it is straightforward to show that the standard techniques developed in [Bc3], especially those in Section 3.3, 4.2 and 4.3 that are used in the parametrization of simple biset functors, still work if one weakens the requirement of an *admissable subcategory*, cf. [Bc3, Definition 4.1.3], to requiring Condition (16), that the morphism sets $\mathsf{C}_{G,H}$ are closed under $G \times H$-conjugation, and that the objects of C are closed under taking sections in the sense of Corollary 7.6. These techniques allow to reduce the determination of simple biset functors on C over R to the case where C is finite. Together with Corollary 7.6 we then obtain the theorem stated below. There we use Bouc's definition of the simple biset functor $S_{(G,T)}$ associated to the pair (G, T), where G is an object of C and T is a simple $R\mathrm{Out}_{\mathsf{C}}(G)$-module. Two pairs (G, T) and (G', T') are called *isomorphic* if G and G' are isomorphic and if for one (or equivalently each) $\tau\colon G' \to G$ with $(G, 1, \tau, G', 1) \in \mathsf{C}_{G,G'}$ the modules T' and T correspond to each other under the map $c_\tau\colon \mathrm{Aut}_{\mathsf{C}}(G') \to \mathrm{Aut}_{\mathsf{C}}(G)$. Note that such an isomorphism τ exists. The parametrization in the following theorem slightly generalizes Bouc's parametrization in [Bc3, Theorem 4.3.10] where $\mathrm{Aut}_{\mathsf{C}}(P/K)$ is assumed to be equal to $\mathrm{Aut}(P/K)$. Unfortunately, our methods require R to be a field of characteristic 0.

7.8 Theorem *Let C be a subcategory of B, not necessarily finite. Assume that C satisfies Condition (16), that for any two objects G and H of C the morphism set $\mathsf{C}_{G,H}$ is closed under $G \times H$-conjugation, and that C is closed under taking sections in the sense of Corollary 7.6. Then the map $(G,T) \mapsto S_{(G,T)}$ induces a bijection between the set of isomorphism classes of pairs (G,T), where G is an object of C and T is a simple $R\mathrm{Out}_{\mathsf{C}}(G)$-module, and the set of isomorphism classes of simple biset functors for C over R.*

8 Cyclic Groups

Throughout this section R denotes a commutative ring and $\mathscr{D}$ denotes a set of cyclic groups such that, for all $G \in \mathscr{D}$, the group order $|G|$ is a unit in R. By φ we denote Euler's totient function. The aim of this section is to show that if, in addition, also $\varphi(|G|)$, for $G \in \mathscr{D}$, are units in R then the R-algebra $\bigoplus_{G,H\in\mathscr{D}} RB(G,H)$ is isomorphic to the category algebra $R\tilde{\mathsf{C}}$, where $\tilde{\mathsf{C}}$ is the finite inverse category associated with the set $\mathscr{D}$ that was introduced in Definition 5.16. Thus we will, in particular, be able to invoke the results quoted in Section 5 to deduce that if R is a field such that, for every $G \in \mathscr{D}$, both $|G|$ and $\varphi(|G|)$ are non-zero in R and if $\mathscr{D}$ is finite then $\bigoplus_{G,H\in\mathscr{D}} RB(G,H)$ is a semisimple R-algebra. In this case we will give an explicit decomposition of this algebra as a direct product of matrix algebras over group algebras.

We set

$$B_R^{\mathscr{D}} := \bigoplus_{G,H\in\mathscr{D}} RB(G,H) \quad \text{and} \quad \tilde{A}_R^{\mathscr{D}} := \bigoplus_{G,H\in\mathscr{D}} R\mathscr{S}_{G\times H}\,.$$

These two R-modules are equipped with the following multiplications: for $G, H, H', K \in \mathscr{D}$, and for $a \in RB(G,H)$, $b \in RB(H',K)$, $x \in R\mathscr{S}_{G\times H}$, and $y \in R\mathscr{S}_{H'\times K}$, we have

$$a \cdot b := \begin{cases} a \cdot_H b & \text{if } H = H', \\ 0 & \text{if } H \neq H', \end{cases}$$

cf. Example 5.15(c), and

$$x \tilde{*}^{\kappa} y := \begin{cases} x \tilde{*}^{\kappa}_H y & \text{if } H = H', \\ 0 & \text{if } H \neq H'. \end{cases}$$

Note that $G \times H$ acts trivially on $R\mathscr{S}_{G\times H}$ when G and H are abelian. Therefore, Propositions 3.2, 3.6 and 3.9 imply the following proposition.

8.1 Proposition *The composition of the maps $(\alpha_{G,H})_{G,H\in\mathscr{D}}\colon B_R^{\mathscr{D}} \to \bigoplus_{G,H\in\mathscr{D}} R\mathscr{S}_{G\times H}$ and $(\zeta_{G,H})_{G,H\in\mathscr{D}}\colon \bigoplus_{G,H\in\mathscr{D}} R\mathscr{S}_{G\times H} \to \tilde{A}_R^{\mathscr{D}}$ defines an isomorphism*

$$\gamma\colon (B_R^{\mathscr{D}}, \cdot) \xrightarrow{\sim} (\tilde{A}_R^{\mathscr{D}}, \tilde{*}^{\kappa})$$

of R-algebras.

Next we will focus on the structure of the R-algebra $(\tilde{A}_R^{\mathscr{D}}, \tilde{*}^{\kappa})$. First we will introduce a new R-basis of $\tilde{A}_R^{\mathscr{D}}$ and then, in Theorem 8.6, show that the product of any two of these basis elements is a multiple of a basis element, or is equal to 0. This will require several steps.

