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Abstract Bivariant Cuntz Semigroups
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ABSTRACT BIVARIANT CUNTZ SEMIGROUPS

RAMON ANTOINE, FRANCESC PERERA, AND HANNES THIEL

Abstract. We show that abstract Cuntz semigroups form a closed symmetric monoidal category. Thus, given Cuntz semigroups $S$ and $T$, there is another Cuntz semigroup $\llbracket S, T \rrbracket$ playing the role of morphisms from $S$ to $T$. Applied to $C^*$-algebras $A$ and $B$, the semigroup $\llbracket Cu(A), Cu(B) \rrbracket$ should be considered as the target in analogues of the UCT for bivariant theories of Cuntz semigroups.

Abstract bivariant Cuntz semigroups are computable in a number of interesting cases. We explore its behaviour under the tensor product with the Cuntz semigroup of strongly self-absorbing $C^*$-algebras and the Jacelon-Razak algebra. We also show that order-zero maps between $C^*$-algebras naturally define elements in the respective bivariant Cuntz semigroup.

1. Introduction

The Cuntz semigroup $Cu(A)$ of a $C^*$-algebra $A$ is an invariant that plays an important role in the structure theory of $C^*$-algebras and the related Elliott classification program. It is defined analogously to the Murray-von Neumann semigroup, $V(A)$, by using equivalence classes of positive elements instead of projections; see [Cun78]. In general, however, the semigroup $Cu(A)$ contains much more information than $V(A)$, and it is therefore also more difficult to compute.

The Cuntz semigroup has been successfully used in the classification program, both in the simple and nonsimple setting. For example, Toms constructed two simple AH-algebras that have the same Elliott invariants, but which are nevertheless not isomorphic, a fact that is detected by the Cuntz semigroup; see [Tom08]. On the other hand, Robert classified (not necessarily simple) inductive limits of one-dimensional NCCW-complexes with trivial $K_1$-group using the Cuntz semigroup; see [Rob12].

The connection of the Cuntz semigroup with the Elliott invariant has been explored in a number of instances; see for instance [PT07], [BPT08] and [Tik11]. In fact, for the class of simple, unital, nuclear $C^*$-algebras that are $Z$-stable (that is, that tensorially absorb the Jiang-Su algebra $Z$), the Elliott invariant and the Cuntz semigroup together with the $K_1$-group determine one another functorially; see [ADPS14]. When dropping the assumption of $Z$-stability, it is not known whether the Elliott invariant together with the Cuntz semigroup provides a complete invariant for classification of simple, unital, nuclear $C^*$-algebras.

It is therefore very interesting to study the structural properties of the Cuntz semigroup of a $C^*$-algebra. This study was initiated by Coward, Elliott and Ivanescu in [CEI08], who introduced a category $Cu$ and showed that the assignment $A \mapsto Cu(A)$ is a sequentially continuous functor from $C^*$-algebras to $Cu$. The objects of $Cu$ are called abstract Cuntz semigroups or $Cu$-semigroups. Working in this category allows one to provide elegant algebraic proofs for structural properties of $C^*$-algebras.

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A systematic study of the category Cu was undertaken in [APT14]. One of the main results obtained is that Cu has a natural structure as symmetric monoidal category (see Paragraph 2.14 for more details). This means, in particular, that Cu admits tensor products and that there is a bifunctor
\[ \otimes : Cu \times Cu \to Cu \]
which is (up to natural isomorphisms) associative, symmetric, and has a unit object, namely the semigroup \( \mathbb{N} = \{0, 1, 2, \ldots, \infty\} \). The basic properties of this construction were studied in [APT14], relating in particular \( Cu(A \otimes B) \) with \( Cu(A) \otimes Cu(B) \) for certain classes of \( C^\ast \)-algebras.

Following the line of thought above, it is very natural to ask whether Cu is also a closed category. This problem was left open in [APT14 Chapter 9]. Given Cu-semigroups \( S \) and \( T \), the question is if there exists a Cu-semigroup \( [S, T] \) that plays the role of morphisms from \( S \) to \( T \). In category theory, this is expressed by requiring that the functor \( [T, -] \) is adjoint to the functor \( - \otimes T \), which means that for any other Cu-semigroup \( P \) we have a natural bijection
\[ Cu(S, [T, P]) \cong Cu(S \otimes T, P), \]
where \( Cu(\cdot, \cdot) \) denotes the set of morphisms in the category Cu. The morphisms in Cu, also called Cu-\( \text{morphisms} \), are order-preserving monoid maps that preserve supremas of increasing sequences and that preserve the so-called way-below relation; see [Definition 2.4].

One of the main objectives of this paper is to construct the Cu-semigroup \( [S, T] \) and to study its basic properties. We call \( [S, T] \) an abstract bivariant Cu-semigroup or a bivariant Cu-semigroup. The construction defines a bifunctor
\[ [\cdot, \cdot] : Cu \times Cu \to Cu, \]
referred to as the \( \text{internal-hom} \) bifunctor; see, for example, [Kel05].

An important motivation for our construction is to find an analogue of the universal coefficient theorem (UCT) for Cuntz semigroups. Recall that a separable \( C^\ast \)-algebra \( A \) is said to satisfy the UCT if for every separable \( C^\ast \)-algebra \( B \) there is a short exact sequence
\[ 0 \to \bigoplus_{i=0,1} \text{Ext} \left(K_i(A), K_i(B)\right) \to KK_0(A, B) \to \bigoplus_{i=0,1} \text{Hom} \left(K_i(A), K_i(B)\right) \to 0.\]
We refer to [Bla98 Chapter 23] for details.

The near goal is then to replace \( KK_0(A, B) \) by a suitable bivariant version of the Cuntz semigroup (for example, along the lines of [BTZ16]), and the Hom-functor in the category of abelian groups by the internal-hom functor \( [Cu(A), Cu(B)] \) alluded to above. In this direction, the construction developed in [BTZ16] uses certain equivalence classes of completely positive contractive (abbreviated c.p.c.) order-zero maps between \( C^\ast \)-algebras, denoted here as \( \text{cpc}_0(A, B) \). In Section 8 we show that every c.p.c. order-zero map \( \varphi : A \to B \) defines an element \( Cu(\varphi) \) in the abstract bivariant Cuntz semigroup \( [Cu(A), Cu(B)] \).

The construction of bivariant Cu-semigroups resorts to the use of a more general class of maps than just Cu-morphisms. A \textit{generalized Cu-morphism} is defined as an order-preserving monoid map that preserves supremas of increasing sequences (but not necessarily the way-below relation); see [Definition 2.3]. We denote the set of such maps by \( Cu[S, T] \). Since every Cu-morphism is also a generalized Cu-morphism, we have an inclusion \( Cu(S, T) \subseteq Cu[S, T] \).

When equipped with pointwise order and addition, \( Cu[S, T] \) has a natural structure as a partially ordered monoid, but it is in general not a Cu-semigroup. Similarly, \( Cu(S, T) \) is usually not a Cu-semigroup. The solution is to consider \textit{paths} in
Cu[S, T], that is, rationally indexed maps \( \mathbb{Q} \cap (0, 1) \to Cu[S, T] \) that are ‘rapidly increasing’ in a certain sense. Equipped with a suitable equivalence relation, these paths define the desired Cu-semigroup \( [S, T] \).

This procedure can be carried out in a much more general setting. In Section 4, we introduce a category \( Q \) of partially ordered semigroups that, roughly speaking, is a weakening of the category Cu, in that the way-below relation is replaced by a possibly different binary relation (called auxiliary relation). We show that Cu is a full subcategory of \( Q \); see Proposition 4.4. The path construction we have delineated above yields a covariant functor

\[ \tau: Q \to Cu, \]

that turns out to be right adjoint to the natural inclusion functor; see Theorem 4.12.

We refer to this functor as the \( \tau \)-construction. This result has numerous advantages. Besides making clearer arguments of the results in the present paper available than going via a direct argument, it allows to transport many categorical properties from \( Q \) to Cu. Some of these constructions will be explored in [APT17]. In our setting, the functor applied to the semigroup of generalized Cu-morphisms Cu[S, T] yields the internal-hom of \( S \) and \( T \). In other words, for Cu-semigroups \( S \) and \( T \), we define

\[ [S, T] := \tau(Cu[S, T]); \]

see Definition 5.3.

We illustrate our results by computing a number of examples, that include the (Cuntz semigroups of the) Jiang-Su algebra \( \mathcal{Z} \), the Jacelon-Razak algebra \( \mathcal{W} \), UHF-algebras of infinite type, and purely infinite simple \( C^* \)-algebras. Interestingly, \( [Cu(W), Cu(W)] \) is isomorphic to the Cuntz semigroup of a \( \Pi_1 \)-factor.

The fact that Cu is a closed category automatically adds additional features well known to category theory. For example, one obtains a composition product given in the form of a Cu-morphism:

\[ \circ: [T, P] \otimes [S, T] \to [S, P]. \]

In the case where \( S = T = P \), the above composition product equips \( [S, S] \) with the structure of a (not necessarily commutative) Cu-semiring.

Although the said features can be derived from general principles, in our setting they become concrete, and this is very useful in applications. In this direction, and bearing in mind that \( [S, T] \) is a semigroup built out of paths of morphisms from \( S \) to \( T \), the composition product can be realized as the composition of paths. Another important example is the evaluation map which, for Cu-semigroups \( S \) and \( T \) is a Cu-morphism \( e_{S,T}: [S, T] \otimes S \to T \) such that \( e_{S,T}(x \otimes s) \) can be interpreted as the evaluation of \( x \in [S, T] \) at \( s \in S \). We therefore also write \( x(s) := e_{S,T}(x \otimes s) \). The evaluation map can be used to concretize the adjunction between the internal-hom bifunctor and the tensor product.

Likewise, the tensor product of generalized Cu-morphisms induces an external tensor product

\[ \boxtimes: [S_1, T_1] \otimes [S_2, T_2] \to [S_1 \otimes S_2, T_1 \otimes T_2], \]

which is associative and, like in \( KK \)-Theory, compatible with the composition product. This means that, for elements \( x_k \in [S_k, T_k] \) and \( y_k \in [T_k, P_k] \) (for \( k = 1, 2 \)), we have

\[ (y_2 \boxtimes y_1) \circ (x_2 \boxtimes x_1) = (y_2 \circ x_2) \boxtimes (y_1 \circ x_1). \]

We also deepen our study of Cu-semirings and their semimodules. Recall that, as shown in [APT14] Chapter 7, the Cuntz semigroup of a strongly self-absorbing \( C^* \)-algebra has a natural product giving it the structure of a Cu-semiring.
A particular instance of the composition product setting arises when we study the semiring $[S, S]$. Any Cu-semigroup $S$ becomes an $[S, S]$-semibimodule. This can also be extended to the more general situation where $T$ has a left $R$-action (for a Cu-semiring $R$ with compact unit) and $S$ is an arbitrary Cu-semigroup to obtain that $[S, T]$ has a left $R$-action as well.

For any Cu-semiring $R$, the internal-hom construction makes it possible to define a left regular representation-like map $\pi_R: R \to [R, R]$, which is always multiplicative (and unital in case the unit of $R$ is a compact element).

Recall that a Cu-semiring $R$ is solid in case the multiplication defines an isomorphism between $R \otimes R$ and $R$ (see [APT14, Chapter 7]). This terminology was inspired by that of solid rings, and reflects the situation of strongly self-absorbing $C^*$-algebras, in the sense that the Cuntz semigroup of any strongly self-absorbing $C^*$-algebra satisfying the UCT is solid.

Under mild assumptions (namely, the so-called axioms (O5) and (O6)), all solid Cu-semirings were classified in [APT14 Theorem 8.3.3]. We show that a Cu-semiring $R$ satisfying (O5) and (O6) is solid if and only if the evaluation map $e_{R,R}: [R, R] \otimes R \to R$ is an isomorphism. The question of whether this holds without assuming (O5) or (O6) remains open [Question 7.26].

We finally specialise to $C^*$-algebras and show that a c.p.c. order-zero map $\varphi: A \to B$ between $C^*$-algebras $A$ and $B$ naturally defines an element $Cu(\varphi)$ in the bivariant Cu-semigroup $[Cu(A), Cu(B)]$; see Theorem 8.3 and Definition 8.4. We then analyse the induced map

$$cpc_{\perp}(A, B) \to [Cu(A), Cu(B)],$$

and show it is surjective in a number of cases; namely for a UHF-algebra of infinite type, the Jiang-Su algebra, or the Jacelon-Razak algebra $W$; see Example 8.9.

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2. Preliminaries

Throughout, $K$ denotes the $C^*$-algebra of compact operators on a separable, infinite-dimensional Hilbert space. Given a $C^*$-algebra $A$, we let $A_+$ denote the positive elements in $A$.

2.1. The category Cu of abstract Cuntz semigroups. In this subsection, we recall the definition of the category Cu of abstract Cuntz semigroups as introduced by Coward, Elliott and Ivanescu in [CEI08].
2.1. Let us first recall the basic theory of the category PoM of positively ordered monoids. We refer to [APT14, Appendix B.2] for details.

A positively ordered monoid is a commutative semigroup $M$, written additively with zero element 0, together with a partial order $\leq$ such that $a \leq b$ implies that $a + c \leq b + c$ for all $a, b, c \in M$, and such that $0 \leq a$ for all $a \in M$. We let PoM denote the category whose objects are positively ordered monoids, and whose morphisms are maps preserving addition, order and the zero element.

Let $M, N$ and $P$ be positively ordered monoids. We denote the set of PoM-morphisms from $M$ to $N$ by $\text{PoM}(M, N)$. A map $\varphi: M \times N \to P$ is called a PoM-bimorphism if it is a PoM-morphism in each variable, that is, for each $m \in M$ the map $N \to P$ given by $n \mapsto \varphi(m, n)$ is a PoM-morphism, and analogously in the first variable. We denote the collection of such maps by $\text{BiPoM}(M \times N, P)$. We equip both $\text{PoM}(M, N)$ and $\text{BiPoM}(M \times N, P)$ with pointwise order and addition, which gives them a natural structure as positively ordered monoids.

Given positively ordered monoids $M$ and $N$, there exists a positively ordered monoid $M \otimes_{\text{PoM}} N$ and a PoM-bimorphism $\omega: M \times N \to M \otimes_{\text{PoM}} N$ with the following universal property: For every positively ordered monoid $P$, the assignment that maps a PoM-morphism $\alpha: M \otimes_{\text{PoM}} N \to P$ to the PoM-bimorphism $\alpha \circ \omega: M \times N \to P$ is a bijection between the following (bi)morphism sets

$$\text{PoM}(M \otimes_{\text{PoM}} N, P) \cong \text{BiPoM}(M \times N, P),$$

which moreover respects the structure of the (bi)morphism sets as positively ordered monoids. We call $M \otimes_{\text{PoM}} N$ together with $\omega$ the tensor product of $M$ and $N$ (in the category PoM).

Recall that a set $\Lambda$ with a binary relation $\prec$ is called upward directed if for all $\lambda_1, \lambda_2 \in \Lambda$ there exists $\lambda \in \Lambda$ such that $\lambda_1, \lambda_2 \prec \lambda$.

Following [GHK03, Definition I-1.11, p.57], we define auxiliary relations on partially ordered sets and monoids:

**Definition 2.2.** Let $X$ be a partially ordered set. An auxiliary relation on $X$ is a binary relation $\prec$ on $X$ satisfying the following conditions for all $x, x', y, y' \in X$:

1. If $x \prec y$ then $x \leq y$.
2. If $x' \leq x \prec y \leq y'$ then $x' \prec y'$.

If $X$ is also a monoid, then an auxiliary relation $\prec$ on $X$ is said to be additive if it satisfies the following conditions for all $x, x_1, x_2, y_1, y_2 \in X$:

1. We have $0 \prec x$.
2. If $x_1 \prec y_1$ and $x_2 \prec y_2$, then $x_1 + x_2 \prec y_1 + y_2$.

An important example of an auxiliary relation is the so called way-below relation, which has its origins in domain theory (see [GHK03]). We recall below its sequential version, which is the one used to define abstract Cuntz semigroups.

**Definition 2.3.** Let $X$ be a partially ordered set, and let $x, y \in X$. We say that $x$ is way-below $y$, or that $x$ is compactly contained in $y$, in symbols $x \ll y$, if whenever $(z_n)_n$ is an increasing sequence in $X$ for which the supremum exists and which satisfies $y \leq \sup_n z_n$, then there exists $k \in \mathbb{N}$ with $x \leq z_k$. We say that $x$ is compact if $x \ll x$. We let $X_c$ denote the set of compact elements in $X$.

The following definition is due to Coward, Elliott and Ivanescu in [CEI08]. See also [APT14, Definition 3.1.2].

**Definition 2.4.** A Cu-semigroup, also called abstract Cuntz semigroup, is a positively ordered semigroup $S$ that satisfies the following axioms (O1)-(O4):

1. Every increasing sequence $(a_n)_n$ in $S$ has a supremum $\sup_n a_n$ in $S$. 


(O2) For every element \(a \in S\) there exists a sequence \((a_n)_n\) in \(S\) with \(a \leq a_{n+1}\) for all \(n \in \mathbb{N}\), and such that \(a = \sup_n a_n\).

(O3) If \(a' \leq a\) and \(b' \leq b\) for \(a', b', a, b \in S\), then \(a' + b' \leq a + b\).

(O4) If \((a_n)_n\) and \((b_n)_n\) are increasing sequences in \(S\), then \(\sup_n (a_n + b_n) = \sup_n a_n + \sup_n b_n\).

Given \(Cu\)-semigroups \(S\) and \(T\), a \(Cu\)-morphism from \(S\) to \(T\) is a map \(f: S \to T\) that preserves addition, order, the zero element, the way-below relation and suprema of increasing sequences. A generalized \(Cu\)-morphism is a \(Cu\)-morphism that is not required to preserve the way-below relation. We denote the set of \(Cu\)-morphisms by \(Cu(S,T)\); and we denote the set of generalized \(Cu\)-morphisms by \(Cu[S,T]\).

We let \(Cu\) be the category whose objects are \(Cu\)-semigroups and whose morphisms are \(Cu\)-morphisms.

Remark 2.5. Let \(S\) be a \(Cu\)-semigroup. Note that \(0 \leq a\) for all \(a \in S\). Thus, (O3) ensures that \(\leq\) is an additive auxiliary relation on \(S\).

2.6. Let \(A\) be a \(C^*\)-algebra, and let \(a, b \in (A \otimes K)_+\). We say that \(a\) is Cuntz subequivalent to \(b\), denoted \(a \preceq b\), if there is a sequence \((x_n)_n\) in \(A \otimes K\) such that \(a = \lim_n x_n b x_n^*\). We say that \(a\) and \(b\) are Cuntz equivalent, written \(a \sim b\), provided \(a \preceq b\) and \(b \preceq a\). The set of equivalence classes

\[Cu(A) = (A \otimes K)_+/\sim\]

is called the (completed) Cuntz semigroup of \(A\). One defines an addition on \(Cu(A)\) by setting \([a] + [b] := [(a \otimes 1)]\) for \(a, b \in (A \otimes K)_+\). (One uses that there is an isomorphism \(M_2(K) \cong K\), and that the definition does not depend on the choice of isomorphism.) The class of \(0 \in (A \otimes K)_+\) is a zero element for \(Cu(A)\), giving it the structure of a commutative monoid. One defines an order on \(Cu(A)\) by setting \([a] \leq [b]\) whenever \(a \preceq b\). This gives \(Cu(A)\) the structure of a positively ordered monoid.

Theorem 2.7 (CEF08). For every \(C^*\)-algebra \(A\), the positively ordered monoid \(Cu(A)\) is a \(Cu\)-semigroup. Furthermore, if \(B\) is another \(C^*\)-algebra, then a \(^*\)-homomorphism \(\varphi: A \to B\) induces a \(Cu\)-morphism \(Cu(\varphi): Cu(A) \to Cu(B)\) by

\[Cu(\varphi)([a]) := [\varphi(a)],\]

for \(a \in (A \otimes K)_+\). This defines a functor from the category of \(C^*\)-algebras with \(^*\)-homomorphisms to the category \(Cu\).

Remark 2.8. Let \(A\) be a \(C^*\)-algebra. In order to show that (O2) holds for \(Cu(A)\) one proves that, for every \(a \in (A \otimes K)_+\) and \(\varepsilon > 0\) we have \([a - \varepsilon] \leq [a]\), and that moreover \([a] = \sup_{\varepsilon \geq 0} ([a - \varepsilon])_+\). One can then derive from this that the sequence \(([(a - 1/n)_+]_n)\)_n satisfies the required properties in (O2).

This suggests the possibility of formally strengthening (O2) for every \(Cu\)-semigroup \(S\) in the following way: Given \(a \in S\), there exists a \((0,1)\)-indexed chain of elements \((a_\lambda)_{\lambda \in (0,1)}\) with the property that \(a = \sup_\lambda a_\lambda\), and \(a_\lambda \preceq a_\lambda'\) whenever \(\lambda' < \lambda\). Next, we show that this property holds for all \(Cu\)-semigroups.

Lemma 2.9. Let \(S\) be a set equipped with a transitive binary relation \(\prec\) that satisfies the following condition:

\((*)\) For each \(a \in S\) there exists a sequence \((a_n)_n\) in \(S\) such that \(a_n \prec a_{n+1} \prec a\) for all \(n\); and such that whenever \(a' \in S\) satisfies \(a' \prec a\) then there exists \(n_0\) with \(a' \prec a_{n_0}\).

Then, for every \(a \in S\), there exists a chain \((a_\lambda)_{\lambda \in (0,1) \cap \mathbb{Q}}\) such that \(a_\lambda \prec a_\lambda'\) whenever \(\lambda', \lambda \in (0,1) \cap \mathbb{Q}\) satisfy \(\lambda' < \lambda\), and such that for every \(a' \in S\) with \(a' \prec a\) there exists \(\mu \in (0,1) \cap \mathbb{Q}\) with \(a' \prec a_\mu\).
Proof. Note that condition (*) implies the following: Whenever $b_1, b_2, b \in S$ satisfy $b_1, b_2 \prec b$, then there exists $b_3 \in S$ with $b_1, b_2 \prec b_3 \prec b$. This property, which we will refer to as the interpolation property, will be used throughout.

Given $a \in S$, first use (*) to fix an increasing sequence $0 \prec a_1 \prec a_2 \prec \cdots \prec a$ which is cofinal in $a^\prec := \{b \mid b \prec a\}$. (This means that, if $a' \in S$ satisfies $a' \prec a$, then there is $k \in \mathbb{N}$ with $a' \prec a_k$.)

Use the interpolation property to find $a_1^{(1)}$ such that $a_1 \prec a_1^{(1)} \prec a$ and consider the chain $0 \prec a_1^{(1)} \prec a$. Now use the interpolation property to refine the above chain as

$$0 \prec a_1^{(1)} \prec a \quad || 
0 \prec a_2^{(2)} \prec a_2^{(2)} \prec a,$$

in such a way that moreover $a_2 \prec a_2^{(2)}$. We now proceed inductively, and thus suppose we have constructed a chain $0 \prec a_1^{(n)} \prec \cdots \prec a_2^{(n)} \prec a$ with $a_n \prec a_2^{(n-1)}$.

Use the interpolation property to construct a new chain

$$0 \prec a_1^{(n+1)} \prec \cdots \prec a_2^{(n+1)} \prec a,$$

such that

$$0 \prec a_1^{(n+1)} \prec a_1^{(n)}, \quad a_1^{(n)} \prec a_2^{(n+1)} \prec a_1^{(n+1)}, \quad a_2^{(n+1)} = a_1^{(n)}, \quad a_2^{(n)} \prec a_2^{(n+1)} \prec a_2^{(n+1)} \prec a,$$

and such that moreover $a_{n+1} \prec a_2^{(n+1)}$. This latter condition will ensure that the set of elements thus constructed is cofinal in $a^\prec$.

The index set $I := \{(n,i) \mid 1 \leq n, 1 \leq i \leq 2^n - 1\}$ can be totally ordered by setting $(n,i) \leq (m,j)$ provided $i2^{-n} \leq j2^{-m}$. It now follows from the construction above that $a_1^{(n)} \prec a_j^{(m)}$ whenever $(n,i) \leq (m,j)$.

The set $I$ is order-isomorphic to the dyadic rationals in $(0,1)$. In fact, $I$ is a countably infinite, totally ordered, dense set with no minimal or maximal element. (Here, dense means that whenever $x < y$ in $I$ there exists $z \in I$ such that $x < z < y$.) By a classical result of G. Cantor (see, for example, [Roitberg 90, Theorem 27]), there is only one such set, up to order-isomorphism. We can therefore choose an order-preserving bijection $\psi: I \to (0,1) \cap \mathbb{Q}$ and, setting $a_\lambda = a_1^{(\psi^{-1}(\lambda))}$ whenever $\psi((n,i)) = \lambda$, the desired conclusion follows.

\[\Box\]

Proposition 2.10. Let $S$ be a Cu-semigroup, and let $a \in S$. Then, there exists a family $(a_\lambda)_{\lambda \in [0,1]}$ in $S$ with $a_1 = a$; such that $a_\lambda \ll a_{\lambda'}$ whenever $\lambda, \lambda' \in (0,1]$ satisfy $\lambda' \prec \lambda$; and such that $a_\lambda = \sup_{\lambda' \ll \lambda} a_{\lambda'}$ for every $\lambda \in (0,1]$.

Proof. Consider $S$ equipped with the transitive relation $\ll$. Then (O2) ensures that condition (*) in Lemma 2.9 is fulfilled with $\ll$ in place of $\prec$. Hence, given $a \in S$ we can apply Lemma 2.9 to choose a $\ll$-increasing chain $(a_\lambda)_{\lambda \in (0,1) \cap \mathbb{Q}}$ with $a = \sup_{\lambda} a_\lambda$. For each $\lambda \in (0,1]$, define $a_\lambda := \sup\{a_{\lambda'} : \lambda' \prec \lambda\}$. It is now easy to see that the chain $(a_\lambda)_{\lambda \in [0,1]}$ satisfies the conclusion.

\[\Box\]

Let $S$ be a Cu-semigroup, and let $a$ be an element in $S$. We say that $a$ is soft if for every $a' \in S$ with $a' \ll a$ we have $a' \ll a$, that is, there exists $k \in \mathbb{N}$ with $(k+1)a' \ll ka$; see [APT14, Definition 5.3.1]. We denote the set of soft elements in $S$ by $S_{\text{soft}}$. Further, we set $S_{\text{soft}}^\circ := S_{\text{soft}} \setminus \{0\}$.

The following result will be used later (in Section 7).

Lemma 2.11. Let $S$ and $T$ be Cu-semigroups, let $\varphi: S \to T$ be a generalized Cu-morphism, and let $a \in S$ be a soft element. Then $\varphi(a)$ is soft.
Proof. To verify that \( \varphi(a) \) is soft, let \( x \in T \) satisfy \( x \ll \varphi(a) \). Using that \( \varphi \) preserves suprema of increasing sequences, we can choose \( a' \in S \) with \( a' \ll a \) and \( x \leq \varphi(a') \). (Indeed, applying (O2) in \( S \), choose \( a \ll a' \) and \( x \leq \varphi(a_n) \).) Since \( a \) is soft, we can choose \( k \in \mathbb{N} \) such that \( (k+1)a' \leq ka \). Then
\[
(k+1)x \leq (k+1)\varphi(a') = \varphi((k+1)a') \leq \varphi(ka) = k\varphi(a),
\]
which shows that \( x <_s \varphi(a) \), as desired.

\[\Box\]

2.12. Given a \( C^* \)-algebra \( A \), it is known that \( \text{Cu}(A) \) satisfies two additional axioms besides (O1)–(O4). The first one is usually referred to the axiom of almost algebraic order or axiom (O5), first considered in \( \text{[RW10]} \) and also in \( \text{[ORT11]} \) and \( \text{[Rob13]} \). The version we use here is a strengthening of the original formulation, introduced in \( \text{[APT14]} \): We say that a \( \text{Cu} \)-semigroup \( S \) satisfies (O5) if, for every \( a', a, b, b', c \in S \) that satisfy \( a + b \leq c, a' \ll a, b' \ll b \), there is \( x \in S \) with \( a' + x \leq c + x \leq a + x \) and \( b' \leq x \).

The second axiom is known as the axiom of almost Riesz decomposition or axiom (O6), and was introduced in \( \text{[Rob13]} \): We say that a \( \text{Cu} \)-semigroup \( S \) satisfies (O6) if, for every \( a', a, b, c \in S \) satisfying \( a' \ll a \leq b + c \), there exist \( b', c' \in S \) such that \( a' \leq b' + c' \) and \( b' \leq a, b' \leq c \).

2.2. Closed, monoidal categories. In this subsection, we recall the basic notions from the theory of closed, monoidal categories. For details we refer to \( \text{[Ko05]} \) and \( \text{[Mac71]} \). See also \( \text{[APT14]} \) Appendix A.

2.13. Recall that a monoidal category \( V \) consists of: a category \( V_0 \) (which we assume is locally small), a bifunctor \( \otimes : V_0 \times V_0 \to V_0 \) (covariant in each variable) and a unit object \( I \) in \( V_0 \) such that, whenever \( X, Y, Z \) are objects in \( V_0 \), there are natural isomorphisms
\[
(X \otimes Y) \otimes Z \cong X \otimes (Y \otimes Z), \quad \text{and} \quad X \otimes I \cong X, \quad \text{and} \quad I \otimes X \cong X,
\]
that are subject to certain coherence axioms. An object or morphism in \( V \) means an object or morphism in \( V_0 \), respectively. In concrete examples, such as the categories \( \text{PoM} \) and \( \text{Cu} \), we will use the same notation for a monoidal category and its underlying category.

A monoidal category \( V \) is called symmetric provided that for each pair of objects \( X \) and \( Y \) there is a natural isomorphism \( X \otimes Y \cong Y \otimes X \).

