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TRAN VAN TAN AND VU VAN TRUONG

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Mathematisches Forschungsinstitut Oberwolfach gGmbH
Oberwolfach Preprints (OWP) ISSN 1864-7596

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Imprint:

Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

The Oberwolfach Preprints (OWP, ISSN 1864-7596) are published by the MFO.

Non-integrated defect relation for meromorphic maps of complete Kähler manifolds into a projective variety intersecting hypersurfaces

Tran Van Tan and Vu Van Truong

Abstract

In 1985, Fujimoto established a non-integrated defect relation for meromorphic maps of complete Kähler manifolds into the complex projective space intersecting hyperplanes in general position. In this paper, we generalize the result of Fujimoto to the case of meromorphic maps into a complex projective variety intersecting hypersurfaces in general position.

1 Introduction and statements

Let f be a meromorphic map of an $\mathfrak{m}-$dimension connected complex manifold M into $\mathbb{C}P^N$, and let p_0 be a positive integer or $+\infty$ and D be a hypersurface in $\mathbb{C}P^N$ with $\mathrm{Im} f \not\subset D$. We denote the intersection multiplicity of the image of f and D at $f(a)$ by $\nu_{(f,D)}(a)$ and the pull-back of the normalized Fubini-Study metric form Ω on $\mathbb{C}P^N$ by Ω_f. The non-integrated defect of f with respect to D cut by p_0 is defined by

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Mathematics Subject Classification 2000: Primary 32H30; Secondary 32H04, 32H25, 14J70.

Key words and phrases: Nevanlinna theory, Second Main Theorem.

$$\delta_f^{[p_0]}(D) := 1 - \inf\{\eta \geq 0 : \eta \text{ satisfies condition } (*)\}.$$

Here, the condition $(*)$ means that there exists a bounded nonnegative continuous function h on M with zeros of order not less than $\min\{\nu_{(f,D)}, p_0\}$ such that

$$(\deg D)\eta\Omega_f + dd^c \log h^2 \geq [\min\{\nu_{(f,D)}, p_0\}],$$

where we mean by $[\nu]$ the $(1,1)-$current associated with a divisor ν. By ([9], pp. 250) if M is a ball in $\mathbb{C}^{\mathfrak{m}}$, then the condition $(*)$ is satisfied if and only if there exists a continuous plurisubharmonic function $u \not\equiv -\infty$ such that $e^u|\phi| \leqslant \|f\|^{(\deg D)\eta}$, where ϕ is a nonzero holomorphic function on M with $\nu_\phi = \min\{\nu_{(f,D)}, p_0\}$. In other words, there exists a continuous plurisubharmonic function $v \not\equiv -\infty$ such that $e^v \leqslant \|f\|^{(\deg D)\eta}$ and $v - \log|\phi|$ is plurisubharmonic, where ϕ is a nonzero holomorphic function on M with $\nu_\phi = \min\{\nu_{(f,D)}, p_0\}$.

It is clear that $0 \leqslant \delta_f^{[p_0+1]}(D) \leqslant \delta_f^{[p_0]}(D) \leqslant 1$, and $\delta_f^{[p_0]}(D) = 1$ if $\mathrm{Im} f \cap D = \varnothing$. Moreover, if $\nu_{(f,D)}(z) \geq p$ for every $z \in f^{-1}(D)$, then

$$\delta_f^{[p_0]}(D) \geq 1 - \frac{p_0}{p}.$$

For $z = (z_1, \ldots, z_{\mathfrak{m}}) \in \mathbb{C}^{\mathfrak{m}}$, we set $\|z\| = \Big(\sum\limits_{j=1}^{\mathfrak{m}} |z_j|^2\Big)^{1/2}$ and define

$$B(r) = \{z \in \mathbb{C}^{\mathfrak{m}} : \|z\| < r\}, \quad S(r) = \{z \in \mathbb{C}^{\mathfrak{m}} : \|z\| = r\},$$
$$d^c = \frac{\sqrt{-1}}{4\pi}(\overline{\partial} - \partial), \quad \mathcal{V}_k = \big(dd^c\|z\|^2\big)^k, \ \sigma = d^c\log\|z\|^2 \wedge \big(dd^c\log\|z\|\big)^{\mathfrak{m}-1}.$$

Let ℓ be a positive integer or $+\infty$ and ν be a divisor on $B(R_0)$ $(0 < R_0 \leqslant +\infty)$. Set $|\nu| := \overline{\{z : \nu(z) \neq 0\}}$ and $\nu^{[\ell]} := \min\{\nu, \ell\}$.

The truncated counting function of ν is defined by

$$N_\nu^{[\ell]}(r, r_0) := \int\limits_{r_0}^{r} \frac{n^{[\ell]}(t)}{t^{2\mathfrak{m}-1}} dt \quad (0 < r_0 < r < R_0),$$

where

$$n^{(\ell)}(t) = \int\limits_{|\nu| \cap B(t)} \nu^{[\ell]} \cdot \mathcal{V}_{\mathfrak{m}-1} \quad \text{for} \quad \mathfrak{m} \geq 2 \text{ and}$$

$$n^{[\ell]}(t) = \sum_{|z| \leqslant t} \nu^{[\ell]} \qquad\qquad \text{for} \quad \mathfrak{m} = 1.$$

For a nonzero meromorphic function φ on $B(R_0)$, we denote by ν_φ the zero divisor of φ and set $N_\varphi^{[\ell]}(r) := N_{\nu_\varphi}^{[\ell]}(r)$.

Let f be a meromorphic map of $B(R_0)$ into $\mathbb{C}P^N$. For arbitrary fixed homogeneous coordinates $(w_0 : \cdots : w_N)$ of $\mathbb{C}P^N$, we take a reduced representation $f = (f_0 : \cdots : f_N)$, which means that each f_i is a holomorphic function on $B(R_0)$ and $f(z) = (f_0(z) : \cdots : f_N(z))$ outside the analytic set $\{z : f_0(z) = \cdots = f_N(z) = 0\}$ of codimension ≥ 2. Set $\|f\| = \max\{|f_0|, \dots, |f_N|\}$.

The characteristic function of f is defined by

$$T_f(r, r_0) := \int\limits_{r_0}^{r} \frac{dt}{t^{2\mathfrak{m}-1}} \int\limits_{B(t)} \Omega_f \wedge \mathcal{V}_{\mathfrak{m}-1}, \quad 0 < r_0 < r < R_0.$$

We have

$$T_f(r, r_0) := \int\limits_{S(r)} \log\|f\|\sigma - \int\limits_{S(r_0)} \log\|f\|\sigma.$$

For a hypersurface D of degree d in $\mathbb{C}P^N$ defined by the homogeneous polynomial $Q \in \mathbb{C}[x_0, \dots, x_N]$, if $Q(f) := Q(f_0, \dots, f_N) \not\equiv 0$ we denote

$$N_f^{[\ell]}(r, r_0, D) := N_{Q(f)}^{[\ell]}(r, r_0) \text{ and } {}^*\delta_f^{[\ell]}(D) := 1 - \lim_{r \to R_0} \sup \frac{N_f^{[\ell]}(r, r_0, D)}{dT_f(r, r_0)}.$$

If $\lim_{r \to R_0} T_f(r, r_0) = +\infty$, then by an argument similar to the proof of Proposition 5.6 in [9], we have

$$0 \leqslant \delta_f^{[\ell]}(D) \leqslant {}^*\delta_f^{[\ell]}(D) \leqslant 1. \tag{1.1}$$

For brevity we will omit the character $^{[\ell]}$ in the counting function, defect number, and divisor if $\ell = +\infty$.

Let $V \subset \mathbb{C}P^N$ be a smooth complex projective variety of dimension $n \geq 1$. Let $D_1, \ldots, D_k$ $(k \geq 1)$ be hypersurfaces in $\mathbb{C}P^N$ of degree d_j. The hypersurfaces $D_1, \ldots, D_k$ are said to be in *general position in* V if for any distinct indices $1 \leqslant i_1 < \cdots < i_s \leqslant k$, $(1 \leqslant s \leqslant n+1)$, there exist hypersurfaces $D'_1, \ldots, D'_{n+1-s}$ in $\mathbb{C}P^N$ such that

$$V \cap D_{i_1} \cap \cdots \cap D_{i_s} \cap D'_1 \cap \cdots \cap D'_{n+1-s} = \varnothing.$$

In particular for hypersurfaces $D_1, \ldots, D_k$ in general position in V, we have $V \not\subseteq D_j$ for all $j = 1, ..., k$.

