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Aperiodic Order and Spectral Properties

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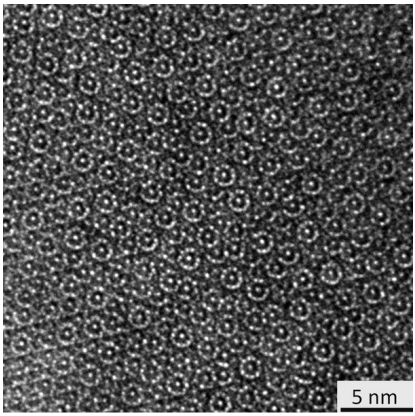
Periodic structures like a typical tiled kitchen floor or the arrangement of carbon atoms in a diamond crystal certainly possess a high degree of order. But what is order *without* periodicity? In this snapshot, we are going to explore highly ordered structures that are substantially nonperiodic, or *aperiodic*. As we construct such structures, we will discover surprising connections to various branches of mathematics, materials science, and physics. Let us catch a glimpse into the inherent beauty of aperiodic order!

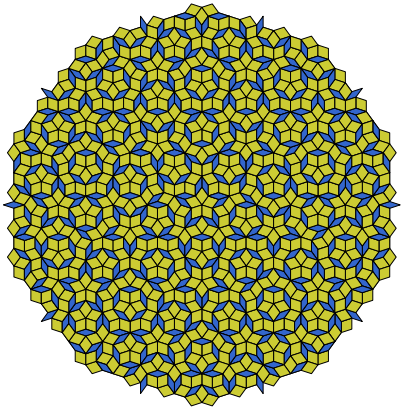
1 Introduction

The concept of *order* is fundamental to human culture. It not only underlies much of art and architecture — the scientific approach to the understanding of our world is based on detecting and describing order in nature.

Although humans instinctively understand what order is, giving a precise definition of order is surprisingly difficult. A perfect crystal, such as a flawless diamond, provides an example of order in nature. In crystals, atoms are ordered in a periodically repeating pattern. But nature can accommodate more complex forms of order: *quasicrystals* are materials with a highly ordered atomic structure, but with no *periodicity* at all in the arrangement of atoms.



(a) Electron microscopy image of a natural quasicrystal with composition $Al_{71}Ni_{24}Fe_5$, found in the Khatyrka meteorite.



(b) A patch of the rhombic version of the Penrose tiling, which was discovered by Roger Penrose in 1974 [14]; see also [2].

Figure 1: Two examples of aperiodic order.

Quasicrystals were discovered by Dan Shechtman in 1982 [17].[1] An electron microscopy image of a natural quasicrystal is shown in Figure 1a.

The *theory of aperiodic order* considers mathematical structures that possess order without periodicity. While quasicrystals provide additional physical context to the research, the concept dates back to the beginning of the twentieth century, with the work of Harald Bohr on almost-periodic functions [7, 8]. It has since developed into a fascinating field of modern mathematics, with links to many areas such as dynamical systems, harmonic analysis, spectral theory, and number theory, to name but a few.

The visual attraction of order becomes apparent in *tilings*. In mathematics, a tiling is a covering of the entire plane or space by tiles with no overlaps or gaps — like in a puzzle.[2] Periodic tilings (such as the ones many people have in their kitchens) may be interesting, but aperiodic tilings bring about many more complex and fascinating phenomena. For example, the aperiodic tiling in Figure 1b has many symmetries, but it has no *translational symmetry*: if you have two identical copies of this infinite tiling and move them with respect to

[1] For his ground-breaking discovery, Dan Shechtman received the Wolf Prize in Physics in 1999, and the Nobel Prize in Chemistry in 2011.

[2] For more information on tilings in mathematics see, for example, Wikipedia: `https://en.wikipedia.org/wiki/Tessellation#In_mathematics`.

each other, there is only one position where the two copies completely agree.

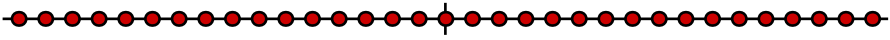
At a more abstract level, an attractive aspect of our field is the fact that one can make seemingly simple statements which are easy to understand but turn out to be difficult to prove.

