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Locally Compact Abelian p-Groups Revisited

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and Linus Kramer

1. Introduction

Even though the structure of locally compact abelian groups is generally considered to be rather thoroughly known through a wealth of publications, one keeps encountering corners that are not elucidated in up-to-date literature. In a study of a particular class of metabelian locally compact groups (see [HHR17]) we encountered some issues about noncompact locally compact abelian groups which do not appear to be discussed in the literature even though some of them were anticipated in Braconnier's article on his local product [Bra48]. Here we treat some of them, notably some aspects of totally disconnected torsion-free locally compact abelian groups which one might consider unexpected if not pathological. However, firstly we deal with some points concerning noncompact locally compact abelian torsion groups. For compact abelian groups we often refer to the monograph [HM13]. It will be convenient to use additive notation for abelian groups.

2. Locally Compact Abelian Torsion Groups

Let $\{G_j : j \in J\}$ be a family of topological abelian groups each of which contains an open subgroup U_j. According to Braconnier (see [Bra48]) we have an important subgroup of $\prod_{j\in J} G_j$ which contains the direct sum $\bigoplus_{j\in G} G_j$, defined as follows:

Definition 2.1. The group

$$\{(g_j)_{j\in J} : (\exists \text{ finite } F \subseteq J)(\forall j{\in}J \setminus F)\, g_j{\in}U_j\},$$

is called the *local product* whenever it carries the group topology for which the subgroup $\prod_{j\in J} U_j$ has its compact product topology and is an *open* subgroup; it will be denoted $\prod^{\mathrm{loc}}_{j\in J}(G_j, U_j)$.

A locally compact group is *periodic* if it is totally disconnected and every element is contained in a compact subgroup. One of the first uses of the local product was the following result of Braconnier's:

Theorem 2.2. *Any periodic locally compact abelian group is a local product*

$$(B) \qquad \prod_{p\in\pi}(G_p, C_p)$$

of its p-Sylow subgroups G_p as p ranges through the set π of all primes, and where every element $g \in G_p$ is contained in a compact p-subgroup (see Definition 8.7 in [HM13]*. (Here C_p is assumed to be the p-Sylow subgroup of a compact open subgroup C of G.)*

(See [Bra48]).

Clearly, a locally compact abelian torsion group G is periodic, and so Braconnier's Local Product Theorem 2.2 applies to it. A compact abelian torsion group has finite exponent by Corollary 8.9(iii) of [HM13]. Then the character group of a compact abelian group of finite exponent is a discrete abelian torsion group of finite exponent and therefore is the direct sum of cyclic groups of bounded order (see [Fuc73]). By duality, therefore, we have the following remark:

Lemma 2.3. *A compact abelian torsion group is a direct product of cyclic groups of bounded order.*

Of course, every locally compact abelian torsion group is periodic and thus has its unique Braconnier-Sylow decomposition (B). By Lemma 2.3 the set $\phi := \{p \in \pi : C_p \neq \{1\}\}$ is finite and so

$$G_{\pi\setminus\phi} := \prod_{p\in\pi\setminus\phi}^{\mathrm{loc}} (G_p, C_p) = \bigoplus_{p\in\pi\setminus\phi} G_p$$

is a discrete torsion group, so that, algebraically and topologically, G is the direct sum of G_π and $G_{\pi\setminus\phi}$. We summarize:

Proposition 2.4. *Firstly, for any locally compact abelian torsion group G we find a finite set $\phi \subseteq \pi$ of primes such that all G_p for $p \in \pi \setminus \phi$ are discrete and*

$$G \cong \prod_{p\in\phi} G_p \oplus \bigoplus_{p\in\pi\setminus\phi} G_p \tag{1}$$

Secondly, for each $p \in \phi$ the group G_p has a compact open subgroup

$$C_p \cong \mathbb{Z}(p)^{I_1} \times \mathbb{Z}(p^2)^{I_2} \times \cdots \times \mathbb{Z}(p^n)^{I_n} \tag{2}$$

for a finite collection of sets I_k, $k = 1, \ldots, n$, and G_p/C_p is discrete.

Corollary 2.5. *For a locally compact abelian torsion group G the following conditions are equivalent:*

(1) *G is discrete.*
(2) *The only compact open subgroups of G are finite.*
(3) *For each prime p, the endomorphism $x \mapsto p{\cdot}x$ of G_p is an open map.*

Proof. Since every locally compact abelian totally disconnected group has a compact open subgroup, (1) and (2) are clearly equivalent.

Trivially, (1) implies (3), and so we have to argue that (3) implies (2). By Proposition 2.4 it suffices to show that for each $p \in \phi$, the compact group C_p is finite. From Proposition 2.4 (2) we know that C_p has exponent p^n for some n and thus is a finite product of powers $\mathbb{Z}(p^k)^{I_k}$ for sets $I_1, \ldots, I_n$. Statement (3) implies that for each $k = 1, \ldots, n$ the power

$$\mathbb{Z}(p^{k-1})^{I_k} = p\cdot\Big(\frac{1}{p^k}\mathbb{Z}/\mathbb{Z}\Big)^{I_k}$$

is open in $\mathbb{Z}(p^k)^{I_k}$, and that implies that I_k is finite. This proves that C_p is finite. □

The preceding results were derived from our knowledge of compact torsion p-groups, notably the fact that a compact abelian torsion group has a finite exponent. What we are challenged to explain at this point are the details of the structure of noncompact and nondiscrete locally compact abelian torsion p-groups.

So let G be a locally compact abelian torsion p-group. Then the socle $S(G) = \{g \in G : p\cdot g = 0\}$ is a well defined closed characteristic subgroup of exponent p. So Proposition 2.4 immediately applies to the socle of a locally compact abelian group and yields:

Corollary 2.6. *Any exponent p locally compact abelian group G is a direct product of a compact and a discrete exponent p-subgroup.*

Specifically, there are sets I_1 and I_2 such that

$$G \cong \mathbb{Z}(p)^{(I_1)} \times \mathbb{Z}(p)^{I_2}.$$

Furthermore,

$$\operatorname{rank}_p G = \operatorname{rank}_p S(G) = \operatorname{card} I_1 + \operatorname{card} I_2.$$

In forming an intuition of locally compact abelian torsion groups the inspection of an example may be helpful right away. Before we enter the details we recall the concept of the divisible hull of an abelian group (see for instance [HM13], notably Proposition A1.1.33 and Corollary A1.36):

Remark 2.7. Any abelian group A is a subgroup of a divisible group D such that every nonzero subgroup of D meets A nontrivially and $\operatorname{card} D = \max\{\aleph_0, \operatorname{card} A\}$, and the p-ranks of A and D agree for all primes p. Such a group is called a *divisible hull* of A. If D_1 and D_2 are divisible hulls of A inside an abelian group G, there is an isomorphism $f\colon D_1 \to D_2$ such that $f|A = \mathrm{id}_A$. If A is torsion-free then $\mathbb{Q} \otimes_{\mathbb{Z}} A$ is a divisible hull of A (up to isomorphism).

Notably in examples, it is convenient to have a notation for the *local power* of a pair (A, B) of groups with $B \leq A$: The local product $\prod^{\mathrm{loc}}_{j\in J}(A_j, B_j)$, where $A_j = A$ and $B_j = B$ for all $j \in J$ we shall write as $(A, B)^{\mathrm{loc},J}$. We observe $(A, A)^{\mathrm{loc},J} = A^J$ and $(A, \{0\})^{\mathrm{loc},J} = A^{(J)}$.

