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The Pseudo-Hyperresolution and Applications

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The Pseudo-hyperresolution and Applications

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Abstract

Resolving objects in an abelian category by injective (projective) resolutions is a fundamental problem in mathematics, and this article aims at introducing a particular solution called "Pseudo-hyperresolutions". This method originates in the category of unstable modules to study the minimal resolution of the reduced singular cohomology of spheres. In particular, for all integers $n \geq 0$, we can describe a large range of the minimal injective resolution of the sphere S^n based on the Bockstein operation of the Steenrod algebra. Moreover, many classical constructions in algebraic topology, such as the algebraic EHP sequence or the Lambda algebra can be recovered using the Pseudo-hyperresolution method. A particular connection between spheres and the projective spaces is also established. Despite its origin, Pseudo-hyperresolutions generalize to all abelian categories. In particular, many classical construction of injective resolutions of strict polynomial functors can be reunified in view of Pseudo-hyperresolutions. As a consequence, we recover the global dimension of the category of homogeneous strict polynomial functors of finite degree as well as the Mac Lane cohomology of finite fields.

1 Introduction

As their name suggests, unstable modules are an important tool in studying unstable homotopy theory, and in particular, the homotopy groups of topological spaces. Among all the interesting spaces in topology, the spheres are the most fundamental and important. However, the homotopy groups of spheres are a hugely intractable object even though their cohomology groups are elementary. One of the most powerful tool in studying the homotopy groups of spheres is the unstable Adams spectral sequence - as it was introduced by Massey and Peterson in [20], generalized by Bousfield and Curtis in [4], and generalized further by Bousfield and Kan in [5] - which passes from homological information to homotopical information. In fact, it reduces the computations of the homotopy groups of spheres to that of certain Ext-groups in the category $\mathcal{U}$ of unstable modules:

$$E_2^{s,t}(S^n) = \mathrm{Ext}^s_{\mathcal{U}}\left(\tilde{\mathrm{H}}^*(S^n;\mathbb{Z}/p), \tilde{\mathrm{H}}^*\left(S^t;\mathbb{Z}/p\right)\right) \Longrightarrow \pi_{t-s}(S^n)^\wedge_p.$$

Therefore, the understanding of the minimal injective resolution of $\tilde{\mathrm{H}}^*(S^t;\mathbb{Z}/p)$ is of vital importance, and is one of the primary goals of this article. Our approach to this problem relies on the simple structure of $\tilde{\mathrm{H}}^*(S^t;\mathbb{Z}/p)$: it is an $\mathbb{N}-$graded $\mathbb{F}_p-$vector space, concentrated in degree t and isomorphic to $\mathbb{F}_p$ in that degree, thus is denoted by $\Sigma^t\mathbb{F}_p$. Moreover, the tensor product with $\Sigma^t\mathbb{F}_p$ defines an exact functor $\Sigma^t : \mathcal{U} \to \mathcal{U}$, called the $t-$th suspension functor (we simply write Σ when $t = 1$ and call it the suspension functor). This suggests that the construction of the minimal injective resolution of $\Sigma^t\mathbb{F}_p$ should be carried out by induction on t, and this is what we are going to do. In fact, as the suspension functor Σ is exact and $\Sigma(\Sigma^t) = \Sigma^{t+1}$, then by applying this functor to the minimal injective resolution of $\Sigma^t\mathbb{F}_p$ we obtain a resolution of $\Sigma^{t+1}\mathbb{F}_p$, which is no longer injective. However, each member of this resolution is the suspension of an injective unstable module, which has a simple injective resolution induced by the Mahowald exact sequences. It turns out that these injective resolutions suffice to construct the minimal injective resolution for $\Sigma^{t+1}\mathbb{F}_p$ with the help of the *Pseudo-hyperresolution* method, which is the solution to the following general question.

Question 1. Let $\mathcal{C}$ be an abelian category with enough injective objects and let $M \in \mathcal{C}$. Resolving M by a resolution $\left(A^k, \partial^k\right)_{k\geq 0}$. For each $k \geq 0$, given an injective resolution of A^k, then is it possible to construct an injective resolution of M from this collection of injective resolutions?

The pseudo-hyperresolution method, as it will be introduced in Section 4, plays an essential role in the present paper. An easy way to describe the pseudo-hyperresolution method is to use the Poincaré power series of (co)chain complexes. That is, given a (co)chain complex $\left(A^k, \partial^k\right)_{k\geq 0}$, we define its power series $\mathrm{P}\left(A^k, \partial^k\right)$ as

$$\mathrm{P}\left(A^k, \partial^k\right) := \sum_{k=0}^{\infty} t^k \left[A^k\right],$$

where $\left[A^k\right]$ denotes the corresponding class of A^k in the Grothendieck group of $\mathcal{C}$. The answer to Question 1 now goes as follows.

Lemma 4.11 (**Pseudo-hyperresolution**). *Denote by $\left(I^{k,\bullet},\partial^{k,\bullet}\right)$ the given injective resolution of A^k for all integers $k\geq 0$. Then, there exists an injective resolution $\left(J^{\bullet},\delta^{\bullet}\right)$ of M such that*

$$\mathrm{P}\left(J^{\bullet},\delta^{\bullet}\right)=\sum_{k=0}^{\infty}t^k\mathrm{P}\left(I^{k,\bullet},\partial^{k,\bullet}\right).$$

Note that, when the length of the (co)chain complex in play is finite, the resulting pseudo-hyperresolution is well-known to algebraists: this resolution is identical to the one obtained using the successive cone technique. However, for infinite-length complexes, the mapping telescope technique [3] yields more complicated resolutions than those induced by the pseudo-hyperresolution method.

Before explaining why the pseudo-hyperresolution method is useful in studying unstable modules, let recall some basic facts about these objects. The mod 2 Steenrod algebra is the quotient of the free associative unital graded $\mathbb{F}_2-$algebra generated by the symbols Sq^k of degree $k\geq 1$, subject to the Adem relations. An unstable module M is an $\mathbb{N}-$graded module over the Steenrod algebra such that for all elements $x\in M$ of degree n, then $Sq^kx=0$ for all integers $k>n$. The category $\mathcal{U}$ of unstable modules is abelian and has enough injectives. The Brown-Gitler modules $J(n),n\geq 0$, representing object of the functor $M\mapsto \mathrm{Hom}_{\mathbb{F}_2}\left(M^n,\mathbb{F}_2\right)$ from $\mathcal{U}$ to the category of $\mathbb{F}_2-$vector spaces, form a system of injective cogenerators for $\mathcal{U}$. Moreover, Brown-Gitler modules are connected by the short Mahowald exact sequences, which amounts to saying that the suspension $\Sigma J(2k)$ is again an injective unstable module as it is isomorphic to $J(2k+1)$, whereas $\Sigma J(2k+1)$ is of injective dimension 1 and its minimal injective resolution is given by $J(2k+2)\twoheadrightarrow J(k+1)$, which is induced by the Steenrod operation Sq^{k+1}. Therefore, in view of pseudo-hyperresolution, if the minimal injective of an unstable module is explicit, then so is that of its suspension. For instance, the reduced singular cohomology of spheres and that of the infinite projective spaces are such objects. Let discuss about the former first. Recall that a morphism between two Brown-Gitler modules $J(n)$ and $J(m)$ is determined by a Steenrod operation θ of degree $n-m$, and we denote this morphism by $\bullet\theta$. As the square power of Sq^1 is trivial, then the sequence $\left(J(n),\bullet Sq^1\right)_{n\geq 1}$ is a complex, which is quite close to being exact.

Proposition 2.6. *Let $n\geq 1$ be an integer. Then, the sequence*

$$J(n+2)\oplus\mathscr{J}(n+2)\xrightarrow{\begin{pmatrix}\bullet Sq^1 & 0\\ \bullet Sq^{\frac{n+2}{2}} & 0\end{pmatrix}}J(n+1)\oplus\mathscr{J}(n+1)\xrightarrow{\begin{pmatrix}\bullet Sq^1 & 0\\ \bullet Sq^{\frac{n+1}{2}} & 0\end{pmatrix}}J(n)\oplus\mathscr{J}(n)$$

is exact, where

$$\mathscr{J}(n)=\begin{cases}J(2k) & \text{if } n+1=4k,\\ 0 & \text{otherwise.}\end{cases}$$

It turns out that this particular long exact sequence agrees with the minimal injective resolution of $\Sigma^t\mathbb{F}_2$ in a large range:

Theorem 8.3. *Let $n,s,t\geq 0$ be three integers such that $s>[n/2]$, where $[-]$ denotes the integral part of a number. Then, we have*

$$\mathrm{Ext}^s_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2,\Sigma^t\mathbb{F}_2\right)\cong\begin{cases}\mathbb{F}_2 & \text{if } n=t-s,\\ \mathbb{F}_2 & \text{if } t-s-1\equiv 0(4) \text{ and } t-s-1=2n,\\ 0 & \text{otherwise.}\end{cases}$$

In other words, in this case, the $s-$th term of the minimal injective resolution of $\Sigma^t\mathbb{F}_2$ is isomorphic to

$$J(t-s)\oplus\mathscr{J}(t-s).$$

An alternative way to compute $\mathrm{Ext}^s_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2,\Sigma^t\mathbb{F}_2\right)$ relies on the fact that $\mathrm{Hom}_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2,J(k)\right)$ is isomorphic to $\mathbb{F}_2$ if $k=n$ and is trivial otherwise. Therefore, $\mathrm{Ext}^s_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2,\Sigma^t\mathbb{F}_2\right)$ is isomorphic to $\mathbb{F}_2^{\oplus d}$, where d is the number of copies of direct summands of the form $J(n)$ in the $s-$th term of the minimal injective resolution of $\Sigma^t\mathbb{F}_2$. Because the pseudo-hyperresolution method passes information from the minimal injective resolution of $\Sigma^t\mathbb{F}_2$ to that of the minimal injective resolution of $\Sigma^{t+1}\mathbb{F}_2$, then by using Mahowald short exact sequences - which is the algebraic analogue of the James fibrations - we recover the algebraic EHP sequence.

Theorem 9.4. *There exists a long exact sequence*

$$E_2^{s-2,t}(S^{2n+1})\xrightarrow{P}E_2^{s,t}(S^n)\xrightarrow{E}E_2^{s,t+1}(S^{n+1})\xrightarrow{H}E_2^{s-1,t}(S^{2n+1})$$

for all integers $n\geq 0$ and $s\geq 2$, where $E_2^{s,t}(S^n)$ stands for $\mathrm{Ext}^s_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2,\Sigma^t\mathbb{F}_2\right)$.

In fact, the algebraic EHP sequence exists in a much more general form. Recall that the suspension functor Σ admits a left adjoint, denoted by Ω. Having observed that the left-derived functors of Ω, denoted by Ω_s for all integers $s \geq 0$, are trivial in homological degrees greater than 1, Bousfield showed that there existed an exact sequence

$$\mathrm{Ext}_{\mathcal{U}}^{s-2}(\Omega_1 M, N) \to \mathrm{Ext}_{\mathcal{U}}^{s}(\Omega M, N) \to \mathrm{Ext}_{\mathcal{U}}^{s}(M, \Sigma N) \to \mathrm{Ext}_{\mathcal{U}}^{s-1}(\Omega_1 M, N)$$

for all integers $s \geq 2$ and all unstable modules M, N. His proof might be well-known to experts, but is not available in the literature. (As pointed out to the author by Hans-Werner Henn, only a compact explanation can be found in [22].) Following Singer's instruction [26], we also recover Bousfield's proof. Thanks to the generous permission of Bousfield, we give a detailed account of his approach in Section 10.

Another application of the pseudo-hyperresolution method that we discuss in this article is the construction of the Lambda algebra. In stead of studying the minimal injective resolution, we construct for each integer $t \geq 0$ a particular injective resolution of $\Sigma^t \mathbb{F}_2$. These resolutions fit together into a direct systems of which the limit is endowed with the structure of a bigraded differential algebra, which is isomorphic to the Lambda algebra. This construction can then be considered as a *special injective resolution of* $\Sigma^\infty \mathbb{F}_2$.

Proposition 6.14. *There exists a bigraded differential algebra* (Λ, d) *such that*

$$\mathrm{H}^{r,s}(\Lambda, d) \cong \mathrm{Ext}_{\mathcal{A}}^{r}(\mathbb{F}_2, \Sigma^s \mathbb{F}_2).$$

We now turn to the case of the reduced singular cohomology of the infinite complex projective space. Recall that it is isomorphic to the augmentation ideal of the polynomial algebra on one variable of degree 2, then we can resolve it by the following resolution

$$\bar{\mathrm{H}} \to \Sigma^1 \mathrm{H} \to \cdots \to \Sigma^{t+k} \mathrm{H} \to \Sigma^{t+k+1} \mathrm{H} \to \cdots, \tag{1.1}$$

where $\bar{\mathrm{H}}$ stands for $\tilde{\mathrm{H}}(B\mathbb{Z}/2; \mathbb{F}_2)$ and H stands for $\mathrm{H}^*(B\mathbb{Z}/2; \mathbb{F}_2)$. As a consequence, we obtain the spectral sequence

$$E_2^{q,k} := \begin{cases} \mathrm{Ext}_{\mathcal{U}}^{q}\left(\Sigma^n \mathbb{F}_2, \Sigma^{t+k} \mathrm{H}\right) & \text{if } k \geq 1, \\ \mathrm{Ext}_{\mathcal{U}}^{q}\left(\Sigma^n \mathbb{F}_2, \Sigma^{t} \bar{\mathrm{H}}\right) & \text{if } k = 0. \end{cases} \Longrightarrow \mathrm{Ext}_{\mathcal{U}}^{q+k}\left(\Sigma^n \mathbb{F}_2, \Sigma^t \tilde{\mathrm{H}}^*(\mathbb{C}P^\infty; \mathbb{F}_2)\right). \tag{1.2}$$

Moreover, because of the characterization of injective unstable modules (see, *e.g.*, [18]), then the tensor product of an injective resolution of $\Sigma^k \mathbb{F}_2$ with H (or $\bar{\mathrm{H}}$) yields an injective resolution of $\Sigma^k \mathrm{H}$ (or $\Sigma^k \bar{\mathrm{H}}$). This particular relation at the level of resolutions show that the spectral sequence (1.2) collapses at the E_2-term, giving rise to the isomorphism:

Theorem 11.3. *For all integers* $s \geq 0$*, we have*

$$\mathrm{Ext}_{\mathcal{U}}^{s}\left(\Sigma^n \mathbb{F}_2, \Sigma^t \tilde{\mathrm{H}}^*(\mathbb{C}P^\infty; \mathbb{F}_2)\right) \cong \bigoplus_{m=1}^{s} \mathrm{Ext}_{\mathcal{U}}^{s-m}\left(\Sigma^n \mathbb{F}_2, \Sigma^{t+m} \mathbb{F}_2\right). \tag{1.3}$$

The intrigued readers might wonder what is the interest in studying such a relation. Here is the reason. Because, on the one hand, there exists a spectral sequence

$$E_2^{s,t} := \mathrm{Ext}_{\mathcal{U}}^{s}\left(\Sigma^n \mathbb{F}_2, \Sigma^t \tilde{\mathrm{H}}^*(\mathbb{C}P^\infty; \mathbb{F}_2)\right) \Longrightarrow \pi_{t-s}\left(\mathrm{Map}_*\left(\mathbb{C}P^\infty, \hat{S^n}\right)\right),$$

where $(-)^\wedge$ denotes the profinite completion of a topological space, and on the other hand, the mapping space $\mathrm{Map}_*\left(\mathbb{C}P^\infty, \hat{S^n}\right)$ is contractible (see, *e.g.*, [29, 21]), then the isomorphism (1.3) is of interest as it might give new information about the groups $\mathrm{Ext}_{\mathcal{U}}^{s}(\Sigma^n \mathbb{F}_2, \Sigma^t \mathbb{F}_2)$.

Despite its origin, the pseudo-hyperresolution method generalizes to all abelian categories. Of particular interest is the category of strict polynomial functors, which is closely related to the category of unstable modules (see, *e.g.*, [16, 23]). Typical examples of strict polynomial functors are given by the symmetric powers S^n, the exterior powers Λ^n, and the divided power Γ^n for all integers $n \geq 0$. The category $\mathcal{P}_{\mathbb{F}_2}$ is well-known for its computability as the injective and the projective objects are well understood. Each injective strict polynomial functor is isomorphic to a direct sum of functors of the form $S^{\lambda_1} \otimes S^{\lambda_2} \otimes \ldots \otimes S^{\lambda_k}$, and each projective one is isomorphic to a direct sum of functors of the form $\Gamma^{\lambda_1} \otimes \Gamma^{\lambda_2} \otimes \ldots \otimes \Gamma^{\lambda_k}$, where $(\lambda_1, \lambda_2, \ldots, \lambda_k) \in \mathbb{N}^{\times k}$. Now, the pseudo-hyperresolution method finds its place in the category of strict polynomial functors because of the Koszul exact sequences, which connect the exterior power functors to the symmetric and the divided ones:

$$0 \longrightarrow \Lambda^n \longrightarrow \Lambda^{n-1} \otimes S^1 \longrightarrow \Lambda^2 \otimes S^2 \longrightarrow \cdots \longrightarrow \Lambda^1 \otimes S^{n-1} \to S^n \longrightarrow 0$$

$$0 \longrightarrow \Gamma^n \longrightarrow \Gamma^{n-1} \otimes \Lambda^1 \longrightarrow \Gamma^2 \otimes \Lambda^2 \longrightarrow \cdots \longrightarrow \Gamma^1 \otimes \Lambda^{n-1} \to \Lambda^n \longrightarrow 0.$$

Therefore, an easy induction on d using the pseudo-hyperresolution method allows to construct for each integer $d \geq 1$ a certain injective resolution of Γ^d. These resolutions allow to show that the injective dimension of a projective strict polynomial functor of degree d is bounded by $2d - 2k$, where k is the number of nontrivial $2-$adic digits of d. This concludes the global dimension of the category $\mathcal{P}_{\mathbb{F}_2,d}$.

Theorem 5.19. *Let $d \geq 1$ be an integer and let $\sum_{i=1}^{k} 2^{n_k}$ be its $2-$adic expression. Then, the global dimension of $\mathcal{P}_d$ is $2d - 2k$.*

One of the most fundamental notion in the theory of strict polynomial functors is the Frobenius twist. Let $I = \Gamma^1$, then we define the Frobenius twist of I, denoted by $I^{(1)}$, as the strict polynomial functor that associates an $\mathbb{F}_2-$vector space V to the $\mathbb{F}_2-$vector space $V^{(1)}$, which is obtained from V by base change along the Frobenius map $\mathbb{F}_2 \to \mathbb{F}_2, x \mapsto x^2$. Then, we define the Frobenius twist of a functor F, denoted by $F^{(1)}$, as the precomposition $F \circ I^{(1)}$. Recursively, we define $F^{(r)}$ as the Frobenius twist of $F^{(r-1)}$. As the Frobenius twist is exact, then it induces the morphism

$$\mathrm{Ext}^*_{\mathcal{P}_{\mathbb{F}_2}}\left(I^{(r)}, I^{(r)}\right) \to \mathrm{Ext}^*_{\mathcal{P}_{\mathbb{F}_2}}\left(I^{(r+1)}, I^{(r+1)}\right) \tag{1.4}$$

for all integers $r \geq 0$. The colimit of $\mathrm{Ext}^*_{\mathcal{P}_{\mathbb{F}_2}}\left(I^{(r)}, I^{(r)}\right)$ with respect to the direct system induced by the morphisms 1.4 is of interest as it is isomorphic to the Mac Lane cohomology $\mathrm{HML}^*(\mathbb{F}_2)$ of the field $\mathbb{F}_2$ (see, *e.g.*, [15]). It follows that the computation of $\mathrm{HML}^*(\mathbb{F}_2)$ can be deduced from that of $\mathrm{Ext}^*_{\mathcal{P}_{\mathbb{F}_2}}\left(I^{(r)}, I^{(r)}\right)$ for all integers $r \geq 0$, which, in turn, can be carried out by constructing injective resolutions of $I^{(r)}$. Note that the Frobenius twist of an injective object of $\mathcal{P}_{\mathbb{F}_2}$ is no longer injective, but admits the following injective resolution:

$$0 \to S^{d(1)} \to S^{2d} \to S^{2d-1} \otimes S^1 \to S^{2d-2} \otimes S^2 \to \cdots \to S^1 \otimes S^{2d-1} \to S^{2d} \to 0.$$

Therefore, with the help of pseudo-hyperresolution method, we can show that

$$\mathrm{Ext}^k_{\mathcal{P}_{\mathbb{F}_2}}\left(I^{(r)}, I^{(r)}\right) \cong \begin{cases} \mathbb{F}_2 & \text{if } 2|k, \text{ and } k \leq 2^{r+1} - 2, \\ 0 & \text{otherwise,} \end{cases}$$

which allows recovery of the Mac Lane cohomology of $\mathbb{F}_2$:

Theorem 5.10. *We have*

$$\mathrm{HML}^k(\mathbb{F}_2, I) \cong \begin{cases} \mathbb{F}_2 & \textit{if } 2|k, \\ 0 & \textit{otherwise.} \end{cases}$$

Note that, most of the methods we use throughout this article works for all prime characteristics. Some particular methods are only treated in characteristic 2, but there is no difficulty extending to all prime characteristics. However, special care is required in dealing with odd prime characteristics. Therefore, they will be studied separately in a subsequent article. Moreover, when it comes to concrete examples and recalls, we always restrict the attention to characteristic 2 for ease of exhibition.

