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Spaces of Riemannian metrics

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Riemannian metrics endow smooth manifolds such as surfaces with intrinsic geometric properties, for example with curvature. They also allow us to measure quantities like distances, angles and volumes. These are the notions we use to characterize the "shape" of a manifold. The space of Riemannian metrics is a mathematical object that encodes the many possible ways in which we can geometrically deform the shape of a manifold.

1 Smooth manifolds

The main characters of this snapshot are smooth manifolds and Riemannian metrics. We will give the formal definition of a smooth manifold in a moment but start with the special case of smooth surfaces. A *smooth surface* is a 2-dimensional subspace of some surrounding space $\mathbb{R}^k$ such that every small enough part of the subspace can be deformed to a 2-dimensional Euclidean disk. This defomation has to be *smooth*, that is, we can take the derivative of the deformation as often as we wish.

However, smooth surfaces are just the special case for $n = 2$ of smooth n-dimensional manifolds. To define smooth n-dimensional manifolds in general, we introduce the notion of diffeomorphisms. A *diffeomorphism* is a bijective mapping which is smooth and whose inverse is also smooth. Two open sets in $\mathbb{R}^n$ are *diffeomorphic* if there exists a diffeomorphism that maps one to the other. We can now define a *smooth n-dimensional manifold M* as a subspace of the Euclidean space $\mathbb{R}^k$ for some large enough number $k \geq n$ with the property that every point of M has an open neighborhood in M which is diffeomorphic to an open subset of $\mathbb{R}^n$ (see Figure 1).

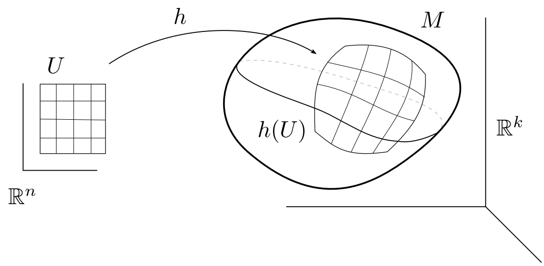

Figure 1: A smooth manifold is made of patches of Euclidean space.

The first example of a smooth n-dimensional manifold is $\mathbb{R}^n$, the Euclidean n-space. To see that $\mathbb{R}^n$ is indeed a smooth manifold, note that $\mathbb{R}^n$ itself serves as an open neighborhood of every point in it, and the identity map is a diffeomorphism.

Another less trivial and important example of a smooth manifold is the n-dimensional sphere $\mathbb{S}^n$. It consists of all points in $\mathbb{R}^{n+1}$ which are at distance 1 from the origin. For example in dimension $n = 2$, it looks like the surface of a 3-dimensional ball (such as a globe) and is formally defined as follows:

$$\mathbb{S}^2 = \{(X, Y, Z) \in \mathbb{R}^3 \mid X^2 + Y^2 + Z^2 = 1\}.$$

Note that $\sqrt{X^2 + Y^2 + Z^2}$ describes the distance of the point (X, Y, Z) to $(0, 0, 0)$ and if this distance is 1 then also $X^2 + Y^2 + Z^2$ is equal to 1. To prove that $\mathbb{S}^2$ is a smooth manifold of dimension 2, we use a language inspired by the one that is used to describe a globe. First, we define the *enlarged southern hemisphere* $\mathcal{S}$ in $\mathbb{S}^2$ to be the whole sphere with the north pole removed. Likewise, the *enlarged northern hemisphere* $\mathcal{N}$ is everything but the south pole in $\mathbb{S}^2$. Note that both $\mathcal{N}$ and $\mathcal{S}$ are open sets in $\mathbb{S}^2$ and every point $(X, Y, Z) \in \mathbb{S}^2$ is contained in at least one of the two enlarged hemispheres. Therefore to prove that $\mathbb{S}^2$ is a smooth 2-dimensional manifold it suffices to show that each of these two enlarged hemispheres is diffeomorphic to an open set in $\mathbb{R}^2$. We can do this by means of the *stereographic projection*: select your favorite point, say $p = (X, Y, Z)$ in $\mathcal{S}$ and draw the line passing through the north pole $N = (0, 0, 1)$ and p (see Figure 2). The stereographic projection of p to the XY-plane is now defined to be the point where this line intersects the XY-plane. The vector equation for this line is $r(t) = N + tv$, where $v = (X, Y, Z - 1)$. Thus the intersection of this line with the XY-plane occurs when $t = 1/(1 - Z)$, which