Example 2.8. Let $\mathbb{Z}(p) := \frac{1}{p}\mathbb{Z}/\mathbb{Z} \subseteq \frac{1}{p^\infty}\mathbb{Z}/\mathbb{Z} = \mathbb{Z}(p^\infty) \cong \frac{\mathbb{Q}_p}{\mathbb{Z}_p}$. Consider $G = \big(\mathbb{Z}(p^\infty), \mathbb{Z}(p)\big)^{\mathrm{loc},\mathbb{N}}$. Then

(a) G is a locally compact abelian torsion p-group.
(b) G has a compact open socle $S(G) = \mathbb{Z}(p)^{\mathbb{N}}$ of rank $\aleph_0$.
(c) The discrete factor group $G/S(G) \cong \mathbb{Z}(p^\infty)^{(\mathbb{N})}$ is a divisible torsion p-group of rank $\aleph_0$.
(d) The unique largest divisible group $D = \mathbb{Z}(p^\infty)^{(\mathbb{N})}$ of G is countable, dense and nonclosed.
(e) The group $D\cap S(G) = S(D) = \mathbb{Z}(p)^{(\mathbb{N})}$ is a countable $\mathrm{GF}(p)$-vector subspace of $S(G)$ and therefore has an algebraic $\mathrm{GF}(p)$-vector space complement C of continuum dimension, that is, *algebraically*, $C \cong \mathbb{Z}(p)^{\mathbb{N}}$.
(f) *Algebraically*, $S(G)/S(D) \cong \mathbb{Z}(p)^{\mathbb{N}}$.

The venue in which Example 2.8 takes place is the abelian divisible torsion group $\Delta = \mathbb{Z}(p^\infty)^{\mathbb{N}}$ in which we consider the subgroup $\mathbf{S} := \mathbb{Z}(p)^{\mathbb{N}}$, a $\mathrm{GF}(p)$-vector space, whose dimension $\dim_{\mathrm{GF}(p)} \mathbf{S}$ equals $\operatorname{card} \mathbf{S} = 2^{\aleph_0}$ = the cardinality of the continuum, say, $\operatorname{card} \mathbb{R}$. We define on Δ again the finest group topology for which $\mathbf{S}$ has its own product topology and is an open subgroup in Δ. Indeed, the group considered in Example 2.8 is an open subgroup of Δ.

We consider the Prüfer group $\mathbb{Z}(p^\infty) = (1/p^\infty){\cdot}\mathbb{Z}/\mathbb{Z}$ and the subgroup $\mathbb{Z}(p) = (1/p){\cdot}\mathbb{Z}/\mathbb{Z}$.

Example 2.9. The group $\Delta = \mathbb{Z}(p^\infty)^{\mathbb{N}}$ is a divisible locally compact abelian group whose topology is defined by declaring the subgroup $\mathbf{S}= \mathbb{Z}(p)^{\mathbb{N}}$ with its compact product topology an open subgroup of Δ. We define inside Δ the subgroup

$$D := \bigcup_{n=0}^{\infty} \big(\frac{1}{p^n}\mathbb{Z}/\mathbb{Z}\big)^{\mathbb{N}} \subseteq \mathbb{Z}(p^\infty)^{\mathbb{N}}.$$

Then D contains precisely the elements $d = (z_1, z_2, \dots)$, $z_k \in \mathbb{Z}(p^\infty)$, $k \in \mathbb{N}$ for which there is an $n \in \mathbb{N}$ such that $p^n{\cdot}d = 0$, that is, D is the torsion subgroup of Δ. Morever, $\mathbf{S} \subseteq D \subseteq \Delta$ and the following statements hold:

(a) D is a nondiscrete locally compact abelian divisible torsion p-group, while $D^* := \Delta/D$ is a discrete torsion-free divisible group, that is, is a $\mathbb{Q}$-vector group failing to be a p-group. Further, $\dim_{\mathbb{Q}} D^* = \operatorname{card} D^*$.
(b) The socle $\mathbf{S}$ of D is a compact open subgroup, and
(c) the cardinality of D, Δ and D^* is the continuum $2^{\aleph_0}$, whence $D^* \cong \mathbb{Q}^{(\mathbb{R})}$.
(d) Both $D/\mathbf{S}$ and $\Delta/\mathbf{S}$ are discrete divisible abelian groups algebraically isomorphic to D, respectively Δ.
(e) The weights $w(D)$ and $w(\Delta)$ agree and equal $2^{\aleph_0}$, while
(f) both D and Δ are first countable.
(g) D is a divisible hull of $\mathbf{S}$.
(h) Since D is an open divisible subgroup of Δ, we have $\Delta \cong D \oplus D^*$ algebraically and topologically.

In particular, D is not sigma-compact.

It should be clear that many nonisomorphic variations of this theme abound.

2.1. **Pure subgroups of compact p-groups.** A subgroup P of an abelian group G is called *pure*, if $P \cap n{\cdot}G \subseteq n{\cdot}P$ for all natural numbers n.

Remark 2.10. Every finite subgroup A of an abelian group G is contained in a pure subgroup P of the same p-rank for all p.

(See also[Fuc73, Corollary 27.8])

For finite abelian groups one knows the following, see [Fuc73, Theorem 27.5]:

Remark 2.11. (Kulikoff's Lemma) A pure subgroup of finite exponent in an abelian group is a direct summand.

Accordingly,

Lemma 2.12. *A finite subgroup of an abelian p-group of finite exponent is contained in a direct summand of the same p-rank.*

We want to generalize this useful fact to finite subgroups of compact abelian torsion groups. The compact Hausdorff space of all closed subgroups of a locally compact group G is $\mathcal{SUB}(G)$. We recall that nowadays it is named the *Chabauty*-space of G. In the proof of the following result it will play an essential role.

Theorem 2.13. *In a compact torsion p-group every finite subgroup is contained in a finite (algebraic and topological) direct summand of the same p-rank.*

Proof. Let G be a compact abelian p-group. Let $\mathcal{N}$ denote the filter basis of all compact open subgroups. Thus

$$\lim \mathcal{N} = 0. \tag{1}$$

Now let H be a finite subgroup of G. Let $\mathcal{N}_H$ denote the set of all $N \in \mathcal{N}$ satisfying

$$H \cap N = \{0\}. \tag{2}$$

By (1) $\mathcal{N}_H$ is cofinal in $\mathcal{N}$.

By Lemma 2.12 applied to $(H+N)/N \cong H$ and G/N for $N \in \mathcal{N}_H$, there are subgroups F_N and B_N of G containing N such that

$$H \subseteq F_N \text{ and } \operatorname{rank}_p(F_N/N) = n \tag{3}$$

for $n = \operatorname{rank}_p H$ and

$$G/N = F_N/N \oplus B_N/N. \tag{4}$$

Now by the compactness of $\mathcal{SUB}(G)$ we find some cofinal function $j \mapsto N_j : J \to \mathcal{N}_H$ for some directed poset J such that $(F,B) = \lim_{j\in J}(F_{N_j}, B_{N_j})$ exists in $\mathcal{SUB}(G) \times \mathcal{SUB}(G)$.

Now we claim

$$H \subseteq F \text{ and } \operatorname{rank}_p F = \operatorname{rank}_p H, \tag{3$'$}$$

moreover,

$$G = F \oplus B. \tag{4$'$}$$

First we prove (3′). Since the graph of the containment relation $\subseteq$, defined on G, is a closed subset of $\mathcal{SUB}(G)\times\mathcal{SUB}(G)$, the containment $H+N/N \subseteq F_N$ for all N implies $H \subseteq F$. Thus $n = \operatorname{rank}_p H \le \operatorname{rank}_p F$. By (3) there is a n-element subset $X_N \subseteq F_N$ such that $\langle X_N + N/N\rangle = F_N$. Let X be a cluster point in the compact space $\mathcal{F}(G)$ of all closed subsets of G (see [Bou63], Chap VIII, Exercises, §5, p.206). Then $F \subseteq \langle X\rangle$. Hence $\operatorname{rank}_p F \le |X| \le |X_N| = n$. Thus (3′) is proved.