In 1983, relating to the study of value distribution of the Gauss maps of a complete minimal surfaces in $\mathbb{R}^m$, Fujimoto [8] introduced the new notion of the non-integrated defect for a holomorphic map of an open Riemann surface into $\mathbb{C}P^n$ and obtained some results analogous to the Nevanlinna-Cartan defect relation. In [9], he generalized his result in 1983 to the case of meromorphic maps of a complete Kähler manifold into the complex projective space intersecting hyperplanes in general position. By using the technique of Diophantine approximation introduced in [2,6,7], recently, Ru [13,14], Dethloff-Tan-Thai [3,4] obtained Nevanlinna-Cartan defect relations for holomorphic maps of $\mathbb{C}^{\mathrm{m}}$ into a complex projective variety intersecting hypersurfaces in general position. The purpose of this paper is to generalize the result of Fujimoto [9] on the non-integrated defect relation to the case of meromorphic maps of a complete Kähler manifold into a complex projective variety intersecting hypersurfaces in general position. A part of our paper is motived by [3,4,13,14]. However, we would like to remark that in the proofs of these papers, the result obtained by Ru in 1997 plays an essential role, but it does not remain valid for our situation. Instead we use the Logarithmic Derivative Lemma. Moreover, whereas in previous papers the truncation level of multiplicity depends on the number of hypersurfaces, in our situation we need a truncation level of multiplicity that does not depend on this number.

Let $V \subset \mathbb{C}P^N$ be a smooth complex projective variety of dimension $n \geq 1$ and let $D_1, \ldots, D_q$ $(q > n+1)$ be hypersurfaces in $\mathbb{C}P^N$ of degree d_j. Assume that $D_1, \ldots, D_q$ are in general position in V. Denote by d the least common multiple of $d_1, \ldots, d_q$. Let ϵ be an arbitrary constant with $0 < \epsilon < 1$. Set

$$m := [4d^{n+1}(2n+1)(n+1)\deg V \cdot \frac{1}{\epsilon}] + 1 \tag{1.2}$$

where $[x] := \max\{k \in \mathbb{Z} : k \leqslant x\}$ for a real number x.

With these notations, we state our main results:

Theorem 1.1. *Let f be an algebraically nondegenerate map of $B(R_0)$ into V. Then, there exists a positive integer $\ell \leqslant \binom{N+md}{md}$ such that*

$$(q-n-1-q\epsilon)T_f(r,r_0) \leqslant \sum_{j=1}^{q} \frac{1}{d_j} N_f^{[\ell]}(r,r_0,D_j) + A(r),$$

where $A(r)$ is evaluated as follows.

i) In the case $R_0 < \infty$,

$$A(r) \leqslant K\big(\log^+ \frac{1}{R_0 - r} + \log^+ T_f(r,r_0)\big)$$

for every $r \in [r_0, R)$ excluding a set E with $\int_E \frac{1}{(R_0-t)} dt < \infty$, where K is a positive constant.

ii) In the case $R_0 = \infty$,

$$A(r) \leqslant K\big(\log r + \log^+ T_f(r,r_0)\big)$$

for every $r \in [r_0, +\infty)$ excluding a set E' with $\int_{E'} dt < \infty$.

As a corollary of Theorem 1.1, we get the following defect relation.

Corollary 1.2. *In the same situation as in Theorem 1.1, if (i) $R_0 < \infty$ and*

$$\lim_{r\to R_0} \sup \frac{T_f(r,r_0)}{\log \frac{1}{R_0 - r}} = \infty$$

or (ii) $R_0 = \infty$, then

$$\sum_{j=1}^{q} {}^*\delta_f^{[\ell]}(D_j) \leqslant n + 1 + q\epsilon.$$

Theorem 1.3. *Let M be a complete Kähler manifold with Kähler form $\omega = \frac{\sqrt{-1}}{2} \sum_{i,j} h_{ij} dz_i \wedge dz_j$. Set*

$$Ric\, \omega = dd^c \log(det(h_{ij})).$$

Assume that the universal covering $\widetilde{M}$ of M is biholomorphically isomorphic to a ball $B(R_0)$ $(0 < R_0 \leqslant \infty)$. Let f be an algebraically nondegenerate *meromorphic map of M into V. For some $\rho \geq 0$, if there exists a bounded continuous function $h \geq 0$ on M such that*

$$\rho\Omega_f + dd^c \log h^2 \geq Ric\,\omega, \tag{1.3}$$

then for any $\epsilon > 0$ we have

$$\sum_{j=1}^{q} \delta_f^{[\ell]}(D_j) \leqslant n + 1 + q\epsilon + \rho T$$

for some posivite integers ℓ, T satisfying

$$\ell \leqslant \binom{N+md}{md} \ \text{ and } \ T \leqslant \frac{(n+1)\binom{N+md}{md}}{d(m-(n+1)(2n+1)d^n \deg V)}.$$

Let $\mathcal{D}_k$ be an arbitrary set of hypersurfaces in $\mathbb{C}P^N$ satisfying following conditions:

i) $1 \leqslant degD \leqslant k$, for any $D \in \mathcal{D}_k$, and

ii) $(\cap_{i=1}^{n+1} D_i) \cap V = \varnothing$, for any (n+1) distinct hypersurfaces $D_1, \ldots, D_{n+1} \in \mathcal{D}_k$.

Under the same assumption of Theorem 1.3, we note that $\delta_f(D) \leqslant \delta_f^{[p]}(D)$ for any positive integer p, the number m given in the formula (1.2) does not depend on q, and the least common multiple of all $\deg D$ ($D \in \mathcal{D}_k$) is not bigger than $k!$. Therefore, according to Theorem 1.3, it is easy to see that for any $\epsilon > 0$, the cardinality of the set $\{D \in \mathcal{D}_k : \delta_f(D) \geq 2\epsilon\}$ is finite. By this fact, we have the following corollary.

Corollary 1.4. *The number of $D \in \mathcal{D}_k$ with $\delta_f(D) > 0$ is at most countable.*

We finally give an application of Theorem 1.3 to the study of the Gauss map of a complete regular submanifold of $\mathbb{C}^\kappa$.

Let $g = (g_1, \ldots, g_\kappa) : M \longrightarrow \mathbb{C}^\kappa$ be a regular submanifold of $\mathbb{C}^\kappa$, namely, M be a connected complex manifold and g be a holomorphic map of M into $\mathbb{C}^\kappa$ such that rank $d_p g = \dim M$ for every point $p \in M$. To each point $p \in M$, we assign the tangent space T_pM of M at p which may be regarded as an $\mathfrak{m}-$dimensional linear subspace of $T_{g(p)}\mathbb{C}^\kappa$, where $\mathfrak{m} = \dim M$. On the other hand, each $T_x\mathbb{C}^\kappa$ is identified with $T_0\mathbb{C}^\kappa = \mathbb{C}^\kappa$ by an parallel translation. Therefore, to each T_pM corresponds a point $G(p)$ in the complex

Grassmannian manifold $G(m, \kappa)$ of all $\mathfrak{m}-$dimensional linear subspace of $\mathbb{C}^\kappa$. We call the map $G : M \longrightarrow G(\mathfrak{m}, \kappa)$ the Gauss map of $g : M \longrightarrow \mathbb{C}^\kappa$. On the other hand, the space $G(\mathfrak{m}, \kappa)$ is canonically imbedded in $\mathbb{C}P^N$, where $N = \binom{\kappa}{\mathfrak{m}} - 1$. Therefore, the Gauss map G may be identified with the holomorphic map of M into $\mathbb{C}P^N$ given as follows.