In many concrete examples of monoidal categories, the tensor product of two objects \( X \) and \( Y \) is the object \( X \times Y \) (unique up to natural isomorphism) that linearizes bilinear maps from \( X \times Y \). This is formalized by considering a functorial association of bimorphisms \( \text{Bimor}(X \times Y, Z) \) (covariant in \( Z \), and contravariant in \( X \) and \( Y \)) such that \( X \otimes Y \) represents the functor \( \text{Bimor}(X \times Y, \_ \_ ) \), that is, for each \( Z \) there is a natural bijection
\[
\text{Bimor}(X \otimes Y, Z) \cong \text{Mor}(X \otimes Y, Z).
\]

One instance of this is the monoidal structure in the category \( \text{Cu} \) of abstract Cuntz semigroups, as introduced in \( \text{[APT14]} \). We recall details in \( \text{Subsection 2.3} \).

Another example is the category \( \text{PoM} \) of positively ordered monoids. There is a natural notion of bimorphisms in \( \text{PoM} \), and the tensor product in \( \text{PoM} \) has the corresponding universal property of linearizing such bimorphisms; see \( \text{Paragraph 2.1} \).

2.14. A monoidal category \( V \) is said to be closed provided that for each object \( Y \), the functor \( 
abla \otimes Y : V_0 \to V_0 \) has a right adjoint, that we will denote by \( [Y, \_ \_ ] \). Thus, in a closed monoidal category, for all objects \( X, Y, Z \), there is a natural bijection
\[
V_0(X \otimes Y, Z) \cong V_0(X, [Y, Z]).
\]
where $\mathcal{V}(\cdot, \cdot)$ denotes the morphisms between two objects $X$ and $Y$.

Let $\mathcal{V}$ be a monoidal category with unit object $I$. An enriched category $\mathcal{C}$ over $\mathcal{V}$ consists of: a collection of objects in $\mathcal{C}$; an object $\mathcal{C}(X, Y)$ in $\mathcal{V}$, for each pair of objects $X$ and $Y$ in $\mathcal{C}$ (playing the role of the morphisms in $\mathcal{C}$ from $X$ to $Y$); a $\mathcal{V}$-morphism $j_X : I \to \mathcal{C}(X, X)$, called the identity on $X$, for each object $X$ in $\mathcal{C}$ (playing the role of the identity morphism on $X$); and for each triple $X$, $Y$ and $Z$ of objects in $\mathcal{C}$, a $\mathcal{V}$-morphism

$$\mathcal{C}(Y, Z) \otimes \mathcal{C}(X, Y) \to \mathcal{C}(X, Z),$$

that plays the role of a composition law and is subject to certain coherence axioms; see [Kel05, Section 1.2] for details.

It follows from general category theory that every closed symmetric monoidal category $\mathcal{V}$ can be enriched over itself. Let us recall some details. Given two objects $X$ and $Y$ in $\mathcal{V}$, the object $[X, Y]$ in $\mathcal{V}$ plays the role of the morphisms from $X$ to $Y$. Given an object $X$, the identity on $X$ (for the enrichment) is defined as the $\mathcal{V}$-morphism $j_X : I \to [X, X]$ that corresponds to the ‘usual’ identity morphism $\text{id}_X \in \mathcal{V}(X, X)$ under the following natural bijections

$$\mathcal{V}(I, [X, X]) \cong \mathcal{V}(I \otimes X, X) \cong \mathcal{V}(X, X).$$

It is easiest to construct the composition map by using the evaluation maps. Given objects $X$, $Y$, and $Z$, the evaluation (or counit) map is defined as the $\mathcal{V}$-morphism $e^X_{Y} : [X, Y] \otimes X \to Y$ that corresponds to the identity morphism in $\mathcal{V}(0([X, Y], [X, Y]))$ under the natural bijection

$$\mathcal{V}(0([X, Y] \otimes X, Y) \cong \mathcal{V}(0([X, Y], [X, Y])).$$

Then, given objects $X$, $Y$, and $Z$, the composition

$$[Y, Z] \otimes [X, Y] \to [X, Z]$$

is defined as the $\mathcal{V}$-morphism that corresponds to the composition

$$[Y, Z] \otimes [X, Y] \otimes X \xrightarrow{id_{[Y,Z]} \otimes e^{Y}_{X}} [Y, Z] \otimes Z \xrightarrow{e^{Z}_{Z}} Z$$

under the natural bijection

$$\mathcal{V}(0([Y, Z] \otimes [X, Y], [X, Z]) \cong \mathcal{V}(0([Y, Z] \otimes [X, Y], [X, Z]).$$

The natural question of whether the monoidal category $\text{Cu}$ is closed was left open in [APT14, Problem 2]. One of the objectives of this paper is to show that this is indeed the case by applying the $\tau$-construction that will be developed in Section 3 to a suitable semigroup of morphisms in the category $\text{Cu}$; see [Definition 5.3 and Theorem 5.11]

2.3. Tensor products in $\text{Cu}$. In this subsection we recall the construction of tensor products of $\text{Cu}$-semigroups as introduced in [APT14]. We first recall the notion of $\text{Cu}$-bimorphisms.

**Definition 2.15 ([APT14, Definition 6.3.1]).** Let $S, T$ and $P$ be $\text{Cu}$-semigroups, and let $\varphi : S \times T \to P$ be a PoM-bimorphism. We say that $\varphi$ is a $\text{Cu}$-bimorphism if it satisfies the following conditions:

1. We have that $\sup_k \varphi(a_k, b_k) = \varphi(\sup_k a_k, \sup_k b_k)$, for every increasing sequences $(a_k)_k$ in $S$ and $(b_k)_k$ in $T$.

2. If $a', a \in S$ and $b', b \in T$ satisfy $a' \ll a$ and $b' \ll b$, then $\varphi(a', b') \ll \varphi(a, b)$.

We denote the set of $\text{Cu}$-bimorphisms by $\text{BiCu}(S \times T, R)$. 
Given Cu-semigroups $S, T$ and $P$, we equip BiCu$(S \times T, R)$ with pointwise order and addition, giving it the structure of a positively ordered monoid. Similarly, we consider the set of Cu-morphisms between two Cu-semigroups as a positively ordered monoid with the pointwise order and addition.

**Theorem 2.16** ([APT14 Theorem 6.3.3]). Let $S$ and $T$ be Cu-semigroups. Then there exists a Cu-semigroup $S \otimes T$ and a Cu-bimorphism $\omega: S \times T \to S \otimes T$ such that for every Cu-semigroup $P$ the following universal properties hold:

1. For every Cu-bimorphism $\varphi: S \times T \to P$ there exists a (unique) Cu-morphism $\tilde{\varphi}: S \otimes T \to P$ such that $\varphi = \tilde{\varphi} \circ \omega$.
2. If $\alpha_1, \alpha_2: S \otimes T \to P$ are Cu-morphisms, then $\alpha_1 \leq \alpha_2$ if and only if $\alpha_1 \circ \omega \leq \alpha_2 \circ \omega$.

Thus, for every Cu-semigroup $P$, the assignment that sends a Cu-morphism $\alpha: S \otimes T \to P$ to the Cu-bimorphism $\alpha \circ \omega: S \times T \to P$ defines a natural bijection between the following (bi)morphism sets

$$\text{Cu}(S \otimes T, P) \cong \text{BiCu}(S \times T, P),$$

Moreover, this bijection respects the structure of the (bi)morphism sets as positively ordered monoids.

2.17. Let $S$ and $T$ be Cu-semigroups, and consider the universal Cu-bimorphism $\omega: S \times T \to S \otimes T$ from Theorem 2.16. Given $s \in S$ and $t \in T$, we set $s \otimes t := \omega(s, t)$. We call $s \otimes t$ a simple tensor.

The tensor product in Cu is functorial in each variable: If $\varphi_1: S_1 \to T_1$ and $\varphi_2: S_2 \to T_2$ are Cu-morphisms, then there is a unique Cu-morphism $\varphi_1 \otimes \varphi_2: S_1 \otimes S_2 \to T_1 \otimes T_2$ with the property that $(\varphi_1 \otimes \varphi_2)(a_1 \otimes a_2) = \varphi_1(a_1) \otimes \varphi_2(a_2)$ for every $a_1 \in S_1$ and $a_2 \in S_2$.

Thus, the tensor product in Cu defines a bifunctor $\otimes: \text{Cu} \times \text{Cu} \to \text{Cu}$. The Cu-semigroup $\mathbb{N} := \{0, 1, 2, \ldots, \infty\}$ (with usual addition and order) is a unit object, that is, for every Cu-semigroup $S$ there are canonical isomorphisms $S \otimes \mathbb{N} \cong S$ and $\mathbb{N} \otimes S \cong S$. Further, for every Cu-semigroups $S, T$ and $P$, there are natural isomorphisms

$$S \otimes (T \otimes P) \cong (S \otimes T) \otimes P \quad \text{and} \quad S \otimes T \cong T \otimes S.$$

It follows that Cu is a symmetric, monoidal category; see also [APT14 6.3.7].

3. The Path Construction

In this section we introduce a functorial construction from a category of monoids with a transitive relation to the category Cu. This construction, when restricted to the category $\mathcal{Q}$ introduced in Section 4 (a category that contains Cu) is a coreflection for the natural inclusion from Cu.

**Definition 3.1.** A $\mathcal{P}$-semigroup is a pair $(S, \prec)$, where $S$ is a commutative monoid and where $\prec$ is a transitive relation on $S$, such that:

1. We have $0 \prec a$ for all $a \in S$.
2. If $a_1, a_2, b_1, b_2 \in S$ satisfy $a_1 \prec b_1$ and $a_2 \prec b_2$, then $a_1 + a_2 \prec b_1 + b_2$.

We often suppress the reference to the relation and denote a $\mathcal{P}$-semigroup $(S, \prec)$ simply by $S$.

A $\mathcal{P}$-morphism is a monoid morphism that preserves the relation. Given $\mathcal{P}$-semigroups $(S, \prec)$ and $(T, \prec)$, we denote the collection of all $\mathcal{P}$-morphisms by $\mathcal{P}((S, \prec), (T, \prec))$, or simply by $\mathcal{P}(S, T)$. We let $\mathcal{P}$ be the category whose objects are $\mathcal{P}$-semigroups and whose morphisms are $\mathcal{P}$-morphisms.
Remark 3.2. Conditions (1) and (2) of Definition 3.1 are the same as the conditions from Definition 2.2 for an auxiliary relation to be additive.

Definition 3.3. Let $I = (I, \prec)$ be a set with an upward directed transitive relation $\prec$. Let $S = (S, \prec)$ be a $\mathcal{P}$-semigroup. An $I$-path (or simply a path) in $S$ is a map $f : I \to S$ such that $f(\lambda') \prec f(\lambda)$ whenever $\lambda', \lambda \in I$ satisfy $\lambda' \prec \lambda$. We set

$$P(I, S) := \{ f : I \to S \text{ such that } f \text{ is a path in } S \}.$$  

Given two paths $f$ and $g$, we define their sum $f + g$ by setting $(f + g)(\lambda) := f(\lambda) + g(\lambda)$ for all $\lambda \in I$. Let $0 \in P(I, S)$ denote the path that satisfies $0(\lambda) = 0$ for all $\lambda \in I$.

We define a binary relation $\preceq$ on $P(I, S)$ by setting $f \preceq g$ for two paths $f$ and $g$ if and only if for every $\lambda \in I$ there exists $\mu \in I$ such that $f(\lambda) \prec g(\mu)$. Finally we antisymmetrize the relation $\preceq$ by setting $f \sim g$ if and only if $f \preceq g$ and $g \preceq f$.

Given $s \in S$ and $f \in P(I, S)$, we write $s \prec f$ if $s \prec f(\lambda)$ for all $\lambda \in I$; and we write $f \prec s$ provided $f(\lambda) \prec s$ for all $\lambda \in I$.

The proof of the following result is straightforward and therefore omitted.

Lemma 3.4. Let $I$ be a set with an upward directed transitive relation, and let $S$ be a $\mathcal{P}$-semigroup. Then the addition and the zero element defined in Definition 3.1 give $P(I, S)$ the structure of a commutative monoid. Moreover, the relation $\preceq$ on $P(I, S)$ is transitive, reflexive and satisfies:

1. For every $f \in P(I, S)$ we have $0 \preceq f$.
2. If $f_1, f_2, g_1, g_2 \in P(I, S)$ satisfy $f_1 \preceq g_1$ and $f_2 \preceq g_2$, then $f_1 + f_2 \preceq g_1 + g_2$.

Further, $\sim$ is an equivalence relation on $P(I, S)$.

Definition 3.5. Let $I$ be a set with an upward directed transitive relation, and let $S$ be a $\mathcal{P}$-semigroup. Let $\sim$ be the equivalence relation on $P(I, S)$ from Definition 3.3.

We define

$$\tau_I(S) := P(I, S)/\sim.$$  

Given a path $f$ in $S$, its equivalence class in $\tau_I(S)$ is denoted by $[f]$.

We define $0 \in \tau_I(S)$ as the equivalence class of the zero-path. We define $+$ and $\leq$ on $\tau_I(S)$ by setting $[f] + [g] := [f + g]$, and by setting $[f] \leq [g]$ provided $f \preceq g$.

The following results follows immediately from Lemma 3.4.

Proposition 3.6. Let $I$ be a set with an upward directed transitive relation, and let $S$ be a $\mathcal{P}$-semigroup. Then the addition, the zero element, and the order defined in Definition 3.5 give $\tau_I(S)$ the structure of a positively ordered monoid.

Remarks 3.7. (1) We call the construction of $\tau_I(S)$ the $\tau$-construction or path construction. We call $I$ the path type.

(2) Given a $\mathcal{P}$-semigroup $S$, the path construction $\tau_I(S)$ depends heavily on the choice of $I$. For instance, using the most simple case $I = \{\{0\}, \leq\}$, we obtain

$$\tau_{\{\{0\}\}}(S) \simeq \{ a \in S : a \prec a \}.$$  

For $I = (\mathbb{N}, <)$, one can show that $\tau_I(S)$ is the (sequential) round ideal completion of $S$ as considered for instance in [APT14] Proposition 3.1.6.

We will not pursue this general constructions further. Rather, motivated by the results in Lemma 2.9 and Proposition 2.10, we will focus on the concrete case where the path type is taken to be $(\mathbb{Q} \cap (0, 1), <)$.

Notation 3.8. We set $I_0 := (\mathbb{Q} \cap (0, 1), <)$. Given a $\mathcal{P}$-semigroup $S$, we denote $P(I_0, S)$ and $\tau_{I_0}(S)$ by $P(S)$ and $\tau(S)$, respectively. If we want to stress the auxiliary relation on $S$, we also write $P(S, \prec)$ and $\tau(S, \prec)$.

Thinking of $I_0$ as an ordered index set, we will often denote a path in $S$ as an indexed family $(a_\lambda)_{\lambda \in I_0}$.  

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Given a $\mathcal{P}$-semigroup $S$, we show in **Theorem 3.15** that $\tau(S)$ is a Cu-semigroup when equipped with the order and addition in **Definition 3.5**. We split the proof into several lemmas.

Recall from **Definition 3.3** that, given paths $f$ and $g$ in $S$, and given $\lambda \in I_Q$, we write $f(\lambda) \prec g$ (respectively, $f \prec g(\lambda)$) if $f(\lambda) \prec g(\mu)$ (respectively, $f(\mu) \prec g(\lambda)$) for every $\mu \in I_Q$.

**Lemma 3.9.** Let $S = (S, \prec)$ be a $\mathcal{P}$-semigroup, let $f$ be a path in $S$, and let $\lambda', \lambda \in I_Q$ satisfy $\lambda' < \lambda$. Then there exists a path $h$ in $S$ such that $f(\lambda') \prec h \prec f(\lambda)$.

**Proof.** Define $h : I_Q \to S$ by

$$h(\gamma) := f(\gamma \lambda + (1 - \gamma)\lambda'),$$

for $\gamma \in I_Q$. Then $h$ is a path satisfying $f(\lambda') \prec h \prec f(\lambda)$, as desired. \qed

**Lemma 3.10.** Let $S = (S, \prec)$ be a $\mathcal{P}$-semigroup. Given a sequence $(f_n)_{n \geq 1}$ of paths in $S$, and given a sequence $(a_n)_{n \geq 1}$ in $S$ such that

$$0 < f_1 < a_1 < f_2 < a_2 < f_3 < a_3 \cdots,$$

there exists a path $h$ in $S$ such that $h(\frac{n}{n+1}) = a_n$ for all $n \geq 1$.

**Proof.** Define $h : I_Q \to S$ as follows:

$$h(\lambda) := \begin{cases} f_n(\lambda), & \text{if } \lambda \in \left(\frac{n-1}{n}, \frac{n}{n+1}\right); \\ a_n, & \text{if } \lambda = \frac{n}{n+1}. \end{cases}$$

It is easy to see that $h$ is a path and that $h(\frac{n}{n+1}) = a_n$, as desired. \qed

**Lemma 3.11.** Let $S$ be a $\mathcal{P}$-semigroup, and let $(\{f_n\})_{n \geq 1}$ be an increasing sequence in $\tau(S)$. Then there exists a strictly increasing sequence $(\lambda_m)_{m \geq 1}$ in $I_Q$ and a path $f$ in $S$ such that the following conditions hold:

1. We have $\sup_m \lambda_m = 1$.
2. We have $f_n(\lambda_m) \prec f_l(\lambda_1)$, whenever $n, m < l$.
3. We have $f(\frac{n}{n+1}) = f_n(\lambda_n)$ for all $n \geq 1$.

Moreover, if $f$ is a path in $S$ for which there exists a strictly increasing sequence $(\lambda_m)_{m \geq 1}$ in $I_Q$ satisfying conditions (1), (2) and (3) above, then $[f] = \sup_n [f_n]$ in $\tau(S)$. In particular, $\tau(S)$ satisfies (O1).

**Proof.** The proof is divided in two parts.

We inductively find $\lambda_m \in I_Q$ and $h_m \in P(S)$ for $m \geq 1$ such that:

1. $\lambda_{m-1} < \lambda_m$ and $\frac{m-1}{m+1} \leq \lambda_m$, for all $m$; and
2. $f_m(\lambda_{m-1}) \prec h_m \prec f_m(\lambda_m)$, for all $n < m$.

Set $\lambda_1 := \frac{1}{2}$, and define the path $h_1$ by $h_1(\lambda) = f_1(\lambda)$. Note that $0 \prec h_1 \prec f(\lambda_1)$.

Assume we have chosen $\lambda_n$ and $h_n$ for all $n < m$. For each $k = 1, \ldots, m - 1$, using that $f_k \lessgtr f_m$, we choose $\lambda_{m,k} \in I_Q$ such that $f_k(\lambda_{m-1}) \prec f_m(\lambda_{m,k})$. Let $\lambda'_m$ be the maximum of $\lambda_{m,1}, \ldots, \lambda_{m,m-1}, \frac{m}{m+1}$. Choose $\lambda_m \in I_Q$ with $\lambda'_m \prec \lambda_m$. Using **Lemma 3.9** we choose a path $h_m$ with $f_m(\lambda'_m) \prec h_m \prec f_m(\lambda_m)$.

Note that in particular we have the following relations:

$$0 \prec h_1 \prec f_1(\lambda_1) \prec h_2 \prec f_2(\lambda_2) \prec h_3 \prec f_3(\lambda_3) \prec h_4 \cdots$$

Applying **Lemma 3.10**, we choose $f \in P(S)$ with $f(\frac{n}{n+1}) = f_n(\lambda_n)$ for all $n \geq 1$. Then it is easy to check that the sequence $(\lambda_m)_{m}$ and the path $f$ satisfy conditions (1), (2) and (3).

For the second part of the proof, let $(\lambda_m)_{m \geq 1}$ be a strictly increasing sequence in $I_Q$, and let $f \in P(S)$ satisfy (1), (2) and (3). We show that $[f] = \sup_n [f_n]$ in $\tau(S)$. 

We first show that \([f_n] \leq [f]\) for each \(n \geq 1\). Fix \(n \geq 1\). To verify that \(f_n \preceq f\), let \(\lambda\) be an element in \(I_Q\). Use (1) to choose \(m \in (n < m\) and \(\lambda < \lambda_m\). Using that \(f_n\) is a path at the first step, using condition (2) at the second step, and using (3) at the last step, we obtain that
\[
f_n(\lambda) \prec f_n(\lambda_m) \prec f_{m+1}(\lambda_{m+1}) = f\left(\frac{m+1}{m+2}\right).
\]
Hence \(f_n \preceq f\), as desired.

Conversely, let \(g \in P(S)\) satisfy \(f_n \preceq g\) for all \(n \geq 1\). To show that \(f \preceq g\), take \(\lambda \in I_Q\). Choose \(m\) such that \(\lambda < \frac{m}{m+1}\). Since \(f_m \preceq g\), there exists \(\mu \in I_Q\) such that \(f_m(\lambda_m) < g(\mu)\). Using this at the last step, using that \(f\) is a path at the first step, and using condition (3) at the second step, we get
\[
f(\lambda) \prec f\left(\frac{m}{m+1}\right) = f_m(\lambda_m) \prec g(\mu).
\]
This shows that \(f \preceq g\), as desired.

**Definition 3.12.** Let \(S\) be a \(\mathcal{P}\)-semigroup, let \(f \in P(S)\), and let \(\varepsilon \in I_Q\). We define \(f_\varepsilon : I_Q \to S\) by
\[
f_\varepsilon(\lambda) := \begin{cases} f(\lambda - \varepsilon), & \text{if } \lambda > \varepsilon \\ 0, & \text{otherwise} \end{cases}
\]
We will refer to \(f_\varepsilon\) as the \(\varepsilon\)-cut down of \(f\).

**Remark 3.13.** It is easy to see that \(f_\varepsilon\) is a path in \(S\). If \(t\) is a real number, we write \(t_\varepsilon\) for \(\max\{0, t\}\). Then, under the convention that \(f(0) = 0\), we have \(f_\varepsilon(\lambda) = f((\lambda - \varepsilon)_+)\) for all \(\lambda \in I_Q\).

**Lemma 3.14.** Let \(S\) be a \(\mathcal{P}\)-semigroup, and let \(f \in P(S)\). Then \([f_\varepsilon] \ll [f_{\varepsilon'}]\) in \(\tau(S)\), for every \(\varepsilon', \varepsilon \in I_Q\) with \(\varepsilon' < \varepsilon\). Moreover, we have \([f] = \sup_{\varepsilon \in I_Q} [f_\varepsilon]\) in \(\tau(S)\).

**Proof.** It is routine to check that \([f] = \sup_{\varepsilon} [f_\varepsilon]\). Given \(\varepsilon', \varepsilon \in I_Q\) with \(\varepsilon' < \varepsilon\), note that \(f_\varepsilon = (f_{\varepsilon'})_{\varepsilon - \varepsilon'}\). Thus it is enough to show that \([f_\varepsilon] \ll [f]\) for every \(\varepsilon > 0\).

Fix \(\varepsilon > 0\). To show that \([f_\varepsilon] \ll [f]\), let \(\langle [g_n] \rangle_n\) be an increasing sequence in \(\tau(S)\) with \([f] \leq \sup_n [g_n]\). By Lemma 3.11 there exists a path \(h \in P(S)\) and an increasing sequence \(\langle \lambda_n \rangle_n\) in \(I_Q\) such that \([h] = \sup_n [g_n]\), and such that \(h(\frac{n}{n+1}) = g_n(\lambda_n)\) for all \(m \geq 1\).

Choose \(m_0 \geq 1\) with \(\frac{1}{m_0} < \varepsilon\). Since \(f \preceq h\), there exists \(\mu \in I_Q\) satisfying \(f(1 - \frac{1}{m_0}) \prec h(\mu)\). Choose \(m_1 \geq 1\) such that \(\mu < \frac{1}{m_1+1}\). Let us show that \(f_\varepsilon \preceq g_{m_1}\).

For every \(\lambda \in I_Q\), we have \(\lambda - \varepsilon < 1 - \frac{1}{m_0}\). Therefore, using that \(f\) and \(h\) are paths at the second and fourth step, respectively, and using that \(f(1 - \frac{1}{m}) \prec h(\mu)\) at the third step, we obtain that
\[
f_\varepsilon(\lambda) = f((\lambda - \varepsilon)_+) \prec f(1 - \frac{1}{m}) \prec h(\mu) \prec h(\frac{m_1}{m_1+1}) = g_{m_1}(\lambda_{m_1}),
\]
for every \(\lambda \in I_Q\). This proves that \([f_\varepsilon] \leq [g_{m_1}]\), as desired. \(\square\)

**Theorem 3.15.** Let \(S\) be a \(\mathcal{P}\)-semigroup. Then \(\tau(S)\) is a Cu-semigroup.

**Proof.** By Proposition 3.9, Lemma 3.11 and Lemma 3.14, \(\tau(S)\) is a positively ordered monoid that satisfies axioms (O1) and (O2). It remains to show that \(\tau(S)\) satisfies (O3) and (O4).

To verify (O3), let \([f'], [f], [g'], [g] \in \tau(S)\) satisfy \([f'] \ll [f]\) and \([g'] \ll [g]\). Using that \([f] = \sup_{\varepsilon} [f_\varepsilon]\), we can choose \(\varepsilon_1 \in I_Q\) such that \([f'] \leq [f_{\varepsilon_1}]\). Similarly we choose \(\varepsilon_2\) for \([g]\). Set \(\varepsilon := \min\{\varepsilon_1, \varepsilon_2\}\). We then have \([f'] \leq [f_\varepsilon]\) and \([g'] \leq [g_\varepsilon]\).

Using that \(f_\varepsilon + g_\varepsilon = (f+g)_\varepsilon\) at the third step, and using Lemma 3.14 at the fourth step, we deduce that
\[
[f'] + [g'] \leq [f_\varepsilon] + [g_\varepsilon] = [f_\varepsilon] + [g_\varepsilon] = [(f + g)_\varepsilon] \ll [f + g] = [f] + [g],
\]
which implies that \([f'] + [g'] \ll [f] + [g]\), as desired.

To prove (O4), let \(\langle (f_n) \rangle_n\) and \(\langle (g_n) \rangle_n\) be two increasing sequences in \(\tau(S)\). It is clear that \(\sup_n ([f_n] + [g_n]) \leq \sup_n [f_n] + \sup_n [g_n]\). Let us prove the converse inequality.

By Lemma 3.11 there exist \(f, g \in P(S)\) and increasing sequences \((\lambda_m)_m\) and \((\mu_m)_m\) in \(I_Q\) such that \([f] = \sup_n [f_n]\) and \([g] = \sup_n [g_n]\), and such that \(f(\frac{m}{n+1}) = f_m(\lambda_m)\) and \(g(\frac{m}{n+1}) = g_m(\mu_m)\) for all \(m \in \mathbb{N}\). Given \(\lambda \in I_Q\), choose \(m \in \mathbb{N}\) with \(\lambda \prec \frac{m}{n+1} \). Choose \(\tilde{\lambda} \in I_Q\) such that \(\lambda_m, \mu_m \prec \tilde{\lambda}\). We deduce that

\[
\begin{align*}
  f(\lambda) + g(\lambda) \prec (f + g)(\frac{m}{n+1}) = f_m(\lambda_m) + g_m(\mu_m) \prec f_n(\tilde{\lambda}) + g_n(\tilde{\lambda}).
\end{align*}
\]

It follows that \([f] + [g] \leq \sup_n ([f_n] + [g_n]), as desired. This verifies (O4) and completes the proof. \(\square\)

The following result provides a useful criterion for compact containment in \(\tau(S)\).

**Lemma 3.16.** Let \(S\) be a \(\mathcal{P}\)-semigroup, and let \(f', f\) be elements in \(P(S)\). Then \([f'] \ll [f]\) in \(\tau(S)\) if and only if there exists \(\mu \in I_Q\) such that \(f' \prec f(\mu)\).

**Proof.** Assume that \([f'] \ll [f]\). Since \([f] = \sup_n [f_n]\), there exists \(\delta \in I_Q\) such that \([f'] \leq [f_\delta]\). Let us show that \(\mu = 1 - \delta\) has the desired properties, that is, \(f' \prec f(\mu)\). Given \(\lambda \in I_Q\), there is \(\mu' \in I_Q\) with \(f'(\lambda) \prec f_\delta(\mu')\). Using that \((\mu' - \delta)_+ < 1 - \delta = \mu\) at the last step, we deduce that

\[
\begin{align*}
  f'(\lambda) \prec f_\delta(\mu') = f((\mu' - \delta)_+) \prec f(\mu),
\end{align*}
\]

as desired.

Conversely, suppose that there exists \(\mu \in I_Q\) with \(f' \prec f(\mu)\). Then, for every \(\mu'\) with \(\mu < \mu' < 1\) we have \([f'] \leq [f_{\mu'}] \ll [f]\), as desired. \(\square\)

**Lemma 3.17.** Let \(S\) and \(T\) be \(\mathcal{P}\)-semigroups, and let \(\alpha \colon S \to T\) be a \(\mathcal{P}\)-morphism. Then, for every \(f \in P(S)\), the map \(\alpha \circ f \colon I_Q \to T\) belongs to \(P(T)\). Moreover, the induced map \(\tau(\alpha) \colon \tau(S) \to \tau(T)\) given by

\[
\tau(\alpha)([f]) := [\alpha \circ f],
\]

for \(f \in P(S)\), is a well-defined \(\mathcal{Cu}\)-morphism.

**Proof.** Given \(f \in P(S)\), it is easy to see that \(\alpha \circ f\) belongs to \(P(T)\). Moreover, given \(f, g \in P(S)\) with \(f \preceq g\) we have \(\alpha \circ f \preceq \alpha \circ g\). This shows that \(\tau(\alpha)\) is well-defined and order-preserving. It is also easy to see that \(\tau(\alpha)\) preserves addition and the zero element.

