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COCYCLE SUPERRIGIDITY AND GROUP ACTIONS ON STABLY FINITE C*-ALGEBRAS

EUSEBIO GARDELLA AND MARTINO LUPINI

Abstract. Let Λ be a countably infinite property (T) group, and let A be UHF-algebra of infinite type. We prove that there exists a continuum of pairwise non (weakly) cocycle conjugate, strongly outer actions of Λ on A. The proof consists in assigning, to any second countable abelian pro-p group G, a strongly outer action of Λ on A whose (weak) cocycle conjugacy class completely remembers the group G. The group G is reconstructed from the action via its (weak) 1-cohomology set endowed with a canonical pairing function. The key ingredient in this computation is Popa's cocycle superrigidity theorem for Bernoulli shifts on the hyperfinite II_1 factor.

Our construction also shows the following stronger statement: the relations of conjugacy, cocycle conjugacy, and weak cocycle conjugacy of strongly outer actions of Λ on A are complete analytic sets, and in particular not Borel. The same conclusions hold more generally when Λ is only assumed to contain an infinite subgroup with relative property (T), and A is a (not necessarily simple) separable, nuclear, UHF-absorbing, self-absorbing C*-algebra with at least one trace.

1. Introduction

Classification of group actions is a fundamental problem in operator algebras, and positive results are both scarce and useful. The subject is far more developed on the von Neumann algebra side, and it was started with Connes' classification of periodic automorphisms on the hyperfinite II_1 factor R; see [8]. Further generalizations to arbitrary automorphisms [7] and finite group actions [22] quickly followed, and these advances culminated in Ocneanu's work on amenable group actions on R [33]. One of his main results is as follows: for every countable amenable group Λ, any two outer actions of Λ on R are cocycle conjugate. A converse to Ocneanu's theorem was proved by Jones: for any nonamenable group Λ, there exist two outer actions of Λ on R that are not cocycle conjugate. Jones' result was considerably strengthened by Brothier and Vaes in [3, Theorem B]—building on [35]—where it is shown that for any nonamenable group Λ, there exists a continuum of pairwise not cocycle conjugate outer actions of Λ on R. In fact, it follows from [3, Theorem B] that the relation of cocycle conjugacy of outer actions of Λ on R is a complete analytic set whenever Λ is not amenable. These results provide a strong dichotomy for the complexity of the classification problem of outer actions of Λ on R up to cocycle conjugacy: the relation is either smooth or not even Borel.

On the side of C*-algebras, far less is known. In this setting, classification results are more difficult to obtain because of K-theoretical restrictions. Another difficulty is the following: while the crossed product of a II_1 factor by an outer action of a discrete group is again a II_1 factor, the crossed product of a simple stably finite C*-algebra by a (strongly) outer action of an infinite amenable group may be purely infinite. For instance, the crossed product of the stabilization of the CAR algebra by the shift automorphism is isomorphic to the stabilization of the Cuntz algebra $\mathcal{O}_2$. There are, however, some positive results, although they are valid mostly for actions of torsion-free abelian groups. (Groups with torsion are particularly difficult to handle, as they generate unexpected phenomena at the level of K-theory.) It was proved by Kishimoto in [27] that any two strongly outer actions of $\mathbb{Z}$ on an infinite type UHF algebra are cocycle conjugate, and this was later generalized by Matui to $\mathbb{Z}^n$-actions in [29]. In the case of the Jiang-Su algebra $\mathcal{Z}$, it was proved by Sato that there exists a unique cocycle conjugacy class of strongly outer actions of $\mathbb{Z}$ on $\mathcal{Z}$ [37]; this was extended to the case of $\mathbb{Z}^2$-actions in [30], and to the case of actions of the Klein bottle group in [31]. Building on these results, Szabó has recently proved the following generalization: for a countable, torsion-free abelian group Λ and for A being

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either a UHF-algebra of infinite type or the Jiang-Su algebra, any two strongly outer actions of Λ on A are cocycle conjugate; see [39].

In the present paper, we initiate the study of actions of nonamenable groups on UHF-algebras. In contrast with the classification results mentioned above, we show that any "sufficiently nonamenable" countable discrete group Λ admits a continuum of pairwise non cocycle conjugate strongly outer actions on any infinite type UHF-algebra. More precisely, such a conclusion holds whenever Λ contains an infinite subgroup with relative property (T), and A is a separable, nuclear, UHF-absorbing unital C*-algebra with a trace satisfying $A \cong A^{\otimes \mathbb{N}}$.

We also consider a strengthening of the notion of strongly outer action—which we call *free action*—based on the notion of free action on a tracial von Neumann algebra considered in [26]. We prove that, even considering such a stronger notion of freeness, under the above assumptions there exists a continuum of pairwise not cocycle conjugate free actions of Λ on A. In fact, we prove that the relations of conjugacy, cocycle conjugacy, and weak cocycle conjugacy of free actions of Λ on A are complete analytic sets. This shows a stark contrast with the classification results mentioned above: when Λ is a torsion-free abelian group and A is a UHF-algebra of infinite type, the relation of cocycle conjugacy of free actions has a single class. Our results can be seen as a C*-algebra counterpart of the results of Jones and Brothier–Vaes in the case of the hyperfinite II_1 factor.

The assertion that the relation of cocycle conjugacy of free actions of Λ on A is a complete analytic set can be interpreted as follows. There does not exist an explicit uniform procedure that, given two free actions of Λ on A, runs for countably many (but possibly transfinitely many) steps, at each step testing membership in some given open sets, and at the end decides whether the given actions are cocycle conjugate or not. In fact, the problem of deciding whether two such actions are cocycle conjugate is as hard as testing membership in any analytic set. Similar conclusions hold for conjugacy and weak cocycle conjugacy. For a more detailed discussion on this interpretation, see [9, Section 2.4].

The proof of our main theorem consists in assigning, to any second countable abelian pro-p group G, a free action of Λ on A whose weak cocycle conjugacy class completely "remembers" the group G. The group G is reconstructed from this action via its (weak) 1-cohomology set, endowed with a canonical (2-sorted) group structure. The computation of this 1-cohomology set will make crucial use of Popa's cocycle superrigidity theorem for weakly mixing malleable actions on the hyperfinite II_1 factor [35, Theorem 4.1]. The starting point of our construction is a canonical model action of G on the UHF-algebra M_{p^∞}, which we construct in Section 3. The rest of the construction can be seen as a C*-algebra analogue of the construction of factors of measure-preserving Bernoulli actions due to Popa [34] and Törnquist [43]; see also [9].

The present paper consists of four sections, in addition to this introduction. In Section 2, we recall some notions about actions of groups on C*-algebras and tracial von Neumann algebras to be used in the rest of the paper. Section 3 contains the construction of the canonical model action of a given second countable abelian pro-p group on the UHF-algebra M_{p^∞}. In Section 4 we describe the construction of free actions associated with abelian pro-p groups, and produce a continuum of such actions which are pairwise not weakly cocycle conjugate. Finally, in Section 5 we discuss the implications of such a construction for the Borel complexity of the relations of conjugacy, cocycle conjugacy, and weak cocycle conjugacy of free actions.

In the following, all topological groups are supposed to be *Hausdorff* and *second countable*. All tensor products of C*-algebras are supposed to be minimal (also called spatial); see [2, Section II.9]. If A is a C*-algebra and S is a finite set, then we let $A^{\otimes S}$ be the (minimal) tensor product of a family of copies of A indexed by S. Similarly, when A is unital and X is a countable set, then we let $A^{\otimes X}$ denote the limit of the direct system $\left(A^{\otimes S}\right)$, where S varies in the collection of finite subsets of X ordered by containment, and the connective maps are the canonical unital *-homomorphisms $\iota_{S,T} \colon A^{\otimes S} \to A^{\otimes T}$ for $S \subset T \subset X$. In the von Neumann-algebraic setting, we will only consider tensor products of tracial von Neumann algebras with respect to distinguished normal tracial states, which we denote by $\overline{\otimes}$; see [2, Section III.3.1].

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2. Preliminary notions on group actions

2.1. **Actions of groups on tracial von Neumann algebras.** We recall some terminology about group actions on von Neumann algebras. A *tracial von Neumann algebra* is a pair (M, τ), where M is a von Neumann algebra and τ is a normal tracial state on τ. We denote by $\mathrm{Aut}(M, \tau)$ the group of τ-preserving automorphisms of M. Let Λ be a discrete group. An *action* of Λ on (M, τ) is a group homomorphism $\alpha \colon \Lambda \to \mathrm{Aut}(M, \tau)$.

An automorphism $\theta \in \mathrm{Aut}(M,\tau)$ is said to be *inner* if there exists a unitary $u \in M$ with $\theta(x) = uxu^*$ for all $x \in M$. It is said to be *outer* if it is not inner, and *properly outer* if for every θ-invariant projection $p \in M$, the restriction of θ to pMp is outer; see [41, Definition XVII.1.1].

Remark 2.1. As it is remarked in [26, Section 4], in the definition of properly outer autorphism one can equivalently only consider θ-invariant *central* projections; see also the comment after Theorem XVII.1.2 in [41]. In particular, an automorphism of a *factor* is properly outer if and only if it is outer.

Let $\theta_0 \in \mathrm{Aut}(M_0,\tau_0)$ and $\theta_1 \in \mathrm{Aut}(M_1,\tau_1)$ be automorphisms of tracial von Neumann algebras. It is shown in [23, Corollary 1.12] that, if either θ_0 or θ_1 is properly outer, then $\theta_0 \otimes \theta_1$ is a properly outer automorphism of $(M_0 \overline{\otimes} M_1, \tau_0 \otimes \tau_1)$.

Definition 2.2. Let (M,τ) be a tracial von Neumann algebra and let Λ be a discrete group. An action $\alpha\colon \Lambda \to \mathrm{Aut}(M,\tau)$ is called:

(1) *ergodic*, if the fixed point algebra $M^\alpha = \{x \in M : \alpha_\gamma(x) = x \text{ for all } \gamma \in \Lambda\}$ contains only the scalar multiples of the identity; see [40, Definition 7.3];
(2) *weakly mixing*, if for any finite subset $F \subseteq M$ and $\varepsilon > 0$, there exists $\gamma \in \Lambda$ such that
$$|\tau(x\alpha_\gamma(y)) - \tau(x)\tau(y)| < \varepsilon$$
for every $x, y \in F$; see [44, Definition D.1];
(3) *mixing*, if for every $a, b \in M$ one has $\tau(a\alpha_\gamma(b)) \to \tau(a)\tau(b)$ for $\gamma \to \infty$; see [44, Definition D.1];
(4) *outer*, if α_γ is not inner for every $\gamma \in \Lambda \setminus \{1\}$;
(5) *free*, if α_γ is properly outer for every $\gamma \in \Lambda \setminus \{1\}$; see [26, Subsection 4.1].

Observe that any free action is, in particular, outer. When M is a factor, the converse holds in view of Remark 2.1.

Remark 2.3. An action α is weakly mixing if and only if the only finite-dimensional vector subspace of $L^2(M,\tau)$ which is invariant under the representation associated with α is the space of scalar multiples of the identity; see [35, Proposition 2.4.2.] and [44, Proposition D.2].

Let α and β be actions of Λ on tracial von Neumann algebras (M_0,τ_0) and (M_1,τ_1), respectively. We let $(M_0 \overline{\otimes} M_1, \tau_0 \otimes \tau_1)$ be the tensor product of M_0 and M_1 with respect to the normal tracial states τ_0, τ_1 [2, Section III.3.1]. Define $\alpha \otimes \beta\colon \Lambda \to \mathrm{Aut}(M_0 \overline{\otimes} M_1, \tau_0 \otimes \tau_1)$ to be the action given by $(\alpha \otimes \beta)_\gamma = \alpha_\gamma \otimes \beta_\gamma$ for $\gamma \in \Lambda$. It is easy to check that $\alpha \otimes \beta$ is (weakly) mixing if both α and β are.

Definition 2.4. Let π a unitary representation of Λ on a Hilbert space H. Following [25], we say that π has *almost invariant vectors*, and write $1_\Lambda \prec \pi$, if for every $\varepsilon > 0$ and finite subset $F \subseteq \Lambda$, there exists a vector $\xi \in H$ such that $\|\pi(\gamma)\xi - \xi\| \leq \varepsilon$ for every $\gamma \in F$. A unitary representation $\pi\colon \Lambda \to U(H)$ is said to be a *c_0-representation* if for every $\xi, \eta \in H$, the function $\gamma \mapsto \langle \pi(\gamma)\xi, \eta\rangle$ belongs to $c_0(\Lambda)$.

Let X be a countable set endowed with an action of Λ. We say that the action is *amenable* if it satisfies the following *Følner condition*: for any finite subset $Q \subseteq \Lambda$ and $\varepsilon > 0$, there exists a finite subset $F \subseteq X$ such that $|\gamma F \triangle F| \leq \varepsilon |F|$ for every $\gamma \in Q$. For an action $\Lambda \curvearrowright X$, we consider the corresponding *left regular representation* $\lambda_X\colon \Lambda \to U(\ell^2(X))$ determined by $\lambda_X(\gamma)(\delta_x) = \delta_{\gamma^{-1}x}$ for $\gamma \in \Lambda$ and $x \in X$. Theorem 1.1 in [25] asserts that $\Lambda \curvearrowright X$ is amenable if and only if $1_\Lambda \prec \lambda_X$.

For a tracial von Neumann algebra (M,τ), we denote by $(M,\tau)^{\overline{\otimes} X}$ the tensor product of copies of M indexed by X with respect to the normal tracial state τ; see [2, III.3.1]. Then $(M,\tau)^{\overline{\otimes} X}$ carries a canonical trace obtained from τ, which we still denote by τ. We denote by $M^{\odot X}$ the algebraic tensor product, which is dense in $(M,\tau)^{\overline{\otimes} X}$. If Y is a subset of X, then we canonically identify $M^{\overline{\otimes} Y}$ with a subalgebra of $(M,\tau)^{\overline{\otimes} X}$, and $M^{\odot Y}$ with a subalgebra of $M^{\odot X}$.

Notation 2.5. *Let X be a countable set endowed with an action $\Lambda \curvearrowright X$, and let (M,τ) be a tracial von Neumann algebra. We denote by $\beta_{\Lambda \curvearrowright X, M}\colon \Lambda \to \mathrm{Aut}((M,\tau)^{\overline{\otimes} X})$ the associated Bernoulli $(\Lambda \curvearrowright X)$-action with base (M,τ), defined by permuting the indices according to the action of Λ on X.*

Example 2.6. *In the context above, when $M = L^\infty(Z,\mu)$ for a probability space (Z,μ) and $\tau(f) = \int f d\mu$, one has $(M,\tau)^{\overline{\otimes} X} = L^\infty(Z^X, \mu^X)$ with trace $\tau(f) = \int f d\mu^X$. The action on $(M,\tau)^{\overline{\otimes} X}$ corresponds in this case to the Bernoulli action of Λ on (Z^X, μ^X) as considered in* [25].

We denote by κ the corresponding Koopman representation of Λ on $L^2(M^{\overline{\otimes} X},\tau)$, and by κ_0 the restriction of κ to the orthogonal complement in $L^2(M^{\overline{\otimes} X},\tau)$ of the space of scalar multiples of the identity.

The following characterization of weakly mixing and mixing Bernoulli actions is well know; see [35, Lemma 2.4.3] and [25, Proposition 2.1 and Proposition 2.3] for the commutative case. We recall here a proof for convenience of the reader.

Proposition 2.7. *Let X be a countable set endowed with an action of Λ, and let (M,τ) be a tracial von Neumann algebra with a projection $p\in M$ such that $0<\tau(p)<1$.*

(1) The following statements are equivalent:
 (1.a) the Bernoulli action $\beta_{\Lambda\curvearrowright X,M}\colon \Lambda\to \mathrm{Aut}((M,\tau)^{\overline{\otimes} X})$ is ergodic;
 (1.b) the Bernoulli action $\beta_{\Lambda\curvearrowright X,M}\colon \Lambda\to \mathrm{Aut}((M,\tau)^{\overline{\otimes} X})$ is weakly mixing;
 (1.c) all orbits of $\Lambda\curvearrowright X$ are infinite.