Now we prove (4′). The relation (4) implies $G = F_N + B_N$ which implies $G = F+B$. Now let $g \in F\cap B$ then $g = \lim_i f_i = \lim_i b_i$ for some $f_i \in F_{N_{j_i}}$ and $b_i \in B_{N_{j_i}}$ for a suitable cofinal function $i \mapsto j_i : I \to J$. Now let $N \in \mathcal{N}_H$. Then N is a compact open neighborhood of 0. Since $\lim_i(b_i - f_i) = g - g = 0$ there is a $i_0 \in I$ such that $i_0 \le i$ implies $b_i - f_i \in N$. Hence $b_i = f_i + n_i$ with some $n_i \in N \subseteq F_N$ and so $b_i \in F_N \cap B_i \subseteq N$. Since N is compact, $g = \lim_i b_i \in N$. But N was arbitrary in $\mathcal{N}_H$, the equation $g = 0$ follows. Hence $F \cap B = \{0\}$. This completes the proof. $\square$

3. Locally Compact Abelian Divisible Groups

Via duality, torsion and divisibility are juxtaposed in the context of compact and discrete abelian groups as is illustrated in [HM13] in the first Section of Chapter 8. We pursue this in the context of locally compact abelian p-groups. Braconnier's Decomposition Theorem 2.2 into primary components tells us that the restriction to p-groups is no restriction of generality.

We recall the divisible hull of an abelian group from Remark 2.7 and observe that for locally compact abelian p-groups there is a topological version.

Proposition 3.1. *Let A be a locally compact abelian p-group and D an algebraic divisible hull containing A. We give D the unique group topology for which A is an open subgroup. Then D is a locally compact abelian p-group.*

Proof. Our definition of the topology on D makes D a locally compact abelian group. In order to show that D is a p-group, we take an arbitrary element $x \in D$ and must show that $H := \overline{\langle x \rangle}$ is a compact p-group. By Remark 2.7 we find a nonzero element $a \in A \cap \langle x \rangle$ such that $n{\cdot}x = a$ for some natural number n.

By Weil's Lemma (see e.g. [HM13], Proposition 7.43), we have the following two cases:

Case (1): $H \cong \mathbb{Z}$ with the discrete topology.

Case (2): H is compact monothetic.

In Case (1), $n{\cdot}H = \langle n{\cdot}x \rangle$ is a nonsingleton discrete infinite cyclic subgroup of A which is impossible since A is a locally compact abelian p-group.

Thus D is periodic and thus by Braconnier's Local Product Theorem 2.2 is of the form

$$D = \prod_{q \in \pi} (D_q, C_q) \tag{B}$$

for the q-Sylow subgroups of D and C is a compact open subgroup of A. We know that $A \leq D_p$ since A is a p-group and D_p is the unique largest p-group in D. Assume that $D_q \neq \{0\}$ for a prime q. We may consider D_q to be a subgroup of D. Then $D_p \cap D_q \supseteq A \cap D_q \neq \{0\}$ by Remark 2.7. But that implies $p = q$, and so $D = D_p$, showing that D is a p-group. In particular H is a compact monothetic p-group. $\square$

Our Examples 2.8 and 2.9 above were early hints that plausible expectations suggested by the discrete or the compact situations may fail in the nondiscrete and noncompact locally compact one. This cautionary remark also applies to the following examples. The verifications of their properties are easy exercises.

Example 3.2. Let J be any set and give $\mathbb{Q}_p^J$ the topology generated by the open sets of the product topology and $\mathbb{Z}_p^J$ as an open subset. Call the resulting topological group G. Then

(a) G is a locally compact abelian group with respect to addition.
(b) $\mathbb{Z}_p^J$ is a compact open p-subgroup.
(c) G is torsion-free divisible.
(d) With respect to componentwise addition, multiplication, and scalar multiplication with p-adic rationals, the abstract group G is in fact a $\mathbb{Q}_p$-algebra.
(e) Let $\mu_p\colon G \to G$ again denote multiplication by p. Then the function $\mu_p\colon G \to G$ is an open map if and only if J is finite.
(f) Let $D := \bigcup_{n=0}^{\infty} \frac{1}{p^n}\mathbb{Z}_p^J$. Then D is an open subgroup of G which is proper if and only if J is infinite. It satisfies conditions (a)–(d) with D in place of G.
(g) The subgroup D is the smallest $\mathbb{Q}$-vector subspace of G containing $\mathbb{Z}_p^J$, namely, the divisible hull $\mathbb{Q}\otimes\mathbb{Z}_p$.
(h) D is a locally compact abelian p-group in the sense that each of its elements is contained in a compact p-group.

In the case of torsion-free groups, the divisible hull of a subgroup A of a divisible group is unique, being (essentially) the group $\mathbb{Q}\otimes A$. A torsion-free divisible group is a $\mathbb{Q}$-vector space. We may and shall consider A as a subgroup of $\mathbb{Q}\otimes A$ via the injection $a \mapsto 1\otimes a : A \to \mathbb{Q}\otimes A$. If A is a locally compact abelian torsion free group, we shall consider $\mathbb{Q}\otimes A$ as a unique locally compact torsionfree divisible group in such a way that A is an open subgroup of $\mathbb{Q}\otimes A$.

Definition 3.3. A locally compact abelian group D which is isomorphic to the divisible hull

$$\mathbb{Q}\otimes\mathbb{Z}_p^J = \bigcup_{n=0}^{\infty} \frac{1}{p^n}\cdot\mathbb{Z}_p^J$$

of $\mathbb{Z}_p^J$ in $\mathbb{Q}_p^J$ for a set J of cardinality $\aleph$ will be called a *locally compact abelian torsion-free divisible p-group of p-rank* $\aleph$.

Theorem 3.4. (i) *A locally compact abelian torsion-free p-group G is an open subgroup of its divisible hull $D(G) = \mathbb{Q}\otimes G$ which is a* *torsion-free divisible p-group of p-rank $\aleph$ for some cardinal $\aleph$. Morover, $D(G) = \mathbb{Q} \otimes C$ for any compact open subgroup C of G with $C \cong \mathbb{Z}_p^{\aleph}$ and $D(G)$ is a subgroup of $\mathbb{Q}_p^{\aleph}$.*

(ii) *The following conditions are equivalent for such a group G:*

(1) $\operatorname{rank}_p G$ *is finite.*
(2) $G \cong \mathbb{Q}_p^n$ *for some $n = 0, 1, 2, \ldots$*
(3) *The scalar multiplication $x \mapsto p{\cdot}x$ is an automorphism of topological groups.*
(4) *G is sigma-compact.*

Proof. We begin by proving (i). We begin with a simple observation: *A nonsingleton discrete torsion-free group is a rational vector space, all monothetic subgroups are isomorphic to $\mathbb{Z}$, and therefore cannot be locally compact p-group.*

Now let G be a nonsingleton locally compact abelian torsion-free p-group. Being nondiscrete and totally disconnected, there exists a compact open subgroup C. Then C is a compact totally disconnected and torsion-free group. Its character group therefore is a discrete divisible torsion group by Corollary 8.5 of [HM13]. As such it is of the form $\mathbb{Z}(p^\infty)^{(J)}$ for some set J by Proposition A1.41 of [HM13]. Accordingly, $C \cong \mathbb{Z}_p^J$. Since $\mathbb{Z}_p$ is divisible by all natural numbers relatively prime to p, it follows, that $\bigcup_{n=0}^\infty \frac{1}{p^n}\mathbb{Z}_p^J = \mathbb{Q}{\cdot}\mathbb{Z}_p^J$ is the divisible hull in $\mathbb{Q}_p^J$. In the torsion free case, divisible hulls are unique. Hence $\bigcup_{n=0}^\infty \frac{1}{p^n}{\cdot}C = \mathbb{Q}{\cdot}C$ is the divisible hull $D(C)$ of C and is an open subgroup of $\bigcup_{n=0}^\infty \frac{1}{p^n}{\cdot}G = D(G)$. But being divisible, $D(C)$ is pure in $D(G)$, whence $D(G)/D(C)$ is a torsion-free discrete group on the one hand and a p-group on the other and so is singleton. This shows $D(C) = D(G)$. We know that $D(C)$ is a torsion-free divisible p-group of p-rank $\aleph = \operatorname{card} J$, and so this applies to $D(G)$ as well. This completes the proof of (i).