To show that \(\tau(\alpha)\) preserves the way-below relation, let \(f', f \in P(S)\) satisfy \([f'] \ll [f]\) in \(\tau(S)\). By Lemma 3.16 there is \(\mu \in I_Q\) with \(f' \prec f(\mu)\). Since \(\alpha\) is a \(\mathcal{P}\)-morphism, we obtain that \(\alpha \circ f' \prec (\alpha \circ f)(\mu)\). A second usage of Lemma 3.16 implies that \([\alpha \circ f'] \ll [\alpha \circ f]\), as desired.

To show that \(\tau(\alpha)\) preserves suprema of increasing sequences, let \(\langle [f_n] \rangle_n\) be such a sequence in \(\tau(S)\). By Lemma 3.11 there exist \(f \in P(S)\) and a strictly increasing sequence \((\lambda_n)_n\) in \(I_Q\) such that the following conditions are satisfied:

1. We have \(\sup_n \lambda_m = 1\).
2. We have \(f_m(\lambda_m) \prec f_1(\lambda_1)\), whenever \(n, m < l\).
3. We have \(f_n(\frac{m}{n+1}) = f_n(\lambda_n)\) for all \(n \geq 1\).

Further, for every \(f \in P(S)\) satisfying these conditions, we have \([f] = \sup_n [f_n]\).

To show that \([\alpha \circ f] = \sup_n [\alpha \circ f_n]\), we verify that the path \(\alpha \circ f\) and the sequence \((\lambda_n)_n\) satisfy the analogs of the above conditions with respect to the sequence \((\alpha \circ f_n)_n\). Condition (1) is unchanged. To verify the analog of condition (2), let
of increasing sequences. We denote the set of generalized
Example 3.19. Consider the
P
simple objects as
defines a covariant functor, as claimed. every pair of composable
P
≺

We define on
N
is a map that preserves addition, order, the zero element and suprema
O4) If (a
O1) Every increasing sequence (a
O2) If (a
≤ l. Since α is a P-morphism, we have (α \circ f_n)(\lambda_m) \prec (\alpha \circ f_l)(\lambda_l), as desired. The analog of (3) holds, since
(a \circ f)(\frac{\alpha}{\pi+1}) = (\alpha \circ f_n)(\lambda_n),
for every n ≥ 1. Thus, the path α \circ f satisfies conditions (1), (2) and (3) for the sequence (α \circ f_n)_n, which implies that [α \circ f] = sup\_n[α \circ f_n]. Using this in the third step, we deduce that
\tau(\alpha)(sup\_n[f_n]) = \tau(\alpha)([f]) = sup\_n[\alpha \circ f_n] = sup\_n[\tau(\alpha)([f_n])],
as desired. Altogether, we have that \tau(\alpha) is a Cu-morphism.

Proposition 3.18. The \tau-construction defines a covariant functor \tau: P \rightarrow Cu by sending a P-semigroup S to the Cu-semigroup \tau(S) (see [Theorem 3.13]), and by sending a P-morphism \alpha: S \rightarrow T to the Cu-morphism \tau(\alpha): \tau(S) \rightarrow \tau(T) (see [Lemma 3.17]).

Proof. It follow easily from the construction that \tau(id_S) = id_{\tau(S)} for every P-semigroup S. It is also straightforward to check that \tau(\alpha \circ \beta) = \tau(\alpha) \circ \tau(\beta) for every pair of composable P-morphisms \alpha and \beta. This shows that the \tau-construction defines a covariant functor, as claimed.

Although the \tau-construction is a useful tool to derive Cu-semigroups from such simple objects as P-semigroups, the next example shows that without additional care the \tau-construction may just produce a trivial object.

Example 3.19. Consider \mathbb{N} = \{0, 1, 2 \ldots\} with the usual structure as a monoid. We define \prec on \mathbb{N} by setting k \prec l if k < l or k = l = 0. It is easy to check that (\mathbb{N}, \prec) is a P-semigroup, and that the only path in P(\mathbb{N}, \prec) is the constant path with value 0. It follows that \tau(\mathbb{N}, \prec) \cong \{0\}.

4. The category Q

The category P introduced in the previous section, though useful in certain situations to construct Cu-semigroups from semigroups with very little structure, is too general to provide a nice categorical relation from P to Cu. In this section we introduce a subcategory of P, which we denote by Q, where Cu can be embedded as a full subcategory, and in such a way that the restriction of the \tau-construction from Section 3 defines a coreflection Q \rightarrow Cu; see [Theorem 4.12]. In Subsection 4.2 we introduce the notion of internal-hom in the category Q, which will be needed for the corresponding construction in Cu.

4.1. The coreflection \tau: Q \rightarrow Cu. Recall the definition of an additive auxiliary relation from [Definition 2.2].

Definition 4.1. A Q-semigroup is a positively ordered monoid S together with an additive, auxiliary relation \prec on S such that the following conditions are satisfied:
1) Every increasing sequence (a\_n)_n in S has a supremum sup\_n a\_n in S.
2) (a\_n)_n and (b\_n)_n are increasing sequences in S, then sup\_n(a\_n + b\_n) = sup\_n a\_n + sup\_n b\_n.

We often drop the reference to the auxiliary relation and simply call S a Q-semigroup.

Given Q-semigroups S and T, a Q-morphism from S to T is a map S \rightarrow T that preserves addition, order, the zero element, the auxiliary relation and suprema of increasing sequences. We denote the set of Q-morphisms by Q(S, T). A generalized Q-morphism is a map that preserves addition, order, the zero element and suprema of increasing sequences. We denote the set of generalized Q-morphisms by Q[S, T].
We let $Q$ be the category whose objects are $Q$-semigroups and whose morphisms are $Q$-morphisms.

**Remarks 4.2.** (1) Axioms (O1) and (O4) in [Definition 4.1](#) are the same as in [Definition 2.3](#). Note that a generalized $Q$-morphism is a $Q$-morphism if and only if it preserves the auxiliary relation. Moreover, generalized $Q$-morphisms are precisely the Scott continuous $\mathcal{P}$-morphisms. (See for example [GHK'03](#) Proposition II-2.1, p.157.)

(2) Let $S, T$ be $Q$-semigroups. The sets $Q[S,T]$ and $Q(S,T)$ of (generalized) $Q$-morphism are positively ordered monoids, when equipped with the pointwise addition and order. It is easy to see that $Q[S,T]$ satisfies (O1) and (O4). In [Definition 4.17](#) we will define an auxiliary relation on $Q[S,T]$ giving it the structure of a $Q$-semigroup; see [Proposition 4.18](#).

**4.3.** We define a functor $\iota: Cu \to Q$ as follows: Given a $Cu$-semigroup $S$, the (sequential) way-below relation $\ll$ is an additive auxiliary relation on $S$. It follows that $(S, \ll)$ is a $Q$-semigroup, and we let $\iota$ map $S$ to $(S, \ll)$.

Further, given $Cu$-semigroups $S$ and $T$, a map $\varphi: S \to T$ is a $Cu$-morphism if and only if $\varphi: (S, \ll) \to (T, \ll)$ is a $Q$-morphism. We let $\iota$ map a $Cu$-morphism to itself, considered as a $Q$-morphism. It is easy to see that this defines a functor from $Cu$ to $Q$.

From our considerations, we clearly have:

**Proposition 4.4.** The functor $\iota: Cu \to Q$ from [Paragraph 4.3](#) embeds $Cu$ as a full subcategory of $Q$.

Every $Q$-semigroup can be considered as a $\mathcal{P}$-semigroup by forgetting its partial order. Therefore, if $S$ is a $Q$-semigroup with auxiliary relation $\prec$, then a path in $S$ is a map $f: I_Q \to S$ such that $f(\lambda') \prec f(\lambda)$ whenever $\lambda', \lambda \in I_Q$ satisfy $\lambda' < \lambda$; see [Definition 3.5](#) and [Notation 3.8](#). Recall that $P(S)$ denotes the set of paths in $S$.

**Definition 4.5.** Let $S$ be a $Q$-semigroup, and let $f \in P(S)$. We define the **endpoint** of $f$, denoted by $f(1)$, as $f(1) := \sup_{\lambda \in I_Q} f(\lambda)$.

**Proposition 4.6.** Let $S$ be a $Q$-semigroup, and let $f, g \in P(S)$. Then:

1. We have $(f + g)(1) = f(1) + g(1)$ in $S$.
2. If $f \preceq g$, then $f(1) \leq g(1)$ in $S$.
3. If $[f] \ll [g]$ in $\tau(S)$, then $f(1) < g(1)$.
4. If $([f_n])_n$ is an increasing sequence in $\tau(S)$ and $[f] = \sup_n [f_n]$, then $f(1) = \sup_n f_n(1)$ in $S$.

**Proof.** (1): This is a consequence of the fact that $S$ satisfies (O4).

(2): Given $\lambda \in I_Q$, using that $f \preceq g$, there is $\mu \in I_Q$ with $f(\lambda) \prec g(\mu) \leq g(1)$. Taking the supremum over $\lambda$, we obtain that $f(1) \leq g(1)$.

(3): Assuming $[f] \ll [g]$, we use [Lemma 3.16](#) to choose $\mu \in I_Q$ with $f \prec g(\mu)$. Then $f(1) \leq g(\mu) \prec g(1)$.

(4): Let $([f_n])_n$ be an increasing sequence in $\tau(S)$, and let $[f] = \sup_n [f_n]$. By (2), the endpoint of a path only depends on its equivalence class with respect to the relation $\sim$ from [Definition 3.3](#).

By [Lemma 3.11](#) there are $f' \in P(S)$ and an increasing sequence $(\lambda_n)_n$ in $I_Q$ such that $\sup_n \lambda_n = 1$ and $[f'] = \sup_n [f_n]$, and such that $f'(\frac{n}{n+1}) = f_n(\lambda_n)$ for all $n \in \mathbb{N}$. Using that $f' \sim f$ at the first step, and using the above property of $f'$ at the fourth step we obtain that

$$f(1) = f'(1) = \sup_{\lambda \in I_Q} f'(\lambda) = \sup \left( (\frac{n}{n+1}) \right) = \sup_n f_n(\lambda_n) \leq \sup_n f_n(1).$$
For each \( n \), we have \( f_n \leq f \) and therefore \( f_n(1) \leq f(1) \) by (2). It follows that \( \sup_n f_n(1) \leq f(1) \), and therefore \( f(1) = \sup_n f_n(1) \), as desired. \( \square \)

By Proposition 4.6 the endpoint of a path only depends on the equivalence class in \( \tau(S) \). Therefore, the following definition makes sense.

**Definition 4.7.** Let \( S \) be a \( \mathbb{Q} \)-semigroup. We define a map \( \varphi_S : \tau(S) \to S \) by
\[ \varphi_S([f]) := f(1), \]
for all \( f \in P(S) \). We refer to \( \varphi_S \) as the endpoint map.

**Proposition 4.8.** Let \( S \) be a \( \mathbb{Q} \)-semigroup. Then the endpoint map \( \varphi_S : \tau(S) \to S \) is a well-defined \( \mathbb{Q} \)-morphism (when considering \( \tau(S) \) as a \( \mathbb{Q} \)-morphism via the inclusion functor \( i \) from Paragraph 4.3).

Moreover, the endpoint map is natural in the sense that \( \alpha \circ \varphi_S = \varphi_T \circ \tau(\alpha) \) for every \( \mathbb{Q} \)-morphism \( \alpha : S \to T \) between \( \mathbb{Q} \)-semigroups \( S \) and \( T \). This means that the following diagram commutes:
\[
\begin{array}{ccc}
\tau(S) & \xrightarrow{\varphi_S} & S \\
\tau(\alpha) \downarrow & & \downarrow \alpha \\
\tau(T) & \xrightarrow{\varphi_T} & T
\end{array}
\]

**Proof.** It follows directly from Proposition 4.6 that \( \varphi_S \) is a well-defined \( \mathbb{Q} \)-morphism. To show the commutativity of the diagram, let \( f \in P(S) \). Using that \( \alpha \) preserves suprema of increasing sequences at the second step, we deduce that
\[ \alpha(\varphi_S([f])) = \alpha \left( \sup_{\lambda \in I_0} f(\lambda) \right) = \sup_{\lambda \in I_0} \alpha(f(\lambda)) = \varphi_T(\tau(\alpha)([f])), \]
as desired. \( \square \)

**Remark 4.9.** The naturality of the endpoint map as formulated in Proposition 4.8 means precisely that the \( \mathbb{Q} \)-morphisms \( \varphi_S \), for \( S \) ranging over the objects in \( \mathbb{Q} \), form the components of a natural transformation from \( i \circ \tau \) to the identity functor on \( \mathbb{Q} \).

In general, the endpoint map is neither surjective nor injective; see Examples 4.13 and 4.14. We now show that \( \varphi_S \) is an order-isomorphism if (and only if) \( S \) is a \( \mathbb{Cu} \)-semigroup.

**Proposition 4.10.** Let \( S \) be a \( \mathbb{Cu} \)-semigroup, considered as a \( \mathbb{Q} \)-semigroup \( (S, \leq) \). Then the endpoint map \( \varphi_S : \tau(S, \leq) \to S \) is an order-isomorphism.

**Proof.** We first prove that \( \varphi_S \) is an order-embedding. Let \([f], [g] \in \tau(S, \leq)\) satisfy \( \varphi_S([f]) \leq \varphi_S([g]) \). Then, by definition, \( \sup_\mu f(\mu) \leq \sup_\mu g(\mu) \). To show that \( f \preceq g \), let \( \lambda \in I_Q \). Choose \( \hat{\lambda} \in I_Q \) with \( \lambda < \hat{\lambda} \). We deduce that
\[ f(\lambda) \leq f(\hat{\lambda}) \leq \sup_\mu f(\mu) \leq \sup_\mu g(\mu). \]
Therefore, there exists \( \mu \in I_Q \) such that \( f(\lambda) \leq g(\mu) \). Choose \( \bar{\mu} \in I_Q \) with \( \mu < \bar{\mu} \). Then \( f(\lambda) \leq g(\bar{\mu}) \). This implies that \( f \preceq g \) and thus \([f] \preceq [g] \), as desired.

To show that \( \varphi_S \) is surjective, let \( s \in S \). Choose a \( \preceq \)-increasing chain \((s_\lambda)_{\lambda \in [0, 1)} \) as in Proposition 2.10. In particular, we have \( s = \sup_\lambda s_\lambda \), and \( s_\lambda \preceq s_\lambda \) whenever \( \lambda', \lambda \in I_Q \) satisfy \( \lambda' < \lambda \). Thus, if we define \( f : I_Q \to S \) by \( f(\lambda) := s_\lambda \), for \( \lambda \in I_Q \), then \( f \) belongs to \( P(S, \leq) \). By construction, \( \varphi_S([f]) = s \), as desired. \( \square \)

Given \( \mathbb{Q} \)-semigroups \( S \) and \( T \), recall that we equip the set of \( \mathbb{Q} \)-morphisms \( \mathbb{Q}(S, T) \) with pointwise order and addition; see Remarks 4.2.
Proposition 4.11. Let \( T \) be a Cu-semigroup, let \( S \) be a \( Q \)-semigroup, and let \( \varphi_S : \tau(S) \to S \) be the endpoint map from Definition 4.7. Then:

(1) For every \( Q \)-morphism \( \alpha : T \to S \) there exists a Cu-morphism \( \bar{\alpha} : T \to \tau(S) \) such that \( \varphi_S \circ \bar{\alpha} = \alpha \).

(2) We have \( \varphi_S \circ \beta \leq \varphi_S \circ \gamma \) if and only if \( \beta \leq \gamma \), for any pair of Cu-morphisms \( \beta, \gamma : T \to \tau(S) \).

Statement (1) means that for every \( \alpha \) one can find \( \bar{\alpha} \) making the following diagram commute:

\[
\begin{array}{ccc}
\tau(S) & \xrightarrow{\varphi_S} & S \\
\downarrow{\alpha} & & \downarrow{\bar{\alpha}} \\
T. & \xrightarrow{{\tau(S)}} & T.
\end{array}
\]

Proof. To show (1), let \( \alpha \) be given. Since \( T \) is a Cu-semigroup, it follows from Proposition 4.10 that \( \varphi_T : \tau(T, \ll) \to T \) is an order-isomorphism. Set \( \bar{\alpha} := \tau(\alpha) \circ \varphi_T \), which is clearly a Cu-morphism. By Proposition 4.8 we have \( \varphi_S \circ \tau(\alpha) = \alpha \circ \varphi_T \). It follows that \( \varphi_S \circ \bar{\alpha} = \alpha \). The maps are shown in the following diagram:

\[
\begin{array}{ccc}
\tau(S) & \xrightarrow{\varphi_S} & S \\
\downarrow{\tau(\alpha)} & & \downarrow{\bar{\alpha}} \\
\tau(T, \ll) & \xrightarrow{\varphi_T} & T.
\end{array}
\]

To show (2), let \( \beta, \gamma : T \to \tau(S) \) be Cu-morphisms. It is clear that \( \beta \leq \gamma \) implies that \( \varphi_S \circ \beta \leq \varphi_S \circ \gamma \). Thus let us assume that \( \varphi_S \circ \beta \leq \varphi_S \circ \gamma \).

To show that \( \beta \leq \gamma \), let \( t \in T \). Using that \( T \) satisfies (O2), choose a \( \ll \)-increasing sequence \( (t_n)_n \) in \( T \) with supremum \( t \). Fix \( n \), and choose paths \( f_n, g_n, \gamma \) in \( P(S) \) with \( \beta(t_n) = [f_n] \), and \( \gamma(t_n) = [g_n] \), and \( \gamma(t) = [g] \). Since \( \gamma \) preserves the way-below relation, we have \( [g_n] \ll [g] \) in \( \tau(S) \). By Lemma 3.16 we can choose \( \mu \in I_Q \) such that \( g_n(\lambda) \leq g(\mu) \) for all \( \lambda \in I_Q \). Passing to the supremum over \( \lambda \), we obtain that \( g_n(1) \leq g(\mu) \). Using this at the last step, and using the assumption that \( \varphi_S \circ \beta \leq \varphi_S \circ \gamma \) at the second step, we deduce that

\[
f_n(\lambda) \ll f_n(1) = \varphi_S(\beta(t_n)) \leq \varphi_S(\gamma(t_n)) = g_n(1) \leq g(\mu),
\]

for every \( \lambda \in I_Q \). By definition, we have that \( f_n \preceq g \), and hence \( \beta(t_n) \leq \gamma(t) \).

Using that \( \beta \) preserves suprema of increasing sequences at the second step, and using the above observation \( \beta(t_n) \leq \gamma(t) \) for each \( n \) at the last step, we deduce that

\[
\beta(t) = \beta \left( \sup t_n \right) = \sup \beta(t_n) \leq \gamma(t),
\]

as desired. \( \square \)

Theorem 4.12. The category Cu is a coreflective, full subcategory of \( Q \); the functor \( \tau : Q \to Cu \) is a right adjoint to the inclusion functor \( i : Cu \to Q \) from Paragraph 4.3.

More precisely, let \( S \) be a \( Q \)-semigroup, let \( \varphi_S : \tau(S) \to S \) be the endpoint map from Definition 4.7, and let \( T \) be a Cu-semigroup. Then the assignment that sends a Cu-morphism \( \beta : T \to \tau(S) \) to the \( Q \)-morphism \( \varphi_S \circ \beta \) defines a natural bijection between the following morphism sets:

\[
Cu(T, \tau(S)) \cong Q(T, S).
\]

Moreover, this bijection respects the structure of the bimorphism sets as positively ordered monoids.
Recall that a path in \( \mathbb{Q} \)-or \( \mathbb{R} \)-construction. We denote these \( \mathbb{C} \)-semigroups by \( M_1 \) and \( M_\infty \) since they turn out to be the Cuntz semigroups of \( \Pi_1 \) and \( \Pi_\infty \)-factors, respectively; see Proposition 4.16. These are also examples where the endpoint map is not injective.

**Example 4.15.** In general, the endpoint map is not surjective. Consider for example \( \mathbb{N} = \{0, 1, 2, \ldots, \infty\} \) with auxiliary relation \( \prec \) given by \( k \prec l \) if and only if \( k < l \) or \( k = l = 0 \). As in Example 3.19 we obtain that the only path in \( (\mathbb{N}, \prec) \) is the constant path with value 0. It follows that \( \tau(\mathbb{N}, \prec) = \{0\} \), and thus the endpoint map of this example is clearly not surjective.

**Example 4.14.** Consider \( \mathbb{F} := [0, \infty] \), with its usual structure as a positively ordered monoid. We define two relations \( \prec_1 \) and \( \prec_\infty \) on \( \mathbb{S} \) as follows: given \( a, b \in \mathbb{S} \) we set \( a \prec_1 b \) if and only if \( a < \infty \) and \( a \leq b \); and we set \( a \prec_\infty b \) if and only if \( a \leq b \). It is easy to deduce that \( (\mathbb{F}, \prec_1) \) and \( (\mathbb{F}, \prec_\infty) \) are \( \mathbb{Q} \)-semigroups. We set

\[
M_1 := \tau(\mathbb{F}, \prec_1) \quad \text{and} \quad M_\infty := \tau(\mathbb{F}, \prec_\infty).
\]

Let us compute the precise structure of \( M_1 \) and \( M_\infty \). For the most part, the argument is the same in both cases, and we use \( \prec_* \) to stand for either \( \prec_1 \) or \( \prec_\infty \). Recall that a path in \( P(\mathbb{F}, \prec_*) \) is a \( \prec_* \)-increasing map \( f : I_Q \to \mathbb{F} \). Given a path \( f \), we let \( f(1) \) denote the endpoint, that is, \( f(1) = \sup_{\lambda \in I_Q}\{ f(\lambda) \} \); see Definition 4.5.

Let \( f, g \) be elements in \( P(\mathbb{F}, \prec_*) \). If \( f \preceq g \), then \( f(1) \leq g(1) \), by Proposition 4.6 (2). Conversely, if \( f(1) < g(1) \), then it is easy to deduce that \( f \npreceq g \). However, if \( f(1) = g(1) \) then we do not necessarily have \( f \sim g \). For example, the paths \( f \) and \( g \) given by \( f(\lambda) = \lambda \) and \( g(\lambda) = 1 \) have the same endpoint but we do not have \( g \npreceq f \).

It is clear though that the equivalence class of a path only depends on its definition in \( (1 - \varepsilon, 1) \cap I_Q \), for some \( \varepsilon > 0 \). Therefore, all eventually constant paths with the same endpoint are equivalent and they majorize any path with the same endpoint. Furthermore, it is clear that two paths with equal endpoint which are not eventually constant are in fact equivalent.

Thus, for every \( a \in (0, \infty) \) there are exactly two equivalence classes of paths with endpoint \( a \): the classes \( [f_a'] \) and \( [f_a] \) with \( f_a' \) and \( f_a \) given by \( f_a'(\lambda) = \lambda a \) and \( f_a(\lambda) = a \), for \( \lambda \in I_Q \).

The endpoints 0 and \( \infty \) are particular: The only path with endpoint 0 is the constant path \( f_0 \) with value 0. So far, there was no difference between the case of auxiliary relation \( \prec_1 \) or \( \prec_\infty \).

The only difference appears now for paths with endpoint \( \infty \). There is no \( \prec_1 \)-increasing path that is (eventually) constant with value \( \infty \). Therefore, all paths in \( P(\mathbb{F}, \prec_1) \) with endpoint \( \infty \) are equivalent to \( f_\infty' \) given by \( f_\infty'(\lambda) = \frac{1}{\lambda} \). On the other hand, \( P(\mathbb{F}, \prec_\infty) \) also contains the constant path \( f_\infty \) with value \( \infty \). We obtain that

\[
M_1 = \{ [f_0], [f_a'], [f_a], [f_\infty'] : a \in (0, \infty) \}, \quad \text{and} \quad M_\infty = M_1 \cup \{ [f_\infty] \}.
\]
Thus, the only difference between $M_1$ and $M_\infty$ is that $M_\infty$ contains an additional infinite element. It is easy to see that the natural map $M_1 \to M_\infty$ is an additive order-embedding. Therefore, it is enough to describe the order and addition in $M_\infty$. We have $[f_a] \leq [f'_b] \leq [f_a] \leq [f'_a] \leq [f_\infty]$ for every $a \in [0, \infty)$. Further, it is easy to see that for every $a, b \in (0, \infty)$ we have $[f'_a] \leq [f_b]$ if and only if $a \leq b$; and we have $[f_a] \leq [f'_a]$ if and only if $a < b$. We have $[f'_a] < [f_\infty]$.

It is straightforward to check that the addition in $M_\infty$ is given by

$$[f_a] + [f_b] = [f_{a+b}], \quad [f'_a] + [f_b] = [f'_a] + [f_b] = [f'_{a+b}],$$

for $a \in [0, \infty]$ and $b \in [0, \infty)$. We have that $[f'_a] + [f_\infty] = [f_\infty]$.

Abusing notation, we use $a'$ and $a$ to denote the classes of $f_a'$ and $f_a$ in $M_\infty$, that is, we set $a' := [f_a']$ and $a := [f_a]$, for $a \in (0, \infty)$. Further, we use 0 to denote the classes of $f_0$.

The compact elements in $M_1$ are 0 and $a$ for $a \in (0, \infty)$. The soft elements in $M_1$ are 0 and $a'$ for $a \in (0, \infty)$. The additional element $\infty$ in $M_\infty$ is both soft and compact.

The endpoint map $M_1 \to \mathbb{P}$ is not injective since it sends both $[f_a]$ and $[f'_a]$ to $a$, for every $a \in (0, \infty)$. Analogously, the endpoint map $M_\infty \to \mathbb{P}$ is not injective either.

**Remark 4.15.** The Cu-semigroups $M_1$ and $M_\infty$ from Example 4.14 are massive objects in the sense that they contain uncountably many compact elements and are therefore not countably-based. (A Cu-semigroup $S$ is countably-based if there exists a countable subset $B \subseteq S$ such that every element of $S$ is the supremum of an increasing sequence from $B$. Every such $B$ must contain all compact elements of $S$; see [APT14] Remarks 3.1.3.)

By [APT14] Proposition 3.2.3, the Cuntz semigroup of a separable C$^*$-algebra is countably-based. This shows that $M_1$ and $M_\infty$ cannot be realized as Cuntz semigroups of separable C$^*$-algebras.

**Proposition 4.16.** We have $\text{Cu}(M) \cong M_1$ for every II$_1$-factor $M$; and we have $\text{Cu}(N) \cong M_\infty$ for every II$_\infty$-factor $N$.

**Proof.** Let $M$ be a II$_1$-factor $M$, let $\tau: M_+ \to [0, 1]$ denote its unique tracial state, and let $\bar{\tau}: (M \otimes \mathbb{K})_+ \to [0, \infty]$ denote the unique extension to a tracial weight on the stabilization. Given projections $p, q \in (M \otimes \mathbb{K})_+$ we have $p \precsim q$ if and only if $\bar{\tau}(p) \leq \bar{\tau}(q)$. Moreover, for every $t \in [0, \infty)$, there exists a projection $p \in (M \otimes \mathbb{K})_+$ with $\bar{\tau}(p) = t$. It follows that the Murray-von Neumann semigroup $V(M)$ is isomorphic to $[0, \infty)$, with the usual structure as a positively ordered monoid.

Recall that an interval in a positively ordered monoid is a nonempty, upward directed, order-hereditary subset. An interval is called countably-generated if it contains a countable cofinal subset. By [ABP11] Theorem 6.4, the Cuntz semigroup of a $\sigma$-unital C$^*$-algebra $A$ with real rank zero can be computed as $\text{Cu}(A) \cong \Lambda_\sigma(V(A))$, the set of countably-generated intervals in $V(A)$, with the natural addition and order given by set inclusion. By [BP91] Proposition 1.3, every von Neumann algebra has real rank zero.

It is easy to see that the (countably-generated) intervals of $[0, \infty)$ are given as: $I_0 := \{0\}; I'_a := [0, a)$ and $I_a := [0, a]$, for $a \in (0, \infty)$; and $I'_\infty := [0, \infty)$. We obtain an order-isomorphism $\Lambda_\sigma([0, \infty)) \cong M_1$ by mapping $I_a$ to $[f_a]$, for $a \in (0, \infty)$, and by mapping $I'_a$ to $[f'_a]$, for $a \in (0, \infty)$. Together, we obtain order-isomorphisms:

$$\text{Cu}(M) \cong \Lambda_\sigma(V(M)) \cong \Lambda_\sigma([0, \infty)) \cong M_1.$$

Let $N$ be a II$_\infty$-factor. The argument runs analogous to the II$_1$-case, with the difference that $N$ contains infinite projections. One obtains that $V(N) \cong [0, \infty]$. It
follows that
\[ \text{Cu}(N) \cong \Lambda_\alpha(V(N)) \cong \Lambda_\alpha([0, \infty]) \cong M_\infty, \]
as desired. \qed

4.2. The internal hom in \( Q \).

**Definition 4.17.** Let \( S \) and \( T \) be \( \mathcal{Q} \)-semigroups. We define a binary relation \( \prec \) on the set of generalized \( \mathcal{Q} \)-morphisms \( \mathcal{Q}[S, T] \) by setting \( \varphi \prec \psi \) if and only if \( \varphi(a') < \psi(a) \) for all \( a', a \in S \) with \( a' \prec a \).

**Proposition 4.18.** Let \( S \) and \( T \) be \( \mathcal{Q} \)-semigroups. Then the relation \( \prec \) on \( \mathcal{Q}[S, T] \), as defined in [Definition 4.17], is an auxiliary relation. Moreover, \( \mathcal{Q}[S, T], \prec \) is a \( \mathcal{Q} \)-semigroup.