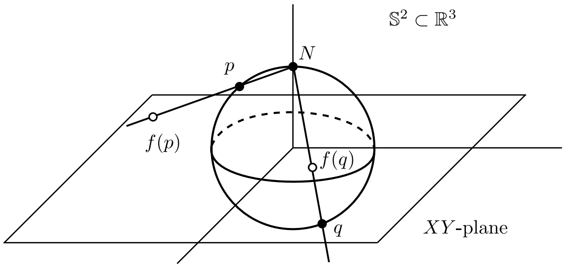

Figure 2: Stereographic projection for two different points p and q, where points on the sphere are depicted solid, while those on the XY-plane are depicted as a ring.

corresponds to the point $P = (x, y, 0) = \left(\frac{X}{1-Z}, \frac{Y}{1-Z}, 0\right)$. In this way, we have defined a smooth map $f\colon \mathcal{S} \to \mathbb{R}^2$ by the equation

$$f(X, Y, Z) = (x, y) = \left(\frac{X}{1-Z}, \frac{Y}{1-Z}\right). \tag{1}$$

It is not hard to obtain an inverse to f just by starting at any point $(x, y, 0)$ on the XY-plane, drawing the line from there to the north pole in $\mathbb{S}^2$ and computing its intersection with the sphere. With this, one easily sees that f is a diffeomorphism between the open neighborhood $\mathcal{S}$ of p and $\mathbb{R}^2$. Analogously one proves that also $\mathcal{N}$ is diffeomorphic to $\mathbb{R}^2$. Consequently $\mathbb{S}^2$ is a smooth 2-dimensional manifold.

2 Riemannian metrics

Now we want to introduce geometry on smooth manifolds. For example, how can we measure lengths of curves on a smooth manifold M or angles between vectors? Let us first take the simplest smooth manifold, namely $\mathbb{R}^n$. A basic operation between vectors in $\mathbb{R}^n$ is the *dot product*, which is a real number

$$vw^T = v_1 w_1 + \cdots + v_n w_n$$

for any two (row) vectors $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$. Once we have the dot product, we can write the formula for the angle between v and w as $\arccos(vw^T/\|v\|\|w\|)$. Furthermore, the length of a curve with velocity $\sigma(t)$ is $\int_{t_0}^{t_1} \|\sigma(t)\| dt$. Here $\|v\| = \sqrt{vv^T} = \sqrt{(v_1)^2 + \ldots + (v_n)^2}$ is the length of the vector v.

In order to make sense of all this on general smooth manifolds we need to develop the notion of a tangent vector to the manifold in question and also its length. In an informal way, we can say that a tangent vector is a vector in the surrounding space $\mathbb{R}^k$ that starts in a point of the manifold and whose direction is tangential to the manifold. Let us make this precise. A *tangent vector* v at a point p in a smooth manifold M can be defined as the derivative at time $t = 0$ of a smooth curve $\alpha : (-\varepsilon, \varepsilon) \to M$ on the manifold M such that $\alpha(0) = p$. Note that if two such smooth curves through p have the same derivative at p then they define the same tangent vector at p. In Figure 3, we depict two curves and the corresponding tangent vectors. The set of all tangent vectors at p is denoted by T_pM and is called the *tangent space of M at p*. For example, the tangent space of a smooth surface at any point can be seen as a plane in $\mathbb{R}^3$. Vector addition turns T_pM into a vector space and a basis for it can be given as follows: by definition of a smooth manifold, there is an open subset $U \subset \mathbb{R}^n$ together with a diffeomorphism $h : U \to h(U) \subset M \subset \mathbb{R}^k$ with $p = h(x)$ for some $x = (x_1, \ldots, x_n) \in U$. Then h can be thought as a vector function depending on variables $x_1, \ldots, x_n$. The tangent vectors $\frac{\partial h}{\partial x_1}\big|_x, \ldots, \frac{\partial h}{\partial x_n}\big|_x$ form a basis for the tangent space T_pM.