8.2 Notation For finite groups G and H and a subgroup $L \leqslant G \times H$, we set

$$\tilde{L} := \sum_{L' \in \mathscr{P}(L)} L' \in R\mathscr{S}_{G\times H}\,,$$

where

$$\mathscr{P}(L) = \{L' \leqslant L \mid p_1(L') = p_1(L),\ p_2(L') = p_2(L)\}\,,$$

cf. Definition 5.16. Note that the elements in $\{\tilde{L} \mid L \leqslant G \times H\}$, form again an R-basis of $R\mathscr{S}_{G\times H}$. We also set

$$s(L) := [p_1(L) : k_1(L)] = [p_2(L) : k_2(L)]\,.$$

Note that $s(L) = 1$ if and only if $L = p_1(L) \times p_2(L)$ is a direct product.

In order to see that the basis elements $\tilde{L}$, $L \leqslant G \times H$, have the desired property we first consider the special case of cyclic p-groups G and H.

8.3 Lemma *Let p be prime, and let G and H be cyclic p-groups.*

(a) *Let $M' \leqslant M \leqslant G \times H$ be such that $p_2(M') = p_2(M)$ and $p_1(M') < p_1(M)$. Then $M = p_1(M) \times p_2(M)$; in particular, $M' \leqslant U \times p_2(M) \leqslant M$, where $U < p_1(M)$ is such that $[p_1(M) : U] = p$.*

(b) *Let $L' \leqslant L \leqslant G \times H$ be such that $p_1(L') = p_1(L)$ and $p_2(L') < p_2(L)$. Then $L = p_1(L) \times p_2(L)$; in particular, $L' \leqslant p_1(L) \times V \leqslant L$, where $V < p_2(L)$ is such that $[p_2(L) : V] = p$.*

Proof We verify Part (a); Part (b) is proved analogously. So let $(g, h) \in M$ be such that $\langle g\rangle = p_1(M)$, thus $g \notin p_1(M')$. Since $h \in p_2(M) = p_2(M')$, there is some $g' \in p_1(M')$ such that $(g', h) \in M'$. Moreover, $\langle g^{-1}g'\rangle = p_1(M)$, and $g^{-1}g' \in k_1(M)$. Hence $k_1(M) = p_1(M)$, that is, $s(M) = 1$. □

8.4 Proposition *Let p be a prime that is invertible in R, let G, H, and K be cyclic p-groups and let $L \leqslant G \times H$ and $M \leqslant H \times K$ be subgroups such that $p_2(L) = p_1(M)$. Then, in $R\mathscr{S}_{G\times K}$, one has*

$$\tilde{L}\,\tilde{*}^{\kappa}_{H}\,\tilde{M} = \begin{cases} \frac{\varphi(|k_2(L)\cap k_1(M)|)}{|H|}\cdot \widetilde{L * M} & \text{if } s(L) = 1 = s(M),\\ \frac{|k_2(L)\cap k_1(M)|}{|H|}\cdot \widetilde{L * M} & \text{otherwise.}\end{cases}$$

Proof By the definition of $-\tilde{*}^{\kappa}_{H}-$ in Definition 3.8 and by Theorem 4.1, we have

$$\tilde{L}\,\tilde{*}^{\kappa}_{H}\,\tilde{M} = \sum_{(L',M')\in\mathscr{P}(L)\times\mathscr{P}(M)} L'\,\tilde{*}^{\kappa}_{H}\,M' = \sum_{(L',M')\in\mathscr{P}(L)\times\mathscr{P}(M)} \Big[\sum_{N\in\mathscr{P}(L'*M')} \big(\sum_{\sigma\in\mathcal{E}^{N}_{L',M'}} (-1)^{|\sigma|}\kappa(\sigma)\big)\cdot N\Big]\,.$$

Note that if $(L', M') \in \mathscr{P}(L) \times \mathscr{P}(M)$ then $p_2(L') = p_1(M')$, since $p_2(L) = p_1(M)$. Moreover, $p_2(L') = p_1(M')$ and $p_2(L) = p_1(M)$ together with Lemma 2.7(i) imply that

$$p_1(L' * M') = p_1(L') = p_1(L) = p_1(L * M) \quad\text{and}\quad p_2(L' * M') = p_2(M') = p_2(M) = p_2(L * M)\,.$$

Therefore we have $\mathscr{P}(L' * M') \subseteq \mathscr{P}(L * M)$ and we can write

$$\tilde{L}\,\tilde{*}^{\kappa}_{H}\,\tilde{M} = \sum_{N\in\mathscr{P}(L*M)} b^{N}_{L,M}\cdot N$$

with uniquely determined elements $b_{L,M}^N \in R$. Moreover, the first equation in the proof implies that

$$b_{L,M}^N = \sum_{(L',M')\in\mathscr{P}(L)\times\mathscr{P}(M)} \Big(\sum_{\sigma\in\mathcal{E}_{L',M'}^N} (-1)^{|\sigma|}\kappa(\sigma)\Big) = \sum_{(L',M')\in\mathcal{Z}_{L,M}^N\cap(\mathscr{P}(L)\times\mathscr{P}(M))} \Big(\sum_{\sigma\in\mathcal{E}_{L',M'}^N} (-1)^{|\sigma|}\kappa(\sigma)\Big)\,.$$

The last equation holds, since $\mathcal{E}_{L',M'}^N$ is empty if $(L',M')\notin\mathcal{Z}_{L',M'}^N$ and since $\mathcal{Z}_{L',M'}^N\subseteq\mathcal{Z}_{L,M}^N$.