(2) The following statements are equivalent:
 (2.a) the Bernoulli action $\beta_{\Lambda\curvearrowright X,M}\colon \Lambda\to \mathrm{Aut}((M,\tau)^{\overline{\otimes} X})$ is mixing;
 (2.b) κ_0 is a c_0-representation;
 (2.c) λ_X is a c_0-representation;
 (2.d) the stabilizers of the action $\Lambda\curvearrowright X$ are finite.

Proof. We begin with (1). Since any weakly mixing action is ergodic, it is enough to prove (1.a)⇒(1.c)⇒(1.b).

(1.a)⇒(1.c) Let F be a finite orbit of $\Lambda\curvearrowright X$. Then $\bigotimes_{x\in F} p\in M^{\overline{\otimes} X}$ is a nontrivial invariant projection.

(1.c)⇒(1.b) Let $S\subseteq M^{\otimes X}$ be a finite subset. Upon perturbing S, we may assume that there exists a finite subset $F\subseteq X$ such that $S\subseteq M^{\odot F}$. By [25, Lemma 2.2] there exists $\gamma\in\Lambda$ such that $\gamma F\cap F=\varnothing$. Therefore $\tau(a\beta_\gamma(a))=\tau(a)\tau(a)$ for every $a,b\in S$.

We prove (2). The equivalence of (2.c) and (2.d) is proved in [25, Proposition 2.3]. The equivalence of (2.a) and (2.b) is easy to prove, so we only prove (2.a)⇔(2.d).

(2.a)⇒(2.d) Suppose that there exists $x\in X$ whose stabilizer is infinite. We regard p as an element of $(M,\tau)^{\overline{\otimes} X}$ satisfying $\beta_\gamma(p)=p$ for infinitely many $\gamma\in\Lambda$. Thus $\tau(p\beta_\gamma(p))=\tau(p^2)=\tau(p)>\tau(p)^2=\tau(p)\,\tau(\beta_\gamma(p))$ for infinitely many $\gamma\in\Lambda$. This shows that β is not mixing.

(2.d)⇒(2.a) Let $a,b\in M^{\odot X}$. Pick finite subsets $F_1,F_2\subseteq X$ such that $a\in M^{\odot F_1}$ and $b\in M^{\odot F_2}$. Set $S=\{\gamma\in\Lambda\colon \gamma F_1\cap F_2\neq\varnothing\}$. Then S is finite and $\tau(a\beta_\gamma(b))=\tau(a)\tau(b)$ for $\gamma\in\Lambda\backslash S$. Hence β is mixing. ☐

2.2. **Actions of groups on C*-algebras.** Let Λ be a discrete group, and let A be a unital C*-algebra. Write $\mathrm{Aut}(A)$ for the automorphism group of A. An *action* of Λ on A is a group homomorphism $\alpha\colon\Lambda\to\mathrm{Aut}(A)$. In this case, we also say that the pair (A,α) is a Λ-C*-algebra. We denote by A^α the *fixed point algebra* $A^\alpha=\{a\in A\colon \alpha_\gamma(a)=a \text{ for all } \gamma\in\Lambda\}$. We say that elements x,y of A are *equivalent modulo scalars*, and write $x=y\mathrm{mod}\mathbb{C}$, if $x=\lambda y$ for some $\lambda\in\mathbb{C}$. We denote by $U(A)$ the unitary group of A.

Definition 2.8. Let $\alpha\colon\Lambda\to\mathrm{Aut}(A)$ be an action of a discrete group Λ on a unital C*-algebra A, and let $u\colon\Lambda\to U(A)$ be a function.

(1) We say that u is a 1-cocycle for α if $u_\gamma\alpha_\gamma(u_\rho)=u_{\gamma\rho}$ for every $\gamma,\rho\in\Lambda$.
(2) We say that u is a *weak* 1-cocycle if $u_\gamma\alpha_\gamma(u_\rho)=u_{\gamma\rho}$ $\mathrm{mod}\mathbb{C}$ for every $\gamma,\rho\in\Lambda$.

The notion of weak 1-cocycles allows one to define the weak 1-cohomology of actions. We will mostly use it for actions on tracial von Neumann algebras, but the definition can be given in general.

Definition 2.9. Let $\alpha\colon\Lambda\to\mathrm{Aut}(A)$ be an action of a discrete group Λ on a unital C*-algebra A. Following [35], we say that two weak 1-cocycles u and u' for α are *weakly cohomologous* (or cohomologous modulo scalars), if there exists a unitary $v\in U(A)$ such that $u'_\gamma=v^*u_\gamma\alpha_\gamma(v)$ $\mathrm{mod}\mathbb{C}$ for every $\gamma\in\Lambda$. We say that u is a *weak coboundary* if it is weakly cohomologous to the weak 1-cocycle constantly equal to 1.

We denote by $Z^1_w(\alpha)$ the set of weak 1-cocycles for α. The relation of being weakly 1-cohomologous is an equivalence relation on $Z^1_w(\alpha)$, and we let $H^1_w(\alpha)$ be the corresponding quotient set, called the *weak cohomology set*. The class of the weak 1-cocycle u will be denoted by $[u]$.

Let us use the notation as in the definition above. If A is abelian, then the product of two weak 1-cocycles for α is again a weak 1-cocycle for α, and thus $H^1_w(\alpha)$ can be given a canonical group structure. In general, however, one can not define a group operation on $H^1_w(\alpha)$ in a similar fashion. To make up for the lack of multiplication in the 1-cohomology set $H^1_w(\alpha)$, we consider a natural “two-sorted group structure” on $H^1_w(\alpha)$, given by a pairing function $H^1_w(\alpha)\times H^1_w(\alpha)\to H^1_w(\alpha\otimes\alpha)$. Such a pairing function will be used to encode the group operation of a given countable group.

Definition 2.10. Let $\alpha \otimes \alpha$ be the diagonal action of Λ on $A \otimes A$. Then there is a canonical function

$$\widetilde{m}^\alpha \colon Z^1_w(\alpha) \times Z^1_w(\alpha) \to Z^1_w(\alpha \otimes \alpha)$$

given by $\widetilde{m}^\alpha(u, w) = u \otimes w$ for $u, w \in Z^1_w(\alpha)$. Observe that if u is weakly cohomologous to w and u' is weakly cohomologous to w', then $u \otimes w$ is weakly cohomologous to $u' \otimes w'$. Therefore, the map $\widetilde{m}^\alpha$ induces pairing function $m^\alpha \colon H^1_w(\alpha) \times H^1_w(\alpha) \to H^1_w(\alpha \otimes \alpha)$.

Given a weak 1-cocycle u for an action $\alpha \colon \Lambda \to \mathrm{Aut}(A)$, one can define the *cocycle perturbation* $\alpha^u \colon \Lambda \to \mathrm{Aut}(A)$ of α by setting $\alpha^u_\gamma = \mathrm{Ad}(u_\gamma) \circ \alpha_\gamma$ for every $\gamma \in \Lambda$. (The weak cocycle condition implies that α^u is also an action.)

Definition 2.11. Let Λ be a countable discrete group, and let α and β be actions of Λ on unital C*-algebras A and B, respectively.

(1) We say that α and β are *conjugate* if there exists an isomorphism $\psi \colon A \to B$ such that $\psi \circ \alpha_\gamma = \beta_\gamma \circ \psi$ for every $\gamma \in \Lambda$,
(2) We say that α and β are *cocycle conjugate* if β is conjugate to α^u for some 1-cocycle u for α,
(3) We say that α and β are *weakly cocycle conjugate* if β is conjugate to α^u for some weak 1-cocycle u for α.

Remark 2.12. Let $\alpha, \beta \colon \Lambda \to \mathrm{Aut}(A)$ be actions of a discrete group Λ on a C*-algebra A. It is easy to check that if α and β are (weakly) cocycle conjugate, then there is a canonical bijection between the (weak) 1-cohomology sets of α and β.

When the actions α and β are conjugate, we also say that the Λ-C*-algebras (A, α) and (B, β) are *equivariantly isomorphic.* An *equivariant unital embedding* from (A, α) to (B, β) is an injective unital *-homomorphism $\phi \colon A \to B$ satisfying $\phi \circ \alpha_\gamma = \beta_\gamma \circ \phi$ for every $\gamma \in \Lambda$.

Suppose that A is a unital C*-algebra. A linear functional τ on A is said to be a *trace* if $\tau(1) = \|\tau\| = 1$ and $\tau(ab) = \tau(ba)$ for every $a, b \in A$. We let $\mathrm{T}(A)$ be the simplex of traces on A. Suppose that τ is a trace on A, θ is an automorphism of A, and α is an action of Λ on A. We say that τ is θ-invariant if $\tau \circ \theta = \tau$, and that it is α-invariant if it is α_γ-invariant for every $\gamma \in \Lambda$. If τ is α-invariant, then we also say that α is τ-preserving. We let $\mathrm{T}(A)^\alpha \subseteq \mathrm{T}(A)$ be the closed convex subset of α-invariant traces. Observe that, if Λ is amenable, then $\mathrm{T}(A)^\alpha$ is nonempty whenever $\mathrm{T}(A)$ is nonempty.

For a trace τ on A, consider the corresponding *left regular representation* $\pi_\tau \colon A \to B(L^2(A, \tau))$ obtained via the GNS construction. We let $\overline{A}^\tau$ be the closure of $\pi_\tau(A)$ inside $B(L^2(A, \tau))$ with respect to the weak operator topology. We regard $\overline{A}^\tau$ as a tracial von Neumann algebra, endowed with the unique extension of τ to $\overline{A}^\tau$. The unit ball of A is dense in the unit ball of $\overline{A}^\tau$ with respect to the 2-norm $\|a\|_\tau = \tau(a^*a)^{1/2}$ defined by τ. If α is a τ-preserving action of Λ on A, then it induces a canonical action $\overline{\alpha}^\tau \colon \Lambda \to \mathrm{Aut}(\overline{A}^\tau, \tau)$.

Notation 2.13. *As in the case of actions on tracial von Neumann algebras, given a unital C*-algebra A, we denote by $\beta_{\Lambda \curvearrowright X, A} \colon \Lambda \to \mathrm{Aut}(A^{\otimes X})$ the Bernoulli $(\Lambda \curvearrowright X)$-action with base A induced by an action $\Lambda \curvearrowright X$ of a countable discrete group Λ on a countable set X.*

The following lemma is well known, and we will use it without further reference.

Lemma 2.14. Let A be a C*-algebra, let Λ be a countable discrete group, let $\Lambda \curvearrowright X$ be an action of Λ on a countable set X, and let τ_0 be a trace on A. Denote by τ the trace $\tau_0^{\otimes X}$ on $A^{\otimes X}$. Then the action $\overline{\beta}^\tau_{\Lambda \curvearrowright X, A}$ is conjugate to the von Neumann-algebraic Bernoulli action $\beta_{\Lambda \curvearrowright X, \overline{A}^{\tau_0}}$.

Definition 2.15. Let Λ be a discrete group, let A be a unital C*-algebra, and let $\tau \in \mathrm{T}(A)$.

(1) We say that α is *strongly outer* if for every $\sigma \in \mathrm{T}(A)^\alpha$, the extension $\overline{\alpha}^\sigma$ is outer;
(2) We say that α is *free* if for every $\gamma \in \Lambda \setminus \{1\}$ and for every α_γ-invariant trace σ on A, the extension $\overline{\alpha}^\sigma_\gamma$ is properly outer;
(3) If α is τ-preserving, then we say that α is (*weakly*) τ*-mixing* if $\overline{\alpha}^\tau \colon \Lambda \to \mathrm{Aut}(\overline{A}^\tau, \tau)$ is (weakly) mixing in the sense of Definition 2.2.
(4) We say that two τ-preserving actions $\alpha, \beta \colon \Lambda \to \mathrm{Aut}(A)$ are τ*-conjugate* (respectively, *cocycle* τ*-conjugate*, or *weakly cocycle* τ*-conjugate*), if $\overline{\alpha}^\tau$ and $\overline{\beta}^\tau$ are conjugate (respectively, *cocycle conjugate*, or *weakly cocycle conjugate*), in the sense of Definition 2.11.

It is clear that any free action on a C*-algebra is in particular strongly outer. In view of Remark 2.1 and Lemma 2.21 below, the converse holds for any separable nuclear C*-algebra with a unique trace, for instance. In particular, this applies to any separable UHF algebra of infinite type.

The notion of free action from Definition 2.15 generalizes the notion of free action on a locally compact Hausdorff space, as the next proposition shows.

Proposition 2.16. *Let $\Lambda \curvearrowright X$ be a topological action of a discrete group Λ on a locally compact Hausdorff space X, and denote by $\alpha\colon \Lambda \to \mathrm{Aut}(C_0(X))$ the induced action. Then α is free in the sense of Definition 2.15 if and only if $\Lambda \curvearrowright X$ is free.*

Proof. Suppose that α is free, and let $\gamma \in \Lambda \setminus \{1\}$. To reach a contradiction, assume that there exists $x \in X$ such that $\gamma \cdot x = x$. Then the Dirac probability measure concentrated on $\{x\}$ is Borel and γ-invariant. This measure induces, via integration, an α_γ-invariant trace τ_x on $C_0(X)$. Since $\overline{C_0(X)}^{\tau_x}$ is isomorphic to $\mathbb{C}$, the weak extension of α_γ cannot be outer. This contradiction implies that $\Lambda \curvearrowright X$ is free.

Conversely, assume that $\Lambda \curvearrowright X$ is free and let $\gamma \in \Lambda \setminus \{1\}$. Let τ be an α_γ-invariant trace on $C_0(X)$. Then τ is given by integration with respect to a Borel probability measure μ on X which satisfies $\mu(\gamma \cdot U) = \mu(U)$ for every open subset $U \subseteq X$. Moreover, $\overline{C_0(X)}^{\tau}$ is isomorphic to $L^\infty(X, \mu)$. Suppose, to reach a contradiction, that there exists a nonzero, $\overline{\alpha}_\gamma$-invariant projection $p \in L^\infty(X,\mu)$ such that the restriction of $\overline{\alpha}_\gamma$ to $pL^\infty(X,\mu)p$ is inner (and hence trivial). Since p is the characteristic function of some measurable subset of the support $\mathrm{supp}(\mu)$ of μ, it follows that there exists $x \in \mathrm{supp}(\mu) \subseteq X$ such that $\gamma \cdot x = x$. This is a contradiction, and the proof is finished. ☐

Let A be a C*-algebra. Then $\mathrm{Aut}(A)$ is a topological group when endowed with the topology of pointwise convergence. An action of a topological group G on A is said to be *continuous* if it is continuous as a group homomorphism $G \to \mathrm{Aut}(A)$. *In the following, all the actions of topological groups are supposed to be continuous.* The following is a natural example of a continuous action:

Notation 2.17. *For a compact group G, let $C(G)$ be the commutative C*-algebra of continuous complex-valued functions on G. We denote by $\mathtt{Lt}^G\colon G \to \mathrm{Aut}(C(G))$ the canonical action by left translation, given by $\mathtt{Lt}^G_g(f)(h) = f(g^{-1}h)$ for $g, h \in G$ and $f \in C(G)$. When the group G is clear from the context, we write* $\mathtt{Lt}$ *instead of* $\mathtt{Lt}^G$ *to lighten the notation.*

We recall the definition of the Rokhlin property for compact group actions on unital C*-algebras from [18, Definition 3.2]. The formulation given here is taken from [15, Lemma 3.7].