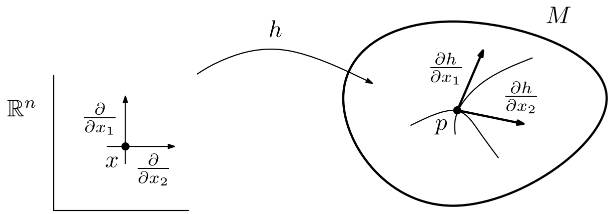

Figure 3: Smooth manifolds have a tangent space at each point.

This basis is used to endow the tangent space T_pM with an inner product. This concept is a generalization of the dot product, which we used earlier. More precisely, an *inner product* on a real vector space V is a symmetric bilinear map $P : V \times V \to \mathbb{R}$ such that $P(v, v) \geq 0$ for all $v \in V$ and $P(v, v) = 0$ if and only if $v = 0$.

The way we do this is by considering the $n \times n$ symmetric matrix

$$g_p = \begin{pmatrix} \frac{\partial h}{\partial x_1}\left(\frac{\partial h}{\partial x_1}\right)^T & \cdots & \frac{\partial h}{\partial x_1}\left(\frac{\partial h}{\partial x_n}\right)^T \\ \vdots & \ddots & \vdots \\ \frac{\partial h}{\partial x_n}\left(\frac{\partial h}{\partial x_1}\right)^T & \cdots & \frac{\partial h}{\partial x_n}\left(\frac{\partial h}{\partial x_n}\right)^T \end{pmatrix}, \tag{2}$$

where $\frac{\partial h}{\partial x_i}\left(\frac{\partial h}{\partial x_j}\right)^T$ is the usual dot product. Then the formula

$$g_p(v,w) = v g_p w^T = (v_1,\ldots,v_n)\begin{pmatrix} \frac{\partial h}{\partial x_1}\left(\frac{\partial h}{\partial x_1}\right)^T & \cdots & \frac{\partial h}{\partial x_1}\left(\frac{\partial h}{\partial x_n}\right)^T \\ \vdots & \ddots & \vdots \\ \frac{\partial h}{\partial x_n}\left(\frac{\partial h}{\partial x_1}\right)^T & \cdots & \frac{\partial h}{\partial x_n}\left(\frac{\partial h}{\partial x_n}\right)^T \end{pmatrix}\begin{pmatrix} w_1 \\ \vdots \\ w_n \end{pmatrix}, \quad (3)$$

where $v = (v_1,\ldots,v_n)$ and $w = (w_1,\ldots,w_n)$ are tangent vectors at $p \in M$, defines an inner product on T_pM. Thus we can define the length of a tangent vector at p by $\|v\| = \sqrt{g_p(v,v)}$. Observe that if M is a Euclidean space $\mathbb{R}^n$ and h is the identity map, then g_p is just the identity matrix, and Formula (3) coincides with the usual dot product.

If we endow every tangent space (one for every point $p \in M$) with an inner product like that, we arrive to the concept of a Riemannian metric:

> A *Riemannian metric* on a smooth manifold M is a smoothly varying assignment of an inner product to every tangent space of M.

As an example, let us measure the length of the great circle on the 2-dimensional sphere $\mathbb{S}^2$, parametrized (in stereographic coordinates) by $\alpha(t) = \left(0, \frac{\sin(t)}{1-\cos(t)}\right)$, $0 < t < 2\pi$. Its velocity at time t is the tangent vector $\dot\alpha(t) = \left(0, \frac{1}{\cos(t)-1}\right)$.