Next we fix an element $N\in\mathscr{P}(L*M)$. To complete the proof of the proposition, it suffices to show that

$$b_{L,M}^N = \begin{cases} \frac{\varphi(|k_2(L)\cap k_1(M)|)}{|H|} & \text{if } s(L)=1=s(M),\\ \frac{|k_2(L)\cap k_1(M)|}{|H|} & \text{otherwise.}\end{cases} \tag{22}$$

Note that by Möbius inversion in the poset $\mathcal{Z}_{L,M}^N$ we have

$$\begin{aligned}\kappa(L,M) &= \sum_{\substack{(L'',M'')\leqslant(L',M')\\ \text{in } \mathcal{Z}_{L,M}^N}} \mu_{(L'',M''),(L',M')}^{\mathcal{Z}_{L,M}^N}\kappa(L'',M'')\\ &= \sum_{(L',M')\in\mathcal{Z}_{L,M}^N}\ \sum_{\sigma\in\mathcal{E}_{L',M'}^N}(-1)^{|\sigma|}\kappa(\sigma) = b_{L,M}^N + c_{L,M}^N\end{aligned} \tag{23}$$

with

$$c_{L,M}^N := \sum_{(L',M')\in\mathcal{Z}_{L,M}^N\smallsetminus(\mathscr{P}(L)\times\mathscr{P}(M))}\ \sum_{\sigma\in\mathcal{E}_{L',M'}^N}(-1)^{|\sigma|}\kappa(\sigma)\,.$$

Claim: If $L'\leqslant L$ with $p_1(L')=p_1(L)$ and $p_2(L')<p_2(L)$ then $k_2(L')<k_2(L)$. In fact, by Equation (1) we have $|L|=|p_1(L)|\cdot|k_2(L)|$ and $|L'|=|p_1(L')|\cdot|k_2(L')|$, and $p_2(L')<p_2(L)$ implies $L'<L$. Similarly, one can see that if $M'\leqslant M$ with $p_2(M')=p_2(M)$ and $p_1(M')<p_1(M)$ then $k_1(M')<k_1(M)$. This settles the claim.

To finish the proof we distinguish three cases:

(i) If $s(L)>1$ or $s(M)>1$ then $\mathcal{Z}_{L,M}^N\smallsetminus(\mathscr{P}(L)\times\mathscr{P}(M))=\emptyset$ by Lemma 8.3. Therefore we obtain $c_{L,M}^N=0$ and Equation (23) implies $b_{L,M}^N=\kappa(L,M)$, as desired.

(ii) If $k_2(L)=1$ or $k_1(L)=1$ then the above claim implies $c_{L,M}^N=0$, and Equation (23) implies $b_{L,M}^N=\kappa(L,M)=1/|H|=\varphi(|k_2(L)\cap k_1(L)|)/|H|$, as desired.

(iii) If $s(L)=1=s(M)$, $k_2(L)>1$ and $k_1(M)>1$ then, by Lemma 8.3, we have $\mathcal{Z}_{L,M}^N\smallsetminus(\mathscr{P}(L)\times\mathscr{P}(M))=\mathcal{Z}_{p_1(L)\times V,V\times p_2(M)}^N$, where $V<p_2(L)$ is the unique subgroup of index p. Thus,

$$c_{L,M}^N = \sum_{(L',M')\in\mathcal{Z}_{p_1(L)\times V,V\times p_2(M)}^N}\ \sum_{\sigma\in\mathcal{E}_{L',M'}^N}(-1)^{|\sigma|}\kappa(\sigma)$$

and this is equal to $\kappa(p_1(L)\times V,V\times p_2(M))=\kappa(L,M)/p$, by the first part of the equations in (23) applied to $(p_1(L)\times V,V\times p_2(M))$. Now Equation (23) implies

$$b_{L,M}^N=\kappa(L,M)-c_{L,M}^N=\kappa(L,M)-\kappa(L,M)/p=\frac{(1-1/p)\cdot|k_2(L)\cap k_1(M)|}{|H|}=\frac{\varphi(|k_2(L)\cap k_1(M)|)}{|H|}\,,$$

and the proof is complete. $\square$

In the sequel we will denote by $\mathbb{P}$ the set of prime numbers. For a finite abelian group G and a prime p we denote by G_p the Sylow p-subgroup of G.

8.5 Definition Assume that G, H, and K are finite abelian groups such that $|H|$ is invertible in R. Let $L \leqslant G \times H$ and $M \leqslant H \times K$ be subgroups. For each $p \in \mathbb{P}$, we define $\lambda_p(L, M) \in R$ as follows: if $p_2(L_p) \neq p_1(M_p)$ then we set $\lambda_p(L, M) := 0$, and if $p_2(L_p) = p_1(M_p)$ then we set

$$\lambda_p(L, M) := \begin{cases} \frac{\varphi(|k_2(L_p)\cap k_1(M_p)|)}{|H_p|} & \text{if } s(L_p) = 1 = s(M_p), \\ \frac{|k_2(L_p)\cap k_1(M_p)|}{|H_p|} & \text{otherwise.} \end{cases}$$

Note that if p does not divide $|H|$ then $\lambda_p(L, M) = 1$. Moreover, we define

$$\lambda(L, M) := \prod_{p\in\mathbb{P}} \lambda_p(L, M) .$$

Thus, $\lambda(L, M) \neq 0$ if and only if $p_2(L) = p_1(M)$, and in this case $\lambda(L, M)$ is invertible in R provided that $\varphi(|H|)$ is also invertible in R.

8.6 Theorem *Assume that G, H and K are finite cyclic groups such that $|H|$ is invertible in R and let $L \leqslant G \times H$ and $M \leqslant H \times K$ be subgroups. Then*

$$\tilde{L} \tilde{*}^{\kappa}_{H} \tilde{M} = \lambda(L, M) \cdot \widetilde{L * M} .$$

In particular, $\tilde{L} \tilde{*}^{\kappa}_{H} \tilde{M} = 0$ if and only if $p_2(L) \neq p_1(M)$.