Taking holomorphic local coordinates $(z_1 \ldots, z_\mathfrak{m})$ defined on an open set $U \subset M$, we consider the map

$$\Lambda := D_1 g \wedge \cdots \wedge D_\mathfrak{m} g : U \longrightarrow \wedge^\mathfrak{m} \mathbb{C}^\kappa \setminus \{0\} = \mathbb{C}^{N+1} \setminus \{0\},$$

where $D_i g := (\frac{\partial g_1}{\partial z_i}, \ldots, \frac{\partial g_\kappa}{\partial z_i})$. Then $G := \pi \cdot \Lambda$ locally, where π is the canonical projection map.

A regular submanifold M of $\mathbb{C}^\kappa$ is considered a Kähler manifold with the metric ω induced from the standard flat metric on $\mathbb{C}^\kappa$. Take $\rho = 1, h \equiv 1$, then by ([9], pp. 259) we have

$$\rho \Omega_G + dd^c \log h^2 = dd^c \|G\| = Ric\, \omega.$$

Therefore, we get the following corollary of Theorem 1.3 (with $\rho = 1, h \equiv 1$).

Corollary 1.5. *Let $g : M \longrightarrow \mathbb{C}^\kappa$ be a complete regular submanifold such that the universal covering of M is biholomorphically isomorphic to $B(R_0)$ $(0 < R_0 \leqslant \infty)$. Let $G : M \longrightarrow \mathbb{C}P^N$ be the Gauss map of g. Let $V \subset \mathbb{C}P^N$ be a smooth complex projective variety of dimension n such that $ImG \subset V$ and $G : M \longrightarrow V$ is algebraically nondegenerate. Then*

$$\sum_{j=1}^{q} \delta_G^{[\ell]}(D_j) \leqslant n + 1 + q\epsilon + T$$

for some posivite integers ℓ, T satisfying

$$\ell \leqslant \binom{N+md}{md} \text{ and } T \leqslant \frac{(n+1)\binom{N+md}{md}}{d(m-(n+1)(2n+1)d^n \deg V)}.$$

Acknowledgements: This work was done during a stay of the first named author at the Mathematisches Forschungsinstitut Oberwolfach, Germany. He wishes to express his gratitude to this organization.

2 Some lemmas

Let $X \subset \mathbb{C}P^N$ be a projective variety of dimension n and degree $\triangle$. Let I_X be the prime ideal in $\mathbb{C}[x_0, \ldots, x_N]$ defining X. Denote by $\mathbb{C}[x_0, \ldots, x_N]_m$ the vector space of homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_N]$ of degree m (including 0). Put $I_X(m) := \mathbb{C}[x_0, \ldots, x_N]_m \cap I_X$.

The Hilbert function H_X of X is defined by

$$H_X(m) := \dim \mathbb{C}[x_0, \ldots, x_N]_m \big/ I_X(m). \tag{2.1}$$

Lemma 2.1 ([4], Lemma 3.1)**.** *For $n \geq 1$, we have $H_X(m) \geq m+1$ for all $m \geq 1$.*

For each tuple $c = (c_0, \ldots, c_N) \in \mathbb{R}^{N+1}_{\geq 0}$, and $m \in \mathbb{N}$, we define the m-th Hilbert weight $S_X(m, c)$ of X with respect to c by

$$S_X(m, c) := \max \sum_{i=1}^{H_X(m)} I_i \cdot c,$$

where $I_i = (I_{i0}, \ldots, I_{iN}) \in \mathbb{N}_0^{N+1}$ and the maximum is taken over all sets $\{x^{I_i} = x_0^{I_{i0}} \cdots x_N^{I_{iN}}\}$ whose residue classes modulo $I_X(m)$ form a basis of the vector space $\mathbb{C}[x_0, \ldots, x_N]_m \big/ I_X(m)$.

Lemma 2.2. *Let $X \subset \mathbb{C}P^N$ be an algebraic variety of dimension n and degree $\triangle$. Let $m > \triangle$ be an integer and let $c = (c_0, \ldots, c_N) \in \mathbb{R}^{N+1}_{\geq 0}$. Let $\{i_0, \ldots, i_n\}$ be a subset of $\{0, \ldots, N\}$ such that $\{x = (x_0 : \cdots : x_N) \in \mathbb{C}P^N : x_{i_0} = \cdots = x_{i_n} = 0\} \cap X = \varnothing$. Then*

$$\frac{1}{mH_X(m)} S_X(m, c) \geq \frac{1}{(n+1)}(c_{i_0} + \cdots + c_{i_n}) - \frac{(2n+1)\triangle}{m} \cdot \max_{0 \leqslant i \leqslant N} c_i.$$

Proof. We refer to [6], Theorem 4.1, and [7], Lemma 5.1 (or [14], Theorem 2.1 and Lemma 3.2). $\square$

Let f be a linearly nondegenerate meromorphic map of $B(R_0)$ to $\mathbb{C}P^N$ with reduced presentation $f = (f_0 : \cdots : f_N)$. Then by Proposition 4.5 in [9], there exist $N+1$ sets $(\alpha_i = (\alpha_{i1}, \ldots, \alpha_{i\mathfrak{m}})$ $(0 \leqslant i \leqslant N)$ of $\mathfrak{m}$ nonnegative integers such that $|\alpha_0| + \cdots + |\alpha_N| \leqslant \frac{N(N+1)}{2}$, and the Wronskian $W_\alpha(f) := \det(D^{\alpha_i} f,\ 0 \leqslant i \leqslant N) \not\equiv 0$.

Lemma 2.3 ([9], Proposition 6.1). *In the above situation, take constants t, p with $0 < t(|\alpha_0| + \cdots + |\alpha_N|) < p < 1$. Let $H_1, \ldots, H_q$ $(q \geq N+1)$ be hyperplanes in $\mathbb{C}P^N$ in general position. Then, for $0 < r_0 < R_0$ there exists a positive constant K such that for $r_0 < r < R < R_0$*

$$\int\limits_{S(r)} |z^{\alpha_0+\cdots+\alpha_N} \frac{W_\alpha(f)}{H_1(f)\cdots H_q(f)}|^t \|f\|^{t(q-N-1)}\sigma \leqslant K(\frac{R^{2\mathfrak{m}-1}}{R-r}T_f(R, r_0))^p.$$

Lemma 2.4 ([9], Proposition 4.10).

$$\nu_{\frac{H_1(f)\cdots H_q(f)}{W_\alpha(f)}} \leqslant \sum_{j=1}^{q} \nu^{[N]}_{H_j(f)}$$

outside an analytic set of codimension ≥ 2.

3 Proof of Theorems 1.1-1.3.

Let P_j $(j = 1, ..., q)$ be homogeneous polynomials in $\mathbb{C}[x_0, \ldots, x_N]$ defining the hypersurfaces D_j, $\deg P_j = d_j$. Set $Q_j := P_j^{\frac{d}{d_j}}$. Then, $\deg Q_j = d$ and

$$\frac{1}{d_j}N^{[\ell]}_f(r, r_0, D_j) = \frac{1}{d_j}N^{[\ell]}_{P_j(f)}(r, r_0) \geq \frac{1}{d}N^{[\ell]}_{Q_j(f)}(r, r_0). \tag{3.1}$$

Since $D_1, \ldots, D_q$ are in general position in V, we have $\cap_{j=1}^{q} D_j \cap V = \varnothing$. We define a map $\Phi : V \longrightarrow \mathbb{C}P^{q-1}$ by $\Phi(x) = (Q_1(x) : \cdots : Q_q(x))$. Then Φ is a finite morphism (see [15], Theorem 8, page 65). We have that $Y := im\Phi$ is a complex projective subvariety of $\mathbb{C}P^{q-1}$ and $\dim Y = n$ and

$$\triangle := \deg Y \leqslant d^n \cdot \deg V. \tag{3.2}$$

This follows, in the same way as [15], Theorem 8, page 65, from the fact that $\Phi : V \longrightarrow \mathbb{C}P^{q-1}$ is the composition of the restriction of the d-uple embedding $\rho_d|_V : V \longrightarrow \mathbb{C}P^{L-1}$ to V (with $L = \binom{N+d}{N}$) with the linear projection $p : \mathbb{C}P^{L-1} \longrightarrow \mathbb{C}P^{q-1}$, defined by the linear forms $Q_1, ..., Q_q$ in the monomials of degree d, since we have:

$$\deg Y = \deg \Phi(V) \leqslant \deg \rho_d|_V(V) \leqslant d^n \cdot \deg V.$$

It is clear that for any $1 \leqslant i_0 < \cdots < i_n \leqslant q$ such that $\cap_{i=0}^n D_{j_i} \cap V = \varnothing$, we have

$$\{y = (y_1 : \cdots : y_q) \in \mathbb{C}P^{q-1} : \ y_{i_0} = \cdots = y_{i_n} = 0\} \cap Y = \varnothing. \tag{3.3}$$

Denote by $\{I_1, \ldots, I_{q_m}\}$ the set of all $I_i := (I_{i1}, \ldots, I_{iq}) \in \mathbb{N}_0^q$ with $I_{i1} + \cdots + I_{iq} = m$.

