Snapshots of modern mathematics
from Oberwolfach

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Towards a Mathematical Theory of Turbulence in Fluids

Jacob Bedrossian

Fluid mechanics is the theory of how liquids and gases move around. For the most part, the basic physics are well understood and the mathematical models look relatively simple. Despite this, fluids display a dazzling mystery to their motion. The random-looking, chaotic behavior of fluids is known as *turbulence*, and it lies far beyond our mathematical understanding, despite a century of intense research.

1 What would it mean to understand turbulence?

> I am an old man now, and when I die and go to heaven there are two matters on which I hope for enlightenment. One is quantum electrodynamics, and the other is the turbulent motion of fluids. And about the former I am rather optimistic.
>
> —*Sir Horace Lamb, 1932*

The startling complexity of fluid motion has intrigued scientists, mathematicians, and artists for years; indeed, even Leonardo da Vinci documented his observations of turbulent flows.[1] Informally speaking, in fluid mechanics turbulence refers to a chaotic, disordered behavior in the fluid which can only be predicted statistically or in an average sense, rather than exactly. Examples include the smoke pillowing out of a chimney, the wake of a vehicle or tennis ball, or the flow of water downstream of rocks, pylons, or propellers. Richard Feynman

[1] In fact, Leonardo da Vinci coined the term *turbulence*; see, for example, [6].

wrote about the notoriously difficult problem of understanding hydrodynamic turbulence, “What we really cannot do is deal with actual, wet water running through a pipe. That is the central problem which we ought to solve some day, and we have not” [5, Volume 1, Chapter 3-7].

There have been many great strides and influential advances over the last century in the form of semi-empirical theories of turbulence, most famously the work of Kolmogorov.[2] *Semi-empirical* means that instead of starting from the basic physics and working forward, the theories start by making assumptions about the behavior based on observations and work backwards from there. For example, in experiments we observe that friction dissipates a surprisingly large amount of energy in turbulence, even when friction is ostensibly very weak (see Section 4 below for a little more discussion). We do not totally understand this process, nor do we know how to use the governing equations of fluid mechanics to describe this. However, one can begin a (non-rigorous) theory by simply assuming that this “anomalous friction” occurs, and then draw further logical deductions from there. While being stuck with unsatisfying theories is a natural and important part of scientific progress, in order to really understand turbulence, we need to know why our assumptions should be true and whether or not our mathematical models really do accurately describe the behavior we observe.

Semi-empirical methods can be brilliantly insightful and practically useful, but we are ultimately after a much deeper understanding: a “predictive” and “logically self-consistent” theory of turbulence. Ideally: a *mathematically rigorous* theory which starts only from the basic physics, and from there correctly predicts the experimental observations. While this is a very tall order which might never be totally achieved, it is one of the primary goals of the mathematical fluid-mechanics community.

Each section below discusses one piece of the larger puzzle and what we, as a community, are doing.

Section 2 regards the basic question of whether or not the fundamental equations even make sense mathematically in the situations we are interested in. Section 3 discusses self-organization and the transition between orderly fluid flow and chaos. Section 4 regards understanding how kinetic energy is transferred from large to small length-scales and vice versa.

The physics community is attacking turbulence from more pragmatic angles, continually refining the successful semi-empirical methods. The dream is that our communities will eventually meet in the middle somewhere.

[2] See, for example, [12]. An excellent, more rigorous introduction to hydrodynamic turbulence can be found in [6].