Using Formula (2) and the inverse of the diffeomorphism (1), it is easy (though rather tedious) to show that a Riemannian metric on $\mathbb{S}^2$ is given, in stereographic coordinates, by the matrix[1]

$$g_{(x,y)} = \frac{4}{(1+x^2+y^2)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad (4)$$

and hence at the point $\left(0, \frac{\sin(t)}{1-\cos(t)}\right)$ the metric is

$$\frac{4}{\left(1+\frac{\sin^2(t)}{(1-\cos(t))^2}\right)^2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

And now using Formula (3) we find that $\|\dot\alpha(t)\| = \sqrt{g_{\left(0,\frac{\sin t}{1-\cos(t)}\right)}(\dot\alpha(t)),\dot\alpha(t))} = 1$. Therefore the length of the curve is

$$\int_0^{2\pi} \|\dot\alpha(t)\|\,dt = \int_0^{2\pi} 1\,dt = 2\pi.$$

[1] Technically this formula gives the inner product for the tangent space at $(x,y) \in \mathbb{S}^2$. The Riemannian metric on $\mathbb{S}^2$ would be the collection of all these inner products.

In general, every smooth manifold $M \subset \mathbb{R}^k$ can be endowed with a Riemannian metric by restricting the dot product in $\mathbb{R}^k$ to tangent spaces of M. If a chosen metric g as above coincides with this induced metric, we say that M is *isometrically embedded* into $\mathbb{R}^k$. It is in this way that we can visualize a particular Riemannian metric (at least for $k \leq 3$) and how all of the following figures of Riemannian metrics should be understood.[2]

Let us sum up what we have so far: the object of study is a smooth manifold, which is nothing but a space locally diffeomorphic to Euclidean space $\mathbb{R}^n$. At every point in a smooth manifold there is the tangent space, which is a vector space that can very naturally be endowed with an inner product. The collection of all these inner products (one for each point of the smooth manifold) is what we call a Riemannian metric. The latter is the device we use to make sense of lengths, angles, etc. From now on we will refer to smooth manifolds endowed with a Riemannian metric as *Riemannian manifolds*.

3 Curvature

Although smooth manifolds have been defined as objects contained in some surrounding (Euclidean) space, one can speak about their *intrinsic* local geometric properties. Roughly speaking, these are quantities that can be perceived in a neighborhood of a point by a creature "living" in the smooth manifold without making reference to the view of the object from the surrounding space. If you think for a while, it is really hard to come up with a *local* property that distinguishes between smooth manifolds and Euclidean space. Just remember that before the Greek scientific cosmology was developed, people in other European cultures thought that the surface of the Earth was flat. But isn't that forgivable when we forget our modern knowledge for a moment and take a look at the vicinity around us? As for the case of the Earth, smooth manifolds in general are, by definition, homogeneous entities in the sense that they look the same in the vicinity of every point, namely they are not different from Euclidean space locally. So in principle, a creature living in a smooth manifold has no means to decide what kind of manifold it lives in just by looking around the point it is standing at. This changes if we take into account the geometry described by a Riemannian metric on the manifold. It was Carl Friedrich Gauss (1777–1855) who first realized that such a structure gives rise to a further intrinsic property called *curvature*. Actually, Gauss studied regular surfaces in $\mathbb{R}^3$ and only later Bernhard Riemann (1826–1866) formalized and generalized

[2] For particular values of k it is not easy to decide if there is an isometric embedding into $\mathbb{R}^k$, although the so-called Nash Embedding Theorem ensures that there exists an isometric embedding for some k much larger than n.

these notions to higher-dimensional manifolds. That curvature is an intrinsic property of Riemannian manifolds is such an amazing and remarkable fact that Gauss himself refered to this discovery as *Theorema Egregium*, which is a Latin expression for *extraordinary theorem*.

The curvature of a Riemannian metric is a measurement of how "intrinsically distorted" the space is with respect to Euclidean space. Indeed, suppose you are standing at some point p in a smooth surface M and turn on a headlight. The headlight would create a light cone [3] on the smooth manifold M determined by two tangent vectors $v, w \in \mathrm{T}_pM$. After a very short time t, the light cone that would have width L in a Euclidean space would appear to you as having width approximately equal to

$$L\left(1 + K_{v,w}\frac{t^2}{6}\right)$$

for some coefficient $K_{v,w}$, which depends on how intrinsically distorted the smooth manifold is at the point p.[4] We call this number $K_{v,w}$ that has to be used in the term above the *sectional curvature of v and w at p*. In a higher-dimensional manifold we can think of light emitted in a 2-dimensional cone spanned by two tangent vectors to introduce the very same notion. Note that if $K_{v,w}$ is positive then the light cone would appear wider than it would be on a Euclidean plane. (This occurs on the sphere with the metric defined in (4). Actually in that case we have $K_{v,w} = 1$.) And if $K_{v,w}$ is negative then the light cone would appear narrower than on a Euclidean plane.