**Proof.** Since addition and order in \( \mathcal{Q}[S, T] \) are defined pointwise, it is easy to verify that \( \mathcal{Q}[S, T] \) is a positively ordered monoid. Given an increasing sequence \( (\varphi_n)_n \) in \( \mathcal{Q}[S, T] \), let \( \varphi : S \to T \) be the pointwise supremum, that is, \( \varphi(s) := \sup_s \varphi_n(s) \), for \( s \in S \). Then clearly \( \varphi \) is a generalized \( \mathcal{Q} \)-morphism and \( \sup_s \varphi_n = \varphi \) in \( \mathcal{Q}[S, T] \). Thus, \( \mathcal{Q}[S, T] \) satisfies (O1). It is also clear that taking suprema is compatible with addition and hence \( \mathcal{Q}[S, T] \) also satisfies (O4).

Next, note that \( \prec \) is an auxiliary relation on \( \mathcal{Q}[S, T] \). To show that \( \prec \) is additive, let \( \varphi', \varphi, \psi', \psi \in \mathcal{Q}[S, T] \) satisfy \( \varphi' \prec \varphi \) and \( \psi' \prec \psi \). Given \( s', s \in S \) with \( s' \prec s \), we use that the auxiliary relation in \( T \) is additive at the second step to deduce that
\[ (\varphi' + \psi')(s') = \varphi'(s') + \psi'(s') \prec \varphi(s) + \psi(s) = (\varphi + \psi)(s). \]
Hence, \( \varphi' + \psi' \prec \varphi + \psi \), as desired. Therefore, \( \mathcal{Q}[S, T], \prec \) is a \( \mathcal{Q} \)-semigroup. \( \square \)

Next, we define bimorphisms in the category \( \mathcal{Q} \) analogous to the definition of \( \text{Cu} \)-bimorphisms; see [Definition 2.15]. Recall the definition of PoM-bimorphisms from [Paragraph 2.1].

**Definition 4.19.** Let \( S, T \) and \( P \) be \( \mathcal{Q} \)-semigroups, and let \( \varphi : S \times T \to P \) be a PoM-bimorphism. We say that \( \varphi \) is a \( \mathcal{Q} \)-bimorphism if it satisfies the following conditions:

1. We have that \( \sup_k \varphi(a_k, b_k) = \varphi(\sup_k a_k, \sup_k b_k) \), for every increasing sequences \( (a_k)_k \) in \( S \) and \( (b_k)_k \) in \( T \).
2. If \( a', a \in S \) and \( b', b \in T \) satisfy \( a' \prec a \) and \( b' \prec b \), then \( \varphi(a', b') \prec \varphi(a, b) \).

We denote the set of \( \mathcal{Q} \)-bimorphisms by \( \text{BiQ}(S \times T, P) \).

Given \( \mathcal{Q} \)-semigroups \( S, T \) and \( P \), we equip \( \text{BiQ}(S \times T, P) \) with pointwise order and addition, giving it the structure of a positively ordered monoid. Similarly, we consider the set of \( \mathcal{Q} \)-morphisms between two \( \mathcal{Q} \)-semigroups as a positively ordered monoid with the pointwise order and addition.

The proof of the following result follows straightforward from the definition of \( \mathcal{Q} \)-bimorphisms and is therefore omitted.

**Lemma 4.20.** Let \( S, T \) and \( P \) be \( \mathcal{Q} \)-semigroups, and let \( \varphi : S \times T \to P \) be a \( \mathcal{Q} \)-bimorphism. For each \( a \in S \), define \( \varphi_a : T \to P \) by \( \varphi_a(b) = \varphi(a, b) \). Then \( \varphi_a \) belongs to \( \mathcal{Q}[T, P] \). Moreover, if \( a', a \in S \) satisfy \( a' \prec a \), then \( \varphi_{a'} \prec \varphi_a \).

**Notation 4.21.** Let \( S, T \) and \( P \) be \( \mathcal{Q} \)-semigroups, and let \( \varphi : S \times T \to P \) be a \( \mathcal{Q} \)-bimorphism. Using [Lemma 4.20] we may define a map \( \tilde{\varphi} : S \to \mathcal{Q}[T, P] \) by \( \tilde{\varphi}(a) = \varphi_a \), for \( a \in S \), which belongs to \( \mathcal{Q}(S, \mathcal{Q}[T, P]) \).

**Theorem 4.22.** Let \( S, T \) and \( P \) be \( \mathcal{Q} \)-semigroups. Then:

1. For every \( \mathcal{Q} \)-morphism \( \alpha : S \to \mathcal{Q}[T, P] \) there exists a \( \mathcal{Q} \)-bimorphism \( \varphi : S \times T \to P \) such that \( \alpha = \tilde{\varphi} \).
(2) If \( \varphi, \psi : S \times T \rightarrow P \) are \( \mathcal{Q} \)-bimorphisms, then \( \varphi \leq \psi \) if and only if \( \tilde{\varphi} \leq \tilde{\psi} \).

Thus, the assignment \( \Phi \) that sends a \( \mathcal{Q} \)-bimorphism \( \varphi : S \times T \rightarrow P \) to the \( \mathcal{Q} \)-morphism \( \tilde{\varphi} : S \rightarrow \mathcal{Q}[T, P] \) defines a natural bijection between the following (bi)morphism sets:

\[
\text{Bi}(\mathcal{Q}[S \times T, P]) \cong \mathcal{Q}(S, \mathcal{Q}[T, P]).
\]

Moreover, this bijection respects the structure of the (bi)morphism sets as positively ordered monoids.

Proof. To verify (1), let \( \alpha : S \rightarrow \mathcal{Q}[T, P] \) be a \( \mathcal{Q} \)-morphism. Define \( \varphi : S \times T \rightarrow P \) by \( \varphi(s, t) = \alpha(s)(t) \). It is straightforward to check that \( \varphi \) is a \( \mathcal{Q} \)-bimorphism satisfying \( \alpha = \tilde{\varphi} \), as desired.

Statement (2) is also easily verified. It follows that \( \Phi \) is an order-isomorphism, and hence a bijection. It is also clear that \( \Phi \) is additive and preserves the zero element. \( \square \)

**Lemma 4.23.** Let \( S_1, S_2 \) and \( T \) be \( \mathcal{Q} \)-semigroups, and let \( \alpha : S_1 \rightarrow S_2 \) be a (generalized) \( \mathcal{Q} \)-morphism. Then the map \( \alpha^* : \mathcal{Q}[S_2, T] \rightarrow \mathcal{Q}[S_1, T] \) given by \( \alpha^*(f) := f \circ \alpha \), for \( f \in \mathcal{Q}[S_2, T] \), is a (generalized) \( \mathcal{Q} \)-morphism.

Analogously, given \( \mathcal{Q} \)-semigroups \( S, T_1 \) and \( T_2 \), and given a (generalized) \( \mathcal{Q} \)-morphism \( \beta : T_1 \rightarrow T_2 \), the map \( \beta_* : \mathcal{Q}[S, T_1] \rightarrow \mathcal{Q}[S, T_2] \) defined by \( \beta_*(f) := \beta \circ f \), for \( f \in \mathcal{Q}[S, T_1] \), is a (generalized) \( \mathcal{Q} \)-morphism.

Proof. It is straightforward to check that \( \alpha^* \) and \( \beta_* \) are generalized \( \mathcal{Q} \)-morphisms. Assume that \( \alpha \) is a \( \mathcal{Q} \)-morphism. To show that \( \alpha^* \) preserves the auxiliary relation, let \( f_1, f_2 \in \mathcal{Q}[S_2, T] \) satisfy \( f_1 \prec f_2 \). To show that \( \alpha^*(f_1) \prec \alpha^*(f_2) \), let \( a', a \in S \) satisfy \( a' \prec a \). Since \( \alpha \) preserves the auxiliary relation, we have \( \alpha(a') \prec \alpha(a) \). Using that \( f_1 \prec f_2 \) at the second step, we deduce that

\[
\alpha^*(f_1)(a') = f_1(\alpha(a')) \prec f_2(\alpha(a)) = \alpha^*(f_2)(a),
\]

as desired. Analogously, one shows that \( \beta_* \) preserves the auxiliary relation whenever \( \beta \) does. \( \square \)

**4.24.** Let \( T \) be a \( \mathcal{Q} \)-semigroup. We let \( \mathcal{Q}[\_, T] : \mathcal{Q} \rightarrow \mathcal{Q} \) be the contravariant functor that sends a \( \mathcal{Q} \)-semigroup \( S \) to the \( \mathcal{Q} \)-semigroup \( \mathcal{Q}[S, T] \) [see Proposition 4.18], and that sends a \( \mathcal{Q} \)-morphism \( \alpha : S_1 \rightarrow S_2 \) to the \( \mathcal{Q} \)-morphism \( \alpha^* : \mathcal{Q}[S_2, T] \rightarrow \mathcal{Q}[S_1, T] \) as in Lemma 4.23.

Analogously, we obtain a covariant functor \( \mathcal{Q}[S, \_] : \mathcal{Q} \rightarrow \mathcal{Q} \) for every \( \mathcal{Q} \)-semigroup \( S \). Thus, we obtain a bifunctor

\[
\mathcal{Q}[\_, \_] : \mathcal{Q} \times \mathcal{Q} \rightarrow \mathcal{Q}.
\]

**Remark 4.25.** Given \( \mathcal{Q} \)-semigroups \( S \) and \( T \), one can construct a \( \mathcal{Q} \)-semigroup \( S \otimes \mathcal{Q} \) together with a \( \mathcal{Q} \)-bimorphism \( \omega : S \times T \rightarrow S \otimes T \) that have the same universal properties as the tensor product in \( \text{Cu} \). One can then show that \( \mathcal{Q} \) is a closed, symmetric monoidal category, with \( \mathcal{Q}[\_, \_] \) as internal-hom bifunctor. We omit the details since for our purpose it is not necessary to show that \( \mathcal{Q} \) is a monoidal category.

**5. Abstract bivariant Cuntz semigroups**

In this section, we use the \( \tau \)-construction developed in Sections 3 and 4 to prove that \( \text{Cu} \) is a closed symmetric monoidal category. Thus, given \( \text{Cu} \)-semigroups \( S \) and \( T \), we construct a \( \text{Cu} \)-semigroup \( [S, T] \), playing the role of the set of morphisms from \( S \) to \( T \), such that this ‘internal-hom’ is adjoint to the tensor product. This means that, given another \( \text{Cu} \)-semigroup \( P \), there is a natural bijection between the morphism sets

\[
\text{Cu}(S, [T, P]) \cong \text{Cu}(S \otimes T, P),
\]
as shown in [Theorem 5.10]. We will call $[S, T]$ the bivariant Cu-semigroup, or the abstract bivariant Cuntz semigroup of $S$ and $T$. The Cu-morphisms $S \to T$ actually correspond to compact elements in $[S, T]$.

In [Subsection 5.3] we compute the first examples of bivariant Cu-semigroups.

We also study the situation for algebraic Cu-semigroups; see [Subsection 5.3] (Recall that a Cu-semigroup $S$ is called algebraic if every element is the supremum of an increasing sequence of compact elements.) Although, in general, it is neither necessary nor sufficient that $S$ and $T$ are algebraic in order that $[S, T]$ is algebraic, the full subcategory of $Cu$ consisting of algebraic Cu-semigroups is also a closed, symmetric, monoidal category.

In [Subsection 5.4] we analyse bivariant Cu-semigroups of ideals and quotients. More concretely, given an ideal $J$ of $T$, we show that $[S, J]$ is naturally identified with an ideal in $[S, T]$. Similarly, given an ideal $J$ of $S$, we can identify $[S/J, T]$ with an ideal in $[S, T]$. We provide examples to the effect that the ideal structure of the abstract bivariant Cu-semigroup $[S, T]$ does not only depend on the ideal structure of $S$ and $T$.

5.1. Construction of abstract bivariant Cuntz semigroups. Recall that a generalized Cu-morphism between Cu-semigroups $S$ and $T$ is a map that preserves order, addition, the zero element, and suprema of increasing sequences, and thus it is not necessarily a Cu-morphism; see [Definition 2.4]. In the language of domain theory, a generalized Cu-morphism is a monoid morphism that is sequentially Scott continuous (see [GHKT 03] Section II.2, p.157ff). We denote the set of generalized Cu-morphisms $S \to T$ by Cu$[S, T]$, and we equip it with the pointwise order and addition, giving it a natural structure as a positively ordered monoid.

Recall from [Paragraph 4.5] that there is functor $\iota: Cu \to Q$ that embeds Cu as a full subcategory of $Q$. This is given by considering a Cu-semigroup $S$ as a $Q$-semigroup for the auxiliary relation $\ll$. In [Definition 4.17] we introduced an auxiliary relation on the set of generalized $Q$-mappings, giving itself the structure of a $Q$-semigroup; see [Proposition 4.18].

Let us transfer this definition to the setting of Cu-semigroups.

**Definition 5.1.** Let $S$ and $T$ be Cu-semigroups. We define a binary relation $\prec$ on the set of generalized Cu-morphisms Cu$[S, T]$ by setting $\varphi \prec \psi$ if and only if $\varphi(a') \ll \psi(a)$ for all $a', a \in S$ with $a' \ll a$.

**Remarks 5.2.** (1) The auxiliary relation $\prec$ on the set of generalized Cu-morphisms was already considered in [APT14 6.2.6].

(2) Every Cu-morphism is also a generalized Cu-morphism, and we therefore consider Cu$(S, T)$ as a subset of Cu$[S, T]$. For $\varphi \in Cu(S, T)$, we have $\varphi \prec \varphi$ if and only if $\varphi$ is a Cu-morphism.

It follows from [Proposition 4.18] that $\prec$ is an auxiliary relation on Cu$[S, T]$ and that $(Cu[S, T], \prec)$ is a $Q$-semigroup. We may therefore apply the $\tau$-construction.

**Definition 5.3.** Let $S$ and $T$ be Cu-semigroups. We define the internal hom from $S$ to $T$ as the Cu-semigroup

$$[S, T] := \tau(Cu[S, T], \prec).$$

We call $[S, T]$ the bivariant Cu-semigroup, or the abstract bivariant Cuntz semigroup of $S$ and $T$.

**Remark 5.4.** Recall that a path in Cu$[S, T]$ is a map $f: I_0 \to Cu[S, T]$ such that $f(\lambda') \prec f(\lambda)$ whenever $\lambda', \lambda \in I_0$ satisfy $\lambda' < \lambda$. We often denote $f(\lambda)$ by $f_\lambda$ and we denote the path by $f = (f_\lambda)_\lambda$. 
By definition then, the elements of \([S, T]\) are equivalence classes of paths in the \(Q\)-semigroup \((Cu[S, T], \prec)\).

**5.5.** Let us show that the internal-hom in \(Cu\) is functorial in both variables: contravariant in the first and covariant in the second variable.

Let \(T\) be a \(Cu\)-semigroup. Considering \(T\) as a \(Q\)-semigroup, we have a contravariant functor \(Q(\cdot, T): Q \to Q\) as in **Paragraph 4.24**. Precomposing with the inclusion \(\iota: Cu \to Q\) from **Paragraph 4.23** and postcomposing with the functor \(\tau: Q \to Cu\), we obtain a functor \([\iota, T]\): \(Cu \to Cu\).

Given \(Cu\)-semigroups \(S_1\) and \(S_2\), and a \(Cu\)-morphism \(\alpha: S_1 \to S_2\), we use \(\alpha^*\) to denote the induced \(Cu\)-morphism \([S_2, T] \to [S_1, T]\). Thus, if we consider \(\alpha\) as a \(Q\)-morphism and if we let \(\alpha_Q^*: Q[S_2, T] \to Q[S_1, T]\) denote the induced \(Q\)-morphism from **Lemma 4.23**, then \(\alpha^*\) is given as \(\alpha^* := \tau(\alpha_Q^*)\).

Analogously, given a \(Cu\)-semigroup \(S\), we define the functor \([S, \cdot]\): \(Cu \to Cu\) as the composition of the functors \(\iota\), the functor \(Q[S, \cdot]\) from **Paragraph 4.24**, and \(\tau\).

Given \(Cu\)-semigroups \(T_1\) and \(T_2\), and a \(Cu\)-morphism \(\beta: T_1 \to T_2\), we use \(\beta_\cdot\) to denote the induced \(Cu\)-morphism \([S, T_1] \to [S, T_2]\). If we consider \(\beta\) as a \(Q\)-morphism and if we let \(\beta_Q^*: Q[S, T_1] \to Q[S, T_2]\) denote the induced \(Q\)-morphism from **Lemma 4.23**, then \(\beta^*\) is given as \(\beta^* := \tau(\beta_Q^*)\).

Thus, the internal-hom in the category \(Cu\) is a bifunctor \([\iota, \cdot]: Cu \times Cu \to Cu\).

In **Subsection 6.3** below we will generalize these basic functoriality properties by describing a composition product \([T, P] \otimes [S, T] \to [S, P]\).

Next, we transfer the concept of the endpoint map from **Definition 4.7** to the setting of bivariant \(Cu\)-semigroups. To simplify notation, we write \(\sigma_{S,T}\) for \(\varphi_{Cu[S,T]}\), the endpoint map associated to the \(Q\)-semigroup \(Cu[S, T]\). The next definition makes this precise.

**Definition 5.6.** Let \(S\) and \(T\) be \(Cu\)-semigroups. We let \(\sigma_{S,T}: [S, T] \to Cu[S, T]\) be defined by

\[
\sigma_{S,T}(\{f\})(a) = \sup_{\lambda \in I_f} f_\lambda(a),
\]

for a path \(f = \{f_\lambda\}\) in \(Cu[S, T]\) and \(a \in S\). We refer to \(\sigma_{S,T}\) as the endpoint map.

**Lemma 5.7.** Let \(S, T\) and \(P\) be \(Cu\)-semigroups, and let \(\alpha: S \to [T, P]\) be a \(Cu\)-morphism. Let \(\sigma_T, P: [T, P] \to Cu[T, P]\) be the endpoint map from **Definition 5.6**. Define \(\bar{\alpha}: S \times T \to P\) by

\[
\bar{\alpha}(a, b) = \sigma_T, P(\alpha(a))(b),
\]

for \(a \in S\) and \(b \in T\). Then \(\bar{\alpha}\) is a \(Cu\)-bimorphism.

**Proof.** We write \(\sigma\) for \(\sigma_T, P\). To show that \(\bar{\alpha}\) is a generalized \(Cu\)-morphism in the first variable, let \(b \in T\). Since \(\alpha\) and \(\sigma\) are both additive and order preserving, we conclude that \(\bar{\alpha}(\cdot, b) = \sigma(\alpha(\cdot))(b)\) is additive and order preserving as well. To show that \(\bar{\alpha}(\cdot, b)\) preserves suprema of increasing sequences, let \((a_n)_n\) be an increasing sequence in \(S\). Set \(a := \sup_n a_n\). Since both \(\alpha\) and \(\sigma\) preserve suprema of increasing sequences, we obtain that

\[
\sigma(\alpha(a)) = \sup_n \sigma(\alpha(a_n)),
\]

in \(Cu[T, P]\). Since the supremum of an increasing sequence in \(Cu[T, P]\) is the pointwise supremum, we get that \(\bar{\alpha}(a, b) = \sup_n \bar{\alpha}(a_n, b)\), as desired.

For each \(a \in S\), we have \(\bar{\alpha}(a, \cdot) = \sigma(\alpha(a))\), which is an element in \(Cu[T, P]\). Therefore, \(\bar{\alpha}\) is a generalized \(Cu\)-morphism in the second variable.
Lastly, to show that \( \bar{\alpha} \) preserves the joint way-below relation, let \( a', a \in S \) and \( b', b \in T \) satisfy \( a' \ll a \) and \( b' \ll b \). Since \( \alpha \) is a \( \Cu \)-morphism we have \( \alpha(a') \ll \alpha(a) \). Using that \( \sigma \) is a \( \mathcal{Q} \)-morphism, it follows that \( \sigma(\alpha(a')) \ll \sigma(\alpha(a)) \). Therefore, applying the definition of the auxiliary relation \( \prec \) at the second step, we obtain that
\[
\bar{\alpha}(a', b') = \sigma(\alpha(a'))(b') \ll \sigma(\alpha(a))(b) = \bar{\alpha}(a, b),
\]
as desired. \( \square \)

We omit the straightforward proof of the following result.

**Lemma 5.8.** Let \( S, T \) and \( P \) be \( \Cu \)-semigroups, and let \( \varphi : S \times T \to P \) be a map. Then \( \varphi \) is a \( \Cu \)-bimorphism if and only if \( \varphi \), considered as a map between \( \mathcal{Q} \)-semigroups, is a \( \mathcal{Q} \)-bimorphism. Thus, we have a canonical bijection
\[
\text{Bi}_\mathcal{Q}(S \times T, P) \cong \text{Bi}_\Cu(S \times T, P).
\]
Moreover, this bijection respects the structure of the bimorphism sets as positively ordered monoids.

**Lemma 5.9.** Let \( S, T \) and \( P \) be \( \Cu \)-semigroups. Then the assignment that sends a \( \Cu \)-morphism \( \alpha : S \to [T, P] \) to the \( \Cu \)-bimorphism \( \bar{\alpha} : S \times T \to P \) given in Lemma 5.7 defines a natural bijection:
\[
\Cu(S, [T, P]) \cong \text{Bi}_\Cu(S \times T, P).
\]
Moreover, this bijection respects the structure of the morphism sets as positively ordered monoids.

**Proof.** By definition, we have \( \Cu(S, [T, P]) = \Cu(S, \tau(\mathcal{Q}[T, P])) \). Further, we have natural bijections, respecting the structure as positively ordered monoids, using Theorem 4.4 at the first step, using Theorem 4.22 at the second step, and using Lemma 5.8 at the last step:
\[
\Cu(S, \tau(\mathcal{Q}[T, P])) \cong \mathcal{Q}(S, \mathcal{Q}[T, P]) \cong \text{Bi}_\mathcal{Q}(S \times T, P) \cong \text{Bi}_\Cu(S \times T, P).
\]
It is straightforward to check that the composition of these bijections identifies a \( \Cu \)-morphism \( \alpha \) with the \( \Cu \)-bimorphism \( \bar{\alpha} \) as defined in Lemma 5.7. \( \square \)

**Theorem 5.10.** Let \( S, T \) and \( P \) be \( \Cu \)-semigroups. Then there are natural bijections
\[
\Cu(S, [T, P]) \cong \text{Bi}_\Cu(S \times T, P) \cong \Cu(S \otimes T, P).
\]
Moreover, these bijection respects the structure of the (bi)morphism sets as positively ordered monoids.

The first bijection is given by assigning to a \( \Cu \)-morphism \( \alpha : S \to [T, P] \) the \( \Cu \)-bimorphism \( \bar{\alpha} : S \times T \to P \) as in Lemma 5.7, that is, \( \bar{\alpha}(a, b) = \sigma_{T, P}(\alpha(a))(b) \), for \( (a, b) \in S \times T \). The second bijection is given by assigning to a \( \Cu \)-morphism \( \beta : S \otimes T \to P \) the \( \Cu \)-bimorphism \( \tilde{\beta} : S \times T \to P \), \( (a, b) \mapsto \beta(a \otimes b) \), for \( (a, b) \in S \times T \).

**Proof.** The first bijection is obtained from Lemma 5.9. The second bijection follows from Theorem 2.16. It is also shown in these results that the bijections respect the structure of the (bi)morphism sets as positively ordered monoids. \( \square \)

Let \( T \) be a \( \Cu \)-semigroup. We consider the functor \( \_ \otimes T : \Cu \to \Cu \) given by tensoring with \( T \). It follows from Theorem 5.10 that the functor \([T, 
\_]\) is a right adjoint of \( \_ \otimes T \). By definition, this shows that the monoidal category \( \Cu \) is closed, and we obtain the following result:

**Theorem 5.11.** The category \( \Cu \) of abstract Cuntz semigroups is a closed, symmetric, monoidal category.
Every closed symmetric monoidal category is enriched over itself, as noted in [Paragraph 2.14]. Let us make this precise for the category $\Cu$. (See [Section 6] for further details.) Given $\Cu$-semigroups $S$ and $T$, the $\Cu$-semigroup $[S,T]$ plays the role of morphisms from $S$ to $T$. First, we show that $\Cu$-morphisms $S \to T$ correspond to compact elements in $[S,T]$.

**Proposition 5.12.** Let $S$ and $T$ be $\Cu$-semigroups. Then there is a natural bijection between $\Cu$-morphisms $S \to T$ and compact elements in $[S,T]$: $$[S,T]_c \cong \Cu(S,T).$$

A $\Cu$-morphism $\varphi : S \to T$ is associated with the class in $[S,T]$ of the constant path with value $\varphi$. Conversely, given a compact element in $[S,T]$ represented by a path $(\varphi_\lambda)_\lambda$, then for $\lambda$ close enough to 1 the map $\varphi_\lambda$ is a $\Cu$-morphism and independent of $\lambda$.

**Proof.** It is straightforward to verify that the described associations are well-defined and inverses of each other. Alternatively, note that for every $\Cu$-semigroup $P$, there is a natural identification of $P$ with $\Cu([N,P])$ by associating to a $\Cu$-morphism $\varphi : [N,P]$ the compact element $(\varphi_\lambda)_\lambda$, then for $\lambda$ close enough to 1 the map $\varphi_\lambda$ is a $\Cu$-morphism and independent of $\lambda$.

In particular, the identity $\Cu$-morphism $\text{id}_S : S \to S$ naturally corresponds to a compact element in $[S,S]$, also denoted by $\text{id}_S$. Further, $\text{id}_S$ also naturally corresponds to a $\Cu$-morphism $j_S : [N,S] \to [S,S]$, which is the identity of $S$ for the enrichment of $\Cu$ over itself.

Given $\Cu$-semigroups $S$ and $T$, recall that the counit map, or evaluation map is the $\Cu$-morphism $\text{ev}_S : [S,T] \to [S,S]$ that corresponds to $\text{id}_{[S,T]}$ under the identification $\Cu([S,T],[S,T]) \cong \Cu([S,T] \otimes S,T)$. Given $\Cu$-semigroups $S, T$ and $P$, consider the following $\Cu$-morphism:

$$([T,P] \otimes [S,T]) \otimes S \xrightarrow{\cong} [T,P] \otimes ([S,T] \otimes S) \xrightarrow{\text{id} \otimes \text{ev}_T} [T,P] \otimes T \xrightarrow{\text{ev}_T} P.$$  

Under the identification $\Cu([T,P] \otimes [S,T],[S,P]) \cong \Cu([T,P] \otimes [S,T] \otimes S,P)$, the above $\Cu$-morphism corresponds to a $\Cu$-morphism

$$\circ : [T,P] \otimes [S,T] \to [S,P],$$

that we will call the composition product. The composition product implements the composition of morphisms when viewing the category $\Cu$ as enriched over itself. (See [Section 6] for further details.) We obtain:

**Theorem 5.13.** The category $\Cu$ of abstract Cuntz semigroups is enriched over itself.

### 5.2. First examples
In this subsection, we compute several examples of bivariant $\Cu$-semigroups $[S,T]$. We mostly consider the case that $S$ and $T$ are the Cuntz semigroups of the Jacelon-Razak algebra $W$, of the Jiang-Su algebra $Z$, of a UHF-algebra of infinite type, or of the Cuntz algebra $O_2$.

Recall that $\mathbb{F}$ denotes the semigroup $[0,\infty]$ with the usual order and addition. It is known that $\mathbb{F} \cong \Cu(W)$, the Cuntz semigroup of the Jacelon-Razak algebra $W$ introduced in [Jac13] (see [Rob13]). The product of real numbers extends to a natural product on $[0,\infty]$ giving $\mathbb{F}$ the structure of a solid $\Cu$-semiring; see [APT14, Definition 7.1.5, Example 7.1.7], and also [Subsection 7.3] The $\Cu$-semiring $\mathbb{F}$ and its $\Cu$-semimodules were studied in [APT14, Section 7.5].
Let $M_1$ be defined as in Example 4.14. By Proposition 4.16, $M_1$ is the Cuntz semigroup of a II$_1$-factor $M$.

**Proposition 5.14.** There is a natural isomorphism $[P, P] \cong M_1$.

**Proof.** We show that the $Q$-semigroup $(\text{Cu}[P, P], \prec)$ is isomorphic to $(P, \prec_1)$, where $\prec_1$ is the auxiliary relation defined in Example 4.14. Applying the $\tau$-construction, and using the arguments in Example 4.14 at the last step, we then obtain

$$[P, P] = \tau(\text{Cu}[P, P], \prec) \cong \tau(P, \prec_1) = M_1.$$  

Since $P$ is a solid $Cu$-semiring, every generalized $Cu$-morphism $P \to P$ is automatically $P$-linear; see Proposition 7.24 and [APT14 Proposition 7.1.6]. Thus, given a generalized $Cu$-morphism $\varphi : P \to P$, we have $\varphi(x) = \varphi(1)x$ for all $x \in P$. This allows us to identify $\text{Cu}[P, P]$ with $P$ by $\varphi \mapsto \varphi(1)$ and this is easily seen to be an additive order-isomorphism. To conclude the argument, we need to show that under this identification, the auxiliary relation $\prec$ on $\text{Cu}[P, P]$ corresponds to the auxiliary relation $\prec_1$ on $P$ as defined in Example 4.14.

Let $\varphi, \psi \in \text{Cu}[P, P]$. Clearly $\varphi \prec \psi$ implies $\varphi(1) \leq \psi(1)$. Moreover, if $\varphi(1) = \psi(1) = \infty$, then $\varphi \neq \psi$, since $\varphi(1) = \infty \ll \infty = \psi(2)$ while $1 \ll 2$. Thus, $\varphi \prec \psi$ implies $\varphi(1) \prec_1 \psi(1)$. Conversely, assume that $\varphi(1) \prec_1 \psi(1)$. By definition, $\varphi(1)$ is finite, and $\varphi(1) \leq \psi(1)$. To show that $\varphi \prec \psi$, let $s, t \in P$ satisfy $s \leq t$. Using that $\varphi(1)$ is finite at the second step, we deduce that

$$\varphi(s) = \varphi(1)s \ll \varphi(1)t \leq \psi(1)t = \psi(t),$$

as desired. \hfill $\square$

We let $Z$ be the disjoint union $\mathbb{N} \sqcup (0, \infty]$, with elements in $\mathbb{N}$ being compact, and with elements in $(0, \infty)$ being soft. It is known that $Z$ is isomorphic to the Cuntz semigroup of the Jiang-Su algebra $Z$ introduced in [JS99] (see [LT07] and also [BT10]). We next recall some details.