Proof of (ii): (1), (2), and (3) are equivalent after Example 3.2. Clearly a finite dimensional $\mathbb{Q}_p$-vector space with its natural topology is sigma-compact since $\mathbb{Q}_p^n/\mathbb{Z}_p^n \cong \mathbb{Z}(p^\infty)^n$ is countable. It remains to show that G fails to be sigma compact if its rank is infinite. In that case we observe that we have

$$G/C \cong \mathbb{Q}{\cdot}\mathbb{Z}_p^J/\mathbb{Z}_p^J \supseteq \frac{\frac{1}{p}{\cdot}\mathbb{Z}_p{}^J}{\mathbb{Z}_p^J} \cong \mathbb{Z}(p)^J,$$

so that G is the disjoint union of at least $2^{\aleph_0}$ copies of C if J is infinite. So G cannot be sigma-compact in this case. $\square$

In the context of the preceding proposition we return briefly to Example 3.2. The group $G = \mathbb{Q}_p^J$ is a locally compact abelian divisible torsion free group containing the divisible hull $D = D(G)$ of the compact open subgroup $\mathbb{Z}_p^J$ of G. Assume now that J is infinite. Then the containment is proper, and G is therefore of the form $G = D \oplus D^*$ with a discrete divisible torsion-free group $D^* \cong G/D$. Let $A = \mathbb{Z}(p^\infty)^J$ and $S = \left(\frac{1}{p}\mathbb{Z}/\mathbb{Z}\right)^J$ its socle. Then $\operatorname{tor} A = \bigcup_{n=1}^\infty \left(\frac{1}{p^n}\mathbb{Z}/\mathbb{Z}\right)^J$, and $D^* \cong A/\operatorname{tor} A \cong \mathbb{Q}^J$. The point that we observe here is that G is *not* a p-group, and therefore fails to be isomorphic to the divisible hull of a compact power of groups $\mathbb{Z}_p$ such as, for instance D.

So we model the next example after our examples in the torsion case in order to show that a maximal divisible subgroup of a torsion-free locally compact abelian group need not be closed:

We let $\Delta = \mathbb{Q}_p^{\mathbb{N}}$, $D = \mathbb{Q} \otimes \mathbb{Z}_p^{\mathbb{N}}$, $C = \mathbb{Z}_p^{\mathbb{N}}$ and interpolate between the compact open subgroup C and its divisible hull D the group

$$P = (\mathbb{Q}_p, \mathbb{Z}_p)^{\mathrm{loc},\mathbb{N}}. \tag{$*$}$$

Example 3.5. The group P in $(*)$ is a locally compact abelian torsion-free group with compact open subgroup C and $D/C \cong \mathbb{Z}(p^\infty)^{(\mathbb{N})}$. But P is not divisible since $(c_n)_{n\in\mathbb{N}} \in C$, $c_n = 1$ for all n does not have a p-th root in P.

The subgroup $E := \mathbb{Q}_p^{(\mathbb{N})}$ is a dense proper subgroup of P which is divisible. Hence the maximal divisible subgroup M_P is dense and proper. We note $P = E + C$ while $E \cap C = \mathbb{Z}_p^{(\mathbb{N})}$. Thus

$$P/E \cong C/(E \cap C) = \frac{\mathbb{Z}_p^{\mathbb{N}}}{\mathbb{Z}_p^{(\mathbb{N})}}. \tag{$**$}$$

This last group we abbreviate by K. The following fact is noteworthy:

Lemma 3.6.

$$K \cong \frac{\mathbb{Z}^{\mathbb{N}}}{\mathbb{Z}^{(\mathbb{N})}}. \tag{$***$}$$

Proof. Since $\mathbb{Z}_p^{\mathbb{N}}$ is compact and hence cotorsion and K is a homomorphic torsion-free image it follows that K is algebraically compact (see [Fuc73, VII]). First of all $K = D \oplus R$ for D the maximal divisible subgroup, and R reduced. By the torsion-freeness and using Kaplansky's Theorem one can provide a cardinal $\mathfrak{m}_0$ for which $D \cong \mathbb{Q}^{(\mathfrak{m}_0)}$ and $R \cong \prod_q A_q$ where, for q any prime, A_q is the q-adic completion of a direct sum $\mathbb{Z}_p^{(\mathfrak{m}_p)}$. One verifies that $\mathbb{Z}^{\mathbb{N}}/\mathbb{Z}^{(\mathbb{N})}$ is a torsion-free algebraically compact pure subgroup of K with corresponding cardinal invariants

$\mathfrak{n}_0 = \mathfrak{c}$ and $\mathfrak{n}_q = \mathfrak{c}$. Then the pureness of this embedding yields estimates of cardinalities from below $\mathfrak{c} \le \mathfrak{m}_0$ and $\mathfrak{c} \le \mathfrak{m}_p$. From this and the fact that P/E has cardinality $\mathfrak{c}$ we deduce the equalities $\mathfrak{c} = \mathfrak{m}_0$ and $\mathfrak{c} = \mathfrak{m}_p$. Hence we have an (abstract) isomorphism $K \cong \mathbb{Z}^{\mathbb{N}}/\mathbb{Z}^{(\mathbb{N})}$. □

Returning to Example 3.5 and denoting by M_K the maximal divisible subgroup of K, and further U its full inverse image in P, we observe that $U/E \cong M_K$ and since divisible subgroups split we have $U = E \oplus M'$ with $M' \cong M_K$, whence $U = M_p$, the maximal divisible subgroup if P.

For giving an explicit description of the maximal divisible subgroup M_K of K we use the convention $p^\infty = 0$ and say that a sequence $\ell \in (\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$ in $\mathbb{N} \cup \{\infty\}$ *has finite sublevel sets* iff

$$(\forall m \in N)\, |\{n \in \mathbb{N} : \ell_n \le m\}| \le \infty.$$

Now let

$$L = \{\ell \in (\mathbb{N} \cup \{\infty\})^{\mathbb{N}} : \ell \text{ has finite sublevel sets}\}.$$

The set L is a lattice in the componentwise partial order. For each $\ell \in L$ set

$$H_\ell = p^{\ell_1}{\cdot}\mathbb{Z}_p \times p^{\ell_2}{\cdot}\mathbb{Z}_p \times \cdots \subseteq \mathbb{Z}_p^{\mathbb{N}}.$$

The function $\ell \mapsto H_\ell$ from L into the lattice of subgroups of $\mathbb{Z}_p^{\mathbb{N}}$ is an order reversing lattice morphism. We claim that

(i) $(H_\ell + \mathbb{Z}_p^{(\mathbb{N})})/\mathbb{Z}_p^{(\mathbb{N})}$ *is divisible.*

For a proof we let $m \in \mathbb{N}$ and $z = (p^{\ell_1}z_1, p^{\ell_2}z_2, \dots) \in H_\ell$. Then $F := \{n \in \mathbb{N} : \ell_n \le m\}$ is finite since ℓ has finite sublevel sets. Now we define

$$\ell' = \begin{cases} \ell'_n = \ell_n & \text{for } n \notin F \\ m & \text{for } n \in F, \end{cases}$$

and

$$z' = (p^{\ell'_1}z_1, p^{\ell'_2}z_2, \dots) = p^m(p^{\ell'_1-m}z_1, p^{\ell'_2-m}z_2, \dots) = p^m z''.$$

But now $z - p^m z'' = z - z' \in \mathbb{Z}_p^{(\mathbb{N})}$, and this proves Claim (i).

Now we set $H = \bigcup_{\ell \in L} H_\ell$. Since $\ell \mapsto H_\ell$ is monotone, H is a subgroup of $\mathbb{Z}_p^{\mathbb{N}}$.

Next we claim that

(ii) $(H + \mathbb{Z}_p^{\mathbb{N}})/\mathbb{Z}_p^{(\mathbb{N})} = M_K$.

The containment $\subseteq$ follows from (i) above. Conversely, assume that an element $z = (z_1, z_2, \dots)$ in $\mathbb{Z}_p^{\mathbb{N}}$ is divisible modulo $\mathbb{Z}_p^{(\mathbb{N})}$. Then we write $z = (p_1^\ell x_1, p_2^\ell x_2, \dots)$ with maximal exponents $\ell_m \in \mathbb{N} \cup \{\infty\}$. Since z is divisible by p^m for all m modulo $\mathbb{Z}_p^{(\mathbb{N})}$ we conclude that for each m we have $\ell_n \geq m$ with at most finitely many exceptions, that is, ℓ has finite sublevel sets and so $z \in H_\ell \subseteq H$ modulo $\mathbb{Z}_p^{(\mathbb{N})}$. This proves Claim (ii).