The paper is organized as follows:

Section 2 recalls basic facts about the Steenrod algebra and unstable modules. We also give a brief account of the Brown-Gitler modules and show how to fit them together into the long exact sequence that we call the *Bockstein sequence*.

Section 3 introduces the notion of a graph representation of complexes, which facilitates the presentation of our exposition.

Section 4 is at the heart of the present paper as it settles down the notion of a pseudo-hyperresolution. We begin this section with a simple example that motivates our study, and generalize the idea to the most general context.

Section 5 aims at providing applications of the pseudo-hyperresolution method in functor homology. It is in this section that we recover the Mac Lane cohomology of finite fields as well as the global dimension of the category of homogeneous strict polynomial functors of finite degree.

Section 6 covers the construction of the Lambda algebra. It will be carried out with the help of the graph representation of complexes that we introduce in Section 3.

Sections 7 and 8 deal with the minimal injective resolution of the reduced singular cohomology of spheres. This will be done by studying the Bockstein sequence introduced in Section 2.

Section 9 provides our approach to the algebraic EHP sequence and Section 10 recalls that of Bousfield.

Section 11 studies the minimal injective resolution of the infinite complex projective space and its relation to that of spheres.

2 The Steenrod algebra and Brown-Gitler modules

The Steenrod algebra. The mod 2 Steenrod algebra $\mathcal{A}_2$ is the quotient of the free associative unital graded $\mathbb{F}_2-$algebra generated by the symbols Sq^k of degree $k \geq 0$, subject to the Adem relations (see [27, 24]). An $\mathbb{N}-$graded $\mathcal{A}_2-$module M is called unstable if $Sq^k x = 0$ for all $x \in M^n$ and all $k > n$. We denote by $\mathcal{U}$ the category of unstable modules. Serre [25] introduced the notions of *admissible* and *excess*. A monomial

$$Sq^{i_1} Sq^{i_2} \dots Sq^{i_k}$$

is called admissible if $i_j \geq 2i_{j+1}$ for all $k-1 \geq j \geq 1$ and $i_k \geq 1$; the excess of this operation is defined by

$$e\left(Sq^{i_1} Sq^{i_2} \dots Sq^{i_k}\right) = 2i_1 - \left(\sum_{j=1}^{k} i_j\right).$$

The set of admissible monomials and Sq^0 is an $\mathbb{F}_2-$basis of $\mathcal{A}_2$.

Projective unstable modules. Recall that the functor that associates an unstable module with the $\mathbb{F}_2-$vector space of its elements of degree n is representable and we denote by $F(n)$ the representing unstable module. Hence, $F(n)$ is freely generated by an element $\imath_n$ of degree n. Therefore, if M is an unstable module, then the morphism

$$\bigoplus_n \bigoplus_{x \in M^n} F(n) \to M, \quad \imath_n \mapsto x$$

is surjective. It follows that the modules $F(n), n \geq 0$, form a system of projective generators of $\mathcal{U}$.

Brown-Gitler modules. The category $\mathcal{U}$ also has enough injective objects. The Brown-Gitler module $J(n)$ is the unstable injective hull of the mod 2 reduced cohomology $\Sigma^n \mathbb{F}_2$ of the sphere S^n. It is a cocyclic unstable module, cogenerated by an element of degree n. The modules $J(n), n \geq 0$, are injective satisfying $\mathrm{Hom}_{\mathcal{U}}(M, J(n)) \cong (M^n)^\sharp$. Here, $(-)^\sharp$ stands for the $\mathbb{F}_2-$linear dual. Moreover, each $J(n)$ is $\mathcal{U}-$indecomposable. Each homogeneous element $x \in M^n$ determines a morphism $i_x : M \to J(n)$. Because the induced morphism

$$i_M : M \to \prod_n \prod_{x \in M^n} J(n)$$

is injective, the modules $J(n), n \geq 0$, form a system of injective co-generators of $\mathcal{U}$.

A morphism $F(n) \to F(m)$ is determined by a Steenrod operation of degree $n-m$. Since $J(n)^m \cong (F(m)^n)^\sharp$, then a morphism $J(n) \to J(m)$ is also determined by a Steenrod operation of degree $n-m$. Let θ be a Steenrod operation of degree $n-m$, we denote by $\bullet\theta$ the morphism $J(n) \to J(m)$ induced by θ.

The Frobenius twist of unstable modules. Let $|-|$ be the degree of a homogeneous element. Denote by Sq_0 the operation which associates a homogeneous element $x \in M^n$ of an unstable module M with the element $Sq^{|x|}x$. Let Φ be the endofunctor of $\mathcal{U}$ which associates an unstable module M with the module ΦM concentrating in even degrees and $(\Phi M)^{2n} = M^n$. If $x \in M^n$, then we denote by Φx the corresponding element in $(\Phi M)^{2n}$; the action of the Steenrod algebra on ΦM is defined by $Sq^k \Phi x = \Phi Sq^{k/2} x$ (with the convention that $Sq^{k/2} = 0$ if k is not divisible by 2). Let λ_M be the morphism defined by:

$$\begin{aligned} \lambda_M : \Phi M &\to M \\ \Phi x &\mapsto \mathrm{Sq}_0 x. \end{aligned} \tag{2.1}$$

The morphism λ_M is natural in M. There exists a right adjoint $\tilde{\Phi}$ of Φ [24]. Therefore, adjoint to λ_M, there exists a morphism $\tilde{\lambda}_M : M \to \tilde{\Phi} M$, natural in M. An unstable module M is reduced if λ_M is injective (this is slightly different for odd prime characteristic, see, *e.g.*, [24]). It is called nilpotent if for all $x \in M^n$, there exists an integer $n_x \geq 1$ such that $\mathrm{Sq}_0^{n_x} x = 0$. By definition, there is no nontrivial morphism from a nilpotent unstable module to a reduced one.

The suspension functor and Mahowald's short exact sequences. Let Σ be the endofunctor of $\mathcal{U}$ defined by $(\Sigma M)^n = M^{n-1}$ for all unstable modules M and all integers $n \geq 0$. If $x \in M^n$, we denote by Σx the corresponding element in ΣM. The action of the Steenrod algebra on ΣM is determined by $Sq^k \Sigma x = \Sigma Sq^k x$. The suspension functor Σ admits a right adjoint, denoted by $\tilde{\Sigma}$. Let $s : \Sigma\tilde{\Sigma} \to Id$ be the counit of the adjunction, and we shorten $s(J(n))$ to s_n for all integers $n \geq 1$.

Theorem 2.1 (Mahowald's short exact sequences)**.** *The sequence*

$$0 \to \Sigma\tilde{\Sigma} J(n) \xrightarrow{s_n} J(n) \xrightarrow{\tilde{\lambda}_{J(n)}} \tilde{\Phi} J(n) \to 0 \tag{2.2}$$

is exact.

Let $n \geq 0$ be an integer, we fix:

$$J\left(\frac{n}{2}\right) = \begin{cases} k & \text{if } n = 2k, \\ 0 & \text{otherwise,} \end{cases} \quad \text{and} \quad Sq^{\frac{n}{2}} = \begin{cases} Sq^k & \text{if } n = 2k, \\ 0 & \text{otherwise.} \end{cases}$$

Proposition 2.2 ([24])**.** *For all integers $n \geq 1$, there are isomorphisms of unstable modules*

$$\tilde{\Sigma} J(n) \cong J(n-1),$$
$$\tilde{\Phi} J(n) \cong J\left(\frac{n}{2}\right).$$

Moreover $\tilde{\lambda}_{J(n)} = \bullet Sq^{\frac{n}{2}}$.

Recall that $\mathrm{H}^*(B\mathbb{Z}/2; \mathbb{F}_2)$ is isomorphic to the polynomial algebra $\mathbb{F}_2[u]$ on one variable of degree 1, and the action of the Steenrod algebra is given by

$$Sq^n u^k = \binom{k}{n} u^{n+k} \pmod 2.$$

Since $F(1)$ can be identified as the submodule of $\mathrm{H}^*(B\mathbb{Z}/2; \mathbb{F}_2)$ generated by u, then it has an $\mathbb{F}_2-$basis consisting of $u^{2^n}, n \geq 0$. As a result, we have

$$\mathrm{Hom}_{\mathcal{U}}(F(1), J(2^n)) \cong \mathbb{F}_2$$

for all integers $n \geq 0$. Denote by x_n the unique generator in degree 1 of $J(2^n)$, and by $\mathscr{M}$ the bigraded algebra

$$\mathbb{F}_2[x_n, n \geq 0, ||x_n|| = (1, 2^n)].$$

The following theorem, due to Miller, describes the $J(n)$.

Theorem 2.3 ([22])**.** *The morphism*

$$\bigoplus_{n \geq 0} J(n) \to \mathscr{M}, \quad x_n \mapsto x_n.$$

is an $\mathcal{A}_2-$isomorphism of bigraded algebras.

Corollary 2.4. *Let $\sum_{i=1}^{d} 2^{j_i}$ be the $2-$adic expression of n. Then, we have*

$$J(n)^n = \mathbb{F}_2 \langle x_0^n \rangle,$$
$$J(n)^d = \mathbb{F}_2 \left\langle \prod_{i=1}^{d} x_{j_i} \right\rangle,$$
$$J(n)^m = 0 \text{ if either } m > n \text{ or } d > m.$$

The Bockstein long exact sequence. Because the square of the Bockstein operation is trivial, the sequence $\left(J(n), \bullet Sq^1 : J(n+1) \to J(n)\right)^{n \geq 1}$ is a complex, which is close to being exact. In fact, we can modify this complex to obtain an exact sequence.

Definition 2.5 (The Bockstein sequence)**.** Let $(\mathscr{B}_k, \beta_k)$ be the following complex:

$$\mathscr{B}_{-n} = \begin{cases} J(4k-1) \oplus J(2k) & \text{if } n = 4k-1, \text{ and } n > 0, \\ J(n) & \text{if } n \not\equiv 0(4), \text{ and } n > 0, \\ 0 & \text{otherwise.} \end{cases}$$

$$\beta_{-n} = \begin{cases} \begin{pmatrix} \bullet Sq^1 & 0 \end{pmatrix} & \text{if } n = 4k-1, \text{ and } n > 0, \\ \begin{pmatrix} \bullet Sq^1 \\ 0 \end{pmatrix} & \text{if } n = 4k, \text{ and } n > 0, \\ 0 & \text{if } 0 \geq n \\ \bullet Sq^1 & \text{otherwise.} \end{cases}$$

We call this complex *the Bockstein sequence of Brown-Gitler modules.*

Proposition 2.6. *The Bockstein sequence is exact.*

Proof. We observe that an element $x \in J(n)$ belongs to $\mathrm{Ker}\left(\bullet Sq^1 : J(n) \to J(n-1)\right)$ if and only if there exists an admissible Steenrod operation

$$\theta = Sq^{2k_1} Sq^{k_2} \ldots Sq^{k_m}$$

such that $e(\theta) \leq |x|$, $4k_1 < n$, and $\theta x = x_0^n$. Remark that $e(\theta) = |x|$ if and only if $n = 4k_1$. Therefore,

$$e(Sq^1\theta) \leq |x| \text{ if } 4 \nmid n,$$
$$e(Sq^1\theta) > |x| \text{ otherwise.}$$

It means that if n is not divisible by 4, then

$$\operatorname{Ker}(\beta_n) = \operatorname{Im}(\beta_{n-1}).$$

It remains to show that

$$\operatorname{Ker}(\beta_{-4k+1}) = \operatorname{Im}(\beta_{-4k})$$

for all integers $k \geq 1$. This amounts to saying that all element of $J(2k) \subset \mathscr{B}_{4k-1}$ is a co-boundary. But this comes from the surjectivity of the map $\bullet Sq^{2k} : J(4k) \to J(2k)$, which is a direct consequence of Theorem 2.1. $\square$

3 Graph representation of complexes

In this section, we introduce the notion of *graph representations of complexes*. In other words, we show how to associate a complex with an appropriate graph with respect to a certain decomposition.

3.1 Graph representations

In all abelian category, a morphism between two direct sums of objects can be represented by matrix: let

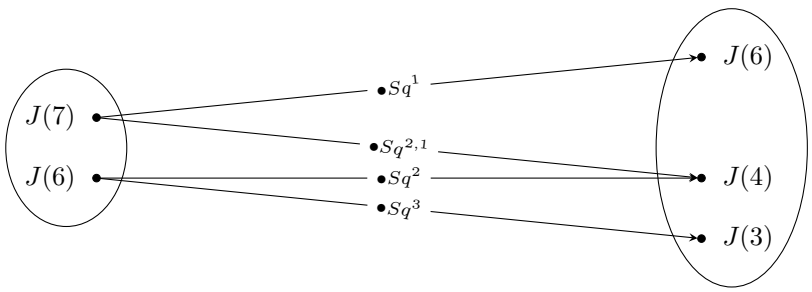
$$f : \bigoplus_{i=1}^{n} M_i \to \bigoplus_{j=1}^{m} N_j$$

be such a morphism, then we represent f as an $m-$by$-n$ matrix $M_f = \{f_{ij}\}$, where f_{ij} denotes the induced morphism $M_j \to N_i$. It is sometimes convenient to respresent such a morphism as a bipartite graph with two disjoint sets of vertices M, N indexed by M_i and N_j respectively, and the edges are determined by the matrix M_f.

Example 3.1. The morphism

$$\begin{pmatrix} \bullet Sq^1 & 0 \\ \bullet Sq^{2,1} & \bullet Sq^2 \\ 0 & \bullet Sq^3 \end{pmatrix} : J(7) \oplus J(6) \longrightarrow J(6) \oplus J(4) \oplus J(3)$$

is represented as follows:



Definition 3.2. Let $\left(C^k, \partial^k\right)$ be a cochain complex. Suppose that each C^k admits a decomposition

$$C^k \cong \bigoplus_{\alpha \in A_k} C^k_\alpha \tag{3.1}$$

and denote by M_{∂^k} the corresponding matrix representing ∂^k. Then, the graph associated with $\left(C^k, \partial^k\right)$ with respect to the decomposition (3.1) is defined as follows:

1. The set of vertices is the disjoint union of V_k indexed by A_k:

$$V_k := \left\{ v^k_\alpha \middle| \alpha \in A_k \right\}.$$

2. The set of edges is the disjoint union of E_k containing edges from V_k to V_{k+1} determined by the matrix M_{∂^k}:

$$E_k := \left\{ \left[v_\alpha^k, v_\beta^{k+1}\right] \middle| 0 \neq \partial_{\beta,\alpha}^k : C_\alpha^k \to C_\beta^{k+1} \right\}.$$

We write $\left[v_\alpha^k, v_\beta^{k+1}\right] = \partial_{\beta,\alpha}^k$, with the convention that $\left[v_\alpha^k, v_\beta^{k+1}\right] = 0$ means there is no edge between v_α^k and v_β^{k+1}.

3.2 The graph representation of resolutions

We now consider some particular complexes of unstable modules and their graph representation.

Definition 3.3 (BG modules and BG complexes)**.** A BG module is a direct sum of Brown-Gitler modules, and a complex of BG modules is called a BG complex.

Definition 3.4. Let $f : \bigoplus_{\alpha\in A} J(n_\alpha) \to \bigoplus_{\beta\in B} J(m_\beta)$ be a morphism between BG modules, and let

$$M_f = \{f_{\beta,\alpha} : J(n_\alpha) \to J(m_\beta)\}$$

be its representing matrix. We define $\bar{f}$ to be the morphism with the same source and target as those of f, but the representing matrix is obtained from that of f by identifying all $f_{\beta,\alpha}$ with 0 whenever $f_{\beta,\alpha}$ is not the identity map $\bullet Sq^0$.

The following lemma is straightforward.

Lemma 3.5. *Let $(I^\bullet, \partial^\bullet)$ be a BG complex, then so is $\left(I^\bullet, \bar{\partial}^\bullet\right)$.*

Definition 3.6. Under the same hypothesis as that of Lemma 3.5, we denote by $G\left(I^\bullet, \bar{\partial}^\bullet\right)$ the associated graph with respect to the decomposition of $I^\bullet$ as direct sum of Brown-Gitler modules. We define $R(I^\bullet, \partial^\bullet)$ to be the bigraded $\mathbb{F}_2-$vector space generated by the vertices of $G\left(I^\bullet, \bar{\partial}^\bullet\right)$, where $v \in V_r\left(G\left(I^\bullet, \bar{\partial}^\bullet\right)\right)$ is of bidegree (r, s) if v is indexed by a direct summand of the form $J(s)$. We call this the *BG degree* of v and denote it by $|v| = (|v|_1, |v|_2)$, where $|v|_1 = r$ and $|v|_2 = s$.

We now show how to relate Ext-groups of unstable modules to the graph representation of resolutions. The following lemma is the key to our construction of the Lambda algebra in Section 6.

Lemma 3.7. *Let M be a finite unstable module. Then, M admits an injective resolution $(I^\bullet, \partial^\bullet)$ of finite length such that each I^k is a finite BG module for all $k \geq 0$. Moreover, we have*

$$\mathrm{H}^{r,s}\left(R\left(I^\bullet, \partial^\bullet\right)\right) \cong \mathrm{Ext}_{\mathcal{U}}^r\left(\Sigma^s\mathbb{F}_2, M\right) \tag{3.2}$$

for all integers $r, s \geq 0$.

Proof. Denote by $d(M)$ the least degree n such that $M^k = 0$ for all $k > n$. The existence of such an injective resolution $(I^\bullet, \partial^\bullet)$ can be proved by induction on the degree $d(M)$. We now show that the isomorphism (3.2) holds. Indeed, for all integers $r, s \geq 0$, we have

$$\begin{aligned}\mathrm{Ext}_{\mathcal{U}}^r\left(\Sigma^s\mathbb{F}_2, M\right) &\cong \mathrm{H}^r\mathrm{Hom}_{\mathcal{U}}\left(\Sigma^s\mathbb{F}_2, (I^\bullet, \partial^\bullet)\right)\\ &\cong \mathrm{H}^r\mathrm{Hom}_{\mathcal{U}}\left(\Sigma^s\mathbb{F}_2, \left(I^\bullet, \bar{\partial}^\bullet\right)\right)\\ &\cong \mathrm{H}^{r,s}\left(R\left(I^\bullet, \partial^\bullet\right)\right),\end{aligned}$$

whence the conclusion. □

4 Pseudo-hyperresolutions

Constructing resolutions is one of the most basic problems in homological algebra. This section aims to study a certain class of objects whose resolutions can be made explicit. We begin with some elementary examples which are the origin of the present paper.

Recall that the groups $\mathrm{Ext}_{\mathcal{U}}^*\left(\Sigma^m\mathbb{F}_2, \Sigma^n\mathbb{F}_2\right)$ are of interest because of the unstable Adams spectral sequence computing the $2-$component of the homotopy groups of spehres. Therefore, it is natural to search for the minimal injective resolutions of $\Sigma^n\mathbb{F}_2$ in the category $\mathcal{U}$. The cases $n = 0$ and $n = 1$ are trivial since $\Sigma^n\mathbb{F}_2 \cong J(n)$ in these cases. As a result, the Mahowald short exact sequence

$$0 \to \Sigma J(1) \xrightarrow{s_1} J(2) \xrightarrow{\bullet Sq^1} J(1) \to 0 \tag{4.1}$$

is an injective resolution of $\Sigma^2\mathbb{F}_2$. Applying Σ to (4.1) we obtain a resolution of $\Sigma^3\mathbb{F}_2$, which is no longer injective. However, we have the following commutative diagram.

$$\begin{tikzcd}
\Sigma^2 J(1) \arrow[r] & \Sigma J(2) \arrow[r, "\Sigma(\bullet Sq^1)"] \arrow[d] & \Sigma J(1) \arrow[r] \arrow[d] & 0 \\
& J(3) \arrow[r, "\bullet Sq^1"] \arrow[d] & J(2) \arrow[d] & \\
& 0 \arrow[r] & J(1) &
\end{tikzcd} \tag{4.2}$$

Consequently, the total complex of the commutative square (4.2) is an injective resolution of $\Sigma^3\mathbb{F}_2$.