For the proof of Theorem 8.6 we will need two lemmas that will reduce the theorem to the case of cyclic p-groups and therefore to Proposition 8.4. For a finite abelian group G, one has the canonical bijection

$$\mathscr{S}_G \xrightarrow{\sim} \bigtimes_{p\in\mathbb{P}} \mathscr{S}_{G_p} , \quad U \mapsto (U_p)_{p\in\mathbb{P}} ,$$

which induces an R-module isomorphism

$$\theta_G \colon R\mathscr{S}_G \xrightarrow{\sim} \bigotimes_{p\in\mathbb{P}} R\mathscr{S}_{G_p} ,$$

where the tensor product is taken over R. The inverse of θ maps the tensor product of a collection U_p, $p \in \mathbb{P}$, of subgroups of G to their product $\prod_{p\in\mathbb{P}} U_p$ in p. If also H is a finite abelian group then $(G \times H)_p = G_p \times H_p$ and we obtain an R-module isomorphism

$$\theta_{G,H} \colon R\mathscr{S}_{G\times H} \xrightarrow{\sim} \bigotimes_{p\in\mathbb{P}} R\mathscr{S}_{G_p\times H_p} .$$

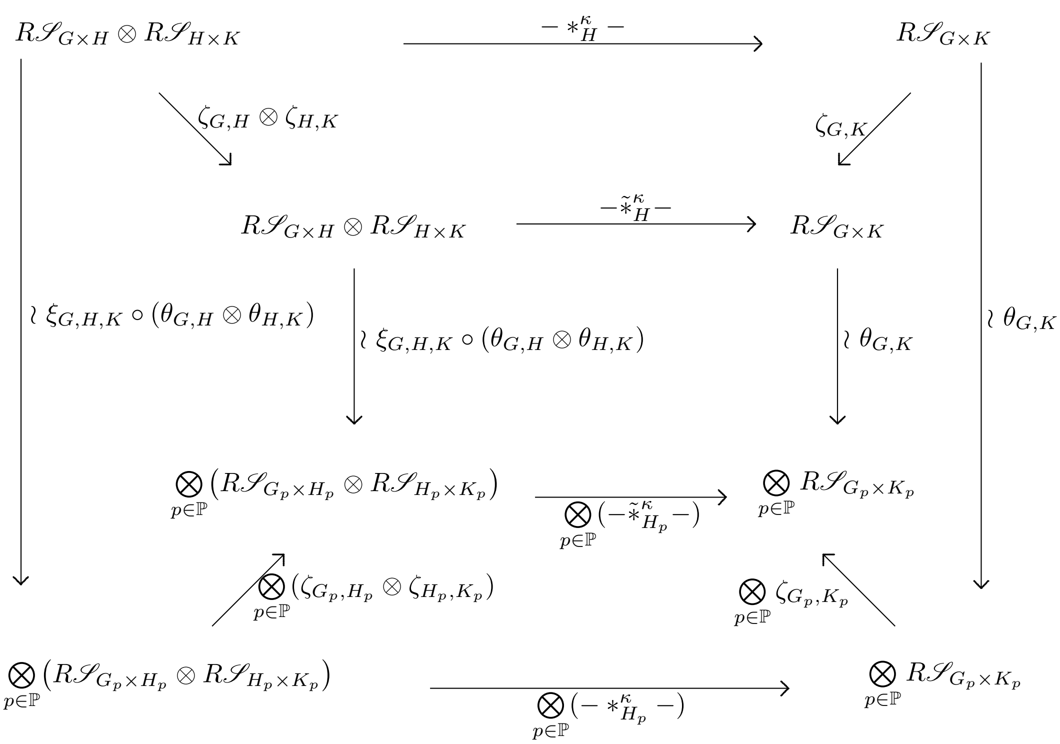
The assertions of the following lemmas are straightforward verifications. The proof of the first one is left to the reader.

8.7 Lemma *Let G and H be finite abelian groups and let $L \leqslant G \times H$ be a subgroup. Then the following equations hold in $\bigotimes_{p\in\mathbb{P}} R\mathscr{S}_{G_p\times H_p}$:*

(a) $\theta_{G,H}(\sum_{L'\leqslant L} L') = \otimes_{p\in\mathbb{P}}(\sum_{L'_p\leqslant L_p} L'_p)$;

(b) $\theta_{G,H}(\tilde{L}) = \otimes_{p\in\mathbb{P}}\widetilde{L_p}$.

8.8 Lemma *Let G, H and K be finite abelian groups such that $|H|$ is invertible in R. Then the diagram*



is commutative. Here

$$\xi_{G,H,K} : \Big(\bigotimes_{p\in\mathbb{P}} R\mathscr{S}_{G_p\times H_p}\Big) \otimes \Big(\bigotimes_{p\in\mathbb{P}} R\mathscr{S}_{H_p\times K_p}\Big) \xrightarrow{\sim} \bigotimes_{p\in\mathbb{P}} \big(R\mathscr{S}_{G_p\times H_p} \otimes R\mathscr{S}_{H_p\times K_p}\big)$$

denotes the canonical isomorphism.

Proof Let $L \leqslant G \times H$ and $M \leqslant H \times K$. The two paths in the outer square map $L \otimes M$ to the same element since

$$\frac{|k_2(L) \cap k_1(M)|}{|H|} = \prod_{p\in\mathbb{P}} \frac{|(k_2(L) \cap k_1(M))_p|}{|H_p|} = \prod_{p\in\mathbb{P}} \frac{|k_2(L_p) \cap k_1(M_p)|}{|H_p|}\,.$$

Furthermore, the top and bottom rectangles of the diagram commute, by Proposition 3.9, and the left and right rectangles of the diagram commute, by Lemma 8.7(a). Lastly, since the maps ζ with various indices are isomorphisms, also the inner square commutes. □