This concludes the analysis of Example 3.5.

For a full understanding of the example we recall the following remark:

Remark 3.7. Let J be an arbitrary set. The group $\mathbb{Z}_p$ and thus also $C = \mathbb{Z}_p^J$ is reduced, that is, does not contain any nonzero divisible subgroup (see Proposition 4.26 of [HHR17]).

So $\mathbb{Z}_p^{\mathbb{N}}$ is reduced while in fact we argued that $\mathbb{Z}_p^{\mathbb{N}}/\mathbb{Z}_p^{(\mathbb{N})}$ has a large divisible subgroup.

3.1. **Pure subgroups.** We recall that in a torsion-free abelian group G, the *pure subgroup* $[C]$ generated in G by a subgroup C is

$$[C] = \{g \in G : (\exists n \in \mathbb{N}) \ \ n{\cdot}g \in C\} = \bigcup_{n=1}^{\infty} (\mu_n^G)^{-1} C.$$

(See [HM13], Proposition A1.25).

Lemma 3.8. *Let G be a torsion-free locally compact abelian p-group and C a compact-open subgroup. Then*

(i) $[C] = G$.

(ii) *G/C is a discrete torsion p-group.*

Proof. (i) Since $[C]$ is pure, $G/[C]$ is torsion-free. The subgroup $[C]$ contains C and thus is open. Hence $G/[C]$ is a discrete torsion-free p-group and thus is singleton.

(ii) By the definition of $[C]$, the factor group $[C]/C$ is always a torsion group. Thus (ii) follows from (i) at once. □

3.2. **Splitting in torsion-free groups.** In spite of an abundance of counterexamples, some splitting results hold in torsion-free locally compact abelian groups.

Recall that any torsion-free compact p-group is isomorphic to $\mathbb{Z}_p^J$ for some et J, and that these groups are the projectives in the category of compact p-groups. This is a consequence of the fact that their divisible duals are injective in the category of discrete abelian p-groups.

Lemma 3.9. *Let $C = \mathbb{Z}_p^J$ and P a closed pure subgroup. Then there is a closed subgroup F such that $C = P \oplus F$, algebraically and topologically.*

Proof. The group C/P is a compact p-group which is torsion-free since P is pure. Since C/P is projective, there is a morphism $j\colon C/P \to C$ such that for the quotient epimorphism $e\colon C \to C/P$ the following diagram is commutative

$$\begin{array}{ccc} C & \xleftarrow[j]{} & C/P \\ {\scriptstyle \mathrm{id}_C}\big\downarrow & & \big\downarrow{\scriptstyle \mathrm{id}_{C/P}} \\ C & \xrightarrow[e]{} & C/P, \end{array}$$

that is, $e \circ j = \mathrm{id}_{C/P}$, saying that $R = j(C/P)$ is a retract. Thus there is a closed subgroup R such that $C = P \oplus R$ in the category of compact abelian p-groups. □

We can reformulated this lemma as follows:

Lemma 3.10. *For any closed pure subgroup P of a compact torsionfree p-group C there is an endomorphism q of C such that $q^2 = q$ and $P = q(C)$.*

Proposition 3.11. *Any closed divisible subgroup of a divisible, torsion-free locally compact abelian p-group is a direct summand, algebraically and topologically.*

Proof. Let D be a divisible, torsionfree locally compact abelian group and V be a closed divisible subgroup. From Theorem 3.4 we know that there is a compact open subgroup C of D such that D may be identified with $\mathbb{Q}\otimes C$. The subgroup $P := V \cap C$ of C satisfies $n{\cdot}C \cap P = n{\cdot}C \cap V = n{\cdot}C \cap n{\cdot}V = n{\cdot}(C \cap V) = n{\cdot}P$ since V is divisible and D is torsionfree, and so P is a pure subgroup of C. Also $C \cong \mathbb{Z}^J$ for some set J since C is a compact torsion free p-group. Hence Lemma 3.9 and Lemma 3.10 apply and produce an endomorphism $q\colon C \to C$ such that $q^2 = q$ and $q(C) = P = C \cap V$. Since D is torsion free divisible and $D = \mathbb{Q}{\cdot}C$, every element $d \in D$ is uniquely of the form $d = p^{-n}{\cdot}c$ for $c = p^n{\cdot}d \in C$, and the endomorphism q of C extends uniquely to an endomorphism $f\colon D \to D$ by $f(d) = p^{-n}{\cdot}q(c)$. It satisfies $f^2 = f$. An element $d' \in D$ is in V iff it is of the form $d' = p^{-n}{\cdot}c'$ with $c' \in P$. That is the case iff there is a $c \in C$ such that $c' = q(c)$ and so $d' = p^{-n}{\cdot}c' = p^{-n}{\cdot}q(c) = f(d)$ for $d = p^{-n}{\cdot}c$. This shows that $f(D) = V$ and therefore that there is a closed subgroup W of D such that $D = V \oplus W$ (algebraically and topologically). This proves the proposition. □

For the category of abelian groups we know for a fact that each divisible subgroup of an abelian group is a direct summand. Now the preceding proposition enables us to prove the following fact for the category of torsion-free locally compact abelian p-groups:

Theorem 3.12. *Every closed divisible subgroup of a torsion-free locally compact abelian p-group is a direct summand (algebraically and topologically).*

Proof. Let V be a closed divisible subgroup of a torsion-free locally compact abelian p-group G. By Theorem 3.4 (i) there is a torsion-free divisible locally compact abelian hull

$$D(G) = \mathbb{Q} \otimes G = \bigoplus_{n=0}^{\infty} p^{-n}{\cdot}G \supseteq V.$$

By Proposition 3.11 there is a closed subgroup W of $D(G)$ such that $D(G) = V \oplus W$ in the category of locally compact abelian groups. By the Modular Law, since $V \subseteq G$ we have $G = V \oplus (W \cap G)$ in the category of locally compact p-groups. □

In particular, we have the following corollary.

Corollary 3.13. *Any subgroup of a torsion-free locally compact abelian p-group splits provided it is isomorphic to $\mathbb{Q}_p^m$ for a natural number m.*

Proof. The group $\mathbb{Q}_p^m$ is locally compact in its natural topology as the topological $\mathbb{Q}_p^m$-vector space is locally compact. Any locally compact subgroup of a Hausdorff topological group is closed. Hence the corollary follows from Theorem 3.12. □

Next we shall exhibit in Example 3.17 a torsion free locally compact abelian p-group with a quotient that is isomorphic to $\mathbb{Q}_p$ but which does not split, and in Proposition 3.20 we shall find a locally compact abelian and sigma-compact p-group with a closed subgroup isomorphic to $\mathbb{Q}_p$ that does not split.

3.3. **Splitting $\mathbb{Q}_p$.** We recall $\mathbb{N}_0 = \{0, 1, 2, \dots\}$.

Lemma 3.14. *We let $k\colon \mathbb{Z}(p^\infty) \to \mathbb{Z}(p^\infty)^{(\mathbb{N}_0)}$ be defined by $k(x) = (x, p{\cdot}x, p^2{\cdot}x, p^3{\cdot}x, \dots)$ and let $\kappa = \widehat{k}\colon \mathbb{Z}_p^{\mathbb{N}_0} \to \mathbb{Z}_p$ be the dual morphism, where we identify $\mathbb{Z}_p$ with the character group of $\mathbb{Z}(p^\infty)$ and $\mathbb{Z}_p^{\mathbb{N}_0}$ with the character group of $\mathbb{Z}(p^\infty)^{(\mathbb{N}_0)}$. Then*

(i) *k is a well defined injective morphism.*
(ii) *κ is a surjective morphism given explicitly by*
$$\kappa((z_0, z_1, z_2, \dots)) = \textstyle\sum_{n=0}^{\infty} z_n p^n.$$

Proof. (i) This is immediate due to the fact that each element z in the Prüfer group $\mathbb{Z}(p^\infty)$ has finite order.