2 Do the equations make sense mathematically?

In fluid mechanics, both gases and liquids are referred to as “fluids”, since provided the gas is not too thin, the two obey very similar mathematical laws. Compressible fluids – such as air – practically do not compress or expand much as long as the motion is significantly slower than the speed of sound. Hence, for slower motions, we can make the approximation that the fluid does not compress or expand at all (called *incompressibility*). This seems reasonable for motions such as a car driving at 100 km/h. For simple fluids like water and air, the equations modeling this case are the *incompressible Navier–Stokes equations*:

$$\varrho\left(\partial_t u_i + \sum_j u_j \partial_{x_j} u_i\right) = -\partial_{x_i} p + \frac{1}{\mathbf{Re}} \sum_j \partial_{x_j x_j} u_i$$

$$\sum_j \partial_{x_j} u_j = 0,$$

for $1 \le i \le 3$. Don’t worry if you don’t understand all the notation above, you don’t need to understand exactly the mathematical meaning of the above to follow the rest of the snapshot. There are actually four equations since the first equation holds for each of $i = 1, 2, 3$. In the equations,

u_i denotes the i-th component of the velocity vector u of the fluid (one component for each of the three dimensions of space),

p is the pressure, which is a real-valued function of space and time,

ϱ is the density. For simplicity, we are going to assume that ϱ is a constant in space and time,

Re is the *Reynolds number*, a parameter that describes how dominant the friction in the fluid is – the smaller **Re**, the more like molasses the fluid is. Turbulence only happens when **Re** is large.[3]

These equations might look a bit complicated,[4] but actually, as far as physical models go, they aren’t so bad. In fact, the equations themselves are basically a fancy version of Newton’s law $ma = F$, mass times acceleration equals force. The $-\partial_{x_i} p$ is the force which stops the fluid from compressing

[3] The Navier–Stokes equations are named after Claude Louis Marie Henri Navier (1785–1836) and George Gabriel Stokes (1819–1903). They were first published in 1822 [8]. The Reynolds number is named after Osborne Reynolds (1842–1912), due to his influential experiments [9].

[4] If you have not taken multi-variable calculus (or you are a bit rusty), $\partial_{x_j} u_i$ is the partial derivative of the function u_i as you move in the direction of x_j.

or expanding and the $\sum_j \partial_{x_j x_j} u_i$ is the force of internal friction. The $\varrho\, \partial_t u_i$ is the mass times acceleration as a function of t and x, but the origin of the all-important *nonlinear* term $\varrho \sum_j u_j \partial_{x_j} u_i$ is less clear. This term arises since the parcels of fluid are themselves being transported along with the flow.

The first question a mathematician could ask is:

> "Do the Navier–Stokes equations make sense mathematically?"

That is, if the initial state of the fluid makes sense physically, do the equations uniquely predict a meaningful answer for future times? This question turns out to be pretty close to the more physical-sounding:[5]

> "If we can neglect the compression and expansion of the fluid at the initial time, do the equations predict that we can neglect it at future times?"

For example, if I drive a car through the air at 100 km/h, it sounds like I could ignore compression of the air surrounding the car. However, maybe in the turbulence in the wake of my car, the air accelerates itself so much that it starts to compress or expand. This sounds silly at first: why would the air ever go faster than 100 km/h? In fact, it does. In order to get out of the way of the car, rather than just pile up on the window, some of the air needs to accelerate up and over the car (see Figure 1 for a little cartoon of this). Hence, a bit of air has to go a little faster than 100 km/h. Then this bit of air slams into other air, but now at, say, 120 km/h. Then, just as near the windshield, maybe in order to avoid compressing, a littler bit of air needs to accelerate to 140 km/h... and so forth.

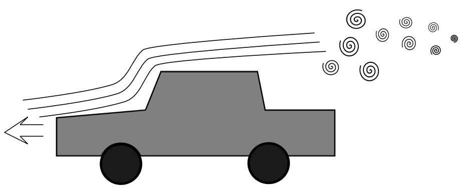

Figure 1: A cartoon depicting air flow over a car in motion with a turbulent wake. The speed of the air in front of the windshield generally exceeds the speed of the car. In the wake, energy can be transferred to smaller and smaller eddies wherein small bits of air may be accelerated faster and faster.

[5] They are not quite the same: the physical question is more stringent than the mathematical question, but for now, it's okay to think of them as almost equivalent.