The above examples lead to the following question.

Question 1. Let $\mathcal{C}$ be an abelian category with enough injective objects and let $M \in \mathcal{C}$. Resolving M by a resolution $\left(A^k, \partial^k\right)_{k\geq 0}$. For each $k \geq 0$, given an injective resolution of A^k, then is it possible to construct an injective resolution of M from this collection of injective resolutions?

The answer for this question is stated in Propositions 4.4 and 4.5.

4.1 Refining resolutions

For all integers $n \geq 0$, we construct a resolution $\left(I^k(n), \partial^k(n)\right)$ of M such that:

1. the term $I^k(n)$ is injective for all $k \leq n$, and
2. we have $I^k(n) = I^k(n+1)$ for all $k \leq n$.

Therefore, by letting n tend to infinity, we obtain an injective resolution of M. Before formulating the principle, we need the following key lemma.

Lemma 4.1. *In an abelian category, if the sequence*

$$E_1 \xrightarrow{f_1} E_2 \oplus M \xrightarrow{\left(\begin{smallmatrix} g_1 & g_2 \\ g_3 & id \end{smallmatrix}\right)} E_3 \oplus M \xrightarrow{f_3} E_4$$

is exact at $E_2 \oplus M$ and $E_3 \oplus M$, then the sequence

$$E_1 \xrightarrow{f_1} E_2 \xrightarrow{g_1 + g_2 \circ g_3} E_3 \xrightarrow{f_3} E_4$$

is exact at E_2 and E_3.

Proof. Note that the diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & M \arrow[r, "id"] \arrow[d, "\binom{0}{id}"] & M \arrow[r] \arrow[d, "\binom{g_2}{id}"] & 0 \arrow[d] \\
E_1 \arrow[r, "f_1"] \arrow[d, "id"] & E_2 \oplus M \arrow[r, "\left(\begin{smallmatrix} g_1 & g_2 \\ g_3 & id \end{smallmatrix}\right)"] \arrow[d, "(id,0)"] & E_3 \oplus M \arrow[r, "f_3"] \arrow[d, "(id,g_2)"] & E_4 \arrow[d, "id"] \\
E_1 \arrow[r, "f_1"] & E_2 \arrow[r, "g_1+g_2\circ g_3"] & E_3 \arrow[r, "f_3"] & E_4
\end{tikzcd}$$

is commutative, and each of its columns is exact. As the first two upper rows are exact at the second and the third terms, then so is the last row. □

The argument now goes as follows.

Lemma 4.2. *In an abelian category with enough injective objects, let M be an object and let $\left(M^k, \tau^k, k \geq 0\right)$ be a resolution of M. Suppose that we can resolve each M^k by a resolution*

$$I^k \xrightarrow{p^k} J^k \to 0$$

of length 1 such that I^k is injective. Then, for all integers $k \geq 0$, there exist morphisms

$$\theta^k : I^k \to I^{k+1}, \quad \delta^k : J^k \to I^{k+2}, \quad \omega^k : J^k \to J^{k+1},$$

such that the morphisms $\partial^k := \left(\begin{smallmatrix} \theta^k & \delta^{k-1} \\ p^k & \omega^{k-1} \end{smallmatrix}\right)$ *for all integers* $k \geq 1$*, and* $\partial^0 := \left(\begin{smallmatrix} \theta^0 \\ p^0 \end{smallmatrix}\right)$ *make the sequence*

$$I^0 \xrightarrow{\partial^0} I^1 \bigoplus J^0 \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^t} I^{t+1} \bigoplus J^t \xrightarrow{\partial^{t+1}} \cdots \tag{4.3}$$

a resolution of M*.*

Proof. For all integers $k \geq 0$, denote by i^k the inclusion $M^k \to I^k$. Because I^k is injective for all integers $k \geq 0$, the τ^k gives rise to a morphism $\theta^k : I^k \to I^{k+1}$ such that

$$\theta^k \circ i^k = i^{k+1}\tau^k.$$

As $\theta^{k+1} \circ \theta^k \circ i^k$ is trivial for all integers $k \geq 0$, there exists a morphism $\delta^k : J^k \to I^{k+2}$ such that

$$\delta^k \circ p^k = \theta^{k+1} \circ \theta^k.$$

Denote by

$$\omega^k : J^k \to J^{k+1}$$

the morphism induced by θ^k for all integers $k \geq 0$. Then, $\left(p^k, k \geq 0\right)$ is a morphism of complexes:

$$\left(I^k, \theta^k, k \geq 0\right) \to \left(J^k, \omega^k, k \geq 0\right).$$

Denote by K^l the direct sum $I^l \oplus I^{l+1}$ for all integers $l \geq 1$, and by K^0 the module I^0. Then, the morphisms

$$\beta^0 = \left(\begin{smallmatrix} \theta^0 \\ \theta^1 \circ \theta^0 \end{smallmatrix}\right),$$
$$\beta^l = \left(\begin{smallmatrix} \theta^l & id \\ \theta^{l+1} \circ \theta^l & \theta^{l+1} \end{smallmatrix}\right), \quad l \geq 1,$$

make $\left(K^l, \beta^l : K^l \to K^{l+1}, l \geq 0\right)$ a complex. Denote by H^l the direct sum $I^{l+1} \oplus J^l$ for all integers $l \geq 1$, and denote by H^0 the module J^0. Then, the morphisms

$$\gamma^0 = \left(\begin{smallmatrix} \theta^0 \\ \omega^0 \end{smallmatrix}\right),$$
$$\gamma^k = \left(\begin{smallmatrix} \theta^{k+1} & \delta^k \\ p^{k+1} & \omega^k \end{smallmatrix}\right), \quad k \geq 1,$$

make $\left(H^k, \gamma^k, k \geq 0\right)$ a complex. For all integers $k \geq 1$, we fix:

$$\eta^0 = i^0, \quad \eta^k = \left(\begin{smallmatrix} i^k \\ 0 \end{smallmatrix}\right),$$
$$\chi^0 = p^0, \quad \chi^k = \left(\begin{smallmatrix} 0 & id \\ p^k & 0 \end{smallmatrix}\right).$$

Then, the sequence

$$0 \to M^k \xrightarrow{\eta^k} K^k \xrightarrow{\chi^k} H^k \to 0$$

is exact for all integers $k \geq 0$. Since $\left(\eta^l, l \geq 0\right)$ and $\left(\chi^l, l \geq 0\right)$ are morphisms of complexes we obtain the double complex

$$\begin{array}{ccccccccc} K^0 & \xrightarrow{\beta^0} & K^1 & \xrightarrow{\beta^1} & \cdots & \xrightarrow{\beta^{n-2}} & K^{n-1} & \xrightarrow{\beta^{n-1}} & \cdots \\ \downarrow{\chi^0} & & \downarrow{\chi^1} & & & & \downarrow{\chi^{n-1}} & & \\ H^0 & \xrightarrow{\gamma^0} & H^1 & \xrightarrow{\gamma^1} & \cdots & \xrightarrow{\gamma^{n-2}} & H^{n-1} & \xrightarrow{\gamma^{n-1}} & \cdots \end{array} \tag{4.4}$$

where all the columns are acyclic. As the sequence $\left(M^k, \tau^k : M^k \to M^{k+1}, k \geq 0\right)$ is also acyclic, using standard spectral sequence arguments for double complexes, it is straightforward that the total complex of the double complex (4.4) is acyclic and its only nontrivial cohomology is M. By applying Lemma 4.1 to this acyclic sequence, we obtain the exact sequence (4.3). □

By the same method, we can prove the following generalized version of Lemma 4.2, which does not require complexes to be bounded below.

Corollary 4.3. *In an abelian category with enough injectives, let* $\left(M^k, \tau^k, k \in \mathbb{Z}\right)$ *be a complex. Suppose that we can resolve each* M^k *by a resolution*

$$I^k \xrightarrow{p^k} J^k \to 0$$

of length 1 *such that* I^k *is injective. Then, for all integers* $k \geq 0$*, there exist morphisms*

$$\theta^k : I^k \to I^{k+1}, \quad \delta^k : J^k \to I^{k+2}, \quad \omega^k : J^k \to J^{k+1},$$

such that the morphisms $\partial^k := \begin{pmatrix} \theta^k & \delta^{k-1} \\ p^k & \omega^{k-1} \end{pmatrix}$ *for all integers* $k \geq 1$*, and* $\partial^0 := \begin{pmatrix} \theta^0 \\ p^0 \end{pmatrix}$ *make the sequence*

$$I^0 \xrightarrow{\partial^0} I^1 \bigoplus J^0 \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^t} I^{t+1} \bigoplus J^t \xrightarrow{\partial^{t+1}} \cdots \tag{4.5}$$

a complex. Moreover, the compositions

$$M^k \hookrightarrow I^k \xrightarrow{\binom{id}{0}} I^k \bigoplus J^{k-1}$$

form a quasi-isomorphism of complexes

4.2 The constructions

The following proposition is a strengthening of Lemma 4.2.

Proposition 4.4. *Given an integer* $k \geq 1$*. In an abelian category with enough injective objects, let* M *be an object and let* $\left(M^l, \tau^l, l \geq 0\right)$ *be a resolution of* M*. Suppose that we can resolve each* M^l *by a resolution*

$$I^{l,0} \xrightarrow{\partial^{l,0}} I^{l,1} \xrightarrow{\partial^{l,1}} \cdots \xrightarrow{\partial^{l,k-1}} I^{l,k} \to 0$$

of length k *such that* $I^{l,t}$ *is injective for all* $0 \leq t < k$*. Then, there exist morphisms*

$$\partial_i^{l,t} : I^{l,t} \to I^{l+i+1,t-i}$$

such that the morphisms

$$\partial^l = \begin{pmatrix} \partial^{0,l} & 0 & \cdots & 0 \\ \partial_0^{0,l} & \partial^{1,l-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{l-1}^{0,l} & \partial_{l-2}^{1,l-1} & \cdots & \partial^{l,0} \\ \partial_l^{0,l} & \partial_{l-1}^{1,l-2} & \cdots & \partial_0^{l,0} \end{pmatrix}$$

make the sequence

$$0 \to M \to I^{0,0} \xrightarrow{\partial^0} I^{0,1} \bigoplus I^{1,0} \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^l} \bigoplus_{\substack{m+n=l+1 \\ 0 \leq n \leq k}} I^{m,n} \xrightarrow{\partial^{l+1}} \cdots \tag{4.6}$$

exact.

Proof. We proceed by induction on k. The case $k = 1$ is proved in Lemma 4.2. Suppose that the lemma is true for $k < q$. We show that it is also true for $k = q$. Denote by $J^{i,q-1}$ the quotient $I^{i,q-2}/\mathrm{Im}\left(\partial^{i,q-3}\right)$. By induction hypothesis, there exists an exact sequence:

$$M \hookrightarrow I^{0,0} \to I^{0,1} \bigoplus I^{1,0} \to \cdots \to \left(\bigoplus_{\substack{m+n=l+1 \\ 0 \leq n < q-1}} I^{m,n} \right) \oplus J^{l-q+2,q-1} \to \cdots$$

Because the sequence

$$0 \to \left(\bigoplus_{\substack{m+n=l+1 \\ 0 \leq n < q-1}} I^{m,n} \right) \oplus J^{l-q+2,q-1} \to \left(\bigoplus_{\substack{m+n=l+1 \\ 0 \leq n < q-1}} I^{m,n} \right) \oplus I^{l-q+2,q-1} \to I^{l-q+2,q} \to 0$$

is exact, then, according to Lemma 4.2, the complex

$$0 \to M \to I^{0,0} \xrightarrow{\partial^0} I^{0,1} \bigoplus I^{1,0} \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^l} \bigoplus_{\substack{m+n=l+1 \\ 0\le n\le k}} I^{m,n} \xrightarrow{\partial^{l+1}} \cdots$$

is exact. We can then conclude the proposition. □

Note that, in the exact sequence (4.6), the first $k-1$ terms are injective. Let k tend to infinity, we obtain an injective resolution of M.

Definition-Proposition 4.5 (Pseudo-hyperresolution)**.** In an abelian category with enough injective objects, let M be an object and let $\left(M^k, \tau^k, k \ge 0\right)$ be a resolution of M. Suppose that we can resolve each M^k by an injective resolution $\left(I^{k,t}, \partial^{k,t}, t \ge 0\right)$. Then, there exist morphisms

$$\partial_i^{k,t} : I^{k,t} \to I^{k+i+1,t-i}$$

such that the morphisms

$$\partial_k = \begin{pmatrix} \partial^{0,k} & 0 & \cdots & 0 \\ \partial_0^{0,k} & \partial^{1,k-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{k-1}^{0,k} & \partial_{k-1}^{1,k-1} & \cdots & \partial^{k,0} \\ \partial_k^{0,k} & \partial_k^{1,k-2} & \cdots & \partial_0^{k,0} \end{pmatrix}$$

make the complex

$$I^{0,0} \xrightarrow{\partial^0} I^{0,1} \bigoplus I^{1,0} \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^k} \bigoplus_{m+n=k+1} I^{m,n} \xrightarrow{\partial^{k+1}} \cdots$$

an injective resolution of M. This resolution, denoted by $\mathscr{P}\left(M, I^{\bullet,\bullet}\right)$, is called the pseudo-hyperresolution of M with respect to the resolutions $\left(I^{k,t}, \partial^{k,t}, t \ge 0\right)$.

Because of Corollary 4.3, we also have a generalized version of Proposition 4.5.

Corollary 4.6. *In an abelian category with enough injectives, let $\left(M^k, \tau^k, k \in \mathbb{Z}\right)$ be a complex. Moreover, for all integers k, given an injective resolution $\left(I^{k,t}, \partial^{k,t}\right)_{t\ge 0}$ of M^k. Then, there exist morphisms*

$$\partial_i^{k,t} : I^{k,t} \to I^{k+i+1,t-i}$$

such that the morphisms

$$\partial_k = \begin{pmatrix} \partial^{0,k} & 0 & \cdots & 0 \\ \partial_0^{0,k} & \partial^{1,k-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{k-1}^{0,k} & \partial_{k-2}^{1,k-1} & \cdots & \partial^{k,0} \\ \partial_k^{0,k} & \partial_{k-1}^{1,k-2} & \cdots & \partial_0^{k,0} \end{pmatrix}$$

make the following sequence

$$\cdots \xrightarrow{\partial^{k-1}} \bigoplus_{m+n=k} I^{m,n} \xrightarrow{\partial^k} \bigoplus_{m+n=k+1} I^{m,n} \xrightarrow{\partial^{k+1}} \cdots$$

a complex, and the compositions

$$M^k \hookrightarrow I^{k,0} \hookrightarrow \bigoplus_{m+n=k} I^{m,n}$$

is a quasi-isomorphism of complexes.

Remark 4.7. The technical point we used in the proof of Lemma 4.2 is similar to the mapping telescope technique in triangulated categories introduced by Böckstedt and Neeman [3]. But, thanks to Lemma 4.1, our method is different from that of Böckstedt and Neeman. In fact, when the length of the complex in play is infinite, the resulting resolution we obtained is more compact and, hence, more computable. To illustrate the value of the pseudo-hyperresolution method, we will give an application for infinite-length complexes in Section 11.

Dually, we can construct the so-called pseudo-hyper projective resolutions.

Corollary 4.8. *In an abelian category with enough projective objects, let M be an object and let $(N_k, \tau_k, k \geq 0)$ be a resolution of N. Suppose that we can resolve each N_k by a projective resolution $(P_{k,t}, \partial_{k,t}, t \geq 0)$. There exist morphisms*

$$\partial^i_{k,t} : P_{k,t} \to P_{k-i-1,t+i}$$

such that the morphisms

$$\partial_k = \begin{pmatrix} \partial_{0,k} & \partial^0_{1,k-1} & \cdots & \partial^{k-2}_{k-1,1} & \partial^{k-1}_{k,0} \\ 0 & \partial_{1,k-1} & \cdots & \partial^{k-3}_{k-1,1} & \partial^{k-2}_{k,0} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & \partial_{k-1,1} & \partial^0_{k,0} \end{pmatrix}$$

make the following sequence

$$\cdots \xrightarrow{\partial_{k+2}} \bigoplus_{m+n=k+1} P_{m,n} \xrightarrow{\partial_{k+1}} \cdots \xrightarrow{\partial_2} P_{0,1} \bigoplus P_{1,0} \xrightarrow{\partial_1} P_{0,0}$$

a projective resolution of N. This resolution, denoted by $\mathscr{P}(N, P_{\bullet,\bullet})$, is called the pseudo-hyperresolution of N with respect to the resolutions $(P_{k,t}, \partial_{k,t}, t \geq 0)$.

Note that each term of the pseudo-hyperresolution of an object in an abelian category is well-understood. Howerver, in general, it is difficult to determine the differentials of this resolution. Therefore, it is convenient to use the notion of a *Poincaré series* of a complex to reformulate pseudo-hyperresolutions.

Definition 4.9 (Poincaré series of complexes)**.** In an abelian category, let $(C_\bullet, \partial_\bullet)$ be a complex. Then, we denote by $\mathrm{H}(t, C_\bullet)$ the series

$$\mathrm{H}(t, C_\bullet) := \sum_i [C_i]\, t^i, \tag{4.7}$$

where $[C_i]$ denotes the corresponding class of C_i in the Grothendieck group of the category.

The following lemmas are straightforward.

Lemma 4.10. *In an abelian category, let $(C_\bullet, \partial_\bullet)$ and $(D_\bullet, \delta_\bullet)$ be two complexes. Then, we have*

$$\mathrm{H}(t, C_\bullet \oplus D_\bullet) = \mathrm{H}(t, C_\bullet) + \mathrm{H}(t, D_\bullet).$$

Moreover, if the category is monoidal, then we have

$$\mathrm{H}(t, C_\bullet \otimes D_\bullet) = \mathrm{H}(t, C_\bullet) \cdot \mathrm{H}(t, D_\bullet).$$

Lemma 4.11. *Under the same hypothesis as that of Corollary 4.8, we have*

$$\mathrm{H}(t, \mathscr{P}(N, P_{\bullet,\bullet})) = \sum_{r,s \geq 0} P_{r,s} t^{r+s}.$$

5 Functor homology

The main purpose of this section is to provide two simple applications of the pseudo-hyperresolution method in functor homology:

1. Constructing explicit resolutions of certain strict polynomial functors and use them to compute the Mac Lane cohomology of $\mathbb{F}_2$ (see Theorem 5.10).

2. Computing the global dimension of the category of homogeneous strict polynomial functors of finite degree (see Theorem 5.19).

5.1 Mac Lane cohomology of finite fields

Following the works [10, 11, 12] of Eilenberg and Mac Lane on the homology of the spaces that bear their names, Mac Lane introduces in [19] the notion of the cohomology of a ring with coefficients in a bimodule. Jibladze and Pirashvili extend this notion for more general coefficients (see [17]). In what follow, we recall the definition of Mac Lane cohomology and show how to use the pseudo-hyperresolution method to compute the Mac Lane cohomology of the finite field $\mathbb{F}_2$.

5.1.1 Ordinary functors and strict polynomial functors

Let R be a ring and denote by $\mathscr{F}(R)$ the category of functors from the category of projective $R-$modules of finite rank to the category of $R-$modules. Denote by I the inclusion functor that associates a projective $R-$modules of finite rank V with itself. Then, following [17], we define the Mac Lane cohomology of R with coefficients in a functor $F \in \mathscr{F}(R)$ as follows:

$$\mathrm{HML}^*(R, F) := \mathrm{Ext}^*_{\mathscr{F}(R)}(I, F).$$

Taking $F = I$, we recover the original Mac Lane cohomology of the ring R with coefficients in R, with the obvious structure of $R - R-$bimodule (see [19]).