Proof *of Theorem 8.6.* Consider the inner square of the diagram in Lemma 8.8, and $\tilde{L} \otimes \tilde{M} \in R\mathscr{S}_{G\times H} \otimes R\mathscr{S}_{H\times K}$ in its top left corner. Using Lemma 8.7(b), the left-hand arrow in this inner square applied to this element yields the tensor product of the elements $\widetilde{L_p} \otimes \widetilde{M_p}$. By Proposition 8.4, this latter element is mapped under the bottom map to the tensor product of the elements $\lambda_p(L,M)\cdot(\widetilde{L_p * M_p})$. Since $L_p * M_p = (L*M)_p$, also $\lambda(L,M)\cdot\widetilde{L*M}$ is mapped to this element under $\theta_{G,K}$, by Lemma 8.7. This completes the proof of the theorem. □

Theorem 8.6 allows us to view the R-algebra $\tilde{A}_R^{\mathscr{D}}$ as a twisted category algebra $R_\lambda\tilde{\mathsf{C}}$: let $\tilde{\mathsf{C}} = \tilde{\mathsf{C}}(\mathscr{D})$ denote the inverse category associated to $\mathscr{D}$ as in Definition 5.16, and recall that we are assuming $|G|$ to be invertible in R, for every $G \in \mathscr{D}$. Recall further that the objects of $\tilde{\mathsf{C}}$ are pairs (G,G') with $G \in \mathscr{D}$ and $G' \leqslant G$. Moreover, for objects (G,G') and (H,H') of $\tilde{\mathsf{C}}$, the set of morphisms from (H,H') to (G,G') is given by $\mathscr{P}(G'\times H')$, and if (K,K') is also an object of $\tilde{\mathsf{C}}$ and $L \in \mathscr{P}(G'\times H')$ and $M \in \mathscr{P}(H'\times K')$ are morphisms in $\tilde{\mathsf{C}}$ then their composition is defined by $L*M$. The identity morphism of (G,G') is equal to $\Delta(G')$. Recall also that, for a morphism L between (H,H') and (G,G'), we will often write ${}_GL_H$ to indicate that it is a morphism between the objects (H,H') and (G,G'). The associativity of the R-algebra $\tilde{A}_R^{\mathscr{D}}$ and Theorem 8.6 imply that $(L,M)\mapsto\lambda(L,M)$ defines a 2-cocycle on C with values in $R^\times$ (if also $\varphi(|G|)$ is invertible in R for all $G\in\mathscr{D}$), and that the maps

$$R\mathscr{S}_{G\times H} \to R_\lambda\tilde{\mathsf{C}}\,,\quad \tilde{L}\mapsto L \in \mathrm{Hom}_{\tilde{\mathsf{C}}}\big((H,p_2(L)),(G,p_1(L))\big)\,,$$

for $G,H\in\mathscr{D}$, induce an R-algebra isomorphism

$$\delta\colon \tilde{A}_R^{\mathscr{D}} \xrightarrow{\sim} R_\lambda\tilde{\mathsf{C}} \qquad (24)$$

onto the twisted category algebra $R_\lambda\tilde{\mathsf{C}}$ of the inverse category $\tilde{\mathsf{C}}$.

Our next goal is to show that $R_\lambda\tilde{\mathsf{C}}$ is isomorphic to the 'untwisted' category algebra $R\tilde{\mathsf{C}}$. The next proposition will achieve this by showing that λ is a coboundary.

8.9 Definition For objects (G,G'), (H,H') of $\tilde{\mathsf{C}}$ and a morphism ${}_GL_H \in \mathscr{P}(G'\times H')$ from (H,H') to (G,G') we set

$$\mu({}_GL_H) := \prod_{p\in\mathbb{P}} \mu_p({}_GL_H)\,,$$

where

$$\mu_p({}_GL_H) := \begin{cases} \frac{\varphi(|p_1(L_p)|)}{|H_p|} & \text{if } s(L_p)=1,\\ \frac{|k_1(L_p)|}{|H_p|} & \text{if } s(L_p)>1.\end{cases}$$

8.10 Proposition *Assume that, for each $G\in\mathscr{D}$, both $|G|$ and $\varphi(|G|)$ are invertible in R. Let $(G,G'),(H,H'),(K,K')\in\mathrm{Ob}(\tilde{\mathsf{C}})$, and let ${}_GL_H\in\mathrm{Hom}_{\tilde{\mathsf{C}}}((H,H'),(G,G'))$ and ${}_HM_K\in\mathrm{Hom}_{\tilde{\mathsf{C}}}((K,K'),(H,H'))$. Then one has*

$$\lambda({}_GL_H,{}_HM_K) = \mu({}_GL_H)\cdot\mu({}_HM_K)\cdot\mu({}_G(L*M)_K)^{-1}\,.$$

Thus the 2-cocycle λ on the category $\tilde{\mathsf{C}}$ is a 2-coboundary, and the twisted category algebra $R_\lambda\tilde{\mathsf{C}}$ is isomorphic to the category algebra $R\tilde{\mathsf{C}}$, via the isomorphism

$$\epsilon\colon R_\lambda\tilde{\mathsf{C}}\to R\tilde{\mathsf{C}}\,,\quad {}_GL_H\mapsto\mu({}_GL_H)\cdot{}_GL_H\,,$$

for ${}_GL_H\in\mathrm{Mor}(\tilde{\mathsf{C}})$.

