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AFFINE SPACE FIBRATIONS

R.V. GURJAR, K. MASUDA AND M. MIYANISHI

Abstract. We discuss various aspects of affine space fibrations. Our interest will be focused in the singular fibers, the generic fiber and the propagation of properties of a given smooth special fiber to nearby fibers.

Contents

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Introduction

In the present article, we will be interested in fibrations of algebraic varieties in order to study structures of higher-dimensional varieties. Fibration structure allows us to study varieties of smaller dimension and their interrelations. In particular, we will deal fibrations whose fibers are affine spaces $\mathbb{A}^n$ or fibrations obtained as the quotient morphisms of varieties with unipotent group actions. There are long-standing problems which are still open in higher dimensions. We will discuss these problems.

Let k be an algebraically closed field of characteristic zero, which we fix as the base field throughout the article. But, we assume without loss of generality that k is the complex filed $\mathbb{C}$ whenever we use topological arguments.

Let $X \to Y$ be a dominant morphism of algebraic varieties X and Y. Let $k(X)$ and $k(Y)$ be the function fields of X and Y, respectively. Then f induces the inclusion of fields $k(Y) \hookrightarrow k(X)$. We thereby identify $k(Y)$ with a subfield of $k(X)$. We say that f is a *fibration* if $k(X)$ is a regular extension of $k(Y)$, or equivalently, if the generic fiber is geometrically integral. Further, the definition is equivalent to saying that $k(Y)$ is algebraically closed in $k(X)$. The following properties are well-known.

(1) If $f : X \to Y$ and $g : Y \to Z$ are fibrations then $g \cdot f : X \to Z$ is a fibration.
(2) If $f : X \to Y$ is a fibration, then the base change $f_{Y'} : X \times_Y Y' \to Y'$ is a fibration for any algebraic variety Y' dominating Y.
(3) Let $f : X \to Y$ be a fibration. Then there exists a non-empty open set U of Y such that the fiber $X_y = X \times_Y \operatorname{Spec} k(y)$ is geometrically integral for every closed point y of U, where $k(y)$ denotes the residue field of Y at y.

We say that a fibration $f : X \to Y$ is an *F-fibration* for an algebraic variety F if X_y is isomorphic to F over k for general (closed) points y of Y. The F-fibration is *generically trivial* if there exists a non-empty open set U of Y such that $f^{-1}(U)$ is isomorphic to $U \times F$ over U. If $f^{-1}(U) \times_U U' \cong U' \times F$ over U' for an open set U of Y and a finite étale covering $U' \to U$, f is called *generically isotrivial.* If we can choose an open neighborhood U_y as U for each closed point y then X is a locally trivial F-*bundle* over Y in the Zariski topology of Y. If we can choose a finite étale covering U'_y of an open neighborhood U_y of y so that $f^{-1}(U_y) \times_{U_y} U'_y \cong F \times U'_y$, we say that X is a locally isotrivial *étale F-bundle* over Y. Let $X_\eta := X \times_Y \operatorname{Spec} k(Y)$ be the *generic fiber* of f. If f is a generically trivial F-fibration then X_η is isomorphic to F over the field $k(Y)$. Similarly, if f is generically isotrivial then X_η is isomorphic to F over a finite separable extension $k(U')$ of $k(U) = k(Y)$, while it is not necessarily isomorphic to F over $k(Y)$. It is a fundamental question about fibrations to ask if an fibration $f : X \to Y$ is generically trivial or generically isotrivial provided general closed fibers of f are isomorphic to an algebraic variety F (*the generic triviality or generic isotriviality problem*).

If k has infinite transcendence degree over $\mathbb{Q}$, F is an affine variety and f is an affine morphism, the generic isotriviality of f follows from the generic equivalence theorem of Kraft-Russell [29] which is stated as:

Generic Equivalence Theorem. *Let k be an algebraically closed field of infinite transcendence degree over the prime field. Let $p : S \to Y$ and $q : T \to Y$ be two affine morphisms where S, T and Y are k-varieties. Assume that for all $y \in Y$ the two (schematic) fibers $S_y := p^{-1}(y)$ and $T_y := q^{-1}(y)$ are isomorphic. Then there is a dominant morphism of finite degree $\varphi : U \to Y$ and an isomorphism $S \times_Y U \cong T \times_Y U$ over U.*

We are interested in the case f is an affine morphism and general fibers are isomorphic to the affine space $\mathbb{A}^n$. An example of $\mathbb{A}^n$-fibration is obtained as the quotient morphism $q : X \to Y := X/G$ when a unipotent algebraic group G acts on an affine variety $X = \operatorname{Spec} A$ so that the ring of invariants $B = A^G$ is finitely generated over k. Then we set $X/G = \operatorname{Spec} B$ and $q : X \to Y$ the morphism induced by the canonical inclusion $B \hookrightarrow A$.