Figure 4 shows some examples of smooth 2-dimensional manifolds with distorted light cones.

4 Deformations

So far we have been dealing with only one Riemannian metric on a smooth manifold M at a time. But the same manifold may be given many more Riemannian metrics. Let us think of the round metric on the sphere defined in (4).[5] We can create a new metric by multiplying the old one by some positive constant, say 100. Then the new metric on $\mathbb{S}^2$ is

$$h_{(x,y)} = 100 \cdot g_{(x,y)} = \frac{400}{(1+x^2+y^2)^2}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}.$$

[3] It is to be understood that "light" in our manifold travels along shortest paths.

[4] This is difficult to verify with the tools we have so far, but refer to do Carmo's book [1, Chapter 5] to see how this is done.

[5] We call a Riemannian metric on the sphere *round* if it has constant sectional curvature. The bottom right manifold in Figure 4 is a round sphere.

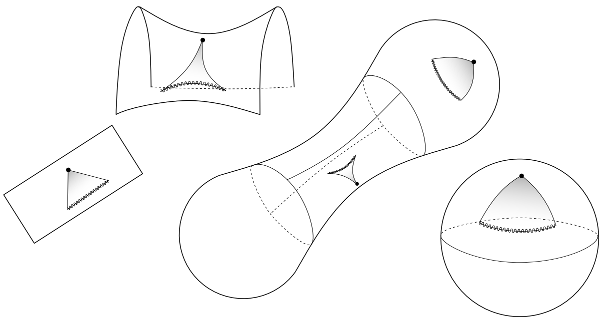

Figure 4: Positive curvature makes the light cone look wider, whereas negative curvature makes it look narrower. Flatness corresponds to zero curvature. The width of the light cones is highlighted with curly lines and the light is emitted from the solid dots.

Viewed with this new metric, the great circle passing through the north pole and the south pole that used to have length 2π will now have length 20π. And not only the distances become larger but the smooth manifold will look less distorted. In fact, in the case at hand the metric h is round with constant sectional curvature decreased to $1/100$.

One question that arises is if this decrease in the curvature can be seen as the consequence of some type of time evolution process that made the metric g mutate in a period of time into the metric h. Actually in this case we can model the change from one metric into another as a continuous process just by "tracing a straight line of metrics" between g and h, that is, for each time $t \in [0,1]$ we can define a new metric g^t on the sphere by

$$g^t = (99t + 1)g. \tag{5}$$

Note that the initial metric ($t = 0$) is g and the final metric ($t = 1$) is h, so that we have just constructed a continuous deformation of one Riemannian metric into another. Some steps in this deformation process are shown in Figure 5.

However, not every Riemannian metric on the sphere is round. In fact, the largest surface in Figure 4 represents a Riemannian metric on the sphere which has positive curvature at some points and negative curvature at other points. We could ask more generally whether *any* Riemannian metric $\widetilde{g}$ on the sphere $\mathbb{S}^2$ can be obtained by a continuous deformation of the round metric g (or vice versa). The answer is yes. How? Well... again, join them by a straight line! That is, for each $t \in [0,1]$ the equation

$$g^t = tg + (1-t)\widetilde{g} \tag{6}$$

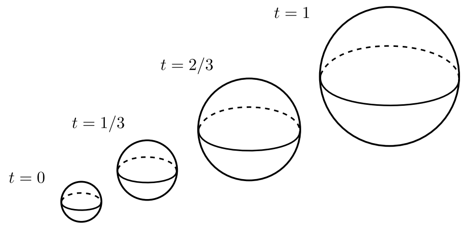

Figure 5: A representation of a "straight line" made of round metrics on the sphere.

defines a 1-parameter family (that is, a continuous deformation) of Riemannian metrics on $\mathbb{S}^2$ such that the initial metric is g and the final metric is $\widetilde{g}$.