In fact, Ext-groups in $\mathscr{F}(R)$ can capture more than just the Mac Lane cohomology. When R is a finite field $\Bbbk$ of characteristic p, we write $\mathscr{F}_{\Bbbk}$ for $\mathscr{F}(\Bbbk)$. If $F \in \mathscr{F}_{\Bbbk}$, then structure morphism

$$F_{\Bbbk^n,\Bbbk^n} : \mathrm{Hom}_{\Bbbk}(\Bbbk^n, \Bbbk^n) \to \mathrm{Hom}_{\Bbbk}(F(\Bbbk^n), F(\Bbbk^n))$$

turns $F(\Bbbk^n)$ into a $\Bbbk[\mathrm{GL}_n(\Bbbk)]-$module. Therefore, the evaluation on $\Bbbk^n$ yields a natural morphism

$$\mathrm{Ext}^*_{\mathscr{F}_{\Bbbk}}(F, G) \to \mathrm{Ext}^*_{\Bbbk[\mathrm{GL}_n(\Bbbk)]}(F(\Bbbk^n), G(\Bbbk^n)).$$

The canonical inclusion

$$\mathrm{GL}_n(\Bbbk) \to \mathrm{GL}_{n+1}(\Bbbk), \quad M \mapsto \begin{pmatrix} M & 0 \\ 0 & 1 \end{pmatrix},$$

along with the splitting projection $\Bbbk^{n+1} \to \Bbbk^n$ onto the first n coordinates, induces a natural map

$$\mathrm{Ext}^*_{\Bbbk[\mathrm{GL}_{n+1}(\Bbbk)]}\left(F\left(\Bbbk^{n+1}\right), G\left(\Bbbk^{n+1}\right)\right) \to \mathrm{Ext}^*_{\Bbbk[\mathrm{GL}_n(\Bbbk)]}(F(\Bbbk^n), G(\Bbbk^n))$$

As n tends to infinity, this map stabilizes, and we denote by $\mathrm{Ext}^*_{\Bbbk[\mathrm{GL}_\infty(\Bbbk)]}(F, G)$ the stable value. Under some mild conditions on F and G (namely, F, G are polynomial in the sense of Eilenberg and Mac Lane [11]), the induced map

$$\mathrm{Ext}^*_{\mathscr{F}_{\Bbbk}}(F, G) \to \mathrm{Ext}^*_{\Bbbk[\mathrm{GL}_\infty(\Bbbk)]}(F, G)$$

is an isomorphism (see, *e.g.*, [2, 13]). Recall that if M, N are $\Bbbk[\mathrm{GL}_n(\Bbbk)]-$modules, then $\mathrm{Hom}_{\Bbbk}(M, N)$ is endowed with an action of $\mathrm{GL}_n(\Bbbk)$, and, moreover, there is an isomorphism

$$\mathrm{Ext}^*_{\Bbbk[\mathrm{GL}_\infty(\Bbbk)]}(F, G) \cong \mathrm{H}^*(\mathrm{GL}_n(\Bbbk), \mathrm{Hom}_{\Bbbk}(M, N)).$$

So, Ext-groups in $\mathscr{F}_{\Bbbk}$ can capture the stable cohomology of $\mathrm{GL}_n(\Bbbk)$. Now, if we consider the affine algebraic group scheme $\mathrm{GL}_{n,\Bbbk}$ instead, *i.e.*, the functor that associates a $\Bbbk-$algebra A with $\mathrm{GL}_n(A)$, then it is natural to ask whether there exists a suitable category that is related to $\mathrm{GL}_{n,\Bbbk}-$modules the way $\mathscr{F}_{\Bbbk}$ does to $\mathrm{GL}_n(\Bbbk)-$modules. In [15], Friedlander and Suslin introduce the category $\mathcal{P}_{\Bbbk}$ of strict polynomial functors, resolving in the affirmative this question. A strict polynomial functor F is a functor from the category of $\Bbbk-$vector spaces of finite dimension to the category of $\Bbbk-$vector spaces such that the structure morphism

$$F_{V,W} : \mathrm{Hom}_{\Bbbk}(V, W) \to \mathrm{Hom}_{\Bbbk}(F(V), F(W)) \tag{5.1}$$

is a polynomial map. Here, a polynomial map between two $\Bbbk-$vector spaces X and Y is a morphism between two schemes $Spec\left(S^*\left(X^\sharp\right)\right)$ and $Spec\left(S^*\left(Y^\sharp\right)\right)$, where $S^*\left(X^\sharp\right)$ is the symmetric algebra on the $\Bbbk-$linear dual $X^\sharp$. Let $Sch/\Bbbk$ denote the category of schemes over $\Bbbk$. Because

$$\mathrm{Hom}_{Sch/\Bbbk}\left(Spec\left(S^*\left(X^\sharp\right)\right), Spec\left(S^*\left(Y^\sharp\right)\right)\right) \cong S^*\left(X^\sharp\right) \otimes Y,$$

then we define a polynomial map $p : X \to Y$ to be homogeneous of degree d if it belongs to $S^d\left(X^\sharp\right) \otimes Y$. And then, a strict polynomial functor is homogeneous of degree d if its structure morphisms (5.1) are homogeneous polynomial maps of the same degree. Denote by $\mathcal{P}_{\Bbbk,d}$ the full subcategory of homogeneous strict polynomial functors of degree d. Then, $\mathcal{P}_{\Bbbk} \cong \bigoplus_{d\geq 0} \mathcal{P}_{\Bbbk,d}$.

As the evaluation on $\Bbbk$ of a morphism of schemes over $\Bbbk$ yields a set theoretic map, a strict polynomial functor F, with structural morphisms $(F(V), F_{V,W})$ gives rise to an ordinary functor $F \in \mathscr{F}_{\Bbbk}$, with structural morphisms $(F(V), F_{V,W}(\Bbbk))$. We obtain in this way an exact forgetful functor $\mathcal{O} : \mathcal{P}_{\Bbbk} \to \mathscr{F}_{\Bbbk}$.

Let $G \in Sch/\Bbbk$, then a representation of G (or a $G-$module) is a $\Bbbk-$vector space, endowed with a natural transformation $G \to \mathrm{GL}_M$, where GL_M is the functor that associates a $\Bbbk-$algebra A with the group $\mathrm{GL}_A(A \otimes_\Bbbk M)$ of invertible $A-$linear endomorphisms of $A \otimes_\Bbbk M$. Evaluation on $\Bbbk$ yields an exact forgetful functor $\mathcal{O} : \mathrm{GL}_{n,\Bbbk} - \mathrm{mod} \to \mathrm{GL}_n(\Bbbk) - \mathrm{mod}$, and we have a commutative diagram

$$\begin{tikzcd} \mathcal{P}_\Bbbk \arrow[r, "\mathcal{O}"] \arrow[d, "\mathrm{ev}"'] & \mathscr{F}_\Bbbk \arrow[d, "\mathrm{ev}"] \\ \mathrm{GL}_{n,\Bbbk} - \mathrm{mod} \arrow[r, "\mathcal{O}"'] & \mathrm{GL}_n(\Bbbk) - \mathrm{mod} \end{tikzcd} \tag{5.2}$$

If $F, G \in \mathcal{P}_\Bbbk$ are homogeneous of degree less than n, then it follows from [15] that the evaluation on $\Bbbk^n$ induces a natural isomorphism

$$\mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F, G) \to \mathrm{Ext}^*_{\mathrm{GL}_{n,\Bbbk}}(F(\Bbbk^n), G(\Bbbk^n)).$$

As the forgetful functors shown in the diagram (5.2) are exact, we obtain the commutative square

$$\begin{tikzcd} \mathrm{Ext}^*_{\mathcal{P}_\Bbbk}(F, G) \arrow[r] \arrow[d] & \mathrm{Ext}^*_{\mathscr{F}_\Bbbk}(F, G) \arrow[d] \\ \mathrm{Ext}^*_{\mathrm{GL}_n(\Bbbk)}(F(\Bbbk^n), G(\Bbbk^n)) \arrow[r] & \mathrm{Ext}^*_{\mathrm{GL}_{n,\Bbbk}}(F(\Bbbk^n), G(\Bbbk^n)) \end{tikzcd} \tag{5.3}$$

According to the work of Cline, Parshall, Scott, and van der Kallen [7], Ext-groups of $\mathrm{GL}_n(\Bbbk)-$modules can be computed via that of $\mathrm{GL}_{n,\Bbbk}-$modules, with the help of the Frobenius twist. The square (5.3) allows to establish similar relations between Ext-groups in $\mathscr{F}_\Bbbk$ and that of strict polynomial functors. Let recall briefly about this fact.

Given an integer $r \geq 1$, we denote by $I^{(r)}$ the strict polynomial functor defined as the intersection of the kernel of $\mathcal{P}_\Bbbk-$morphisms $S^{p^r} \to S^k \otimes S^{p^r-k}$ for all $0 < k < p^r$, induced by the comultiplication of the graded Hopf algebra S^* (to be recalled in Subsection 5.1.2). The nontrivial strict polynomial functor $I^{(r)}$ is homogeneous of degree p^r. Therefore, $I^{(r)}$ and $I^{(s)}$ are not isomorphic if $r \neq s$. However, if $\Bbbk$ is perfect, then the forgetful functor sends $I^{(r)}$ to the inclusion functor I for all $r \geq 1$. The Frobenius twist of a strict polynomial functor F, denoted by $F^{(r)}$, is defined as $F \circ I^{(r)}$. The relation between Ext-groups in $\mathscr{F}_\Bbbk$ and in $\mathcal{P}_\Bbbk$ now goes as follows.

Theorem 5.1 ([13])**.** *Let $\Bbbk$ be a perfect field, and let $F, G \in \mathcal{P}_\Bbbk$ be two homogeneous strict polynomial functors of degree d, which is less than the cardinal of $\Bbbk$. If r is big enough with respect to i, then the natural morphism*

$$\mathrm{Ext}^i_{\mathcal{P}_\Bbbk}\left(F^{(r)}, G^{(r)}\right) \to \mathrm{Ext}^i_{\mathscr{F}_\Bbbk}(F, G)$$

is an isomorphism.

In particular, we have the following corollary.

Corollary 5.2. *If r is big enough with respect to i, then the natural morphism*

$$\mathrm{Ext}^i_{\mathcal{P}_{\mathbb{F}_2}}\left(I^{(r)}, I^{(r)}\right) \to \mathrm{Ext}^i_{\mathscr{F}_{\mathbb{F}_2}}(I, I)$$

is an isomorphism.

Therefore, we can compute the Mac Lane cohomology $\mathrm{HML}^*(\mathbb{F}_2, I)$ *via* $\mathrm{Ext}^i_{\mathcal{P}_{\mathbb{F}_2}}\left(I^{(r)}, I^{(r)}\right)$. In [15], Friedlander and Suslin compute these groups by constructing injective resolutions for $I^{(r)}$, using the method pioneered by Franjou, Lannes, and Schwartz [14]. We now show that the pseudo-hyperresolution method also allows to construct explicit resolutions of $I^{(r)}$ and leads to the same result.

5.1.2 Injective resolutions of twisted strict polynomial functors

We begin with some basic homological algebra of strict polynomial functors. Throughout this subsection, we restrict our attention to the case $\Bbbk = \mathbb{F}_2$, and we shorten $\mathcal{P}_{\mathbb{F}_2,n}$ to $\mathcal{P}_n$.

Definition 5.3. An *exponential strict polynomial functor* is a graded functor $F^* = \left(F^0, F^1, F^2, \ldots, F^n, \ldots\right)$, where $F^n \in \mathcal{P}_n$, together with natural isomorphisms

$$F^0(V) = \mathbb{F}_2, \quad F^n(V \oplus W) \cong \bigoplus_{i=0}^{n} F^i(V) \otimes F^{n-i}(W)$$

for all integers $n > 0$ and all $\mathbb{F}_2-$vector spaces V, W.

Typical examples of exponential functors are given by the symmetric algebra S^*, the exterior algebra Λ^*, and the divided power algebra Γ^*. For all exponential strict polynomial functor F^* and all $\mathbb{F}_2-$vector space V, the addition map $+ : V \oplus V \to V$ and the diagonal map $\Delta : V \to V \oplus V$ give rise to the natural maps

$$F^n(V) \otimes F^m(V) \hookrightarrow F^{n+m}(V \oplus V) \xrightarrow{F^{n+m}(+)} F^{n+m}(V),$$
$$F^{n+m}(V) \xrightarrow{F^{n+m}(\Delta)} F^{n+m}(V \oplus V) \twoheadrightarrow F^n(V) \otimes F^m(V),$$

and then define natural product and coproduct operations

$$\text{mult} : F^n \otimes F^m \to F^{n+m}, \quad \text{comult} : F^{n+m} \to F^n \otimes F^m.$$

Remark that $S^1 = \Lambda^1 = \Gamma^1 = I$. Let $F^*, G^* \in \{S^*, \Lambda^*\Gamma^*\}$, then we define $\kappa^{d,e}_{F^*,G^*}$ as the composition map

$$F^d \otimes G^e \xrightarrow{\text{comult}\otimes Id} F^{d-1} \otimes F^1 \otimes G^e = F^{d-1} \otimes G^1 \otimes G^e \xrightarrow{Id\otimes\text{mult}} F^{d-1} \otimes G^{e+1},$$

Following [14], we have two exact sequences:

Theorem 5.4. *The complexes* $\left(\Gamma^n \otimes \Lambda^{d-n}, \kappa^{n,d-n}_{\Gamma^*,\Lambda^*} \middle| 0 \le n \le d\right)$ *and* $\left(\Lambda^n \otimes S^{d-n}, \kappa^{n,d-n}_{\Lambda^*,S^*} \middle| 0 \le n \le d\right)$ *are exact.*

Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{N}^m$, and let $F^* = \left(F^0, F^1, F^2, \ldots, F^n, \ldots\right)$, where $F^n \in \mathcal{P}_n$, be an exponential strict polynomial functor. We denote by F^λ the tensor product

$$F^{\lambda_1} \otimes F^{\lambda_2} \otimes \cdots \otimes F^{\lambda_m}.$$

Recall that the symmetric power functors $S^d, d \ge 0$, are injective in $\mathcal{P}_{\mathbb{F}_2}$, and the tensor product of injective strict polynomial functors remains injective. Moreover, the set $\left\{ S^\lambda \middle| \lambda \in \mathbb{N}^m, m \ge 1\right\}$ forms a system of injective cogenerators of $\mathcal{P}_{\mathbb{F}_2}$. According to [14], the Frobenius twist of S^d admits the following explicit injective resolution.

Theorem 5.5 ([14, Section 2])**.** *For all integers $d \ge 1$, the complex* $\left(S^n \otimes S^{d-n}, \kappa^{n,d-n}_{S^*,S^*} \middle| 0 \le d \le n\right)$ *is an injective resolution of $S^{d(1)}$. Throughout this article, we denote this complex by $\mathscr{S}(d,1)$.*

Because the tensor product of injective strict polynomial functors is again injective, then $S^{\lambda(1)}$ admits

$$\mathscr{S}(\lambda,1) := \mathscr{S}(\lambda_1,1) \otimes \mathscr{S}(\lambda_2,1) \otimes \cdots \otimes \mathscr{S}(\lambda_m,1)$$

as an injective resolution. Since the Frobenius twist is exact, the pseudo-hyperresolution method is a suitable way to construct injective resolutions of $S^{\lambda(r)}$ for all integers $r \ge 1$.

Let $\mathscr{S}$ denote the class of all resolutions that are direct sum of resolutions of the form $\mathscr{S}(\lambda,1)$. Hence, for all injective strict polynomial functors J, $J^{(1)}$ has a unique injective resolution belonging to $\mathscr{S}$. Given an injective resolution $J^\bullet$ of a strict polynomial functor F, in this paragraph, we denote by $\mathscr{P}\left(F^{(1)}, J^{\bullet(1)}\right)$ the pseudo-hyperresolution of $F^{(1)}$ with respect to the collection of resolutions of $J^{\bullet(1)}$ coming from $\mathscr{S}$.

We define:

$$\mathscr{S}(d,r) := \begin{cases} \mathscr{S}(d,1) & \text{if } r = 1, \\ \mathscr{P}\left(S^{d(r)}, \mathscr{S}(d,r-1)^{(1)}\right) & \text{if } r > 1. \end{cases}$$

The following lemma is straightforward.

Lemma 5.6. *The complex $\mathscr{S}(d,r)$ is an injective resolution of $S^{d(r)}$ for all integers $r \ge 1$. Moreover, the length of $\mathscr{S}(d,r)$ is $\left(2^{r+1} - 2\right) d$.*

We say that $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{N}^m$ is divisible by n if λ_k is divisible by n for all $1 \le k \le m$. Remark that, if λ is divisible by 2^r, then the $s-$th term of the resolution $\mathscr{S}(\lambda,1)$ contains a factor of the form S^α such that $2^{r+1}|\alpha$ if and only if $2^{r+1}|s$. Otherwise, no factor of this form can appear in $\mathscr{S}(\lambda,1)$. A simple induction on n using the pseudo-hyperresolution method shows that a similar property for $\mathscr{S}(d,n)$ also holds.

Lemma 5.7. *If d is divisible by 2^r, then the $s-$th term of the resolution $\mathscr{S}(d,n)$ contains a direct summand of the form S^λ such that $2^{n+r}|\lambda$ if and only if $2^{n+r}|s$. Otherwise, no direct summand of this form can appear in $\mathscr{S}(d,n)$.*

In particular, when $d = 1$, we obtain the following result.

Corollary 5.8. *The resolution $\mathscr{S}(1,n)$ is of length $2^{n+1}-2$. Denote by $\mathscr{S}(1,n)^s$ the $s-$th term of this resolution. Then, $\mathscr{S}(1,n)^{2k}$ contains a unique direct summand of the form S^{2^n} for all $0 \le k \le 2^n - 1$, and $\mathscr{S}(1,n)^{2k+1}$ contains no direct summand of this form.*

Recall that, for all $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m) \in \mathbb{N}^m$, we have

$$\mathrm{Hom}_{\mathcal{P}_{\mathbb{F}_2}}\left(F^{(r)}, S^{\lambda}\right) \cong \begin{cases} \mathrm{Hom}_{\mathcal{P}_{\mathbb{F}_2}}\left(F, S^{\left(\frac{\lambda_1}{2^r}, \frac{\lambda_2}{2^r}, \ldots, \frac{\lambda_m}{2^r}\right)}\right) & \text{if } 2^r|\lambda, \\ 0 & \text{otherwise.} \end{cases}$$

Therefore, if F is a homogeneous strict polynomial functor of degree d, then there is no nontrivial differentials in the complex $\mathrm{Hom}_{\mathcal{P}_{\mathbb{F}_2}}\left(F^{(r)}, \mathscr{S}(d,r)\right)$.

Corollary 5.9. *For all homogeneous strict polynomial functor F of degree d, we have*

$$\mathrm{Hom}_{\mathcal{P}_{\mathbb{F}_2}}\left(F^{(r)}, \mathscr{S}(d,r)\right) \cong \mathrm{Ext}^*_{\mathcal{P}_{\mathbb{F}_2}}\left(F^{(r)}, S^{d(r)}\right).$$

In particular, we have

$$\mathrm{Ext}^k_{\mathcal{P}_{\mathbb{F}_2}}\left(I^{(r)}, I^{(r)}\right) \cong \begin{cases} \mathbb{F}_2 & \textit{if } 2|k, \textit{ and } k \leq 2^{r+1}-2, \\ 0 & \textit{otherwise.} \end{cases}$$

Following Corollary 5.2, we obtain the main result of this subsection.

Theorem 5.10 ([6, 14, 15])**.** *We have:*

$$\mathrm{HML}^k\left(\mathbb{F}_2, I\right) \cong \begin{cases} \mathbb{F}_2 & \textit{if } 2|k, \\ 0 & \textit{otherwise.} \end{cases}$$

5.2 Global dimension of homogeneous strict polynomial functors

It is known that each homogeneous strict polynomial functor of finite degree is of finite projective (injective) dimension (see [1, 9, 28]). In this subsection, we will use the pseudo-hyperresolution method to show that the global dimension of the cateogry $\mathcal{P}_d$ is $2d - 2\alpha(d)$ for all $d \geq 0$. (Here $\alpha(d)$ is the sum of the coefficients of the $2-$adic expansion of d.)

