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WOLFGANG HERFORT, KARL H. HOFMANN, LINUS KRAMER AND FRANCESCO G. RUSSO

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Mathematisches Forschungsinstitut Oberwolfach gGmbH (MFO)
Schwarzwaldstrasse 9-11
77709 Oberwolfach-Walke
Germany

Tel +49 7834 979 50
Fax +49 7834 979 55
Email admin@mfo.de
URL www.mfo.de

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# THE SYLOW STRUCTURE OF SCALAR AUTOMORPHISM GROUPS

WOLFGANG HERFORT, KARL H. HOFMANN, LINUS KRAMER,
AND FRANCESCO G. RUSSO

Abstract. For any locally compact abelian periodic group $A$ its automorphism group contains as a subgroup those automorphisms that leave invariant every closed subgroup of $A$, to be denoted by $\mathrm{SAut}(A)$. This subgroup is again a locally compact abelian periodic group in its natural topology and hence allows a decomposition into its $p$-primary subgroups for $p$ the primes for which topological $p$-elements in this automorphism subgroup exist. The interplay between the $p$-primary decomposition of $\mathrm{SAut}(A)$ and $A$ can be encoded in a bipartite graph, the *mastergraph* of $A$. Properties and applications of this concept are discussed.

## Introduction

This text deals with periodic locally compact abelian groups. A topological group is called *periodic* if it is locally compact and totally disconnected and if it is the union of compact subgroups. The ring $\mathbb{Z}$ of integers acts on every abelian group $A$ via scalar multiplication. The ring $\mathbb{Z}$ has a universal compactification to a compact totally disconnected topological ring $\widetilde{\mathbb{Z}} \supseteq \mathbb{Z}$, and if $A$ is a periodic locally compact abelian group, then the scalar multiplication of $A$ by $\mathbb{Z}$ extends to a continuous scalar multiplication

$$(z, a) \mapsto z{\cdot}a : \widetilde{\mathbb{Z}} \times A \to A.$$

The automorphism group $\mathrm{Aut}(A)$ is of considerable interest to group theoreticians. Its center contains all automorphisms of the form $a \mapsto r{\cdot}a$ for any (multiplicatively) invertible element $r \in \widetilde{\mathbb{Z}}$. Such elements are called *units* and they form a compact multiplicative subgroup $\widetilde{\mathbb{Z}}^\times$ of $\widetilde{\mathbb{Z}}$. The profinite abelian group $\widetilde{\mathbb{Z}}^\times$ has a remarkably rich structure. So, for each prime number $p$ the compact ring $\mathbb{Z}_p$ is a subring of $\widetilde{\mathbb{Z}}$, and so its group of units $\mathbb{Z}_p^\times$ is a subgroup of $\widetilde{\mathbb{Z}}^\times$. It contains a compact open multiplicative subgroup which is isomorphic to the additive group $\mathbb{Z}_p$, but is also contains a finite cyclic group of order $p-1$ of roots of unity which therefore contains elements of order $q^{\nu_p(q-1)}$ if $q|(p-1)$, where

$\nu_p(q-1)$ is the largest natural number $n$ such that $p = q^n m + 1$ for a natural number $m$. The simple task of finding the $p$-Sylow subgroups of $\widetilde{\mathbb{Z}}^\times$ appears to be a mind boggling problem at first sight.

We solve this problem by describing a countably infinite bipartite labeled graph that is easily depicted and imagined as drawn in the real plane. It supplies a very good organization of the set of all procyclic (and cyclic) subgroups of $\widetilde{\mathbb{Z}}$ that are compact $p$-groups, which allows us to find the maximal $p$-subgroups. Indeed, the essential cyclic and procyclic subgroups are lucidly indexed by the labeled edges of the graph, which we call *the mastergraph.* With the help of the tools that it provides it is, for instance, possible to argue that the multiplicative group $\widetilde{\mathbb{Z}}^\times$ is isomorphic to the additively written group $\widetilde{\mathbb{Z}} \times \overline{\mathrm{tor}(\widetilde{\mathbb{Z}}^\times)}$ and that the group $\overline{\mathrm{tor}(\widetilde{\mathbb{Z}})}^\times$ contains a dense subgroup algebraically isomorphic to the large torsion-free group $(\widetilde{\mathbb{Z}},+)^{(\mathbb{N})}$. (See Corollary 23.)

Given a periodic locally compact abelian group $A$ we let $\mathrm{SEnd}(A) \subseteq \mathrm{End}(A)$ denote the subring of all endomorphisms implemented by scalar multiplication. Then the natural homomorphism $\zeta\colon \widetilde{\mathbb{Z}} \to \mathrm{SEnd}(A)$ defined by $\zeta(r)(a) = r{\cdot}a$ will be shown to be a quotient morphism of profinite rings, and we call the ring $\mathcal{R}(A) := \widetilde{\mathbb{Z}}/\ker(\zeta)$ the *ring of scalars of* $A$. Then $\zeta$ factors through $\mathcal{R}(A)$ with an isomorphism $\mathcal{R}(A) \to \mathrm{SEnd}(A)$ of rings. The group of units of $\mathrm{SEnd}(A)$ is denoted $\mathrm{SAut}(A)$, and we have $\mathcal{R}(A)^\times \cong \mathrm{SAut}(A)$. We shall clarify the structure of $\mathcal{R}(A)^\times$ completely in the way it depends on the exponents of the $A_p$. (See Theorem 41)

Let $G$ be a locally compact group with a closed normal subgroup $A$. Let $\mathrm{Int}(A)$ denote the group of all inner automorphisms. There is a natural representation $G \to \mathrm{Int}(A)$ sending $g$ to the inner automorphism $a \mapsto gag^{-1}$ whose kernel is the centralizer of $A$ in $G$.

**Proposition 1.** *For a locally compact group $G$ with a periodic abelian closed normal subgroup $A$ the following statements are equivalent:*

(i) $\mathrm{Int}(A) \subseteq \mathrm{SAut}(A)$.
(ii) *Every inner automorphism induced on $A$ is a scalar automorphism.*
(iii) *Every closed subgroup of $A$ is normal in $G$.*
(iv) *There is a morphism $\rho\colon G \to \mathcal{R}(A)^\times$ such that*

$$(\forall g \in G,\, a \in A)\ gag^{-1} = \rho(g){\cdot}a$$

For a proof of this proposition see Proposition 36.

We emphasize here again that in Theorem 41 we shall give an explicit structure theory of $\mathcal{R}(A)^\times \cong \mathrm{SAut}(A)$.

Now let us formalize the situation described in the preceding proposition.

**Definition 2.** If $(G, A)$ is a pair consisting of a topological group $G$ and a closed normal subgroup $A$, then we call it a *special extension of* $A$ if $G$ is a locally compact group and the equivalent conditions of Proposition 1 are satisfied.

For any special extension $(G, A)$ of a periodic abelian group $A$ we have a subgraph

$$\mathcal{G}(U(G,A), V(G,A), \mathcal{E}(G,A), \lambda)$$

of the mastergraph $\mathcal{G}$ as follows:

(i) $U(G, A)$ contains all vertices $(p, 1)$ for which $\rho(G_p) \neq \{1\}$ in $\mathcal{R}(A)^\times$.

(ii) $V(G, A)$ contains all vertices $(q, 0)$ such that $(\mathcal{R}(A_q)^\times) \neq \{1\}$.

(iii) $\mathcal{E}(G, A)$ contains all edges from $p$ to $q$ such that $\rho(G_p) \cap \mathcal{R}(A_q)^\times \neq \{1\}$.

All labels $\lambda(e)$ are those of the mastergraph.

As a sample of the efficiency of the graphs we show

**Theorem 3.** *Let $(G, A)$ be a special extension of a periodic locally compact abelian group. Then for each sloping edge $e \in \mathcal{E}(G, A)$ from some $p$ to some $q$, all of $A_q$ consists of commutators. In particular, $A_q \subseteq G'$.*

(See Theorem 43 below.)

## 1. The Sylow Structure of the Compactified Ring of Integers

By the "ring of compactified integers" we mean the profinite completion of the ring $\mathbb{Z}$ and we denote it by $\widetilde{\mathbb{Z}}$. Technically, if $B = \alpha(G)$ is the Bohr compactification of a topological group $G$, then $B/B_0$ (with the identity component $B_0$ of $B$) is the *zero dimensional compactification* or the *profinite completion* of $G$. The profinite ring $\widetilde{\mathbb{Z}}$ is at the focus of the present discussion.