(ii) As a consequence of the duality between discrete and compact abelian groups, κ is surjective since k is injective. We may identify the module action $(z,x) \mapsto z{\cdot}x : \mathbb{Z}_p \times \mathbb{Z}(p^\infty) \to \frac{1}{p^\infty} Z/\mathbb{Z} \to \mathbb{R}/\mathbb{Z}$ and the bilinear map

$$\begin{aligned} ((z_0, z_1, \dots), (x_0, x_1, \dots)) &\mapsto \sum_{n=0}^{\infty} z_n{\cdot}x_n, \\ \mathbb{Z}_p^{N_0} \times \mathbb{Z}(p^\infty)^{(\mathbb{N}_0)} &\to \mathbb{Z}(p^\infty) \subseteq \mathbb{R}/\mathbb{Z} \end{aligned}$$

the dual pairings of duality, Then for $z = (z_0, z_1, \dots) \in \mathbb{Z}_p^{(\mathbb{N}_0)}$ and $x \in \mathbb{Z}(p^\infty)$ we have $\kappa(z){\cdot}x = \sum_{n=0}^{\infty} p^n z_n{\cdot}x$ on the one hand and
$z{\cdot}k(x) = (z_0, z_1, z_2 \dots){\cdot}(x, p{\cdot}x, p^2{\cdot}x, \dots) = \sum_{n=0}^{\infty} z_n{\cdot}(p^n{\cdot}x)$
on the other. The right hand sides agree, and this shows that k and κ are adjoint under the duality. □

Lemma 3.15. *Let $C = \prod_{n\in\mathbb{N}_0} p^{2n}\mathbb{Z}_p \cong \mathbb{Z}_p^{\mathbb{N}_0}$. There is a morphism $\eta\colon C \to \mathbb{Z}_p$ defined by*

$$\eta((x_0, x_1, \dots)) = \sum_{n=0}^{\infty} x_n p^{-n}.$$

Proof. By the definition of C, for each $n = 0, 1, \dots$ there is a y_n such that $x_n = p^{2n} y_n$. Therefore $\sum_{n=0}^{\infty} x_n p^{-n} = \sum_{n=0}^{\infty} y_n p^n$ which converges in $\mathbb{Z}_p$ so that η is well defined. Let $\alpha\colon \mathbb{Z}_p^{\mathbb{N}_0} \to C$ be the isomorphism given by
$$\alpha((y_0, y_1, \dots)) = (y_0, p^2 y_1, \dots, p^{2n} y_n, \dots) = (x_0, x_1, \dots).$$
Then

$$(\eta \circ \alpha)((y_0, y_1, \dots)) = \eta((x_0, x_1, \dots)) = \sum_{n=0}^{\infty} y_n p^n = \kappa((y_0, y_1, \dots)),$$

that is $\eta = \kappa \circ \alpha^{-1}$. We saw in Lemma 3.14 that κ is a morphism and so η is a morphism. □

Lemma 3.16. *Let G be the torsion-free locally compact abelian group $\prod_{n\in\mathbb{N}_0}^{\mathrm{loc}} (\mathbb{Z}_p, p^{2n}\mathbb{Z}_p)$, and let C denote its compact open subgroup*

$$\prod_{n\in\mathbb{N}_0} p^{2n}\mathbb{Z}_p \cong \mathbb{Z}_p^{\mathbb{N}_0}.$$

Then the morphism $\eta\colon C \to \mathbb{Z}_p$ of Lemma 3.15 *extends to a continuous open surjective morphism $\widetilde{\eta}\colon G \to \mathbb{Q}_p$.*

Proof. (a) We let $S = \mathbb{Z}_p^{(\mathbb{N}_0)} \subseteq \mathbb{Z}_p^{\mathbb{N}_0}$. Then $G = C + S$. Now $\eta'\colon S \to \mathbb{Q}_p$ by $\eta'((z_0, z_1, \dots, z_n, \dots)) = \sum_{n\in\mathbb{N}_0} z_n p^{-n} \in \mathbb{Q}_p$, as a finite sum, is a well defined algebraic homomorphism. On $C \cap S$ the definitions of η and η' agree.

(b) Now we define $\eta^*\colon C \times S \to \mathbb{Q}_p$ by $\eta^*(c,s) = \eta(c) - \eta'(s)$. We also have a surjective morphism $\delta\colon C \times S \to G$ defined by $\delta(c,s) = c - s$. Then $\ker \delta = \{(c,c) : c \in C \cap S\}$, and by Part (a) of the proof, η^* vanishes on $\ker \delta$. Hence there is a morphism $\widetilde{\eta}\colon G \to \mathbb{Q}_p$ such that $\eta^* = \widetilde{\eta} \circ \delta$. Moreover, for $c \in C$ we have $\widetilde{\eta}(c) = \widetilde{\eta}(\delta(c,0)) = \eta^*(c,0) = \eta(c)$, that is, $\widetilde{\eta}$ extends η which is a continuous open morphism of the open subgroup C of G. Therefore, $\widetilde{\eta}$ is a continuous open and surjective morphism. $\square$

Example 3.17. (i) The quotient morphism $\widetilde{\kappa}\colon G \to \mathbb{Q}_p$ with

$$(*) \qquad G = \prod_{n\in\mathbb{N}_0}^{\text{loc}} (\mathbb{Z}_p, p^{2n}\mathbb{Z}_p)$$

constructed in Lemma 3.16 does not split, that is, there is no morphism $f\colon \mathbb{Q}_p \to G$ such that $\widetilde{\kappa} \circ f = \mathrm{id}_{\mathbb{Q}_p}$.

(ii) The group G is a locally compact, sigma-compact, torsion-free, p-group such that $\overline{pG} \neq G$. In fact, there is a compact open subgroup $C \cong \mathbb{Z}_p^{\mathbb{N}}$ such that G/C is a (discrete!) countable torsion group of infinite exponent.

Proof. (i) The subgroup $C = \mathbb{Z}_p^{\mathbb{N}_0}$ is reduced by Remark 3.7, and $G/C \cong \bigoplus_{n=0}^{\infty} \mathbb{Z}(p^{2n})$ is reduced as well. Hence G is reduced and so a splitting morphism f cannot exist.

(ii) The assertions in (ii) are straightforward. $\square$

We now wish to record the dual situation of Example 3.17 and for this purpose we record Braconnier's Theorem on the dual of a local product of locally compact abelian groups. (See [Bra48], Theorem 1, p. 10.)

Lemma 3.18. *Let $\{(A_j, B_j) : j \in J\}$ be a family of locally compact abelian groups A_j with compact open subgroups B_j. Then the dual group of*

$$(1) \qquad G = \prod_{j\in J}^{\text{loc}} (A_j, B_j)$$

may be identified with

$$(2) \qquad \widehat{G} = \prod_{j\in J}^{\text{loc}} (\widehat{A_j}, B_j^{\perp}),$$

where, as usual, $B_j^{\perp}$ is the annihilator of B_j in $\widehat{A_j}$.

Lemma 3.19. *Let G be as in $(*)$ in Lemma 3.17. Then the character group $\widehat{G}$ may be identified with the group $\mathbf{G}$ described as follows:*

$$(**)\qquad \mathbf{G} = \prod_{n\in\mathbb{N}_0}^{\mathrm{loc}} (\mathbb{Q}_p/\mathbb{Z}_p, p^{-2n}\mathbb{Z}_p/\mathbb{Z}_p).$$

Then $\mathbf{G}$ is a locally compact abelian p-group isomorphic to

$$\prod_{n\in\mathbb{N}_0}^{\mathrm{loc}} (\mathbb{Z}(p^\infty), \mathbb{Z}(p^{2n}))$$

where we write $\mathbb{Z}(p^k) = \frac{1}{p^k}\cdot\mathbb{Z}/\mathbb{Z} \subseteq \frac{1}{p^\infty}\cdot\mathbb{Z}/\mathbb{Z} = \mathbb{Z}(p^\infty)$.