To distinguish elements in both parts, we write $a'$ (with a prime symbol) for the soft element of value $a$. For example, the compact one, denoted 1, corresponds the class of the unit in $Z$; and the soft one, denoted $1'$, corresponds to the class of a positive element $x$ in $Z$ that has spectrum $[0, 1]$ and with $\lim_{n \to \infty} \tau(x^{1/n}) = 1$, for the unique trace $\tau$ on $Z$.

Order and addition are the usual inside the components $\mathbb{N}$ and $(0, \infty]$ of $Z$. Given $a \in \mathbb{N}$ and $b' \in (0, \infty]$, we have $a + b' = (a + b)'$ (the soft part is absorbing), and we have $a \leq b'$ if and only if $a' \leq b'$. We have a natural commutative product in $Z$, extending the natural products in the components $\mathbb{N}$ and $(0, \infty]$, and such that $0a = 0$ for every $a \in Z$. Note that $1 \in \mathbb{N}$ (the compact one) is a unit for this semiring, but $1'$ is not. Indeed, we have $1'1 = 1'$.

This gives $Z$ the structure of a solid $Cu$-semiring. The $Cu$-semiring $Z$ and its $Cu$-semimodules were studied in [APT14 Section 7.3].

Given a supernatural number $q$ satisfying $q = q^2 \neq 1$, we let $\mathbb{N}_{[q]}$ denote the set of nonnegative rational numbers whose denominators divide $q$, with usual addition. Let $R_q = \mathbb{N}_{[q]} \sqcup (0, \infty]$, with elements in $\mathbb{N}_{[q]}$ being compact, and with elements in $(0, \infty]$ being soft. The interplay of addition and order with the two components of $R_q$ is analogous to the situation for $Z$. If we now let $M_q$ denote the UHF-algebra of type $q$, then it is known that $Cu(M_q) \cong R_q$.

Analogous to the case for $Z$, we can define a commutative multiplication on $R_q$, giving it the structure of a solid $Cu$-semiring. The $Cu$-semiring $R_q$ and its $Cu$-semimodules were studied in [APT14 Section 7.4].
We exclude zero as a supernatural number. However, 1 is supernatural number that agrees with its square. It is consistent to let $R_1$ denote the Cuntz semigroup of the Jiang-Su algebra $Z$. Thus, we set $R_1 := Z$, which simplifies the statement of Proposition 5.15 below.

Given supernatural numbers $p$ and $q$ satisfying $p = p^2$ and $q = q^2$, we have $R_p \otimes R_q \cong R_{pq}$. In particular, $Z \cong R_1 \otimes R_p \cong R_p$. Moreover, if we let $Q = \mathbb{Q}^+ \cup \{0, \infty\}$, then $Q$ is isomorphic to the Cuntz semigroup of the universal UHF-algebra (whose $K_0$-group is isomorphic to the rational numbers). We have $Q \otimes R_p \cong Q$.

**Proposition 5.15.** Let $p$ and $q$ be supernatural numbers with $p = p^2$ and $q = q^2$. If $p$ divides $q$, then $[R_p, R_q] \cong R_q$. If $p$ does not divide $q$, then $\text{Cu}(R_p, R_q) = \{0\}$ and $[R_p, R_q] \cong \mathbb{F}$.

Proof. First, assume that $p$ divides $q$. Then $R_p \cong R_p \otimes R_p$ and $R_q \cong R_p \otimes R_p$. Let $\varphi : R_p \to R_q$ be a generalized Cu-morphism. It follows from Proposition 7.1.6 that $\varphi$ is $R_p$-linear. Thus, $\varphi$ is determined by the image of the unit. Moreover, for every $a \in R_q$, there is a generalized Cu-morphism $\varphi : R_p \to R_q$ with $\varphi(1) = a$, given by $\varphi(t) = ta$ for $t \in R_p$. Thus, there is a bijection $\text{Cu}[R_p, R_q] \cong R_q$ given by identifying $\varphi$ with $\varphi(1)$. It is straightforward to check that under this identification, the relation $\prec$ on $\text{Cu}[R_p, R_q]$ corresponds precisely to the way-below relation on $R_q$. It follows that

$$[R_p, R_q] = \tau(\text{Cu}[R_p, R_q], \prec) \cong \tau(R_q, \ll) \cong R_q,$$

as desired.

Assume now that $p$ does not divide $q$. Let $r$ be a prime number dividing $p$ but not $q$. Every element of $R_p$ is divisible by arbitrary powers of $r$.

On the other hand, we claim that only the soft elements of $R_q$ are divisible by arbitrary powers of $r$. Indeed, every element of $R_q$ is either compact or nonzero and soft. Moreover, the sum of a nonzero soft element with any other element in $R_q$ is soft. Thus, if $x \in R_q$ is compact, and if $x = ky$ for some $k \in \mathbb{N}$ and $y \in R_q$, then $y$ is necessarily compact. Thus, if a compact element of $R_q$ is divisible in $R_q$ then it is also divisible in the monoid of compact elements of $R_q$, which we identify with $\mathbb{N}[[\frac{1}{r}]]$. However, since $r$ does not divide $q$, the only element in $\mathbb{N}[[\frac{1}{r}]]$ that is divisible by arbitrary powers of $r$ is the zero element, which is soft.

It follows that every generalized Cu-morphism $R_p \to R_q$ has its image contained in the soft part of $R_q$.

In particular, if $\varphi : R_p \to R_q$ is a Cu-morphism, then every compact element of $R_p$ is sent to zero by $\varphi$. Using that $R_p$ is simple, it follows that $\varphi$ is the zero map. Thus, $\text{Cu}(R_p, R_q) = \{0\}$, as desired.

We identify $\mathbb{F}$ with the soft part of $R_p$, and similarly for $R_q$. Let $\varphi : R_p \to R_q$ be a generalized Cu-morphism. We have seen that $\varphi(1)$ belongs to $\mathbb{F} = (R_q)_{\text{soft}}$. Moreover, for every $a \in \mathbb{F}$ there is a generalized Cu-morphism $\varphi : R_p \to \mathbb{F} \subseteq R_q$ with $\varphi(1) = a$, given by $\varphi(t) = ta$ for $t \in R_p$. Thus, there is a bijection $\text{Cu}[R_p, R_q] \cong \mathbb{F}$ given by identifying $\varphi$ with $\varphi(1)$. It is straightforward to check that under this identification, the relation $\prec$ on $\text{Cu}[R_p, R_q]$ corresponds to the way-below relation on $\mathbb{F}$. As above, it follows that

$$[R_p, R_q] = \tau(\text{Cu}[R_p, R_q], \prec) \cong \tau(\mathbb{F}, \ll) \cong \mathbb{F},$$

as desired. \[\square\]

**Example 5.16.** Recall that $Z$ and $Q$ are isomorphic to the Cuntz semigroups of the Jiang-Su algebra and the universal UHF-algebra, respectively. By Proposition 5.15
there are natural isomorphisms
\[
\llbracket Z, Z \rrbracket \cong Z \quad \text{and} \quad \llbracket Q, Q \rrbracket \cong Q.
\]
More generally, there are natural isomorphisms
\[
\llbracket Z, R_q \rrbracket \cong R_q \quad \text{and} \quad \llbracket R_q, Q \rrbracket \cong Q,
\]
for every supernatural number \(q\) with \(q = q^2\).

**Example 5.17.** Let \(q\) be a supernatural number with \(q = q^2\). Then there are natural isomorphisms
\[
\llbracket R_q, \mathbb{F} \rrbracket \cong \mathbb{F} \quad \text{and} \quad \llbracket \mathbb{F}, R_q \rrbracket \cong M_1,
\]
which can be proved similarly as Proposition 5.15 and Proposition 5.14. In particular, we have \(\llbracket Z, \mathbb{F} \rrbracket \cong \mathbb{F}\) and \(\llbracket \mathbb{F}, Z \rrbracket \cong M_1\).

Given \(k \in \mathbb{N}\), we set \(E_k := \{0, 1, \ldots, k, \infty\}\), equipped with the natural order and addition as a subset of \(\mathbb{N}\), with the convention that \(a + b = \infty\) whenever \(a + b > k\) in \(\mathbb{N}\). With the obvious multiplication, \(E_k\) is a solid Cu-semiring (see, e.g. [APT14, Example 8.1.2]). Note that \(E_0 = \{0, \infty\}\) is the Cuntz semigroup of the Cuntz algebra \(O_2\) (or of any other simple, purely infinite \(C^*\)-algebra). The Cu-semiring \(E_0 = \{0, \infty\}\) and its Cu-semimodules were studied in [APT14, Section 7.2].

**Proposition 5.18.** Let \(k, l\) be natural numbers. Let \([\frac{k+1}{k+l}]\) denote the smallest natural number larger than or equal to \(\frac{k+1}{k+l}\). Then \(\llbracket E_k, E_l \rrbracket\) is isomorphic to the sub-Cu-semigroup \(\{0, [\frac{k+1}{k+l}], \ldots, l, \infty\}\) of \(E_l\).

**Proof.** Let \(\varphi: E_k \to E_l\) be a generalized Cu-morphism. Then \(\varphi\) is determined by the image of 1, which can be zero or any element \(a \in E_l\) such that \((k + 1)a = \infty\). Thus, for every \(a \geq \frac{k+1}{k+l}\) there is a unique generalized Cu-morphism \(E_k \to E_l\) given by \(x \mapsto ax\). Moreover, each such a map preserves the way-below relation and is therefore a Cu-morphism. The desired result follows. \(\square\)

**Example 5.19.** It follows from Proposition 5.18 that there is a natural isomorphism
\[
\llbracket [0, \infty], [0, \infty] \rrbracket \cong \{0, \infty\},
\]
and more generally \(\llbracket E_k, E_k \rrbracket \cong E_k\) for every \(k \in \mathbb{N}\).

Recall the definition of axiom (O5) (almost algebraic order) from Paragraph 2.12. Note that \(E_k\) satisfies (O5) for every \(k \in \mathbb{N}\). On the other hand, \(\llbracket E_k, E_k \rrbracket\) does not satisfy (O5) whenever \(l > k > 1\). For example, in \([E_2, E_3] = \{0, 2, 3, \infty\}\) we have \(2 \ll 2 \leq 3\), yet there is no element \(x\) with \(2 + x = 3\). This shows that (O5) does not pass to abstract bivariant Cuntz semigroups.

Probably, (O6) (almost Riesz decomposition) does not pass to bivariant Cu-semigroups either, but we are currently not aware of any example.

5.3. **Algebraic Cu-semigroups.**

5.20. Recall that a Cu-semigroup \(S\) is algebraic if the subset \(S_c\) of compact elements is sup-dense in \(S\), that is, if every element in \(S\) is the supremum of an increasing sequence of compact elements. We always endow \(S_c\) with the addition and partial order inherited from \(S\), giving it the structure of a positively ordered monoid.

A natural source of algebraic semigroups comes from \(C^*\)-algebras of real rank zero. More concretely, the Cuntz semigroup of a \(C^*\)-algebra with real rank zero is always algebraic. Conversely, if a \(C^*\)-algebra \(A\) has stable rank one and \(\text{Cu}(A)\) is algebraic, then \(A\) has real rank zero; see [CE108]. Another natural source comes from positively ordered monoids. Specifically, given any positively ordered monoid \(M\), then its Cu-completion \(\gamma(M, \leq)\) as constructed in [APT13, Proposition 3.1.6] (see also [APT14, 5.5.3]) is always an algebraic Cu-semigroup.
By [APT14, Proposition 5.5.5], there is an equivalence of the following categories: the category PoM of positively ordered monoids, and the full subcategory of Cu, consisting of algebraic Cu-semigroups. This means in particular that there is a natural bijection \( \text{Cu}(S, T) \cong \text{PoM}(S_e, T_e) \), whenever \( S \) and \( T \) are algebraic Cu-semigroups.

It is natural to ask whether \( \llbracket S, T \rrbracket \) is an algebraic Cu-semigroup whenever \( S \) and \( T \) are. In general, this is not the case. Consider for examples the Cuntz semigroups \( R_{2\infty} \) and \( R_{3\infty} \) of the UHF-algebras of type \( 2\infty \) and \( 3\infty \), respectively. Note that \( R_{2\infty} \) and \( R_{3\infty} \) are algebraic. However, by [Proposition 5.15] we have \( [R_{2\infty}, R_{3\infty}] \cong \mathbb{F} \), which is not algebraic.

Conversely, if \( \llbracket S, T \rrbracket \) is algebraic, then it need not follow that \( S \) or \( T \) is algebraic. Indeed, while \( \mathbb{F} \) is not algebraic, we showed in [Proposition 5.14] that \( \llbracket \mathbb{F}, \mathbb{F} \rrbracket \) is isomorphic to the algebraic Cu-semigroup \( M_1 \) from [Example 4.14].

Thus, the natural problem to determine when an abstract bivariant Cuntz semigroup is algebraic has probably no simple answer.

Let \( \text{Cu}_{\text{alg}} \) denote the full subcategory of \( \text{Cu} \) whose objects are algebraic Cu-semigroups. The tensor product of two algebraic Cu-semigroups is again algebraic, as shown in [APT14, Corollary 6.4.8]. Thus, \( \text{Cu}_{\text{alg}} \) has a natural symmetric, monoidal structure.

**Theorem 5.21.** The category \( \text{Cu}_{\text{alg}} \) is closed. Given algebraic Cu-semigroups \( S \) and \( T \), the internal-hom of \( S \) and \( T \) in \( \text{Cu}_{\text{alg}} \) is the algebraic Cu-semigroup \( \llbracket S, T \rrbracket_{\text{alg}} \) given as the Cu-completion of the positively ordered monoid \( \text{Cu}(S, T) \):

\[ \llbracket S, T \rrbracket_{\text{alg}} = \gamma(\text{Cu}(S, T)) \]

**Proof.** Let \( S, T \) and \( R \) be algebraic Cu-semigroup. Then \( R \otimes S \) is also algebraic, with \( (R \otimes S)_e \cong R_e \otimes_{\text{PoM}} S_e \), the latter tensor product being the one in the category of partially ordered monoids; see [APT14, Proposition 6.4.7 and Corollary 6.4.8]. We deduce that there are natural bijections

\[ \text{Cu}(R \otimes S, T) \cong \text{PoM}(R_e \otimes S_e, T_e) \cong \text{PoM}(R_e, \text{PoM}(S_e, T_e)) \]

\[ \cong \text{PoM}(R_e, \text{Cu}(S, T)) \cong \text{Cu}(R, \gamma(\text{Cu}(S, T))) \]

Thus, the functor \( - \otimes S \colon \text{Cu}_{\text{alg}} \to \text{Cu}_{\text{alg}} \) has a right adjoint given by the functor \( \gamma(\text{Cu}(S, -)) \colon \text{Cu}_{\text{alg}} \to \text{Cu}_{\text{alg}} \), as desired. \( \square \)

### 5.4. Bivariant Cuntz semigroups of ideals and quotients.

Let \( T \) be a Cu-semigroup, and let \( S \subseteq T \) be a submonoid. We call \( S \) a sub-Cu-semigroup of \( T \) if \( S \) is a Cu-semigroup for the partial order inherited from \( T \) and such that the inclusion \( S \to T \) is a Cu-morphism. It is easy to see that \( S \) is a sub-Cu-semigroup of \( T \) if and only if \( S \) is closed under passing to suprema of increasing sequences and if the way-below relation in \( S \) and \( T \) agree.

**Lemma 5.22.** Let \( S \) and \( T \) be Cu-semigroups, and let \( T' \subseteq T \) be a sub-Cu-semigroup. Then the inclusion \( \iota \colon T' \to T \) induces an order-embedding \( \iota_* \colon \llbracket S, T' \rrbracket \to \llbracket S, T \rrbracket \).

**Proof.** Let \( f = (f_\lambda)_\lambda \) be a path in \( \text{Cu}[S, T'] \). Then \( \tilde{f} = (\iota \circ f_\lambda)_\lambda \) is a path in \( \text{Cu}[S, T] \). It follows from Paragraph 5.5 that \( \iota_*(\tilde{f}) = \tilde{f} \).

To show that \( \iota_* \) is an order-embedding, let \( x, y \in \llbracket S, T' \rrbracket \) with \( \iota_*(x) \leq \iota_*(y) \). Choose paths \( f \) and \( g \) in \( \text{Cu}[S, T'] \) representing \( x \) and \( y \), respectively. We have \( (\iota \circ f_\lambda)_\lambda \preceq (\iota \circ g_\lambda)_\lambda \). Thus, for every \( \lambda \in I_Q \), there is \( \mu \in I_Q \) such that \( \iota \circ f_\lambda \preceq \iota \circ g_\mu \).

Using that \( T' \subseteq T \) is a sub-Cu-semigroup, for such \( \lambda \) and \( \mu \) we deduce that \( f_\lambda \preceq g_\mu \).

(We use that for \( a', a \in T' \) we have \( a' \ll a \) in \( T' \) if and only if \( \iota(a') \ll \iota(a) \) in \( T \).)

It follows that \( f \preceq g \), and hence \( x \leq y \), as desired. \( \square \)
Recall that $J \subseteq S$ is called an ideal (of $S$) if $J$ is a submonoid of $S$ that is closed under passing to suprema of increasing sequences and that is downward-hereditary (if $a, b \in S$ satisfy $a \leq b$ and $b \in J$, then $a \in J$). We write $J \triangleleft S$ to mean that $J$ is an ideal of $S$. Note that every ideal is in particular a sub-$Cu$-semigroup. (See [APT14] Section 5.1 for an account on ideals and quotients.)

**Proposition 5.23.** Let $S$ and $T$ be $Cu$-semigroups, and let $J$ be an ideal of $T$. Let $\iota : J \to T$ denote the inclusion map. Then the induced $Cu$-morphism $\iota_* : [S,J] \to [S,T]$ is an order-embedding that identifies $[S,J]$ with an ideal of $[S,T]$. Moreover, $x \in [S,T]$ belongs to $[S,J]$ if and only if for some (equivalently, for every) path $(f_\lambda)_\lambda$ representing $x$, each $f_\lambda$ takes image in $J$.

**Proof.** By Lemma 5.22, $\iota_*$ is an order-embedding. It follows that $\iota_*$ identifies $[S,J]$ with a submonoid of $[S,T]$ that is closed under passing to suprema of increasing sequences.

Let $x \in [S,T]$ be represented by a path $f = (f_\lambda)_\lambda$ in $Cu[S,T]$. If each $f_\lambda$ takes values in $J$, then we can consider $f$ as a path in $Cu[S,J]$ whose class is an element $x' \in [S,J]$ satisfying $\iota_*(x') = x$. Conversely, assume that $x$ belongs to $[S,J]$. Then there is a path $g = (g_\mu)_\mu$ in $Cu[S,J]$ with $\iota_*(g) = x$. Let $\lambda \in I_G$. Since $f \not\leq g$, we can choose $\mu \in I_G$ with $f_\lambda \not\leq g_\mu$. Since $g_\mu$ takes values in $J$, and since $J$ is downward-hereditary, it follows that $f_\lambda$ takes values in $J$, as desired.

A similar argument shows that $[S,J]$ is downward-hereditary in $[S,T]$. □

**5.24.** Given $S$, let us study whether the functor $[S,-] : Cu \to Cu$ is exact. More precisely, let $J \triangleleft T$ be an ideal, with inclusion map $\iota : J \to T$ and with quotient map $\pi : T \to T/J$. This induces the following $Cu$-morphism:

$$[S,J] \xrightarrow{\iota_*} [S,T] \xrightarrow{\pi_*} [S,T/J].$$

By Proposition 5.23, $\iota_*$ identifies $[S,J]$ with an ideal in $[S,T]$. Since $\pi \circ \iota$ is the zero map, so is $\pi_* \circ \iota_*$. Thus, $\pi_*$ vanishes on the ideal $[S,J] \triangleleft [S,T]$. It follows that $\pi_*$ induces a $Cu$-morphism

$$\bar{\pi}_* : [S,T]/[S,J] \to [S,T/J].$$

**Problem 5.25.** Study the order-theoretic properties of the $Cu$-morphism $\bar{\pi}_*$ from Paragraph 5.24. In particular, when is $\bar{\pi}_*$ an order-embedding, when is it surjective?

We are currently not aware of any example for $S$ and $J \triangleleft T$ such that the map $\bar{\pi}_* : [S,T]/[S,J] \to [S,T/J]$ is not an isomorphism.

**Remark 5.26.** There are several possible concepts of projectivity for objects in $Cu$. One could say that a $Cu$-semigroup $S$ is projective, if for every $Cu$-semigroup $T$ and every ideal $J \triangleleft T$, with induced quotient map $\pi : T \to T/J$, and for every $Cu$-morphism $\varphi : S \to T/J$, there exists a $Cu$-morphism $\tilde{\varphi} : S \to T$ (called a lift) such that $\varphi = \pi \circ \tilde{\varphi}$. However, this concept seems very restrictive, since not even $\NN$ is projective in this sense.

A more promising approach is to say that $S$ is projective, if for every $T$ and every ideal $J \triangleleft T$, with induced quotient map $\pi : T \to T/J$, and for every $x \in [S,T/J]$, there exists $\tilde{x} \in [S,T]$ such that $x = \pi \circ \tilde{x}$. This means precisely that the map $\bar{\pi}_* : [S,T]/[S,J] \to [S,T/J]$ from Paragraph 5.24 is surjective for every $J \triangleleft T$.

In this sense, $\NN$ is projective. More generally, one can show that the product $\prod_i \NN$ is projective. The theory of products and, more generally, limits in $Cu$ is developed in [APT17]. It follows that there are many projective objects in $Cu$, meaning that for every $Cu$-semigroup $T$ there is a projective $Cu$-semigroup $P$ and a surjective $Cu$-morphism $P \to T$. This could be the starting point to develop a theory of projective resolutions and derived functors, in particular an ext-functor in $Cu$. We will not pursue this line of thought here.
5.27. Let $J$ be an ideal of $T$, and assume that $J$ has a largest element $z_J$. (Every simple and every countably-based Cu-semigroup has a largest element. For Cuntz semigroups of $C^*$-algebras this is connected to the question whether the $C^*$-algebra contains a full element. See [APT14, Section 5.1] for more details.) Then, every element $a \in T/J$ has a largest preimage in $T$, given as $x + z_J$ for any choice of preimage $x$. Let $\omega : T/J \to T$ be the map that sends an element in $T/J$ to its largest preimage in $T$. It is straightforward to check that $\omega$ is a generalized Cun-morphism. (However, in general, it does not preserve the way-below relation.) It is a natural question, closely related to Problem 5.25, to determine whether $\omega$ is the endpoint of a path in $\text{Cu}(T/J, T)$.

The following result and its proof are analogous to Proposition 5.23.

**Proposition 5.28.** Let $S$ and $T$ be Cu-semigroups, let $J \triangleleft S$, and let $\pi : S \to S/J$ denote the quotient map. Then the induced Cun-morphism $\pi^* : [S/J, T] \to [S, T]$ is an order-embedding that identifies $[S/J, T]$ with an ideal in $[S, T]$. Moreover, $x \in [S, T]$ belongs to $[S/J, T]$ if and only if for some (equivalently, for every) path $(f_\lambda)_\lambda$ representing $x$, each $f_\lambda$ vanishes on $J$.

5.29. By Propositions 5.23 and 5.28, ideals in $S$ and $T$ naturally induce ideals in $[S, T]$. More precisely, if $J \triangleleft S$ and $K \triangleleft T$, we can identify $[S/J, K]$ with an ideal in $[S, T]$. Let $\text{Lat}(P)$ denote the ideal lattice of a Cu-semigroup $P$. We therefore obtain a map

$$\text{Lat}(S)^{\text{op}} \times \text{Lat}(T) \to \text{Lat}([S, T]).$$

However, this map need not be injective. For example, consider $S = Z$ and $T = \mathbb{N} \oplus Z$ with the ideal $J = 0 \oplus Z$. Note that every generalized Cu-morphism $Z \to \mathbb{N} \oplus Z$ necessarily takes values in the ideal $0 \oplus Z$. It follows that in this case $[S, J] = [S, T]$.

The following example shows that the above map is also not surjective in general. In fact, the example shows that there exists a simple Cu-semigroup $S$ such that $[S, S]$ is not simple.

**Example 5.30.** Let $S := [0, 1] \cup \{\infty\}$, considered with order and addition as a subset of $\mathbb{F}$, with the convention that $a + b = \infty$ whenever $a + b > 1$ in $\mathbb{F}$. It is easy to check that $S$ is a simple Cu-semigroup.

Given $t \in \{0\} \cup [1, \infty]$, let $\varphi_t : S \to S$ be the map given by $\varphi_t(a) := ta$, where $ta$ is given by the usual multiplication in $\mathbb{F}$ applying the above convention that an element is $\infty$ as soon as it is larger than 1. Then $\varphi_t$ is a generalized Cu-morphism. One can show that every generalized Cu-morphism $S \to S$ is of this form.

We deduce that $\text{Cu}(S, S)$ is isomorphic to $\{0\} \cup [1, \infty]$, identifying $\prec$ with $\leq$. It follows that

$$[S, S] = \tau(\text{Cu}(S, S), \prec) \cong \tau(\{0\} \cup [1, \infty], \leq) \cong \{0\} \cup [1, \infty] \cup (1, \infty),$$

which is a disjoint union of compact elements corresponding to $\{0\} \cup [1, \infty]$ and nonzero soft elements corresponding to $(1, \infty)$. (Similar to the decomposition of $Z$ and $R_\eta$.)

In particular, $[S, S]$ contains a compact infinite element $\infty$, and a noncompact infinite element $\infty'$. The set $J := \{x : x \leq \infty'\}$ is an ideal in $[S, S]$. We have $\infty \notin J$, which shows that $[S, S]$ is not simple.

**Problem 5.31.** Characterize when $[S, T]$ is simple. In particular, given simple Cu-semigroups $S$ and $T$, give necessary and sufficient criteria for $[S, T]$ to be simple.
6. Concretization of categorical constructions for Cu

In this section, we give concrete pictures of general constructions in closed, symmetric, monoidal categories for the category Cu. This will be used in the next section. In Subsection 6.1 we study unit and counit maps, which are natural Cu-morphisms
\[ d_{S,T}: S \to [T, S \otimes T] \quad \text{and} \quad e_{S,T}: [S, T] \otimes S \to T, \]
respectively. In the particular case that \( T = \mathbb{N} \), and after applying the isomorphism \( S \otimes \mathbb{N} \cong S \), the unit map \( d_{S,\mathbb{N}} \) takes the form \( S \to [\mathbb{N}, S] \) and is denoted by \( i_S \). We will see that \( i_S \) is a natural isomorphism between \( S \) and \( [\mathbb{N}, S] \).

In Subsection 6.2 we generalize the tensor product of Cu-morphisms (as defined in Paragraph 2.17) by introducing the external tensor product map
\[ \boxtimes: [S_1, T_1] \otimes [S_2, T_2] \to [S_1 \otimes S_2, T_1 \otimes T_2]. \]

In Subsection 6.3, we study the composition product, which is the generalization of the composition of morphisms in a category to a notion of composition between internal-hom objects in a closed category; see the comments after Proposition 6.12.

We then show that the external tensor product and the composition product are compatible; see Proposition 6.22.

In Subsection 6.4, we show how the unit and counit maps, the tensor and the composition product can be used to give concrete formulas for the correspondence
\[ \text{Cu}(S, [T, P]) \cong \text{Cu}(S \otimes T, P), \]
proved in Theorem 5.10, see Proposition 6.23.

We also study functorial properties of the endpoint map, and of the unit and counit maps. Finally, similar to KK-theory for C*-algebras, we have a general form of the product that simultaneously generalizes the composition product and the external tensor product; see Paragraph 6.30.

6.1. Unit and counit.

Definition 6.1. Given Cu-semigroups \( S \) and \( T \), we define the unit map as the Cu-morphism \( d_{S,T}: S \to [T, S \otimes T] \) that under the identification
\[ \text{Cu}(S, [T, S \otimes T]) \cong \text{Cu}(S \otimes T, S \otimes T) \]
corresponds to the identity map on \( S \otimes T \).

Proposition 6.2. Let \( S \) and \( T \) be Cu-semigroups, and let \( s \in S \). Let \( (s_\lambda)_{\lambda \in I_S} \) be a path in \( (S, \preccurlyeq) \) with endpoint \( s \). Then for each \( \lambda \in I_S \), the map \( s_\lambda \otimes \cdot : T \to S \otimes T \), sending \( t \in T \) to \( s_\lambda \otimes t \), is a generalized Cu-morphism. Moreover, \( (s_\lambda \otimes \cdot)_{\lambda \in I_S} \) is a path in \( \text{Cu}(T, S \otimes T), \preccurlyeq \), and we have \( d_{S,T}^\lambda(s) = [(s_\lambda \otimes \cdot)_{\lambda}] \).

Proof. The map \( \omega: S \times T \to S \otimes T \), given by \( \omega(a, b) = a \otimes b \), is a Cu-bimorphism. This implies that \( a \otimes \cdot : T \to S \otimes T \) is a generalized Cu-morphism for each \( a \in S \). Moreover, using that \( \omega \) preserves the joint way-below relation, we obtain that \( a' \otimes \cdot \preccurlyeq a \otimes \cdot \) for \( a' \in S \) satisfying \( a' \preccurlyeq a \). In particular, if \( (a_\lambda)_{\lambda \in I_S} \) is a path in \( S \), then \( (a_\lambda \otimes \cdot)_{\lambda \in I_S} \) is a path in \( \text{Cu}(T, S \otimes T), \preccurlyeq \). We define \( \alpha: S \to [T, S \otimes T] \) by sending \( a \in S \) to \([((a_\lambda \otimes \cdot)_{\lambda})_\lambda]\) for some choice of path \( (a_\lambda)_{\lambda} \) in \( S \) with endpoint \( a \). It is straightforward to check that \( \alpha \) is a well-defined Cu-morphism.