The generic triviality problem for $F = \mathbb{A}^n$ is called the *Dolgachev-Weisfeiler problem*, and was treated in many references including [2, 6, 22, 23, 24, 25, 26, 39, 45]. If $f : X \to Y$ is generically trivial, X contains an open set $f^{-1}(U) \cong U \times \mathbb{A}^n$. We call such an open set an $\mathbb{A}^n$-*cylinder* or a *cylinderlike open set* if $n = 1$. The answer to Dolgachev-Weisfeiler problem is positive if $n = 1, 2$ and not known for

$n \geq 3$. The problem is related to the triviality of forms of $\mathbb{A}^n$ and the structure of the automorphism group $\mathrm{Aut}(\mathbb{A}^n)$.

An F-fibration $f : X \to Y$ has *singular fiber* which is, by definition, a closed fiber X_y which is not isomorphic to F. There are four possibilities for which the fiber X_y is not isomorphic to F.

(1) The fiber X_y is integral, i.e., irreducible and reduced, but not isomorphic to F.
(2) The fiber X_y is not integral. Hence either X_y has two or more irreducible components (*reducible fiber*), or X_y is irreducible but non-reduced (*multiple fiber*).
(3) Each irreducible component Z_i of X_y has right dimension $\dim F$, but has multiplicity $\mathrm{length}\mathcal{O}_{X_y,\xi_i}$ which is a multiple of some integer $d > 1$ (*multiple fiber*), where ξ is the generic point of Z_i.
(4) Some irreducible component Z_i has dimension bigger than $\dim F$.

In section one, we consider first an $\mathbb{A}^1$-fibration $f : S \to C$ on an integral affine surface S over a curve C without assuming that S is smooth or normal, and consider singular fibers of f. This situation occurs when S is embedded into a bigger variety X with milder singularity as $S = \widetilde{f}^{-1}(C)$ with C embedded into Y, where $\widetilde{f} : X \to Y$ is an $\mathbb{A}^1$-fibration. If the $\mathbb{A}^1$-fibration f is obtained as the quotient morphism $q : S \to C$ of a G_a-action on S, we have a multiple fiber contained in the fixed-point locus S^{G_a}. But we have a chance to retrieve the G_a-action near the multiple fiber by looking at the infinitesimal neighborhoods of the multiple fiber (Lemmas 1.2.1 and 1.2.2). In [14], we observed singular fibers of the quotient morphism $q : X \to Y$ when G_a acts on a smooth affine threefold X. A singular fiber might contain a two-dimensional irreducible component S. We here, in subsection 1.3, show under a mild assumption that S contains a cylinderlike open set (Theorem 1.3.1). In subsection 1.4 we observe singular fibers of a $\mathbb{P}^1$-fibration on a normal surface S or a smooth algebraic threefold X. Our observation is limited to showing the simply-connectedness of the fiber F or its irreducible components.

Section two is devoted to an equivariant Abhyankar-Sathaye conjecture in dimension three. The conjecture in general asks if an embedded affine plane $\mathbb{A}^2$ in $\mathbb{A}^3$ is a fiber of an $\mathbb{A}^2$-bundle on $\mathbb{A}^3$. Contrary to the case of the affine line embedded into $\mathbb{A}^2$ (cf. Theorem of Abhyankar-Moh-Suzuki), this is far more difficult. We consider a G_a-action on $\mathbb{A}^3$ for which the embedded plane is stable. If we put two more conditions, the answer is positive (Theorem 2.2.1). These additional conditions will suggest what kind of observations must be made in the non-equivariant case.

Section three is for a study of forms of $\mathbb{A}^n$ with unipotent group actions. Consider a $\mathbb{A}^n$-fibration $f : X \to Y$. The following two are major questions in our mind.