Proof. We have to show that we have a dual pairing of the groups G and $\mathbf{G}$ in $(*)$ and $(**)$, respectively. For each $n \in \mathbb{N}_0$ we have a pairing $\langle -, -\rangle$:

$$(z, q + \mathbb{Z}_p) \mapsto zq + \mathbb{Z}_p : \mathbb{Z}_p \times \mathbb{Q}_p/\mathbb{Z}_p \to \mathbb{Q}_p/\mathbb{Z}_p \cong \frac{1}{p^\infty}\cdot\mathbb{Z}/\mathbb{Z} \subseteq \mathbb{R}/\mathbb{Z}$$

such that $p^{2n}\mathbb{Z}_p$ and $p^{-2n}\mathbb{Z}_p/\mathbb{Z}_p$ are annihilators of each other. Fix $n \in \mathbb{N}$; we determine for which $m \in \mathbb{N}$ the group $p^{2m}\mathbb{Z}_p$ is annihilated by $p^{-2n}\mathbb{Z}_n$. An element $z \in \mathbb{Z}_p$ is in $p^{2m}\mathbb{Z}_p$ for an $m \in \mathbb{Z}$ iff there is an $x \in \mathbb{Z}_p^\times$, the group of units of $\mathbb{Z}_p$, such that $z = p^{2m}x$; similarly an element $q + \mathbb{Z}_p \in \mathbb{Q}_p/\mathbb{Z}_p$ is in $p^{-2n}\mathbb{Z}_p/\mathbb{Z}_p$ iff there is a $y \in \mathbb{Z}_p^\times$ such that $q = p^{-2n}y$. Now the relation $\langle z, q + \mathbb{Z}_p\rangle = zq + \mathbb{Z}_p = 0$ holds iff $p^{2m}x\cdot p^{-2n}y = zq \in \mathbb{Z}_p$ iff $m - n \geq 0$, that is, $m \geq n$. That proves that $p^{2n}\mathbb{Z}_p$ is indeed the annihilator of $p^{-2n}\mathbb{Z}_p$. Then Lemma 3.18 completes the proof of the lemma. ∎

It is understood that the character group of $\mathbb{Q}_p$ may be identified with $\mathbb{Q}_p$ under the dual pairing

$$(r, s) \mapsto rs + \mathbb{Z}_p : \mathbb{Q}_p \times \mathbb{Q}_p \to \mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\iota} \mathbb{R}/\mathbb{Z},$$

where ι is the embedding morphism

$$\mathbb{Q}_p/\mathbb{Z}_p \xrightarrow{\cong} \frac{1}{p^\infty}\cdot\mathbb{Z}/\mathbb{Z} \xrightarrow{\mathrm{incl}} \mathbb{R}/\mathbb{Z}.$$

In Lemma 3.16 we had a quotient morphism $\widetilde{\eta}\colon G \to \mathbb{Q}_p$ for which we now obtain a dual injection $\iota\colon \mathbb{Q}_p \to \mathbf{G}$ so that

$$(\forall q \in \mathbb{Q}_p, (z_0, z_1, \dots) \in G)$$

$$(3.1)\qquad \langle \iota(q), (z_0, z_1, \dots)\rangle \;=\; \langle q, \eta((z_0, z_1, \dots))\rangle = \sum_{n=0}^{\infty} q z_n p^{-n}.$$

(Recall here that almost all $z_n \in \mathbb{Z}_p$ are of the form $z_n = p^{2n}x_n$ for some x_n so that almost all summands of the infinite series in equation (3.1) read qx_np^n. Thus the convergence of the series in $\mathbb{Z}_p$ is never in question.) We now easily verify that equation (3.1) is satisfied if and only if

$$\iota(q) \in \mathbf{G} = \prod_{n\in\mathbb{N}_0}^{\mathrm{loc}} (\mathbb{Q}_p/\mathbb{Z}_p, p^{-2n}\mathbb{Z}_p/\mathbb{Z}_p)$$

is of the form

$$\iota(q) = (q + \mathbb{Z}_p, qp^{-1} + \mathbb{Z}_p, qp^{-2} + \mathbb{Z}_p, \dots) = \left(\frac{q}{p^n} + \mathbb{Z}_p\right)_{n\in\mathbb{N}_0}.$$

Thus we have the following proposition:

Proposition 3.20. *The group* $\mathbf{G}$= $\prod_{n\in\mathbb{N}_0}^{\mathrm{loc}}(\mathbb{Q}_p/\mathbb{Z}_p, p^{-2n}\mathbb{Z}_p/\mathbb{Z}_p)$ *is a locally compact and sigma-compact* p*-group with a closed subgroup*

$$\left\{\left(\frac{q}{p^n} + \mathbb{Z}_p\right)_{n\in\mathbb{N}_0} : q \in \mathbb{Q}_p\right\}$$

isomorphic to $\mathbb{Q}_p$ *which is not a direct summand in the category of locally compact abelian groups.*

We note that any subgroup that is isomorphic to $\mathbb{Q}_p$ is divisible and thus is a direct summand in the category of abelian groups. We also remark explicitly that $\mathbf{G}$ itself is not divisible:

Lemma 3.21. *The group* $\mathbf{G}$ *is not divisible, but* $p{\cdot}\mathbf{G}$ *is dense in* $\mathbf{G}$*.*

Proof. Let $C = \prod_{n\in\mathbb{N}_0} p^{-2n}\mathbb{Z}_p/\mathbb{Z}_p \cong \prod_{n\in\mathbb{N}} \mathbb{Z}(p^{2n})$. Then $S := C/pC \cong \mathbb{Z}(p)^{\mathbb{N}}$ and we consider the quotient

$$H := \mathbf{G}/pC \cong (\mathbb{Z}(p^\infty), \mathbb{Z}(p))^{\mathrm{loc},\mathbb{N}}.$$

This group has the subgroups $E = \mathbb{Z}(p^\infty)^{(\mathbb{N})}$ and $S = \mathbb{Z}(p)^{\mathbb{N}}$ so that $H = E + S$ and so $H/E \cong S/(S \cap E) \cong \mathbb{Z}(p)^{\mathbb{N}}/\mathbb{Z}(p)^{(\mathbb{N})}$. This group is a quotient of a GF(p)-vector space of dimension $2^{\aleph_0}$ modulo a vector subspace of dimension $\aleph_0$ and thus is isomorphic to $\mathbb{Z}(p)^{\mathbb{N}}$. So we have seen that $\mathbf{G}$ has a nondivisible quotient and therefore is nondivisible.

The character group G of $\mathbf{G}$ is torsion free. This implies that μ_p^G is injective, and so $\mu_p^{\mathbf{G}} = (\mu_p^G)\widehat{}$ has a dense image. ☐

4. Locally compact abelian p-groups of finite rank

Let us give an ad-hoc definition of the p-rank of a locally compact abelian p-group G.

Definition 4.1. For a compact abelian p-group C let its p-rank be the minimal cardinality of a subset S of C topologically generating C. When G is an arbitrary locally compact abelian p-group then its p-rank will be defined as

$$\operatorname{rank}_p(G) := \sup\{\operatorname{rank}_p(C) : C \text{ compact subgroup of } G\}.$$

A few remarks and examples may illuminate this notion and connect it with rank definitions from the literature.

Remark 4.2.

(a) For G any discrete abelian p-group our definition agrees with the one given in [Fuc73, page 85]. Note that Fuchs considers every abelian p-group in a natural fashion as a discrete $\mathbb{Z}_p$-module.

(b) When G is a finite abelian p-group then $\operatorname{rank}_p(G) = \dim_{F_p} G/pG$, that is, $\operatorname{rank}_p(G)$ is just the minimal number of generators of G.

(c) For an abelian compact p-group, i.e., a *pro-p* group G it turns out that

$$\operatorname{rank}_p(G) = \operatorname{rank}_p(G/pG).$$

Our definition of p-rank agrees then with the definition of *rank* of the pro-p group G as given in [RZ10, page 90] for *free pro-p groups.*

In particular, for a cardinal $\aleph$, the p-rank of $\mathbb{Z}_p^{\aleph}$ agrees with the one of $\mathbb{Z}(p)^{\aleph}$ and amounts to $\aleph$.