Definition 2.18. Let G be a compact group, let A be a unital C*-algebra, and let $\alpha\colon G \to \mathrm{Aut}(A)$ be an action. We say that α has the *Rokhlin property* if for every $\varepsilon > 0$, for every finite subset $S \subseteq C(G)$, and every finite subset $F \subseteq A$, there exists a unital completely positive linear map $\psi\colon C(G) \to A$ satisfying

- $\|(\psi \circ \mathtt{Lt}_g)(f) - (\alpha_g \circ \psi)(f)\| < \varepsilon$ for all $f \in S$ and all $g \in G$;
- $\|\psi(f)a - a\psi(f)\| < \varepsilon$ for all $f \in S$ and all $a \in F$;
- $\|\psi(f_0 f_1) - \psi(f_0)\psi(f_1)\| < \varepsilon$ for any $f, f_0, f_1 \in S$.

2.3. **Direct and inverse limit constructions.**

Definition 2.19. An *inverse system of topological groups* is a family $(G_i, \pi_{i,j})_{i,j\in I}$, where I is an ordered set, G_i are topological groups, and $\pi_{i,j}\colon G_j \to G_i$, for $i \leq j$, is a surjective continuous group homomorphism. Given such a countable inverse system, we denote by $G = \varprojlim(G_i, \pi_{i,j})$ the inverse limit, together with the canonical continuous surjective group homomorphisms $\pi_{i,\infty}\colon G \to G_i$ for $i \in I$.

Similarly, a *direct system of unital C*-algebras* is a family $(A_i, \iota_{i,j})_{i,j\in I}$, where I is an ordered set, A_i is a unital C*-algebra, and $\iota_{i,j}\colon A_i \to A_j$, for $i \leq j$, is an injective unital *-homomorphism. We denote by $A = \varinjlim(A_i, \iota_{i,j})$ the corresponding direct limit, together with the canonical injective unital *-homomorphisms $\iota_{i,\infty}\colon A_i \to A$ for $i \in I$.

Next, we will see that one can construct actions of inverse limits of groups on direct limits of C*-algebras in a natural way.

Lemma 2.20. Let I be an ordered set, let $(A_i, \iota_{i,j})_{i,j\in I}$ be a direct system of unital C*-algebras with limit A, and let $(G_i, \pi_{i,j})_{i,j\in I}$ be an inverse system of topological groups with limit G. For every $i \in I$, let $\alpha^{(i)}\colon G_i \to \mathrm{Aut}(A_i)$ be an action satisfying
$$\alpha^{(j)}_g \circ \iota_{i,j} = \iota_{i,j} \circ \alpha^{(i)}_{\pi_{i,j}(g)} \tag{1}$$
for every $i, j \in I$ with $i \leq j$ and every $g \in G_j$. Then there exists a unique action $\alpha\colon G \to \mathrm{Aut}(A)$ such that
$$\alpha_g \circ \iota_{i,\infty} = \iota_{i,\infty} \circ \alpha^{(i)}_{\pi_{i,\infty}(g)} \tag{2}$$
for every $i \in I$ and $g \in G$.

Proof. It is clear that Equation (2) defines a unique action of G on A in view of Equation (1). We check that such an action is continuous. For every $i \in I$, we identify A_i with its image under $\iota_{i,\infty}$.

Fix $\varepsilon > 0$, and a finite subset $F \subseteq A$. By perturbing F within ε, we may assume that there exists $i \in I$ such that $F \subseteq A_i$. Since $\alpha^{(i)}$ is continuous, there exists a neighborhood U of the identity of G_i such that $\|\alpha_g^{(i)}(x) - x\| < \varepsilon$ for every $x \in F$ and $g \in U$. Set $V = \pi_{i,\infty}^{-1}(U)$, which is a neighborhood of the identity of G. For every $g \in V$ and $x \in F$, we have

$$\left\|(\alpha_g^{(i)} \circ \iota_{i,\infty})(x) - \iota_{i,\infty}(x)\right\| \leq \left\|\alpha^{(i)}_{\pi_{i,\infty}(g)}(x) - x\right\| \leq \varepsilon.$$

This concludes the proof. □

The definition of an amenable trace on a unital C*-algebra A can be found in [5, Definition 6.2.1]. Observe that the set $\mathrm{T}_{\mathrm{am}}(A)$ of amenable traces on A is a face of the simplex $\mathrm{T}(A)$ of traces on A. Particularly, any extreme point of $\mathrm{T}_{\mathrm{am}}(A)$ is also an extreme point of $\mathrm{T}(A)$. Recall that every trace on a nuclear C*-algebra is amenable [4, Theorem 4.2.1]. The notion of locally reflexive C*-algebra can be found in [4, Definition 4.3.1]. Every exact C*-algebra is locally reflexive [5, Corollary 9.4.1]. The following result is folklore, and we thank Stuart White for suggesting this formulation.

Lemma 2.21. Let A be a separable, locally reflexive *C*-algebra*, and let τ be a nonzero trace on A. Then $\overline{A}^{\tau}$ is isomorphic to the hyperfinite II_1-factor with separable predual R if and only if τ is amenable and extreme, and A is infinite dimensional.

Proof. It is well known that a trace is extreme if and only if $\overline{A}^{\tau}$ is a factor (in which case it will be of type II_1 or I_n for some $n \in \mathbb{N}$). If τ is amenable, then $\overline{A}^{\tau}$ is hyperfinite by [4, Corollary 4.3.4], because A is locally reflexive. Finally, since A is infinite dimensional, $\overline{A}^{\tau}$ must be isomorphic to R by its uniqueness. Conversely, assume that $\overline{A}^{\tau} \cong R$. Since the trace on R is amenable, its restriction to A, which agrees with τ, must also be amenable. Infinite dimensionality of A is clear, so the proof is complete. □

2.4. **Subgroups with relative property (T).** In this subsection, we recall the definition of relative property (T) for a subgroup Δ of a discrete group Λ.

Definition 2.22. Let Λ be a discrete group and let Δ be a subgroup. We say that Δ has *relative property* (T) (of Kazhdan–Margulis), if there exist a finite subset $F \subseteq \Lambda$ and $\varepsilon > 0$ such that whenever $u\colon \Lambda \to U(H)$ is a unitary representation of Λ on a Hilbert space H, and $\xi \in H$ is a unit vector satisfying $\|u_\gamma(\xi) - \xi\| < \varepsilon$ for all $\gamma \in F$, then H has a nonzero vector which is fixed by the restriction of u to Δ.

For $\Lambda = \Delta$, the definition above recovers the notion of property (T) group. More generally, it is clear that if either Λ *or* Δ has property (T), then $\Delta \subseteq \Lambda$ has relative property (T). There also exist inclusions of groups with relative property (T), for which neither the subgroup nor the containing group have property (T). One such example is $\mathbb{Z}^2 \subseteq \mathbb{Z}^2 \rtimes \mathrm{SL}_2(\mathbb{Z})$. (One can also replace $\mathrm{SL}_2(\mathbb{Z})$ with any nonamenable subgroup of it, by a result of Burger.) Subgroups with relative property (T) have been studied, among others, by Margulis [28], Burger [6], and Jolissaint [21]. Also, relative property (T) is one of the main ingredients in Popa's superrigidity results on cocycles for Bernoulli actions from [35], which we will use below.

3. Model action for profinite abelian groups

3.1. **Profinite groups.** Let $\mathcal{C}$ be a class of groups closed under quotients, finite products, and subgroups. A *pro-$\mathcal{C}$* group is a topological group G that can be realized as the inverse limit of groups from $\mathcal{C}$. Particularly, a group G is said to be

- *profinite* if it is pro-$\mathcal{C}$ for the class $\mathcal{C}$ of finite groups;
- *pro-p* if it is a pro-$\mathcal{C}$ for the class $\mathcal{C}$ of finite p-groups.

It is clear that a profinite group is abelian if and only if it is pro-$\mathcal{C}$ for the class $\mathcal{C}$ of finite abelian groups. Similarly, a pro-p group is abelian if and only if it is pro-$\mathcal{C}$ for the class of finite abelian p-groups. Equivalent characterizations of pro-$\mathcal{C}$ groups can be found in [36, Theorem 2.1.3]. In particular, these characterizations show that a topological group is profinite if and only if it is totally disconnected, if and only if the identity of G has a basis of neighborhoods made of open subgroups [36, Theorem 2.1.3]. Recall that, by [36, Lemma 2.1.2], a subgroup of a profinite abelian group is open if and only if it is closed and has finite index.

Denote by $\mathcal{P}$ the set of prime numbers. A *supernatural number* is a function $\boldsymbol{n}\colon \mathcal{P} \to \{0, 1, 2, \ldots, \infty\}$. Recall also that a separable UHF-algebra is a unital C*-algebra that is obtained as the direct limit of a countable direct system of full matrix algebras. By a fundamental result of Glimm, any separable UHF-algebra has the form

$\otimes_{p\in\mathcal{P}} M_p^{\otimes \boldsymbol{n}(p)}$ for some supernatural number $\boldsymbol{n}$. The supernatural number can be obtained intrinsically from the given separable UHF-algebra, and it is a complete invariant for separable UHF-algebras up to *-isomorphism. Next, we associate to each second countable profinite group, a canonical supernatural number.

Definition 3.1. Let G be a profinite group. The *supernatural number associated with* G is defined by

$$\boldsymbol{n}_G(p) = \begin{cases} \infty & \text{if } p \text{ divides the index of an open subgroup of } G, \\ 0 & \text{otherwise.} \end{cases}$$

We let D_G be the UHF-algebra $\bigotimes_{p\in\mathcal{P}} M_p^{\otimes \boldsymbol{n}_G(p)}$ corresponding to $\boldsymbol{n}_G$.

It is clear that G is a pro-p group if and only if $\boldsymbol{n}_G = p^\infty$ or, equivalently, $D_G = M_{p^\infty}$.

3.2. **Model action.** The goal of this subsection is to construct a model action δ^G of G on D_G with the Rokhlin property; see Theorem 3.5. This action will be crucial in the next section, where for certain nonamenable groups, we construct many non weakly cocycle conjugate strongly outer actions on UHF-algebras.

Remark 3.2. Suppose that G is finite. Then $D_G = M_{|G|^\infty}$, and the model action in this case is rather easy to describe. If $\lambda^G\colon G \to U(\ell^2(G))$ denotes the left regular representation, then the model action $\delta^G\colon G\to \mathrm{Aut}(D_G)$ is given by $\delta^G_g = \mathrm{Ad}(\lambda^G_g)^{\otimes\mathbb{N}}$.

The following is folklore; see, for example, [19, Subsection 2.4] (but note that the reference given there only proves the statement about fixed point algebras for $G = \mathbb{Z}_p$). Since we have not been able to find a reference, we include a short proof for the convenience of the reader. (The proof given below uses the classification results of [20], but a direct and elementary, although longer, proof can also be given.)

Lemma 3.3. Let G be a finite group. Then the action $\delta^G\colon G \to \mathrm{Aut}(D_G)$ described in the remark above has the Rokhlin property. When G is abelian, then $D_G \rtimes_{\delta^G} G$ is naturally isomorphic to $D_{\widehat{G}}$, in such a way that the dual action $\widehat{\delta^G}\colon \widehat{G} \to \mathrm{Aut}(D_G \rtimes_{\delta^G} G)$ is conjugate to $\delta^{\widehat{G}}$.

Proof. It is easy to see that δ^G has the Rokhlin property, since there is a unital and equivariant embedding $C(G) \to B(\ell^2(G)) \cong M_{|G|}$ as multiplication operators. The crossed product $D_G \rtimes_{\delta^G} G$ is a UHF-algebra by [14, Corollary 3.11]. Since there are unital inclusions

$$D_G \subseteq D_G \rtimes_{\delta^G} G \subseteq M_{|G|}(D_G) \cong D_G,$$

it follows that D_G and $D_G \rtimes_{\delta^G} G$ have the same corresponding supernatural number, and hence they are isomorphic. In particular, $D_G \rtimes_{\delta^G} G$ is isomorphic to $D_{\widehat{G}}$.

It remains to identify the dual action of δ^G. Observe that δ^G is approximately representable in the sense of [20, Definition 3.6], as one may take the unitaries $u(g)$ appearing in said definition to be $u(g) = \lambda_g^{\otimes n}$ for a large enough $n\in\mathbb{N}$. By [20, part (2) of Lemma 3.8], it follows that the dual action $\widehat{\delta^G}$ has the Rokhlin property as an action of $\widehat{G}$ on $D_G \rtimes_{\delta^G} G \cong D_{\widehat{G}}$. Since for $\chi\in\widehat{G}$, the automorphisms $\delta^{\widehat{G}}_\chi$ and $\widehat{\delta^G_\chi}$ are both approximately inner, it follows from [20, Theorem 3.5] that $\widehat{\delta^G}$ is conjugate to $\delta^{\widehat{G}}$, and the proof is finished. ∎

The model action δ^G for an arbitrary profinite abelian group G will be constructed from its finite quotients using Lemma 3.3. The following proposition is the inductive step in the construction.

Proposition 3.4. *Let H be a finite abelian group, let N be a subgroup of H, and set $Q = H/N$ with quotient map $\pi\colon H \to Q$. Denote by $\delta^H\colon H \to \mathrm{Aut}(D_H)$ and $\delta^Q\colon Q\to\mathrm{Aut}(D_Q)$ the actions described in Remark 3.2. Then there is an injective unital homomorphism $\iota\colon D_Q\to D_H$ satisfying*

$$\delta^H_h \circ \iota = \iota\circ \delta^Q_{\pi(h)}$$

for all $h\in H$.

Proof. For a finite group K, we denote by $\{\xi^K_k\}_{k\in K}$ the canonical basis of $\ell^2(K)$. Also, when K is abelian, we write $\widehat{K}$ for its dual group, and an element of $\widehat{K}$ will be denoted, with a slight abuse of notation, by $\widehat{k}$. Observe that $\widehat{Q}$ is a subgroup of $\widehat{H}$, and that $\widehat{H}/\widehat{Q}\cong\widehat{N}$. Fix a section $s\colon \widehat{N}\to\widehat{H}$. Then s induces a unitary $U\colon \ell^2(\widehat{Q})\otimes\ell^2(\widehat{N})\to\ell^2(\widehat{H})$ given by

$$U(\xi^{\widehat{Q}}_{\widehat{q}}\otimes\xi^{\widehat{N}}_{\widehat{n}}) = \xi^{\widehat{H}}_{\widehat{q}s(\widehat{n})}$$

for every $\widehat{q}\in\widehat{Q}$ and every $\widehat{n}\in\widehat{N}$. Define a unital embedding $\varphi\colon B(\ell^2(\widehat{Q}))\to B(\ell^2(\widehat{H}))$ by

$$\varphi(a)(\xi^{\widehat{H}}_{\widehat{q}s(\widehat{n})}) = U(a(\xi^{\widehat{Q}}_{\widehat{q}})\otimes\xi^{\widehat{N}}_{\widehat{n}})$$

for every $a \in B(\ell^2(\widehat{Q}))$, for every $\widehat{q} \in \widehat{Q}$ and every $\widehat{n} \in \widehat{N}$. Let $\widehat{q} \in \widehat{Q}$. We claim that $\varphi(\lambda^{\widehat{Q}}_{\widehat{q}}) = \lambda^{\widehat{H}}_{\widehat{q}}$. To see this, let $\widehat{p} \in \widehat{Q}$ and let $\widehat{n} \in \widehat{N}$. Then

$$\varphi(\lambda^{\widehat{Q}}_{\widehat{q}})(\xi^{\widehat{H}}_{\widehat{p}s(\widehat{n})}) = U(\lambda^{\widehat{Q}}_{\widehat{q}}(\xi^{\widehat{Q}}_{\widehat{p}}) \otimes \xi^{\widehat{N}}_{\widehat{n}}) = U(\xi^{\widehat{Q}}_{\widehat{q}\widehat{p}} \otimes \xi^{\widehat{N}}_{\widehat{n}}) = \xi^{\widehat{H}}_{\widehat{q}\widehat{p}s(\widehat{n})} = \lambda^{\widehat{H}}_{\widehat{q}}(\xi^{\widehat{H}}_{\widehat{p}s(\widehat{n})}).$$

This proves the claim. It follows that φ induces, upon taking its infinite tensor product, a unital injective homomorphism $\psi\colon D_{\widehat{Q}} \to D_{\widehat{H}}$, which moreover satisfies

$$\psi \circ \delta^{\widehat{Q}}_{\widehat{q}} = \delta^{\widehat{H}}_{\widehat{q}} \circ \psi$$

for all $\widehat{q} \in \widehat{Q}$, by the claim above. After taking crossed products by $\widehat{Q}$ and $\widehat{H}$, and using Lemma 3.3, we obtain a unital embedding $\iota\colon D_Q \to D_H$, which satisfies $\delta^H_h \circ \iota = \iota \circ \delta^Q_{\pi(h)}$ for all $h \in H$. This completes the proof. □

Here is the main result of this section.