Proof For convenience, set $L := {}_GL_H$ and $M := {}_HM_K$. By the definitions of λ and μ, it suffices to show that

$$\lambda_p(L_p, M_p)\mu_p(L_p)^{-1}\mu_p(M_p)^{-1} = \mu_p(L_p * M_p)^{-1}\,, \tag{25}$$

for all $p \in \mathbb{P}$. To this end, let $p \in \mathbb{P}$, and set $l_p := \lambda_p(L_p, M_p)\mu_p(L_p)^{-1}\mu_p(M_p)^{-1}$ and $r_p := \mu_p(L_p * M_p)^{-1}$. We distinguish four cases:

(i) If $s(L_p) = 1 = s(M_p)$ then we have $L_p * M_p = p_1(L_p) \times p_2(M_p)$, and thus

$$l_p = \frac{|G_p|}{\varphi(|p_1(L_p)|)} = \frac{|G_p|}{\varphi(|p_1(L_p * M_p)|)} = r_p\,.$$

(ii) If $s(L_p) = 1$ and $s(M_p) > 1$ then $L_p * M_p = p_1(L_p) \times p_2(M_p)$, and

$$l_p = \frac{|G_p| \cdot |H_p| \cdot |k_1(M_p)|}{|H_p| \cdot \varphi(|p_1(L_p)|) \cdot |k_1(M_p)|} = \frac{|G_p|}{\varphi(|p_1(L_p * M_p)|)} = r_p\,.$$

(iii) If $s(L_p) > 1$ and $s(M_p) = 1$ then $L_p * M_p = p_1(L_p) \times p_2(M_p)$, and

$$l_p = \frac{|G_p| \cdot |H_p| \cdot |k_2(L_p)|}{|H_p| \cdot \varphi(|p_2(L_p)|) \cdot |k_1(L_p)|}\,.$$

Since $s(L_p) > 1$, we also have $|p_i(L_p)| > 1$, for $i = 1, 2$, and hence $\varphi(|p_i(L_p)|) = |p_i(L_p)| - |p_i(L_p)|/p$, for $i = 1, 2$. Thus

$$\frac{\varphi(|p_2(L_p)|)}{|k_2(L_p)|} = \frac{|p_2(L_p)|}{|k_2(L_p)|} - \frac{|p_2(L_p)|}{p \cdot |k_2(L_p)|} = \frac{|p_1(L_p)|}{|k_1(L_p)|} - \frac{|p_1(L_p)|}{p \cdot |k_1(L_p)|} = \frac{\varphi(|p_1(L_p)|)}{|k_1(L_p)|}\,.$$

This shows that

$$l_p = \frac{|G_p|}{\varphi(|p_1(L_p)|)} = \frac{|G_p|}{\varphi(|p_1(L_p * M_p)|)} = r_p\,.$$

(iv) If $s(L_p) > 1$ and $s(M_p) > 1$ then, by Lemma 2.7, $s(L_p * M_p) = \min\{s(L_p), s(M_p)\} > 1$, since $p_2(L_p) = p_1(M_p)$. Thus we have

$$l_p = \frac{|G_p| \cdot |H_p| \cdot |k_2(L_p) \cap k_1(M_p)|}{|H_p| \cdot |k_1(M_p)| \cdot |k_1(L_p)|} = \frac{|G_p| \cdot |k_1(L_p)| \cdot |k_1(M_p)|}{|k_1(L_p * M_p)| \cdot |k_1(L_p)| \cdot |k_1(M_p)|} = \frac{|G_p|}{|k_1(L_p * M_p)|} = r_p\,.$$

This settles Equation (25), and the assertion of the proposition follows. □

In consequence of Proposition 8.1, Equation (24), Proposition 8.10, Theorem 5.13, and Proposition 5.17 we have established the following theorem.

8.11 Theorem *Let $\mathscr{D}$ be a finite set of cyclic groups, let $\tilde{\mathsf{C}}$ be the finite inverse category associated with $\mathscr{D}$ in Definition 5.16, and let R be a commutative ring. Suppose that, for all $G \in \mathscr{D}$, both $|G|$ and $\varphi(|G|)$ are units in R. Then there is a sequence of R-algebra isomorphisms*

$$\bigoplus_{G,H\in\mathscr{D}} RB(G,H) = B_R^{\mathscr{D}} \xrightarrow{\ \gamma\ } \tilde{A}_R^{\mathscr{D}} \xrightarrow{\ \delta\ } R_\lambda\tilde{\mathsf{C}} \xrightarrow{\ \epsilon\ } R\tilde{\mathsf{C}} \xrightarrow{\ \omega\ } \bigtimes_k \mathrm{Mat}_{n(k)}(R\mathrm{Aut}(C_k))\,;$$

here k varies over all divisors of all group orders $|G|$ $(G \in \mathscr{D})$, C_k denotes a cyclic group of order k, and $n(k)$ the number of sections of groups in $\mathscr{D}$ isomorphic to C_k. In particular, if R is a field then $B_R^{\mathscr{D}}$ is a semisimple R-algebra.

8.12 Remark Suppose that R is a field.

(a) Specializing to the case where the set $\mathscr{D}$ consists of one cyclic group G only, Theorem 8.11 implies that the double Burnside algebra $RB(G,G)$ is semisimple, provided $|G|$ and $\varphi(|G|)$ are non-zero in R. Suppose that, whenever p is a prime divisor of $|G|$, also p^2 is a divisor of $|G|$. In this case $\varphi(|G|) \in R^\times$ implies $|G| \in R^\times$, and we recover one half of [Bc3, Proposition 6.1.7], characterizing the fields R and finite groups G such that $RB(G,G)$ is a semisimple R-algebra. In addition, Theorem 8.11 yields an explicit decomposition.

(b) Assume now that R has characteristic 0. Note that the R-algebra $R\tilde{\mathsf{C}}$ and the corresponding isomorphism ω from Theorem 5.13 are defined for an arbitrary finite set $\mathscr{D}$ of groups. However, if $\mathscr{D}$ does contain a non-cyclic group G then $R\tilde{\mathsf{C}}$ cannot be isomorphic to $\tilde{A}_R^{\mathscr{D}}$, since otherwise $RB(G,G)$ (which can be considered as a Schur algebra of $\tilde{A}_R^{\mathscr{D}}$) would be semisimple, contradicting Bouc's characterization, cf. [Bc3, Proposition 6.1.7].