The set of all prime numbers is denoted $\pi$. For each profinite abelian group $A$ the $p$-primary component or $p$-Sylow subgroup $A_p$ is the largest $p$-subgroup of $A$, and one has the Sylow decomposition $A = \prod_{p\in\pi} A_p$. Note, however that the ring $\mathbb{Z}$ of integers in its discrete topology is not profinite, allowing the standard notation $\mathbb{Z}_p$ for the ring of $p$-adic integers to be an exception to this convention. Accordingly, we shall formulate the equation $(\widetilde{\mathbb{Z}})_p = \mathbb{Z}_p$. The compact ring $\widetilde{\mathbb{Z}}$ then satisfies

$$\widetilde{\mathbb{Z}} = \prod_{p\in\pi} (\widetilde{\mathbb{Z}})_p \cong \prod_{p\in\pi} \mathbb{Z}_p. \tag{1}$$

This is the Sylow decomposition (or primary decomposition) of $\widetilde{\mathbb{Z}}$.

## 2. The Group of $p$-Adic Units $\mathbb{Z}_p^\times$

For a unital commutative ring $R$ we denote by $R^\times$ the multiplicative group of its units, i.e., invertible elements. We clarify this concept for $R = \mathbb{Z}_p$ by a reminder of some elementary structural information of $\mathbb{Z}_p$. Recall that under suitable circumstances in a topological ring $R$ the sequence $1 + x + \frac{1}{2}{\cdot}x + \frac{1}{3!}x^3 \cdots$ converges for $x$ from a suitable domain $D$ and defines a function

$$\exp\colon D \to 1 + D, \quad 1 + D \subseteq R^\times.$$

If $p \in \pi$ is a prime and $m \in \mathbb{N}$, then

$$\nu_p(m) = \max\{n \in \mathbb{N}_0 : p^n | m\} \tag{2}$$

is that unique nonnegative integer $n$ for which $m = p^n m'$ and $(m', p) = 1$.

For the following information on the ring $\mathbb{Z}_p$ of $p$-adic integers see e.g. [3].

**Lemma 4.** (i) *For each prime $p \neq 2$, the function*

$$\exp\colon p{\cdot}\mathbb{Z} \to (1 + p\mathbb{Z}_p), \quad 1 + p\mathbb{Z}_p \subseteq \mathbb{Z}_p^\times$$

*is an isomorphism of profinite groups and $1 + p\mathbb{Z}$ is an open subgroup of $\mathbb{Z}_p^\times$. In particular,*

$$z \mapsto \exp pz : (\mathbb{Z}_p, +) \to (1 + p\mathbb{Z}_p, \times) \tag{3}$$

*is an isomorphism of profinite groups.*

(ii) *The factor ring $\mathbb{Z}_p/p\mathbb{Z}_p$ is the field $\mathrm{GF}(p)$ of $p$ elements, and so $(\mathbb{Z}_p/p\mathbb{Z}_p)^\times$ is a cyclic group of $p - 1$ elements. The ring $\mathbb{Z}_p^\times$ contains a cyclic group $C_p$ of $p - 1$ elements (called roots of unity) such that*

$$(x, c) \mapsto xc : (1 + p\mathbb{Z}_p) \times C_p \to \mathbb{Z}_p^\times \text{ is an isomorphism,}$$

*and*

$$(\mathbb{Z}_p^\times, \times) \cong \left(\mathbb{Z}_p \oplus \bigoplus_{q\in\pi} \mathbb{Z}\left(q^{\nu_q(p-1)}\right), +\right). \tag{4}$$

In particular, for $q \in \pi$ the $q$-Sylow subgroup of $\mathbb{Z}_p^\times$ is procyclic and

$$(\mathbb{Z}_p^\times)_q \cong \begin{cases} \mathbb{Z}\left(q^{\nu_q(p-1)}\right) & \text{if } q < p, \\ \mathbb{Z}_p & \text{if } q = p, \\ \{0\} & \text{if } p < q. \end{cases}$$

From a formalistic point of view it is regrettable that the case $p = 2$ is not exactly subordinate to the scheme. However, here it is:

**Lemma 5.** (i) *The function*

$$\exp\colon 4\mathbb{Z}_2 \to (1 + 4\mathbb{Z}_2), \quad 1 + 4\mathbb{Z}_2 \subseteq \mathbb{Z}_2^\times$$

*is an isomorphism of profinite groups and* $1 + 4\mathbb{Z}$ *is an open subgroup of* $\mathbb{Z}_2^\times$*. In particular,*

$$z \mapsto \exp 4z : (\mathbb{Z}_2, +) \to (1 + 4\mathbb{Z}_p, \times) \tag{5}$$

*is an isomorphism of profinite groups.*

(ii) *The factor ring* $\mathbb{Z}_2/2\mathbb{Z}_2$ *is the field* $\mathrm{GF}(2)$ *of* 2 *elements, and the group of units of* $(\mathbb{Z}_2/4\mathbb{Z}_2)^\times$ *is a group of* 2 *elements. The group* $\mathbb{Z}_2^\times$ *contains a cyclic group* $C_2$ *of* 2 *elements (called roots of unity) such that*

$$(x, c) \mapsto xc : (1 + 4\mathbb{Z}_2) \times C_2 \to \mathbb{Z}_2^\times \text{ is an isomorphism,}$$

*and*

$$(\mathbb{Z}_2^\times, \times) \cong (\mathbb{Z}_2 \oplus \mathbb{Z}(2), +). \tag{6}$$

In particular, $\mathbb{Z}_2^\times$ is a nonprocyclic 2-group.

The product representation (1) $\widetilde{\mathbb{Z}}_p = \prod_{p\in\pi} \mathbb{Z}_p$ immediately yields

$$\widetilde{\mathbb{Z}}^\times = \prod_{p\in\pi} \mathbb{Z}_p^\times. \tag{7}$$

Since for $p \neq 2$ the profinite group $\mathbb{Z}_p^\times$ is not a $p$-group, the product representation of the profinite group $\widetilde{\mathbb{Z}}^\times$ in $(1^\times)$ is not its Sylow decomposition. Our first and foremost goal is now to determine the Sylow decomposition of $\widetilde{\mathbb{Z}}^\times$ and to describe it in an intuitive and useful form.

## 3. Some Helpful Facts on Groups and Numbers

The information contained in (1) through (6) suggests rather clearly that products $G = \prod_{j\in J} \mathbb{Z}(p^{n_j})$ (for families $(n_j)_{j\in J}$ of natural numbers and for a fixed prime number $p$ will play a role in the structure of $\widetilde{\mathbb{Z}}^\times$. Lemma 3.9 of [1] provides the following standard information:

**Lemma 6.** *The group* $\prod_{j\in J} \mathbb{Z}(p^{n_j})$ *is a torsion group if and only if* $(n_j)_{j\in J}$ *is a bounded family.*

Accordingly we first collect some general facts on groups $G$ for general families $(n_j)_{j\in J}$ and keep in mind as a special example the family $\mathbb{N} = (1, 2, 3, \dots)$ and, accordingly, the group

$$P = \mathbb{Z}(p) \times \mathbb{Z}(p^2) \times \mathbb{Z}(p^3) \times \cdots \tag{8}$$

**Definition 7.** *Let $p \in \pi$. Given $G = \prod_{j\in J} \mathbb{Z}(p^{n_j})$ for a family $(n_j)_{j\in J}$, for each $m \in \mathbb{N}$ we define*

$$n_{jm} = \begin{cases} n_j & \textit{if } n_j \le m, \\ 0 & \textit{otherwise.} \end{cases}$$

*Now we set $G_m = \prod_{j\in J} \mathbb{Z}(p^{n_{jm}})$. For any finite subset $F \subseteq J$ we let*

$$n_{jF} = \begin{cases} n_j & \textit{if } j \in F, \\ 0 & \textit{otherwise} \end{cases}$$

*and set*

$$G_F = \prod_{j\in J} \mathbb{Z}(p^{n_{jF}}) \cong \bigoplus_{j\in F} \mathbb{Z}(p^{n_j}).$$

We see that $m \le n$ implies $G_m \le G_n$, and for any finite subset $F \subseteq J$ there is an $m$ such that $G_F \le G_m$. Since $\bigcup_{F\subseteq J,\ F \text{ finite}} G_F$ is dense in $\prod_{j\in J} \mathbb{Z}(p^{n_j})$, we have

*Remark* 8. For any family $(n_j)_{j\in J}$ of natural numbers, the profinite $p$-group $G = \prod_{j\in J} \mathbb{Z}(p^{n_j})$ has the dense torsion subgroup $\bigcup_{m\in\mathbb{N}} G_m$ of the ascending *sequence* $G_m$, $m = 1, 2, \dots$ of compact torsion subgroups.

Let us consider the character group $A := \widehat{G}$ of $G$. Then $A \cong \bigoplus_{j\in J} \mathbb{Z}(p^{n_j})$.

From Propositions 8.2 and 8.3 in [2] we cite

**Lemma 9.** *If $\Gamma$ is any compact or any discrete group, then*

$$\overline{\operatorname{tor}\Gamma} = \operatorname{Div}(\widehat{\Gamma})^{\perp}, \tag{9}$$

*the annihilator of the group of all divisible elements of the character group of $\Gamma$.*

We consider the special group

$$\Sigma_p = \mathbb{Z}(p) \oplus \mathbb{Z}(p^2) \oplus \mathbb{Z}(p^3) \oplus \cdots, \tag{10}$$

the character group of the group $P$ in (8) above. In [1] $\Sigma_p$ emerges as the torsion subgroup of the remarkable locally compact $p$-group $\nabla_p$ (see [1], Theorem 3.16) and it shows some surprising features itself.