Let us show that \( \alpha = d_{S,T}^S \). Consider the bijections
\[ \text{Cu}(S, [T, S \otimes T]) \cong \text{BiCu}(S \times T, S \otimes T) \cong \text{Cu}(S \otimes T, S \otimes T) \]
from Theorem 5.10. Under the first bijection, \( \alpha \) corresponds to the Cu-bimorphism \( \bar{\alpha} \) given by
\[ \bar{\alpha}(a, b) = \sigma_{T,S\otimes T}(\alpha(a))(b), \]
for \( a \in S \) and \( b \in T \), where \( \sigma_{T,S \otimes T} \) is the endpoint map. We compute

\[
\bar{\alpha}(a,b) = \sigma_{T,S \otimes T}(\alpha(a))(b) = \sup_{\lambda \in \mathfrak{I}_2}(a\lambda \otimes b)(b) = \sup_{\lambda \in \mathfrak{I}_2}(a \otimes b) = a \otimes b,
\]

for every path \((a\lambda)_\lambda\) with endpoint \( a \in S \), and every \( b \in T \). It follows that \( \bar{\alpha} \) corresponds to \( id_{S \otimes T} \) under the second bijection. By definition of \( d_{S,T} \), this show that \( \alpha = d_{S,T} \), as desired.

**Remark 6.7.** □

**Notation 6.3.** Given Cu-semigroups \( S, T \) and \( P \), and a Cu-bimorphism \( \alpha : S \times T \to P \), we shall often use the notation \( \bar{\alpha} : S \to [T, P] \) to refer to the Cu-morphism that corresponds to \( \alpha \) under the identification in [Theorem 5.10](#).

**Corollary 6.4.** Let \( S \) and \( T \) be Cu-semigroups. Then the composition

\[
\sigma_{T,S \otimes T} \circ d_{S,T} : S \xrightarrow{d_{S,T}} [T, S \otimes T] \xrightarrow{\sigma_{T,S \otimes T}} Cu[T, S \otimes T].
\]

satisfies \( (\sigma_{T,S \otimes T} \circ d_{S,T}(s)) = s \otimes \_ \), for every \( s \in S \). In particular

\[
(\sigma_{T,S \otimes T} \circ d_{S,T}(s))(t) = s \otimes t,
\]

for \( s \in S \) and \( t \in T \).

**Proof.** Let \( s \in S \) and \( t \in T \). Choose a path \((s_\lambda)_\lambda\) in \( S \) with endpoint \( s \). Then \( d_{S,T}(s) = ([s_\lambda \otimes \_])_\lambda \) by [Proposition 6.2](#). The supremum of the maps \( s_\lambda \otimes \_ \) in \( Cu[S, T \otimes S] \) is the map \( s \otimes \_ \). Thus, \( (\sigma_{T,S} \circ T \otimes d_{S,T}(s)) = s \otimes \_ \), as desired. □

**Definition 6.5.** Given Cu-semigroups \( S \) and \( T \), recall that the counit map (also called the evaluation map) is defined as the Cu-morphism \( e_{S,T} : [S, T] \otimes S \to T \) that under the identification

\[
Cu([S, T] \otimes S, T) \cong Cu([S, T], [S, T])
\]

corresponds to the identity map on \([S, T]\). (See [Paragraph 2.14](#) and the comments after [Proposition 5.12](#).) Given \( x \in [S, T] \) and \( s \in S \), we denote \( e_{S,T}(x \otimes s) \) by \( x(s) \).

**Proposition 6.6.** Let \( S \) and \( T \) be Cu-semigroups, let \( x \in [S, T] \), and let \( s \in S \). Then \( e_{S,T}(x \otimes s) = \sigma_{S,T}(x)(s) \). Thus, if \( f := (f_\lambda)_\lambda \) is a path in \( Cu[S, T] \), then

\[
[f](s) = e_{S,T}([f] \otimes s) = \sup_{\lambda \in \mathfrak{I}_1} f_\lambda(s).
\]

**Proof.** Consider the bijections

\[
Cu([S, T], [S, T]) \cong BiCu([S, T] \times S, T) \cong Cu([S, T] \otimes S, T)
\]

from [Theorem 5.10](#). To simplify notation, we denote the identity map on \([S, T]\) by \( id \). Under the first bijection, \( id \) corresponds to the Cu-bimorphism \( id \) satisfying

\[
\hat{id}(y, b) = \sigma_{S,T}(id(y))(b),
\]

for all \( y \in [S, T] \) and \( b \in S \). We obtain that

\[
e_{S,T}(x \otimes s) = \hat{id}(x, s) = \sigma_{S,T}(id(x))(s) = \sigma_{S,T}(x)(s),
\]

as desired. □

**Remark 6.7.** Let \( \varphi : S \to T \) be a Cu-morphism, and let \( s \in S \). Considering \( \varphi \) as an element of \([S, T]\), the notation \( \varphi(s) \) for \( e_{S,T}(\varphi \otimes s) \) is consistent with the usual notation of \( \varphi(s) \) for the evaluation of \( \varphi \) at \( s \).

**Lemma 6.8.** Let \( S \) be a Cu-semigroup. Let \( ev_1 : Cu[\mathbb{N}, S] \to S \) be given by \( ev_1(f) = f(1) \) for \( f \in Cu[\mathbb{N}, S] \). Then \( ev_1 \) is an isomorphism of \( \mathbb{Q} \)-semigroups. That is, \( ev_1 \) is an additive order-isomorphism and we have \( f < g \) if and only if \( ev_1(f) < ev_1(g) \), for \( f, g \in Cu[\mathbb{N}, S] \).

It follows that \((Cu[\mathbb{N}, S], \prec)\) is a Cu-semigroup (naturally isomorphic to \( S \) via \( ev_1 \)). Moreover, the endpoint map \( \sigma_{\mathbb{N}, S} : [\mathbb{N}, S] \to Cu[\mathbb{N}, S] \) from [Definition 5.6](#) is
an isomorphism. We obtain a commutative diagram of \(\text{Cu}\)-semigroups and isomorphisms:

\[
\begin{array}{ccc}
\tau(\text{Cu}[\mathbb{N}, S]) & \xrightarrow{\sigma_{\mathbb{N}, S}} & \text{Cu}[\mathbb{N}, S] \\
\tau(\text{ev}_1) & \cong & \text{ev}_1 \\
\tau(S) & \cong & S
\end{array}
\]

**Proof.** It is straightforward to prove that \(\text{ev}_1\) is an isomorphism of \(Q\)-semigroups. By Proposition 4.10 the endpoint map of a \(\text{Cu}\)-semigroup is an isomorphism. Thus, the endpoint maps \(\varphi_S\) and \(\varphi_{\text{Cu}[\mathbb{N}, S]}\) are isomorphisms. By definition, \(\sigma_{\mathbb{N}, S} = \varphi_{\text{Cu}[\mathbb{N}, S]}\). Since \(\text{ev}_1\) is an isomorphism, so is \(\tau(\text{ev}_1)\). By Proposition 4.8 the endpoint map is natural, which implies that the diagram is commutative. \(\square\)

**Definition 6.9.** Given a \(\text{Cu}\)-semigroup \(S\), we let \(i_S : S \rightarrow [\mathbb{N}, S]\) be the \(\text{Cu}\)-morphism that under the identification

\[
\text{Cu}(S, [\mathbb{N}, S]) \cong \text{Cu}(S \otimes \mathbb{N}, S)
\]

corresponds to the natural isomorphism \(r_S : S \otimes \mathbb{N} 
arrow S\).

We leave the proof of the following result to the reader.

**Proposition 6.10.** Let \(S\) be a \(\text{Cu}\)-semigroup. Then the map \(i_S : S \rightarrow [\mathbb{N}, S]\) from Definition 6.9 is an isomorphism. The inverse of \(i_S\) is \(\text{ev}_1 \circ \sigma_{\mathbb{N}, S}\), where \(\text{ev}_1\) is evaluation at 1 as in Lemma 6.8 and where \(\sigma_{\mathbb{N}, S} : [\mathbb{N}, S] \rightarrow \text{Cu}[\mathbb{N}, S]\) denotes the endpoint map from Definition 5.6. Moreover, we have \(i_S = (r_S)_\ast \circ d_{S, \mathbb{N}}\).

6.2. **External tensor product.**

Let \(S_k\) and \(T_k\) be \(\text{Cu}\)-semigroups, and let \(\varphi_k : S_k \rightarrow T_k\) be (generalized) \(\text{Cu}\)-morphisms, for \(k = 1, 2\). Recall from Paragraph 2.17 that the map \(\varphi_1 \times \varphi_2 : S_1 \times S_2 \rightarrow T_1 \otimes T_2\), defined by

\[
(\varphi_1 \times \varphi_2)(s_1, s_2) := f_1(s_1) \otimes f_2(s_2),
\]

for \(s_1 \in S_1\) and \(s_2 \in S_2\), is (generalized) \(\text{Cu}\)-bimorphism. We denote the induced (generalized) \(\text{Cu}\)-bimorphism by \(\varphi_1 \otimes \varphi_2 : S_1 \otimes S_2 \rightarrow T_1 \otimes T_2\) and we call the map \(\varphi_1 \otimes \varphi_2\), as customary, the tensor product of \(\varphi_1\) and \(\varphi_2\).

Next, we generalize this construction and define an external tensor product between elements of internal-homs.

**Definition 6.12.** Given \(\text{Cu}\)-semigroups \(S_1, S_2, T_1\) and \(T_2\), we define the external tensor product map \(\boxtimes : [S_1, T_1] \otimes [S_2, T_2] \rightarrow [S_1 \otimes S_2, T_1 \otimes T_2]\) as the \(\text{Cu}\)-morphism that under the identification

\[
\text{Cu}([S_1, T_1] \otimes [S_2, T_2], [S_1 \otimes S_2, T_1 \otimes T_2]) \\
\cong \text{Cu}([S_1, T_1] \otimes [S_2, T_2] \otimes S_1 \otimes S_2, T_1 \otimes T_2),
\]

corresponds to the composition

\[
(e_{S_1, T_1} \otimes e_{S_2, T_2}) \circ (\text{id}_{[S_1, T_1]} \otimes \sigma \otimes \text{id}_{S_2}),
\]

where \(\sigma : [S_2, T_2] \otimes S_1 \rightarrow S_1 \otimes [S_2, T_2]\) denotes the flip isomorphism.

Given \(x_1 \in [S_1, T_1]\) and \(x_2 \in [S_2, T_2]\), we denote the image of \(x_1 \otimes x_2\) under this map by \(x_1 \boxtimes x_2\), and we call it the external tensor product of \(x_1\) and \(x_2\).

**Remark 6.13.** Let \(\varphi_1 : S_1 \rightarrow T_1\) and \(\varphi_2 : S_2 \rightarrow T_2\) be \(\text{Cu}\)-morphisms. Using Proposition 5.12 we identify \(\varphi_1\) with a compact element in \([S_1, T_1]\), and similarly for \(\varphi_2\). It is easy to see that the element \(\varphi_1 \boxtimes \varphi_2\) from Definition 6.12 agrees with
the compact element in $[S_1 \otimes S_2, T_1 \otimes T_2]$ that is identified with the tensor product map $\varphi_1 \otimes \varphi_2: S_1 \otimes S_2 \to T_1 \otimes T_2$ from Paragraph 6.11.

Notice that there is a certain ambiguity with the notation $\varphi_1 \otimes \varphi_2$, in that it may refer to a Cu-morphism (identified with a compact element in $[S_1 \otimes S_2, T_1 \otimes T_2]$), and also to an element in $[S_1, T_1] \otimes [S_2, T_2]$. However, the precise meaning will be clear from the context.

**Proposition 6.14.** Let $S_1, S_2, T_1$ and $T_2$ be Cu-semigroups, and let $f = (f_\lambda)_{\lambda}$ and $g = (g_\lambda)_{\lambda}$ be paths in $\text{Cu}[S_1, T_1]$ and $\text{Cu}[S_2, T_2]$, respectively. For each $\lambda$, consider the generalized Cu-morphism $f_\lambda \otimes g_\lambda: S_1 \otimes S_2 \to T_1 \otimes T_2$. Then $(f_\lambda \otimes g_\lambda)$ is a path in $\text{Cu}[S_1 \otimes S_2, T_1 \otimes T_2]$ and we have

$$[f] \otimes [g] = [(f_\lambda \otimes g_\lambda)_{\lambda}].$$

**Proof.** To show that $(f_\lambda \otimes g_\lambda)_{\lambda}$ is a path, let $\lambda', \lambda \in I_Q$ satisfy $\lambda' < \lambda$. To show that $f_{\lambda'} \otimes g_{\lambda'} < f_{\lambda} \otimes g_{\lambda}$, let $t', t \in S_1 \otimes S_2$ satisfy $t' < t$. By properties of the tensor product in Cu, we can choose $n \in \mathbb{N}$, elements $a_k', a_k \in S_1$ and $b_k', b_k \in S_2$ satisfying $a_k' \ll a_k$ and $b_k' \ll b_k$ for $k = 1, \ldots, n$, and such that

$$t' \leq \sum_{k=1}^{n} a_k' \otimes b_k', \quad \text{and} \quad \sum_{k=1}^{n} a_k \otimes b_k \leq t.$$ 

We have $f_{\lambda'} \ll f_{\lambda}$ and $g_{\lambda'} \ll g_{\lambda}$, and therefore $f_{\lambda'}(a_k') \ll f_{\lambda}(a_k)$ and $g_{\lambda'}(b_k') \ll g_{\lambda}(b_k)$ for $k = 1, \ldots, n$. Using this at the third step we deduce that

$$(f_{\lambda'} \otimes g_{\lambda'})(t') \leq (f_{\lambda} \otimes g_{\lambda}) \left( \sum_{k=1}^{n} a_k' \otimes b_k' \right)$$

$$= \sum_{k=1}^{n} f_{\lambda'}(a_k') \otimes g_{\lambda'}(b_k')$$

$$\ll \sum_{k=1}^{n} f_{\lambda}(a_k) \otimes g_{\lambda}(b_k)$$

$$= (f_{\lambda} \otimes g_{\lambda}) \left( \sum_{k=1}^{n} a_k \otimes b_k \right) \leq (f_{\lambda} \otimes g_{\lambda})(t),$$

as desired.

Thus, given paths $p = (p_\lambda)_{\lambda}$ and $q = (q_\lambda)_{\lambda}$ in $\text{Cu}[S_1, T_1]$ and $\text{Cu}[S_2, T_2]$, respectively, then $(p_\lambda \otimes q_\lambda)_{\lambda}$ is a path in $\text{Cu}[S_1 \otimes S_2, T_1 \otimes T_2]$. Moreover, it is tedious but straightforward to check that the map $[S_1, T_1] \times [S_2, T_2] \to [S_1 \otimes S_2, T_1 \otimes T_2]$ that sends a pair $(p_\lambda \otimes q_\lambda)$ to $(p_\lambda \otimes q_\lambda)_{\lambda}$ is a well-defined Cu-bimorphism. We let $\alpha: [S_1, T_1] \otimes [S_2, T_2] \to [S_1 \otimes S_2, T_1 \otimes T_2]$ be the induced Cu-morphism.

To show that $[f] \otimes [g] = [(f_\lambda \otimes g_\lambda)_{\lambda}]$, we will prove that the external tensor product $\otimes$ and the map $\alpha$ correspond to the same Cu-morphism under the bijection

$$\text{Cu}([S_1, T_1] \otimes [S_2, T_2], [S_1 \otimes S_2, T_1 \otimes T_2]) \cong \text{Cu}([S_1, T_1] \otimes [S_2, T_2] \otimes [S_1 \otimes S_2, T_1 \otimes T_2])$$

from Theorem 5.10.

Let $p = (p_\lambda)_{\lambda}$ and $q = (q_\lambda)_{\lambda}$ be paths in $\text{Cu}[S_1, T_1]$ and $\text{Cu}[S_2, T_2]$, respectively, and let $s_i$ be elements in $S_i$, for $i = 1, 2$. By definition of $\otimes$ (see Definition 6.12 and Notation 6.3), we have

$$\otimes([p] \otimes [q] \otimes s_1 \otimes s_2) = [p](s_1) \otimes [q](s_2) = f_1(s_1) \otimes q_1(s_2).$$
Using Theorem 5.10 at the first step, we obtain that
\[
\bar{\sigma}(\bar{[p]} \otimes \bar{[q]} \otimes s_1 \otimes s_2) = \sigma_{S_1 \otimes S_2, T_1 \otimes T_2}(\alpha(\bar{[p]} \otimes \bar{[q]}))(s_1 \otimes s_2) \\
= \sigma_{S_1 \otimes S_2, T_1 \otimes T_2}([p_1 \otimes q_1, \lambda])(s_1 \otimes s_2) \\
= (p_1 \otimes q_1)(s_1 \otimes s_2) = p_1(s_1) \otimes q_1(s_2).
\]
It follows that \( \boxtimes = \alpha \) and therefore
\[
\bar{[f]} \boxtimes \bar{[g]} = \bar{\alpha}(\bar{[f]} \otimes \bar{[g]}) = \alpha([f] \otimes [g]) = ([f_\lambda \otimes g_\lambda], \lambda),
\]
as desired.

The following result shows that the external tensor product is associative.

**Proposition 6.15.** Let \( S_1, S_2, T_1, T_2, P_1 \) and \( P_2 \) be Cu-semigroups, let \( s \), \( u \), \( x \) and \( y \) be given by
\[
S_1 \otimes T_1 \otimes P_1 \rightarrow S_2 \otimes T_2 \otimes P_2.
\]
Then
\[
(x \boxtimes y) \boxtimes z = x \boxtimes (y \boxtimes z).
\]

**Proof.** Given \( f \in Cu[S_1, S_2] \), \( g \in Cu[T_1, T_2] \) and \( h \in Cu[P_1, P_2] \), it is straightforward to check that
\[
(f \otimes g) \otimes h = f \otimes (g \otimes h),
\]
as generalized Cu-morphisms \( S_1 \otimes T_1 \otimes P_1 \rightarrow S_2 \otimes T_2 \otimes P_2 \). The result follows by applying Proposition 6.14.

**Problem 6.16.** Study the order-theoretic properties of the external tensor product map \( \boxtimes : [S_1, T_1] \otimes [S_2, T_2] \rightarrow [S_1 \otimes S_2, T_1 \otimes T_2] \). In particular, when is this map an order-embedding, when is it surjective?

### 6.3. Composition product.

**Definition 6.17.** Given Cu-semigroups \( S, T \) and \( P \), we define the composition product
\[
\circ : [T, P] \otimes [S, T] \rightarrow [S, P]
\]
as the Cu-morphism that under the identification
\[
Cu([T, P] \otimes [S, T], [S, P]) \cong Cu([T, P] \otimes [S, T] \otimes S, P)
\]
corresponds to the composition \( e_{T,P} \circ (\text{id}_{T, P} \otimes \circ_{S, T}) \). Given \( x \in [S, T] \) and \( y \in [T, P] \), we denote the image of \( y \circ x \) under the composition product by \( y \circ x \).

Given \( x \in [S, T] \), we let \( x^* : [T, P] \rightarrow [S, P] \) be given by \( x^*(y) := y \circ x \) for \( y \in [T, P] \). Analogously, given \( y \in [T, P] \), we let \( y_* : [S, T] \rightarrow [S, P] \) be given by \( y_*(x) := y \circ x \) for \( x \in [S, T] \).

**Proposition 6.18.** Let \( S, T \) and \( P \) be Cu-semigroups, and let \( f = (f_\lambda) \) and \( g = (g_\lambda) \) be paths in \( Cu[S, T] \) and \( Cu[T, P] \), respectively. For each \( \lambda \), consider the generalized Cu-morphism \( g_\lambda \circ f_\lambda : S \rightarrow P \). Then \( (g_\lambda \circ f_\lambda) \) is a path in \( Cu[S, P] \) and
\[
[g] \circ [f] = [(g_\lambda \circ f_\lambda) \lambda].
\]

**Proof.** It is easy to check that \( (g_\lambda \circ f_\lambda) \lambda \) is a path. Moreover, it is tedious but straightforward to check that the map \([T, P] \times [S, T] \rightarrow [S, P] \) that sends a pair \(([p], [q])\) to \([g_\lambda \circ p_\lambda]) \lambda \) is a well-defined Cu-bimorphism. We let \( \alpha : [T, P] \otimes [S, T] \rightarrow [S, P] \) be the induced Cu-morphism.

To show that \( [g] \circ [f] = [(g_\lambda \circ f_\lambda) \lambda] \), we will prove that the composition product \( \circ \) and the map \( \alpha \) correspond to the same Cu-morphism under the bijection
\[
Cu([T, P] \otimes [S, T], [S, P]) \cong Cu([T, P] \otimes [S, T] \otimes S, P)
\]
from [Theorem 5.10]

Let $p = (p_{\lambda})_{\lambda}$ and $q = (q_{\lambda})_{\lambda}$ be paths in $Cu[S,T]$ and $Cu[T,P]$, respectively, and let $s \in S$. Set $p_1 := \text{sup}_{\lambda \leq 1} p_{\lambda}$ and $q_1 := \text{sup}_{\lambda \leq 1} q_{\lambda}$. By definition, we have

$$\tilde{\alpha}([q] \otimes [p] \otimes s) = e_{T,P} \circ (id_{[T,P]} \otimes e_{S,T})([q] \otimes [p] \otimes s)$$

$$= e_{T,P}([q] \otimes e_{S,T}([p] \otimes s))$$

$$= e_{T,P}([q] \otimes p_1(s))$$

$$= q_1(p_1(s)).$$

On the other hand, using [Theorem 5.10] at the first step, we obtain that

$$\tilde{\alpha}([q] \otimes [p] \otimes s) = \sigma_{S,P}(\alpha([q] \otimes [p]))(s)$$

$$= \sigma_{S,P}((q_{\lambda} \circ p_{\lambda})_{\lambda})(s)$$

$$= (q_1 \circ p_1)(s)$$

It follows that $\circ = \alpha$ and therefore

$$[g] \circ [f] = \alpha([g] \otimes [f]) = \alpha([g] \otimes [f]) = ([g_{\lambda} \otimes f_{\lambda})_{\lambda}},$$

as desired. \qed

Note that, in [Proposition 6.18], the composition product of two Cu-morphisms, viewed as compact elements in the internal-hom set, is the usual composition of morphisms as maps.

The following result shows that the composition product is associative and that the identity element $id_S \in Cu(S,S) \subseteq [S,S]$ acts as a unit for the composition product (as expected). It follows that $[S,S]$ and $[[T,T]]$ are (not necessarily commutative) Cu-semirings and that $[S,T]$ has a natural left $[[S,S]]$- and right $[[T,T]]$-semimodule structure; see Propositions [7.1] and [7.7] in the next section.

**Proposition 6.19.** Let $S, T, P$ and $Q$ be Cu-semigroups, let $x \in [S,T]$, $y \in [T,P]$, and let $z \in [P,Q]$. Then

$$(z \circ y) \circ x = z \circ (y \circ x).$$

Further, for the identity Cu-morphisms $id_S \in Cu(S,S)$ and $id_T \in Cu(T,T)$, we have

$$id_T \circ x = x \circ id_S.$$ 

**Proof.** Given $f \in Cu[S,T]$, $g \in Cu[T,P]$ and $h \in Cu[P,Q]$, it is straightforward to check that

$$(h \circ g) \circ f = h \circ (g \circ f),$$

in $Cu[S,Q]$. The result follows by applying [Proposition 6.18]. The statement about the composition with $id_S$ and $id_T$ follows also directly from [Proposition 6.18]. \qed

Next we show that the composition product is compatible with the evaluation map in the expected way. It will follow later that the evaluation map $e^S_S \colon [S,S] \otimes S \to S$ defines a natural left $[S,S]$-semimodule structure on $S$; see [Proposition 7.3]

**Lemma 6.20.** Let $S, T$ and $P$ be Cu-semigroups, let $x \in [S,T]$, and let $y \in [T,P]$. Then

$$\sigma_{S,P}(y \circ x) = \sigma_{T,P}(y) \circ \sigma_{S,T}(x).$$
Proof. Let $f = (f_{\lambda})_\lambda$ be a path in $\text{Cu}[S,T]$ representing $x$, and let $g = (g_{\lambda})_\lambda$ be a path in $\text{Cu}[T,P]$ representing $y$. Let $s \in S$. By Proposition 6.18, we have $y \circ x = [(g_{\lambda} \circ f_{\lambda})_\lambda]$. Using this at the second step, and using Proposition 6.14 and Proposition 6.18 at the first step, we obtain that

$$
\sigma_{S,P}(y \circ x)(s) = \sup_{\lambda \in I_2} (g_{\lambda} \circ f_{\lambda})(s) = \sup_{\mu \in I_2} g_{\mu} \left( \sup_{\lambda \in I_2} f_{\lambda}(s) \right) = \sigma_{T,P}(y)(\sigma_{S,T}(x)(s)),
$$
as desired. \qed

By combining Lemma 6.20 with Proposition 6.6, we obtain:

**Proposition 6.21.** Let $S, T$ and $P$ be Cu-semigroups, let $x \in [S,T]$, let $y \in [T,P]$, and let $s \in S$. Then

$$(y \circ x)(s) = y(x(s)).$$

Moreover, for the identity Cu-morphism $id_S \in \text{Cu}(S,S)$, we have $id_S(s) = s$.

The following result shows that the external tensor product and the composition product commute.

**Proposition 6.22.** Let $S_1, S_2, T_1$ and $T_2$ be Cu-semigroups. Given $x_k \in [S_k,T_k]$ and $y_k \in [T_k,P_k]$ for $k = 1, 2$, we have

$$(y_2 \boxtimes y_1) \circ (x_2 \boxtimes x_1) = (y_2 \circ x_2) \boxtimes (y_1 \circ x_1).$$

Proof. Let $f^{(k)} = (f^{(k)}_{\lambda})_\lambda$ be a path in $\text{Cu}[S_k,T_k]$ representing $x_k$, for $k = 1, 2$, and let $g^{(k)} = (g^{(k)}_{\lambda})_\lambda$ be a path in $\text{Cu}[T_k,P_k]$ representing $y_k$, for $k = 1, 2$. Given $\lambda$, it is straightforward to check that

$$(g^{(2)}_{\lambda} \otimes g^{(1)}_{\lambda}) \circ (f^{(2)}_{\lambda} \otimes f^{(1)}_{\lambda}) = (g^{(2)}_{\lambda} \circ f^{(2)}_{\lambda}) \otimes (g^{(1)}_{\lambda} \circ f^{(1)}_{\lambda}).$$

Using this at the second step, and using Proposition 6.14 and Proposition 6.18 at the first and last step, we obtain that

$$(y_2 \boxtimes y_1) \circ (x_2 \boxtimes x_1) = \left[ (g^{(2)}_{\lambda} \otimes g^{(1)}_{\lambda}) \circ (f^{(2)}_{\lambda} \otimes f^{(1)}_{\lambda}) \right]_\lambda = \left[ (g^{(2)}_{\lambda} \circ f^{(2)}_{\lambda}) \otimes (g^{(1)}_{\lambda} \circ f^{(1)}_{\lambda}) \right]_\lambda = (y_2 \circ x_2) \boxtimes (y_1 \circ x_1),$$
as desired. \qed

6.4. Implementation of the adjunction using unit and counit.

**Proposition 6.23.** Let $S, T$ and $P$ be Cu-semigroups. Then the bijection

$$\text{Cu}(S,[T,P]) \cong \text{Cu}(S \otimes T, P)$$

from Theorem 5.10 identifies a Cu-morphism $f : S \to [T,P]$ with

$$e_{T,P} \circ (f \otimes \text{id}_T) : S \otimes T \overset{f \otimes \text{id}_T}{\longrightarrow} [T,P] \otimes T \overset{e_{T,P}}{\longrightarrow} P.$$ 

Conversely, a Cu-morphism $g : S \otimes T \to P$ is identified with

$$g_\ast \circ d_{S,T} : S \overset{d_{S,T}}{\longrightarrow} [T,S \otimes T] \overset{g_\ast}{\longrightarrow} [T,P].$$

In particular, we have

$$f = (e_{T,P} \circ (f \otimes \text{id}_T))_\ast \circ d_{S,T}, \quad \text{and} \quad g = e_{T,P} \circ ((g_\ast \circ d_{S,T}) \otimes \text{id}_T).$$

Proof. Let $f : S \to [T,P]$ be a Cu-morphism. Under the natural bijection from Theorem 5.10 $f$ corresponds to the Cu-morphism $\tilde{f} : S \otimes T \to P$ with

$$\tilde{f}(s \otimes t) = \sigma^T_P(f(s))(t),$$

where $\sigma^T_P$ is the transition function associated with the Cu-morphism $f$. Conversely, given a Cu-morphism $g : S \otimes T \to P$, we have

$$g_\ast \circ d_{S,T} = (e_{T,P} \circ (g \otimes \text{id}_T))_\ast \circ d_{S,T}.$$
for a simple tensor \( s \otimes t \in S \otimes T \). On the other hand, we have
\[
(e_{T,P} \circ (f \otimes \text{id}_T))(s \otimes t) = e_{T,P}(f(s) \otimes t) = \sigma_{T,P}(f(s))(t),
\]
for a simple tensor \( s \otimes t \in S \otimes T \). Thus \( f \) and \( e_{T,P} \circ (f \otimes \text{id}_T) \) agree on simple tensors, and consequently \( \bar{f} = e_{T,P} \circ (f \otimes \text{id}_T) \), as desired.