5.2.1 Resolutions of classical strict polynomial functors

In this paragraph, we will construct various injective (projective) resolutions of the classical exponential strict polynomial functors Λ^*, Γ^*, and S^*. First, we fix the following notations. Let $F^* = \left(F^0, F^1, F^2, \ldots, F^n, \ldots\right)$, where $F^n \in \mathcal{P}_n$, be an exponential strict polynomial functor. We denote:

$$A(d,s) := \left\{ (n_1, n_2, \ldots, n_s) \middle| n_i \geq 1, \sum_{i=1}^{s} n_i = d \right\}, \tag{5.4}$$

$$F(d,s) := \bigoplus_{A(d,s)} F^{(n_1, n_2, \ldots, n_s)}. \tag{5.5}$$

Recall that the divided power functors $\Gamma^d, d \geq 0$, are projective in $\mathcal{P}_{\mathbb{F}_2}$, and the tensor product of projective strict polynomial functors remains projective. Moreover, the set $\left\{\Gamma^{\lambda} \middle| \lambda \in \mathbb{N}^m, m \geq 1\right\}$ forms a system of projective generators of $\mathcal{P}_{\mathbb{F}_2}$. We will use Theorem 5.4 to construct explicit injective and projective resolutions of Λ^d. In fact, for $d = 1$ and $d = 2$, the complex $\left(\Gamma^n \otimes \Lambda^{d-n}, \kappa^{n,d-n}_{\Gamma^*,\Lambda^*} \middle| 0 < n \leq d\right)$ is a projective resolution of Λ^d. In general, we construct a canonical projective resolution of Λ^d as follows. We denote by Λ_d the set of canonical projective resolution $C_{proj}(\Lambda^n)$ of Λ^n for all $n \leq d$, which is defined recursively as follows:

$$\Lambda_2 := \left\{ \left(\Gamma^n \otimes \Lambda^{d-n}, \kappa^{n,d-n}_{\Gamma^*,\Lambda^*} \middle| 0 < n \leq d\right) \middle| 1 \leq d \leq 2 \right\},$$

$$\Lambda_d := \Lambda_{d-1} \bigsqcup \left\{ \mathscr{P}\left(\Lambda^d, \left\{\Gamma^{d-s} \otimes C_{proj}(\Lambda^s) \middle| 0 \leq s \leq d-1\right\}\right) \right\}.$$

A simple induction on d yields the following result.

Lemma 5.11. *For all integers $d \geq 1$, we have*

$$\mathrm{H}\left(t, C_{proj}\left(\Lambda^d\right)\right) = \sum_{s=0}^{d-1} \Gamma(d, d-s) t^s.$$

(See (5.5) *for the definition of* $\Gamma(d, d-s)$*.)*

Similarly, we construct a canonical injective resolution for Λ^d as follows. We denote by Inj_d the set of canonical injective resolution $C_{inj}(\Lambda^n)$ of Λ^n for all $n \leq d$, which is defined recursively by:

$$\mathrm{Inj}_2 := \left\{ \left(\Lambda^n \otimes S^{d-n}, \kappa^{n,d-n}_{\Lambda^*,S^*} \middle| 0 \leq n \leq d \right) \middle| 1 \leq d \leq 2 \right\},$$
$$\mathrm{Inj}_d := \mathrm{Inj}_{d-1} \bigsqcup \left\{ \mathscr{P}\left(\Lambda^d, \left\{ C_{inj}(\Lambda^s) \otimes S^{d-s} \middle| 0 \leq s \leq d-1 \right\}\right) \right\}.$$

A simple induction on d yields the following result.
Lemma 5.12. *For all $d \geq 1$, we have*

$$\mathrm{H}\left(t, C_{inj}\left(\Lambda^d\right)\right) = \sum_{s=0}^{d-1} S(d, d-s) t^s.$$

(See (5.4) *for the definition of $S(d, d-s)$.)*

Now, using the complex $\left(\Gamma^n \otimes \Lambda^{d-n}, \kappa^{n,d-n}_{\Gamma^*,\Lambda^*} \middle| 0 < n \leq d \right)$, the set Inj_d, and the pseudo-hyperresolution method, we can define a set Γ_d of canonical injective resolution $C_{inj}(\Gamma^n)$ of Γ^n for all $n \leq d$ as follows:

$$\Gamma_1 := \left\{ \left(\Gamma^n \otimes \Lambda^{d-n}, \kappa^{n,d-n}_{\Gamma^*,\Lambda^*} \middle| 0 < n \leq d \right) \middle| d = 1 \right\},$$
$$\Gamma_d := \Gamma_{d-1} \bigsqcup \left\{ \mathscr{P}\left(\Gamma^d, \left\{ C_{inj}(\Gamma^s) \otimes C_{inj}\left(\Lambda^{d-s}\right) \middle| 0 \leq s \leq d-1 \right\}\right) \right\}.$$

Induction on d gives rise to the following lemma.
Lemma 5.13. *For all integers $d \geq 1$, we have*

$$\mathrm{H}\left(t, C_{inj}\left(\Gamma^d\right)\right) = \sum_{s=0}^{d-1} S(d, d-s) t^{2s} (1+t)^{d-s-1}.$$

Similarly, using the complex $\left(\Lambda^n \otimes S^{d-n}, \kappa^{n,d-n}_{\Lambda^*,S^*} \middle| 0 \leq n \leq d \right)$, the set Γ_d, and the pseudo-hyperresolution method, we can define a set S_d of canonical projective resolution $C_{proj}(S^n)$ of S^n for all $n \leq d$ as follows:

$$S_1 := \left\{ \left(\Lambda^n \otimes S^{d-n}, \kappa^{n,d-n}_{\Lambda^*,S^*} \middle| 0 < n \leq d \right) \middle| d = 1 \right\},$$
$$S_d := S_{d-1} \bigsqcup \left\{ \mathscr{P}\left(S^d, \left\{ C_{proj}\left(\Lambda^{d-s}\right) \otimes C_{proj}(S^s) \middle| 0 \leq s \leq d-1 \right\}\right) \right\}.$$

Induction on d gives rise to the following lemma.
Lemma 5.14. *For all $d \geq 1$, we have:*

$$\mathrm{H}\left(t, C_{proj}\left(S^d\right)\right) = \sum_{s=0}^{d-1} \Gamma(d, d-s) t^{2s} (1+t)^{d-s-1}.$$

Remark 5.15. Lemma 5.13 and 5.14 signify that

$$\dim_{inj}\left(\Gamma^d\right) \leq 2d-2, \quad \dim_{proj}\left(S^d\right) \leq 2d-2.$$

5.3 Global dimension of strict polynomial functors

The following lemma is a direct consequence of Corollary 5.14.
Lemma 5.16. *For all integers $n \geq 1$, we have*

$$\mathrm{Ext}^{2^{n+1}-2}_{\mathcal{P}_{2^n}}\left(S^{2^n}, \Gamma^{2^n}\right) \cong \mathbb{F}_2.$$

Hence, S^{2^n} is of projective dimension $2^{n+1}-2$, and Γ^{2^n} is of injective dimension $2^{n+1}-2$.

Proof. Denote 2^n by d. In view of Corollary 5.14, $\mathrm{Ext}^{2d-2}_{\mathcal{P}_d}\left(S^d, \Gamma^d\right)$ is isomorphic to the cokernel of the map

$$f : \mathrm{Hom}_{\mathcal{P}_d}\left(\Gamma(d,2), \Gamma^d\right) \to \mathrm{Hom}_{\mathcal{P}_d}\left(\Gamma^d, \Gamma^d\right).$$

If this cokernel was trivial, then Γ^d would be a direct summand of $\Gamma(d,2)$, whence a contradiction. Therefore, $\mathrm{Coker}(f)$ is nontrivial. But, on the other hand, $\mathrm{Hom}_{\mathcal{P}_d}\left(\Gamma^d, \Gamma^d\right) \cong \mathbb{F}_2$, then the lemma follows from Remark 5.15. $\square$

Corollary 5.17. *Let $d \geq 1$ be an integer and let $\sum_{i=1}^{k} 2^{n_k}$ be its $2-$adic expansion. Let $I \in \mathcal{P}_d$ be an injective strict polynomial functors, and let $P \in \mathcal{P}_d$ be a projective one. Then, we have*

$$\dim_{inj} P \leq 2d - 2k, \quad \dim_{proj} I \leq 2d - 2k.$$

In particular, let $\lambda = (2^{n_1}, 2^{n_2}, \ldots, 2^{n_k})$, then we have:

$$\dim_{inj} \Gamma^{\lambda} = 2d - 2k, \quad \dim_{proj} S^{\lambda} = 2d - 2k.$$

Proof. The corollary follows from Lemma 5.16 and from the fact that the morphism $\Gamma^{\lambda} \to \Gamma^{d}$ induced by the multiplication of Γ^* is a splitting projection. □

Corollary 5.17 can be extended to all strict polynomial functors.

Lemma 5.18. *Let $d \geq 1$ be an integer and let $\sum_{i=1}^{k} 2^{n_k}$ be its $2-$adic expression. Let $F \in \mathcal{P}_d$ be an arbitrary strict polynomial, then*

$$\dim_{inj} F \leq 2d - 2k, \quad \dim_{proj} F \leq 2d - 2k.$$

Proof. It is sufficient to show that

$$\begin{aligned} \mathrm{Ext}^{s}_{\mathcal{P}_d}(G, F) = 0, \text{ for all } s > 2d - 2 \text{ and all } G \in \mathcal{P}_d, \\ \mathrm{Ext}^{s}_{\mathcal{P}_d}(F, G) = 0, \text{ for all } s > 2d - 2 \text{ and all } G \in \mathcal{P}_d. \end{aligned}$$

But these computations can be carried out by induction on injective and projective dimensions of F, using Corollary 5.17. □

Theorem 5.19 ([28])**.** *Let $d \geq 1$ be an integer and let $\sum_{i=1}^{k} 2^{n_k}$ be its $2-$adic expression. Then, the global dimension of $\mathcal{P}_d$ is $2d - 2k$.*

6 The Lambda algebra

This section aims at giving a new construction of the Lambda algebra, using the pseudo-hyperresolution method.

6.1 Iterated suspensions of Brown-Gitler modules

For all integers $n \geq 0$, the Mahowald short exact sequence

$$0 \to \Sigma J(n) \xrightarrow{s_n} J(n+1) \xrightarrow{\bullet Sq^{\frac{n+1}{2}}} J\left(\frac{n+1}{2}\right) \to 0$$

yields an injective resolution for $\Sigma J(n)$. Since the functor Σ is exact, applying it to an injective resolution of $\Sigma^m J(n)$ induces a resolution of $\Sigma^{m+1} J(n)$. Therefore, the pseudo-hyperresolution method allows to construct an injective resolution of $\Sigma^{m+1} J(n)$ from that of $\Sigma^m J(n)$.

Definition 6.1. Let $B = \bigoplus_A J(n_\alpha)$ be a BG module, we define:

$$\mathrm{Aug}\,(B) := \bigoplus_A J(1 + n_\alpha).$$

A reformulation of Lemma 4.2 gives rise to the following proposition.

Proposition 6.2 (Pseudo-hyperresolution for the suspension of BG complexes)**.** *Let $\left(B^k, \tau^k : B^k \to B^{k+1}, \kappa \geq 0\right)$ be a BG complex. Denote by p^k the canonical projection $\mathrm{Aug}\left(B^k\right) \to \tilde{\Phi}\left(\mathrm{Aug}\left(B^k\right)\right)$ induced by the natural transformation $Id \to \tilde{\Phi}$. Then, for all integers $k \geq 0$, there exist morphisms*

$$\begin{aligned} \theta^k &: \mathrm{Aug}\left(B^k\right) \to \mathrm{Aug}\left(B^{k+1}\right), \\ \delta^k &: \tilde{\Phi}\left(\mathrm{Aug}\left(B^k\right)\right) \to \mathrm{Aug}\left(B^{k+2}\right), \\ \omega^k &: \tilde{\Phi}\left(\mathrm{Aug}\left(B^k\right)\right) \to \tilde{\Phi}\left(\mathrm{Aug}\left(B^{k+1}\right)\right) \end{aligned}$$

such that the morphisms $\partial^k := \begin{pmatrix} \theta^k & \delta^{k-1} \\ p^k & \omega^{k-1} \end{pmatrix}$ *for all* $k \geq 1$,, *and* $\partial^0 := \begin{pmatrix} \theta^0 \\ p^0 \end{pmatrix}$ *make the sequence*

$$0 \to \mathrm{Aug}\left(B^0\right) \xrightarrow{\partial^0} \mathrm{Aug}\left(B^1\right) \bigoplus \tilde{\Phi}\left(\mathrm{Aug}\left(B^0\right)\right) \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^k} \mathrm{Aug}\left(B^{k+1}\right) \bigoplus \tilde{\Phi}\left(\mathrm{Aug}\left(B^k\right)\right) \xrightarrow{\partial^{k+1}} \cdots \tag{6.1}$$

a complex that we denote by $\mathscr{G}\left(B^k, \tau^k\right)$. *Moreover, if* $\left(B^k, \tau^k\right)$ *is an injective resolution of some unstable module* M, *then* $\mathscr{G}\left(B^k, \tau^k\right)$ *is the injective resolution of* ΣM.

Corollary 6.3. *Denote by* $J_n^\bullet$ *the acyclic complex where* $J_n^0 = J(n)$ *and* J_n^k *is trivial for all integers* $k \geq 1$. *Then,* $\mathscr{G}^m\left(J_n^\bullet\right)$ *is an injective resolution of* $\Sigma^m J(n)$. *Hereafter, we denote this resolution by* $\mathscr{G}(m,n)$.

Definition 6.4. For all integers $m, n \geq 0$, each term of $\mathscr{G}(m,n)$ is a direct sum of Brown-Gitler modules. Denote by $G(m,n)$ the graph associated with $\mathscr{G}(m,n)$ with respect to this decomposition, and $V_r\left(G(m,n)\right), E_r\left(G(m,n)\right)$ are shortened to $V_r(m,n), E_r(m,n)$ respectively. (See Section 3 for the definition of graph representations.)

Definition-Proposition 6.5. For all integers $m, n \geq 0$, we have:

1. There exists a canonical inclusion of graphs $i_{m,n} : G(m,n) \to G(m+1,n)$ corresponding to the inclusion of complexes $\Sigma\mathscr{G}(m,n) \to \mathscr{G}(m+1,n)$.

2. The set $V_0(m,n)$ contains a unique vertex, denoted by $p_{m,n}$, corresponding to $J(m+n)$.

3. For all integers $r \geq 0$ and all $v \in V_r(m,n), w \in V_{r+1}(m,n)$, then either $[v,w] = \bullet Sq^k$ for some integer $k \geq 0$, or $[v,w] = 0$. Hereafter, for the sake of simplicity, we write $[v,w] = Sq^k$ instead of $[v,w] = \bullet Sq^k$.

4. For all vertices $v \in V_r(m,n)$, there exists a unique collection of vertices

$$\left\{ v_i \;\middle|\; \begin{array}{l} v_i \in V_i(m,n), 0 \leq i \leq r, \\ v_0 = p_{m+n}, v_r = v, \\ [v_i, v_{i+1}] = Sq^{k_i}, \\ [v_i, w] \neq Sq^{k_i}, \forall v_{i+1} \neq w, \\ 2k_0 > n, \\ 2k_{i+1} > k_i, r-2 \geq i \geq 0, \\ m+n-\sum_{i=0}^{r-1} k_i \geq k_{r-1}, \end{array} \right\}$$

   forming a path from p_{m+n} to v, and such a path is called an *admissible path*. In this case, we say that v is of bidegree $\left(r, \sum_{i=0}^{r-1} k_i\right)$. We call this the *Lambda bidegree* of v and denote it by $\|v\| = \left(\|v\|_1, \|v\|_2\right)$, where $\|v\|_1 = r$ and $\|v\|_2 = \sum_{i=0}^{r-1} k_i$.

5. The set $E(m,n)$ of edges of $G(m,n)$ can be defined as the colimit of the increasing sequence of sets

$$E^0(m,n) \subset E^1(m,n) \subset \cdots \subset E^r(m,n) \subset \ldots$$

   as follows. Denote by $E^0(m,n)$ the set of all edges induced by the admissible paths. Suppose that we have defined $E^r(m,n)$. We now define $E^{r+1}(m,n)$. For all vertices $v \in V_i(m,n)$ and $w \in V_{i+2}(m,n)$, consider the sum of all products $[v,u]\,[u,w]$ over all $u \in V_{i+1}(m,n)$, where $[v,u], [u,w]$ belong to $E^r(m,n)$. Recall that this is a sum of products of Steenrod operations of the form Sq^k. Then, we can write this sum in terms of admissible monomials $\sum_{(i_1,i_2)\in A} Sq^{i_1} Sq^{i_2}$. Let $z \in V_{i+1}(m,n)$ such that $[v,z] = Sq^{i_1}$ belongs to $E^0(m,n)$, then we connect z to w by the edge $[z,w] = Sq^{i_2}$. We now define $E^{r+1}(m,n)$ as the union of $E^r(m,n)$ with these new edges.

Proof. The proof is carried out by induction on m. The case $m = 1$ is straightforward as $G(1,n)$ is the graph corresponding to the Mahowald short exact sequence (2.2) of $\Sigma J(n)$. Suppose that the proposition is proved for all $m < k$, where $k \geq 2$ is a certain integer. We now verify the case $m = k$. By definition, we have

$$\mathscr{G}(k,n)^s \cong \mathrm{Aug}\left(\mathscr{G}(k-1,n)\right)^s \oplus \tilde{\Phi}\left(\mathrm{Aug}\left(\mathscr{G}(k-1,n)\right)\right)^{s-1}$$

for all integers $s \geq 0$, with the convention that $\tilde{\Phi}\left(\mathrm{Aug}\left(\mathscr{G}(k-1,n)\right)\right)^{-1} = 0$. Here, the upper index s signifies the $s-$th term of the complex. Therefore, $\mathscr{G}(k,n)^0 = \mathrm{Aug}\left(\mathscr{G}(k-1,n)\right)^0$, and hence, following the induction hypothesis, the set $V_0(k,n)$ contains a unique vertex, denoted by $p_{k,n}$, corresponding to $J(k+n)$. **This verifies the point** (2). Moreover, the morphisms

$$\Sigma\mathscr{G}(k-1,n)^s \to \mathrm{Aug}\left(\mathscr{G}(k-1,n)\right)^s \hookrightarrow \mathscr{G}(k,n)^s,$$

for all $s \geq 0$, induce a canonical inclusion $\Sigma\mathscr{G}(k-1,n) \to \mathscr{G}(k,n)$. As the graph of $\Sigma\mathscr{G}(k-1,n)$ and that of $\mathscr{G}(k-1,n)$ are identical, we get an inclusion $G(k-1,n) \hookrightarrow G(k,n)$. **This concludes the proof of the point** (1).

Remark that, in view of Proposition 6.2, if τ^s is represented by a matrix with coefficients of the form $\bullet Sq^s$, then so are θ^s, δ^s, and ω^s. Therefore, each edge of $E_r(k,n)$ is also labeled by a Steenrod operation of the form Sq^s for some integer s. **Then, the point** (3) **holds**.

We also observe that the vertices of $G(k,n)$ that do not belong to the image of $G(k-1,n)$ correspond to $\tilde{\Phi}(\mathrm{Aug}(\mathscr{G}(k-1,n)))$. **Then, the induction hypothesis and the Mahowald short exact sequences take care of the points** (4) **and** (5). □

Remark 6.6. 1. The bigrading structure of $G(m,n)$ is different from that of the general associated graph of a BG complex (see Definition 3.6). We make such a modification so that the canonical inclusion $i_{m,n}$ from $G(m,n)$ to $G(m+1,n)$ is of bidegree $(0,0)$.

2. Let $k \geq 1$ be an integer. Then, for all integers m,n such that $m+n \geq 2k$, the inclusion $i_{m,n}$ of $G(m,n)$ into $G(m+1,n)$ is an isomorphism on the area of bidegrees (r,s) for all $s \leq k$. It is obvious that this is an isomorphism on the set of vertices. What is less evident is that it is also bijective on the set of edges. But, this is a consequence of the instability condition: the morphism $\bullet Sq^q : J(t+q) \to J(t)$ is nontrivial if and only if $q \leq t$. In fact, Proposition 6.2 shows that all the edges of $G(m+1,n)$ that are not in the image of $i_{m,n}$ must have a vertex belonging to $\tilde{\Phi}(\mathrm{Aug}(\mathscr{G}(m,n)))$, which is not in the area of consideration.

Therefore, using Lemma 3.7 with an appropriate changing of bidegrees, we obtain the following stabilization result:

Proposition 6.7. *Let $s \geq 1$ be an integer. Then, for all for all integers m such that $m > s+1$, the morphism*

$$\mathrm{Ext}^*_{\mathcal{U}}\left(\Sigma^m\mathbb{F}_2, \Sigma^{m+s}J(n)\right) \to \mathrm{Ext}^*_{\mathcal{U}}\left(\Sigma^{m+1}\mathbb{F}_2, \Sigma^{m+s+1}J(n)\right)$$

induced by the suspension Σ is an isomorphism. Moreover, in this case, we have an isomorphism

$$\mathrm{Ext}^*_{\mathcal{U}}\left(\Sigma^m\mathbb{F}_2, \Sigma^{m+s}J(n)\right) \cong \mathrm{Ext}^*_{\mathcal{A}}\left(\mathbb{F}_2, \Sigma^s J(n)\right)$$

of $\mathbb{F}_2-$vector spaces.