Firstly we cite Lemma 3.17 of [1] known to Prüfer:

**Lemma 10.** *Let $e_n$ be the generator of $\mathbb{Z}(p^n)$ in $\Sigma_p$ and let $\phi\colon \Sigma_p \to \Sigma_p$ be the endomorphism defined by $\phi(e_n) = e_n - p{\cdot}e_{n+1}$. Then $\phi$ is injective and its cokernel $\Sigma_p/\phi(\Sigma_p)$ is, up to isomorphism, the Prüfer group $\mathbb{Z}(p^\infty)$. That is, the following sequence is exact:*

$$0 \to \Sigma_p \xrightarrow{\phi} \Sigma_p \to \mathbb{Z}(p^\infty) \to 0.$$

We can iterate $\phi$ and set $S_n = \phi^n(\Sigma_p)$, $n = 0, 1, 2, \ldots$. Then $\Sigma_p = S_0 \supseteq S_1 \supseteq S_2 \supseteq \cdots$. Since $\phi$ is injective, all $S_n$ are isomorphic to $\Sigma_p$.

**Proposition 11.** *The countable torsion group $\Sigma_p$ is filtered by a sequence $S_0 = \Sigma_p \supseteq S_1 \supseteq S_2 \supseteq \cdots$ of isomorphic subgroups such that*
(i) *$S_{n-1}/S_n \cong \mathbb{Z}(p^\infty)$ for $n \in \mathbb{N}$, and*
(ii) *$\bigcap_{n\in\mathbb{N}} S_n = \{0\}$.*

*Proof.* We have to prove (i) and (ii). For each $n \in \mathbb{N}$, set $K_n = S_n/S_{n-1}$; in particular $K_0 = \mathbb{Z}(p^\infty)$. The injective endomorphism $\phi\colon S_0 \to S_0$ leaves $S_n$ invariant and induces an injective endomorphism $\phi_n\colon S_n \to S_n$ with cokernel $K_n$. We have the commutative diagram

$$\begin{array}{ccccccccc}
0 & \to & S_0 & \xrightarrow{\phi} & S_0 & \to & K_0 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
0 & \to & S_1 & \xrightarrow{\phi_2} & S_1 & \to & K_1 & \to & 0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots \\
0 & \to & S_n & \xrightarrow{\phi_n} & S_n & \to & K_n & \to & 0 \\
\vdots & & \vdots & & \vdots & & \vdots & & \vdots
\end{array}$$

in which all rows are exact and the vertical morphisms $S_{n-1} \to S_n$ $n \in \mathbb{N}$ are the isomorphisms induced by $\phi|S_n$ Since the downarrows $S_{n-1} \to S_n$ are isomorphisms inductively, and $K_0 = \mathbb{Z}(p^\infty)$, it follows, inductively, that $K_n \cong \mathbb{Z}(p^\infty)$ for all $n \in \mathbb{N}$.

(ii) By the definition of $\phi$ in Lemma 10 we have $\phi(e_n) = e_n - p{\cdot}e_{n+1}$. We define $\ell\colon \Sigma_p \to \mathbb{N}$ as follows: let $x = \sum_{n\in\mathbb{N}} x_m$ with $x_m \in \mathbb{Z}(p^m)$. Then

$$\ell(x) = \begin{cases} 0 & \text{if } x = 0, \\ \max\{m \in \mathbb{N} |\ 0 \neq x_m \in \mathbb{Z}(p^m)\} & \\ -\min\{m \in \mathbb{N} |\ 0 \neq x_m \in \mathbb{Z}(p^m)\} + 1 & \text{otherwise.} \end{cases}$$

In the definition of $\phi$ in Lemma 10 we set $\phi(e_n) = e_n - p{\cdot}e_{n+1}$. Thus let $y = \sum_{n\in\mathbb{N}} y_n$ be $\phi(x)$ and assume $x \neq 0$. Then

$$\min\{m \in \mathbb{N} |\ 0 \neq y_m \in \mathbb{Z}(p^m)\} = \min\{m \in \mathbb{N} |\ 0 \neq x_m \in \mathbb{Z}(p^m)\}$$

and

$$\max\{m \in \mathbb{N} |\ 0 \neq y_m \in \mathbb{Z}(p^m)\} = \max\{m \in \mathbb{N} |\ 0 \neq x_m \in \mathbb{Z}(p^m)\} + 1.$$

Thus

$$\ell\big(\phi^n(x)\big) = \ell(x) + n. \tag{11}$$

Now assume that $y \in \bigcap_{m\in\mathbb{N}} S_m$. Suppose that $y \neq 0$ and set $n = \ell(y) \in \mathbb{N}$. Then $y \in \bigcap_{m\in\mathbb{N}} S_m \subseteq S_n$, and so there is an $x \neq 0$ such that $\phi^n(x) = y$. Thus (11) shows that $\ell(y) = \ell\big(\phi^n(x)\big) = \ell(x) + n = \ell(x) + \ell(y)$, that is, $\ell(x) = 0$ and hence $x = 0$ which is impossible. □

This proposition dualizes comfortably according to the Annihilator Mechanism of locally compact abelian groups (see [2], 7.17 ff., notably Corollary 7.22, all of which fully applies to locally compact abelian groups). So let $P$ of (8) be the dual of $\Sigma_p$ and let $H_n \leq P$ be the annihilator $(S_n)^\perp$ of $S_n \leq \Sigma_p$. Since the $S_n$ are descending, the $H_n$ are ascending, and since $\bigcap_{n\in\mathbb{N}} S_n = \{0\}$ we know that

$$P = \overline{\bigcup_{n\in\mathbb{N}} H_n}. \tag{12}$$

For all $n \in \mathbb{N}$ we deduce via duality from $S_{n-1}/S_n \cong \mathbb{Z}(p^\infty)$ that $H_n/H_{n-1} \cong \mathbb{Z}_p$ for $n \in \mathbb{N}$. However, at this point we can utilize the fact that in the category of compact $p$-groups, the group $\mathbb{Z}_p$ is projective (since its dual $\mathbb{Z}(p^\infty)$ is divisible hence injective in the category of discrete $p$-groups; see also [2], Theorem 8.78.) Therefore, for each $n \in \mathbb{N}$, the compact group $H_n$ contains a compact subgroup $K_n \cong \mathbb{Z}_p$ such that

$$(\forall n \in \mathbb{N})\, H_n = H_{n-1}K_n \cong H_n \times K_n. \tag{13}$$

By induction we conclude at once that

$$(\forall n \in \mathbb{N})\, H_n = C_1 \cdots C_n = C_1 \times \cdots \times C_n \cong \mathbb{Z}_p^n \tag{14}$$

and

$$\bigcup_{n\in N} H_n = \left\langle \bigcup_{n\in\mathbb{N}} C_n \right\rangle \cong \bigoplus_{n\in\mathbb{N}} C_n \cong \mathbb{Z}_p^{(\mathbb{N})} \tag{15}$$

Let us collect this information:

**Corollary 12.** *The group $P = \mathbb{Z}(p) \times \mathbb{Z}(p^2) \times \mathbb{Z}(p^3) \times \cdots$ contains a dense $\mathbb{Z}_p$-submodule which is algebraically isomorphic to the $\mathbb{Z}_p$-module $\mathbb{Z}_p^{(\mathbb{N})}$.*

**Corollary 13.** *For any family $(n_j)_{j\in J}$ of natural numbers, the profinite $p$-group $G=\prod_{j\in J}\mathbb{Z}(p^{n_j})$ is either a torsion group or else it contains a $\mathbb{Z}_p$-submodule isomorphic to $\mathbb{Z}_p^{(\mathbb{N})}$ whose closure is isomorphic to $P=\mathbb{Z}(p)\times\mathbb{Z}(p^2)\times\mathbb{Z}(p^3)\times\cdots$*

*Proof.* Either the family $(n_j)_{j\in J}$ is bounded, in which case $G$ is a torsion group, or else it is unbounded. In that case there is an increasing unbounded subsequence $(n_{j(m)})_{m\in\mathbb{N}}$. Set $k_m=n_{j(m)}$. Since the $k_n$ are increasing, we have $n\le k_n$. The cyclic group $\mathbb{Z}(p^{k_m})=\mathbb{Z}(p^{n_{j(m)}})$ contains a subgroup $B_m\cong\mathbb{Z}(p^m)$. Then group $B_1\times B_2\times B_3\times\cdots$ is clearly isomorphic to a subgroup $B$ of $G$ which is isomorphic to $\mathbb{Z}(p)\times\mathbb{Z}(p^2)\times\mathbb{Z}(p^3)\times\cdots$. Then it follows from Corollary 12 that $B$ contains a dense $\mathbb{Z}_p$-submodule algebraically isomorphic to $\mathbb{Z}_p^{(\mathbb{N})}$, as asserted. ☐

We shall need the following pieces of information. The first one is number theoretical. As in (2), for a prime $p$ and a natural number $r$, let $\nu_p(r)$ be the exponent of the largest $p$-power dividing $r$.