Theorem 3.5. *Let G be a second countable, abelian, profinite group. Let D_G denote the UHF-algebra associated with G as in Definition 3.1. Then there exists a canonical action $\delta^G\colon G \to \mathrm{Aut}(D_G)$ with the following properties.*

(1) There exists an equivariant unital embedding $(C(G), \mathtt{Lt}^G) \to (D_G, \delta^G)$.
(2) $(D_G^{\otimes\mathbb{N}}, (\delta^G)^{\otimes\mathbb{N}})$ is equivariantly isomorphic to (D_G, δ^G).
(3) δ^G has the Rokhlin property.
(4) The fixed point algebra $D_G^{\delta^G}$ is isomorphic to D_G.

Moreover, δ^G is—up to conjugacy—the unique *action of G on D_G with the Rokhlin property. Furthermore, if E is a unital C*-algebra with $D_G \otimes E \cong E$ and $\beta\colon G \to \mathrm{Aut}(E)$ is an action with the Rokhlin property, then β is conjugate to $\delta^G \otimes \beta$.*

Proof. Since the group G is fixed, we drop the subscript G from all algebras and actions, in order to lighten the notation. We first construct the action, and then show that it has the desired properties. Let $\mathcal{V}$ be the collection of open subgroups of G, and observe that $\mathcal{V}$ is countable. Define an inverse system $(G_i, \pi_{i,j})_{i,j\in I}$ of finite groups as follows. Set $I = \mathcal{V}$ ordered by *reverse inclusion.* For $i \in I$, let $G_i = G/i$, and for $i, j \in I$ with $i \le j$, let $\pi_{i,j}\colon G_j \to G_i$ be the canonical quotient map. Then $G = \varinjlim(G_i, \pi_{i,j})$. By Proposition 3.4, for every $i, j \in I$ with $i \le j$, there exists a unital embedding $\iota_{i,j}\colon D_{G_i} \to D_{G_j}$ satisfying $\delta^{G_j}_g \circ \iota_{i,j} = \iota_{i,j} \circ \delta^{G_i}_{\pi_{i,j}(g)}$ for all $g \in G_j$. Observe that D can be identified with the direct limit of the UHF-algebras D_{G_i}, for $i \in I$, with connective maps $\iota_{i,j}$ for $i, j \in I$ with $i \le j$. By Lemma 2.20, there exists an induced action $\delta\colon G \to \mathrm{Aut}(D)$ given by

$$\delta_g(\iota_{i,\infty}(a)) = \iota_{i,\infty}(\delta^{G_i}_{\pi_{i,\infty}(g)}(a))$$

for all $g \in G$, for all $i \in I$, and all $a \in D_{G_i}$.

(1): Let $i \in I$. Observe that the restriction of δ^{G_i} to $\mathbb{C} \rtimes \widehat{G_i} \cong C(G_i)$ is naturally conjugate to the left translation action $\mathtt{Lt}^{G_i}$. In particular, there is a unital equivariant embedding $\phi_i\colon (C(G_i), \mathtt{Lt}^{G_i}) \to (D_{G_i}, \delta^{G_i})$. For $i, j \in I$ with $i \le j$, denote by $\pi^*_{i,j}\colon C(G_i) \to C(G_j)$ the injective unital *-homomorphism given by $\pi^*_{i,j}(f) = f \circ \pi_{i,j}$ for all $f \in C(G_i)$. Then the maps ϕ_i are easily seen to satisfy $\iota_{i,j} \circ \phi_i = \phi_j \circ \pi^*_{i,j}$ for all $i, j \in I$ with $i \le j$. By the universal property of direct limits, it follows that there exists a unital equivariant embedding $(C(G), \mathtt{Lt}) \to (D, \delta)$, as desired.

(2): This is an easy consequence of the fact that δ^{G_i} is conjugate to $(\delta^{G_i})^{\otimes\mathbb{N}}$ for every $i \in I$.

(3): This is an immediate consequence of (1) and (2).

(4): By part (1) of Corollary 3.11 in [14], the fixed point algebra D^δ is a UHF-algebra, and it absorbs D by [14, Theorem 4.3]. Since it is obviously unitally embedded in D, it follows from [42, Proposition 5.12] that D^δ is isomorphic to D. The last part of the theorem is a consequence of [16, Theorem X.4.5]. □

4. Uncountably many actions

In this section, given a countable group Λ containing an infinite subgroup Δ with relative property (T), and given a UHF-algebra A of infinite type, we construct uncountably many strongly outer (in fact, free) actions of Λ on A, which are not weakly cocycle conjugate; see Theorem 4.6. In fact, we perform the construction for an arbitrary separable unital C*-algebra A satisfying the following properties: A is locally reflexive, M_{p^∞}-absorbing for some prime p, has an amenable trace, and is isomorphic to its infinite tensor product $A^{\otimes\mathbb{N}}$.

We fix in the following such a C*-algebra A. We also fix a prime p such that $A \cong A \otimes M_{p^\infty}$, and an infinite relative property (T) subgroup Δ of Λ. We will frequently use the notation for Bernoulli actions from

Notation 2.13. We write D for M_{p^∞}. We also fix an isomorphism $\phi\colon A \to A^{\otimes\Lambda}$. Using this isomorphism, we let $\sigma\colon \Lambda \to \mathrm{Aut}(A)$ denote the action given by

$$\sigma_\gamma = \phi^{-1} \circ (\beta_{\Lambda\curvearrowright\Lambda,A})_\gamma \circ \phi$$

for all $\gamma \in \Lambda$.

Let G be a second countable abelian pro-p group, and let $\delta^G\colon G \to \mathrm{Aut}(D)$ be the action constructed in Theorem 3.5. Consider the diagonal action $(\delta^G)^{\otimes\Lambda}\colon G \to \mathrm{Aut}(D^{\otimes\Lambda})$, and denote by E_G its fixed point algebra, which, by parts (2) and (4) of Theorem 3.5, is isomorphic to D. Since $(\delta^G)^{\otimes\Lambda}$ and $\beta_{\Lambda\curvearrowright\Lambda,D}$ commute, $\beta_{\Lambda\curvearrowright\Lambda,D}$ restricts to an action $\beta_{\Lambda\curvearrowright\Lambda,D}|_{E_G}\colon \Lambda \to \mathrm{Aut}(E_G)$.

Definition 4.1. For each pro-p group G, we choose an isomorphism $\xi_G\colon A \to E_G \otimes A$. Now, we define an action $\alpha^G\colon \Lambda \to \mathrm{Aut}(A)$ by

$$\alpha^G_\gamma = \xi_G^{-1} \circ (\beta_{\Lambda\curvearrowright\Lambda,D}|_{E_G} \otimes \sigma)_\gamma \circ \xi_G$$

for all $\gamma \in \Lambda$. We also define $\theta^G\colon \Delta \to \mathrm{Aut}(A)$ to be the restriction of α^G to Δ.

It will be shown in Theorem 4.5 that for non-isomorphic pro-p groups G_0 and G_1, the actions α^{G_0} and α^{G_1} are not weakly cocycle conjugate. In order to do this, we will need to study the weak extensions of these actions with respect to certain invariant traces. Our next result provides us with a canonical subset of $T(A)$ consisting of traces that are α^G-invariant for every pro-p group G. Later, in Proposition 4.3, we will show that for any of these traces and for any G, the weak extension of any of α^G is mixing. For a pro-p group G, we denote by τ_{E_G} the (unique) trace on E_G.

Proposition 4.2. *Adopt the notation from the previous discussion, and define a continuous, affine map $\iota\colon T(A) \to T(A)$ by $\iota(\tau) = \tau^{\otimes\Lambda} \circ \phi$ for all $\tau \in T(A)$. If τ is extreme and amenable, then so is $\iota(\tau)$.*

Moreover, if G is any pro-p group, then $\iota(\tau) = (\tau_{E_G} \otimes \iota(\tau)) \circ \xi_G$ for all $\tau \in T(A)$. In particular, $\iota(\tau)$ is α^G-invariant for all $\tau \in T(A)$.

Proof. The first assertion is standard (and in our case, it follows from Lemma 2.21, since the weak closure of $A^{\otimes\Lambda}$ with respect to $\tau^{\otimes\Lambda}$ is canonically isomorphic to $(\overline{A}^\tau)^{\otimes\Lambda} \cong R^{\otimes\Lambda} \cong R$.)

Let G be a pro-p group and let $\tau \in T(A)$. Observe that $(\tau_{E_G} \otimes \iota(\tau)) \circ \xi_G$ is an α^G-invariant trace on A. Hence, it suffices to show that this trace equals $\iota(\tau)$.

Observe that since E_G is a UHF-algebra of infinite type, the isomorphism $\xi_G\colon A \to E_G \otimes A$ is approximately unitarily equivalent to the first tensor factor embedding $\kappa\colon A \to E_G \otimes A$ given by $\kappa(a) = 1_{E_G} \otimes a$ for all $a \in A$; see [42, Corollary 1.12]. It follows that ξ_G and κ induce the same map at the level of traces. Using this at the first step in the following computation, we conclude that

$$(\tau_{E_G} \otimes \iota(\tau)) \circ \xi_G = (\tau_{E_G} \otimes \iota(\tau)) \circ \kappa = \iota(\tau)$$

for all traces $\tau \in T(A)$, as desired. ☐

Our next goal is to establish a number of properties for α^G; this will be done in Proposition 4.3. In order to do this, we need an alternative description of α^G. Since the group G will be fixed from now on and until Theorem 4.5, we will drop it from the notation for the actions δ^G, α^G and θ^G, as well as from the notation for the algebra E_G. In Theorem 4.5, we will show that for nonisomorphic G_0 and G_1, the actions constructed above are not weakly cocycle conjugate. Until then, we will work with a fixed pro-p group G, which agrees with the restriction of $\widetilde{\alpha}$ to Δ.

Observe that the Bernoulli action $\beta_{\Lambda\curvearrowright\Lambda,D\otimes A}$ commutes with the diagonal action

$$(\delta \otimes \mathrm{id}_A)^{\otimes\Lambda}\colon G \to \mathrm{Aut}((D \otimes A)^{\otimes\Lambda}).$$

Thus, with B denoting the fixed point algebra of $(\delta \otimes \mathrm{id}_A)^{\otimes\Lambda}$, the action $\beta_{\Lambda\curvearrowright\Lambda,D\otimes A}$ restricts to an action $\widetilde{\alpha}\colon \Lambda \to \mathrm{Aut}(B)$. We also define $\widetilde{\theta}$ be the restriction of $\beta_{\Delta\curvearrowright\Lambda,D\otimes A}$ to B.

Fix an amenable extreme trace τ_0 on $D \otimes A$ and let $\widetilde{\tau}$ be the trace $\tau_0^{\otimes\Lambda}$ on $(D \otimes A)^{\otimes\Lambda}$. Then $\overline{D \otimes A}^{\tau_0}$ is isomorphic to R by Lemma 2.21, and the extension of τ_0 to $\overline{D \otimes A}^{\tau_0}$ is the unique trace on R. We identify $\overline{\beta}^{\widetilde{\tau}}_{\Lambda\curvearrowright\Lambda,D\otimes A}$ with the von Neumann-algebraic Bernoulli action $\beta_{\Lambda\curvearrowright\Lambda,R}\colon \Lambda \to \mathrm{Aut}(R^{\otimes\Lambda})$, and $\overline{\beta}^{\widetilde{\tau}}_{\Delta\curvearrowright\Lambda,D\otimes A}$ with $\beta_{\Delta\curvearrowright\Lambda,R}\colon \Delta \to \mathrm{Aut}(R^{\otimes\Lambda})$. Similarly, the extension of $(\delta \otimes \mathrm{id}_A)^{\otimes\Lambda}$ to the weak closure with respect to $\widetilde{\tau}$ can be identified with $(\overline{\delta}^{\tau_D} \otimes \mathrm{id}_R)^{\overline{\otimes}\Lambda}$, where τ_D is the unique trace on D. Furthermore, since G is compact, $\overline{B}^{\widetilde{\tau}}$ can be identified with the fixed point algebra of $(\overline{\delta}^{\tau_D} \otimes \mathrm{id}_R)^{\overline{\otimes}\Lambda}$, and the weak extension of $\widetilde{\alpha}$ can be identified with the restriction of $\overline{\beta}^{\widetilde{\tau}}$ to $\overline{B}^{\widetilde{\tau}}$.

In the next proposition, we first show that α and θ are conjugate to $\widetilde{\alpha}$ and $\widetilde{\theta}$, respectively. Then we use these alternative descriptions to verify some properties of α and θ.

Proposition 4.3. *Adopt the notation of the discussion above. Let τ_0 be a trace on A, and τ be the image of τ_0 under the map ι from Proposition 4.2. Define $\widetilde{\tau}$ to be the trace $(\tau_D \otimes \tau_0)^{\otimes\Lambda}$ on $(D\otimes A)^{\otimes\Lambda}$. Then:*

(1) There is a Λ-equivariant trace-preserving isomorphism $(A,\tau,\alpha) \cong (B,\widetilde{\tau},\widetilde{\alpha})$, which restricts to a Δ-equivariant trace-preserving isomorphism $(A,\tau,\theta) \cong (B,\widetilde{\tau},\widetilde{\theta})$.
(2) There is a Λ-equivariant isomorphism $(A,\alpha) \cong (A\otimes M_p^{\otimes\Lambda}, \alpha\otimes\beta_{\Lambda\curvearrowright\Lambda,M_p})$.
(3) There is a Δ-equivariant trace-preserving isomorphism $(A,\tau,\theta)\cong(A\otimes A,\tau\otimes\tau,\theta\otimes\theta)$.
(4) The action α is free;
(5) The actions $\overline{\alpha}^{\tau}$ and $\overline{\theta}^{\tau}$ are mixing.