Proof. The first isomorphism follows from Lemma 3.7. We only verify the second one. Recall that the forgetful functor from the category $\mathcal{U}$ to the category of $\mathcal{A}-$modules admits a left adjoint D that we call the *destabilization functor*. Therefore, there is a Grothendieck spectral sequence:

$$\mathrm{Ext}^s_{\mathcal{U}}\left(D_t M, N\right) \longrightarrow \mathrm{Ext}^{s+t}_{\mathcal{U}}\left(M, N\right),$$

where D_t denote the $t-$th left derived functor of D. For an unstable module M, denote by $conn\,(M)$ the supremum of i such that $M^j = 0$ for all $j \leq i$. We also write $\mathrm{d}\,(M)$ the infimum of i such that $M^j = 0$ for all $j > i$. On the one hand, we have

$$\mathrm{d}\left(\Sigma^{m+s}J\left(n\right)\right) = m+n+s,$$

and on the other hand, we have

$$conn\left(D_t M\right) \geq 2conn\left(M\right) + t$$

for all $t > 0$. Therefore, for $m > s+n+2$, we have

$$conn\left(D_t\Sigma^m\mathbb{F}_2\right) > \mathrm{d}\left(\Sigma^{m+s}J(n)\right).$$

It follows that

$$\mathrm{Ext}^*_{\mathcal{U}}\left(D_t\Sigma^m\mathbb{F}_2, \Sigma^{m+s}J(n)\right) = 0$$

for all $t > 0$. Thus, the Grothendieck spectra sequence for the destabilization functor collapses at the E_2-term giving rise to the isomorphism

$$\mathrm{Ext}^*_{\mathcal{U}}\left(\Sigma^m\mathbb{F}_2, \Sigma^{m+s}J(n)\right) \cong \mathrm{Ext}^*_{\mathcal{A}}\left(\mathbb{F}_2, \Sigma^s J(n)\right)$$

of $\mathbb{F}_2-$vector spaces. □

Definition 6.8. For all integers $n \geq 1$, we denote by $G(n)$ the limit of the system

$$\cdots \xrightarrow{i_{m-1,n}} G(m,n) \xrightarrow{i_{m,n}} G(m+1,n) \xrightarrow{i_{m+1,n}} G(m+2,n) \xrightarrow{i_{m+2,n}} \cdots .$$

The following description of $G(n)$ follows from Proposition 6.5.

Lemma 6.9. *For all integers $n \geq 0$, the set of vertices of $G(n)$ is the disjoint union of $V_r(n)$, and the set of edges is the disjoint union of $E_r(n)$, where $r \geq 0$, such that:*

1. *The set $V_0(n)$ contains a unique vertex, denoted by p_n.*
2. *For all integers $r \geq 1$, each edge of $E_r(n)$, from $v \in V_r(n)$ to $w \in V_{r+1}(n)$, corresponds to a morphism of the form $\bullet Sq^k$ for some integer $k \geq 0$, and we label this edge by Sq^k. In this case, we write $[v,w] = Sq^k$. If there is no edge from v to w, then we write $[v,w] = 0$.*
3. *For all vertices $v \in V_r(n)$, there exists a unique collection of vertices*

$$\left\{ v_i \;\middle|\; \begin{array}{l} v_i \in V_i(n), 0 \leq i \leq r, \\ v_0 = p_{m+n}, v_r = v, \\ [v_i, v_{i+1}] = Sq^{k_i}, \\ [v_i, w] \neq Sq^{k_i}, \forall w \neq v_{i+1}, \\ 2k_0 > n, \\ 2k_{i+1} > k_i, 0 \leq i \leq r-2, \\ m+n-\sum_{i=0}^{r-1} k_i \geq k_{r-1}, \end{array} \right\}$$

forming a path from p_{m+n} to v, and such a path is called an admissible path. We also say that v is of Lambda bidegree $\|v\| = (\|v\|_1, \|v\|_2) = \left(r, \sum_{i=0}^{r-1} k_i\right)$.

In particular, $V_1(1)$ consists of vertices $\left\{ \lambda_i | \, 0 \leq i, [p_1, \lambda_i] = Sq^{i+1} \right\}$.

Remark 6.10. In Lemma 6.9, we do not describe the set of edges because it can be obtained by taking the colimit of the sets $E(m,n)$. And thanks to the point (2) of Remark 6.6, in practice, the computations concerning $G(n)$ will always be carried out using the model $G(m,n)$ for some suitable integer $m \geq 0$.

6.2 The Lambda algebra

In this paragraph, we will show that there exists an appropriate product $\times$ on $G(1)$ such that $(G(1), \times)$ is the Lambda algebra.

Definition 6.11. We define $\mathbb{T}$ to be the free bigraded algebra over $\mathbb{F}_2$ generated by the symbols λ_i of bidegree $(1, i+1)$ for all integers $i \geq 0$, and Λ to be the bigraded $\mathbb{F}_2-$vector space generated by the set of vertices of $G(1)$, where we take into account the Lambda bidegree as defined in Lemma 6.9.

Lemma 6.12. *The $\mathbb{F}_2-$vector space Λ is endowed with a structure of right $\mathbb{T}-$module, defined as follows:*

$$x\lambda_i = \sum_{[x,y]=Sq^{i+1}} y \tag{6.2}$$

for all vertices x of $G(1)$. Moreover, the $\mathbb{T}-$linear morphism

$$\begin{aligned} g : \mathbb{T} &\longrightarrow \Lambda \\ \alpha &\longmapsto p_1\alpha \end{aligned}$$

is an epimorphism, and the set

$$\{ p_1\lambda_{k_1}\lambda_{k_2}\ldots\lambda_{k_n} | \, 2k_i \leq k_{i+1}, 1 \leq i \leq n-1 \}$$

forms an $\mathbb{F}_2-$basis of Λ.

Proof. It is straightforward that the action law (6.2) defines the structure of a right $\mathbb{T}-$module for Λ. We now show that g is surjective. Indeed, for all vertices $v \in V_r(1)$, there exists a unique admissible path

$$\left\{ v_i \;\middle|\; \begin{array}{l} v_i \in V_i(1), 0 \leq i \leq r, \\ v_0 = p_1, v_r = v, \\ [v_i, v_{i+1}] = Sq^{k_i}, \\ [v_i, w] \neq Sq^{k_i}, \forall w \neq v_{i+1}, \\ 2k_0 > n, \\ 2k_{i+1} > k_i, r-2 \geq i \geq 0 \end{array} \right\}$$

connecting p_1 and v. Hence, we have

$$v = p_1\lambda_{k_0-1}\lambda_{k_1-1}\lambda_{k_2-1}\dots\lambda_{k_{r-1}-1}.$$

Then, this concludes that g is surjective. □

Lemma 6.13. *The kernel of $g : \mathbb{T} \to \Lambda$ is the two-sided ideal generated by:*

$$\lambda_a\lambda_b = \sum_{2a+2b>3i+1\geq 6b+4} \binom{a-2i-1}{i-2b-1}\lambda_i\lambda_{a+b-i} \tag{6.3}$$

for all integers a, b such that $a \geq 2b+1$.

Proof. Recall that the set

$$\{p_1\lambda_{k_1}\lambda_{k_2}\dots\lambda_{k_n} \,|\, 2k_i \leq k_{i+1}, 1 \leq i \leq n-1\}$$

forms an $\mathbb{F}_2-$basis of Λ. We now show how to express $p_1\lambda_a\lambda_b$ as a linear combination of this basis for all $a \geq 2b+1$. Let A be the set

$$\left\{v \in V_2(1) \,\middle|\, [p_1\lambda_a, v] = Sq^{b+1}\right\},$$

then

$$p_1\lambda_a\lambda_b = \sum_{v\in A} v.$$

By definition of $G(1)$ as the colimit of $G(m,1), m \geq 1$, then $p_1\lambda_a \in G(m,1)$ if and only if $m \geq 2a+1$. Moreover, for all $w \in V_2(1)$ such that $[p_1\lambda_a, w] \neq 0$, then $w \in V_2(m,1)$ for all $m \geq 2a+1$, and $[p_1\lambda_a, w] \neq 0$ in $G(m,1)$. In particular, when $m = 2a+1$, then $p_1\lambda_a$ corresponds to a direct summand of the form $J(a+1)$ of $\tilde{\Phi}\left(\mathrm{Aug}\left(\mathscr{G}\left(k-1,n\right)\right)\right)^0$. Therefore, an element $w \in V_2(2a+1,1)$, with $[p_1\lambda_a, w] = Sq^{b+1}$, where $a \geq 2b+1$, exists if and only if w is defined by the admissible form $p_1\lambda_i\lambda_j$ such that $Sq^{a+1}Sq^{b+1}$ appears in the expression of $Sq^{i+1}Sq^{j+1}$ as linear combination of admissible Steenrod monomials. Hence, it follows from the Adem relations that we have

$$p_1\lambda_a\lambda_b = \sum_{2a+2b>3i+1\geq 6b+4} \binom{a-2i-1}{i-2b-1} p_1\lambda_i\lambda_{a+b-i}$$

for all $a \geq 2b+1$. Now, let $x = p_1\lambda_{a_1}\lambda_{a_2}\dots\lambda_{a_k}$, and denote by $m < k$ the first index such that $a_m \geq 2a_{m+1}+1$. Then, by the same method, we have

$$p_1\lambda_{a_1}\lambda_{a_2}\dots\lambda_{a_k} = p_1\lambda_{a_1}\lambda_{a_2}\dots\lambda_{a_{m-1}}\left(\sum_{3i=6a_{m+1}+3}^{2a_m+2a_{m+1}-2} \binom{a_m-2i-1}{i-2a_{m+1}-1}\lambda_i\lambda_{a+b-i}\right)\lambda_{a_{m+2}}\lambda_{a_{m+3}}\dots\lambda_{a_k}.$$

As a result, the kernel of g is the two-sided ideal generated by the relations (6.3). □

It follows from Lemma 3.7 and Proposition 6.7 that Λ is endowed with a differential graded module structure with explicit homology.

Definition-Proposition 6.14. Let $d : \Lambda \to \Lambda$ be the $\mathbb{F}_2-$linear map defined by:

$$d(x) = \sum_{[x,v]=Sq^0} v$$

for all vertices $x \in V_r(1)$. Then, (Λ, d) is a differential graded $\mathbb{F}_2-$vector space. Moreover, we have

$$\mathrm{H}^{r,s}(\Lambda, d) \cong \varinjlim_n \mathrm{Ext}^r_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2, \Sigma^{n+s}\mathbb{F}_2\right) \cong \mathrm{Ext}^r_{\mathcal{A}}\left(\mathbb{F}_2, \Sigma^s\mathbb{F}_2\right).$$

We will actually prove that (Λ, d) is a differential graded algebra.

Definition 6.15. Denote by $X^\bullet$ the exact sequence where $X^0 = X^1 = J(1)$, $\partial^0 = \bullet Sq^0$ and X^k is trivial for all $k \geq 2$. Then $\mathscr{G}^m\left(X^\bullet\right)$ is an exact sequence that we denote by X_m. We denote by $G\left(X_m\right)$ the associated graph with respect to the decomposition of the terms of X_m as direct sum of Brown-Gitler modules.

Remark 6.16. 1. For all integers $m \geq 1$, we have

$$V_r\left(G\left(X_m\right)\right) = V_r(m,1) \bigsqcup V_{r-1}(m,1),$$
$$E_r\left(G\left(X_m\right)\right) = E_{r,r+1}(m,1) \bigsqcup E_{r-1}(m,1) \bigsqcup F_r,$$

where

$$F_r := \left\{ [v,v] = Sq^0 \middle| \, v \in V_r(m,1) \subset V_r\left(G\left(X_m\right)\right) \right\}.$$

2. For all integers $m \geq 1$, we have

$$\mathrm{H}\left(t, G\left(X_m\right)\right) = (1+t)\mathrm{H}\left(t, G(m,1)\right).$$

Similar to the construction of Λ, we have the following result.

Proposition 6.17. *There exist canonical inclusions of graphs $G(X_m) \to G(X_{m+1})$ corresponding the the inclusion of complexes $\Sigma X_m \to X_{m+1}$. Denote by X the colimit of*

$$\cdots \to G(X_{m-1}) \to G(X_m) \to G(X_{m+1}) \to \cdots,$$

and by $\mathbb{L}$ the $\mathbb{F}_2-$vector space generated by the vertices of X. Then, $\mathbb{L}$ is an associative bigraded algebra generated by the symbols λ_i of bidegree $(1, i+1)$ for all integers $i \geq -1$, subject to the relations

$$\lambda_a \lambda_b = \sum_{2a+2b > 3i+1 \geq 6b+4} \binom{a-2i-1}{i-2b-1} \lambda_i \lambda_{a+b-i},$$

for all integers a, b such that $a \geq 2b+1$.

We now show that (Λ, d) is a differential algebra.

Lemma 6.18. *The algebra Λ is a subalgebra of $\mathbb{L}$. Moreover, we have*

$$d(\alpha) = \alpha\lambda_{-1} - \lambda_{-1}\alpha.$$

for all $\alpha \in \Lambda \subset \mathbb{L}$. Hence, (Λ, d) is a differential algebra.

Proof. For all $\alpha \in \Lambda$ of bidegree (r,s), we have:

$$\begin{aligned}
\alpha\lambda_{-1} &= \sum_{[\alpha,v]=Sq^0} v \\
&= \left(\sum_{\substack{[\alpha,v]=Sq^0 \\ v \in V_{r+1}(1)}} v \right) + \left(\sum_{\substack{[\alpha,v]=Sq^0 \\ v \in V_r(1) \subset V_{r+1}(X)}} v \right) \\
&= d(\alpha) + \lambda_{-1}\alpha.
\end{aligned}$$

The lemma follows. □

Definition 6.19. For all integers $m, n \geq 0$, let $\Lambda(m,n)$ denote the bigraded $\mathbb{F}_2-$vector space generated by the vertices $V(m,n)$ of the graph $G(m,n)$, where the bidegree of the generator associated with a vertex v of bidegree (r,s) in $G(m,n)$ is $(r, m+n-s)$.

Remark 6.20. 1. It is obvious that the law

$$x\lambda_i = \sum_{[x,y]=Sq^{i+1}} y$$

for all vertices x of $G(m,n)$ defines an action of $\mathbb{T}$ on $\Lambda(m,n)$. Moreover, this action is unstable in the following sense: $x\lambda_i = 0$ for all generators x of bidegree (r,s) such that $2i+2 > s$. Using a similar argument as that in the proof of Lemma 6.13, we can show that this action passes to the quotient Λ and yields a structure of right $\Lambda-$module for $\Lambda(m,n)$.

2. We can also define a $\mathbb{F}_2-$linear map $\partial^{m,n} : \Lambda(m,n) \to \Lambda(m,n)$ by:

$$\partial^{m,n}(x) = \sum_{[x,v]=Sq^0} v$$

for all vertices $x \in V(m,n)$. In fact, we have

$$\partial^{m,n}\left(p_{m,n}\theta\right) = p_{m,n} d\left(\theta\right),$$

where d is the differential of Λ.

7 Minimal resolutions and finite exact sequences of BG modules

The Bockstein sequence, as it is defined in Definition 2.5, plays an important role in the rest of the present paper. This section is devoted to study the interaction between the pseudo-hyperresolution method and the iterated suspensions of the Bockstein sequence. In fact, we will prove that this sequence is *saturated* with respect to the pseudo-hyperresolution method, in the sense of Proposition 7.8.

Let $\mathcal{C}$ be an abelian category with enough injective objects, and let M be an object of $\mathcal{C}$. Recall that an injective resolution $\left(I^i, \partial^i, i \geq 0\right)$ of M is minimal if I^0 is the injective hull of M, I^1 is the injective hull of I^0/M, and I^j is that of $I^{j-1}/\mathrm{Im}\left(\partial^{j-2}\right)$ for all $j \geq 2$. The following lemma is classical and is left to the readers.

Lemma 7.1. *Let M and N be two objects of an abelian category $\mathcal{C}$. Moreover, let $\left(I^i, \partial^i, i \geq 0\right)$ be a minimal injective resolution of M, and let $\left(J^i, \delta^i, i \geq 0\right)$ be an injective resolution of $M \oplus N$. Then a split projection $M \oplus N \to M$ gives rise to a split projection of complexes $\left\{\alpha^i : J^i \to I^i, i \geq 0\right\}$.*

As a consequence, minimal injective resolutions are unique up to isomorphisms.

Let denote by $\mathrm{Ch}(\mathcal{C})$ the category of cochain complexes $\left(C^i, \partial^i, i \geq 0\right)$ of objects in $\mathcal{C}$.

Definition 7.2 (Extension of complexes)**.** If

$$0 \to \left(C^i, \partial^i, i \geq 0\right) \xrightarrow{\alpha^i} \left(D^i, \delta^i, i \geq 0\right) \xrightarrow{\beta^i} \left(E^i, \eta^i, i \geq 0\right) \to 0$$

is a short exact sequence of cochain complexes, then $\left(D^i, \delta^i, i \geq 0\right)$ is called an extension of $\left(C^i, \partial^i, i \geq 0\right)$ by $\left(E^i, \eta^i, i \geq 0\right)$.

Definition 7.3. A complex $\left(C^i, \partial^i, i \geq 0\right)$ is called $0-$*simple* if there exists an integer n such that ∂^n is an isomorphism, and C^i is trivial for all $i \neq n, n+1$. A complex $\left(C^i, \partial^i, i \geq 0\right)$ is called $n-$simple if it is an extension of a $(n-1)-$simple complex by another one of type $0-$simple. A complex is called simple if it is $k-$simple for some k.

An easy induction on the height of BG modules yields:

Lemma 7.4. *If*

$$0 \to C^0 \xrightarrow{\partial^0} C^1 \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^k} C^{k+1} \to 0$$

is an exact sequence of BG modules of finite height, then it is simple.

In what follows, we characterize minimal resolutions of finite unstable modules.

Proposition 7.5. *Let M be a finite unstable module and $\left(I^k, \partial^k, k \geq 0\right)$ be an injective resolution of M. Then this resolution is minimal if and only if every induced morphism*

$$J(r) \hookrightarrow I^n \xrightarrow{\partial^n} I^{n+1} \twoheadrightarrow J(s)$$

is not an isomorphism.

Proof. Suppose that there exist an isomorphism

$$J(r) \hookrightarrow I^n \xrightarrow{\partial^n} I^{n+1} \twoheadrightarrow J(r),$$

then following Lemma 4.1, the resolution $\left(I^k, \partial^k, k \geq 0\right)$ is an extension of another resolution of M by an exact $0-$simple sequence and hence cannot be minimal.

Vice versa, let $\left(J^k, \delta^k, k \geq 0\right)$ be the minimal resolution of M. Then there exists a split projection of complexes $\alpha^i : I^i \to J^i$. The kernel of this projection is therefore a finite exact sequence of BG modules. Hence Lemma 7.4 and the proposition hypothesis imply the triviality of this kernel. Then, $\left(\alpha^i, i \geq 0\right)$ is an isomorphism of complexes. $\square$

Corollary 7.6. *Let M be a finite unstable module and $I := \left(I^k, \partial^k, k \geq 0\right)$ be an injective resolution of M. Denote by C its maximal simple sub-complex. Then the quotient I/C is the minimal injective resolution of M.*
Definition 7.7. Let $I := \left(I^k, \partial^k\right)$ be a complex of BG modules. We denote by $A(I)$ the quotient I/C of I by its maximal simple sub-complex.