**Lemma 14.** *Let $p\in\pi$ be an arbitrary prime number and $n$ an arbitrary natural number. Then there is a prime number $q$ such that $n\le\nu_p(q-1)$. Accordingly, $p^n|(q-1)$. In particular $p|(q-1)$.*

*Proof.* Fix $p\in\pi$ and an arbitrary natural number $n$. The numbers $a=p^n$ and $b=1$ are relatively prime. Hence the arithmetic progression $(am+b)_{m\in\mathbb{N}}$ contains infinitely many primes $q$ by the Dedekind Prime Number Theorem. Let $q$ be one of them. Then $q-1=p^nm$, that is $\nu_p(q-1)\ge n$. In particular, $p|(q-1)$. ☐

Lemmas 4 and 5 imply via (7) that $\widetilde{\mathbb{Z}}^\times$ contains for each fixed prime $p$ a product

$$E:=\prod_{q\in\pi}\mathbb{Z}(p^{\nu_p(q-1)}),$$

where we note that $\nu_p(q-1)=0$ if $p$ fails to divide $q-1$. Therefore the following conclusion of the preceding Lemma 14 is relevant:

**Proposition 15.** *Let $p\in\pi$ be an arbitrary prime number. Then the group $E$ contains a subgroup isomorphic to*

$$P=\mathbb{Z}(p)\times\mathbb{Z}(p^2)\times\mathbb{Z}(p^3)\times\cdots$$

*which in turn contains a dense subgroup and $\mathbb{Z}_p$-module $D\cong\mathbb{Z}_p^{(\mathbb{N})}$.*

*Proof.* By Lemma 14 for each $n$ there is a $q \in \pi$ such that $n \leq \nu_p(q-1)$. Hence the group $\mathbb{Z}(p^{\nu_p(q-1)})$ contains a subgroup $B_n \cong \mathbb{Z}(p^n)$. Thus $E$ contains an isomorphic copy of

$$B = \prod_{n \in \mathbb{N}} B_n \cong \mathbb{Z}(p) \times \mathbb{Z}(p^2) \times \mathbb{Z}(p^3) \times \cdots$$

The remainder then follows from Corollary 12. □

## 4. The Mastergraph

We introduce a bipartite edge-labeled graph $\mathcal{G}$ as follows:

**Definition 16.** *A* bipartite graph *consists of two disjoint sets $U$ and $V$ and a binary relation $\mathcal{E} \subseteq (U \cup V)^2$ such that $(u, v) \in \mathcal{E}$ implies $u \in U$ and $v \in V$. The elements of $U \cup V$ are called* vertices *and the elements of $\mathcal{E}$ are called* edges. *Any triple $(U, V, \mathcal{E})$ of this type is called a* bipartite graph.

*An* edge labeled *graph is a quadruple $(U, V, \mathcal{E}, \lambda)$ such that $(U, V, \mathcal{E})$ is a bipartite graph and $\lambda$ is a function $\lambda \colon \mathcal{E} \to L$ for some set $L$ of labels.*

Labels could be numbers, or symbols like $\infty$.

Now we define a particular edge labeled graph $\mathcal{G}$. Recall the definition of $\nu_p(m)$ from (2) above.

**Definition 17.** *The following bipartite edge labeled graph*

$$\mathcal{G} = (U, V, \mathcal{E}, \lambda), \quad \mathcal{E} \subseteq U \times V,$$

*will be called the* prime mastergraph *or* mastergraph *for short:*

(i) $U = \pi \times \{1\} \subseteq \pi \times \{0, 1\}$,
(ii) $V = \pi \times \{0\} \subseteq \pi \times \{0, 1\}$,
(iii) $\mathcal{E} = \{\big((p, 1), (q, 0)\big) : p = q \text{ or } p | (q-1)\}$,
(iv) $\lambda \colon \mathcal{E} \to \mathbb{N} \cup \{\infty\},\ \lambda\big(((p, 1), (q, 0))\big) = \begin{cases} \infty, & \text{if } p = q, \\ \nu_p(q-1), & \text{if } p < q. \end{cases}$

*We shall call the vertices in $U$ the* upper *and those in $V$ the* lower *vertices. The edges $\big((p, 1), (p, 0)\big)$, $p \in \pi$ are said to be* vertical, *all others are called* sloping. *We say that $e = ((p, 1), (q, 0))$ is* the edge from $p$ to $q$.

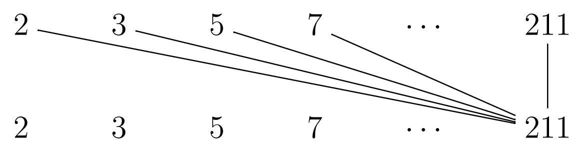


Figure 2. The five edges in $U$ connected to the lower edge numbered "211" in $V$.

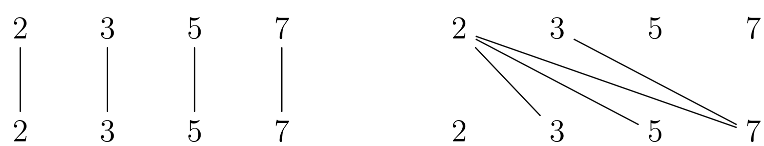


Figure 1. Vertical and sloping edges

The labels of the sloping edges are all $= 1$ in this example.

The "geometric" terminology is chosen because $\mathcal{G}$ has an intuitive representation in the plane $\mathbb{R}^2$ preserving the order:

**Proposition 18.** *Let $\omega\colon \pi \to \mathbb{N}$ be the bijection inverse to the usual enumeration $n \mapsto p_n$ of primes according to their natural ordering according to their size. Let* id *be the identity map of the set $\{0,1\}$. There is a faithful representation of the configuration of $\mathcal{G}$ into the plane $\mathbb{R}^2$ preserving the componentwise order which is induced by the injection*

$$\pi \times \{0,1\} \xrightarrow{\omega\times\mathrm{id}} \mathbb{N} \times \{0,1\} \xrightarrow{\mathrm{incl}} \mathbb{R} \times \mathbb{R} = \mathbb{R}^2$$

*and taking $U$ to $\mathbb{N} \times \{1\}$ and $V$ to $\mathbb{N} \times \{0\}$.*

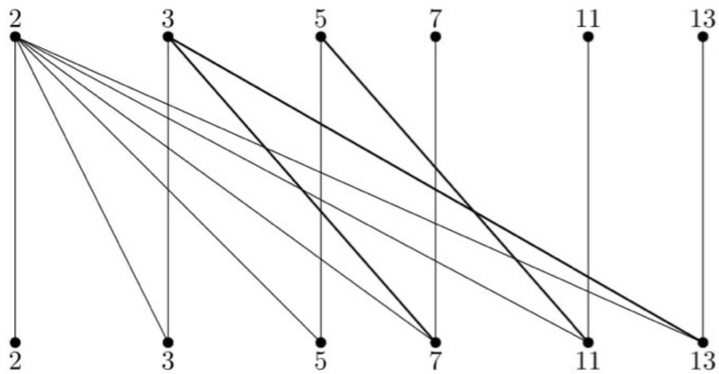


FIGURE 3. The initial part of the master-graph.

The label of the edge from $(2,1)$ to $(13,0)$ is 2.

**Definition 19.** Let $p$ and $q$ be any primes. Then

$$\mathcal{E}_p = \{e : e = ((p,1),(p',0)) \in \mathcal{E} \text{ such that } p = p' \text{ or } p|(p'-1)\},$$

the set of all edges emanating downwards from the vertex $(p,1) \in U$ will be called the *cone peaking at* $p$. Further the set

$$\mathcal{F}_q = \{e : e = ((q',1),(q,0)) \in \mathcal{E} \text{ such that } q'|(q-1)\},$$

the set of edges ending below in the vertex $(q,0) \in V$, is called the *funnel pointing to* $q$.

Both the cones and the funnels provide a partition of the set of edges. It is instantly clear that each funnel is finite, and so the funnels, are not as important as the cones. The structure of a cone is more interesting than that of a funnel, as the following proposition shows.