For example: is there a planar shape that can tile the entire plane without gaps or overlaps but does not admit any periodic tiling (note that the tiling in Figure 1b is built from *two* different planar shapes)? The answer to this question is still open, although there has been some recent progress towards an answer by the Australian mathematician Joan Taylor (see [19], [2, Example 6.6] and references therein).

In this snapshot, we introduce the general idea behind aperiodic order by means of simple but instructive examples, and provide a hint of why so-called spectral properties are of interest in this context. In doing so, we will gloss over any technical details. For a gentle introduction to aperiodic order, we refer to [3, 4, 1]; a proper mathematical account is given in [2], which also serves as our master reference for details that we cannot include here.

2 Point sets

Let us introduce the notion of periodic and aperiodic order by considering simple examples of point sets on the real line $\mathbb{R}$, that is, in one dimension of space. Sometimes we would like to distinguish different types of points, say by assigning colours. Each point is then characterised by its position $x \in \mathbb{R}$ and by its colour. Imagine placing a red point at each integer position to obtain the point set P_0 of red points at all positions $n \in \mathbb{Z}$. This looks like



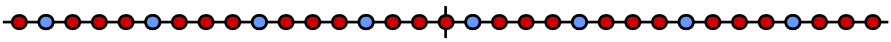
where the small vertical line denotes the position $n = 0$. This point set, which you have to imagine to continue indefinitely in both directions, is *periodic* with *period* 1, because shifting all positions by 1 reproduces the same point set. We express this as

$$P_0 + 1 = P_0,$$

where $P_0 + 1$ means adding 1 to the position of each point in the coloured point set P_0. Of course, if shifting by 1 maps the point set onto itself, so does shifting by 2, or indeed by any integer $n \in \mathbb{Z}$. We denote the set of all possible periods of P_0 by $\mathrm{per}(P_0)$. These periods clearly must be integers, and by what we just said, we have $\mathrm{per}(P_0) = \mathbb{Z}$. Our reasoning holds true for any periodic point set: once a point set possesses a non-zero period, it automatically possesses an infinite set of periods. The smallest positive period is called the *fundamental* period. The fundamental period of P_0 is 1.

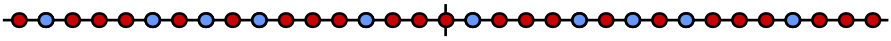
In general, the set of all periods forms a lattice, consisting of all integer linear combinations of a set of fundamental periods; see [2, Definition 2.4] for more on this.

Now, let us take the points at positions n with $n \equiv 1 \bmod 4$. This means, all n which give a remainder of 1 when divided by 4. We change the colour of these points to blue, and call the corresponding point set P_1. It looks like this:



Again you have to imagine the point set to continue in both directions, so we have changed the colour of infinitely many points. What is the periodicity of the new point set? We now need to shift P_1 by multiples of 4 to respect the positions and the colourings of points, so $\mathrm{per}(P_1) = 4\,\mathbb{Z} = \{\ldots, -8, -4, 0, 4, 8, \ldots\}$.

As the next step, let us look at all points at positions n with $n \equiv 7 \bmod 16$. You can convince yourself that all these points are currently red, so let us change them to blue to obtain the next point set, which we call P_2. The result



is still periodic, but now only under shifts by multiples of 16, so we have $\mathrm{per}(P_2) = 16\,\mathbb{Z}$.

We can continue this game, for example by defining P_{k+1} as the point set obtained from P_k by changing the colour of all points at positions

$$n \equiv (2 \cdot 4^k - 1) \bmod 4^{k+1}.$$

All these points are still red in P_k and become blue in P_{k+1}. The point set P_{k+1} is periodic under shifts by multiples of 4^{k+1}. In this way, we obtain point sets P_k for all integers $k \geq 0$, which are periodic with $\mathrm{per}(P_k) = 4^k\mathbb{Z}$. With increasing k, the periods get sparser and sparser. Indeed, in order to actually see this happening even for the next step ($k = 3$), you would need to consider a longer part of the point set than we displayed above, because this step affects points at positions $n \equiv 31 \bmod 64$ only. If we keep on performing this process indefinitely for all k and consider all the resulting blue points, we eventually end up with a point set P that no longer has any periods at all: if it had a period, this period would have to be an integer greater than 4^k for all k, but of course such an integer does not exist. For the underlying notion of convergence of series of point sets, the reader is invited to consult [2, Section 4].