Let \( g: S \otimes T \to P \) be a Cu-morphism. Set \( \alpha := g_* \circ \text{d}_{S,T} \). Under the natural bijection from Theorem 5.10, \( \alpha \) corresponds to the Cu-morphism \( \bar{\alpha}: \otimes T \to P \) with
\[
\bar{\alpha}(s \otimes t) = \sigma_{T,P}(\alpha(s))(t),
\]
for a simple tensor \( s \otimes t \in S \otimes T \). It is straightforward to verify that \( \sigma_{T,P} \circ g_* = g_* \circ \sigma_{T,S \otimes T} \). (See Proposition 6.25) Using this at the third step, and using Corollary 6.4 at the fourth step, we deduce that
\[
\bar{\alpha}(s \otimes t) = \sigma_{T,P}(\alpha(s))(t) = (\sigma_{T,P} \circ g_*)(s)(t) = (g_* \circ \sigma_{T,S \otimes P} \circ \text{d}_{S,T})(s)(t) = g(s \otimes t),
\]
for every simple tensor \( s \otimes t \in S \otimes T \). Thus, \( \bar{\alpha} = g \), as desired.

Applying the previous result to the identity morphisms, we obtain:

**Corollary 6.24.** Let \( S \) and \( T \) be Cu-semigroups. Then
\[
\text{id}_{[S,T]} = (e_{S,T}^*)_* \circ \text{d}_{S,T}^S, \quad \text{and} \quad \text{id}_{S \otimes T} = e_{T \otimes T}^S \circ (\text{d}_{S \otimes T}^S \otimes \text{id}_T).
\]

6.5. **Functorial properties.** Next, we study the functorial properties of the endpoint map, the unit map, and the counit map.

**Proposition 6.25.** Let \( S, S', T \) and \( T' \) be Cu-semigroups, let \( x \in [S,T] \), and let \( f: T \to T' \) and \( g: S' \to S \) be Cu-morphisms. Then
\[
\sigma_{S,T'} \circ f_* = f_* \circ \sigma_{S,T}, \quad \text{and} \quad \sigma_{S,T} \circ g^* = g^* \circ \sigma_{S,T}
\]
which means that the following diagrams commute:

\[
\begin{array}{ccc}
[S,T] & \xrightarrow{f} & [S,T'] \\
\downarrow_{\sigma_{S,T}} \quad & & \downarrow_{\sigma_{S,T'}} \\
\text{Cu}[S,T] & \xrightarrow{f_*} & \text{Cu}[S,T']
\end{array}
\quad
\begin{array}{ccc}
[S',T] & \xrightarrow{g^*} & [S,T] \\
\downarrow_{\sigma_{S,T}} \quad & & \downarrow_{\sigma_{S,T'}} \\
\text{Cu}[S,T'] & \xrightarrow{g^*} & \text{Cu}[S,T]
\end{array}
\]

**Proof.** This follows from the definition of the abstract bivariant Cu-semigroup (see Definition 5.3) and the naturality of the endpoint map (see Proposition 4.8). \( \square \)

The proofs of the following results are straightforward and left to the reader.

**Proposition 6.26.** Let \( S_1, S_2 \) and \( T \) be Cu-semigroups, and let \( f: S_1 \to S_2 \) be a Cu-morphism. Then
\[
\text{d}_{S_1,T} \circ f = (f \otimes \text{id}_T)_* \circ \text{d}_{S_1,T},
\]
which means that the left diagram below commutes.

Analogously, if \( S, T_1 \) and \( T_2 \) are Cu-semigroups, and if \( g: T_1 \to T_2 \) is a Cu-morphism, then
\[
g \circ e_{S,T_1} = e_{S,T_2} \circ (g_* \otimes \text{id}_S),
\]
which means that the right diagram below commutes.

\[
\begin{array}{ccc}
S_1 & \xrightarrow{\text{d}_{S_1,T}} & [T,S_1 \otimes T] \\
\downarrow_f \quad & & \downarrow_{(f \otimes \text{id}_T)_*} \\
S_2 & \xrightarrow{\text{d}_{S_2,T}} & [T,S_2 \otimes T]
\end{array}
\quad
\begin{array}{ccc}
[S,T_1] \otimes S & \xrightarrow{e_{S,T_1}} & T_1 \\
\downarrow_{g \circ e_{S,T_1}} \quad & & \downarrow_{g} \\
[S,T_2] \otimes S & \xrightarrow{e_{S,T_2}} & T_2
\end{array}
\]
6.6. General form of unit and product. Given Cu-semigroups $S$ and $T$, we consider the unit map $d_{S,T} : S \to [T,S \otimes T]$ from Definition 6.1. Next, we introduce a more general form of the unit map.

**Definition 6.27.** Let $S, T$ and $T'$ be Cu-semigroups. We define the general left unit map $S \otimes [T', T] \to [T', S \otimes T]$ as the Cu-morphism that under the identification
\[
\text{Cu}(S \otimes [T', T], [T', S \otimes T]) \cong \text{Cu}(S \otimes [T', T] \otimes T', S \otimes T)
\]
corresponds to the map $\text{id}_S \otimes e_{T',T}$. Given $a \in S$ and $x \in [T', T]$, we denote the image of $a \otimes x$ under this map by $a_x$.

Analogously, we define the general right unit map $[T', T] \otimes S \to [T', T \otimes S]$ as the Cu-morphism that under the identification
\[
\text{Cu}([T', T] \otimes S, [T', T \otimes S]) \cong \text{Cu}([T', T] \otimes S \otimes T', T \otimes S)
\]
corresponds to the map $(e_{T',T} \otimes \text{id}_S) \circ (\text{id}_{T',T} \otimes \sigma)$, where $\sigma$ denotes the flip isomorphism. Given $a \in S$ and $x \in [T', T]$, we denote the image of $x \otimes a$ under this map by $x_a$.

We leave the proofs of the following results to the reader.

**Proposition 6.28.** Let $S, T$ and $T'$ be Cu-semigroups, let $a$ be an element in $S$, and let $x$ be an element in $[T', T]$. Let $i_S : S \to [\mathbb{N}, S]$ be the isomorphism from Definition 6.9 and let $1_{\mathbb{N}} : \mathbb{N} \to T'$ and $r_{T'} : T' \otimes \mathbb{N} \to T'$ be the natural Cu-isomorphism. Then
\[
_a x = (i_S(a) \otimes x) \circ l_T^{-1} = d_{S,T'}(a) \circ x = (\text{id}_S \otimes x) \circ d_{S,T}(a).
\]
and analogously $x_a = (x \otimes i_S(a)) \circ r_T^{-1}$. Further, for the unit map $d_{S,T} : S \to [T, S \otimes T]$, we have $d_{S,T}(a) = a_{1_T}$ for every $a \in S$.

**Lemma 6.29.** Let $S, T, T'$ and $T''$ be Cu-semigroups. Given $x \in [T, T']$, $y \in [T', T''']$, and $a \in S$, we have
\[
a(y \circ x) = (\text{id}_S \otimes y) \circ a_x = a_y \circ x, \quad \text{and} \quad (y \circ x)_a = (\text{id}_S \otimes y) \circ x_a = y_a \circ x,
\]
in $[T, S \otimes T''']$ and $[T, T' \otimes S]$, respectively.

Given another Cu-semigroup $S'$ and $a' \in S'$, we have
\[
a'(a_x) = a' \circ a_x, \quad \text{and} \quad (x_a)_{a'} = x_{a \circ a'},
\]
in $[T, S' \otimes S \otimes T']$ and $[T, T' \otimes S \otimes S']$, respectively.

6.30. As in $KK$-theory, one can define a ‘general form of the product’ which generalizes both the composition product and the external tensor product; see [Bla98] Section 18.9, p.180f.

Let $P, S_1, S_2, T_1$ and $T_2$ be Cu-semigroups. We let
\[
\boxtimes_P : [S_1 \otimes P, T_1] \otimes [S_2, P \otimes T_2] \to [S_1 \otimes S_2, T_1 \otimes T_2],
\]
be the Cu-morphism that under the identification
\[
\text{Cu}([S_1 \otimes P, T_1] \otimes [S_2, P \otimes T_2], [S_1 \otimes S_2, T_1 \otimes T_2])
\]
\[
\cong \text{Cu}([S_1 \otimes P, T_1] \otimes [S_2, P \otimes T_2] \otimes S_1 \otimes S_2, T_1 \otimes T_2)
\]
corresponds to the composition
\[
(e_{S_1 \otimes P, T_1} \otimes \text{id}_{T_2}) \circ (\text{id}_{S_1 \otimes P, T_1} \otimes e_{S_2, P \otimes T_2}) \circ (\text{id}_{S_1 \otimes P, T_1} \otimes \sigma_{S_2, P \otimes T_2}, S_1 \otimes \text{id}_{T_2}),
\]
where $\sigma_{S_2, P \otimes T_2}, S_1$ denotes the flip isomorphism.

Given $x \in [S_1 \otimes P, T_1]$ and $y \in [S_2, P \otimes T_2]$, we have
\[
x \boxtimes_P y = (x \boxtimes \text{id}_{T_2}) \circ (\text{id}_{S_1} \boxtimes y).
\]
Specializing to the case $P = \mathbb{N}$, we obtain the external tensor product, after applying the usual isomorphisms $S_1 \otimes \mathbb{N} \cong S_1$ and $\mathbb{N} \otimes T_2 \cong T_2$.

Specializing to the case $T_2 = S_1 = \mathbb{N}$, we obtain the composition product, after applying the natural isomorphisms $\mathbb{N} \otimes P \cong P \cong P \otimes \mathbb{N}$.

**Remark 6.31.** The order of the product in $KK$-theory is reversed from the one used here for the category Cu, that is, given $C^*$-algebras $A$, $B$ and $D$, the product in $KK$-theory is as a bilinear map

$$KK(A, D) \times KK(D, B) \to KK(A, B);$$

see [Blau98, Section 18.1, p.166] and [JT91, Before Lemma 2.2.9, p.73].

We have mainly two reasons for our choice of ordering for the composition product in the category Cu: First, the composition product extends the usual composition of Cu-morphisms and our choice is compatible with the standard notation for composition of maps. Second, our ordering agrees with that of the composition law of internal-homs in closed categories; see [Kel05, Section 1.6, p.15].

### 7. Cu-semirings and Cu-semimodules

In [Subsection 7.1] we first recall the definitions of Cu-semirings and of left and right Cu-semimodules. Given a Cu-semigroup $S$, the abstract bivariant Cuntz semigroup $[S, S]$ has a natural Cu-semiring structure; see [Proposition 7.1]. In Example 7.9 we will see that $[S, S]$ is noncommutative in general.

The evaluation map $e_{S, S} : [S, S] \otimes S \to S$ defines a natural left $[S, S]$-semimodule structure on $S$; see [Proposition 7.3]. Given a Cu-semigroup $R$ with compact unit and a Cu-semigroup $T$ with left $R$-action, for every Cu-semigroup $S$ the bivariant Cu-semigroup $[S, T]$ has a natural left $R$-action; see [Proposition 7.6]. It follows that $[S, T]$ has a natural left action by the Cu-semiring $[T, T]$, and a compatible right $[S, S]$-action; see [Proposition 7.7].

In [Subsection 7.2] we study the connection between a Cu-semiring $R$ and the associated Cu-semiring $[R, R]$. We show that there is a natural multiplicative Cu-morphism $\pi_R : R \to [R, R]$ that is an order-embedding; see [Definition 7.10] and [Theorem 7.13] and [Proposition 7.17]. If the unit of $R$ is compact, then $\pi_R$ is unital. Thus, $R$ naturally is a (unital) sub-Cu-semiring of $[R, R]$.

In [Subsection 7.3] we study the situation for solid Cu-semirings. We first show that a (not necessarily commutative) Cu-semiring $R$ is solid whenever the map $\mu : R \otimes R \to R$ (induced by the multiplication in $R$) is injective; see [Lemma 7.20]. In [Theorem 7.26] we relate the property of $R$ being solid with other natural properties of Cu-semirings. In particular, a Cu-semiring $R$ with compact unit is solid if and only if the evaluation map $e_{R, R} : [R, R] \otimes R \to R$ is an isomorphism; see [Remark 7.27].

If $R$ is a solid Cu-semiring with compact unit, and if $T$ is an $R$-stable Cu-semigroup, then $[R \otimes S, T] \cong [S, T]$ for any $S$; see [Proposition 7.30]. In particular, we have $[R, T] \cong T$ for every $R$-stable Cu-semigroup $T$; see [Corollary 7.31].

#### 7.1. The Cu-semiring $[S, S]$ and the Cu-semibimodule $[S, T]$. A (unital) Cu-semiring is a Cu-semigroup $R$ together with a Cu-bimorphism $R \times R \to R$, denoted by $(r_1, r_2) \mapsto r_1r_2$, and a distinguished element $1 \in R$, called the unit of $R$, such that $r_1(r_2r_3) = (r_1r_2)r_3$ and $r1 = r = 1r$ for all $r, r_1, r_2, r_3 \in R$. This concept was introduced and studied in [APT14, Chapter 7], where it is further assumed that the product in $R$ be commutative. We will not make this assumption here.

We often let $\mu : R \otimes R \to R$ denote the Cu-morphism induced by the multiplication in a Cu-semiring $R$. 


Proposition 7.1. Let $S$ be a Cu-semigroup. Then $[S,S]$ is a Cu-semiring with product given by the composition product $\circ : [S,S] \otimes [S,S] \to [S,S]$, and with unit element given by the identity map $\text{id}_S \in [S,S]$.

Proof. It follows directly from Proposition 6.19 that the composition product on $[S,S]$ is associative, and that $\text{id}_S$ is a unit element for $[S,S]$. □

Remark 7.2. Let $S$ be a Cu-semigroup. The identity map $\text{id}_S : S \to S$ is a Cu-morphism. Therefore, the unit of the Cu-semiring $[S,S]$ is compact.

In Example 7.9 we will see that $[S,S]$ is noncommutative in general.

Given a Cu-semiring $R$, a left Cu-semimodule over $R$ is a Cu-semigroup $S$ together with a Cu-bimorphism $R \times S \to S$, denoted by $(r,a) \mapsto ra$, such that for all $r_1,r_2 \in R$ and $a \in S$, we have $(r_1r_2)a = r_1(r_2a)$ and $1a = a$. We also say that $S$ has a left action on $R$ if $S$ is a left Cu-semimodule over $R$. Right Cu-semimodules are defined analogously. If $R_1$ and $R_2$ are Cu-semirings, we say that a Cu-semigroup $S$ is a $(R_1,R_2)$-Cu-semibimodule if it has a left $R_1$-action and a right $R_2$-action that satisfy $r_1(ar_2) = (r_1a)r_2$ for all $r_1 \in R_1$, $r_2 \in R_2$ and $a \in S$.

We refer the reader to [APT14, Chapter 7] for a discussion on commutative Cu-semirings and their Cu-semimodules.

Proposition 7.3. Let $S$ be a Cu-semigroup. Then $e_{S,S} : [S,S] \otimes [S,S] \to S$ defines a left action of $[S,S]$ on $S$.

Proof. It follows directly from Proposition 6.21 that the action of $[S,S]$ on $S$ is associative and that $\text{id}_S$ acts as a unit. □

Assume that $R$ is a Cu-semiring with a compact unit. Next, we show that a left action of $R$ on a Cu-semigroup $T$ induces a left action of $R$ on $[S,T]$, for every Cu-semigroup $S$.

7.4. Let $R$ be a Cu-semiring with multiplication given by $\mu : R \otimes R \to R$, let $T$ be a Cu-semigroup with a left $R$-action $\alpha : R \otimes T \to T$, and let $S$ be a Cu-semigroup. Consider the general left unit map $R \otimes [S,T] \to [S,R \otimes T]$ from Definition 6.27. Postcomposing with $\alpha_* : [S,R \otimes T] \to [S,T]$ we obtain a Cu-morphism that we denote by $\alpha_* : [S,R \otimes T] \to [S,T]$.

Let $r \in R$ and $x \in [S,T]$. We denote $\alpha_*(r \otimes x)$ by $rx$. Applying Proposition 6.28 at the third step, we have

$$rx = \alpha_*(r \otimes x) = \alpha_* \circ (r,x) = \alpha_* \circ d_{R,T}(r) \circ x.$$

Lemma 7.5. We retain the notation from Paragraph 7.4. Let $r \in R$, and let $f = (f_\lambda)_\lambda$ be a path in $\text{Cu}[S,T]$. Choose a path $(r_\lambda)_\lambda$ in $R$ with endpoint $r$. For each $\lambda$, let $r_\lambda f_\lambda : S \to T$ be given by $s \mapsto r_\lambda f_\lambda(s)$. Then $(r_\lambda f_\lambda)_\lambda$ is a path in $\text{Cu}[S,T]$ and

$$r[f] = [(r_\lambda f_\lambda)_\lambda].$$

Proof. Using the equation at the end of Paragraph 7.3 at the first step, using Proposition 6.2 at the second step, and using Proposition 6.18 at the third step, we deduce that

$$r[f] = \alpha_* \circ d_{R,T}(r) \circ [f] = \alpha_* \circ [(r_\lambda \otimes \_)_\lambda] \circ [(f_\lambda)_\lambda] = \alpha_* \circ [(r_\lambda \otimes f_\lambda(\_))_\lambda] = [(r_\lambda f_\lambda)_\lambda],$$

as desired. □

Proposition 7.6. Let $R$ be a Cu-semiring with compact unit, let $S$ be a Cu-semigroup, and let $T$ be a Cu-semigroup with a left $R$-action $\alpha : R \otimes T \to T$. Then the map $\alpha_* : R \otimes [S,T] \to [S,T]$ from Paragraph 7.4 defines a left $R$-action on $[S,T]$.
\textbf{Proof.} Let \( r, r' \in R \) and \( x \in [S, T] \). Choose a path \( f = (f_\lambda)_\lambda \) in \( \text{Cu}[S, T] \) representing \( x \). Choose paths \( (r_\lambda)_\lambda \) and \( (r'_\lambda)_\lambda \) in \( R \) with endpoints \( r \) and \( r' \), respectively.

Then \( (r_\lambda r'_\lambda)_\lambda \) is a path in \( R \) with endpoint \( rr' \). Using [Lemma 7.5] at the first, third and last step, we deduce that

\[
(rr')x = [(r_\lambda r'_\lambda)f_\lambda]_\lambda = [(r_\lambda f_\lambda)]_\lambda = r'(r_\lambda f_\lambda)_\lambda = r(r'_\lambda x),
\]
as desired.

Let 1 denote the unit element of \( R \). For every \( f \in \text{Cu}[S, T] \), we have \( 1f = f \).

Since 1 is compact, the constant function with value 1 is a path in \( R \) with endpoint 1. Using [Lemma 7.5] at the first step, we deduce that

\[
1x = [(1f_\lambda)]_\lambda = [(f_\lambda)]_\lambda = x,
\]
as desired. \qed

\textbf{Proposition 7.7.} Let \( S \) and \( T \) be \( \text{Cu} \)-semigroups. Then the composition product \( \circ: [T, T] \otimes [S, T] \to [S, T] \) defines a left action of the \( \text{Cu} \)-semiring \( [T, T] \) on \( [S, T] \). Analogously, we obtain a right action of \( [S, S] \) on \( [S, T] \). These actions are compatible and thus \( [S, T] \) is a \(([T, T], [S, S])\)-\( \text{Cu} \)-semimodule.

\textbf{Proof.} This follows directly from the associativity of the composition product; see Proposition 6.19. \qed

\textbf{Remark 7.8.} Let \( S \) and \( T \) be \( \text{Cu} \)-semigroups. By [Proposition 7.3], the evaluation map \( e_{T,S}: [T, T] \otimes T \to T \) from Definition 6.5 defines a left action of \([T, T]\) on \( T \). By Proposition 7.6 this induces a left action of \([T, T]\) on \([S, T]\). This action agrees with that from Proposition 7.7.

\textbf{Example 7.9.} Given \( k \in \mathbb{N} \), we let \( \overline{\mathbb{N}}^k \) denote the Cuntz semigroup of the \( C^* \)-algebra \( \mathcal{C}^k \). We think of an element \( v \in \overline{\mathbb{N}}^k \) as a tuple \((v_1, \ldots, v_k)^T\) with \( k \) entries in \( \mathbb{N} \). We let \( e^{(1)}, \ldots, e^{(k)} \) denote the ‘standard basis vectors’ of \( \mathbb{N}^k \), such that \( v = \sum_{i=1}^k v_1 e^{(i)} \).

Let \( k, l \in \mathbb{N} \). Let us show that \([\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^l] \) can be identified with \( M_{l,k}(\mathbb{N}) \), the \( l \times k \)-matrices with entries in \( \mathbb{N} \), with order and addition defined entrywise. Thus, as a \( \text{Cu} \)-semigroup, \([\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^l] \) is isomorphic to \( \overline{\mathbb{N}}^{kl} \). However, the presentation as matrices allows to expatiate the composition product.

First, let \( \varphi: \overline{\mathbb{N}}^k \to \overline{\mathbb{N}}^l \) be a generalized \( \text{Cu} \)-morphism. For each \( j \in \{1, \ldots, k\} \), we consider the vector \( \varphi(e^{(j)}) \) in \( \mathbb{N}^l \) and we let \( x_{i,j}, \ldots, x_{l,j} \) denote its coefficients.

This defines a matrix \( x = (x_{i,j})_{i,j} \) with \( l \times k \) entries in \( \mathbb{N} \). It is then readily verified that the coefficients of \( \varphi(v) \) are obtained by multiplication of the matrix \( x \) with the vector of coefficients of \( v \). We identify \( \varphi \) with the associated matrix \( x \) in \( M_{l,k}(\mathbb{N}) \).

Let \( \varphi, \psi: \overline{\mathbb{N}}^k \to \overline{\mathbb{N}}^l \) be generalized \( \text{Cu} \)-morphisms with associated matrices \( x \) and \( y \) in \( M_{l,k}(\mathbb{N}) \). It is straightforward to check that \( \varphi \circ \psi \) if and only if \( x_{i,j} \) is finite and \( x_{i,j} \leq y_{i,j} \) for each \( i, j \).

It follows that \([\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^l] \) can be identified with \( M_{l,k}(\mathbb{N}) \), with addition and order defined entrywise.

Given \( k, l, m \in \mathbb{N} \), consider the composition product

\[
[\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^l] \otimes [\overline{\mathbb{N}}^l, \overline{\mathbb{N}}^m] \to [\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^m].
\]

After identifying \([\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^l] \) with \( M_{l,k}(\mathbb{N}) \), identifying \([\overline{\mathbb{N}}^l, \overline{\mathbb{N}}^m] \) with \( M_{m,l}(\mathbb{N}) \), and identifying \([\overline{\mathbb{N}}^k, \overline{\mathbb{N}}^m] \) with \( M_{m,k}(\mathbb{N}) \), the composition product is given as a map

\[
M_{m,l}(\mathbb{N}) \otimes M_{l,k}(\mathbb{N}) \to M_{m,k}(\mathbb{N}).
\]
It is straightforward to check that this map is induced by matrix multiplication. In particular, the Cu-semiring $[\mathbb{N}^k, \mathbb{N}]$ can be identified with $M_{k,k}(\mathbb{N})$. Thus, for $k \geq 2$, the Cu-semiring $[\mathbb{N}^k, \mathbb{N}]$ is not commutative.

The left action of the Cu-semiring $[\mathbb{N}, \mathbb{N}] = M_{1,1}(\mathbb{N})$ on the Cu-semigroup $[\mathbb{N}^k, \mathbb{N}] = M_{k,k}(\mathbb{N})$ is given by matrix multiplication, and similarly for the right action of $[\mathbb{N}, \mathbb{N}]$.

7.2. The embedding of a Cu-semiring $R$ in $[R, R]$. Given a Cu-semiring $R$, recall that $\mu : R \otimes R \to R$ denotes the Cu-morphism induced by multiplication in $R$.

**Definition 7.10.** Given a Cu-semiring $R$, we let $\pi_R : R \to [R, R]$ be the Cu-morphism that corresponds to $\mu$ under the identification

$$\text{Cu}(R, [R, R]) \cong \text{Cu}(R \otimes R, R).$$

**Remark 7.11.** The Cu-morphism $\pi_R$ plays the role of the left regular representation. In a similar way, one might define an analogue of the right regular representation as the Cu-morphism $\bar{\pi}_R : R \to [R, R]$ corresponding to $\mu \sigma$, where $\sigma$ is the flip automorphism.

**Lemma 7.12.** We have $\pi_R = \mu_\ast \circ d_{R,R}$ and $e_{R,R} \circ (\pi_R \otimes \text{id}_R) = \mu$.

**Proof.** The first equality follows from Proposition 6.23. By Proposition 6.26 we have $e_{R,R} \circ (\mu_\ast \otimes \text{id}_R) = \mu e_{R,R,R}$. Further, we have $e_{R,R} \circ (\mu_\ast \otimes \text{id}_R) = \text{id}_{R \otimes R}$ by Corollary 6.24. Using these equations at the second and third step, we deduce that

$$e_{R,R} \circ (\pi_R \otimes \text{id}_R) = e_{R,R} \circ (\mu_\ast \circ d_{R,R} \otimes \text{id}_R) = e_{R,R} \circ (\mu_\ast \circ (\mu_\ast \circ d_{R,R} \otimes \text{id}_R)) = \mu \circ e_{R,R} \circ (d_{R,R} \otimes \text{id}_R) = \mu,$$

as desired.

**Theorem 7.13.** Let $R$ be a Cu-semiring. Then the map $\pi_R : R \to [R, R]$ from Definition 7.10 is multiplicative. If the unit element of $R$ is compact, then $\pi_R$ is unital.

**Proof.** Let $M : [R, R] \otimes [R, R] \to [R, R]$ denote the composition map. We need to show that $M \circ (\pi_R \otimes \pi_R) = \pi_R \circ \mu$.

Given $r$ and $s \in R$, choose paths $r = (r_\lambda)_{\lambda}$ and $s = (s_\lambda)_{\lambda}$ in $(R, \ll)$ with endpoints $r$ and $s$, respectively. For each $\lambda$, let $f_\lambda : R \to R$ and $g_\lambda : R \to R$ be the generalized Cu-morphism given by left multiplication with $r_\lambda$ and $s_\lambda$, respectively. By Proposition 6.2, we have $d_{R,R}(r) = [(r_\lambda \otimes s_\lambda)]$, where $r_\lambda \otimes s_\lambda : R \to R \otimes R$ is the map sending $t \in R$ to $r_\lambda \otimes t$. We also have $\mu \circ (r_\lambda \otimes s_\lambda) = f_\lambda$. Since $\pi_R = \mu_\ast \circ d_{R,R}$ by Lemma 7.12 it follows that $\pi_R(r) = [(f_\lambda)]$. Likewise, we deduce $\pi_R(s) = [(g_\lambda)]$.

By Proposition 6.18 we obtain $M(\pi_R(r) \otimes \pi_R(s)) = [(f_\lambda \circ g_\lambda)]$. As the product in $R$ is associative, the composition $f_\lambda \circ g_\lambda$ is the generalized Cu-morphism $h_\lambda$ defined by left multiplication with $r_\lambda s_\lambda$. Notice that $(r_\lambda s_\lambda)$ is a path in $(R, \ll)$ with endpoint $rs$. Therefore, $\pi_R(rs) = [(h_\lambda)]$. Altogether, this implies

$$M(\pi_R(r) \otimes \pi_R(s)) = [(f_\lambda \circ g_\lambda)] = [(h_\lambda)] = \pi_R(\mu(r \otimes s)),$$

as desired.

To show the second statement, let us assume that the unit $1_R$ of $R$ is compact. Then the constant function with value $1_R$ is a path in $(R, \ll)$ with endpoint $1_R$. 

Then it follows easily as in the first part of the proof that \( \pi_R(1_R) = [(\text{id}_R)_\lambda] = \text{id}_R. \)

**Definition 7.14.** Given a Cu-semiring \( R \), we let \( \varepsilon_R : [R, R] \to R \) be the generalized Cu-morphism given by

\[
\varepsilon_R(f) = \sup_{\lambda} f_\lambda(1),
\]

for a path \( f = (f_\lambda)_{\lambda} \) in \( \text{Cu}[R, R] \).

**Remark 7.15.** Let \( \sigma_{R,R} : [R, R] \to \text{Cu}[R, R] \) denote the endpoint map as introduced in [Definition 5.6] and let \( 1 \) denote the unit of \( R \). Then \( \varepsilon_R(x) = \sigma_{R,R}(x)(1) \) for every \( x \in [R, R] \).

**Lemma 7.16.** We have \( \varepsilon_R \circ \pi_R = \text{id}_R \).

**Proof.** Let \( r \) be an element in \( R \). Proceeding as in [Theorem 7.13] we know that, if \( (r_\lambda)_\lambda \) is a path in \( (R, \lll) \) with endpoint \( r \) and \( f_\lambda : R \to R \) is given by left multiplication by \( r_\lambda \), then \( \pi_R(r) = [(f_\lambda)_\lambda] \). Given \( r \in R \), we deduce that

\[
\varepsilon_R(\pi_R(r)) = \varepsilon_R((f_\lambda)_\lambda) = \sup_{\lambda} f_\lambda(1) = \sup_{\lambda}(r_\lambda 1) = r,
\]

as desired. \( \square \)

**Proposition 7.17.** Let \( R \) be a Cu-semiring. Then \( \pi_R : R \to [R, R] \) is a multiplicative order-embedding. Thus, in a natural way, \( R \) is a sub-semiring of \( [R, R] \). If the unit of \( R \) is compact, then \( R \) is even a unital sub-semiring of \( [R, R] \). (One could call this a sub-Cu-semiring.)