The point now is that the deformation defined in (6) may not be satisfactory if we want to control the curvature during the deformation. For instance, it sounds natural to require that the deformation of the round sphere stay positively curved at every time. But can this be achieved for *any two positively curved* Riemannian metrics on $\mathbb{S}^2$?

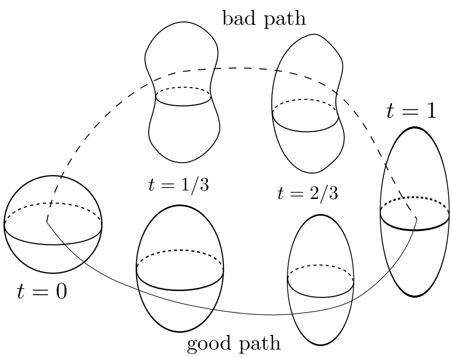

Figure 6: Paths joining two positively curved metrics on $\mathbb{S}^2$. One passes through metrics with some negative curvature (bad path), the other passes through metrics of positive curvature (good path).

First of all, there is no reason to think that our naive deformation defined in (6) will retain any curvature conditions. This is due in part to the non-linear dependence of the curvature on the Riemannian metric.

As a matter of fact, it really takes a lot of effort to answer this question. One way the problem can be approached is by considering a given Riemannian metric $\widetilde{g}$ as the initial condition of certain systems of partial differential equations, known as the Ricci flow – named after Gregorio Ricci-Curbastro (1853–1925). Richard Hamilton [3] showed in 1986 that there exist solutions to these equations. These solutions would be some Riemannian metrics g^t on $\mathbb{S}^2$, one for every time $t \geq 0$. In other words, the Ricci flow naturally gives us a continuous deformation of the initial metric $\widetilde{g}$. What is even more interesting and was also shown by Hamilton, is that if the initial condition $\widetilde{g}$ on $\mathbb{S}^2$ was a positively curved metric then all the "deformed" metrics g^t would stay positively curved! (Like the good path in Figure 6.) Furthermore, Hamilton showed that as time goes by, the curvature of these metrics not only stays positive but also evolves so that at the end it is evenly distributed throughout the sphere; or in other words, the sphere eventually becomes round! In summary:

> Every positively curved Riemannian metric on $\mathbb{S}^2$ can be deformed, through positively curved metrics, to one with constant curvature.

We discussed only the very special case of the 2-dimensional sphere $\mathbb{S}^2$. But the general question we try to answer is the following:

> Given a smooth manifold M and two Riemannian metrics g and $\widetilde{g}$ on it having some geometric constraint (for example, that their curvatures have some specific upper or lower bound), is it possible to obtain $\widetilde{g}$ from g by a continuous deformation through Riemannian metrics that maintain the same geometric constraint?

For most smooth manifolds and geometric constraints, this is a broadly open question (which has been discussed several times at the Mathematisches Forschungsinstitut Oberwolfach!). Let us just point out that the situation in higher dimensions is somewhat complicated, in part because the methods from partial differential equations (like Ricci flow) are very difficult to handle. Nevertheless, some results have been obtained using tools from algebraic and geometric topology and index theory. The interested reader should consult the book by Wilderich Tuschmann and David Wraith [5], wherein the authors collect and explain many of the known results regarding this question. But just to let you go with an idea of how different things are when we escape the lower dimensional range and impose different curvature constraints, we mention that Francis Thomas Farrell and Pedro Ontaneda showed in [2] that there exist smooth 10-dimensional manifolds which can be given an infinite number of Riemannian metrics with constant negative curvature -1 which *cannot* be deformed one into another through *negatively* curved metrics! Also, Hitchin was able to exhibit in [4] two metrics of positive curvature on an 8-dimensional sphere such that they cannot be joined by a deformation of positively curved metrics!

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