**Proposition 20.** *Let $p$ be any prime. Accordingly, in the graph $\mathcal{G}$, the cone $\mathcal{E}_p$ is peaking at the upper vertex $(p,1)$, and for each natural number $n$, it contains an edge $e = ((p,1),(q,0))$ labeled $\nu_p(q-1) \geq n$. In particular, $\mathcal{E}$ contains infinitely many edges.*

*Proof.* There is nothing to prove – this is just a translation of Lemma 14 into the language of the mastergraph $\mathcal{G}$. □

## 5. THE SYLOW DECOMPOSITION OF $\widetilde{\mathbb{Z}}$ INDEXED BY $\mathcal{G}$

We recall that $\mathcal{E}$ is the set of all edges of the mastergraph $\mathcal{G} = (U, V, \mathcal{E}, \lambda)$. We start the indexing by attaching to each edge $e =$

$((p,1),(q,0)) \in \mathcal{E}$ a profinite group $\mathbb{S}_e$ being, up to a natural isomorphism, a subgroup of $\widetilde{\mathbb{Z}}$:

**Definition 21.** *For each edge $e \in \mathcal{E}$ from $p$ to $q$ we set*

$$\mathbb{S}_e = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}(2), & \text{if } p = q = 2, \\ \mathbb{Z}_p, & \text{if } 2 < p = q, \\ \mathbb{Z}(p^{\nu_p(q-1)}), & \text{if } p < q. \end{cases} \tag{16}$$

We noted in (7) that $\widetilde{\mathbb{Z}}^\times = \prod_{p\in\pi} \mathbb{Z}_p^\times$ and in Lemmas 4 and 5 a procyclic $p$-group occurs precisely as a subgroup of some $\mathbb{S}_e$ for an edge $e$ with upper vertex $p$. Therefore the $p$-Sylow subgroup of $\widetilde{\mathbb{Z}}^\times$ is represented by the cone $\mathcal{E}$ peaking in $p$:

$$(\widetilde{\mathbb{Z}}^\times)_p \cong \prod_{e\in\mathcal{E}_p} \mathbb{S}_e,$$

$$p \mapsto \mathcal{E}_p : \pi \to \mathcal{C}$$

is a bijection from the set of primes to the set $\mathcal{C}$ of cones such that $\mathcal{C} = \bigcup_{p\in\pi} \mathcal{E}_p$ in the mastergraph.

Taking these matters and Proposition 15 into account, we can summarize:

**Theorem 22.** (i) *The group $\widetilde{\mathbb{Z}}^\times$ of units of the universal procyclic compactification $\widetilde{\mathbb{Z}}$ of the ring of integers $\mathbb{Z}$ is the product*

$$\widetilde{\mathbb{Z}}^\times \cong \prod_{e\in\mathcal{E}} \mathbb{S}_e \tag{17}$$

*extended over the set $\mathcal{E}$ of all edges of the mastergraph, where $\mathbb{S}_e$ is the profinite group given in (16) above.*

(ii) *Its $p$-Sylow subgroup is the subproduct extended over the cone peaking in $p$:*

$$(\widetilde{\mathbb{Z}}^\times)_p \cong \prod_{e\in\mathcal{E}_p} \mathbb{S}_e = \begin{cases} \mathbb{Z}_2 \oplus \mathbb{Z}(2) \oplus \prod_{q>2} \mathbb{Z}(2^{\nu_2(q-1)}), & \text{if } p = 2, \\ \mathbb{Z}_p \oplus \prod_{q>p} \mathbb{Z}(p^{\nu_p(q-1)}), & \text{otherwise.} \end{cases} \tag{18}$$

(iii) *For each $p \in \pi$ fixed,*

$$(\widetilde{\mathbb{Z}}^\times)_p \cong \mathbb{Z}_p \oplus T_p, \qquad \text{where } T_p = \overline{\operatorname{tor}(\widetilde{\mathbb{Z}}^\times)_p}, \tag{19}$$

*and where $T_p$ contains a $\mathbb{Z}_p$-submodule algebraically isomorphic to $\mathbb{Z}_p^{(\mathbb{N})}$ whose closure is isomorphic to $\prod_{n\in\mathbb{N}} \mathbb{Z}(p^n)$.*

Let $T = \overline{\mathrm{tor}(\widetilde{\mathbb{Z}})}$. For each prime $p$, define

$$\mathbb{ZP}_p = \prod_{n\in\mathbb{N}} \mathbb{Z}(p^n), \qquad \mathbb{ZP} = \prod_{p\in\pi} \mathbb{ZP}_p = \left(\prod_{(p,n)\in\pi\times\mathbb{N}} \mathbb{Z}(p^n).\right) \tag{20}$$

**Corollary 23.** (i) $\mathbb{ZP}$ *contains a dense copy of the torsion-free* $\widetilde{\mathbb{Z}}$-*module* $M := \widetilde{\mathbb{Z}}^{(\mathbb{N})}$.

(ii) *The closure* $T$ *of the torsion subgroup of* $\widetilde{\mathbb{Z}}^\times$ *contains a copy of* $M$.

*Proof.* (i) The group $\mathbb{ZP}_p$ contains a dense copy of $\mathbb{Z}_p^{(\mathbb{N})}$ (see Theorem 22 (iii) above). Hence $\mathbb{ZP} = \prod_{p\in\pi} \mathbb{ZP}_p$ contains a dense copy of $\prod_{p\in\pi} \mathbb{Z}_p^{(\mathbb{N})}$ which contains a copy of $\widetilde{\mathbb{Z}}^{(\mathbb{N})} \cong \left(\prod_{p\in\pi} \mathbb{Z}_p\right)^{(\mathbb{N})}$ and this copy is still dense in $\mathbb{ZP}$.

(ii) From Theorem 22 (iii) implies that for each prime, $T_p$ contains a copy of $\mathbb{ZP}_p$. Hence $T$ contains a copy of $\mathbb{ZP}$. □

## 6. The Sylow decomposition of $\mathbb{Z}(n)^\times$ Indexed by $\mathcal{G}$

We record $n = \prod_{p|n} p^{\nu(n)}$ (finite product: almost all $\nu_p(n) \neq 0$ only if $p|n$) and accordingly $\mathbb{Z}(n) = \prod_{p|n} \mathbb{Z}(p^{\nu_p(n)})$. Hence $\mathbb{Z}(n)^\times = \prod_{p|n} \mathbb{Z}(p^{\nu_p(n)})^\times$, and it suffices to recall the case that $n = p^m$. This we assume for the remainder of this section, and we fix a prime $\mathbf{p}$.

Here we have $\mathbb{Z}(\mathbf{p}^m) = \mathbb{Z}_\mathbf{p}/\mathbf{p}^m{\cdot}\mathbb{Z}_\mathbf{p}$. Let $\mu\colon \mathbb{Z}_\mathbf{p} \to \mathbb{Z}_\mathbf{p}$ denote the scalar endomorphism given by $\mu(x) = \mathbf{p}^m x$. Then

$$0 \to \mathbb{Z}_\mathbf{p} \xrightarrow{\mu} \mathbb{Z}_\mathbf{p} \to \mathbb{Z}(\mathbf{p}^m) \to 0$$

is exact and $\mu$ induces a quotient morphism $\mu^\times\colon \mathbb{Z}_\mathbf{p}^\times \to \mathbb{Z}(\mathbf{p}^m)^\times$. We recall that the morphism $\mathbb{Z}_\mathbf{p} \to \mathbb{Z}_\mathbf{p}/\mathbf{p}\mathbb{Z}_\mathbf{p} \cong \mathrm{GF}(\mathbf{p})$ maps $C_\mathbf{p}$ of Lemmas 4 and 5 faithfully because $\mathbf{p}^m\mathbb{Z}_\mathbf{p} \subseteq \mathbf{p}\mathbb{Z}_\mathbf{p}$ unless $\mathbf{p} = 2$ and $m \leq 2$, in which case $\mathbf{p}^m = 2$ or $= 4$, in which case we have $\mathbb{Z}(2)^\times = \{1\}$, respectively, $\mathbb{Z}(4)^\times = \{\pm 1\}$. If $\mathbf{p} > 2$ then we know that

$$\exp\colon (\mathbf{p}\mathbb{Z}_\mathbf{p}, +) \to (1 + \mathbf{p}\mathbb{Z}_\mathbf{p}, \times) \text{ is an isomorphism,}$$

whence by applying $\mu$

$$\exp\colon \left(\frac{\mathbf{p}\mathbb{Z}_\mathbf{p}}{\mathbf{p}^m\mathbb{Z}_\mathbf{p}}, +\right) \to (\mu(1 + \mathbf{p}\mathbb{Z}_\mathbf{p}), \times) \text{ is an isomorphism.}$$

Since $\frac{\mathbf{p}\mathbb{Z}_\mathbf{p}}{\mathbf{p}^m\mathbb{Z}_\mathbf{p}} \cong \mathbb{Z}(\mathbf{p}^{m-1})$ in view of Lemma 4 we have

$$\mathbb{Z}(\mathbf{p}^m)^\times \cong \mathbb{Z}(\mathbf{p}^{m-1}) \oplus \mathbb{Z}(\mathbf{p} - 1).$$

Analogously, for $\mathbf{p} = 2$ and $m > 2$, from Lemma 5 we obtain

$$\mathbb{Z}(2^m)^\times \cong \mathbb{Z}(2^{m-2}) \oplus \mathbb{Z}(2) \tag{21}$$