**Proof.** By [Lemma 7.16] we have \( \varepsilon_R \circ \pi_R = \text{id}_R \), which implies that \( \pi_R \) is an order-embedding. By [Theorem 7.13] \( \pi_R \) is a (unital) multiplicative Cu-morphism. \( \square \)

Recall that \( \mathbb{P} = [0, \infty] \) is isomorphic to the Cuntz semigroup of the Jacelon-Razak algebra. The usual multiplication of real numbers extends to \( \mathbb{P} \). This gives \( \mathbb{P} \) the structure of a commutative Cu-semiring.

**Example 7.18.** Recall that \( M_1 \) denotes the Cuntz semigroup of a II_1-factor; see [Example 4.14] and [Proposition 4.16]. Note that \( M_1 \) is the disjoint union of compact elements \( [0, \infty] \) and nonzero soft elements \( (0, \infty] \). We identify \( \mathbb{P} = [0, \infty] \) with the sub-Cu-semigroup of soft elements in \( M_1 \). We define a Cu-morphism \( \varphi : M_1 \to \mathbb{P} \subseteq M_1 \) by fixing all soft elements and by sending a compact to the soft element of the same value.

We define a product on \( M_1 \) as follows: We equip the compact part \( [0, \infty] \) with the usual multiplication of real numbers, and similarly for the product in \( (0, \infty] \). The product of any element with 0 is 0. Given a nonzero compact element \( a \) and a nonzero soft element \( b \), their product is defined as the soft element \( ab : = \varphi(a)b \).

This gives \( M_1 \) the structure of a commutative Cu-semiring. Moreover, we may identify \( \mathbb{P} \) with the (nonunital) sub-Cu-semiring of soft elements in \( M_1 \). The map \( \varphi : M_1 \to \mathbb{P} \) is multiplicative. One can show that the map \( \pi_{M_1} : M_1 \to [M_1, M_1] \) is an isomorphism.

**Example 7.19.** We have \( [\mathbb{P}, \mathbb{P}] \cong M_1 \). The map \( \pi_{\mathbb{P}} : \mathbb{P} \to [\mathbb{P}, \mathbb{P}] \) embeds \( \mathbb{P} \) as the sub-Cu-semiring of soft elements in \( M_1 \). In particular, \( \pi_{\mathbb{P}} \) is not unital.

**Proof.** We have \( [\mathbb{P}, \mathbb{P}] \cong M_1 \) by [Proposition 5.14]. By [Proposition 7.17] \( \pi_{\mathbb{P}} \) is a multiplicative order-embedding. Note that every element of \( \mathbb{P} \) is soft. By [Lemma 2.11] a generalized Cu-morphism maps soft elements to soft elements. Thus, the image of \( \pi_{\mathbb{P}} \) is contained in the soft elements of \( M_1 \). It easily follows that \( \pi_{\mathbb{P}} \) identifies \( \mathbb{P} \) with the soft elements in \( M_1 \). Since the unit of \( M_1 \) is compact, it also follows that \( \pi_{\mathbb{P}} \) is not unital. \( \square \)
7.3. **Solid Cu-semirings.** Throughout this subsection, $R$ denotes a (not necessarily commutative) Cu-semiring, with multiplication given by the Cu-morphism $\mu: R \otimes R \to R$.

Recall from [APT14, Definition 7.1.5] that $R$ is said to be **solid** if $\mu: R \otimes R \to R$ is an isomorphism. In [APT14], all Cu-semirings were required to be commutative, and thus a solid Cu-semiring was assumed to be commutative. Next, we show that this assumption is not necessary since a Cu-semiring is automatically commutative as soon as $\mu$ is injective.

**Lemma 7.20.** Let $R$ be a (not necessarily commutative) Cu-semiring such that $\mu: R \otimes R \to R$ is injective. Then $R$ is commutative and $\mu$ is an isomorphism (and consequently $R$ is solid.)

**Proof.** To show that $R$ is commutative, let $a, b \in R$. We have

$$\mu(1 \otimes a) = a = \mu(a \otimes 1),$$

and therefore $1 \otimes a = a \otimes 1$ in $R \otimes R$. Consider the shuffle Cu-morphism $\alpha: R \otimes R \otimes R \to R \otimes R \otimes R$ that satisfies

$$\alpha(x \otimes y \otimes z) = y \otimes x \otimes z$$

for every $x, y, z \in R$. It follows that

$$1 \otimes b \otimes a = \alpha(b \otimes 1 \otimes a) = \alpha(b \otimes a \otimes 1) = a \otimes b \otimes 1$$

in $R \otimes R \otimes R$. By the associativity of the product in $R$, this implies that $ba = ab$, as desired.

Thus, if $\mu$ is injective, then $R$ is commutative and $1 \otimes a = a \otimes 1$ in $R \otimes R$, for every $a \in R$. Using [APT14, Proposition 7.1.6], this implies that $R$ is solid. □

**7.21.** Let $R$ be a solid Cu-semiring, and let $S$ be a Cu-semigroup. It was shown in [APT14, Corollary 7.1.8] that any two $R$-actions on $S$ agree. (Since $R$ is commutative, we need not distinguish between left and right $R$-actions.) Thus, $S$ either has a (unique) $R$-action, or it does not admit any $R$-action. Thus, having an $R$-action is a property rather than an additional structure for $S$, which justifies the following definition.

**Definition 7.22.** Let $R$ be a solid Cu-semiring, and let $S$ be a Cu-semigroup. We say that $S$ is **$R$-stable** if $S$ has an $R$-action.

**Remark 7.23.** In [APT14], we said that $S$ ‘has $R$-multiplication’ if it has an $R$-action. Given a solid ring $R$, it was shown [APT14, Theorem 7.1.12] that $S$ is $R$-stable if and only if $S \cong R \otimes S$.

Recall that a $C^*$-algebra $A$ is said to be **$Z$-stable** if $A \cong Z \otimes A$, and similarly one defines being UHF-stable and $O\infty$-stable. Thus, the terminology of being ‘$R$-stable’ for Cu-semigroups is analogous to the terminology used for $C^*$-algebras.

The following fact about solid Cu-semirings will be used in the sequel.

**Proposition 7.24** ([APT14, Proposition 7.1.6]). Let $R$ be a solid Cu-semiring, and let $S$ and $T$ be $R$-stable Cu-semigroups. Then every generalized Cu-morphism $\varphi: S \to T$ is automatically $R$-linear, that is, we have $\varphi(ra) = r\varphi(a)$ for all $r \in R$ and $a \in S$.

**Theorem 7.25.** Given a Cu-semiring $R$, consider the following statements:

1. $R$ is solid, that is, $\mu: R \otimes R \to R$ is an isomorphism.
2. The map $\varepsilon_{R,R}: [R, R] \otimes R \to R$ is an isomorphism.
3. The map $\pi_R \otimes \text{id}_R: R \otimes R \to [R, R] \otimes R$ is an isomorphism.
4. The map $\pi_R: R \to [R, R]$ is an isomorphism.
5. The map $\varepsilon_R: [R, R] \to R$ is an isomorphism.
Then the following implications hold:

\[(1) \iff (2) \Rightarrow (3) \iff (4) \iff (5).\]

Further, if \( R \) satisfies (1) and (3), then it satisfies (2). The \( \text{Cu}-\)semiring \( \mathcal{P} \) satisfies (1), (2) and (3), but not (4); see Example 7.19. The \( \text{Cu}-\)semiring \( M_1 \) from Example 4.14 satisfies (3) and (4) but neither (1) nor (2); see Example 7.18.

**Proof.** By Lemma 7.16 we have \( \varepsilon_R \circ \pi_R = \text{id}_R \). It follows that \( \varepsilon_R \) is an isomorphism if and only if \( \pi_R \) is, which shows the equivalence of (4) and (5). We also deduce that

\[(\varepsilon_R \circ \text{id}_R) \circ (\pi_R \circ \text{id}_R) = \text{id}_R \circ \text{id}_R.\]

Therefore, the map \( \pi_R \circ \text{id}_R \) is always an order-embedding. It is obvious that (4) implies (3).

To show that (2) implies (1), assume that \( e_{R,R} \) is an isomorphism. Then the composition \( e_{R,R} \circ (\pi_R \circ \text{id}_R) \) is an order-embedding. By Lemma 7.12 we have \( e_{R,R} \circ (\pi_R \circ \text{id}_R) = \mu \), which shows that \( \mu \) is an order-embedding. By Lemma 7.20 this implies that \( R \) is solid.

Using again that \( e_{R,R} \circ (\pi_R \circ \text{id}_R) = \mu \), if any two of the three maps \( e_{R,R}, \pi_R \circ \text{id}_R \) and \( \mu \) are isomorphisms, then so is the third. This shows that (2) implies (3), and that the combination of (1) and (3) implies (2). \( \square \)

**Question 7.26.** Given a solid \( \text{Cu}-\)semiring \( R \), is the evaluation map \( e_{R,R} : [R, R] \otimes R \rightarrow R \) an isomorphism?

**Remark 7.27.** Let \( R \) be a solid \( \text{Cu}-\)semiring. The answer to Question 7.26 is ‘yes’ in the following cases:

1. If the unit of \( R \) is compact; see Remark 7.32 below.
2. If \( R \) satisfies (O5) and (O6). This follows from the classification of solid \( \text{Cu}-\)semirings with (O5) obtained in [APT14, Theorem 8.3.13] which shows that each such \( \text{Cu}-\)semiring is either isomorphic to \( \mathcal{P} \) or has a compact unit. In either case, Question 7.26 has a positive answer.

In particular, a \( \text{Cu}-\)semiring \( R \) with compact unit is solid if and only if the evaluation map \( e_{R,R} : [R, R] \otimes R \rightarrow R \) is an isomorphism.

**Theorem 7.28.** Let \( R \) be a solid \( \text{Cu}-\)semiring with compact unit, and let \( S \) and \( T \) be \( \text{Cu}-\)semigroups. Assume that \( T \) is \( R \)-stable. Then \( [S, T] \) is \( R \)-stable, and hence \( [S, T] \cong R \circ [S, T] \).

**Proof.** Since the unit of \( R \) is compact, it follows from Proposition 7.6 that \( [S, T] \) has a left \( R \)-action. Since \( R \) is solid, this implies that \( [S, T] \) is \( R \)-stable. \( \square \)

**Lemma 7.29.** Let \( R \) be a solid \( \text{Cu}-\)semiring, let \( S \) and \( T \) be \( \text{Cu}-\)semigroups, and let \( f, g : R \otimes S \rightarrow T \) be a generalized \( \text{Cu}-\)morphisms. Assume that \( T \) is \( R \)-stable.

Then \( f \leq g \) if and only if \( f(1 \otimes a) \leq g(1 \otimes a) \) for all \( a \in S \).

If the unit of \( R \) is compact, then \( f \prec g \) if and only if \( f(1 \otimes a') \ll g(1 \otimes a) \) for all \( a', a \in S \) with \( a' \ll a \).

**Proof.** The forward implications are obvious. To show the converse of the first statement, assume that \( f(1 \otimes a) \leq g(1 \otimes a) \) for all \( a \in S \). To verify \( f \leq g \), it is enough to show that \( f(r \otimes a) \leq g(r \otimes a) \) for all \( r \in R \) and \( a \in S \). Note that \( R \otimes S \) and \( T \) are \( R \)-stable. Since \( R \) is solid, every generalized \( \text{Cu}-\)morphism between \( R \)-stable \( \text{Cu}-\)semigroups is automatically \( R \)-linear; see Proposition 7.24. Thus, given \( r \in R \) and \( a \in S \), we obtain

\[ f(r \otimes a) = f(r(1 \otimes a)) = rf(1 \otimes a) \leq rg(1 \otimes a) = g(r \otimes a), \]

as desired.
To show the converse of the second statement, assume that \( f(1 \otimes a') \ll g(1 \otimes a) \) for all \( a', a \in S \) with \( a' \ll a \). To verify \( f \prec g \), it is enough to show that \( f(r' \otimes a') \ll g(r \otimes a) \) for all \( r', r \in R \) and \( a', a \in S \) with \( r' \ll r \) and \( a' \ll a \). Given such \( r', r, a', a \) and \( \tau \), we use at the second step that multiplication in \( R \) preserves the joint way-below relation, to deduce

\[
f(r' \otimes a') = r'f(1 \otimes a') \ll rg(1 \otimes a) = g(r \otimes a),
\]
as desired. \( \square \)

**Proposition 7.30.** Let \( R \) be a solid Cu-semiring with compact unit, and let \( S \) and \( T \) be Cu-semigroups. Assume that \( T \) is \( R \)-stable. Let \( \alpha : S \to R \otimes S \) be the Cu-morphism given by \( \alpha(a) = 1 \otimes a \), for \( a \in S \). Then the induced map \( \alpha^* : [R \otimes S, T] \to [S, T] \) is an isomorphism.

**Proof.** Consider the map \( \alpha_Q^* : Cu[R \otimes S, T] \to Cu[S, T] \) given by sending a generalized Cu-morphism \( f : R \otimes S \to T \) to the generalized Cu-morphism \( \alpha_Q^*(f) \) given by

\[
\alpha_Q^*(f)(a) = f(1 \otimes a),
\]
for \( a \in S \). It follows from **Lemma 7.29** that \( \alpha_Q^* \) is an isomorphism of \( \mathbb{Q} \)-semigroups.

Since \( \alpha^* \) is obtained by applying the functor \( \tau \) to \( \alpha_Q^* \) (see **Paragraph 5.5**), it follows that \( \alpha^* \) is an isomorphism, as desired. \( \square \)

**Corollary 7.31.** Let \( R \) be a solid Cu-semiring with compact unit, and let \( T \) be an \( R \)-stable Cu-semigroup. Then there is a natural isomorphism \([R, T] \cong T\).

**Proof.** Applying **Proposition 7.30** for \( S = \mathbb{N} \), we obtain \([R, T] \cong [\mathbb{N}, T]\). By **Proposition 6.10**, we have a natural isomorphism \([\mathbb{N}, T] \cong T\). \( \square \)

**Remark 7.32.** Let \( R \) be a solid Cu-semiring with compact unit. Since \( R \) is \( R \)-stable itself, it follows from **Corollary 7.31** that \([R, R] \cong R\). It follows that the evaluation map \( \varepsilon_{R,R} : [R, R] \otimes R \to R \) is an isomorphism.

For the solid Cu-semiring \( \mathbb{F} \), we have seen in **Proposition 5.14** that \([\mathbb{F}, \mathbb{F}] \cong M_1 \neq \mathbb{F}\). This shows that **Proposition 7.30** and **Corollary 7.31** cannot be generalized to solid Cu-semirings without compact unit.

8. **Applications to \( C^\ast \)-algebras**

Given \( C^\ast \)-algebras \( A \) and \( B \), recall that a map \( \varphi : A \to B \) is called **completely positive contractive** (abbreviated c.p.c.) if it is linear, contractive and for each \( n \in \mathbb{N} \) the amplification to \( n \times n \)-matrices \( \varphi \otimes \text{id} : A \otimes M_n \to B \otimes M_n \) is positive. Every c.p.c. map \( \varphi : A \to B \) induces a contractive, positive map \( \varphi \otimes \text{id} : A \otimes K \to B \otimes K \).

Two elements \( a \) and \( b \) in a \( C^\ast \)-algebra are called **orthogonal**, denoted \( a \perp b \), if \( ab = a^*b = ba = a^*b^* = 0 \). If \( a \) and \( b \) are self-adjoint, then \( a \perp b \) if and only if \( ab = 0 \). A c.p.c. map \( \varphi \) is said to have **order-zero** if for all \( a, b \in A \) we have that \( a \perp b \) implies \( \varphi(a) \perp \varphi(b) \). We denote the set of c.p.c. order-zero maps by \( \text{cpc}_0(A, B) \).

The concept of c.p.c. order-zero maps was studied by Winter and Zacharias, **[WZ09]**, who also gave a useful structure theorem for such maps. We present their result in a slightly different way.

**Theorem 8.1** (Winter and Zacharias, **[WZ09** Theorem 3.3]). Let \( A \) and \( B \) be \( C^\ast \)-algebras, and let \( \varphi : A \to B \) be a c.p.c. order-zero map. Set \( C := C^\ast(\varphi(A)) \), the sub-\( C^\ast \)-algebra of \( B \) generated by the image of \( \varphi \). Then there exist a unital *-homomorphism \( \pi_{\varphi} : A \to M(C) \), from the minimal unitization of \( A \) to the multiplier algebra of \( C \), such that

\[
\varphi(ab) = \varphi(a)\pi_{\varphi}(b) = \pi_{\varphi}(a)\varphi(b),
\]
for all \(a, b \in A\).

In particular, the element \(h := \varphi(1_A)\) is contractive, positive, it commutes with the image of \(\pi_x\), and we have \(\varphi(a) = h\pi_x(a) = \pi_x(a)h\) for all \(a \in A\).

This structure theorem has many interesting applications. For instance, it implies that c.p.c. order-zero maps induce generalized Cu-morphisms. Let us recall some details. Let \(\varphi : A \rightarrow B\) be a c.p.c. order-zero map. Then the amplification \(\varphi \otimes \text{id} : A \otimes K \rightarrow B \otimes K\) is a c.p.c. order-zero map as well; see [WZ09 Corollary 4.3]. Define \(\text{Cu}[\varphi] : \text{Cu}(A) \rightarrow \text{Cu}(B)\) by

\[
\text{Cu}[\varphi](\{a\}) := \{[\varphi \otimes \text{id}](a)\},
\]

for \(a \in (A \otimes K)_+\). Then \(\text{Cu}[\varphi]\) is a generalized Cu-morphism; see [WZ09 Corollary 4.5] and [APT14, 2.2.7, 3.2.5]. We thus obtain a natural map

\[
\text{cpc}_f((A, B) \rightarrow \text{Cu}[\text{Cu}(A), \text{Cu}(B))].
\]

Below, we will show that this map factors through \([\text{Cu}(A), \text{Cu}(B)]\).

8.2. The theorem of Winter and Zacharias also allows us to define functional calculus for order-zero maps: Let \(\varphi : A \rightarrow B\) be a c.p.c. order-zero map. Choose \(C, \pi_x\), and \(h\) as in [Theorem 8.1]. Given a continuous function \(f : [0, 1] \rightarrow [0, 1]\) with \(f(0) = 0\), we define \(f(\varphi) : A \rightarrow B\) by \(f(\varphi)(a) := f(h)\pi_x(a)\) for \(a \in A\); see [WZ09 Corollary 4.2].

In particular, this allows us to define ‘cut-downs’ of c.p.c. order-zero maps: Given \(\varepsilon > 0\), we may apply the function \((\cdot - \varepsilon)_+\) to \(\varphi\). To simplify notation, we set \(\varphi_\varepsilon := (\varphi - \varepsilon)_+\). Thus, for \(a \in A\) we have

\[
\varphi_\varepsilon(a) = (h - \varepsilon)_+\pi_x(a).
\]

Theorem 8.3. Let \(A\) and \(B\) be \(C^*\)-algebras, and let \(\varphi : A \rightarrow B\) be a c.p.c. order-zero map. For each \(\varepsilon > 0\), let \(f_\varepsilon : \text{Cu}(A) \rightarrow \text{Cu}(B)\) be the generalized Cu-morphism induced by the c.p.c. order-zero map \(\varphi_\varepsilon : A \rightarrow B\). Then \(f = (f_{1-\varepsilon})_1\) is a path in \([\text{Cu}(A), \text{Cu}(B)]\). Moreover, the endpoint of \(f\) is \(\text{Cu}[\varphi]\), the generalized Cu-morphism induced by \(\varphi\).

Proof. We have already observed that every \(f_\varepsilon\) is a generalized Cu-morphism. To verify that \((f_{1-\varepsilon})_1\) is a path, we need to show that \(f_\varepsilon < f_{\varepsilon'}\) for \(\varepsilon' > \varepsilon > 0\). Since \(f_{\varepsilon+\delta} = (f_\varepsilon)_\delta\), it is enough to show the following:

Claim: We have \(f_\varepsilon \prec f\). To show the claim, let \(a, b \in (A \otimes K)_+\) such that \([a] \ll [b] \in \text{Cu}(A)\). Recall that two positive elements \(x\) and \(y\) in a \(C^*\)-algebra satisfy \([x] \ll [y]\) if and only if there exists \(\delta > 0\) with \([x] \leq [y - \delta]_+\). Thus, we can choose \(\delta > 0\) such that \([a] \leq [(b - \delta)]_+\). Note that if \(x\) and \(y\) are commuting positive elements in a \(C^*\)-algebra, then \((x - \varepsilon)_+(y - \delta)_+ \leq (xy - \varepsilon\delta)_+\). Using this at the last step, we deduce that

\[
\varphi_\varepsilon(a) \ll \varphi_\varepsilon((b - \delta)_+) = (h - \varepsilon)_+\pi_x((b - \delta)_+) = (h - \varepsilon)_+(\pi_x(b - \delta)_+) \leq (h\pi_x(b) - \varepsilon\delta)_+ = (\varphi(b) - \varepsilon\delta)_+,
\]

which implies that

\[
f_\varepsilon([a]) = [\varphi_\varepsilon(a)] \ll [\varphi(b)] = f([b]),
\]

as desired. This proves the claim and shows that \(f\) is a path.

Let \(f\) be the generalized Cu-morphism induced by \(\varphi\). To show that the endpoint of \(f\) is \(f\), let \(a \in (A \otimes K)_+\). We have

\[
\lim_{\lambda \rightarrow 1} \varphi_{1-\lambda}(a) = \lim_{\varepsilon \rightarrow 0} \varphi_\varepsilon(a) = \lim_{\varepsilon \rightarrow 0} (h - \varepsilon)_+\pi_x(a) = h\pi_x(a) = \varphi(a).
\]

This implies that \(\sup_{\lambda < 1} f_\lambda([a]) = f([a])\) in \(\text{Cu}(A)\), as desired. \[\square\]
Definition 8.4. Let $A$ and $B$ be $C^*$-algebras, and let $\varphi: A \to B$ be a c.p.c. order-zero map. We let $\text{Cu}(\varphi)$ be the element in $[[\text{Cu}(A), \text{Cu}(B)]]$ that is the class of the path $(\text{Cu}(\varphi_{1-\lambda}))_\lambda$ as constructed in [Theorem 8.3].

Remark 8.5. Let $\varphi: A \to B$ be a $^*$-homomorphism. In the definition of the functor $\text{Cu}: C^* \to \text{Cu}$ we denoted $\text{Cu}(\varphi)$ as the $\text{Cu}$-morphism $\text{Cu}(A) \to \text{Cu}(B)$ given by $\text{Cu}(\varphi)([a]) = ([\varphi \otimes \text{id}](a))$ for $a \in (A \otimes K)_+$. On the other hand, in [Definition 8.4] we defined $\text{Cu}(\varphi)$ as the class of the path $(\text{Cu}(\varphi_{1-\lambda}))_\lambda$ as constructed in [Theorem 8.3]. Given $\varepsilon > 0$, it is easy to verify that $\varphi_\varepsilon = (1 - \varepsilon)_+ \varphi$. It follows that $\text{Cu}([\varphi_\varepsilon]) = \text{Cu}([\varphi])$ for $\varepsilon \in [0, 1)$. Thus, the path $(\text{Cu}(\varphi_{1-\lambda}))_\lambda$ is constant with value $\text{Cu}([\varphi])$.

We identify a $\text{Cu}$-morphism $f: \text{Cu}(A) \to \text{Cu}(B)$ with the compact element in $[[\text{Cu}(A), \text{Cu}(B)]]$ given by the constant path with value $f$; see [Proposition 5.12]. It follows that the notation $\text{Cu}(\varphi)$ for a $^*$-homomorphism $\varphi$ is unambiguous.

8.6. The functor $C^* \to \text{Cu}$ defines a map

$$\text{Cu}: \text{Hom}(A, B) \to \text{Cu}(\text{Cu}(A), \text{Cu}(B)).$$

By [Definition 8.4] we obtain a well-defined map

$$\text{cpc}_\lambda(A, B) \to [[\text{Cu}(A), \text{Cu}(B)]].$$

As noticed in [Remark 8.5], these assignments are compatible, which means that the following diagram commutes:

$$\begin{array}{ccc}
\text{cpc}_\lambda(A, B) & \xrightarrow{\text{Cu}} & [[\text{Cu}(A), \text{Cu}(B)]] \\
\downarrow & & \downarrow \\
\text{Hom}(A, B) & \xrightarrow{\text{Cu}} & \text{Cu}(\text{Cu}(A), \text{Cu}(B))
\end{array}$$

Problem 8.7. Study the properties of the map $\text{cpc}_\lambda(A, B) \to [[\text{Cu}(A), \text{Cu}(B)]].$ In particular, when is this map surjective?

Example 8.8. Recall that $W$ denotes the Jacelon-Razak algebra. We know that $\text{Cu}(W) \cong \mathbb{P}$. By [Proposition 5.14], we have $[[\mathbb{P}, \mathbb{P}]] \cong M_1$, and recall that $M_1 = [0, \infty) \sqcup (0, \infty]$. We claim that the map

$$\text{cpc}_\lambda(W, W) \to [[\text{Cu}(W), \text{Cu}(W)] \cong [[\mathbb{P}, \mathbb{P}]] \cong M_1$$

is surjective.

The idea is to choose a unital, simple, AF-algebra $A$ with unique tracial state and a suitable element $x \in (A \otimes K)_+$ and consider the map $W \to W \otimes A$, given by $y \mapsto y \otimes x$, followed by a $^*$-isomorphism $W \otimes A \cong W$.

Let $A$ be a unital, simple AF-algebra with unique tracial state. We claim that $W \otimes A \cong W$. By construction, $W$ is an inductive limit of the building blocks considered by Razak in [Raz02]. Since $A$ is an AF-algebra, $W \otimes A$ is an inductive limit of Razak building blocks as well. Since $A$ is simple and has a unique tracial state, $W$ and $W \otimes A$ have the same invariant used for the classification [Raz02, Theorem 1.1], which gives the desired $^*$-isomorphism $W \otimes A \cong W$.

Given $a \in M_1$, let us define a c.p.c. order-zero map $W \to W$ corresponding to $a$. We distinguish two cases:

Case 1: Assume that $a$ is nonzero and soft. Let $U$ denote the universal UHF-algebra. We have $\text{Cu}(U) \cong \mathbb{Q}_+ \sqcup (0, \infty]$. We consider $a$ as a soft element in $\text{Cu}(U)_\text{soft} = [0, \infty]$. Choose $x_a \in (U \otimes K)_+$ with Cuntz class $a$. (For example, let $x_a$ be a positive element with spectrum $[0, 1]$ - ensuring that its Cuntz class is soft - and such that for the unique normalized extended trace $\tau: (U \otimes K)_+ \to [0, \infty]$ we have $\lim_{n \to \infty} \tau(x_a^{1/n}) = a.$)
Consider the map \( \varphi_a : W \to W \otimes U \) given by \( \varphi_a(y) = y \otimes x_a \) for \( y \in W \). It is easy to see that \( \varphi_a \) is a c.p.c. order-zero map. Let \( \psi : W \otimes U \to W \) be an isomorphism. Then \( \psi \circ \varphi_a \) is a c.p.c. order-zero map \( W \to W \) with the desired properties.

Case 2: Assume that \( a \) is compact. We claim that there exists a unital, simple AF-algebra \( A \) with unique normalized trace \( \tau : (A \otimes K)_+ \to [0, \infty) \) and a projection \( p_a \in (A \otimes K)_+ \) with \( \tau(p_a) = a \). Indeed, if \( a \) is rational, then we can take \( A = U \). If \( a \) is irrational, then we use that \( \mathbb{Z} + a\mathbb{Z} \) is a dimension group for the order and addition inherited as a subgroup of \( \mathbb{R} \). Moreover, \( \mathbb{Z} + a\mathbb{Z} \) has a unique normalized state. It follows that there is a unique unital AF-algebra \( A \) such that \((K_0(A), K_0(A)_+, [1])\) is isomorphic to \((\mathbb{Z} + a\mathbb{Z}, (\mathbb{Z} + a\mathbb{Z}) \cap [0, \infty), 1)\). By construction, there exists a projection \( p_a \in A \otimes K \) with \( \tau(p_a) = a \).

Define \( \varphi_a : W \to W \otimes A \) by \( \varphi_a(y) = y \otimes p_a \) for \( y \in W \). Then \( \varphi_a \) is a \( * \)-homomorphism. Postcomposing with a \( * \)-isomorphism \( W \otimes R_\theta \cong W \), we obtain a \( * \)-homomorphism \( W \to W \) with the desired properties.

**Example 8.9.** With similar methods as in **Example 8.8** one can show that the map \( \text{cpc}_\text{c}(A, B) \to [\text{Cu}(A), \text{Cu}(B)] \) is surjective whenever \( A \) and \( B \) are any of the following \( C^* \)-algebras: a UHF-algebra of infinite type, the Jiang-Su algebra, the Jaelon-Razak algebra \( W \).

**Remark 8.10.** In [BTZ16, Definition 2.27], Bosa, Tornetta and Zacharias introduced a bivariant Cuntz semigroup, denoted \( WW(A, B) \), as suitable equivalence classes of c.p.c. order-zero maps \( A \otimes K \to B \otimes K \). It would be interesting to study if the map from **Problem 8.7** factors through \( WW(A, B) \), that is, if the following diagram can be completed to be commutative:

\[
\text{cpc}_\text{c}(A, B) \longrightarrow [\text{Cu}(A), \text{Cu}(B)] \nonumber
\]

\[
\downarrow
\]

\[
WW(A, B)
\]

Observe that, in order for this to be satisfied, one needs to show that, given \( \varphi \) and \( \psi \) in \( \text{cpc}_\text{c}(A, B) \) such that \( \varphi \preceq \psi \) in the sense of [BTZ16] then, for \( \epsilon > 0 \), there is \( \delta > 0 \) such that \( \text{Cu}[\varphi_{1-\epsilon}] \prec \text{Cu}[\psi_{1-\delta}] \).

**References**


Abstract Bivariant Cuntz Semigroups


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