Proof. (1): By rearranging the tensor factors, it is clear that there exists a Λ-equivariant trace-preserving isomorphism

$$((D\otimes A)^{\otimes\Lambda},\widetilde{\tau},\beta_{\Lambda\curvearrowright\Lambda,D\otimes A}) \cong (D^{\otimes\Lambda}\otimes A^{\otimes\Lambda}, \tau_D^{\otimes\Lambda}\otimes\tau_0^{\otimes\Lambda}, \beta_{\Lambda\curvearrowright\Lambda,D}\otimes\beta_{\Lambda\curvearrowright\Lambda,A}).$$

Upon identifying $A^{\otimes\Lambda}$ with A via the isomorphism ϕ, we obtain a Λ-equivariant isomorphism

$$((D\otimes A)^{\otimes\Lambda},\widetilde{\tau},\beta_{\Lambda\curvearrowright\Lambda,D\otimes A}) \cong (D^{\otimes\Lambda}\otimes A, \tau_D^{\otimes\Lambda}\otimes\tau, \beta_{\Lambda\curvearrowright\Lambda,D}\otimes\sigma).$$

This isomorphism can be regarded as a G-equivariant isomorphism

$$((D\otimes A)^{\otimes\Lambda},\widetilde{\tau},(\delta\otimes\mathrm{id}_A)^{\otimes\Lambda}) \cong (D^{\otimes\Lambda}\otimes A, \tau_D^{\otimes\Lambda}\otimes\tau, \delta^{\otimes\Lambda}\otimes\mathrm{id}_A).$$

Upon taking G-fixed point algebras, and recalling that E denotes the fixed point algebra of $\delta^{\otimes\Lambda}$, we obtain a trace-preserving isomorphism $\psi\colon (B,\widetilde{\tau})\to(E\otimes A,\tau_E\otimes\tau)$. Moreover, ψ can be regarded as a Λ-equivariant trace-preserving isomorphism

$$\psi\colon (B,\widetilde{\tau},\widetilde{\alpha})\to(E\otimes A,\tau_E\otimes\tau,\beta_{\Lambda\curvearrowright\Lambda,D}|_E\otimes\sigma).$$

Since $\xi\colon(A,\alpha)\to(E\otimes A,\beta_{\Lambda\curvearrowright\Lambda,D}|_E\otimes\sigma)$ is an equivariant isomorphism by definition, and $(\tau_E\otimes\tau)\circ\xi=\tau$ by Proposition 4.2, it follows that $\xi^{-1}\circ\psi$ is a Λ-equivariant trace-preserving isomorphism $(B,\widetilde{\tau},\widetilde{\theta})\to(A,\tau,\alpha)$. Finally, it is clear that $\xi^{-1}\circ\psi$ restricts to a Δ-equivariant trace-preserving isomorphism $(A,\tau,\theta)\cong(B,\widetilde{\tau},\widetilde{\theta})$.

(2): By (1), it is enough to prove that there is a Λ-equivariant isomorphism

$$(B,\widetilde{\alpha})\cong(B\otimes M_p^{\otimes\Lambda},\widetilde{\alpha}\otimes\beta_{\Lambda\curvearrowright\Lambda,M_p}).$$

Using that A is isomorphic to $A\otimes M_p$, it is clear that there is an equivariant isomorphism

$$\left((D\otimes A)^{\otimes\Lambda},\beta_{\Lambda\curvearrowright\Lambda,D\otimes A}\right)\cong\left((D\otimes A)^{\otimes\Lambda}\otimes M_p^{\otimes\Lambda},\beta_{\Lambda\curvearrowright\Lambda,D\otimes A}\otimes\beta_{\Lambda\curvearrowright\Lambda,M_p}\right).$$

This isomorphism intertwines the actions

$$(\delta\otimes\mathrm{id}_A)^{\otimes\Lambda}\colon G\to\mathrm{Aut}\left((D\otimes A)^{\otimes\Lambda}\right)\quad\text{and}\quad(\delta\otimes\mathrm{id}_A)^{\otimes\Lambda}\otimes\mathrm{id}_{M_p^{\otimes\Lambda}}\colon G\to\mathrm{Aut}\left((D\otimes A)^{\otimes\Lambda}\otimes M_p^{\otimes\Lambda}\right).$$

The fixed point algebra of the second action is isomorphic to $B\otimes M_p^{\otimes\Lambda}$, in such a way that the restriction of $\beta_{\Lambda\curvearrowright\Lambda,D\otimes A}\otimes\beta_{\Lambda\curvearrowright\Lambda,M_p}$ to this algebra is conjugate to $\widetilde{\alpha}\otimes\beta_{\Lambda\curvearrowright\Lambda,M_p}$. Thus $(B,\widetilde{\alpha})$ is equivariantly isomorphic to $(B\otimes M_p^{\otimes\Lambda},\widetilde{\alpha}\otimes\beta_{\Lambda\curvearrowright\Lambda,M_p})$, as desired.

(3): This is similar to (2), using the fact that $(A^{\otimes\Lambda},\tau_0^{\otimes\Lambda})\cong(A,\tau)$ via ϕ.

(4): By (1) and (2), it suffices to show that $\widetilde{\alpha}\otimes\beta_{\Lambda\curvearrowright\Lambda,M_p}$ is free. Let $\gamma\in\Lambda\setminus\{1\}$ and let $\hat{\tau}$ be an $(\widetilde{\alpha}\otimes\beta_{\Lambda\curvearrowright\Lambda,M_p})_\gamma$-invariant trace on $B\otimes M_p^{\otimes\Lambda}$. Since $M_p^{\otimes\Lambda}\cong D$ has a unique trace τ_D, we deduce that $\hat{\tau}$ has the form $\tau_B\otimes\tau_D$ for some α_γ-invariant trace τ_B on B. The weak extension of $(\widetilde{\alpha}\otimes\beta_{\Lambda\curvearrowright\Lambda,M_p})_\gamma$ with respect to $\hat{\tau}$ is conjugate to $(\overline{\widetilde{\alpha}}^{\tau_B}\otimes\overline{\beta}_{\Lambda\curvearrowright\Lambda,M_p})_\gamma$, where $\overline{\beta}_{\Lambda\curvearrowright\Lambda,M_p}$ is the von Neumann-algebraic Bernoulli $(\Lambda\curvearrowright\Lambda)$-action with base M_p. It is shown in [26, Section 4.1] that $(\overline{\beta}_{\Lambda\curvearrowright\Lambda,M_p})_\gamma$ is properly outer for every $\gamma\in\Lambda\setminus\{1\}$. We deduce from [23, Corollary 1.12] that the weak extension of $\left(\widetilde{\alpha}\otimes\beta_{\Lambda\curvearrowright\Lambda,M_p}\right)_\gamma$ with respect to $\hat{\tau}$ is properly outer. Therefore we conclude that $\widetilde{\alpha}\otimes\beta_{\Lambda\curvearrowright\Lambda,M_p}$ (and hence $\widetilde{\alpha}$ and α) is free.

(5): Since $\overline{\theta}^{\tau}$ is the restriction of $\overline{\alpha}^{\tau}$ to Δ, it is enough to check that $\overline{\alpha}^{\tau}$ is mixing. By (1), it suffices to check that $\overline{\widetilde{\alpha}}^{\widetilde{\tau}}$ is mixing. Observe that $\overline{\widetilde{\alpha}}^{\widetilde{\tau}}$ is the restriction to $\overline{B}^{\widetilde{\tau}}$ of $\overline{\beta}^{\widetilde{\tau}}_{\Lambda\curvearrowright\Lambda,D\otimes A}$. The latter action is conjugate to the von Neumann-algebraic Bernoulli action $\beta_{\Lambda\curvearrowright\Lambda,\overline{D\otimes A}^{\tau_0}}$, which is mixing by Lemma 2.7. Therefore $\overline{\widetilde{\alpha}}^{\widetilde{\tau}}$ is mixing. ☐

We retain the notation from before Proposition 4.3. In the next theorem, we will make crucial use of Popa's superrigidity theorem for noncommutative Bernoulli actions—which is one of the main results of [35]—in order to compute the weak 1-cohomology of the weak extension $\overline{\theta}^{\tau}$ of θ with respect to any trace τ which belongs to the image of the map ι from Proposition 4.2. For any such trace τ, we will use the pairing function $m^{\overline{\theta}^{\tau}}\colon H^1_w(\overline{\theta}^{\tau})\times H^1_w(\overline{\theta}^{\tau})\to H^1_w(\overline{\theta}^{\tau}\otimes\overline{\theta}^{\tau})$ from Definition 2.10. We write Γ for the Pontryagin dual of G, and we let $m^{\Gamma}\colon\Gamma\times\Gamma\to\Gamma$ be the multiplication operation. Recall also that $\mathtt{Lt}\colon G\to\mathrm{Aut}(C(G))$ denotes the action by left translation.

Theorem 4.4. *Adopt the notation from Definition 4.1. Let $\tau_0\in T(A)$, and set $\tau=\iota(\tau_0)$. Then there exist bijections $\eta\colon H^1_w(\overline{\theta}^{\tau})\to\Gamma$ and $\eta^{(2)}\colon H^1_w(\overline{\theta}^{\tau}\otimes\overline{\theta}^{\tau})\to\Gamma$ satisfying*

$$\eta^{(2)}\circ m^{\overline{\theta}^{\tau}}=m^{\Gamma}\circ(\eta\times\eta).$$

Proof. Let τ_D denote the unique trace on D. For the trace τ_0 in the statement, set $\widetilde{\tau}=(\tau_D\otimes\tau_0)^{\otimes\Lambda}$, which is naturally a trace on $(D\otimes A)^{\otimes\Lambda}$. Recall the definition of $\widetilde{\theta}$ from before Proposition 4.3. By part (1) of Proposition 4.3 the action θ is conjugate to $\widetilde{\theta}$ via an isomorphism that maps τ to $\widetilde{\tau}$. In particular, the weak extension of θ with respect to τ is conjugate to the weak extension of $\widetilde{\theta}$ with respect to $\widetilde{\tau}$. Therefore it is enough to prove the corresponding statement for $\widetilde{\theta}$ and $\widetilde{\tau}$.

In what follows, and since all weak extensions will be taken with respect to $\widetilde{\tau}$, we will omit this trace from the notation. Also, we will regard $\widetilde{\theta}$ as an alternative description of θ, and, with a slight abuse of notation, we will write θ to mean $\widetilde{\theta}$. In particular, the symbol $\overline{\theta}$ will always represent the weak extension of $\widetilde{\theta}$ with respect to the trace $\widetilde{\tau}$. We let ζ be the Bernoulli $(\Delta\curvearrowright\Lambda)$-action with base $D\otimes A$. Observe that θ is the restriction of ζ to $B\subseteq(D\otimes A)^{\otimes\Lambda}$. We abbreviate $\overline{(D\otimes A)}^{\otimes\Lambda}$ to N (omitting the trace $\tau_D\otimes\tau_0$ from the notation), and the G-action $(\overline{\delta}\otimes\mathrm{id}_R)^{\overline{\otimes}\Lambda}$ on N to just $\rho\colon G\to\mathrm{Aut}(N)$. (In particular, $\overline{B}=N^{\rho}$.)

Let $u\colon\Delta\to U(\overline{B})$ be a weak 1-cocycle for $\overline{\theta}$. Since $\overline{\theta}$ is the restriction of $\overline{\zeta}$ to $\overline{B}$, we deduce that u is also a 1-cocycle for $\overline{\zeta}$. It is shown in [35, Section 3] that the von Neumann-algebraic Bernoulli $(\Delta\curvearrowright\Lambda)$-action $\overline{\zeta}$ satisfies the assumptions of [35, Theorem 4.1]. Applying [35, Theorem 4.1] in the case of weak 1-cocycles, with $S=S_1=\{1\}$ in the notation of [35], we conclude that u is a weak coboundary for $\overline{\zeta}$. Therefore, there exist a unitary $v\in U(N)$ and a function $\mu\colon\Delta\to\mathbb{T}$ satisfying $u_\gamma=\mu_\gamma v^*\overline{\zeta}_\gamma(v)$ for every $\gamma\in\Delta$. Fix $g\in G$. Applying ρ_g to the previous identity, and using that $u_\gamma\in N^\rho=\overline{B}$, yields

$$\mu_\gamma v^*\overline{\zeta}_\gamma(v)=u_\gamma=\rho_g(u_\gamma)=\mu_\gamma\rho_g(v)^*\overline{\zeta}_\gamma(\rho_g(v))$$

for every $\gamma\in\Delta$. Hence $\rho_g(v)v^*$ is fixed by $\overline{\zeta}$. Now, since $\overline{\theta}$ is mixing by part (5) of Proposition 4.3, and $\overline{\theta}$ is the restriction of $\overline{\zeta}$ to an invariant subalgebra, we deduce that $\overline{\zeta}$ is mixing as well. By Remark 2.3, we conclude that $\rho_g(v)v^*$ is a scalar. Define a function $\chi_u\colon G\to\mathbb{T}$ by $\chi_u(g)=\rho_g(v)v^*$ for $g\in G$.

Claim. *χ_u is well-defined (that is, independent of the choice of μ and v).*

Proof of claim. Fix $v,v'\in U(N)$ and $\mu,\mu'\colon\Delta\to\mathbb{T}$ satisfying

$$u_\gamma=\mu_\gamma v^*\overline{\zeta}_\gamma(v)=\mu'_\gamma(v')^*\overline{\zeta}_\gamma(v')$$

for every $\gamma\in\Delta$. We want to show that $\rho_g(v)v^*=\rho_g(v')(v')^*$ for all $g\in G$. The above identity implies that

$$\overline{\zeta}_\gamma(v'v^*)=\mu_\gamma\overline{\mu}'_\gamma v'v^*$$

for all $\gamma\in\Lambda$. In particular, the 1-dimensional subspace of $L^2(N)$ spanned by $v'v^*$ is invariant by $\overline{\zeta}$. As explained above, part (5) of Proposition 4.3 implies that $\overline{\zeta}$ is (weakly) mixing. Hence, it follows from Remark 2.3 that $v'v^*$ is a scalar, which we abbreviate to $z\in\mathbb{T}$. Thus,

$$\rho_g(v')(v')^*=\rho_g(zv)(zv)^*=z\overline{z}\rho_g(v)v^*=\rho_g(v)v^*$$

for all $g\in G$, as desired. ☐

Claim. *χ_u is a character on G*

Proof of claim. First, observe that χ_u is a continuous function, since ρ_g is a continuous action. To check the character condition, let $g,h\in G$. Then

$$\chi_u(gh)=\rho_{gh}(v)v^*=\rho_g(\rho_h(v)v^*)\rho_g(v)v^*=\chi_u(g)\chi_u(h),$$

so the claim is proved. ☐

Claim. *For $u\in Z^1_w(\overline{\theta})$, the character χ_u only depends on the weak cohomology class of u.*

Proof of claim. Let $u' \in Z^1_w(\overline{\theta})$ be weakly cohomologous to u, and let $w \in U(\overline{B})$ satisfy $u'_\gamma = w^* u_\gamma \overline{\theta}_\gamma(w) \mathrm{mod}\mathbb{C}$ for every $\gamma \in \Delta$. Let $v \in U(\overline{B})$ be an eigenvector for ρ with eigenvalue χ_u, such that $u_\gamma = v^* \overline{\zeta}_\gamma(v) \,\mathrm{mod}\mathbb{C}$ for every $\gamma \in \Delta$. Then

$$u'_\gamma = \mu_\gamma(vw)^* \overline{\zeta}_\gamma(vw) \mathrm{mod}\mathbb{C}$$

for every $\gamma \in \Delta$, and $vw \in U(\overline{B})$ is an eigenvector for ρ with eigenvalue χ_u. Therefore $\chi_{u'} = \chi_u$. □

In view of the previous claims, we can define a function $\eta\colon H^1_w(\overline{\theta}) \to \Gamma$ by $\eta([u]) = \chi_u$ for all $[u] \in H^1_w(\overline{\theta})$.