What mathematicians worry about in such a situation are *singularities*: the equations predicting that a tiny parcel of fluid will move infinitely fast. Physically, this is obviously impossible, so it *really* means that the equations at some point stopped being a good description of the fluid, and that near singularities we need to find a better mathematical description of the physical world. To clear up a common misconception: the Navier–Stokes equations seem to be a "decent physical model" in our experience. However, many good physical models develop singularities in physically relevant situations. Hence, simply being a mathematical model from physics tells you essentially *nothing* about whether or not you can expect singularities. In Section 2.1 below, I give an example of a simple differential equation which displays a singularity.

In the grand goal of "understanding turbulence", figuring out if the Navier–Stokes equations have singularities is basically a little stepping-stone along the way.[6]

The mathematical community has developed over the last century an impressive array of powerful and sophisticated tools for finding or ruling out singularities in nonlinear equations. In two-dimensional fluids,[7] singularities were ruled out in the 1930s [7]. However, despite our best efforts, in three dimensions the question is still open, and moreover is considered so absurdly hard that it is posed as a Clay Millennium Problem with a *1 million dollar cash prize* [4]! If such a notoriously hard problem is just a "little stepping-stone", you can imagine just how far we have to go still.[8]

The mathematics community is more or less stumped on the problem of singularities – we do not even have a consensus about what the answer should be.[9] What we have done instead is try to classify what kinds of singularities might be possible. For example, we now know that any singularity causes a massive pressure drop – in water, little bubbles of steam would appear. We also know that this would be accompanied by the formation of a very intense, tiny, tornado-like structure, and we know some specific things about the possible geometries of such a tornado. We can also say that in a general turbulent flow,

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[6] It has generally been believed that one must answer this question before one can understand turbulence; however, strictly speaking, this might not be the case. For example, one could potentially build a theory which only rules out singularities in a statistical sense. For brevity and clarity, I have not discussed the very important statistical aspects of turbulence, as this can be a little too subtle for this setting.

[7] Studying two-dimensional fluids sounds a bit theoretical and useless at first, but actually various two-dimensional fluid models are building blocks for models in atmospheric sciences and plasma physics.

[8] Indeed, we ruled out singularities in two dimensions almost a century ago now and we are still very far from a complete theory of fluid turbulence in two dimensions. For this snapshot, I will be mostly (implicitly) concerned with three-dimensional turbulence.

[9] If you are curious, I believe that singularities are possible.

there are not "very many" singularities, although they could still have a large influence. We have also been making steady progress towards understanding singularity formation, or ruling it out, in easier contexts; for example, in certain models from atmospheric sciences.

2.1 An example of a singularity in a differential equation

For those of you with a good calculus background, let us now see a simple example of a nonlinear, quadratic differential equation which develops a singularity (notice that the Navier–Stokes equations are also quadratic nonlinear):

$$\begin{aligned} f'(t) &= (f(t))^2 \\ f(0) &= 1. \end{aligned}$$

This can be considered a toy model of an autocatalytic chemical reaction (with an infinite fuel approximation).[10] We can solve this by the following neat trick. First divide both sides of the equation and then use the chain rule

$$\begin{aligned} \frac{1}{(f(t))^2} f'(t) &= 1 \\ -\frac{d}{dt}\left(\frac{1}{f(t)}\right) &= 1. \end{aligned}$$

Now, we can integrate both sides from 0 to T and use the fundamental theorem of calculus:

$$\begin{aligned} -\int_0^T \frac{d}{dt}\left(\frac{1}{f(t)}\right) dt &= \int_0^T 1 \; dt \\ \frac{1}{f(0)} - \frac{1}{f(T)} &= T. \end{aligned}$$

Using that $f(0) = 1$, we can solve this as

$$f(T) = \frac{1}{1-T}.$$

This has a vertical asymptote at $T = 1$! Hence the equation predicts that f will reach infinity in *finite time* – this phenomenon is aptly named *finite-time blow-up*.[11] In order to "resolve" this singularity, we would need to bring in more

[10] "Autocatalytic" refers to the fact that the product of a chemical reaction is itself a catalyst of the same reaction. For more information, see for example [11].