Summarizing, we have

**Lemma 24.** *The group of units of* $\mathbb{Z}(\mathbf{p}^m)$ *is*

$$\mathbb{Z}(\mathbf{p}^m)^\times \cong \begin{cases} \{0\}, & \text{if } \mathbf{p}^m = 2, \\ \mathbb{Z}(2), & \text{if } \mathbf{p}^m = 4, \\ \mathbb{Z}\left(2^{m-2}\right) \oplus \mathbb{Z}(2), & \text{if } \mathbf{p} = 2,\ m > 2, \\ \mathbb{Z}\left(\mathbf{p}^{m-1}\right) \oplus \mathbb{Z}(\mathbf{p}-1), & \text{if } \mathbf{p} > 2. \end{cases} \tag{22}$$

We may use $\mathcal{G}$ as index set for describing the $p$-Sylow decomposition of $A = \mathbb{Z}(\mathbf{p}^m)^\times$ as follows:

We index subgroups $\mathbb{S}_e \leq A$ by attaching again to each edge $e = ((p,1),(q,0)) \in \mathcal{E}$ a profinite group $\mathbb{S}_e$ being, up to a natural isomorphism, a subgroup of $\widetilde{\mathbb{Z}}$:

**Definition 25.** *For each edge* $e \in \mathcal{E}$ *from* $p$ *to* $q$ *we set*

$$\mathbb{S}_e = \begin{cases} \{0\}, & \text{if } \mathbf{p}^m = 2 \text{ or } q > \mathbf{p}^m, \\ \mathbb{Z}(2), & \text{if } \mathbf{p}^m = 4 \text{ and } p = q = 2, \\ \mathbb{Z}_2 \oplus \mathbb{Z}(2), & \text{if } p = q = \mathbf{p} = 2, \\ \mathbb{Z}\left(p^{m-2}\right), & \text{if } 2 < \mathbf{p} \text{ and } q \leq \mathbf{p}, \\ \mathbb{Z}\left(p^{\nu_p(q-1)}\right), & \text{if } p < q \leq \mathbf{p}. \end{cases} \tag{23}$$

With this indexing we can formulate

**Theorem 26.** *For a fixed prime* $\mathbf{p}$ *and a fixed natural number* $m$,

(i) *the group* $\mathbb{Z}(\mathbf{p}^m)^\times$ *of units of the universal cyclic group* $\mathbb{Z}(\mathbf{p}^m)$ *is*

$$\mathbb{Z}(\mathbf{p}^m)^\times \cong \prod_{e \in \mathcal{E}} \mathbb{S}_e \tag{24}$$

*extended over the set* $\mathcal{E}$ *of all edges of the mastergraph, where* $\mathbb{S}_e$ *is the profinite group given in* (23) *above.*

(ii) *Its* $p$*-Sylow subgroup is the subproduct extended over the cone peaking in* $\mathbf{p}$*:*

$$\left(\mathbb{Z}(\mathbf{p}^m)^\times\right)_p \cong \prod_{e \in \mathcal{E}_p} \mathbb{S}_e = \begin{cases} \mathbb{Z}(4) \oplus \mathbb{Z}(2) \oplus \bigoplus_{\mathbf{p} \geq q > 2} \mathbb{Z}\left(2^{\nu_2(q-1)}\right), & \text{if } p = 2, \\ \mathbb{Z}(p^{m-2}) \oplus \bigoplus_{\mathbf{p} \geq q > p} \mathbb{Z}\left(p^{\nu_p(q-1)}\right), & \text{otherwise.} \end{cases} \tag{25}$$

## 7. The Mastergraph of a Periodic Abelian Group

Recall that for a locally compact group $G$ an element $g$ is called *compact* if it is contained in a compact subgroup. The set of compact elements is called $\mathrm{comp}(G)$. If $G$ is abelian, then $\mathrm{comp}(G)$ is a fully characteristic subgroup. For details see [2], Chapter 7 and [1]. The identity component of a topological group is written $G_0$.

**Definition 27.** *A locally compact group $G$ is said to be* periodic*, if it satisfies the following conditions:*

(i) $G = \mathrm{comp}(G)$,
(ii) $G_0 = \{0\}$.

In other words, $G$ is the union of its compact subgroups and is totally disconnected. In fact, if $G$ is abelian, then $G$ is the directed union of its compact open subgroups, and if $C$ and $K$ are two of them, then $C$ and $K$ are *commensurable*, that is both $C/(C\cap K)$ and $K/(C\cap K)$ are finite.

If $(G_j)_{j\in J}$ is a family of topological groups and $C_j \leq G_j$ is a compact open subgroup for each $j$, then the set of all $(g_j)_{j\in J} \in T = \prod_{j\in J} G_j$ such that $\{j \in J \mid g_j \notin C_j\}$ is finite forms a subgroup $L \leq T$ of the product containing $C = \prod_{j\in J} C_j$, and $L$ is a locally compact topological group for the topology generated by all $C \cap U$ with the open sets $U$ of $T$ in the Tychonov product topology. This group $L$ is called the *local product* of the family $(G_j, C_j)_{j\in J}$ and is written

$$L = \prod_{j\in J}^{\mathrm{loc}} (G_j, C_j).$$

We shall write abelian groups additively in general, unless the context demands otherwise, e.g. in the case of the group of units of a ring, such as $\mathbb{Z}_p$.

With this notation it is easy to reproduce Braconnier's theorem on the Sylow decomposition of a periodic locally compact abelian group $A$ into its $p$-Sylow subgroups $A_p$, $p \in \pi$:

**Theorem 28.** (J. Braconnier) *Let $A$ be a periodic locally compact abelian group and $C$ any compact open subgroup of $A$. Then $A$ is isomorphic to the local product*

$$\prod_{p}^{\mathrm{loc}} (A_p, C_p). \tag{26}$$

If $A$ is a periodic locally compact abelian group, then every endomorphism $\alpha$ leaves the Sylow subgroup $A_p$ invariant. We write

$\alpha_p = \alpha|G_p : A_p \to A_p$. If $C$ is a compact open subgroup, let $\mathrm{End}(G, C)$ denote the subring of the endomorphism ring $\mathrm{End}(G)$ of all endomorphisms leaving $C$ invariant.

In view of Theorem 28 we may identify $A$ with its canonical local product decomposition of the pair $(A, C)$.

Every locally compact abelian $p$-group $A$ is a $\mathbb{Z}_p$-module for a multiplication $(r_p, g_p) \mapsto r_p{\cdot}g_p$. If we identify $\widetilde{\mathbb{Z}}$ and $\prod_{p\in\pi} \mathbb{Z}_p$ by (1) and a periodic locally compact abelian group $A$ with $\prod^{\mathrm{loc}}_{p\in\pi}(A_p, C_p)$ for any compact open subgroup $C$, we see at once that we have a continuous module multiplication

$$(r, g) = ((r_p)_p, (g_p)_p) \to (r_p{\cdot}g_p)_p = r{\cdot}g : \widetilde{\mathbb{Z}} \times A \to A. \tag{27}$$

In a similar vein we observe

**Proposition 29.** *For a periodic locally compact abelian group $A$, the componentwise application $\kappa$ defined by*

$$\alpha \mapsto (\alpha_p)_p : \mathrm{End}(A, C) \to \prod_p \mathrm{End}(A_p, C_p)$$

*is an isomorphism of groups, and $\alpha((g_p)_p) = (\alpha_p(g_p))_p$.*

*Proof.* After identifying $(A, C)$ and $\prod^{\mathrm{loc}}_p (A_p, C_p)$ according to Theorem 28, it is straightforward to verify that $\kappa$ is an injective morphism of groups. Moreover, if

$$(\phi_p)_p \in \prod_p \mathrm{End}(A_p, C_p),$$

then the morphism

$$\phi\colon \prod_p A_p \to \prod_p A_p \text{ defined by } \phi((g_p)_p) = (\phi_p(g_p))_p$$

leaves $C = \prod_p C_p$ fixed as a whole and does the same with $\prod^{\mathrm{loc}}_p (A_p, C_p)$ and so $\kappa(\phi) = (\phi_p)_p$. Thus $\kappa$ is surjective as well. ☐

We noted in (27) that every $r \in \widetilde{\mathbb{Z}}$ yields an endomorphism $a \mapsto r{\cdot}a$ of the periodic locally compact abelian group $A$, giving us a morphism of rings $\zeta\colon \widetilde{\mathbb{Z}} \to \mathrm{End}(A)$. In particular, since scalar multiplication $\widetilde{\mathbb{Z}} \times A \to A$ is continuous, $\ker(\zeta)$ is a closed ideal of $\widetilde{\mathbb{Z}}$.

**Definition 30.** *For a locally compact abelian group $A$ we denote the factor ring $\widetilde{\mathbb{Z}}/\ker(\zeta)$ by $\mathcal{R}(A)$ and call it* the ring of scalars *of $A$. There is an obvious scalar multiplication $\mathcal{R}(A) \times A \to A$.*

The ring morphism $\zeta$ factors through an isomorphism of rings

$$\mathcal{R}(A) \xrightarrow{\cong} \mathrm{End}(A). \tag{28}$$

We note that

$$\mathcal{R}(A) \cong \prod_p \mathcal{R}(A)_p, \tag{29}$$

and scalar multiplication operates componentwise on $A \cong \prod_p (A_p, C_p)$.