(1) Is the fibration generically isotrivial? Namely, is the generic fiber X_η isomorphic to $\mathbb{A}^n$ after a finite algebraic extension K of $k(Y)$?
(2) If the question (1) is affirmative, X_η is a $k(Y)$-form of $\mathbb{A}^n$. Is it then isomorphic to $\mathbb{A}^n$ over $k(Y)$?

If these two questions are answered affirmatively, then f is generically trivial. In fact, if $n = 1, 2$ the answer of these questions are positive. If $n \geq 3$, however, we do not know the answer to these questions. Our purpose is to consider the question (2) with a suitable action of unipotent group. First we prove that a form X of $\mathbb{A}^3$ is trivial if there is a fixed-point free G_a-action on X (Theorem 3.2.4) or an effective action of a unipotent group of dimension two (hence necessarily commutative) (Theorem 3.3.2)[1]. Second, we show that if X is a k-form of $\mathbb{A}^n$ with a *proper* action of a commutative unipotent group G of dimension $n-2$ then the k-form is trivial, where k is a non-closed field of characteristic zero (Remark 3.4.7). But the proof is only given in the case $n = 4$. Assuming that a given action of a unipotent group G on a k-form be proper is seemingly quite technical, but the properness implies nice conditions like fixed-point freeness of the action. Third, we consider a k-form of $\mathbb{A}^n \times \mathbb{A}^1_*$, where $\mathbb{A}^1_*$ is the affine line with one point punctured. Then a k-form has a bundle structure over $\mathbb{A}^1_*$ or its k-form with fiber isomorphic to a k-form of $\mathbb{A}^n$ (Lemma 3.5.3).

In section four, we apply the arguments in subsection 3.4 to prove a special case of the cancellation theorem in dimension three where the variety in question has a proper G_a-action (Theorem 4.1.1). We also consider the geometric meaning of a proper G_a-action in comparison with geometric quotient.

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1. Singular fibers of $\mathbb{A}^1$- and $\mathbb{P}^1$-fibrations

1.1. **G_a-actions on affine surfaces.** Let $S = \operatorname{Spec} A$ be an integral affine surface defined over k. Assume that S has a nontrivial G_a-action. Let $B = A^{G_a}$. Then $B = A \cap K^{G_a}$, where $K = Q(A)$. By

[1] The latter is a theorem of Daigle-Kaliman [5].

Zariski's lemma, B is an affine domain of dimension one over k. [2]. Let $C = \text{Spec}\, B$ and let $q : S \to C$ be the quotient morphism induced by the inclusion $B \hookrightarrow A$.

Problem 1.1.1. *With the notations and assumptions as above, is every fiber $F := q^{-1}(P)$ for $P \in C$ a disjoint union of the affine lines?*

What we can say so far is the following result which will be proved below.

Theorem 1.1.2. *With the above notations and assumptions, the following assertions hold:*

(1) *F is a disjoint union of the irreducible components, each of which is an affine rational curve with one place at infinity.*
(2) *If an irreducible component Z_i of F is reduced in F, then Z_i is isomorphic to the affine line.*
(3) *If S is normal, every irreducible component of F is isomorphic to the affine line.*

We have the following related result [33, Theorem 2.1].

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[2]We prove by topological arguments a part of Zariski's lemma.
Theorem. *Let A be a normal affine domain defined over $\mathbb{C}$ with the quotient field K, let L be a subfield of K with* $\text{tr.deg}_{\mathbb{C}} L = 1$ *and let* $B = A \cap L$. *Then B is finitely generated over $\mathbb{C}$.*
Proof. If $B = \mathbb{C}$, we have nothing to show. So, we assume that $\mathbb{C} \subsetneqq B$. Since A is normal, B is integrally closed in K. Furthermore, since $\text{tr.deg}_{\mathbb{C}} L = 1$, the quotient field of B is equal to L. In fact, suppose that $Q(B) \neq L$. Let ξ be an element of $L \setminus Q(B)$. Since K is algebraic over $Q(B)$, there exist an element $b \in B \setminus \{0\}$ such that $b\xi$ is integral over B. Since it then follows that $b\xi \in A$, we have $b\xi \in B$. Hence $\xi \in Q(B)$. A contradiction.