[11] Indeed, depending on how large the chemical reaction is, you may not want to be standing nearby at $T = 1$!

physics or chemistry; for example, in the case of a runaway chemical reaction, perhaps we could keep track of the limited fuel with a second differential equation. The question then is: do the Navier–Stokes equations do something similar?

3 Self-organization and the transition from order to chaos

Before we try to attack turbulence, perhaps we should first make sure we understand the simpler dynamics of fluids. We can also try to at least understand the transition from orderly motion to turbulence. In fact, this leads to many questions of practical relevance, so this theory, known as *hydrodynamic stability*, has attracted the attention of engineers, physicists, and mathematicians since the 1880s.

A basic, but very revealing, example of self-organization in fluids can be seen simply by carefully stirring a cup of coffee. Stir the coffee in a circle, and then pull the spoon out. The fluid will first very rapidly settle into a pretty smooth, nice vortex, which then takes a much longer time to relax. What is going on

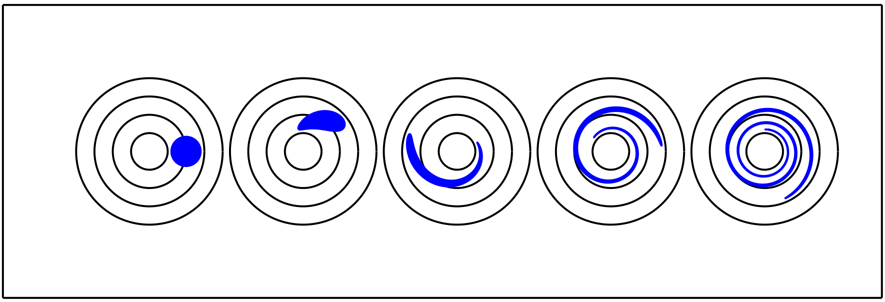

Figure 2: A patch of fluid mixing around a vortex.

here is that the mixing due to the vortex in the cup creates small features in the fluid, see Figure 2. What is "small" is the width of the blue regions, not the length.

One of the most important things to know about fluids is that friction dissipates small features of the flow faster than it dissipates big, boring features. To understand why, here is some intuition. The term modeling friction in the Navier–Stokes equations is $\frac{1}{\mathbf{Re}}\partial_{x_j x_j} u_i$, which has units of Velocity/Length^2. Hence, this term in the equation will be very large near features with relatively small lengths (because being a small feature means that the velocity changes a lot within a small region). In turn, the force due to friction will be very large near small features of the flow, which tends to wipe them out while leaving larger features of the flow relatively intact. In the coffee cup, the small features generated by the mixing are then dissipated by friction quickly, leaving behind the large vortex. There are more subtle things going on, too.

These kinds of self-organization principles are thought to guide the motion of hurricanes, tornadoes, and many other types of phenomena. Moreover, it turns out they are very subtle mathematically and difficult to understand definitively through computer simulation, experimentation, or casual approximations. This makes it an ideal target for the mathematics community, and in fact, this is the current focus of my own research.

Understanding the transition to turbulence has proved to be an especially tricky goal which dates all the way back to famous experiments of Reynolds in 1883, in which he pumped fluid through a pipe and recorded for what values of **Re** the fluid flowed in an orderly manner versus in a turbulent manner. As it happens, self-organization principles similar to the relaxation in the cup of coffee come into play. Experiments and computer simulations have suggested that near the transition to turbulence, the fluid tends to self-organize into relatively simple, coherent structures. However, because these phenomena are notoriously subtle and sensitive to small perturbations (in the sense that small changes can make big effects), experiments and simulations failed to precisely describe the transition to turbulence. In particular, as pointed out by Feynman, we have *still* not provided a complete theoretical explanation for the experiments of Reynolds – even after over 120 years of research! My collaborators and I recently gave a reasonable description of the transition process in the simplest of cases (easier than Reynolds' experiments). The description is not exactly complete, but nonetheless it is a very solid start, so perhaps in a few years we will be a little closer to understanding the experiments. In the next short subsection, we briefly give an overview of this work. It is slightly technical, so feel free to skip it.