A point set that does not admit a period is called *non-periodic*. In fact, our point set P is not just non-periodic, but actually *aperiodic*. To understand the difference between non-periodicity and aperiodicity, consider the following example: Take our original single-colour point set P_0 and change the colour

of a single point, say at 0, to blue. The resulting point set P' is non-periodic, because any non-zero shift moves the blue point at 0 to a red position and hence changes the point set. However, if you keep shifting the blue point further and further away, the point set will look more and more like the original point set P_0 around the origin, and in the limit where the blue point has been "moved off to infinity", the periodicity of the original point set P_0 is restored.

Thus, the point set P' would not be considered as aperiodic, because periodicity is violated only locally, not globally. Because, in our construction of the point sets P_k, we change the colour of infinitely many points in each step, periodicity is ultimately violated globally, and P possesses the stronger aperiodicity property.[3]

Although the point set P is aperiodic, it is clearly ordered in some sense, because it is built from an explicit construction which determines the colour for each position uniquely. Even if you did not know where the origin was located, you can still recognise this order. For instance, if you pick a red point which is located between two blue points, you know immediately that every second point along either direction will be red as well, because all points at even positions stay red in our construction. This also shows that you can never find two blue points next to each other. However, like for the simple periodic set P_0, if you do not know where the origin is located, you cannot decide where it would have been from looking at an arbitrarily large finite part of the set. This is because any local arrangement of colours occurs at infinitely many positions along the line (but not in a periodic manner). This property is called *recurrence* and arises here as a consequence of the systematic way in which we performed the colour changes, affecting points in the same way anywhere along the line.[4]

So, our point set P is an example of a structure that is both ordered and aperiodic. It is closely related to a class of sequences known as Toeplitz sequences [20]. The theory of aperiodic order is concerned with understanding such point sets (and more general ones) and analysing their properties.

3 Substitution and inflation

You may wonder why we changed colours specifically in the way we did when we constructed the point sets P_k above. Clearly, there are lots of ways to produce aperiodic point sets in a similar way. We chose this particular approach because P can also be obtained by a *substitution* or *inflation* rule.

[3] A mathematical explanation of the difference between the notions of non-periodicity versus aperiodicity requires a more careful definition of limits of shifted point sets; see [2, Section 4].

[4] In fact, in our example, we have a stronger form of recurrence, known as *uniform recurrence* or *repetitivity*, which means that the distance between consecutive occurrences of any particular local arrangement of colours is uniformly bounded.

To see how this works, let us denote the sequence of the two-coloured points by letters r for red and b for blue, and consider the rule S that maps $r \mapsto rb$ and $b \mapsto rr$. Applying this rule repeatedly, starting from a single letter, gives

$$r \overset{S}{\longmapsto} rb \overset{S}{\longmapsto} rbrr \overset{S}{\longmapsto} rbrrrbrb \overset{S}{\longmapsto} rbrrrbrbrbrrrbrr \overset{S}{\longmapsto} \dots$$

In each step, every letter is replaced by a pair of letters according to the rule S. You can repeat this procedure as often as you like, producing longer and longer words in the two letters r and b. In the limit, you obtain an infinite word v, which is mapped onto itself: $Sv = v$. This word is thus invariant under the application of the rule S, and in this sense possesses a symmetry under this operation, sometimes referred to as an *inflation symmetry*.

Extending our procedure slightly, we now start from the two-letter seed $r|r$, and apply the rule S to both sides of the vertical bar:

$$\begin{aligned} r|r &\overset{S}{\longmapsto} rb|rb \overset{S}{\longmapsto} rbrr|rbrr \overset{S}{\longmapsto} rbrrrbrb|rbrrrbrb \\ &\overset{S}{\longmapsto} rbrrrbrbrbrrrbrr|rbrrrbrbrbrrrbrr \overset{S}{\longmapsto} \dots \end{aligned}$$

When we now iterate S^2 rather than S, in the limit we obtain a sequence w, infinite in both directions, that satisfies $S^2 w = w$. Moreover, the only difference between w and Sw is at the first position left of the bar, which is red (in w) or blue (in Sw).

The surprising result is that the infinite sequence w exactly reproduces the sequence of colours in P, if you identify the first r after the vertical bar with the red point at 0. The mathematical proof of this identity requires some work. If you are interested, you can find the argument in [2, Chapter 4.5.1], where this example is referred to as the *period doubling substitution*.