We shall be mostly interested in the scalar multiplication by units $r \in \widetilde{\mathbb{Z}}$. In this context it is clear that (29) induces an isomorphism

$$\mathcal{R}(A)^\times \cong \prod_p (\mathcal{R}(A)_p)^\times. \tag{30}$$

where $(R(A)_p)^\times$ is isomorphic to a quotient group of $(\mathbb{Z}_p)^\times$.

One verifies easily the following piece of information:

*Example* 31. Let $A$ be a locally compact abelian $p$-group. Then

$$\mathcal{R}(A) = \begin{cases} \mathbb{Z}_p/p^m\mathbb{Z}_p \cong \mathbb{Z}(p^m), & \text{if } A \text{ has finite exponent } p^m, \\ \mathbb{Z}_p, & \text{otherwise.} \end{cases} \tag{31}$$

Lemma 6 shows that among the *compact* abelian groups $A$ the torsion groups are exactly the ones having finite exponent.

## 8. Scalar Multiplication on a Periodic Locally Compact Abelian Group

The following lemma is straightforward:

**Lemma 32.** *For a continuous endomorphism $\alpha$ of a locally compact group $G$ the following conditions are equivalent:*

(i) $\alpha(H) \subseteq H$ *for all closed subgroups $H$ of $G$.*

(ii) $\alpha(\overline{\langle g\rangle}) \subseteq \overline{\langle g\rangle}$ *for all $g \in G$.*

(iii) $\alpha(g) \in \overline{\langle g\rangle}$ *for all $g \in G$.*

**Definition 33.** An endomorphism $\alpha$ of a locally compact group $G$ is called *scalar* if it satisfies the equivalent conditions of Lemma 32.

In [3] it is shown that on a compact abelian $p$-group $G$, for any automorphism $\alpha$ which is scalar in the sense of Definition 33 there is an $r \in \mathbb{Z}_p^\times$ such that $\alpha(g) = r{\cdot}g$ for all $g \in G$. The proof through Lemma 2.21 and Proposition 2.22 in [3] works for endomorphisms as well and thus yields

**Lemma 34.** *Let $A$ be a compact abelian $p$-group. Then for any scalar endomorphism $\alpha$ there is an $r \in \mathbb{Z}_p$ such that $\alpha(a) = r{\cdot}a$ for all $a \in A$. Accordingly, $\alpha$ is an automorphism iff $r \in \mathbb{Z}_p^\times$.*

In [1], Lemma 4.6 it is shown for a locally compact abelian $p$-group $A$ for any automorphism $\alpha$ for which any restriction to a compact-open subgroup is scalar, there is an $r \in \mathbb{Z}_p$ such that $\alpha(a) = r{\cdot}a$ for all $a \in A$. Again this proof works for endomorphisms as well as for automorphisms. Therefore we have

**Lemma 35.** *Let $A$ be a locally compact abelian $p$-group. Then for any scalar endomorphism $\alpha$ there is an $r \in \mathbb{Z}_p$ such that $\alpha(a) = r{\cdot}a$ for all $a \in A$. Accordingly, $\alpha$ is an automorphism iff $r \in \mathbb{Z}_p^\times$.*

Finally, if $A = \prod_p^{\mathrm{loc}}(A_p, C_p)$ is any periodic locally compact abelian group we observe that every closed subgroup $H$ is of the form $\prod_p^{\mathrm{loc}}(H_p, C_p \cap H_p)$, and so an endomorphism $\alpha$ of $A$ is scalar iff every restriction $\alpha_p$ to $A_p$ is scalar. If this is the case, then for every $p$ there is an $r_p \in \mathbb{Z}_p$ such that $\alpha_p(a_p) = r_p{\cdot}a_p$ for all $a_p \in A_p$. So if $r = (r_p)_p$ in $\widetilde{\mathbb{Z}} = \prod_p \mathbb{Z}_p$, for the scalar endomorphism $\alpha$ we have an $r \in \widetilde{\mathbb{Z}}$ such that $\alpha(a) = r{\cdot}a$ for $a \in A$. Thus we have the following classification of scalar endomorphisms, justifying the nomenclature:

**Proposition 36.** *Let $A$ be a periodic locally compact abelian group and $\alpha\colon A \to A$ an endomorphism of locally compact abelian groups such that $\alpha(H) \subseteq H$ for all closed subgroups of $A$. Then there is an $r \in \widetilde{\mathbb{Z}}$ such that $\alpha(a) = r{\cdot}a$ for all $a \in A$.*

**Definition 37.** *The group of scalar automorphisms of a locally compact group $G$ is denoted by* $\mathrm{SAut}(G)$.

*If $A$ is abelian and is written additively, then the subgroup*

$$\{\mathrm{id}_A, -\mathrm{id}_A\} \subseteq \mathrm{SAut}(A)$$

*is said to consist of* trivial scalar automorphisms. *All other scalar automorphisms are called* nontrivial.

Notice that we shall not only call the identity automorphism, but also the inversion automorphism "$-\mathrm{id}_G$" *trivial*.

For periodic locally compact abelian groups $A$ we have seen in Proposition 36 that all scalar automorphisms are indeed scalar multiplications in the traditional sense (see [1], Proposition 4.15):

**Proposition 38.** *Let the locally compact abelian group $G$ be periodic. Then we have the following conclusions:*

(i) *The natural map $\zeta\colon \widetilde{\mathbb{Z}}^\times \to \mathrm{SAut}(G)$ (such that $\zeta(r)(g) = r{\cdot}g$) is surjective. In particular,* $\mathrm{SAut}(G)$ *is a profinite group and a homomorphic image of $\widetilde{\mathbb{Z}}^\times$.*

(ii) *The subsequent two statements are equivalent:*

(a) $\mathrm{SAut}(G) = \{\mathrm{id}_G, -\mathrm{id}_G\}$.

(b) *The exponent of $G$ is 2, 3, or 4.*

*Notably: The exponent of $G$ is 2 if and only if $-\mathrm{id}_G = \mathrm{id}_G$.*

Indeed, periodicity and the existence of nontrivial scalar multiplications are related as follows (see [1], Theorem 4.16):

**Theorem 39.** *For a locally compact abelian group $G$, we consider the following statements:*

(i) *$G$ has nontrivial scalar automorphisms.*
(ii) *$G$ is periodic.*

*Then (i) implies (ii), and if $G$ does not have exponent 2, 3, or 4, then both statements are equivalent.*

The Sylow decomposition of $\mathrm{SAut}(G)$ is described in the following theorem (see [1], Theorem 4.17)

**Theorem 40** (Mukhin, Theorem 2 in [4])**.** *Let $G$ be a locally compact abelian group written additively.*

(a) $\mathrm{SAut}(G)$ *is a homomorphic image of* $\widetilde{\mathbb{Z}}^\times$.
(b) *If $G$ is not periodic, then* $\mathrm{SAut}(G) = \{\mathrm{id}, -\mathrm{id}\}$.
(c) *If $G$ is periodic, then* $\mathrm{SAut}(G) = \prod_p \mathrm{SAut}(G_p)$, *where* $\mathrm{SAut}(G_p)$ *may be identified with the group of units of the ring $\mathcal{R}(G_p)$ of scalars of $G_p$, namely, $\mathcal{R}(G_p)^\times$ is isomorphic to*

$$\begin{cases} \mathbb{Z}_p \times \mathbb{Z}(p-1), & \text{if } p > 2 \text{ and the exponent of } G_p \text{ is infinite,} \\ \mathbb{Z}(p^{m-1}) \times \mathbb{Z}(p-1), & \text{if } p > 2 \text{ and the exponent of } G_p \text{ is } p^m, \\ \mathbb{Z}_2 \times \mathbb{Z}(2), & \text{if } p = 2 \text{ and the exponent of } G_2 \text{ is infinite,} \\ \mathbb{Z}(2^{m-2}) \times \mathbb{Z}(2), & \text{if } p = 2 \text{ and the exponent of } G_2 \text{ is } 2^m > 2, \\ \{0\}, & \text{if } p = 2 \text{ and the exponent of } G_2 \text{ is } 2. \end{cases}$$

(d) *An $\alpha \in \mathrm{Aut}(G)$ is in $\mathrm{SAut}(G)$ iff there is a unit $z \in \widetilde{\mathbb{Z}}^\times$ such that*
$$(\forall g \in G)\, \alpha(g) = z\cdot g = \textstyle\prod_p z_p\cdot g_p \text{ for } z = \prod_p z_p,\ g = \prod_p g_p.$$

## 9. The Prime Graph of a Periodic Locally Compact Abelian Group

Now let $A$ be a periodic locally compact abelian group; the Sylow structure of $\mathrm{SAut}(A)$ is now easily discussed: The quotient morphism $\zeta\colon \widetilde{\mathbb{Z}}^\times \to \mathrm{SAut}(A)$ of Proposition 38, preserving the Sylow structures, and the structure of $\mathrm{SAut}(A)$ described so far in Theorem 40 allow a precise description of the Sylow structure of $\mathrm{SAut}(A)$.