3.1 A short discussion of subcritical transition

Recently, in [1, 2, 3], my collaborators and I studied the stability of the equilibrium configuration[12] given by the specific velocity field $u = (y, 0, 0)$ with $(x, y, z) \in \mathbb{T} \times \mathbb{R} \times \mathbb{T}$. Here $\mathbb{T}$ represents a circle and hence denotes periodic boundary conditions.

The linearization of the Navier–Stokes equations about this equilibrium predicts that small perturbations will not de-stabilize the configuration, regardless of how large **Re** is. However, in computer simulations and physical

[12] Equilibrium configuration means that the solution does not depend on time.

experiments, instability is nonetheless observed for **Re** sufficiently large. In fluid mechanics, this phenomenon is usually called *subcritical transition.* In 1887, Kelvin suggested the idea that, while the flow is technically stable for all values of **Re**, it is becoming increasingly sensitive as $\mathbf{Re} \longrightarrow \infty$.

We would like to get a better idea of exactly how sensitive. We consider the velocity field

$$u = (y, 0, 0) + u_p(t),$$

where u_p is a small perturbation. The goal becomes the following: given a norm[13] $\|\cdot\|_X$, find a γ (dependent on the norm) such that

$$\begin{aligned} \|u_p(0)\|_X \ll \mathbf{Re}^{-\gamma} &\quad \text{implies stability,} \\ \|u_p(0)\|_X \gg \mathbf{Re}^{-\gamma} &\quad \text{implies potential instability.} \end{aligned}$$

Moreover, one would like to classify and understand all the possible instabilities above the threshold. One would like to study this kind of question for a great deal of different fluid equillibria; for example those relevant to Reynolds' experiments. The case we studied in [1, 2, 3] appears to be the simplest. The choice of $\|\cdot\|_X$, the way in which we measure "size", seems to be important, both from numerical and analytic considerations (see the discussion and references in [3]). For example, one could measure the (square-root of the) total kinetic energy in the perturbation u_p, which would be

$$\|u_p\|_X = \left(\int |u_p(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2}.$$

Alternatively, one could take the kinetic energy and a measure of how many small scale features there are. For example,

$$\|u_p\|_X = \left(\int |u_p(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2} + \left(\int |\nabla u_p(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2}.$$

One can imagine many different choices of $\|\cdot\|_X$, each of which is physically and mathematically distinct. You can understand why the presence of small-scale features might be important by noting that Figure 2 could just as easily be read right-to-left, rather than left-to-right as originally intended. That is, the background flow can transfer information from large scale to small scale and vice-versa. It turns out that this has important implications for understanding the nonlinear effects over long time-scales, which can interact with this energy

[13] For non-experts reading this, a *norm* is a mathematical notion which measures "size" or "length", and is usually denoted $\|\cdot\|$.

transfer in a somewhat pathological way (see [1] and the references therein for discussions). In [1], we used the following special norm:

$$\|u\|_X = \sum_{n=0}^{\infty} \frac{\lambda^n}{(n!)^p} \left(\int |\nabla^n u(\mathbf{x})|^2 \, d\mathbf{x} \right)^{1/2} \quad \text{for any } \lambda > 0, \quad p \in (1,2).$$

Due to the infinite sum over derivatives, this norm is very sensitive to small scale features of the flow. Hence, in order for a vector field to be small in $\|\cdot\|_X$, nearly all of the energy must be at large scales. We proved in [1] that for this norm, $\gamma = 1$. Moreover, in [2] we proved that a little bit above the threshold, there is only one instability. This is naturally the instability most often observed in experiments (see [2] for references and discussion). In [3], we considered the same problem but for a much less stringent choice of $\|\cdot\|_X$, and showed that $\gamma \leq 3/2$ (an estimate which is also suggested by numerical simulations; see [3] for references and discussions). However, we are still lacking a classification of the instabilities above the threshold (except of course, those studied already in [2]). All of these works depend on understanding the subtle balances between the slightly mysterious nonlinearity in the Navier–Stokes equations and the self-organization principles imposed by the background flow alluded to above.