Claim. *The map $\eta\colon H^1_w(\overline{\theta}) \to \Gamma$ is surjective.*

Proof of claim. Fix $\omega \in \Gamma$. Since ω is a continuous function $\omega\colon G \to \mathbb{C}$, we can regard ω as a (unitary) element in $C(G)$. Observe that ω is an eigenvector for $\mathtt{Lt}$ with eigenvalue ω. By part (1) of Theorem 3.5, there exists an equivariant unital embedding $(C(G), \mathtt{Lt}) \to (D, \delta)$. Furthermore, there exists an equivariant unital embedding

$$(D, \delta) \to \left((D \otimes A)^{\otimes\Lambda}, \delta \otimes \mathrm{id}_A)^{\otimes\Lambda}\right).$$

Composing these maps, one can conclude that there exists an equivariant unital embedding

$$(C(G), \mathtt{Lt}) \to \left((D \otimes A)^{\otimes\Lambda}, (\delta \otimes \mathrm{id}_A)^{\otimes\Lambda}\right).$$

Identifying $C(G)$ with its image inside $(D \otimes A)^{\otimes\Lambda}$, we can regard ω as an element of $(D \otimes A)^{\otimes\Lambda}$, which is an eigenvector for $(\delta \otimes \mathrm{id}_A)^{\otimes\Lambda}$ with eigenvalue ω. In turn, this gives an element v of the weak closure N of $(D \otimes A)^{\otimes\Lambda}$ which is an eigenvector for ρ with eigenvalue ω. Define a function $u\colon \Delta \to N$ by $u_\gamma = v^* \overline{\zeta}_\gamma(v)$ for all $\gamma \in \Delta$. For every $g \in G$, we have

$$\rho_g(u_\gamma) = \rho_g(v)^* \overline{\zeta}_\gamma(\rho_g(v)) = v^* \overline{\zeta}_\gamma(v) = u_\gamma$$

for all $\gamma \in \Delta$. It follows that u takes values in $\overline{B} = N^\rho$. On the other hand, given $\gamma \in \Delta$, we have

$$u_\gamma \overline{\theta}_\gamma(u_\rho) = v^* \overline{\zeta}_\gamma(v) \overline{\theta}_\gamma(v^* \overline{\zeta}_\rho(v)) = v^* \overline{\zeta}_{\gamma\rho}(v) = u_{\gamma\rho} \mathrm{mod}\mathbb{C}.$$

Therefore u is a weak 1-cocycle, and $\chi_u = \omega$. It follows that η is surjective, as desired. □

Claim. *The map $\eta\colon H^1_w(\overline{\theta}) \to \Gamma$ is injective (and hence a bijection).*

Proof of claim. Let $u_0, u_1 \in Z^1_w(\overline{\theta})$ satisfy $\chi_{u_0} = \chi_{u_1}$. Denote by ω this character. Find eigenvectors $v_0, v_1 \in U(\overline{B})$ for ρ with eigenvalue ω such that $u_{j,\gamma} = v_j^* \overline{\zeta}_\gamma(v_j) \mathrm{mod}\mathbb{C}$ for all $\gamma \in \Delta$ and $j = 0, 1$. Set $w = v_0^* v_1$, which is a unitary in $\overline{B}$. For every $\gamma \in \Delta$, we have

$$w^* u_{0,\gamma} \overline{\theta}_\gamma(w) = (v_0^* v_1)^* u_{0,\gamma} \overline{\theta}_\gamma(v_0^* v_1) = v_1^* v_0 (v_0^* \overline{\zeta}_\gamma(v_0)) \overline{\zeta}_\gamma(v_0^* v_1) = v_1^* \overline{\zeta}_\gamma(v_1) = u_{1,\gamma} \ \mathrm{mod}\mathbb{C}.$$

Therefore w witnesses the fact that u_0 and u_1 are weakly cohomologous. Thus $[u_0] = [u_1]$, and η is injective. □

We now turn to the construction of the map $\eta^{(2)}\colon H^1_w(\overline{\theta} \otimes \overline{\theta}) \to \Gamma$. Observe that $\overline{\theta} \otimes \overline{\theta}$ is conjugate to $\overline{\theta}$ by part (3) of Proposition 4.3. Let $u \in Z^1_w(\overline{\theta} \overline{\otimes} \overline{\theta})$, and choose a unitary $v \in U(N \overline{\otimes} N)$ satisfying $u_\gamma = v^* (\overline{\zeta}_\gamma \overline{\otimes} \overline{\zeta}_\gamma)(v)$ for all $\gamma \in \Delta$. As before, one checks that v is an eigenvector for $\overline{\theta} \otimes \overline{\theta}$, and its eigenvalue κ_u is a character in Γ, which is independent of v. Similarly to what was done above, one shows that there is a well-defined map $\eta^{(2)}\colon H^1_w(\overline{\theta} \otimes \overline{\theta}) \to \Gamma$ given by $\eta^{(2)}([u]) = \kappa_u$ for all $[u] \in H^1_w(\overline{\theta} \otimes \overline{\theta})$.

It remains to prove the identity $\eta^{(2)} \circ m^{\overline{\theta}} = m^\Gamma \circ (\eta \times \eta)$. Let $[u], [u'] \in H^1_w(\overline{\theta})$, and set $\omega = \eta([u])$ and $\omega' = \eta([u])$. Find eigenvectors $v, v' \in U(\overline{B})$ for $\overline{\zeta}$ with eigenvalues ω and ω', respectively, satisfying

$$u_\gamma = v^* \overline{\zeta}_\gamma(v) \mathrm{mod}\mathbb{C} \quad \text{and} \quad u'_\gamma = v'^* \overline{\zeta}_\gamma(v') \mathrm{mod}\mathbb{C}$$

for all $\gamma \in \Delta$. Hence $(u \otimes u')_\gamma = (v \otimes v')^* (\overline{\theta}_\gamma \otimes \overline{\theta}_\gamma)(v \otimes v')$. Since $v \otimes v'$ is an eigenvector for $\rho \otimes \rho$ with eigenvalue $\omega\omega'$, this shows that

$$(\eta^{(2)} \circ m^{\overline{\theta}})([u], [u']) = \omega\omega' = \eta([u])\eta([u']) = (m^\Gamma \circ (\eta \times \eta))([u], [u']).$$

This concludes the proof of the theorem. □

Using the previous result, we will show below that if one starts with two non-isomorphic abelian pro-p groups G_0 and G_1, then the actions α^{G_0} and α^{G_1} of Λ on A, as in Definition 4.1, are not weakly cocycle conjugate. Since the pro-p group is no longer fixed, we again use superscripts (for the actions) and subscripts (for the algebras) to keep track of which pro-p group they come from.

Theorem 4.5. *Let the notation be as in Definition 4.1 and Proposition 4.2. Fix $\tau_0 \in T(A)$ and set $\tau = \iota(\tau)$. Let G_0 and G_1 be second countable abelian pro-p groups. The following assertions are equivalent:*

(1) The groups G_0 and G_1 are topologically isomorphic;
(2) The Λ-actions α^{G_0} and α^{G_1} are conjugate;
(3) The Λ-actions α^{G_0} and α^{G_1} are cocycle conjugate;
(4) The Λ-actions α^{G_0} and α^{G_1} are weakly cocycle conjugate;
(5) The Λ-actions α^{G_0} and α^{G_1} are τ-conjugate;
(6) The Λ-actions α^{G_0} and α^{G_1} are cocycle τ-conjugate;
(7) The Λ-actions α^{G_0} and α^{G_1} are weakly cocycle τ-conjugate;
(8) The Δ-actions θ^{G_0} and θ^{G_1} are weakly cocycle τ-conjugate.

Proof. Since G_0 and G_1 are pro-p groups, there are isomorphisms $D_{G_0} \cong D_{G_1} \cong M_{p^\infty}$, and we denote this algebra by D. Any isomorphism $G_0 \cong G_1$ is immediately seen to induce an equivariant *-isomorphism $(D, \delta^{G_0}) \cong (D, \delta^{G_1})$. From this, it is easy to construct an equivariant *-isomorphism $(B_{G_0}, \alpha^{G_0}) \cong (B_{G_1}, \alpha^{G_1})$. This proves the implication (1)⇒(2). It is clear that (2) implies (3) and (4). It also implies (5), (6), and (7) because τ is α^{G_0}- and α^{G_1}-invariant by Proposition 4.2. Furthermore, either of (3), (4), (5), (6), (7) implies (8). Therefore, it only remains to prove that (8) implies (1).

Suppose that θ^{G_0} and θ^{G_1} are weakly cocycle τ-conjugate. Let τ_D denote the unique trace on D, and set $\widetilde{\tau} = (\tau_D \otimes \tau_0)^{\otimes\Lambda}$, which is a trace on $(D \otimes A)^{\otimes\Lambda}$. By part (1) of Proposition 4.3, for $i = 0, 1$, the weak extension of θ^{G_i} with respect to τ can be identified with the weak extension of $\widetilde{\theta}^{G_i}$ with respect to $\widetilde{\tau}$. (The action $\widetilde{\theta}^{G_i}$ is described before Proposition 4.3.) Hence, it suffices to show that $\widetilde{\theta}^{G_0}$ and $\widetilde{\theta}^{G_1}$ are not weakly cocycle $\widetilde{\tau}$-conjugate. Since the trace $\widetilde{\tau}$ is fixed, we will omit it from the notation for weak closures and weak extensions. With a slight abuse of notation, we will write θ^{G_i} to mean $\widetilde{\theta}^{G_i}$. This way, the symbol $\overline{\theta}^{G_i}$ will always represent the weak extension of $\widetilde{\theta}^{G_i}$ with respect to the trace $\widetilde{\tau}$. Let ζ be the Bernoulli $(\Delta \curvearrowright \Lambda)$-action with base $D \otimes A$. We also denote the algebra $\overline{(D \otimes A)}^{\otimes\Lambda}$ by N (omitting the trace $\tau_D \otimes \tau_0$), and abbreviate the G_i-action $(\overline{\delta}^{G_i} \otimes \mathrm{id}_R)^{\overline{\otimes}\Lambda}$ on N to $\rho^{(i)}\colon G_i \to \mathrm{Aut}(N)$. (In particular, $\overline{B}_{G_i} = N^{\rho_i}$.) We let Γ_i be the dual group of G_i.

Let $\psi\colon \overline{B}_{G_0} \to \overline{B}_{G_1}$ be an isomorphism and let $w\colon \Delta \to U(\overline{B}_{G_1})$ be a weak 1-cocycle satisfying

$$\mathrm{Ad}(w_\gamma) \circ \overline{\theta}^{G_1}_\gamma = \psi \circ \overline{\theta}^{G_0}_\gamma \circ \psi^{-1} \tag{3}$$

for every $\gamma \in \Delta$. Using Popa's superrigidity theorem [35, Theorem 4.1] in the case of weak 1-cocycles as in the proof of Theorem 4.4, one can find unitaries $z \in U(\overline{B}_{G_1})$ and $v \in U(N)$, and a character $\chi \in \Gamma_1$ such that

$$w_\gamma = z^* v^* \overline{\zeta}_\gamma(v) \overline{\theta}_\gamma(z) \ \mathrm{mod}\mathbb{C} \quad \text{and} \quad \rho^{(1)}_g(v) = \chi(g) v$$

for every $\gamma \in \Delta$ and for every $g \in G_1$. Therefore, upon replacing ψ with $\psi \circ \mathrm{Ad}(z^*)$, we can assume that $z = 1$ and $w_\gamma = v^* \overline{\zeta}_\gamma(v)$ for every $\gamma \in \Delta$.

Next, we want to define a bijection $\varphi\colon H^1_w(\overline{\theta}^{G_0}) \to H^1_w(\overline{\theta}^{G_1})$. Given a function $u\colon \Delta \to U(\overline{B}_{G_1})$, define $\psi(u)w\colon \Delta \to U(\overline{B}_{G_1})$ to be the function given by $(\psi(u)w)_\gamma = \psi(u_\gamma)w_\gamma$ for all $\gamma \in \Delta$.

Claim. *If $u \in Z^1_w(\overline{\theta}^{G_0})$, then $\psi(u)w \in Z^1_w(\overline{\theta}^{G_1})$.*

Proof of claim. Let $\gamma, \sigma \in \Delta$. In the following computation (where all equalities are up to scalars), we use the fact that w is a weak 1-cocycle for $\overline{\theta}^{G_1}$ at the first step, and equation (3) at the second step, to get

$$\psi(u_{\gamma\sigma}) = \psi(u_\gamma)(\psi \circ \overline{\theta}^{G_0}_\gamma \circ \psi^{-1})(\psi(u_\sigma)) = \psi(u_\gamma)(\mathrm{Ad}(w_\gamma) \circ \overline{\theta}^{G_1}_\gamma)(\psi(u_\sigma)) = \psi(u_\gamma) w_\gamma \overline{\theta}^{G_1}_\gamma(\psi(u_\sigma)) w_\gamma^* \ \mathrm{mod}\mathbb{C}$$

Therefore, using the above identity and the fact that u is a weak 1-cocycle for $\overline{\theta}^{G_0}$, we deduce that

$$\psi(u_{\gamma\sigma}) w_{\gamma\sigma} = \psi(u_\gamma) w_\gamma \overline{\theta}^{G_1}_\gamma(\psi(u_\sigma)) w_\gamma^* w_\gamma \overline{\theta}^{G_1}_\gamma(w_\sigma) = \psi(u_\gamma) w_\gamma \overline{\theta}^{G_1}_\gamma(\psi(u_\sigma) w_\sigma) \ \mathrm{mod}\mathbb{C}.$$

This shows that $\psi(u)w$ is a weak 1-cocycle for $\overline{\theta}^{G_1}$, proving the claim. □

It follows that there is a well-defined map $\widehat{\psi}\colon Z^1_w(\overline{\theta}^{G_0}) \to Z^1_w(\overline{\theta}^{G_1})$ given by $\widehat{\psi}(u) = \psi(u)w$ for $u \in Z^1_w(\overline{\theta}^{G_0})$.

Claim. *If $u, u' \in H^1_w(\overline{\theta}^{G_0})$ are cohomologous, then so are $\widehat{\psi}(u)$ and $\widehat{\psi}(u')$.*

Proof of claim. Find a unitary $z \in U(\overline{B}_{G_0})$ satisfying $u'_\gamma = z^* u_\gamma \overline{\theta}^{G_0}_\gamma(z)$ mod$\mathbb{C}$. Then

$$\psi(u'_\gamma) w_\gamma = \psi(z)^* \psi(u_\gamma)(\psi \circ \overline{\theta}^{G_0}_\gamma \circ \psi^{-1})(z) w_\gamma = \psi(z)^* \psi(u_\gamma)(\mathrm{Ad}(w_\gamma) \circ \overline{\theta}^{G_1}_\gamma)(z) w_\gamma = \psi(z)^* \psi(u_\gamma) w_\gamma \overline{\theta}^{G_1}_\gamma(z) \ \mathrm{mod}\mathbb{C}$$

for all $\gamma \in \Delta$. This shows that $\psi(u')w$ and $\psi(u)w$ are weakly cohomologous. □

It follows that $\widehat{\psi}$ induces a well-defined map $\varphi\colon H^1_w(\overline{\theta}^{G_0}) \to H^1_w(\overline{\theta}^{G_1})$.