Substitution rules like S have been studied extensively, and produce many well-known examples of interesting sequences. The most famous such sequence is named after Leonardo of Pisa, also known as Fibonacci, who (implicitly) introduced it in his book Liber Abaci already in 1202 [18], although it was apparently familiar to Indian mathematicians even earlier. The sequence was motivated by studying the evolution of a rabbit population, with the rule that, in one step, an adult rabbit produces one offspring, and a juvenile rabbit matures to an adult rabbit. This is, of course, a very simplified model in which rabbits live and reproduce eternally. And the total population grows exponentially!

Let us denote the adult rabbits by ℓ (for large) and the young rabbits by s (for small). The Fibonacci rule F is $\ell \mapsto \ell s$ and $s \mapsto \ell$. Applying the rule repeatedly, starting with a single adult, gives

$$\ell \overset{F}{\longmapsto} \ell s \overset{F}{\longmapsto} \ell s \ell \overset{F}{\longmapsto} \ell s \ell \ell s \overset{F}{\longmapsto} \ell s \ell \ell s \ell s \ell \overset{F}{\longmapsto} \ell s \ell \ell s \ell s \ell \ell s \ell \ell s \overset{F}{\longmapsto} \dots$$

which, when repeating the process indefinitely, produces an infinite word v, which satisfies $Fv = v$, and is known as the *Fibonacci sequence*.

Each finite word in the iteration above is the concatenation of the two previous words. Thus, the number of letters of any one of these words is the sum of the number of letters of the two previous words. The numbers of letters of the words form the sequence $1, 2, 3, 5, 8, 13, 21, 34, \dots$ known as *Fibonacci numbers*. The Fibonacci numbers satisfy the recursion relation $f_{k+1} = f_k + f_{k-1}$ for $k \geq 1$, with initial conditions $f_0 = 0$ and $f_1 = 1$ (in which case the list above starts with $f_2 = 1$, and $f_3 = f_2 + f_1 = 1 + 1 = 2$ and so on). The Fibonacci number f_k thus gives the total number of rabbits after k generations. Counting the numbers of adult or young rabbits (that is ℓ or s) in each of the generations again produces the same sequence, shifted by 1 in the index. Put differently, in a word of length f_{k+1} there are exactly f_k letters ℓ and f_{k-1} letters s. Using this observation, it is not difficult to show that the ratio of letters ℓ and s (the ratio of adult to young rabbits), as the number of generations grows, approaches the limit

$$\lim_{k \to \infty} \frac{f_k}{f_{k-1}} = \frac{1 + \sqrt{5}}{2} = 1.6180339887\dots$$

This number, often denoted by τ, is known as the *golden ratio*.[5] The number τ is irrational, which shows that the Fibonacci sequence v cannot be periodic: indeed, assuming that v repeated periodically after N letters, the ratio of letters in v would have to be the same as their ratio in a finite word of length N, and hence a rational number with a denominator of at most N.

There is a natural way to interpret the Fibonacci sequence as a point set on the real line $\mathbb{R}$. The rule F then becomes an *inflation* rule in the following sense: let us associate to the two letters ℓ and s two interval lengths, a long one (for ℓ, to fit the adult rabbits in) and a short one (for s, to fit the young rabbits in). A natural way to choose the length is given by the golden ratio again, so let us choose the length of the interval ℓ to be τ and the length of the interval s to be 1 (for the mathematical reason for this choice see the discussion of geometric inflation rules in [2, Chapter 4]).

[5] The number τ plays an important role in art and architecture, representing an "ideal way" of dissecting an interval into two parts. For more information on the golden ratio see, for example, Wikipedia: `https://en.wikipedia.org/wiki/Golden_ratio`.

Then, the geometric interpretation of the rule F is

which consists of a scaling of the intervals by a factor τ, followed by the dissection of the long interval into a long and short one (according to the rule $\ell \mapsto \ell s$, which is geometrically consistent because $\tau^2 = \tau + 1$) and interpreting the scaled short interval as a long one (according to the rule $s \mapsto \ell$). The geometric version of the infinite word v becomes a series of intervals

which is invariant under the geometric inflation map.