8.13 Remark Recall from Remark 5.14 the explicit decomposition of $R\tilde{\mathsf{C}}$ as a direct sum of two-sided ideals I_e that correspond to the matrix algebras under the above isomorphism. We close this section by determining the R-basis of $R\tilde{\mathsf{C}}$ corresponding under ω to the obvious R-basis in the direct product of matrix rings over group algebras, consisting of matrices that have only one non-zero entry, namely a group element in the group algebra. We emphasize that in Remark 5.14, which explicitly described the isomorphism ω, R could be any commutative ring; the additional assumptions in Theorem 8.11 were only necessary to establish the isomorphism $\epsilon \circ \delta \circ \zeta \circ \alpha \colon B_R^{\mathscr{D}} \cong R\tilde{\mathsf{C}}$. For this purpose it is convenient to use a more number-theoretic notation for the subgroups L of $G \times H$ in the case that G and H are cyclic groups.

8.14 Notation Let $G = \langle x \rangle$ and $H = \langle y \rangle$ be cyclic groups with $|G| = n$ and $|H| = m$. Suppose that $a, b, k \in \mathbb{N}$ are such that $a \mid n$, $b \mid m$, and $k \mid \gcd\{a,b\}$. Then $x^{n/a}\langle x^{nk/a}\rangle$ and $y^{m/b}\langle y^{mk/b}\rangle$ generate subquotients of order k of G and H, respectively. Moreover, let $i \in \mathbb{Z}$ be such that $\gcd\{k,i\} = 1$. Then the map

$$\alpha_i : \langle y^{m/b}\rangle/\langle y^{mk/b}\rangle \to \langle x^{n/a}\rangle/\langle x^{nk/a}\rangle\,, \quad y^{m/b}\langle y^{mk/b}\rangle \mapsto x^{in/a}\langle x^{nk/a}\rangle\,,$$

defines a group isomorphism. We denote the group $(\langle x^{n/a}\rangle, \langle x^{nk/a}\rangle, \alpha_i, \langle y^{m/b}\rangle, \langle y^{mk/b}\rangle)$ of $G \times H$ by

$${}_G(k; a, i, b)_H\,.$$

Thus, if $L = {}_G(k; a, i, b)_H$ then one has $|p_1(L)| = a$, $|k_1(L)| = a/k$, $|p_2(L)| = b$, $|k_2(L)| = b/k$, $s(L) = k$. Clearly, each subgroup L of $G \times H$ can be written in this way with k, a, and b, uniquely determined by L. Moreover, ${}_G(k; a, i, b)_H = {}_G(k; a, j, b)_H$ if and only if $i, j \in \mathbb{Z}$ satisfy $i \equiv j \pmod{k}$.

Note that the subgroups $\langle y^{m/b}\rangle$, $\langle y^{mk/b}\rangle$, $\langle x^{n/a}\rangle$, and $\langle x^{nk/a}\rangle$ do not depend on the choice of the generators x and y. However, the isomorphism α_i does depend on these choices. Thus, when we will use the notation ${}_G(k; a, i, b)_H$ we assume that the groups G and H are equipped with a fixed choice of generators.

With this notation the following lemma is an immediate consequence of Lemma 2.7; we thus omit its proof.

8.15 Lemma *With the notation from 8.14, let G, H, and K be cyclic groups, and let $L \leqslant G \times H$ and $M \leqslant H \times K$ with $p_2(L) = p_1(M)$ be given by*

$$L = {}_G(k; a, i, b)_H \quad \text{and} \quad M = {}_H(l; b, j, c)_K\,,$$

for divisors $a \mid |G|$, $b \mid |H|$, $c \mid |K|$, $k \mid \gcd\{a,b\}$, $l \mid \gcd\{b,c\}$, *and integers* $i,j \in \mathbb{Z}$ *with* $\gcd\{i,k\} = 1 = \gcd\{j,l\}$. *Then*

$$L * M = {}_G(\gcd\{k,l\}; a, ij, c)_K\,.$$

8.16 Remark Recall from Proposition 5.17 that the idempotent endomorphisms in $\tilde{\mathsf{C}}$ are of the form ${}_G(k;a,1,a)_G$, for $G \in \mathscr{D}$, $a \mid |G|$ and $k \mid a$. Moreover, idempotent endomorphisms ${}_G(k;a,1,a)_G$ and ${}_H(l;b,1,b)_H$ in $\tilde{\mathsf{C}}$ are equivalent if and only if $l = k$.

Let $\mu\colon \mathbb{N} \to \mathbb{N}$ denote the number-theoretic Möbius function given by $\mu(n) = 0$ if n is divisible by p^2 for some prime p, and $\mu(n) = (-1)^r$ if n is the product of r pairwise distinct primes. Recall that if one considers the divisors l of a fixed natural number k as a partially ordered set under the divisibility relation then the Möbius function for this poset, applied to (l,k), is equal to $\mu(k/l)$.

8.17 Lemma (a) *For every idempotent endomorphism* $e := {}_G(k;a,1,a)_G$ *in* $\tilde{\mathsf{C}}$, *one has*

$$\{s \in \mathrm{Mor}(\tilde{\mathsf{C}}) \mid \hat{s} \circ s = e\} = \{{}_H(k;b,i,a)_G \mid H \in \mathscr{D}, k \mid b, b \mid |H|, \gcd\{i,k\} = 1\}\,.$$

(b) *For every* $s := {}_G(k;a,i,b)_H \in \mathrm{Mor}(\tilde{\mathsf{C}})$, *one has*

$$\{u \leqslant s \mid u \in \mathrm{Mor}(\tilde{\mathsf{C}})\} = \{{}_G(l;a,i,b)_H \mid l \mid k\}\,.$$

In particular, for $u := {}_G(l;a,i,b)_H$ *with* $u \leqslant s$ *in* $\mathrm{Mor}(\tilde{\mathsf{C}})$, *one has* $\mu_{u,s}^{\mathrm{Mor}(\tilde{\mathsf{C}})} = \mu(k/l)$.