Claim. *The map φ is invertible.*

Proof of claim. It follows from equation (3) that

$$\overline{\theta}^{G_0}_\gamma = \psi^{-1} \circ \mathrm{Ad}(w_\gamma) \circ \overline{\theta}^{G_1}_\gamma \circ \psi = \mathrm{Ad}(\psi^{-1}(w_\gamma)) \circ \psi^{-1} \circ \overline{\theta}^{G_0}_\gamma \circ \psi$$

for all $\gamma \in \Delta$. Therefore, the same argument as before shows that the function that assigns to the cocycle u for $\overline{\theta}^{G_1}$ the cocycle $\gamma \mapsto \psi^{-1}(u_\gamma w_\gamma)$ for $\overline{\theta}^{G_0}$ induces a well-defined function $H^1_w(\overline{\theta}^{G_1}) \to H^1_w(\overline{\theta}^{G_0})$, which is easily seen to be the inverse of φ. This proves the claim. ☐

Similarly as above, we define a bijection $\varphi^{(2)}\colon H^1_w(\overline{\theta}^{G_0} \otimes \overline{\theta}^{G_0}) \to H^1_w(\overline{\theta}^{G_1} \otimes \overline{\theta}^{G_1})$, by

$$\varphi^{(2)}([u]) = [(\psi \otimes \psi)(u)(w \otimes w)]$$

for all $u \in Z^1_w(\overline{\theta} \otimes \overline{\theta})$, where $(\psi \otimes \psi)(u)(w \otimes w)\colon \Delta \to U(\overline{B}_{G_1})$ is the weak 1-cocycle for $\overline{\theta}^{G_1} \otimes \overline{\theta}^{G_1}$ given by $\gamma \mapsto (\psi \otimes \psi)(u_\gamma)(w_\gamma \otimes w_\gamma)$ for all $\gamma \in \Delta$. Moreover, a routine calculation shows that $\varphi^{(2)} \circ m^{\overline{\theta}^{G_0}} = m^{\overline{\theta}^{G_1}} \circ \varphi$. For $i \in \{0, 1\}$, let $\eta_{G_i}\colon H^1_w(\overline{\theta}^{G_i}) \to \Gamma_i$ and $\eta^{(2)}_{G_i}\colon H^1_w(\overline{\theta}^{G_i} \otimes \overline{\theta}^{G_i}) \to \Gamma_i$ be the maps from Theorem 4.4, and set

$$\pi = \eta_{G_1} \circ \varphi \circ \eta^{-1}_{G_0}\colon \Gamma_0 \to \Gamma_1 \quad \text{and} \quad \pi^{(2)} = \eta^{(2)}_{G_1} \circ \varphi^{(2)} \circ (\eta^{(2)}_{G_0})^{-1}\colon \Gamma_0 \to \Gamma_1.$$

By Theorem 4.4, the following diagram is commutative:

$$\begin{array}{ccccccc}
\Gamma_0 & \xleftarrow{\eta_{G_0} \times \eta_{G_0}} & H^1_w(\overline{\theta}^{G_0}) \times H^1_w(\overline{\theta}^{G_0}) & \xrightarrow{m^{\overline{\theta}^{G_0}}} & H^1_w(\overline{\theta}^{G_0} \otimes \overline{\theta}^{G_0}) & \xrightarrow{\eta^{(2)}_{G_0}} & \Gamma_0 \\
\downarrow{\scriptstyle \pi \times \pi} & & \downarrow{\scriptstyle \varphi \times \varphi} & & \downarrow{\scriptstyle \varphi^{(2)}} & & \downarrow{\scriptstyle \pi^{(2)}} \\
\Gamma_1 & \xleftarrow[\eta_{G_1} \times \eta_{G_1}]{} & H^1_w(\overline{\theta}^{G_1}) \times H^1_w(\overline{\theta}^{G_1}) & \xrightarrow[m^{\overline{\theta}^{G_1}}]{} & H^1_w(\overline{\theta}^{G_1} \otimes \overline{\theta}^{G_1}) & \xrightarrow{\eta^{(2)}_{G_1}} & \Gamma_1.
\end{array}$$

Recall that χ denotes the character of G_1 associated with the weak 1-cocycle w for $\overline{\alpha}^{G_1}$. Then $\pi^{(2)}(\omega\omega') = \pi(\omega)\pi(\omega')$ and $\pi(1_{\Gamma_0}) = \chi$. It follows that $\pi^{(2)}(\omega) = \pi(\omega)\chi$ for every $\omega \in \Gamma_0$. Therefore the map $\widetilde{\pi}\colon \Gamma_0 \to \Gamma_1$ given by $\widetilde{\pi}(\omega) = \pi(\omega)\chi^{-1}$ for all $\omega \in \Gamma_0$, is a group isomorphism. Indeed, we have

$$\pi(\omega)\chi^{-1}\pi(\omega')\chi^{-1} = \pi^{(2)}(\omega\omega')\chi^{-2} = \pi(\omega\omega')\chi^{-1}$$

for $\omega, \omega' \in \Gamma_0$. Since clearly $\widetilde{\pi}$ is a bijection, we conclude that $\widetilde{\pi}$ is a group isomorphism, and hence $\Gamma_0 \cong \Gamma_1$. By Pontryagin duality, we conclude that $G_0 \cong G_1$, and the proof is finished. ☐

We now arrive at the main result of this section. Its conclusion will be significantly strengthened in Corollary 5.10.

Theorem 4.6. *Let Λ be a countable discrete group with an infinite relative property* (T) *subgroup, let p be a prime number, and let A be separable, locally reflexive, M_{p^∞}-absorbing, unital C*-algebra admitting an amenable trace, and such that $A \cong A^{\otimes \mathbb{N}}$. Then there exists a continuum $(\alpha^{(t)})_{t \in \mathbb{R}}$ of pairwise not weakly cocycle conjugate, free (and, in particular, strongly outer) actions of Λ on A. In fact, there exists an amenable, extreme trace τ that is invariant under $\alpha^{(t)}$ for every $t \in \mathbb{R}$, and such that the actions $\alpha^{(t)}$ are all τ-mixing and pairwise not weakly cocycle τ-conjugate.*

Proof. Let $(G_t)_{t \in \mathbb{R}}$ be a continuum family of pairwise nonisomorphic abelian pro-p groups. For $t \in \mathbb{R}$, set $\alpha^{(t)} := \alpha^{G_t}$, where $\alpha^{G_t} : \Lambda \to \mathrm{Aut}(A)$ is the action of Λ on A given by Definition 4.1. By part (4) of Proposition 4.3, $\alpha^{(t)}$ is free. Since A has an amenable trace and $T_{\mathrm{am}}(A)$ is a face in the simplex $T(A)$, there exists an extreme, amenable trace τ_0 on A. Let $\iota\colon T(A) \to T(A)$ be the map from Proposition 4.2. Then $\tau = \iota(\tau_0)$ is extreme and amenable, and it is $\alpha^{(t)}$-invariant for every $t \in \mathbb{R}$ by Proposition 4.2. By part (5) of Proposition 4.3, $\alpha^{(t)}$ is τ-mixing for every $t \in \mathbb{R}$. Finally, Theorem 4.5 implies that the weak extensions of the $\alpha^{(t)}$ to $\overline{A}^\tau$ are pairwise not weakly cocycle conjugate. This concludes the proof. ☐

We make some comments on the assumptions of the theorem above. First, subgroups with relative property (T) are abundant: if either Λ or Δ has property (T), then the inclusion $\Delta \subseteq \Lambda$ has relative property (T). On the other hand, it is easy to find many C*-algebras satisfying the assumptions of Theorem 4.6. Indeed, if A_0 is any separable, unital, exact C*-algebra with an amenable trace, then $A = M^\infty_p \otimes A_0^{\otimes \mathbb{N}}$ satisfies the assumptions of said theorem. In particular, A_0 and A need not be simple. We also remark that every trace on a nuclear C*-algebra is necessarily amenable.

To end this section, we explicitly state our result for UHF-algebras, to highlight the contrast with the results in [27, 29, 38].

Corollary 4.7. *Let D be a UHF-algebra of infinite type, and let Λ be a countable group with an infinite subgroup with relative property (T). Then there exists a continuum of pairwise non (weakly) cocycle-conjugate, strongly outer actions of Λ on D.*

5. Conjugacy, cocycle conjugacy, and weak cocycle conjugacy are not Borel

In this section, we discuss how the construction from Section 4 can be used to prove that, under the assumptions of Theorem 4.6, conjugacy, cocycle conjugacy, and weak cocycle conjugacy of free actions of Λ on A are complete analytic sets.

5.1. **Borel complexity of equivalence relations.** We recall here some notions from Borel complexity theory. In this setting, a *classification problem* is identified with an equivalence relation E on a Polish space X. Virtually any concrete classification problem in mathematics is of this form, perhaps after a suitable parameterization. For example, a countable discrete group can be identified with a set of triples of natural numbers, coding a group operation on $\mathbb{N}$. The space of such sets of triples is a G_δ subset of the compact metrizable space $\{0,1\}^{\mathbb{N}^3}$ endowed with the product topology. (A G_δ subspace of a Polish space is Polish by [24, Theorem 3.11].)

Definition 5.1. (See [13, Definitions 5.1.1 and 5.1.2]). A *Borel reduction* from an equivalence relation E on a Polish space X to an equivalence relation F on a Polish space Y is a Borel function $f\colon X \to Y$ such that $[x]_E \mapsto [f(x)]_F$ is a well-defined injective function from the space X/E of E-classes to the space Y/F of F-classes. The equivalence relation E is said to be *Borel reducible* to F, in formulas $E \leq_B F$, if there exists a Borel reduction from E to F.

Remark 5.2. When E is Borel reducible to F, the objects of X up to E can be *explicitly* classified using F-classes as complete invariants. In other words, the classification problem represented by F is *at least as complex* as the classification problem represented by E. (Observe that this notion does not depend on the topologies of X and Y, but only on the standard Borel structures that they induce.)

The notion of Borel reducibility can be used to measure the complexity of a given classification problem. The first natural measure of complexity is simply the *number of classes* of the corresponding equivalence relation. Theorem 4.6 addresses this problem in the case of conjugacy, cocycle conjugacy, and weak cocycle conjugacy of free actions of Λ on A: they have a continuum of equivalence classes.

The natural next step in the study of the complexity of a classification problem consists in determining whether the classes can be *explicitly* parameterized as the points of a Polish space. This is equivalent to the corresponding equivalence relation being *smooth*, that is, Borel reducible to the relation of *equality* in some Polish space. As an example, Glimm's classification of separable UHF-algebras implies that the relation of *-isomorphism for these algebras is smooth (even though there exists a continuum of isomorphism classes). Similarly, the orbit equivalence relation of a continuous action of a compact group on a Polish space is smooth. However, isomorphism of countable rank-one torsion-free abelian groups, for instance, is not smooth. Another canonical example of a nonsmooth equivalence relation is the relation of *tail equivalence* for binary sequences.

A more generous notion of being well-behaved for an equivalence relation E on X is being *Borel* as a subset of $X \times X$. For instance, isomorphism of countable rank-one torsion-free abelian groups is Borel. Similarly, tail equivalence of binary sequences is Borel and, more generally, the orbit equivalence relation of a *free* continuous action of a Polish group on a Polish space is Borel. (The orbit equivalence relation of a continuous action of a Polish group G on a Polish pace X is Borel if and only if the map that assigns to each point x of X the corresponding *stabilizer subgroup* G_x of G is Borel; see [1, 7.1.2].) Since the relation of equality on any Polish space is clearly Borel, any smooth equivalence relation is, in particular, Borel.

One can also define a similar notion of comparison among *sets*, rather than equivalence relations.

Definition 5.3. (See [24, Section 14.A and Definition 26.7].) A subset A of a Polish space X is said to be *analytic*, or $\boldsymbol{\Sigma}^1_1$, if there exist a Polish space Z and a Borel function $f\colon Z \to X$ such that A is the image under f of a Borel subset of Z. A *complete analytic set* (also called $\boldsymbol{\Sigma}^1_1$-complete set) is an analytic subset A of a Polish space X such that, for any other analytic subset B of a Polish space Y, there exists a Borel function $f\colon Y \to X$ such that $f^{-1}(A) = B$.

We recall here the fundamental fact that a complete analytic set is not Borel. The canonical example of a complete analytic set is the set of ill-founded trees on $\mathbb{N}$; see [24, Section 27.A].

As above, we regard an equivalence relation E on a Polish space X as a subset of the product space $X \times X$ endowed with the product topology. Consistently, we say that E is a complete analytic set if it is complete analytic as a subset of $X \times X$. It is clear that if E is Borel reducible to an equivalence relation F, and E is a complete analytic set, then F is a complete analytic set as well.

In Theorem 5.9, we will prove that the construction of actions from profinite groups described in Section 4 can be used to show that, under the assumption of Theorem 4.6, the relations of conjugacy, cocycle conjugacy, and weak cocycle conjugacy of free actions of Λ on A are complete analytic sets. This is a significant strengthening of the conclusions of Theorem 4.6.

5.2. **Parametrizing actions.** For the rest of this section, we fix a countable discrete group Λ and a separable unital C*-algebra A. We proceed to explain how the classification problem for free actions of Λ on A can be naturally regarded as equivalence relations on a Polish space. We regard $\mathrm{T}(A)$ as a compact metrizable space endowed with the w*-topology. The set $\mathrm{Act}_\Lambda(A)$ of actions of Λ on A is a closed subset of the product space $\mathrm{Aut}(A)^\Lambda$ endowed with the product topology, giving it the structure of a Polish space.

Notation 5.4. *Let τ be a trace on A.*

- *We denote by $\mathrm{Act}_\Lambda(A,\tau)$ the set of τ-preserving actions of Λ on A;*
- *We denote by $\mathrm{WM}_\Lambda(A,\tau)$ the set of τ-preserving weakly τ-mixing actions of Λ on A;*
- *We denote by $\mathrm{F}_\Lambda(A)$ the set of free actions of Λ on A;*
- *We denote by $\mathrm{FWM}_\Lambda(A,\tau)$ the set of τ-preserving weakly τ-mixing free actions of Λ on A.*

It is easy to see that $\mathrm{Act}_\Lambda(A,\tau)$ and $\mathrm{WM}_\Lambda(A,\tau)$ are G_δ subsets of $\mathrm{Act}_\Lambda(A)$. We will show below that $\mathrm{F}_\Lambda(A)$ and $\mathrm{FWM}_\Lambda(A,\tau)$ are also G_δ subsets of $\mathrm{Act}_\Lambda(A)$.

Given a C*-algebra A, we let A_{sa} be the set of selfadjoint elements of A. An element a of A_{sa} is a contraction if $\|a\| \leq 1$. Given a trace τ on A, we let $\|a\|_\tau = \sqrt{\tau(a^*a)}$ be the 2-norm induced by τ on A and $\overline{A}^\tau$. Using Borel functional calculus [2, Section I.4.3], we fix a continuous function $\varpi\colon [0,+\infty) \to [0,+\infty)$ satisfying the following properties:

- $\varpi(0) = 0$ and $\varpi(t) \geq t$ for all $t \in [0,\infty)$;
- Let (M,τ) be a tracial von Neumann algebra, let $a \in M_{\mathrm{sa}}$ be a contraction, and let $\varepsilon > 0$. If $\|a^2 - a\|_\tau < \varepsilon$, then there exists a projection $p \in M$ such that $\|p - a\|_\tau < \varpi(\varepsilon)$.

Lemma 5.5. Let A be a unital, separable C*-algebra, let $\theta \in \mathrm{Aut}(A)$ be an automorphism of A, and let τ be a θ-invariant trace on A. Fix a countable dense subset A_0 of the unit ball of A_{sa}. Then $\overline{\theta}^\tau$ is properly outer if and only if for every $a \in A_0$, and every $\varepsilon \in (0,+\infty) \cap \mathbb{Q}$ satisfying $\|a^2 - a\|_\tau \leq \varepsilon$, there exists a contraction $b \in A_{\mathrm{sa}}$ satisfying

$$\|b^2 - b\|_\tau < \varepsilon, \quad \|ab - b\|_\tau < \varpi(\varepsilon), \quad \|b\theta(b)\|_\tau < \varepsilon, \quad \text{and} \quad \tau(b) > \frac{1}{3}\tau(a) - \varepsilon.$$

Proof. Let $\pi_\tau : A \to B\left(L^2(A,\tau)\right)$ be the GNS representation associated with τ. Suppose that $\overline{\theta}^\tau$ is properly outer. Let $\varepsilon \in (0,\infty) \cap \mathbb{Q}$, and let $a \in A_0$ satisfy $\|a^2 - a\|_\tau < \varepsilon$. By the choice of ϖ, there exists a projection $p \in \overline{A}^\tau$ such that $\|\pi_\tau(a) - p\|_\tau < \varpi(\varepsilon)$. By (1)⇒(4) in [26, Lemma 4.2], there exists a projection $q \in \overline{A}^\tau$ such that

$$q \leq p, \quad \|q\overline{\theta}^\tau(q)\|_\tau < \varepsilon, \quad \text{and} \quad \tau(q) \geq \frac{1}{3}\tau(p) > \frac{1}{3}\tau(a) - \varepsilon.$$

Therefore, $\|q\pi_\tau(a) - q\|_\tau \leq \|q - qp\|_\tau + \|qp - q\pi_\tau(a)\|_\tau < \varpi(\varepsilon)$, and similarly $\|\pi_\tau(a)q - q\|_\tau < \varpi(\varepsilon)$. Since the norm-unit ball of A is $\|\cdot\|_\tau$-dense in the unit ball of $\overline{A}^\tau$, there exists a contraction $b \in A_{\mathrm{sa}}$ satisfying the conditions in the statement.