4 Cut-and-project sets

Now, we can produce the Fibonacci sequence in a seemingly very different way: the long and short intervals are obtained by projecting specific points of a two-dimensional periodic lattice onto the horizontal axis. The construction is sketched in Figure 2, where the blue points form a two-dimensional periodic lattice. We project all lattice points within the yellow and green strips onto the horizontal axis. Via this projection, points within the yellow strip become left endpoints of long intervals ℓ, and points within the green strip become left endpoints of short intervals s.

The one-dimensional tiling of the horizontal axis we just obtained turns out to be exactly the same as the one from the inflation rule discussed in the previous section.[6]

This interpretation of the Fibonacci case as a point set on the real line is called a *cut-and-project set* or *model set* [13].

It points to the inherent order that is "hidden" in the aperiodic sequence: although the tiling is not periodic, it is very closely related to a periodic structure, albeit in two dimensions rather than in one. This is an important property, and the cut-and-project construction can be generalised and applied in a quite general setting, as described in [2, Chapter 7].

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[6] Note that we have suppressed the important detail how to treat the lattice points that lie on the boundary of the strip – one way to deal with them is described in [2, Example 7.3].

Furthermore, note that there are other lattices leading to the same projected one-dimensional tiling. Our choice is motivated by an interesting connection to number theory, the Minkowski embedding of the ring $\mathbb{Z}[\tau] = \{m + n\tau \mid m, n \in \mathbb{Z}\}$; see [2, Chapter 3.4] for details.

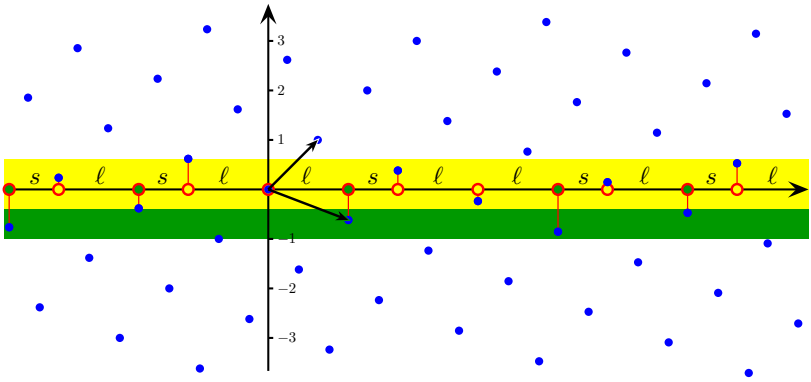

Figure 2: Cut-and-project description of the Fibonacci sequence from a two-dimensional lattice (blue dots). The lattice is generated by the two vectors $(1, 1)$ and $(\tau, 1-\tau)$, which are indicated by black arrows. Lattice points within the yellow strip become left endpoints of long intervals ℓ, while points within the green strip become left endpoints of short intervals s. The green strip intersects the vertical axis between -1 and $\tau - 2$, the yellow strip then takes over until $\tau - 1$. The resulting sequence of long and short intervals gives the Fibonacci sequence. The cut-and-project set comprises the projected lattice points indicated by the red circles.

The underlying higher-dimensional periodicity provides the resulting cut-and-project sets with a "quasi-periodic" order (a particular case of the general notion of "almost-periodic"). These cut-and-project sets are now quite well understood.

The fact that the Fibonacci sequence allows both an inflation and a projection description should not mislead you to assume that this happens in general. Indeed, we are looking at a very special situation here, although many of the "nice" examples are of this kind. A given inflation rule does not automatically allow for an embedding into a periodic lattice in a higher-dimensional space. It turns out that the sequence of red and blue points discussed in Section 2 does in fact also have a projection description, but only in a setting where the periodic lattice lies in a more general space, which is not a finite-dimensional Euclidean space. But in general even this is not guaranteed. Conversely, given a cut-and-project description, the projected structure need not possess an inflation description. In the Fibonacci setup shown in Figure 2, this is only true if the strip is chosen appropriately; see [2, Example 7.3 and Remark 7.6] for details.

5 Spectral properties: Diffraction

The one-dimensional examples discussed above should provide an intuitive idea about the type of structures that we have in mind when we talk about aperiodic order. We were able to understand the order in these examples because we knew how to construct them from explicit rules. But how can we characterise order in aperiodic structures without referring explicitly to construction rules? This is where spectral properties enter, inspired by applications in crystallography, physics, and materials science.