In the rest of this section, we study an interesting interaction of general exact sequences of BG modules with the pseudo-hyperresolution procedure. Recall that the Bockstein sequence is defined in Definition 2.5.
Proposition 7.8. *For all integers $n \leq -1$, denote by $\left(I_k^{n,\bullet}, \partial^{n,\bullet}\right)$ the minimal injective resolution of $\Sigma^k \mathscr{B}_n$. Then, we have an equivalence of exact sequences:*

$$A\left(\mathscr{P}\left(\Sigma^k \mathscr{B}_\bullet, I_k^{\bullet,\bullet}\right)\right)_\bullet \cong \mathscr{B}_{\bullet-k}.$$

Proof. The case $k = 1$ is straight forward as the minimal injective resolution of $\Sigma^k \mathscr{B}_n$ is given by Mahowald short exact sequences. Suppose that the result is true for all $k \leq m$, we prove that it still holds for $k = m+1$. For all $s \leq -1$, denote by $J^{s,\bullet}$ the minimal injective resolution of $\Sigma A\left(\mathscr{P}\left(\Sigma^m \mathscr{B}_\bullet, I_m^{\bullet,\bullet}\right)\right)_s$. We will prove that

$$A\left(\mathscr{P}\left(\Sigma A\left(\mathscr{P}\left(\Sigma^m \mathscr{B}_\bullet, I_m^{\bullet,\bullet}\right)\right), J^{\bullet,\bullet}\right)\right) \cong A\left(\mathscr{P}\left(\Sigma^{m+1} \mathscr{B}_\bullet, I_{m+1}^{\bullet,\bullet}\right)\right), \tag{7.1}$$

Let $\alpha \leq -1$, then

$$A\left(\mathscr{P}\left(\Sigma A\left(\mathscr{P}\left(\Sigma^m \mathscr{B}_{\bullet\geq\alpha}, I_m^{\bullet,\bullet}\right)\right), J^{\bullet,\bullet}\right)\right) \cong A\left(\mathscr{P}\left(\Sigma^{m+1} \mathscr{B}_{\bullet\geq\alpha}, I_{m+1}^{\bullet,\bullet}\right)\right),$$

as they are both isomorphic to the minimal injective resolution of $\Sigma^{m+1}\mathrm{Ker}\left(\beta_\alpha\right)$. Let α tends to $-\infty$, we get the isomorphism (7.1). Therefore, we have:

$$\begin{aligned} A\left(\mathscr{P}\left(\Sigma^{m+1} \mathscr{B}_\bullet, I_{m+1}^{\bullet,\bullet}\right)\right) &\cong A\left(\mathscr{P}\left(\Sigma A\left(\mathscr{P}\left(\Sigma^m \mathscr{B}_\bullet, I_m^{\bullet,\bullet}\right)\right), J^{\bullet,\bullet}\right)\right), \\ &\cong A\left(\mathscr{P}\left(\Sigma A\left(\mathscr{B}_{\bullet-m}\right), J^{\bullet,\bullet}\right)\right), \\ &\cong A\left(\mathscr{P}\left(\Sigma \mathscr{B}_{\bullet-m}, J^{\bullet,\bullet}\right)\right), \\ &\cong \mathscr{B}_{\bullet-m-1}. \end{aligned}$$

The proposition follows. □

8 Resolutions of spheres

Recall that the Ext-groups $\mathrm{Ext}_{\mathcal{U}}^s\left(\Sigma^n \mathbb{F}_2, \Sigma^t \mathbb{F}_2\right)$, where $s, t, n \geq 0$, are of interest because of the unstable Adams spectral sequence converging to the homotopy groups of spheres. One of the most basic way to compute these groups relies on the construction of the minimal injective resolution of $\Sigma^t \mathbb{F}_2$ for all integers $t \geq 0$. That is what we are going to do in this section. In fact, we will compare the minimal injective resolution of $\Sigma^t \mathbb{F}_2$ with the Bockstein sequence, and show that they agree in a certain specific area. The keys to this result are Proposition 7.8 and the injective dimension of $\Sigma^m J(n)$ for all integers $m, n \geq 0$.

8.1 Injective dimension

This section aims at studying the injective dimension of $\Sigma^m J(n)$ for all integers $m \geq 0$ and $n \geq 2$. To be more precise, we will show that this dimension is bounded by $[(m+n)/2]$, where $[-]$ denotes the integral part of a number. The intrigued reader might wonder why we do not consider the case $n = 1$. Here is the reason: this case does not follow the same rule as that of the cases $n \geq 2$. The injective resolution $\mathscr{G}_{m,1}$ of $\Sigma^m J(1)$ is of length m, and, actually, we will show that m is the injective dimension of $\Sigma^m J(1)$.
Lemma 8.1. *For all integers $m \geq 0, n \geq 2$, then the injective dimension of $\Sigma^m J(n)$ is bounded by $[(m+n)/2]$.*

Proof. We proceed by induction on m. The case $m = 0$ is trivial. Suppose that we have proved the cases $m < k$, we now verify the case $m = k$. But, it follows from the exact sequence

$$0 \to \Sigma^k J(n) \to \Sigma^{k-1} J(n+1) \to \Sigma^{k-1} J\left(\frac{n+1}{2}\right) \to 0$$

that

$$\dim_{inj} \Sigma^k J(n) = \dim_{inj} \Sigma^{k-1} J(n+1) \leq \left[\frac{n+k}{2}\right]$$

if n is even, and that

$$\begin{aligned}\dim_{inj}\Sigma^k J(n) &\leq \max\left\{\dim_{inj}\Sigma^{k-1}J(n+1), \dim_{inj}\Sigma^{k-1}J\left(\frac{n+1}{2}\right)\right\}\\ &\leq \max\left\{\left[\frac{n+k}{2}\right], \left[\frac{k-1+\frac{n+1}{2}}{2}\right]\right\} \leq \left[\frac{n+k}{2}\right]\end{aligned}$$

if n is odd. This concludes the proof of the lemma. □

8.2 Resolution of spheres

In this subsection, we will prove that $\Sigma^n\mathbb{F}_2$ is of injective dimension $n-1$, and we also describe a large part of its minimal injective resolution.

Theorem 8.2. *Given an integer $n \geq 0$ and denote by $\left(I^k, \partial^k, k \geq 0\right)$ the minimal injective resolution of $\Sigma^n\mathbb{F}_2$. Then, $I^k = 0$ for all $k > n-1$, and for all $k > [n/2]$, the resolution coincides with the Bockstein sequence as follows:*

$$I^k \cong \mathscr{B}_{k-n}.$$

Proof. This is a direct consequence of Proposition 7.8. Recall that, for all integers $m \leq -1$, we denote by $\left(I_k^{m,\bullet}, \partial^{m,\bullet}\right)$ the minimal injective resolution of $\Sigma^k\mathscr{B}_m$. Then, we have an equivalence of exact sequences:

$$A\left(\mathscr{P}\left(\Sigma^k\mathscr{B}_\bullet, I_k^{\bullet,\bullet}\right)\right)_\bullet \cong \mathscr{B}_{\bullet-k}.$$

Now, it follows from Lemma 8.1 that

$$\dim_{inj}\mathscr{B}_m \leq \left[\frac{k-m}{2}\right]$$

for all $m \leq -1$. Hence, we have

$$\begin{aligned}\mathscr{B}_{s-k} &\cong A\left(\mathscr{P}\left(\Sigma^k\mathscr{B}_\bullet, I_k^{\bullet,\bullet}\right)\right)_s\\ &\cong I_k^{-1,s}\end{aligned}$$

for all $s > [k+1/2]$. Remark that $I_k^{-1,\bullet}$ is the minimal injective resolution of $\Sigma^k J(1)$, which is isomorphic to $\Sigma^{k+1}\mathbb{F}_2$, then replacing k by $n-1$ concludes the proof of the theorem. □

Corollary 8.3. *Let n, t be nonnegative integers and $s > [n/2]$. Then:*

$$\mathrm{Ext}^s_{\mathcal{U}}\left(\Sigma^t\mathbb{F}_2, \Sigma^n\mathbb{F}_2\right) \cong \begin{cases}\mathbb{F}_2 & \text{if } t = n-s,\\ \mathbb{F}_2 & \text{if } n-s-1 \equiv 0(4) \text{ and } n-s-1 = 2t,\\ 0 & \text{otherwise.}\end{cases}$$

9 Algebraic EHP sequences

In this section, we focus on computing extensions groups of unstable modules using the pseudo-hyper resolution method. We then show how to connect these groups by long exact sequences.

9.1 Extensions groups of unstable modules

The $J(n)$ form a system of co-generators for the category $\mathcal{U}$. But we can even say more about the classification of injective unstable modules after the work of Lannes and Schwartz. Before going further, recall that there exist other interesting injective unstable modules beside Brown-Gitler modules. These are given by the cohomologies $\mathrm{H}^*\left(B\left(\mathbb{Z}/2\right)^{\oplus d};\mathbb{F}_2\right)$. Lannes and Zarati observe that the tensor products between these modules and Brown-Gitler modules remain injective. Latter, Lannes and Schwartz prove that an indecomposable injective module must be a tensor product of the form $L \otimes J(n)$, where L denotes a certain indecomposable direct summand of $\mathrm{H}^*\left(B\left(\mathbb{Z}/2\right)^{\oplus d};\mathbb{F}_2\right)$. Since the category $\mathcal{U}$ is locally noetherian, we get:

Theorem 9.1 ([18]). *Each indecomposable injective unstable module is a tensor product of the form $L \otimes J(n)$ between an indecomposable direct summand of a certain $\mathrm{H}^*\left(B\left(\mathbb{Z}/2\right)^{\oplus d};\mathbb{F}_2\right)$ and a Brown-Gitler module $J(n)$. Every injective unstable module splits into a direct sum between a BG module and a direct sum of modules of the form $L \otimes J(n)$ where L is a direct summand of $\tilde{\mathrm{H}}^*\left(B\mathbb{Z}/2;\mathbb{F}_2\right)^{\otimes d}$ for some d.*

The module $\mathrm{H}^*\left(B\mathbb{Z}/2;\mathbb{F}_2\right)$ has several amazing properties that we will recall now. It is reduced. The functor $-\otimes \mathrm{H}^*\left(B\mathbb{Z}/2;\mathbb{F}_2\right)$ admits a left adjoint in $\mathcal{U}$ which is denoted by T. Because

$$\mathrm{H}^*\left(B\mathbb{Z}/2;\mathbb{F}_2\right) \cong \tilde{\mathrm{H}}^*\left(B\mathbb{Z}/2;\mathbb{F}_2\right) \oplus \mathbb{F}_2$$

then $T \cong \bar{T} \oplus \mathbb{F}_2$ where $\bar{T}$ is left adjoint to $-\otimes \tilde{\mathrm{H}}^*\left(B\mathbb{Z}/2;\mathbb{F}_2\right)$. Moreover, if M is a finite unstable module then $\bar{T}M = 0$. Therefore:

Lemma 9.2. *Let M be a finite unstable module and L be a direct summand of $\tilde{\mathrm{H}}^*\left(B\mathbb{Z}/2;\mathbb{F}_2\right)^{\otimes d}$ for some d. Then*

$$\mathrm{Hom}_{\mathcal{U}}\left(M, L \otimes J(n)\right) = 0$$

for very n.

Remark 9.3. Let M be an unstable module. Denote by $(I^\bullet, \partial^\bullet)$ its minimal injective resolution. For each I^j, denote by B^j the BG module part and by R^j the other. It follows from Lemma 9.2 that $(B^\bullet, \partial^\bullet)$ is a sub-complex of $(I^\bullet, \partial^\bullet)$. It also follows that if N is a finite unstable module then

$$\mathrm{Ext}^s_{\mathcal{U}}\left(N, M\right) \cong \mathrm{H}^s\mathrm{Hom}_{\mathcal{U}}\left(N, B^\bullet\right).$$

We will now study the extension groups $\mathrm{Ext}^s_{\mathcal{U}}\left(N, \Sigma^k M\right)$ using the pseudo-hyper resolution method. Let us recall how to use this method to construct an injective resolution of $\Sigma^k M$ from $(I^\bullet, \partial^\bullet)$. The minimal injective resolution of $\Sigma^k J(n)$ is denoted by $B(k,n)$. If L is a direct summand of $\tilde{\mathrm{H}}^*\left(B\mathbb{Z}/2;\mathbb{F}_2\right)^{\otimes d}$ then $L \otimes B(k,n)$ is the minimal injective resolution for $L \otimes \Sigma^k J(n)$. Hence there is no BG module part for this resolution. Now using the pseudo-hyper resolution method with respect to these minimal injective resolutions, we get an injective resolution for $\Sigma^k M$. Denote this resolution by $M(k)$ and by $MB(k)$ the pseudo-hyper complex obtain from $\Sigma^k B^\bullet$ using the same method. What we have shown so far is that $MB(k)$ is the BG module part of $M(k)$. Therefore, we have

$$\mathrm{Ext}^s_{\mathcal{U}}\left(N, \Sigma^k M\right) \cong \mathrm{H}^s\mathrm{Hom}_{\mathcal{U}}\left(N, MB(k)\right).$$

We write $B(k)$ for the quotient of $MB(k)$ by its maximal simple sub-complex. Hence:

$$\mathrm{Ext}^s_{\mathcal{U}}\left(\Sigma^t\mathbb{F}_2, \Sigma^k M\right) \cong \mathrm{Hom}_{\mathcal{U}}\left(\Sigma^t\mathbb{F}_2, B(k)^s\right).$$

We now show how to connect these groups in long exact sequences.

9.2 Algebraic EHP sequences

Fix M and $B(t)$ as in the previous sub-section. We index all the direct summands isomorphic to $J(n)$ of $B(t)^s$ by the set $A_{(t,n,s)}$, then:

$$\mathrm{Ext}^s_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2, \Sigma^t M\right) \cong \mathrm{Hom}_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2, \bigoplus_{A_{(t,n,s)}} J(n)\right). \tag{ext}$$

From the construction of $\mathscr{G}\left(B(t)\right)$ (see Proposition 6.2), the sum of all the direct summands $J(n)$ of $\mathscr{G}\left(B(t)\right)^s$ is

$$B_{(n,t+1,s)} := C_{(n,t+1,s)} \bigoplus D_{(n,t+1,s)}$$

where

$$C_{(n,t+1,s)} := \bigoplus_{A_{(t,n-1,s)}} J(n),$$
$$D_{(n,t+1,s)} := \bigoplus_{A_{(t,2n-1,s-1)}} J(n).$$

Let $X \in \left\{C_{(n,t+1,s)}, D_{(n,t+1,s)}\right\}$ and $Y \in \left\{C_{(n,t+1,s+1)}, D_{(n,t+1,s+1)}\right\}$, we denote by $\partial^s_{X,Y}$ the composition

$$X \hookrightarrow \mathscr{G}\left(B(t)\right)^s \xrightarrow{\partial^s} \mathscr{G}\left(B(t)\right)^{s+1} \twoheadrightarrow Y.$$

Therefore the maps

$$\partial^s_{DA} := \begin{pmatrix} \partial^s_{C_{(n,t+1,s)},C_{(n,t+1,s+1)}} & \partial^s_{D_{(n,t+1,s)},C_{(n,t+1,s+1)}} \\ \partial^s_{C_{(n,t+1,s)},D_{(n,t+1,s+1)}} & \partial^s_{D_{(n,t+1,s)},D_{(n,t+1,s+1)}} \end{pmatrix}$$

make $\left\{B_{(n,t+1,s)}, \partial^s_{DA}\right\}$ a complex. Remark that the BG algorithm shows that

$$\partial^s_{DA} := \begin{pmatrix} 0 & \partial^s_{D_{(n,t+1,s)},C_{(n,t+1,s+1)}} \\ 0 & 0 \end{pmatrix}.$$

Hence

$$\begin{aligned} \mathrm{Ext}^s_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2, \Sigma^{t+1}M\right) &\cong \mathrm{H}_s\mathrm{Hom}_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2, \mathscr{G}\left(B(t)\right)\right), \\ &\cong \mathrm{H}_s\mathrm{Hom}_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2, B_{(n,t+1,s)}\right), \\ &\cong \mathrm{Ker}\left(\left(\partial^s_{D_{(n,t+1,s)},C_{(n,t+1,s+1)}}\right)_*\right) \bigoplus \mathrm{Coker}\left(\left(\partial^{s-1}_{D_{(n,t+1,s-1)},C_{(n,t+1,s)}}\right)_*\right). \end{aligned}$$

We will now explicit the morphisms $\left(\partial^s_{D_{(n,t+1,s)},C_{(n,t+1,s+1)}}\right)_*$. The source of this morphism is described as follows:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2, D_{(n,t+1,s)}\right) &\cong \mathrm{Hom}_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2, \bigoplus_{A_{(t,2n-1,s-1)}} J(n)\right), \\ &\cong \mathrm{Hom}_{\mathcal{U}}\left(\Sigma^{2n-1}\mathbb{F}_2, \bigoplus_{A_{(t,2n-1,s-1)}} J(2n-1)\right), \\ &\cong \mathrm{Ext}^{s-1}_{\mathcal{U}}\left(\Sigma^{2n-1}\mathbb{F}_2, \Sigma^t M\right). \end{aligned}$$

And the target is:

$$\begin{aligned} \mathrm{Hom}_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2, C_{(n,t+1,s+1)}\right) &\cong \mathrm{Hom}_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2, \bigoplus_{A_{(t,n-1,s+1)}} J(n)\right), \\ &\cong \mathrm{Hom}_{\mathcal{U}}\left(\Sigma^{n-1}\mathbb{F}_2, \bigoplus_{A_{(t,n-1,s+1)}} J(n-1)\right), \\ &\cong \mathrm{Ext}^{s+1}_{\mathcal{U}}\left(\Sigma^{n-1}\mathbb{F}_2, \Sigma^t M\right). \end{aligned}$$

Therefore

$$\left(\partial^s_{D_{(n,t+1,s)},C_{(n,t+1,s+1)}}\right)_* : \mathrm{Ext}^{s-1}_{\mathcal{U}}\left(\Sigma^{2n-1}\mathbb{F}_2, \Sigma^t M\right) \to \mathrm{Ext}^{s+1}_{\mathcal{U}}\left(\Sigma^{n-1}\mathbb{F}_2, \Sigma^t M\right).$$

We denote by P^s the morphisms $\left(\partial^s_{D_{(n,t+1,s)},C_{(n,t+1,s+1)}}\right)_*$ and then we obtain the following exact sequences:

$$0 \to \mathrm{Ker}\left(P^{s-1}\right) \to \mathrm{Ext}^{s-2}_{\mathcal{U}}\left(\Sigma^{2n+1}\mathbb{F}_2, \Sigma^t M\right) \to \mathrm{Ext}^s_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2, \Sigma^t M\right) \to \mathrm{Coker}\left(P^{s-1}\right) \to 0,$$

$$0 \to \mathrm{Coker}\left(P^{s-1}\right) \to \mathrm{Ext}^s_{\mathcal{U}}\left(\Sigma^{n+1}\mathbb{F}_2, \Sigma^{t+1} M\right) \to \mathrm{Ker}\left(P^s\right) \to 0.$$

We can now conclude:

Theorem 9.4 (Algebraic EHP sequences)**.** *There exist a long exact sequence for each n:*

$$\cdots \xrightarrow{H} E_2^{s-2,t}(S^{2n+1}, M) \xrightarrow{P} E_2^{s,t}(S^n, M) \xrightarrow{E} E_2^{s,t+1}(S^{n+1}, M) \xrightarrow{H} E_2^{s-1,t}(S^{2n+1}, M) \xrightarrow{P} \cdots \tag{9.1}$$

where $E_2^{s,t}(S^n, M)$ stands for $\mathrm{Ext}^s_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2, \Sigma^t M\right)$.

9.3 A special case

This section studies the special case of EHP sequence for $n = 2^k$. In particular we show that in this case the morphisms P are trivial.

Lemma 9.5. *If $n = 2^k$ then the morphism $\left(\partial^s_{D_{(n,t+1,s)},C_{(n,t+1,s+1)}}\right)_*$ is trivial.*

Proof. Recall that

$$\partial^s_{D_{(2^k,t+1,s)},C_{(2^k,t+1,s+1)}} : \bigoplus_{A_{(t,2^{k+1}-1,s-1)}} J(2^k)_\omega \to \bigoplus_{A_{(t,2^k-1,s+1)}} J(2^k)_\alpha.$$

Suppose that this morphism is not trivial then there must exist an identity coefficient in the matrix form. The first item of the BG algorithm assures that in this case, the Steenrod square Sq^{2^k} can be factorized as a product of other Steenrod operations. This cannot be true since Sq^{2^k} is indecomposable. Therefore $\partial^s_{D_{(2^k,t+1,s)},C_{(2^k,t+1,s+1)}}$ is trivial. □

We get the following consequence:

Theorem 9.6 (James splitting)**.** *There are short exact sequences*

$$0 \to E_2^{s,t}\left(S^{2^k-1}, M\right) \to E_2^{s,t+1}\left(S^{2^k}, M\right) \to E_2^{s-1,t}\left(S^{2^{k+1}-1}, M\right) \to 0$$

for every k.

10 Bousfield's proof on the existence of algebraic EHP sequences

> *An interesting fact about the algebraic EHP sequence: it can be derived in a completely abstract way. That is, it can be derived without the construction of special projective or injective resolutions and without any computation whatsoever. Bousfield explained to me how to do this, about 45 years ago. Here is the key idea. One has a "loop functor" on the category of unstable Steenrod modules. It is left adjoint to the suspension. This functor is right exact, and has non-trivial left-derived functors. The key is to notice that these left-derived functors are zero, in homological degrees greater than one. The existence of the long-exact EHP sequence follows immediately.*
>
> William M. Singer, *Private communication* [26]

Let M and N be two unstable modules. In this section, we prove that there exists a long exact sequence of Ext-groups

$$\mathrm{Ext}^{s-2}_{\mathcal{U}}(\Omega_1 M, N) \to \mathrm{Ext}^{s}_{\mathcal{U}}(\Omega M, N) \to \mathrm{Ext}^{s}_{\mathcal{U}}(M, \Sigma N) \to \mathrm{Ext}^{s-1}_{\mathcal{U}}(\Omega_1 M, N)$$

for all integers $s \geq 2$, where Ω is the left adjoint of the suspension functor Σ and Ω_k is the $k-$th left-derived functor of Ω.