We prove the converse. We want to prove that $\overline{\theta}^\tau$ is properly outer. Fix a nonzero projection $p \in \overline{A}^\tau$ and $\varepsilon, \varepsilon_0 > 0$. By (4)⇒(1) in [26, Lemma 4.2], it is enough to prove that there exists a projection $q \in \overline{A}^\tau$ such that

$$q \leq p, \quad \|q\overline{\theta}^\tau(q)\|_\tau < \varepsilon, \quad \text{and} \quad \tau(q) \geq \frac{1}{3}\tau(p) > \frac{1}{3}\tau(a) - \varepsilon.$$

Let $a \in A_0$ satisfy $\|\pi_\tau(a) - p\|_\tau < \varepsilon$ and $\left\|a^2 - a\right\|_\tau < \varepsilon$. By assumption, there exists a contraction $b \in A_{\mathrm{sa}}$ with

$$\|b^2 - b\|_\tau < \varepsilon, \quad \|ab - b\|_\tau < \varpi(\varepsilon) \quad \|b\theta(b)\|_\tau < \varepsilon, \quad \text{and} \quad \tau(b) > \frac{1}{3}\tau(a) - \varepsilon.$$

By the choice of ϖ, there exists a projection $r \in \overline{A}^\tau$ such that $\|\pi_\tau(r) - b\|_\tau < \varpi(\varepsilon)$. By choosing ε small enough, one can ensure that

$$\|pr - r\|_\tau < \varepsilon_0, \quad \|rp - p\|_\tau < \varepsilon_0, \quad \left\|r\overline{\theta}^\tau(r)\right\|_\tau < \varepsilon_0, \quad \text{and} \quad \tau(r) > \frac{1}{3}\tau(a) - \varepsilon_0.$$

By choosing ε_0 small enough, one can then find a projection $q \in \overline{A}^{\tau}$ satisfying the conditions in item (4) of [26, Lemma 4.2] mentioned above. This concludes the proof. ☐

For convenience, we record the following easy lemma. For a relation $\mathcal{R} \subset X \times Y$, its *projection onto* X is

$$\mathrm{proj}_X(\mathcal{R}) = \{x \in X \colon \text{ there is } y \in Y \text{ with } (x,y) \in \mathcal{R}\}.$$

Lemma 5.6. Let X be a Polish space, let Y be a compact metrizable space, and let $\mathcal{R} \subset X \times Y$ be a subset. If $\mathcal{R}$ is closed, then $\mathrm{proj}_X(\mathcal{R})$ is closed. If $\mathcal{R}$ is F_σ, then $\mathrm{proj}_X(\mathcal{R})$ is F_σ.

Proof. It is enough to prove the first assertion, so assume that $\mathcal{R}$ is closed. Let $(x_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{proj}_X(\mathcal{R})$ converging to $x \in X$. Our goal is to prove that $x \in \mathrm{proj}_X(\mathcal{R})$. For every $n \in \mathbb{N}$, let $y_n \in Y$ satisfy $(x_n, y_n) \in \mathcal{R}$. Since Y is compact, after passing to a subsequence, we can assume that the sequence $(y_n)_{n\in\mathbb{N}}$ converges to some $y \in Y$. Since $\mathcal{R}$ is closed, we have $(x,y) \in \mathcal{R}$ and hence $x \in \mathrm{proj}_X(\mathcal{R})$, as desired. ☐

Recall the definitions of the sets $\mathrm{F}_\Lambda(A)$ and $\mathrm{FWM}_\Lambda(A)$ from Notation 5.4.

Proposition 5.7. *Let A be a unital, separable C*-algebra, let Λ be a countable group, and let τ be a trace on A. Then the sets $\mathrm{F}_\Lambda(A)$ and $\mathrm{FWM}_\Lambda(A,\tau)$ are G_δ subsets of $\mathrm{Act}_\Lambda(A)$.*

Proof. Fix $\gamma \in \Lambda$. Let $\mathcal{R}_\gamma$ be the set of pairs $(\alpha, \tau) \in \mathrm{Act}_\Lambda(A) \times \mathrm{T}(A)$ such that τ is α_γ-invariant and $\overline{\alpha}^{\tau}_{\gamma}$ is *not* properly outer. By Lemma 5.5, $\mathcal{R}_\gamma$ is an F_σ subset of 5.5. By 5.6, its projection $\mathcal{P}_\gamma$ onto $\mathrm{Act}_\Lambda(A)$ is F_σ as well. Let $\mathcal{C}_\gamma$ be the complement of $\mathcal{P}_\gamma$ in $\mathrm{Act}_\Lambda(A)$, which is G_δ. We have that $\mathrm{F}_\Lambda(A)$ is the intersection of $\mathcal{C}_\gamma$ for $\gamma \in \Lambda$, and hence G_δ. We have already observed that $\mathrm{WM}_\Lambda(A,\tau)$ is G_δ. Therefore $\mathrm{FWM}_\Lambda(A,\tau) = \mathrm{WM}_\Lambda(A,\tau) \cap \mathrm{F}_\Lambda(A)$ is G_δ as well. ☐

Adopt the notation of the lemma above. We regard $\mathrm{F}_\Lambda(A)$ as the Polish space of free actions of Λ on A. Consistently, we regard the classification problems for free actions of Λ on A up to conjugacy, cocycle conjugacy, or weak cocycle conjugacy, as equivalence relations on $\mathrm{F}_\Lambda(A)$. Similarly, if τ is a trace on A, we regard $\mathrm{FWM}_\Lambda(A,\tau)$ as the space of τ-preserving weakly τ-mixing free actions of Λ on A. On the latter space we can also consider the relations of τ-conjugacy, cocycle τ-conjugacy, and outer τ-conjugacy.

5.3. **Parametrizing abelian pro-p groups.** Fix a prime number p. In this subsection, we define a compact metrizable space parametrizing in a canonical way all second countable abelian pro-p groups. The construction is analogous to the one from [32, Section 2.2].

Let $\mathbb{Z}^\infty$ be the free abelian group on a countably infinite set $\{x_k \colon k \in \mathbb{N}\}$ of generators. Let $\mathcal{N}$ be the (*countable*) collection of finite index subgroups of $\mathbb{Z}^\infty$ whose index is a multiple of p, and which contain all but finitely many of the generators of $\mathbb{Z}^\infty$. We consider $\mathbb{Z}^\infty$ as a topological group having the elements of $\mathcal{N}$ as basis of neighborhoods of the identity. Define $\hat{\mathbb{Z}}_p^\infty$ to be the completion of $\mathbb{Z}^\infty$ with respect to such a topology, which is a second countable abelian pro-p group. In the terminology of [36, Section 3.3], $\hat{\mathbb{Z}}_p^\infty$ is the free abelian pro-p group on a sequence of generators $(x_k)_{k\in\mathbb{N}}$ converging to 1.

Suppose that G is a second countable abelian pro-p group. By [36, Proposition 2.4.4 and Proposition 2.6.1] G has a generating sequence converging to the identity. It therefore follows from [36, Section 3.3.16] that there exists a surjective continuous group homomorphism $\pi \colon \hat{\mathbb{Z}}_p^\infty \to G$. In other words, G is isomorphic to the quotient of $\hat{\mathbb{Z}}_p^\infty$ by a closed subgroup. Conversely, any quotient of $\hat{\mathbb{Z}}_p^\infty$ by a closed subgroup is a second-countable abelian pro-p group. Thus the closed subgroups of $\hat{\mathbb{Z}}_p^\infty$ naturally parametrize all second-countable abelian pro-p groups.

We let $\mathcal{K}(\hat{\mathbb{Z}}_p^\infty)$ be the space of closed subsets of $\hat{\mathbb{Z}}_p^\infty$ endowed with the Vietoris topology [24, Section 4.F], which turns it into a compact metrizable space. Let also $\mathcal{S}(\hat{\mathbb{Z}}_p^\infty) \subseteq \mathcal{K}(\hat{\mathbb{Z}}_p^\infty)$ be the (closed) subset of closed subgroups of $\hat{\mathbb{Z}}_p^\infty$. Then $\mathcal{S}(\hat{\mathbb{Z}}_p^\infty)$ is a compact metrizable space with the relative topology. We regard isomorphism of second-countable abelian pro-p groups as an equivalence relation on $\mathcal{S}(\hat{\mathbb{Z}}_p^\infty)$.

Proposition 5.8. *Let p be a prime number. The relation of topological isomorphism of second-countable abelian pro-p groups is a complete analytic set.*

Proof. As it is observed in [32], Pontryagin's duality theorem in the special case of profinite *abelian* groups is witnessed by a Borel map. Furthermore, the duals of abelian pro-p groups are precisely the countable abelian p-groups; see [36, Theorem 2.9.6 and Lemma 2.9.3]. Therefore, the relation of isomorphism of countable abelian p-groups is Borel reducible (in fact, Borel isomorphic) to the relation of isomorphism of second-countable abelian pro-p groups. Since the relation of isomorphism of countable abelian p-groups is a complete analytic set [12, Theorem 6], the result follows. ☐

5.4. **Reducing groups to actions.** In this last subsection, we obtain the main results of this work. Recall that for a discrete group Λ, a separable C*-algebra A, and a trace τ on A, we denote by $\mathrm{FWM}_\Lambda(A,\tau)$ the Polish space of τ-preserving free weakly τ-mixing actions of Λ on A. Below, we will assume all the C*-algebras to be *separable*. In the proof of the following theorem we will tacitly use the fact—proved in [10, 11, 17]—that tensor products, direct limits, and crossed products of C*-algebras and C*-dynamical systems are given by Borel functions with respect to the parametrizations of C*-algebras and C*-dynamical systems considered in [10, 11, 17]. It is also not difficult to see that fixed point algebras of actions of *compact* groups on C*-algebras can be computed in a Borel way.

Theorem 5.9. *Let Λ be a countable group containing an infinite relative property* (T) *subgroup. Fix a prime number p. Let A be a separable, locally reflexive, M_{p^∞}-absorbing, unital C*-algebra with an amenable trace, satisfying $A \cong A^{\otimes \mathbb{N}}$. Then there exists an extreme, amenable trace τ on A such that the relation of isomorphism of second-countable abelian pro-p groups is Borel reducible to the following equivalence relations on* $\mathrm{FWM}_\Lambda(A,\tau)$*:*

(1) conjugacy;
(2) cocycle conjugacy;
(3) weak cocycle conjugacy;
(4) τ-conjugacy;
(5) cocycle τ-conjugacy;
(6) weak cocycle τ-conjugacy

Proof. In view of Theorem 4.5, and parts (4) and (5) of Proposition 4.3, it is enough to prove that the function $G \mapsto \alpha^G$ that assigns to a second-countable abelian pro-p group G the action $\alpha^G : \Lambda \to \mathrm{Aut}\,(A)$ from Definition 4.1, is given by a Borel function with respect to the parametrization of second-countable abelian pro-p groups and free actions of Λ on A described in Subsection 5.2 and Subsection 5.3.

Recall that the action α^G is defined by

$$\alpha^G_\gamma = \xi_G^{-1} \circ (\beta_{\Lambda \curvearrowright \Lambda, M_p^\infty}|_{E_G} \otimes \sigma)_\gamma \circ \xi_G$$

for $\gamma \in \Lambda$, for some choice of isomorphism $\xi_G : E_G \otimes A \to A$. It is therefore enough to show that

(1) the assignment $G \mapsto E_G$ is given by a Borel function, and
(2) the isomorphism $\xi_G : E_G \otimes A \to A$ can be chosen in a Borel fashion from E_G.

We address the second assertion first. Recall that the K-theory of a C*-algebra can be computed in a Borel fashion [10, Section 3.3]. Furthermore, given a countable dimension group L one can construct in a Borel fashion a canonical AF algebra whose K_0-group is isomorphic to L. It follows from these considerations that, given a C*-algebra E abstractly isomorphic to M_{p^∞}, one can choose in a Borel fashion an isomorphism $\psi_E : E \to M_{p^\infty}$. Since $M_{p^\infty} \otimes A$ is isomorphic to A, by fixing an isomorphism $M_{p^\infty} \otimes A \cong A$ beforehand, one can choose in a Borel fashion an isomorphism $\xi_E \colon E \otimes A \to A$. This justifies the second assertion.

We now justify the first assertion. Recall that E_G is the fixed point algebra of the action $(\delta^G)^{\otimes \Lambda} \colon G \to \mathrm{Aut}(D_G^{\otimes \Lambda})$. Furthermore, $(\delta^G)^{\otimes \Lambda}$ is conjugate to canonical model action $\delta^G : G \to \mathrm{Aut}\,(D_G)$ of G constructed in Theorem 3.5. Therefore, it is enough to show that δ^G can be constructed in a Borel fashion from G. This is clear when G is finite in view of Remark 3.2. In the general case, consider the following. In our parametrization, a second-countable abelian pro-p group G is given as the quotient $\hat{\mathbb{Z}}_p^\infty/N$ of $\hat{\mathbb{Z}}_p^\infty$ by some closed subgroup N of $\hat{\mathbb{Z}}_p^\infty$. The finite-index closed subgroups of G correspond to finite-index closed subgroups H of $\hat{\mathbb{Z}}_p^\infty$ that contain N. By the Kuratowski–Ryll-Nardzewski selection theorem [24, Theorem 12.13], the collection $\mathcal{V}$ of finite-index closed subgroups H of $\hat{\mathbb{Z}}_p^\infty$ that contain N can be chose in a Borel fashion starting from N. Since the relation of inclusion between closed subgroups is closed with respect to the Vietoris topology, the order on $\mathcal{V}$ given by containment is Borel. This shows that the canonical inverse system $(G_i, \pi_{i,j})_{i,j\in\mathcal{V}}$ of finite groups having G as inverse limit considered in the proof of Theorem 3.5 depends on G in a Borel way. The G-C*-algebra (D_G, δ_G) is obtained in the proof of Theorem 3.5 as the direct limit of the direct system $\left((D_{G_i}, \delta_{G_i}), \iota_{ij}\right)_{i,j\in\mathcal{V}}$, where (D_{G_i}, δ_{G_i}) is the model action of the finite group G_i. It remains to observe now that the direct system $\left((D_{G_i}, \delta_{G_i}), \iota_{ij}\right)_{i,j\in\mathcal{V}}$ can be computed in a Borel fashion from $(G_i, \pi_{i,j})_{i,j\in\mathcal{V}}$. This concludes the proof. □

Corollary 5.10. *Under the hypotheses of Theorem 5.9, the relations of conjugacy, cocycle conjugacy, and weak cocycle conjugacy of free actions of Λ on A are complete analytic sets. Furthermore, there exists an amenable, extreme trace τ on A such that the relations of conjugacy, cocycle conjugacy, weak cocycle conjugacy, τ-conjugacy, cocycle τ-conjugacy, and weak cocycle τ-conjugacy of τ-preserving free weakly τ-mixing actions of Λ on A are complete analytic sets, and in particular not Borel.*

As mentioned after Theorem 4.6, it is easy to construct algebras A satisfied the hypotheses of Corollary 5.10. Indeed, if A_0 is any separable, unital, exact C*-algebra with an amenable trace, we may take $A = M_p^\infty \otimes A_0^{\otimes \mathbb{N}}$.

As we did after Theorem 4.6, we state the case of UHF-algebras separately, to highlight the contrast with the main results of [27, 29, 38].

Corollary 5.11. *Let D be a UHF-algebra of infinite type, and let Λ be a countable group with an infinite subgroup with relative property* (T)*. Then the relations of conjugacy, cocycle conjugacy, and weak cocycle conjugacy of strongly outer actions of Λ on D are complete analytic sets, and in particular not Borel. The same applies to the relations of being conjugate, cocycle conjugate, and weakly cocycle conjugate in the weak closure with respect to the (necessarily Λ-invariant) unique trace on D.*

In fact, the same conclusions hold for any finite, strongly self-absorbing C*-algebra containing a nontrivial projection; see [42, Definition 1.3] and [42, Theorem 1.7].

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