Proof Both parts of the lemma are easy consequences of 5.10, Proposition 5.17, and Lemma 8.15. □

8.18 Notation Suppose that $k \in \mathbb{N}$ is such that k divides $|G|$, for some $G \in \mathscr{D}$. Let $\{e_1, \ldots, e_{n(k)}\}$ be the equivalence class of the idempotent endomorphism ${}_G(k;k,1,k)_G$ in $\tilde{\mathsf{C}}$. Thus, for each $i = 1, \ldots, n(k)$, one has a pair (G_i, a_i), where $G_i \in \mathscr{D}$ and $a_i \in \mathbb{N}$ such that a_i divides $|G_i|$, k divides a_i, and $e_i = {}_{G_i}(k;a_i,1,a_i)_{G_i}$.

With this notation, the following result is now a consequence of Lemma 8.17 and Remark 5.14. It makes the map ω in Theorem 8.11 as explicit as possible.

8.19 Corollary *Let* $\mathscr{D}$ *be a finite set of cyclic groups, let* $\tilde{\mathsf{C}}$ *be the finite inverse category associated with* $\mathscr{D}$ *in Definition 5.16, and let* R *be any commutative ring. Moreover, let* k *be a divisor of the order of some* $G \in \mathscr{D}$ *and assume the notation from 8.18. For* $i,j \in \{1,\ldots,n(k)\}$, *and for an integer* t *with* $\gcd\{t,k\} = 1$, *set*

$$b(k;j,t,i) := \sum_{l \mid k} \mu(k/l) \cdot {}_{G_i}(l;a_j,t,a_i)_{G_i} \in R\tilde{\mathsf{C}}\,.$$

Then $\omega(b(k;i,j,t))$ *is the standard basis element in* $\times_k \mathrm{Mat}_{n(k)}(R\mathrm{Aut}(C_k))$ *that has zero entry everywhere except in the* k*-th component, and whose* k*-th component is equal to the matrix whose only non-zero entry is located in the* (i,j)*-th position and is equal to the element in* $\mathrm{Aut}(C_k)$ *that corresponds to* t. *In particular, the central idempotent* e_k *of* $R\tilde{\mathsf{C}}$, *corresponding under* ω *to the tuple consisting of the identity matrix in the* k*-th component and the zero matrix in all other components, is equal to*

$$\sum_{i=1}^{n(k)} \sum_{l \mid k} \mu(k/l) \cdot {}_{G_i}(l;a_i,1,a_i)_{G_i}\,.$$

References

[BD] R. BOLTJE, S. DANZ: A ghost ring for the left-free double Burnside ring and an application to fusion systems. Adv. Math. **229** (2012), 1688–1733.

[BK] R. BOLTJE, B. KÜLSHAMMER: Central idempotents in double Burnside rings, in preparation.

[Bc1] S. BOUC: Foncteurs d'ensembles munis d'une double action. J. Algebra **183** (1996), 664–736.

[Bc2] S. BOUC: The functor of units of Burnside rings for p-groups. Comment. Math. Helv. **82** (2007), 583–615.

[Bc3] S. BOUC: Biset functors for finite groups. Lecture Notes in Mathematics, 1990. Springer-Verlag, Berlin, 2010.

[BST] S. BOUC, R. STANCU, J. THÉVENAZ: Simple biset functors and double Burnside rings. Preprint 2012. arXiv:1203.0195.

[BT1] S. BOUC, J. THÉVENAZ: The group of endo-permutation modules. Invent. Math. **139** (2000), 275–349.

[BT2] S. BOUC, J. THÉVENAZ: Stabilizing bisets. Preprint 2010.

[BLO] C. BROTO, R. LEVI, B. OLIVER: The homotopy theory of fusion systems. J. Amer. Math. Soc. **16** (2003), 779–856.

[D] A. W. M. DRESS: A characterization of solvable groups. Math. Z. **110** (1969), 213–217.

[Gr1] J. A. GREEN: On the structure of semigroups. Ann. of Math. (2) **54** (1951), 163–172.

[Gr2] J. A. GREEN: Polynomial representations of GL_n. Second corrected and augmented edition. With an appendix on Schensted correspondence and Littelmann paths by K. Erdmann, Green and M. Schocker. Lecture Notes in Mathematics, 830. Springer-Verlag, Berlin, 2007.

[Hup] B. HUPPERT: Endliche Gruppen I. Die Grundlehren der Mathematischen Wissenschaften, Band 134. Springer-Verlag, Berlin-New York, 1967.

[Ka] J. KASTL: Inverse categories. in: Algebraische Modelle, Kategorien und Gruppoide (ed. H.-J. Hoehnke), Akademie-Verlag, Berlin (1979), 51–60.

[L] M. LINCKELMANN: On inverse categories and transfer in cohomology. to appear in Skye conference proceedings.

[LS] M. LINCKELMANN, M. STOLORZ: On simple modules over twisted finite category algebras. preprint, 2011.

[MP] J. MARTINO, S. PRIDDY: Stable homotopy classification of $BG_p^\wedge$. Topology **34** (1995), 633–649.

[RS] K. RAGNARSSON, R. STANCU: Saturated fusion systems as idempotents in the double Burnside ring. Preprint. arXiv:0911.0085.

[St] B. STEINBERG: Möbius functions and semigroup representation theory. II. Character formulas and multiplicities. Adv. Math. **217** (2008), 1521–1557.

[T] J. Thévenaz: Endo-permutation modules, a guided tour. In: Group representation theory, 115–147, EPFL Press, Lausanne, 2007.

[W] P. Webb: Stratifications and Mackey functors. II: globally defined Mackey functors. J. K-Theory **6** (2010), 99–170.

[Y] T. Yoshida: On the unit groups of Burnside rings. J. Math. Soc. Japan **42** (1990), 31-64.