The loop functor of unstable modules

Recall that the Frobenius twist of unstable modules exists under the form of the double functor Φ, defined in Section 2. The operator Sq_0, which associates an element x of degree n of a certain unstable module M with $Sq^n x$, defines an $\mathcal{A}_2-$linear morphism $\lambda_M : \Phi M \to M$. It is classical that both kernel and cokernel of this morphism are a suspension of an unstable module, which can be described explicitly according to the following proposition.

Proposition 10.1. *Let M be an unstable module. Then, the cokernel of λ_M is a suspension, and we denote by ΩM the unstable module such that $\Sigma\Omega M \cong \mathrm{Coker}(\lambda_M)$. Then, Ω defines an endofunctor of unstable modules which is also a left adjoint of Σ. Let $\Omega_k M$ be the $k-$left-derived functor of Ω for all integers $k \geq 0$, then we have*

$$\Sigma\Omega_k M \cong \begin{cases} \mathrm{Ker}(\lambda_M) & \text{if } k = 1, \\ 0 & \text{if } k > 1. \end{cases}$$

Proof. An unstable module N is isomorphic to ΣQ for some $Q \in \mathcal{U}$ if and only if Sq_0 acts trivially on N. Then, it is evident that the cokernel of λ_M is a suspension of some unstable module that we denote by ΩM. This is a well-defined endofunctor of $\mathcal{U}$ as λ_M is natural on M. And since both Φ and Id are exact, it follows that Ω is right exact. In order to show that Ω is left adjoint to Σ, it suffices to show that

$$\mathrm{Hom}_{\mathcal{U}}(\Omega P, N) \cong \mathrm{Hom}_{\mathcal{U}}(P, \Sigma N)$$

for all unstable modules P, N such that P is projective. But this is a direct consequence of the following well-known short exact sequence (see, e.g., [24]).

$$0 \to \Phi F(n) \xrightarrow{\lambda(F(n))} F(n) \xrightarrow{\sigma(F(n))} \Sigma F(n-1) \to 0.$$

where $\sigma(F(n))$ is defined by $\imath_n \mapsto \Sigma\imath_{n-1}$.

Now, to study the left-derived functors of Ω, let $(P_i, \partial_i : P_{i+1} \to P_i)_{i\geq 0}$, abbreviated as $P_\bullet$, be a projective resolution of M. Because, on the one hand, $\Sigma\Omega P_\bullet$ fits in the short exact sequence of complexes

$$0 \to \Phi P_\bullet \xrightarrow{\lambda_{P_\bullet}} P_\bullet \xrightarrow{\sigma_{P_\bullet}} \Sigma\Omega P_\bullet \to 0,$$

and, on the other hand, Φ is exact, then the homology groups of $\Sigma\Omega P_\bullet$ are trivial, in degrees greater than one, and there is a short exact sequence,

$$0 \to \mathrm{H}_1(\Sigma\Omega P_\bullet) \to \Phi M \xrightarrow{\lambda_M} M \xrightarrow{\sigma_M} \Sigma\Omega M \to 0, \tag{10.1}$$

connecting the homology groups of degrees 0 and 1. As Σ is exact, we have:

$$\mathrm{Ker}(\lambda_M) \cong \Sigma\Omega_1 M.$$

The proposition follows. □

Remark 10.2.

- For all unstable module M, the morphism $\lambda_{\Sigma M}$ is trivial. Therefore,

$$\Omega\Sigma M \cong M, \quad \forall M \in \mathcal{U}.$$

- The loop of σ_M is the identity of ΩM. As the loop functor Ω is right exact, then $\Omega\lambda_M$ is trivial.
- Fix $\{P_i, \partial_i : P_{i+1} \to P_i, i \geq 0\}$ a projective resolution of M. Denote by C the co-kernel $\mathrm{Coker}(\Omega\partial_1)$. Then $\{\Omega P_i, \Omega\partial_i : \Omega P_{i+1} \to \Omega P_i, i \geq 1\}$ is a projective resolution of C. Moreover, C fits in the short exact sequence:

$$0 \to \Omega_1 M \to C \to \frac{\Omega P_1}{\mathrm{Ker}(\Omega\partial_0)} \to 0.$$

  Because Ω is right exact, $\Omega P_1/\mathrm{Ker}(\Omega\partial_0)$ is isomorphic to the kernel of the morphism $\Omega P_0 \twoheadrightarrow \Omega M$.
- If $\Omega_1 M$ is trivial, then $\Omega P_\bullet$ is a projective resolution of ΩM.

Bousfield's sequences

Let N be an unstable module. Let M be an unstable module and $\{P_i, \partial_i : P_{i+1} \to P_i, i \geq 0\}$, abbreviated as $P_\bullet$, be a projective resolution of M. Since ΩP_0 is projective, the long exact sequence of Ext-groups associated with the short exact sequence

$$0 \to \frac{\Omega P_1}{\mathrm{Ker}(\Omega\partial_0)} \to \Omega P_0 \to \Omega M \to 0$$

splits into an exact sequence

$$0 \to \mathrm{Hom}_{\mathcal{U}}(\Omega M, N) \to \mathrm{Hom}_{\mathcal{U}}(\Omega P_0, N) \to \mathrm{Hom}_{\mathcal{U}}\left(\frac{\Omega P_1}{\mathrm{Ker}(\Omega\partial_0)}, N\right) \to \mathrm{Ext}^1_{\mathcal{U}}(\Omega M, N) \to 0,$$

and isomorphisms

$$\mathrm{Ext}^s_{\mathcal{U}}\left(\frac{\Omega P_1}{\mathrm{Ker}(\Omega\partial_0)}, N\right) \xrightarrow{\sim} \mathrm{Ext}^{s+1}_{\mathcal{U}}(\Omega M, N),$$

for all $s \geq 1$. Now, because $\{\Omega P_i, \Omega\partial_i : \Omega P_{i+1} \to \Omega P_i, i \geq 1\}$ is a projective resolution of C (see Remark 10.2), then for every $s \geq 1$ we have:

$$\begin{aligned}\mathrm{Ext}^s_{\mathcal{U}}(C, N) &\cong \mathrm{H}^{s+1}\left(\mathrm{Hom}_{\mathcal{U}}(\Omega P_\bullet, N), (\Omega\partial_\bullet)^*\right)\\ &\cong \mathrm{Ext}^{s+1}_{\mathcal{U}}(M, \Sigma N).\end{aligned}$$

Therefore, the long exact sequence of Ext-groups associated with the short exact sequence

$$0 \to \Omega_1 M \to C \to \frac{\Omega P_1}{\mathrm{Ker}(\Omega\partial_0)} \to 0$$

is the general algebraic long-exact EHP sequence.

Theorem 10.3 (Bousfield's proof)**.** *For all unstable modules M and N, there exists a long exact sequence*

$$\cdots \to \mathrm{Ext}_{\mathcal{U}}^{s-2}(\Omega_1 M, N) \to \mathrm{Ext}_{\mathcal{U}}^{s}(\Omega M, N) \to \mathrm{Ext}_{\mathcal{U}}^{s}(M, \Sigma N) \to \mathrm{Ext}_{\mathcal{U}}^{s-1}(\Omega_1 M, N) \to \cdots$$

of Ext-groups, where $s \geq 2$.

Let M be $\Sigma^n \mathbb{F}_2$ and N be $\Sigma^t \mathbb{F}_2$. If $n \geq 1$, then the morphism $\lambda_M : \Phi M \to M$ is trivial. Therefore,

$$\Omega M \cong \Sigma^{n-1}\mathbb{F}_2,$$
$$\Omega_1 M \cong \Sigma^{2n-1}\mathbb{F}_2.$$

A reformulation of Bousfield's long exact sequence, in this case, yields the algebraic EHP sequence for S^n.

Theorem 10.4. *For every positive integer n, there exists a long exact sequence*

$$\cdots \xrightarrow{H} E_2^{s-2,t}(S^{2n+1}) \xrightarrow{P} E_2^{s,t}(S^n) \xrightarrow{E} E_2^{s,t+1}(S^{n+1}) \xrightarrow{H} E_2^{s-1,t}(S^{2n+1}) \xrightarrow{P} \cdots,$$

where $E_2^{s,t}(S^n) := \mathrm{Ext}_{\mathcal{U}}^{s}(\Sigma^n \mathbb{F}_2, \Sigma^t \mathbb{F}_2)$.

Application

In this subsection, we use the loop functor Ω to study a special case of the algebraic EHP sequence.

If $\{C_i, \partial_i : C_{i+1} \to C_i, i \geq 0\}$ is a complex, denote by $C_\bullet[1]$ the complex:

$$C_\bullet[1]_i = \begin{cases} C_{i-1} & \text{if } i \geq 1, \\ 0 & \text{if } i = 0, \end{cases}$$
$$\partial[1]_i = \begin{cases} \partial_{i-1} & \text{if } i \geq 1, \\ 0 & \text{if } i = 0. \end{cases}$$

Let M be an unstable module such that $\Omega_1 M$ is trivial. Fix $\{P_\bullet, \partial_i : P_{i+1} \to P_i, i \geq 0\}$, abbreviated as $P_\bullet$, a projective resolution of M, and fix $\{Q_\bullet, \delta_i : Q_{i+1} \to Q_i, i \geq 0\}$, abbreviated as $Q_\bullet$, a projective resolution of ΦM. The natural transformation $\lambda : \Phi \to Id$ gives rise to a morphism of complexes: $\lambda_{P_\bullet} : \Phi P_\bullet \to P_\bullet$. On the other hand, the identity of ΦM yields a morphism of complexes: $\omega : Q_\bullet \to \Phi P_\bullet$. Therefore, the composition map $\omega \circ \lambda_{P_\bullet}$ makes the following diagram commute.

$$\begin{tikzcd} Q_\bullet \arrow[r, "\omega\circ\lambda_{P_\bullet}"] \arrow[d] & P_\bullet \arrow[d] \\ \Phi M \arrow[r, "\lambda_M"'] & M \end{tikzcd}$$

Now, we can consider $\omega \circ \lambda_{P_\bullet} : Q_\bullet \to P_\bullet$ as a double complex with two non-trivial columns $Q_\bullet$ and $P_\bullet$. Denote by $T_\bullet$ the total complex of this double complex. As these columns, $Q_\bullet$ and $P_\bullet$, are acyclic, the homology groups of $T_\bullet$ are computed as follows.

$$\mathrm{H}_i(T_\bullet) \cong \begin{cases} \mathrm{Coker}(\lambda_M) & \text{if } i = 0, \\ \mathrm{Ker}(\lambda_M) & \text{if } i = 1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $\Omega_1 M$ is trivial, $T_\bullet$ is a projective resolution of $\Sigma\Omega M$. We now compute $\Omega T_\bullet$. It follows from Remark 10.2 that the morphism $\Omega(\omega \circ \lambda_{P_\bullet})$ is trivial. This fact has two simple consequences:

$$\Omega T_\bullet \cong \Omega P_\bullet \bigoplus \Omega Q_\bullet[1],$$
$$\mathrm{H}^s\left(\mathrm{Hom}_{\mathcal{U}}\left(T_\bullet, \Sigma^t\mathbb{F}_2\right)\right) \cong \mathrm{H}^s\left(\mathrm{Hom}_{\mathcal{U}}\left(P_\bullet, \Sigma^t\mathbb{F}_2\right)\right) \oplus \mathrm{H}^{s-1}\left(\mathrm{Hom}_{\mathcal{U}}\left(Q_\bullet, \Sigma^t\mathbb{F}_2\right)\right)$$

For all integers $s \geq 0, t \geq 1$. As it follows from Remark 10.2 that $\Omega P_\bullet$ is a projective resolution of ΩM, and $\Omega Q_\bullet$ is a projective resolution of $\Omega\Phi M$, we have:

Lemma 10.5. *Let M be an unstable module such that $\Omega_1 M$ is trivial. For all unstable module N, we have an isomorphism of Ext-groups*

$$\mathrm{Ext}_{\mathcal{U}}^{s}(\Sigma\Omega M, \Sigma N) \cong \mathrm{Ext}_{\mathcal{U}}^{s}(\Omega M, N) \bigoplus \mathrm{Ext}_{\mathcal{U}}^{s-1}(\Omega\Phi M, N),$$
$$\mathrm{Ext}_{\mathcal{U}}^{s}\left(\Sigma\Omega M, \Sigma^t\mathbb{F}_2\right) \cong \mathrm{Ext}_{\mathcal{U}}^{s}\left(M, \Sigma^t\mathbb{F}_2\right) \bigoplus \mathrm{Ext}_{\mathcal{U}}^{s-1}\left(\Phi M, \Sigma^t\mathbb{F}_2\right)$$

for all integers $s \geq 0, t \geq 1$. (Here, by convention, the Ext-groups of degree -1 are trivial.)

Proof. The Ext-groups $\mathrm{Ext}^*_{\mathcal{U}}(\Sigma\Omega M, \Sigma N)$ can be computed as follows.

$$\begin{aligned}\mathrm{Ext}^s_{\mathcal{U}}(\Sigma\Omega M, \Sigma N) &\cong \mathrm{H}^s(\mathrm{Hom}_{\mathcal{U}}(T_\bullet, \Sigma N))\\ &\cong \mathrm{H}^s(\mathrm{Hom}_{\mathcal{U}}(\Omega T_\bullet, N))\\ &\cong \mathrm{H}^s\left(\mathrm{Hom}_{\mathcal{U}}\left(\Omega P_\bullet \bigoplus \Omega Q_\bullet[1], N\right)\right)\\ &\cong \mathrm{Ext}^s_{\mathcal{U}}(\Omega M, N) \bigoplus \mathrm{Ext}^{s-1}_{\mathcal{U}}(\Omega\Phi M, N).\end{aligned}$$

We can then conclude the lemma. □

Remark that, following the exact sequence

$$\Phi^2 M \to \Phi M \to \Phi(\Sigma\Omega M) \to 0,$$

we have

$$\Omega\Phi M \cong \Sigma\Phi\Omega M$$

for all unstable modules M. Then, applying Lemma 10.5 to $M = \Phi^n F(1)$, we recover Theorem 9.6.

11 Infinite complex projective space

This section aim at studying a particular relation between the infinite complex projective and spheres, which is a direct consequence of the following result in view of pseudo-hyperresolution method.

Lemma 11.1. *Denote by* H *the cohomology* $\mathrm{H}^*(B\mathbb{Z}/2; \mathbb{F}_2)$ *of the classifying space of* $\mathbb{Z}/2$*. Then, we have*

$$\begin{aligned}\Phi\mathrm{H} &\cong \mathrm{H}^*(\mathbb{C}P^\infty; \mathbb{F}_2),\\ \Omega\mathrm{H} &\cong \Phi\mathrm{H}.\end{aligned}$$

Proof. The lemma follows from the fact that H is isomorphic to the polynomial algebra $\mathbb{F}_2[u]$ on one variable of degree 1 whereas $\mathrm{H}^*(\mathbb{C}P^\infty; \mathbb{F}_2)$ is the polynomial algebra $\mathbb{F}_2[t]$ on one variable of degree 2. □

Remark 11.2. Denote by $\bar{\mathrm{H}}$ the reduced cohomology $\tilde{\mathrm{H}}^*(B\mathbb{Z}/2; \mathbb{F}_2)$. Then, we have two short exact sequences

$$\begin{aligned}0 &\longrightarrow \Phi\bar{\mathrm{H}} \longrightarrow \bar{\mathrm{H}} \longrightarrow \Sigma\Phi\mathrm{H} \longrightarrow 0,\\ 0 &\longrightarrow \Phi\mathrm{H} \longrightarrow \mathrm{H} \longrightarrow \Sigma\Phi\mathrm{H} \longrightarrow 0.\end{aligned}$$

As a result, we have the following long exact sequences

$$\begin{aligned}0 &\longrightarrow \Phi\bar{\mathrm{H}} \longrightarrow \bar{\mathrm{H}} \longrightarrow \Sigma\mathrm{H} \longrightarrow \Sigma^2\mathrm{H} \longrightarrow \cdots \longrightarrow \Sigma^d\mathrm{H} \longrightarrow \cdots,\\ 0 &\longrightarrow \Phi\mathrm{H} \longrightarrow \mathrm{H} \longrightarrow \Sigma\mathrm{H} \longrightarrow \Sigma^2\mathrm{H} \longrightarrow \cdots \longrightarrow \Sigma^d\mathrm{H} \longrightarrow \cdots.\end{aligned}$$

Theorem 11.3. *For all integers* $s \geq 0$*, we have*

$$\mathrm{Ext}^s_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2, \Sigma^k\Phi\mathrm{H}\right) \cong \bigoplus_{m+q=s} \mathrm{Ext}^q_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2, \Sigma^{k+m}\mathbb{F}_2\right). \tag{11.1}$$

Proof. Following Remark 11.2, there is a long exact sequence

$$0 \longrightarrow \Sigma^k\Phi\mathrm{H} \longrightarrow \Sigma^k\mathrm{H} \longrightarrow \Sigma^{k+1}\mathrm{H} \longrightarrow \Sigma^{k+2}\mathrm{H} \longrightarrow \cdots \longrightarrow \Sigma^{k+d}\mathrm{H} \longrightarrow \cdots$$

for all integers $k \geq 0$. In accordance with Section 9, we denote by $B(s-1,1)$ the minimal injective resolution of $\Sigma^s\mathbb{F}_2$ for all integers $s \geq 0$. As $\mathrm{H}\otimes J(n)$ is an injective unstable module for all integers $n \geq 0$, then $\mathrm{H}\otimes B(s-1,1)$ is the minimal injective resolution of $\Sigma^s\mathrm{H}$ for all integers $s \geq 0$. Let $B(s-1,1)^t$ denote the $t-$th term of the resolution $B(s-1,1)$, and let $\partial^{l,t}$ be the differential

$$\mathrm{H}\otimes B(k-1+l,1)^t \to \mathrm{H}\otimes B(k-1+l,1)^{t+1}.$$

Then, in view of Proposition 4.5, there exist morphisms

$$\partial_i^{l,t} : \mathrm{H}\otimes B(k-1+l,1)^t \to \mathrm{H}\otimes B(k-1+l+i+1,1)^{t-i}$$

such that the morphisms

$$\partial^l = \begin{pmatrix} \partial^{0,l} & 0 & \cdots & 0 \\ \partial_0^{0,l} & \partial^{1,l-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \partial_{l-1}^{0,l} & \partial_{l-2}^{1,l-1} & \cdots & \partial^{l,0} \\ \partial_l^{0,l} & \partial_{l-1}^{1,l-2} & \cdots & \partial_0^{l,0} \end{pmatrix} \tag{11.2}$$

make the sequence

$$\mathrm{H}\otimes B\left(k-1,1\right)^0 \xrightarrow{\partial^0} \mathrm{H}\otimes B\left(k-1,1\right)^1 \bigoplus \mathrm{H}\otimes B\left(k,1\right)^0 \xrightarrow{\partial^1} \cdots \xrightarrow{\partial^l} \bigoplus_{m+n=l+1} \mathrm{H}\otimes B\left(k-1+m,1\right)^n \xrightarrow{\partial^{l+1}} \cdots \tag{11.3}$$

an injective resolution of $\Sigma^k\Phi\mathrm{H}$. It follows from Remark 9.3 that

$$\mathrm{Ext}^s_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2, \Sigma^k\Phi\mathrm{H}\right) \cong \mathrm{Ext}^s_{\mathcal{U}}\left(\Sigma^n\mathbb{F}_2, B^\bullet\right),$$

where $B^\bullet$ is the BG part of the resolution (11.3). It is evident that

$$B^s \cong \bigoplus_{m+n=s} B\left(k-1+m,1\right)^n.$$

By abuse of notation, we adopt the notation (11.2) for the induced differentials of $B^\bullet$. Because a morphism between two BG modules cannot factorize via a direct sum of tensor products of the form $\bar{\mathrm{H}}\otimes J(n)$, then the induced differential ∂^l of $B^\bullet$ has the following simple form

$$\partial^l = \begin{pmatrix} \partial^{0,l} & 0 & \cdots & 0 \\ 0 & \partial^{1,l-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \partial^{l,0} \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

Therefore, $B^\bullet$ is the direct sum of the resolution $B\left(s,1\right)$ with some appropriate shifting. This conclude the isomorphism (11.1). □

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