Experimentally, the atomic order in crystals is probed by looking at the pattern formed by radiation (such as X-rays) scattered by the material.[7] The pattern of scattered radiation is called the *diffraction pattern*. It provides information about the order and symmetry of the atomic arrangement. Diffuse diffraction patterns indicate a deviation from order in the material, whereas diffraction patterns consisting of discernible points indicate crystal-like order. Mathematically, the diffraction pattern encodes information about the *spatial autocorrelation* of the structure.

To understand what autocorrelation means, let us again consider our point set P from Section 2. We interpret all blue points b as holes (all radiation passes through them without scattering), and all red points r as scatterers (they scatter the radiation shone onto them). To formalize this notion, we introduce a function u, the *scattering strength*, with $u(r) = 1$ and $u(b) = 0$.

The autocorrelation for a given distance m is then the average over the product of scattering strengths of points at distance $m \in \mathbb{Z}$:

$$a(m) \;=\; \lim_{N\to\infty} \frac{1}{2N+1} \sum_{n=-N}^{N} u(w_n)u(w_{n+m}),$$

where w_n denotes the letter (r or b) at position n in the point set P. The *autocorrelation coefficient* $a(m)$ equals the proportion of times you find two scatterers at distance m along the line (by construction, $0 \le a(m) \le 1$).

The value of $a(m)$ can be explicitly calculated for the point set P. Writing $m = (2\ell+1)2^r$ with $r \ge 0$ and $\ell \in \mathbb{Z}$, we can calculate that $a(0) = 2/3$ and

$$a(m) \;=\; \frac{2}{3}\left(1 - \frac{1}{2^{r+1}}\right) \quad \text{for } m \neq 0.$$

[7] *X-ray crystallography* was invented by Max von Laue in 1912. For his discovery, he received the Nobel Prize in Physics in 1914. Further important developments came from William Henry Bragg and William Lawrence Bragg, father and son, which won them the 1915 Nobel Prize in Physics. X-ray crystallography is now one of the most important tools in many scientific fields. For more information see, for example, Wikipedia: `https://en.wikipedia.org/wiki/X-ray_crystallography`.

The details of this calculation are a little more involved, but can be found in [2, Remark 9.16].

The *diffraction* (or *diffraction intensity*) is obtained from the autocorrelation by what is called a "Fourier transform", which essentially means that we interpret the autocorrelation as a weighted sum of periodic functions — somewhat similar to the frequency analysis of sounds.[8] We measure order by analysing the contributions of each frequency to the autocorrelation [11].

Roughly speaking, the occurrence of a contribution on the Fourier side (that is, non-zero diffraction) reflects the coherent repetition of a motive or pattern in our structure.

For the example at hand, the increasingly long periods of powers of 4 in the point set P give rise to frequencies at rational numbers with powers of 2 in the denominator. The diffraction intensity in this case is given by the following function:

$$I(k) \coloneqq \begin{cases} 4/9, & \text{if } k \in \mathbb{Z}, \\ \frac{1}{9 \cdot 4^{r-1}}, & \text{if } k = \frac{2n+1}{2^r} \text{ with } n \in \mathbb{Z}, r \in \mathbb{N}, \\ 0, & \text{otherwise.} \end{cases}$$

The function $I(k)$ has one crucial property: on the one hand, the set

$$\mathcal{I}_0 \coloneqq \{k \,|\, I(k) > 0\}$$

is a *dense subset* of $\mathbb{R}$, meaning that any $x \in \mathbb{R}$ can be approximated arbitrarily well by some element of $\mathcal{I}_0$. But on the other hand, for any $t > 0$, the set

$$\mathcal{I}_t \coloneqq \{k \,|\, I(k) > t\},$$

consists of *isolated points* only, meaning that each element of $\mathcal{I}_t$ has a neighbourhood that does not contain any other element of $\mathcal{I}_t$.

In any experiment where diffraction intensity is measured, you would see the set $\mathcal{I}_t$ for some value of t (because no detector can detect arbitrarily small intensities). Hence, you would see a discrete pattern of spots of different intensity. A sketch of the diffraction intensity pattern of P, which is periodic with period 1, is shown in Figure 3. See [2, Chapter 9.4.4] for more details on this example.