Let F be a holomorphic mapping of $B(R_0)$ into $\mathbb{C}P^{q_m-1}$ with the reduced representation $F = \big(Q_1^{I_{11}}(f) \cdots Q_q^{I_{1q}}(f) : \cdots : Q_1^{I_{q_m1}}(f) \cdots Q_q^{I_{q_mq}}(f)\big)$, (note that $Q_1^m(f), \ldots, Q_q^m(f)$ have no common zero point outside the analytic set $\{f_0 = \cdots = f_N = 0\}$).

Define an isomorphism between vector spaces, $\Psi : \mathbb{C}[z_1, \ldots, z_{q_m}]_1 \longrightarrow \mathbb{C}[y_1, \ldots, y_q]_m$ by $\Psi(z_i) := y^{I_i} \quad (i = 1, \ldots, q_m)$. Consider the vector space $\mathcal{H} := \{H \in \mathbb{C}[z_1, \ldots, z_{q_m}]_1 : H(F) \equiv 0\ \}$. Then F is a linearly nondegenerate mapping of $\mathbb{C}$ into the complex projective space $P := \cap_{H\in\mathcal{H}}\{H = 0\} \subset \mathbb{C}P^{q_m-1}$, and we will from now on, by abuse of notation, consider F to be this linearly nondegenerate map $F : B(R_0) \to P$. By the definition of F, it is clear that $\dim P \leqslant \left(\begin{array}{c} N+md \\ md \end{array} \right)$.

For any linear form $H \in \mathbb{C}[z_1, \ldots, z_{q_m}]_1$, since f is algebraically nondegenerate, we have that $H \in \mathcal{H}$ if only if

$$H(Q_1^{I_{11}}(x) \cdots Q_q^{I_{1q}}(x), \cdots, Q_1^{I_{q_m1}}(x) \cdots Q_q^{I_{q_mq}}(x)) \equiv 0 \ on\ V.$$

This is possible if and only if $\Psi(H)(y) := H(y^{I_1}, \cdots, y^{I_{q_m}}) \equiv 0$ on Y. Therefore, we get that $\Psi(\mathcal{H}) = (I_Y)_m$. On the other hand Ψ is an isomorphism. Hence, we have

$$\begin{aligned} \left(\begin{array}{c} N+md \\ md \end{array} \right) \geq \dim P = \dim \bigcap_{H\in\mathcal{H}} \{H = 0\} &= q_m - 1 - \dim \mathcal{H} \\ &= q_m - 1 - \dim (I_Y)_m = H_Y(m) - 1. \end{aligned} \tag{3.4}$$

Let α be a family of $H_Y(m)$ sets $\alpha_i = (\alpha_{i1}, \ldots, \alpha_{i\mathfrak{m}})$ $(0 \leqslant i \leqslant H_Y(m) - 1)$ such that $W_\alpha(F) \not\equiv 0$ and $|\alpha_0| + \cdots + |\alpha_{H_Y(m)-1}| \leqslant \frac{(H_Y(m)-1)H_Y(m)}{2}$.

We define hyperplanes H_j $(j = 1, \ldots, q_m)$ in the complex projective space P by $H_j := \{(z_1 : \cdots : z_{q_m}) \in \mathbb{C}P^{q_m-1} : \ z_j = 0\} \cap P$, (these intersections are not empty by Bézout's theorem, and they are proper algebraic subsets of P since $V \not\subset D_k,\ 1 \leqslant k \leqslant q$).

Denote by $\mathcal{L}$ the set of all subsets J of $\{1,\dots,q_m\}$ such that $\#J = H_Y(m)$ and the hyperplanes $H_j, j \in J$, are in general position in P. Since Ψ is an isomorphism and $\Psi(\mathcal{H}) = I_Y(m)$, $\mathcal{L}$ is also the set of all subsets J of $\{1,\dots,q_m\}$ such that $\{y^{I_j},\ j \in J\}$ is a basis of $\mathbb{C}[y_1,\dots,y_q]_m \diagup I_Y(m)$.

For each $j \in \{1,\dots,q\}$ and $k \in \{1,\dots,q_m\}$, we put

$$E_{Q_j}(f) = \log \frac{\|f\|^d \cdot \|Q_j\|}{|Q_j(f)|} \geq 0 \ \text{ and } \ E_{H_k}(F) = \log \frac{\|F\| \cdot \|H_k\|}{|H_k(F)|} \geq 0,$$

where $\|Q_j\|$ (respectively $\|H_k\|$) is the maximum of absolute values of the coefficients of Q_j (respectively H_k). They are continuous functions with values in $\mathbb{R}_{\geq 0} \cup \{+\infty\}$ which take the value $+\infty$ only on analytic subsets of codimension 1 in $B(R_0)$.

Let $\mathcal{N}$ be the set of all subsets $J \subset \{1,\dots,q\}$ with $\#J = n+1$. For any $J \in \mathcal{N}$, since the hypersurfaces D_j $(j = 1,\dots,q)$ are in general position in V, the function $\lambda_J(x) := \frac{\max_{j\in J}|Q(x)|}{\|x\|^d}$ is continuous on V and $\lambda_J(x) > 0$ for all $x \in V$. On the other hand, V is compact, so there exist positive constants c_J, c'_J such that $c'_J \geq \lambda_J(f(z)) \geq c_J$ for all $z \in B(R_0)$. This implies that

$$d \cdot \log\|f\| = \max_{j\in J} \log|Q(f)| + O(1), \text{ for all } J \in \mathcal{N}. \tag{3.5}$$

Therefore, there exists a positive constant c such that

$$\min_{\{j_1,\dots,j_{q-n-1}\}} \sum_{i=1}^{q-n-1} E_{Q_{j_i}}(f) \leqslant c.$$

Then, we have

$$\sum_{j=1}^{q} E_{Q_j}(f) \leqslant \max_{J\in\mathcal{N}} \sum_{j\in J} E_{Q_j}(f) + O(1). \tag{3.6}$$

This implies that

$$\begin{aligned} qd\log\|f\| - \sum_{j=1}^{q} \log|Q_j(f)| &\leqslant \sum_{j=1}^{q} E_{Q_j}(f) + O(1) \\ &\leqslant \max_{J\in\mathcal{N}} \sum_{j\in J} E_{Q_j}(f) + O(1). \end{aligned} \tag{3.7}$$

Applying integration on the both sides of (3.7), and using Jensen's formula, we get

$$dqT_f(r,r_0) - \sum_{j=1}^{q} N_{Q_j(f)}(r,r_0) \leqslant \int_{S(r)} \max_{J\in\mathcal{N}} \sum_{j\in J} E_{Q_j}(f)\sigma + O(1). \tag{3.8}$$