We associate with $A$ the bipartite graph $\mathcal{G}(A) = (U, V, \mathcal{E}(A), \lambda)$ with $U$ and $V$ as in the mastergraph and with

$$\mathcal{E}(A) = \{e \in E : e = ((p,1),(q,0)) \text{ such that } \mathrm{SAut}(A_q)_p \neq \{\mathrm{id}_A\}\},$$

and for fixed $p$ we define

$$\mathcal{E}_p(A) \subseteq \{e \in \mathcal{E}(A) : e = ((p,1),(q,0)) \text{ such that } p|(q-1)\},$$

is the set of all edges in $\mathcal{G}(A)$ from $p$ to $q$ such that $\mathrm{SAut}(A_q)_p$ is nontrivial, and for $q$

$$\mathcal{F}_q(A) \subseteq \{e \in \mathcal{E}(A) : e = ((p,1),(q,0)) \text{ such that } p|(q-1)\},$$

is the set of all edges in $\mathcal{G}(A)$ from $p$ to $q$ such that $\mathrm{SAut}(A_q)_p$ is nontrivial.

Finally, for $e \in \mathcal{E}(A)$ from $p$ to $q$ the label is

$$\lambda(e) = \begin{cases} \infty, & \text{if } p = q, \\ \nu_p(q-1), & \text{if } p|(q-1). \end{cases} \tag{32}$$

Now let $A$ be a periodic locally compact abelian group; the Sylow structure of $\mathrm{SAut}(A)$ is then easily discussed: The quotient morphism $\zeta\colon \widetilde{\mathbb{Z}}^\times \to \mathrm{SAut}(A)$ of Proposition 38, preserving the Sylow structures, and the structure of $\mathrm{SAut}(A)$ described so far in Theorem 40 allow a precise description of the Sylow structure of $\mathrm{SAut}(A)$.

**Theorem 41.** (The Sylow Structure of $\mathrm{SAut}(A)$) *Let $A$ be a periodic locally compact abelian group and $\mathrm{SAut}(A) = \prod_{p\in\pi} \mathrm{SAut}(A)_p$ the $p$-primary decomposition of the profinite group $\mathrm{SAut}(A) = \prod_{e\in\mathcal{E}(A)} \mathbb{S}_e$. Then*

(i) *The $p$-primary decomposition of $\mathrm{SAut}(A_q)$ is (additive notation assumed)*

$$\prod_{e\in\mathcal{F}_q} \mathrm{SAut}(A_q)_{p_e} \cong \prod_{e\in\mathcal{F}_q} \mathbb{S}_e(A),$$

*and this group is equal, in case $p = 2$, to*

$$\begin{cases} \{0\}, & \textit{if } A_2 \textit{ has exponent } \leq 2, \\ \mathbb{Z}(2^{r-2}) \oplus \mathbb{Z}(2), & \textit{if } A_2 \textit{ has finite exponent } 2^r > 2, , \\ \mathbb{Z}_2 \oplus \mathbb{Z}(2), & \textit{if } A_2 \textit{ has infinite exponent}, \end{cases}$$

*and in case $p > 2$, to*

$$\begin{cases} \mathbb{Z}(q^{r-1}) \oplus \bigoplus_{e\in\mathcal{F}_q,\ \mathrm{sloping}} \mathbb{Z}(p_e^{\lambda(e)}), & \textit{if } A_q \textit{ has finite exponent } q^r, \\ \mathbb{Z}_q \oplus \bigoplus_{e\in\mathcal{F}_q,\ \mathrm{sloping}} \mathbb{Z}(p_e^{\lambda(e)}), & \textit{if } A_q \textit{ has infinite exponent.} \end{cases}$$

(ii) *The structure of the p-primary component* $\mathrm{SAut}(A)_p$ *of* $\mathrm{SAut}(A)$ *is*

$$\prod_{e\in\mathcal{E}_p}(\mathrm{SAut}(A_{q_e})_p \cong \prod_{e\in\mathcal{E}_p}\mathbb{S}_e(A) =$$

$$\begin{cases}\prod_{e\in\mathcal{E}_p,\ \text{sloping}}\mathbb{Z}(2^{\lambda(e)}), & \textit{if } p=2 \textit{ and } A_2 \textit{ has exponent} \le 2,\\ \mathbb{Z}(2^{r-2})\oplus\mathbb{Z}(2)\oplus\prod_{e\in\mathcal{E}_p,\ \text{sloping}}\mathbb{Z}(2^{\lambda(e)}), & \textit{if } p=2 \textit{ and } A_2 \textit{ has fin.exp. } 2^r{>}2,\\ \mathbb{Z}_2\oplus\mathbb{Z}(2)\oplus\prod_{e\in\mathcal{E}_p,\ \text{sloping}}\mathbb{Z}(2^{\lambda(e)}), & \textit{if } p=2 \textit{ and } A_2 \textit{ has inf. exponent},\\ \mathbb{Z}(p^{r-1})\oplus\prod_{e\in\mathcal{E}_p,\ \text{sloping}}\mathbb{Z}(p_e^{\lambda(e)}), & \textit{if } 2<p \textit{ and } A_p \textit{ has exponent } p^r,\\ \mathbb{Z}_p\oplus\prod_{e\in\mathcal{E}_p,\ \text{sloping}}\mathbb{Z}(p_e^{\lambda(e)}), & \textit{if } 2<p \textit{ and } A_p \textit{ has infinite index.}\end{cases}$$

## 10. An application

For easy reference we repeat the following definition from the introduction

**Definition 42.** *If* $(G, A)$ *is a pair consisting of a topological group* $G$ *and a closed normal subgroup* $A$*, then we call it a* special extension of $A$ *if* $G$ *is a locally compact group and the equivalent conditions of* Proposition *1 are satisfied.*

We now prove the following result as an example of the methods we are proposing.

**Theorem 43.** *Let* $(G, A)$ *be a special extension of a periodic locally compact abelian group. Then for each sloping edge* $e \in \mathcal{E}(G, A)$ *from some* $p$ *to some* $q$*, all of* $A_q$ *consists of commutators. In particular,* $A_q \subseteq G'$*.*

*Proof.* By definition the existence of $e$ that there is a $p$-element $g \in G_p$ such that $1 \neq r = \rho(g) \in (\mathcal{R}(A_q)^\times)_p$. Since $e$ is sloping, $p < q$. By Theorem 40 we know that $(\mathcal{R}(A_q)^\times)_p$ is a cyclic group of order $p^{\lambda(e)} = p^{\nu_p(q-1)}$.

We claim that $1 - r$ is a unit in the ring $\mathcal{R}(A_q)$ of scalars which is isomorphic to $\mathbb{Z}_q$ or quotient ring thereof depending as $A_q$ has infinite or finite exponent. By way of contradiction suppose that $r - 1$ is not a unit. Since $\mathbb{Z}_q^\times = \mathbb{Z}_q \setminus q\mathbb{Z}_q$, there is an element $u \in \mathcal{R}(A_q)$ such that $1 - r = qu$. Then $r = 1 - qu \in 1 + q\mathcal{R}(A_q)$ which, according to the structure of $\mathbb{Z}_q^\times$ in (4), respectively, of $\mathbb{Z}(q^m)^\times$ in (22), is the $q$-Sylow subgroup of $\mathcal{R}(A_q)^\times$. But $r$ is a $p$-element with $p < q$ and this is a contradiction.

Now let $a \in A_q$. For the purpose of this proof we write $G$ additively. Then the commutator of $g$ and $a$ is $[g, a] = \rho(g)(a) - a = r{\cdot}a - a = (r - 1){\cdot}a$. Since $1 - r$ is invertible, we set $b = (r - 1)^{-1}{\cdot}a \in A_q$ and obtain

$$a = \rho(g)(b) - b = gbg^{-1} - b = [g, b].$$

This shows that every element of $A_q$ is a commutator and thus proves the theorem. □

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Institute for Analysis and Scientific Computation
Technische Universität Wien
Wiedner Hauptstrasse 8-10/101
Vienna, Austria
*E-mail address*: wolfgang.herfort@tuwien.ac.at

Fachbereich Mathematik
Technische Universität Darmstadt
Schlossgartenstr. 7
64289 Darmstadt, Germany
*E-mail address*: hofmann@mathematik.tu-darmstadt.de

Mathematisches Institut
Universität Münster
Einsteinstrass 62
48149 Münster, Germany
*E-mail address*: linus.kramer@uni-muenster.de

Department of Mathematics and Applied Mathematics
University of Cape Town
Private Bag X1, Rondebosch 7701
Cape Town, South Africa
*E-mail address*: francescog.russo@yahoo.com