4 Energy distribution and friction

The idea behind the semi-empirical theories of turbulence in three-dimensional space is that kinetic energy in the fluid is transferred from large length-scales to small length-scales. Ideas about how to start understanding this go back a long way, to Richardson, Taylor, and Kolmogorov in the 1930s and 1940s. In three-dimensional space, large features in the flow tend to break down into smaller pieces and carry kinetic energy down to smaller scale features of the flow (as shown in Figure 1) where the energy is then dissipated by friction. For example, a big vortex breaks into small vortices which break into smaller vortices and so forth (see Figure 3). This is known in fluid mechanics as an *energy cascade*. In fact, this effect is so important, that in experiments, even when the effect of friction should be very weak, the fluid can *still* dissipate a large amount of kinetic energy – completely messing up the energy balance you would naively guess from the equations. This is related to the way the coffee cup above was relaxing in Figure 2, but in real turbulence it is *much* more complicated. Ultimately, thoroughly understanding the energy cascade will lie at the heart of any turbulence theory – we have many predictions from the semi-empirical theories, but not yet a deep understanding. Recently, the mathematical community has made a little progress. For example, in a sequence of recent works, researchers constructed crazy, nonsensical-looking

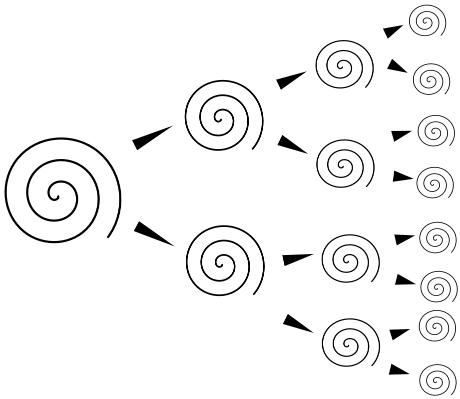

Figure 3: A so-called “energy cascade”.

solutions to some fundamental equations of fluid mechanics which can dissipate energy arbitrarily fast or create energy out of nowhere. Taken literally, of course these solutions are not really physical, however, their existence and their qualitative properties shed light on the process by which energy is dissipated and transported from large to small length-scales (and vice-versa) in real fluids. Moreover, in turbulent flows the fluid might briefly and occasionally resemble these solutions.

After some contemplation, you will see that Sections 2, 3, and 4, are in fact all connected. Consider again the car example in Figure 1. Understanding the process of how the orderly flow over the car breaks down into turbulence in the wake fits neatly into Section 3.[14] The turbulence in the wake is characterized by energy going to smaller and smaller length-scales as in Figure 3 – this leads to a lot of energy dissipation but it could also lead to the singularity we are worried about in Section 2. Moreover, the self-organization principles of Section 3 are also in part based on a simple version of the energy cascade discussed in Section 4. Indeed those works provided the first mathematically rigorous proof of something like an energy cascade in the Navier–Stokes equations. Although we are very far from our ultimate goals, it is encouraging to watch as each branch of mathematical fluid mechanics slowly starts to converge to a common point – and that common point is *turbulence*.

[14] The cartoon also hints at another branch of fluid mechanics we did not discuss: the study of *boundary layers* around objects.

Acknowledgments

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Image credits

All images were created by the author.

Further reading

We recommend the following three snapshots and a TED-Ed Original lesson for some further interesting discussions regarding fluid mechanics:

- Snapshot 10/2014 *Drugs, herbicides, and numerical simulation* by Peter Benner, Hermann Mena, and René Schneider,
- Snapshot 5/2015 *Chaos and chaotic fluid mixing* by Tom Solomon,
- Snapshot 7/2015 *Darcy's law and groundwater flow modelling* by Ben Schweizer,
- TED-Ed Original lesson *The unexpected math behind Van Gogh's "Starry Night"* by Natalya St. Clair [10].

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