Since $ImF \subset P$ and $\{Q_1^{I_{i1}}(f)\cdots Q_q^{I_{iq}}(f),\ 1 \leqslant i \leqslant q_m\}$ have no common zero point, for every $J \in \mathcal{L}$, the holomorphic functions $\{Q_1^{I_{i1}}(f)\cdots Q_q^{I_{iq}}(f),\ i \in J\}$ also have no common zero point.
Then, for every $J \in \mathcal{L}$, we have

$$\|F\| = \max_{i\in J} |H_i(F)| + O(1) = \max_{i\in J} |Q_1^{I_{i1}}(f)\cdots Q_q^{I_{iq}}(f)| + O(1) \leqslant \|f\|^{dm} + O(1).$$

This implies that

$$T_F(r) \leqslant dm \cdot T_f(r) + O(1). \tag{3.9}$$

For every $J \in \mathcal{L}$ and $i \in J$, we have

$$\begin{aligned} E_{H_i}(F) &= \log \frac{\|F\|\cdot\|H_i\|}{\mid H_i(F)\mid} = \log \frac{\|F\|}{|Q_1^{I_{i1}}(f)\cdots Q_q^{Iiq}(f)|} + O(1) \\ &= \log \frac{\|f\|^{dm}}{|Q_1^{I_{i1}}(f)\cdots Q_q^{Iiq}(f)|} - dm\log\|f\| + \log\|F\| + O(1) \\ &= \sum_{1\leqslant j\leqslant q} I_{ij}E_{Q_j}(f) - dm\log\|f\| + \log\|F\| + O(1). \end{aligned} \tag{3.10}$$

Let $c_z := (E_{Q_1}(f(z)), \cdots, E_{Q_q}(f(z)))$ for every $z \in B(R_0)\setminus D$, where D denotes the thin analytic subset where one of these functions takes the value $+\infty$. By the definition of the Hilbert weight, there exists a subset $J_z \in \mathcal{L}$ such that

$$S_Y(m,c_z) = \sum_{i\in J_z} I_i \cdot c_z. \tag{3.11}$$

On the other hand by (1.2) and (3.2) we have $m > \triangle$. Hence, by (3.3) and Lemma 2.2, for every $J \in \mathcal{N}$, we have

$$\frac{S_Y(m,c_z)}{mH_Y(m)} \geq \frac{1}{n+1}\sum_{j\in J} E_{Q_j}(f(z)) - \frac{(2n+1)\triangle}{m} \max_{1\leqslant j\leqslant q} E_{Q_j}(f(z)). \tag{3.12}$$

Then, by (3.10), (3.11) and (3.12), for every $J \in \mathcal{N},\ z \in B(R_0) \setminus D$, we have

$$\begin{aligned}
\frac{1}{(n+1)} \sum_{j \in J} E_{Q_j}(f(z)) &\leqslant \frac{S_Y(m, c_z)}{mH_Y(m)} + \frac{(2n+1)\triangle}{m} \max_{1 \leqslant j \leqslant q} E_{Q_j}(f(z)) \\
&= \frac{\sum_{i \in J_z} I_i \cdot c_z}{mH_Y(m)} + \frac{(2n+1)\triangle}{m} \max_{1 \leqslant j \leqslant q} E_{Q_j}(f(z)) \\
&= \frac{1}{mH_Y(m)} \sum_{\substack{i \in J_z \\ 1 \leqslant j \leqslant q}} I_{ij} E_{Q_j}(z) + \frac{(2n+1)\triangle}{m} \max_{1 \leqslant j \leqslant q} E_{Q_j}(f(z)) \\
&= \frac{1}{mH_Y(m)} \sum_{i \in J_z} E_{H_i}(F(z)) + d \log \|f(z)\| - \frac{1}{m} \log \|F(z)\| \\
&\qquad + \frac{(2n+1)\triangle}{m} \max_{1 \leqslant j \leqslant q} E_{Q_j}(f(z)) + O(1) \\
&\leqslant \frac{1}{mH_Y(m)} \max_{L \in \mathcal{L}} \sum_{i \in L} E_{H_i}(F(z)) + d \log \|f(z)\| - \frac{1}{m} \log \|F(z)\| \\
&\qquad + \frac{(2n+1)\triangle}{m} \sum_{1 \leqslant j \leqslant q} E_{Q_j}(f(z)) + O(1). \qquad (3.13)
\end{aligned}$$

This implies that, for every $z \in B(R_0) \setminus D$,

$$\begin{aligned}
\max_{J \in \mathcal{N}} \frac{1}{(n+1)} \sum_{j \in J} E_{Q_j}(f(z)) &\leqslant \frac{1}{mH_Y(m)} \max_{L \in \mathcal{L}} \sum_{i \in L} E_{H_i}(F(z)) + d \log \|f(z)\| \\
&\qquad - \frac{1}{m} \log \|F(z)\| + \frac{(2n+1)\triangle}{m} \sum_{1 \leqslant j \leqslant q} E_{Q_j}(f(z)) + O(1),
\end{aligned} \qquad (3.14)$$

and by continuity this then holds for all $z \in B(R_0)$. So, by integrating and by (3.8), we get

$$\begin{aligned}
dqT_f(r, r_0) - \sum_{j=1}^{q} N_{Q_j(f)}(r, r_0) &\leqslant \frac{n+1}{mH_Y(m)} \int_{S(r)} \max_{L \in \mathcal{L}} \sum_{i \in L} E_{H_i}(F) \sigma \\
&\qquad + d(n+1) T_f(r, r_0) - \frac{n+1}{m} T_F(r, r_0) \\
&\qquad + \frac{(2n+1)(n+1)\triangle}{m} \sum_{1 \leqslant j \leqslant q} \int_{S(r)} E_{Q_j}(f) \sigma + O(1).
\end{aligned} \qquad (3.15)$$

We have

$$\int_{S(r)} \max_{L\in\mathcal{L}} \sum_{i\in L} E_{H_i}(F)\sigma = \int_{S(r)} \max_{L\in\mathcal{L}} \log \frac{\|F\|^{H_Y(m)+1}}{|\prod_{i\in L} H_i(F)|}\sigma + O(1)$$

$$\leqslant (H_Y(m)+1)T_F(r,r_0) + \int_{S(r)} \max_{L\in\mathcal{L}} \log \Big|\frac{W_\alpha(F)}{\prod_{i\in L} H_i(F)}\Big|\sigma$$

$$- N_{W_\alpha(F)}(r,r_0) + O(1). \tag{3.16}$$

For an integrable function $h \geq 0$ on $S(r)$, by Lemma 3.5 in [1] we have

$$\int_{Sr} \log^+ h\sigma \leqslant \log^+ \int_{S(r)} h\sigma + \log 2.$$

Therefore, for any t, p with $0 < t(|\alpha_0| + \cdots + |\alpha_N|) < p < 1$, by Lemma 2.3 we have

$$t\int_{S(r)} \max_{L\in\mathcal{L}} \log \Big|\frac{W_\alpha(F)}{\prod_{i\in L} H_i(F)}\Big|\sigma + t\int_{S(r)} \log |z^{\alpha_0+\cdots+\alpha_N}|\sigma$$

$$= \int_{S(r)} \max_{L\in\mathcal{L}} \log \Big|\frac{z^{\alpha_0+\cdots+\alpha_N} W_\alpha(F)}{\prod_{i\in L} H_i(F)}\Big|^t\sigma$$

$$\leqslant \int_{S(r)} \sum_{L\in\mathcal{L}} \log^+ \Big|\frac{z^{\alpha_0+\cdots+\alpha_N} W_\alpha(F)}{\prod_{i\in L} H_i(F)}\Big|^t\sigma$$

$$\leqslant \sum_{L\in\mathcal{L}} \log^+ \int_{S(r)} \Big|\frac{z^{\alpha_0+\cdots+\alpha_N} W_\alpha(F)}{\prod_{i\in L} H_i(F)}\Big|^t\sigma + K_1$$

$$\leqslant \sum_{L\in\mathcal{L}} \log^+ K_L\Big(\frac{R^{2\mathfrak{m}-1}}{R-r} \log T_F(R,r_0)\Big)^p + K_1$$

$$\leqslant K\Big(\log^+ \frac{R^{2\mathfrak{m}-1}}{R-r} + \log^+ T_f(R,r_0)\Big) \tag{3.17}$$

where K_L, K, K_1 are positive constants.
On the other hand, by Lemma 2.4 in [10], we have