[8] A periodic function with a short period is said to have a *high frequency*, whereas a periodic function with a long period is said to have a *low frequency*. For an introduction to the Fourier transform see, for example, Wikipedia: `https://en.wikipedia.org/wiki/Fourier_transform`. How to use it in our context is explained in [2, Chapters 8 and 9].

As mentioned previously, the point set P admits an interpretation as a cut-and-project set. The pure-point nature[9] of its diffraction-intensity spectrum is in line with the general result that such sets, under rather general assumptions, are pure-point diffractive [16].

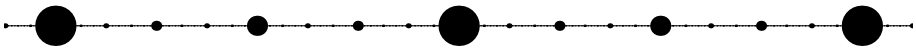

Figure 3: Sketch of the diffraction pattern associated to the point set P. Here, a contribution at k is represented by a disc, centred at k, of an area that is proportional to the diffraction intensity $I(k)$.

6 Advanced topics and recent developments

Diffraction is only one of several "spectral measures" that we could use to study our point set. We shall briefly mention two other spectral measures that have been studied extensively: the *dynamical spectrum* and the *Schrödinger spectrum*. This section assumes more mathematical knowledge than the other sections.

The *dynamical spectrum* is related to the action of translations on our point set, and to the space of all point sets obtained by such translations and appropriate limits of translates. The idea of associating a spectral measure to this dynamical system goes back to Koopman [12] and von Neumann [21]; see also [15].

Rather than looking at the action of translations on the space of all point sets itself, we consider the induced unitary action on a space of suitable functions on this space, in this case the Hilbert space of square-integrable functions. The dynamical spectrum then consists of the spectrum of this unitary operator, which contains its eigenvalues and suitable generalisations. It can detect order beyond the two-point correlations that diffraction "sees".

The diffraction spectrum and the dynamical spectrum, which is usually richer, are closely connected. This connection has been known for a long time: indeed, the original proof that cut-and-project sets (as introduced in Section 4) are, under rather general assumptions, pure-point diffractive was based on an argument linking the two spectra in the pure-point case [16]. Recent work has further elucidated the connection between the dynamical and diffraction spectra, which now is reasonably well understood [5].

[9] The term "pure-point nature" refers to the fact that the entire diffraction image consists of point-like concentrations of scattered intensity, and does not show diffuse components. A detailed understanding requires concepts from measure theory.

Finally, solid-state physics motivated yet another spectral measure, the *Schrödinger spectrum*. In a periodic material such as a crystal, electrons move "freely", whereas in a disordered material, electrons remain localised in suitable bounded regions for all times. Aperiodically ordered structures are somewhere in between these two. On the one hand, they have motives that keep repeating throughout the system, but on the other hand they lack the periodicity which would allow for electrons to move freely.

Indeed, it turns out that in the presence of aperiodic order, electrons behave in a very peculiar way, they are neither localised nor can they move freely. Associated to this behaviour are certain properties of the solutions to the difference equation

$$u(n+1) + u(n-1) + V(n)\,u(n) \;=\; E\,u(n)$$

where V is a given sequence of real numbers and E is a real number. Free motion of electrons corresponds to the presence of so-called "extended states", that is, solutions u that neither decay nor increase at infinity. On the other hand, localisation in space of electrons corresponds to the presence of square-summable solutions u.[10] The first scenario occurs for periodic V, whereas the second scenario occurs for random V, obtained for example by repeatedly tossing a coin to determine the values of the sequence V. If instead we consider a V that is aperiodically ordered (for example, V may be given by the Fibonacci sequence), then the solutions u are neither extended, nor square-summable, which in turn is the signature of the peculiar, anomalous electron motion alluded to above. We refer to [9] and references therein for a recent comprehensive review of the results in this area, and to [10] for a detailed analysis of the Fibonacci case.

Although quite a bit is now known about the spectral properties of Schrödinger operators for large classes of one-dimensional examples, there is currently no satisfactory understanding of the relation between the spectral properties of these systems and the other two spectral properties discussed earlier, if indeed such a relation exists. In some respect, these spectral measures behave in opposite ways; in diffraction and dynamical spectra, point spectra indicate order in a system, while in the Schrödinger case, a periodically ordered system shows a continuous spectrum. It is our hope that further investigation of aperiodically ordered systems may shed some light on this open question.