We have used *this* definition of p-rank during Section 2.

(d) For natural numbers k and m the groups

$$\mathbb{Q}_p^k \text{ and } \mathbb{Z}(p^\infty)^m$$

have respectively p-ranks equal to k and m.

(e) Whenever H is a closed subgroup of the locally compact abelian p-group G then

$$\operatorname{rank}_p(H) \le \operatorname{rank}_p(G) \quad \text{and} \quad \operatorname{rank}_p(G/H) \le \operatorname{rank}_p(G).$$

These statements are direct consequences of the definition of p-rank.

We now prove a closedness result for the maximal divisible subgroup of locally compact abelian p-groups of finite rank.

Lemma 4.3. *Let G be a locally compact abelian p-group of finite p-rank with its divisible subgroup D torsion-free. Then D is closed and is algebraically and topologically isomorphic to $\mathbb{Q}_p^k$ for some natural number k.*

Proof. Since D is torsion-free by assumption, it is algebraically isomorphic to a direct sum of groups isomorphic to $\mathbb{Q}$. We shall inductively construct a finite number of closed subgroups $Q_i \cong \mathbb{Q}_p$ of G, all contained in D, and

$$D = \bigoplus_{i=1}^{k} Q_i.$$

Provided this is done, it will follow that D is closed.

Indeed, given an open compact subgroup U of G, note first that $Q_i \cap U$ is closed in U, and so is

$$D \cap U = \bigoplus_{i=1}^{k} Q_i \cap U.$$

Starting our inductive proof, remark that if $D = \{0\}$, we are done. Suppose now that the in G closed subgroup

$$W_l := \bigoplus_{i=1}^{l} Q_i$$

inside D has already be found. If $D = W_l$ then $k = l$ and we are done. Else there must be a subgroup $X \cong \mathbb{Q}$ of D not contained in W_l. Using Lemma 4.3 let us pass to the closure $\bar{X}$ and find that $Q_{l+1} := \bar{X} \cong \mathbb{Q}_p$ is contained in D. Our proof is finished, once we show that $Q_{l+1} \cap W_l = \{0\}$. Suppose that there is

$$x \in Q_{l+1} \cap W_l.$$

Then, as Q_{l+1} and W_l are divisible, there are suitable elements $y \in Q_{l+1} \setminus W_l$ and $w \in W_l \setminus Q_{l+1}$ such that

$$x = p^\mu y = p^\nu w,$$

for natural numbers $\mu \geq 1$ and $\nu \geq 1$. We can choose x and w such that $\mu + \nu$ is minimal. Then the resulting equality

$$p(p^{\mu-1} y - p^{\nu-1} w) = 0$$

and the torsion freeness of D imply that either $y \in \overline{\langle w \rangle}$ or $w \in \overline{\langle y \rangle}$, both a contradicton to the choice of w and y. $\square$

For locally compact abelian p-groups of finite rank we have a complete description, which we present now. Our next result is due to V. S. Čarin, see [Čar66, Theorem 5].

Proposition 4.4. *A locally compact abelian p-group G is of finite p-rank iff it is of the form $G = \mathbb{Q}_p^k \oplus \mathbb{Z}(p^\infty)^m \oplus \mathbb{Z}_p^n \oplus F$ for some nonnegative integers k, m, n and a finite p-group F, and the p-rank of G is precisely*

$$\operatorname{rank}_p G = k + m + n + \operatorname{rank}_p(F).$$

Proof. Assume first that G has the form

$$G = \mathbb{Q}_p^k \oplus \mathbb{Z}(p^\infty)^m \oplus \mathbb{Z}_p^n \oplus F$$

with k, m, n nonnegative integers and F a finite p-group. Since passing to a subgroup or a quotient does not increase the rank, we find that

$$\operatorname{rank}_p G = k \operatorname{rank}_p \mathbb{Q}_p + m \operatorname{rank}_p \mathbb{Z}(p^\infty) + n \operatorname{rank}_p \mathbb{Z}_p + \operatorname{rank}_p F.$$

Since the p-groups $\mathbb{Q}_p$, $\mathbb{Z}(p^\infty)$ and $\mathbb{Z}_p$ have rank 1 and F is finite, it follows that G has finite rank equal to $k + m + n + \operatorname{rank}_p F$.

Conversely, assume now that G is a locally compact abelian p-group of finite p-rank. We tacitly shall make use of Remark 4.2 in the sequel, in particular the fact that the p-rank of a closed subgroup of a finite p-rank group, as well as any factor group, is finite.

Claim: We claim first that the torsion subgroup $T := \operatorname{tor}(D)$ of the maximal divisible subgroup D is closed and that $G = T \oplus R$ for a closed subgroup R of G topologically and algebraically.

Indeed, for some index set I one has

$$T = \bigoplus_{i \in I} P_i$$

with each $P_i \cong \mathbb{Z}(p^\infty)$. Let U be a compact open subgroup of G. Then $\overline{U \cap T}$ is a finite rank compact group. Therefore its torsion subgroup has finite rank as well so that $U \cap T$ turns out to be finite, whence it is closed. Thus T itself is a closed subgroup of G. Then T is a discrete subgroup of G and hence there is a compact open subgroup, say V, with $T \cap V = \{0\}$.

By a Theorem of Baer, see [Fuc73, Theorem 21.3], T is a direct summand of G and, in a decomposition

$$G = T \oplus R$$

one can stipulate R to contain the open subgroup V. It follows that R is an open subgroup, and hence closed, so that the direct decomposition is algebraic and topological.

From now on we may assume that the maximal divisible subgroup D of G is torsion-free. Then, as G has finite rank, Lemma 4.3 implies that D is closed. The factor group $R := G/D$ is a finite rank reduced locally compact abelian p-group. It contains an open compact subgroup, say U. Since R/U has finite rank and is discrete, conclude that R itself is compact of finite rank. Therefore, see [RZ10, Theorem 4.3.4], there are a nonnegative integer n and a finite subgroup F of R such that

$$R \cong \mathbb{Z}_p^n \oplus F.$$

As G admits, as an abstract group, a decomposition

$$G = D \oplus S$$

with S abstractly isomorphic to $R = \mathbb{Z}_p^n \times F$, we deduce that there is a finite subgroup of G, we again denote it by F, with $D \cap F = \{0\}$. Since F is finite, it is closed in G and hence $L := D + F$ is a closed subgroup of G, topologically and algebraically isomorphic to $D \oplus F$. Let X be a minimal set of topological generators of $G/L \cong \mathbb{Z}_p^n$. Lift it to a subset $\tilde{X}$ of G of the same cardinality n. The topologically generated $\mathbb{Z}_p$-submodule $\overline{\langle \tilde{X} \rangle}$ maps then onto G/L.

Claim that $\overline{\langle \tilde{X} \rangle} \cap L = \{0\}$. Indeed, if

$$g = \sum_{\tilde{x} \in \tilde{X}} \lambda_{\tilde{x}} \tilde{x} = d + f$$

with $\lambda_{\tilde{x}} \in \mathbb{Z}_p$, $d \in D$, and, $f \in F$ belongs to $\overline{\langle \tilde{X} \rangle} \cap (D \oplus F)$ then, passing to factor group G/L implies in G/L the relation

$$\sum_{\tilde{x}} \lambda_{\tilde{x}} (\tilde{x} + L) = 0.$$

Since $X = \tilde{X} + L/L$ is a basis of the free $\mathbb{Z}_p$-module $G/L = \bigoplus_{x \in X} \mathbb{Z}_p$, deduce for every $\tilde{x} \in \tilde{X}$ that $\lambda_{\tilde{x}} = 0$. Therefore $g = 0$ showing that

$$G = (D \oplus F) \oplus \overline{\langle \tilde{X} \rangle} = D \oplus F \oplus \mathbb{Z}_p^n. \tag{4.1}$$

□

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