$$T_f\Big(r + \frac{R_0 - r}{eT_f(r,r_0)}, r_0\Big) \leqslant 2T_f(r,r_0)$$

outside a set E of r such that $\int_E \frac{1}{R_0-r}dr < \infty$ in the case $R_0 < \infty$ and

$$T_f\big(r + \frac{1}{T_f(r,r_0)}, r_0\big) < 2T_f(r,r_0)$$

outside a set E' of r such that $\int_{E'} dr < \infty$ in the case $R_0 = \infty$.
Take $R = r + \frac{R_0-r}{eT_f(r,r_0)}$ if $R_0 < \infty$ and $R = r + \frac{1}{T_f(r,r_0)}$ if $R_0 = \infty$, then by (3.17) we get

$$\int_{S(r)} \max_{L\in\mathcal{L}} \log\Big|\frac{W_\alpha(F)}{\prod\limits_{i\in L} H_i(F)}\Big|\sigma \leqslant A(r) \tag{3.18}$$

where

$$A(r) \leqslant K\big(\log^+ \frac{1}{R_0-r} + \log^+ T_f(r,r_0)\big)$$

outside a set E of r such that $\int_E \frac{1}{R_0-r}dr < \infty$ in the case $R_0 < \infty$ and

$$A(r) \leqslant K\big(\log r + \log^+ T_f(r,r_0)\big)$$

outside a set E' of r such that $\int_{E'} dr < \infty$ in the case $R_0 = \infty$.

By (3.15), (3.16), and (3.18) we have

$$\begin{aligned}
dqT_f(r,r_0) &- \sum_{j=1}^{q} N_{Q_j(f)}(r,r_0) \\
&\leqslant \frac{(n+1)(H_Y(m)+1)}{mH_Y(m)}T_F(r,r_0) - \frac{n+1}{mH_Y(m)}N_{W_\alpha}(r,r_0) + A(r) \\
&+ d(n+1)T_f(r,r_0) - \frac{n+1}{m}T_F(r,r_0) + \frac{(2n+1)(n+1)\triangle}{m}\sum_{1\leqslant j\leqslant q}\int_{S(r)} E_{Q_j}(f)\sigma \\
&\leqslant \frac{n+1}{mH_Y(m)}T_F(r,r_0) - \frac{n+1}{mH_Y(m)}N_{W_\alpha}(r,r_0) + A(r) \\
&+ d(n+1)T_f(r,r_0) + \frac{(2n+1)(n+1)\triangle}{m}\sum_{1\leqslant j\leqslant q}\int_{S(r)} E_{Q_j}(f)\sigma + O(1).
\end{aligned} \tag{3.19}$$

For each $j \in \{1, \dots, q\}$, by Jensen's formula, we have

$$\int_{S(r)} E_{Q_j}(f)\sigma \leqslant d\int_{S(r)} \log \|f\|\sigma - \int_{S(r)} \log |Q_j(f)|\sigma + O(1)$$
$$\leqslant dT_f(r, r_0) - N_{Q_j(f)}(r, r_0) + O(1) \leqslant dT_f(r, r_0) + O(1). \tag{3.20}$$

By (1.2), (3.2), and by Lemma 2.1 we have

$$\frac{(2n+1)(n+1)d\triangle}{m} < \frac{\epsilon}{4} \text{ and } \frac{(n+1)d}{H_Y(m)} < \frac{\epsilon}{4}. \tag{3.21}$$

Therefore, by (3.9), (3.19), and (3.20), we get

$$dqT_f(r, r_0) - \sum_{j=1}^{q} N_{Q_j(f)}(r, r_0) \leqslant \big((n+1)d + q\frac{\epsilon}{2}\big)T_f(r, r_0)$$
$$- \frac{n+1}{mH_Y(m)} N_{W(F)}(r, r_0) + A(r).$$

This implies that

$$d(q - n - 1 - q\frac{\epsilon}{2})T_f(r, r_0) \leqslant \sum_{j=1}^{q} N_{Q_j(f)}(r, r_0)$$
$$- \frac{n+1}{mH_Y(m)} N_{W_\alpha(F)}(r, r_0) + A(r). \tag{3.22}$$

For each $J := \{j_1, \dots, j_{H_Y(m)}\} \in \mathcal{L}$, then there exists a constant $\gamma_J \in \mathbb{C}, \gamma_J \neq 0$ such that

$$W_\alpha(F) = \gamma_J \cdot W_\alpha(Q_1^{I_{j_1 1}}(f) \cdots Q_q^{I_{j_1 q}}(f), \dots, Q_1^{I_{j_{H_Y(m)} 1}}(f) \cdots Q_q^{I_{j_{H_Y(m)} q}}(f)). \tag{3.23}$$

On the other hand, by (3.4) and Lemma 2.4,

$$\nu_{\frac{Q_1^{I_{j_1 1}}(f) \cdots Q_q^{I_{j_1 q}}(f) \cdots Q_1^{I_{j_{H_Y(m)} 1}}(f) \cdots Q_q^{I_{j_{H_Y(m)} q}}(f)}{W_\alpha\left(Q_1^{I_{j_1 1}}(f) \cdots Q_q^{I_{j_1 q}}(f), \dots, Q_1^{I_{j_{H_Y(m)} 1}}(f) \cdots Q_q^{I_{j_{H_Y(m)} q}}(f)\right)}} \leqslant \sum_{1 \leqslant i \leqslant H_Y(m)} \nu^{[H_Y(m)-1]}_{Q_1^{I_{j_i 1}}(f) \cdots Q_q^{I_{j_i q}}(f)}. \tag{3.24}$$

Hence, for all $J \in \mathcal{L}$, we have

$$\begin{aligned}
\nu_{W_\alpha(F)} &\geq \nu_{Q_1^{I_{j_1 1}}(f) \cdots Q_q^{I_{j_1 q}}(f) \cdots Q_1^{I_{j_{H_Y(m)}1}}(f) \cdots Q_q^{I_{j_{H_Y(m)}q}}(f)} - \sum_{1 \leqslant i \leqslant H_Y(m)} \nu^{[H_Y(m)-1]}_{Q_1^{I_{j_i 1}}(f) \cdots Q_q^{I_{j_i q}}(f)} \\
&\geq \sum_{1 \leqslant j \leqslant q} \sum_{i \in J} I_{ij} \big(\nu_{Q_j(f)} - \nu^{[H_Y(m)-1]}_{Q_j(f)} \big). \qquad (3.25)
\end{aligned}$$

For every $z \in B(R_0)$, let $c_z := (c_{1,z}, \ldots, c_{q,z})$ where $c_{j,z} := \nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z)$. Then, by definition of the Hilbert weight, there exists $J_z \in \mathcal{L}$ such that

$$S_Y(m, c_z) = \sum_{i \in J_z} I_i \cdot c_z = \sum_{1 \leqslant j \leqslant q} \sum_{i \in J_z} I_{ij} \big(\nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z) \big).$$

Then, by (3.3) and Lemma 2.2, for every $K \in \mathcal{N}$ we have

$$\begin{aligned}
\frac{1}{m H_Y(m)} &\sum_{1 \leqslant j \leqslant q} \sum_{i \in J_z} I_{ij} \big(\nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z) \big) \\
&\geq \frac{1}{n+1} \sum_{j \in K} \big(\nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z) \big) \\
&\quad - \frac{(2n+1)\triangle}{m} \max_{1 \leqslant j \leqslant q} \big(\nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z) \big) \\
&\geq \frac{1}{n+1} \sum_{j \in K} \big(\nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z) \big) \\
&\quad - \frac{(2n+1)\triangle}{m} \sum_{1 \leqslant j \leqslant q} \big(\nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z) \big).
\end{aligned}$$

Combining with (3.25), for every $K \in \mathcal{N}$ and $z \in B(R_0)$, we have

$$\begin{aligned}
\frac{1}{m H_Y(m)} \nu_{W_\alpha(F)(z)} \geq \frac{1}{n+1} &\sum_{j \in K} \big(\nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z) \big) \\
&- \frac{(2n+1)\triangle}{m} \sum_{1 \leqslant j \leqslant q} \big(\nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z) \big).
\end{aligned}$$