[10] Square-summable means $\sum_{n=-\infty}^{\infty} |u(n)|^2 < \infty$.

7 Summary

In this snapshot, we essentially limited the discussion to one-dimensional point sets — that is, tilings by intervals. However, aperiodic order is not limited to one dimension. Quite on the contrary: some of the beauty of the subject becomes apparent in higher-dimensional aperiodic tilings. The most famous example is Penrose's tiling [14], of which there exist a number of variants; see [2, Chapter 6.2]. One of the versions featuring two rhombic tiles is shown in Figure 1b. Similar to the one-dimensional Fibonacci system, Penrose's tiling can be described either as a two-dimensional inflation tiling or as a cut-and-project set, in this case using a lattice in at least four-dimensional space. Its diffraction is of pure-point nature and shows perfect tenfold symmetry, which is incompatible with periodicity by what is called the "crystallographic restriction", which specifies the rotational symmetries that are compatible with periodicity; see [2, Chapter 3.2].

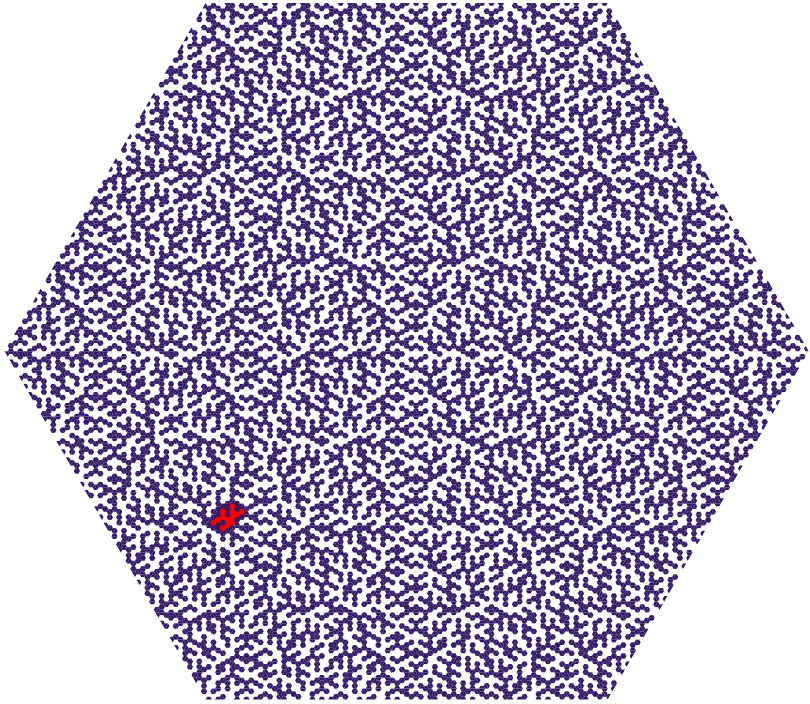

Figure 4: A patch of Joan Taylor's llama tiling. One llama cluster has been highlighted in red.

A more recent, stunning example is Joan Taylor's llama tiling shown in Figure 4. It is related to the open question mentioned at the beginning, of whether there is a planar shape that tiles the plane but does not allow any periodic tiling. In the llama tiling, the tiles are hexagons of two different types or "chirality" (one of which is kept white in the figure), and the name refers to the fact that the smallest connected cluster of tiles of one colour resembles the outline of a llama. For more details about this tiling, we refer to [2, Example 6.6].

Aperiodic order is a fascinating area of mathematics with applications in the world of crystals, which has also inspired works in arts and architecture. It highlights deep questions about the concept of order which lies at the heart of our scientific approach to understand nature. Spectral properties provide one approach to understand and quantify order in a system, but we are still far from a complete classification of ordered structures.

Further reading

We recommend the following snapshots for some further interesting discussions on tilings and Fibonacci numbers

- Snapshot 4/2015 *Friezes and tilings* by Thorsten Holm,
- Snapshot 2/2016 *Random sampling of domino and lozenge tilings* by Éric Fusy.

Image credits

Figure 1a is cropped and resized from [6]. This file is licensed under the Creative Commons Attribution 4.0 International license.

All other images were created by the authors.

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