This implies that

$$\frac{n+1}{mH_Y(m)}\nu_{W_\alpha(F)} \ge \max_{K\in\mathcal{N}}\sum_{j\in K}\big(\nu_{Q_j(f)} - \nu^{[H_Y(m)-1]}_{Q_j(f)}\big)$$
$$- \frac{(n+1)(2n+1)\triangle}{m}\sum_{1\leqslant j\leqslant q}\big(\nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z)\big). \tag{3.26}$$

On the other hand, since the hypersurfaces D_j $(j = 1,\dots,q)$ are in general position in V, we have that for any $z \in B(R_0)$ there are at least (q-n) indices j of $\{1,\dots,q\}$ such that $\nu_{Q_j(f)}(z) = 0$. Thus, we have

$$\sum_{j=1}^{q}\big(\nu_{Q_j(f)} - \nu^{[H_Y(m)-1]}_{Q_j(f)}\big) = \max_{J\in\mathcal{N}}\sum_{j\in J}\big(\nu_{Q_j(f)} - \nu^{[H_Y(m)-1]}_{Q_j(f)}\big).$$

Therefore, by (3.26) we have

$$\frac{n+1}{mH_Y(m)}\nu_{W_\alpha(F)} \ge \sum_{j=1}^{q}\big(\nu_{Q_j(f)} - \nu^{[H_Y(m)-1]}_{Q_j(f)}\big)$$
$$- \frac{(n+1)(2n+1)\triangle}{m}\sum_{1\leqslant j\leqslant q}\big(\nu_{Q_j(f)}(z) - \nu^{[H_Y(m)-1]}_{Q_j(f)}(z)\big). \tag{3.27}$$

So, by integrating and by Jensen's formula, we get

$$\begin{aligned}
\frac{n+1}{mH_Y(m)}N_{W_\alpha(F)}(r) &\ge \sum_{j=1}^{q}\big(N_{Q_j(f)}(r,r_0) - N^{[H_Y(m)-1]}_{Q_j(f)}(r,r_0)\big)\\
&\quad - \frac{(n+1)(2n+1)\triangle}{m}\sum_{1\leqslant j\leqslant q}N_{Q_j(f)}(r,r_0)\\
&\ge \sum_{j=1}^{q}\big(N_{Q_j(f)}(r,r_0) - N^{[H_Y(m)-1]}_{Q_j(f)}(r,r_0)\big)\\
&\quad - \frac{(n+1)(2n+1)dq\triangle}{m}\sum_{1\leqslant j\leqslant q}T_f(r) - O(1)\\
&\overset{(3.21)}{\ge} \sum_{j=1}^{q}\big(N_{Q_j(f)}(r,r_0) - N^{[H_Y(m)-1]}_{Q_j(f)}(r,r_0)\big) - \frac{q\epsilon}{4}T_f(r).
\end{aligned}$$

Combining with (3.22) we get

$$d(q-n-1-q\epsilon)T_f(r,r_0) \leqslant \sum_{j=1}^{q} N^{[H_Y(m)-1]}_{Q_j(f)}(r,r_0) + A(r).$$

Combinning with (3.1) we have

$$(q-n-1-q\epsilon)T_f(r,r_0) \leqslant \sum_{j=1}^{q} \frac{1}{d_j} N^{[H_Y(m)-1]}_{f}(r,r_0,D_j) + A(r).$$

Combining with (3.4), we complete the proof of Theorem 1.1. □

We next prove Theorem 1.3.

Let $\widetilde{\omega}: \widetilde{M} \to M$ be the universal covering of M. Then $\widetilde{f} = f\widetilde{\omega}: \widetilde{M} \to V$ is also algebraically nondegenreate. Moreover, it holds that $\delta^{[\ell]}_f(D_j) \leqslant \delta^{[\ell]}_{\widetilde{f}}(D_j)$. Hence, if Theorem 1.3 is true for $\widetilde{f}$ then it is also true for f. Therefore, we may assume that $M = B(R_0)$ for some R_0 $(0 < R_0 \leqslant \infty)$. According to (1.1) and Corollary 1.2, it suffices to prove Theorem 1.3 for the case $R_0 = 1$ and

$$\limsup_{r\to 1} \frac{T_f(r,r_0)}{\log\frac{1}{1-r}} < \infty.$$

Then, by (3.9) we also have

$$\limsup_{r\to 1} \frac{T_F(r,r_0)}{\log\frac{1}{1-r}} < \infty. \tag{3.28}$$

We now prove that

$$\sum_{j=1}^{q} \delta^{[H_Y(m)-1]}_f(D_j) \leqslant n+1+q\epsilon+\rho T. \tag{3.29}$$

where $T := \frac{(n+1)(H_Y(m)-1)}{d(m-(n+1)(2n+1)\triangle)} \overset{(3.4)}{\leqslant} \frac{(n+1)\binom{N+dm}{dm}}{d(m-(n+1)(2n+1)d^n \deg V)}$.

Indeed, assume that the inequality (3.29) does not hold. Then, by definition of the non-integrated defect, there exist nonnegative constants η_j and continuous plurisubharmonic functions $u_j \not\equiv -\infty$ $(1 \leqslant j \leqslant q)$ such that

$$\sum_{j=1}^{q}(1-\eta_j) > n+1+q\epsilon+\rho T,\ e^{u_j} \leqslant \|f\|^{d_j\eta_j} \tag{3.30}$$

and $u_j - log|\phi_j|$ is plurisubharmonic, where ϕ_j is a nonzero holomorphic function with $\nu_{\phi_j} = \min\{\nu_{(f,D_j)}, H_Y(m)-1\} = \nu^{[H_Y(m)-1]}_{P_j(f)} \geq \frac{d_j}{d}\nu^{[H_Y(m)-1]}_{Q_j(f)}$.
On the other hand, by (3.27) we have

$$\nu_{\frac{\left(\prod_{j=1}^q Q_j(f)\right)^A}{W_\alpha(F)}} \leqslant A \sum_{1 \leqslant j \leqslant q} \nu^{[H_Y(m)-1]}_{Q_j(f)}.$$

where $A := (1 - \frac{(n+1)(2n+1)\triangle}{m})\frac{mH_Y(m)}{n+1}$.
Therefore,

$$v := \log|z^{\alpha_0+\cdots+\alpha_N}\frac{W_\alpha(F)}{\left(\prod_{j=1}^q Q_j(f)\right)^A}| + A\sum_{j=1}^q \frac{d}{d_j}u_j$$

is plurisubharmonic on $B(1)$.

By (cf. [9], pp. 252), the condition (1.3) is satisfied if and only if there exists a continuous plurisubharmonic function $w \not\equiv -\infty$ on $B(1)$ such that

$$e^w dV \leqslant \|f\|^{2\rho}\mathcal{V}_{\mathfrak{m}}$$

where dV denotes the volume form of $B(1)$.

Set

$$t := \frac{2\rho}{d\left((q-n-1-\frac{n+1}{H_Y(m)} - \frac{(n+1)(2n+1)q\triangle}{m})B - (\eta_1+\cdots+\eta_q)A\right)}$$

and $u := w + tv$,

where $B := \frac{mH_Y(m)}{n+1}$. Then u is plurisubharmonic and so subharmonic on the Kähler manifold M.
By (3.21), (3.30) we have

$$\begin{aligned} t(|\alpha_0|+\cdots+|\alpha_N|) &\leqslant \frac{2\rho}{d\left((q-n-1-q\epsilon)A - (\eta_1+\cdots+\eta_q)A\right)} \cdot \frac{(H_Y(m)-1)H_Y(m)}{2} \\ &\leqslant \frac{\rho T}{\sum_{j=1}^q(1-\eta_j) - n - 1 - q\epsilon} < 1 \qquad (3.31) \end{aligned}$$

(note that $0 < A < B$). Then, we have

$$\begin{aligned}
e^u dV &\leqslant e^{tv}\|f\|^{2\rho}\mathcal{V}_{\mathfrak{m}}\\
&= |z^{\alpha_0+\cdots+\alpha_N}\frac{W_\alpha(F)}{\big(\prod_{j=1}^q Q_j(f)\big)^A}|^t e^{t\sum_{j=1}^q A\frac{d}{d_j}u_j}\|f\|^{2\rho}\mathcal{V}_{\mathfrak{m}}\\
&\overset{(3.30)}{\leqslant} |z^{\alpha_0+\cdots+\alpha_N}\frac{W_\alpha(F)}{\big(\prod_{j=1}^q Q_j(f)\big)^A}|^t \|f\|^{t\sum_{j=1}^q dA\eta_j}\|f\|^{2\rho}\mathcal{V}_{\mathfrak{m}}\\
&= |z^{\alpha_0+\cdots+\alpha_N}\frac{W_\alpha(F)}{\big(\prod_{j=1}^q Q_j(f)\big)^A}|^t \|f\|^{td\big(q-n-1-\frac{n+1}{H_Y(m)}-\frac{(n+1)(2n+1)q\triangle}{m})B\big)}\mathcal{V}_{\mathfrak{m}}.
\end{aligned}$$

Therefore, by the help of the identity $\mathcal{V}_{\mathfrak{m}} = 2\mathfrak{m}\|z\|^{2\mathfrak{m}-1}\sigma\wedge d\|z\|$ we get that

$$\begin{aligned}
\int_{B(1)} e^u dV &\leqslant \int_{B(1)} |z^{\alpha_0+\cdots+\alpha_N}\frac{W_\alpha(F)}{\big(\prod_{j=1}^q Q_j(f)\big)^A}|^t \|f\|^{td\big(q-n-1-\frac{n+1}{H_Y(m)}-\frac{(n+1)(2n+1)q\triangle}{m})B\big)}\mathcal{V}_{\mathfrak{m}}\\
&\leqslant 2\mathfrak{m}\int_0^1 \xi^{2\mathfrak{m}-1}\int_{S(\xi)} |z^{\alpha_0+\cdots+\alpha_N}\frac{W_\alpha(F)}{\big(\prod_{j=1}^q Q_j(f)\big)^A}|^t \|f\|^{td\big(q-n-1-\frac{n+1}{H_Y(m)}-\frac{(n+1)(2n+1)q\triangle}{m})B\big)}\sigma d\xi.
\end{aligned} \tag{3.32}$$

By (3.7) and (3.14) we have

$$\begin{aligned}
\log\frac{\|f\|^{dq}}{|\prod_{j=1}^q Q_j(f)|} &\leqslant \frac{n+1}{mH_Y(m)}\max_{L\in\mathcal{L}}\sum_{i\in L}E_{H_i}(F) + d(n+1)\log\|f\|\\
&\quad - \frac{n+1}{m}\log\|F\| + \frac{(2n+1)(n+1)\triangle}{m}\sum_{1\leqslant j\leqslant q}E_{Q_j}(f) + O(1)\\
&= \frac{n+1}{mH_Y(m)}\max_{L\in\mathcal{L}}\log\frac{\|F\|^{H_Y(m)+1}}{|\prod_{i\in L}H_i(F)|} + d(n+1)\log\|f\|\\
&\quad - \frac{n+1}{m}\log\|F\| + \frac{(2n+1)(n+1)\triangle}{m}\log\frac{\|f\|^{dq}}{|\prod_{j=1}^q Q_j(f)|} + O(1)\\
&= \frac{n+1}{mH_Y(m)}\log\max_{L\in\mathcal{L}}\frac{1}{|\prod_{i\in L}H_i(F)|} + d(n+1)\log\|f\|\\
&\quad + \frac{n+1}{mH_Y(m)}\log\|F\| + \frac{(2n+1)(n+1)\triangle}{m}\log\frac{\|f\|^{dq}}{|\prod_{j=1}^q Q_j(f)|} + O(1)
\end{aligned}$$

$$\overset{(3.9)}{\leqslant} \frac{n+1}{mH_Y(m)} \log \sum_{L\in\mathcal{L}} \frac{1}{|\prod_{i\in L}(F,H_i)|} + \big(d(n+1) + \frac{d(n+1)}{H_Y(m)}\big) \log \|f\|$$

$$+ \frac{(2n+1)(n+1)\triangle}{m} \log \frac{\|f\|^{dq}}{|\prod_{j=1}^q Q_j(f)|} + O(1)$$

Therefore,

$$|W_\alpha(F)| \cdot \Big(\frac{\|f\|^{d\left(q-n-1-\frac{n+1}{H_Y(m)}-\frac{q(n+1)(2n+1)\triangle}{m}\right)}}{|\prod_{j=1}^q Q_j(f)|^{1-\frac{(n+1)(2n+1)\triangle}{m}}}\Big)^{\frac{mH_Y(m)}{n+1}} \leqslant \sum_{L\in\mathcal{L}} \frac{|W_\alpha(F)|}{|\prod_{i\in L}(F,H_i)|}.$$

This implies that

$$|W_\alpha(F)| \cdot \frac{\|f\|^{d\left(q-n-1-\frac{n+1}{H_Y(m)}-\frac{q(n+1)(2n+1)\triangle}{m}\right)B}}{|\prod_{j=1}^q Q_j(f)|^A} \leqslant \sum_{L\in\mathcal{L}} \frac{|W_\alpha(F)|}{|\prod_{i\in L} H_i(F)|}. \qquad (3.33)$$

By (3.31), there exists p' such that $t(|\alpha_0| + \cdots + |\alpha_N|) < p' < 1$. Then by (3.32), (3.33) and by Lemma 2.3, we have

$$\begin{aligned}
\int_{B(1)} e^u dV &\leqslant 2\mathfrak{m} \int_0^1 \xi^{2\mathfrak{m}-1} \int_{S(\xi)} \sum_{L\in\mathcal{L}} |z^{\alpha_0+\cdots+\alpha_N}| \frac{|W_\alpha(F)|^t}{|\prod_{i\in L} H_i(F)|^t} \sigma d\xi \\
&\leqslant 2\mathfrak{m} \int_0^1 \xi^{2\mathfrak{m}-1} \sum_{L\in\mathcal{L}} K_L \big(\frac{R^{2\mathfrak{m}-1}}{R-\xi} T_F(R,r_0)\big)^{p'} d\xi \\
&= 2\mathfrak{m} \sum_{L\in\mathcal{L}} K_L \int_0^1 \xi^{2\mathfrak{m}-1} \big(\frac{R^{2\mathfrak{m}-1}}{R-\xi} T_F(R,r_0)\big)^{p'} d\xi \qquad (3.34)
\end{aligned}$$

for $r_0 < \xi < R < 1$.
According to Lemma 2.4 in [10], if we choose $R = \xi + \frac{1-\xi}{eT_F(\xi,r_0)}$, then

$$T_F(R,r_0) \leqslant 2T_F(\xi,r_0)$$

outside a set E with $\int_E \frac{1}{1-\xi} d\xi < \infty$.
Therefore, by (3.28) and (3.34) we have

$$\int_{B(1)} e^u dV \leqslant \sum_{L\in\mathcal{L}} K'_L \int_0^1 \frac{\xi^{2\mathfrak{m}-1}}{(1-\xi)^{p'}} \big(\log \frac{1}{1-\xi}\big)^{p'} d\xi < \infty.$$

On the other hand, by the result of Yau [16] and Karp [11], we have necessarily

$$\int_{B(1)} e^u dV = \infty$$

because $B(1)$ has infinite volume with respect to the given complete Kähler metric (cf. [16], Theorem B). This is a contradiction. Therefore, we get (3.29). This completes the proof of Theorem 1.3. □

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Tran Van Tan
Department of Mathematics
Hanoi National University of Education
136-Xuan Thuy street, Cau Giay, Hanoi, Vietnam
e-mail: tranvantanhn@yahoo.com

Vu Van Truong
Department of Mathematics
Hoa Lu University
Ninh Nhat, Ninh Binh city, Vietnam
e-mail: vvtruong.dnb@moet.edu.vn