We can find elements $b_1, \ldots, b_r \in B$ such that the subring $B_1 := \mathbb{C}[b_1, \ldots, b_r]$ of B is normal and birational to B. Let $C_1 = \text{Spec}\, B_1$. Then C_1 is a smooth affine curve. Let $X = \text{Spec}\, A$ and let $q_1 : X \to C_1$ be the dominant morphism induced by the inclusion $B_1 \hookrightarrow A$. Then the homomorphism of homology groups $(q_1)_* : H_1(X; \mathbb{Q}) \to H_1(C_1; \mathbb{Q})$ is surjective. Suppose that $B_1 \subsetneqq B$. Then we find an affine subalgebra B_2 of B containing strictly B_1 such that B_2 is normal. Let $C_2 = \text{Spec}\, B_2$. Then the inclusion $B_1 \subsetneqq B_2$ induces a birational morphism $C_2 \to C_1$, which makes C_2 a strictly smaller open set of C_1, i.e., $C_2 \subsetneqq C_1$. The morphism $q_2 : X \to C_2$ induces a surjection $(q_2)_* : H_1(X; \mathbb{Q}) \to H_1(C_2; \mathbb{Q})$, where $\text{rank} H_1(C_2; \mathbb{Q}) > \text{rank} H_1(C_1; \mathbb{Q})$ because $C_2 \subsetneqq C_1$. Continuing this way, we obtain a sequence of smooth affine curves $C_1 \supsetneqq C_2 \supsetneqq C_3 \supsetneqq \cdots$, where $C_{i+1} \subsetneqq C_i$ is an open immesrsion. Hence $\text{rank} H_1(C_{i+1}; \mathbb{Q}) > \text{rank} H_1(C_i; \mathbb{Q})$. But, since $\text{rank} H_1(X; \mathbb{Q}) \geq H_1(C_i; \mathbb{Q})$ for every i, this decreasing series of open sets must stop at some i. This implies that B is finitely generated over $\mathbb{C}$. $\square$

Lemma 1.1.3. *Let A be an affine k-algebra of dimension one with a nontrivial locally nilpotent derivation (lnd, in short) δ, and let $R = \operatorname{Ker}\delta$. Assume that the associated G_a-action has no fixed points on $\operatorname{Spec} A$, i.e., $\delta(A)A = A$. Then $A \cong R[t]$, where R is an Artin ring.*

Lemma 1.1.3 implies that if the G_a-action on an irreducible component Z_i of F is nontrivial, then the scheme structure of F on Z_i is $(\operatorname{Spec} R_i) \times \mathbb{A}^1$, where R_i is an Artin local ring and represents the "thickning" of Z_i in F. Hence $\operatorname{length}(R_i)$ is the multiplicity of Z_i in F.

We begin with the following result [34, Chap.3, Lemma 1.4.2].

Lemma 1.1.4. *Let $\widetilde{S}$ be a normal affine surface with a nontrivial G_a-action and let $\widetilde{q} : \widetilde{S} \to C$ be the quotient morphism, where C is a normal affine curve. Let $P \in C$ be a closed point and let $\widetilde{F} = \widetilde{q}^{-1}(P)$. Then $\widetilde{F}$ is a disjoint union of the affine lines.*

In [34], it is not shown that each irreducible component of $\widetilde{F}$ is isomorphic to the affine line. But this follows from the fact that the irreducible component (plus one smooth point) is obtained from a part of a degenerate fiber of a $\mathbb{P}^1$-fibration on a smooth projective surface by contracting a linear chain such that it meets the (proper transform) of the irreducible component at an end component of the linear chain and that the linear chain is the exceptional locus of a cyclic quotient singularity. The assertion (3) of Theorem 1.1.2 follows from this result.

In order to prove the other two assertions of Theorem 1.1.2, we set our notations as follows. With A as given above, let $\widetilde{A}$ be the normalization of A and let $\widetilde{S} = \operatorname{Spec} \widetilde{A}$.

Lemma 1.1.5. *The following reduction of the settings is possible.*

(1) *We may assume that the curve $C = \operatorname{Spec} B$ is normal.*

(2) *The G_a-action on $S = \operatorname{Spec} A$ lifts to $\widetilde{S} = \operatorname{Spec} \widetilde{A}$. Namely, the lnd δ associated with the G_a-action on S extends to an lnd $\widetilde{\delta}$ on $\widetilde{A}$ in such a way that $\operatorname{Ker}\delta = \operatorname{Ker}\widetilde{\delta}$. Hence we have a commutative diagram*

$$\begin{array}{ccc} \widetilde{S} & \xrightarrow{\nu} & S \\ {\scriptstyle\widetilde{q}}\downarrow & & \downarrow{\scriptstyle q} \\ C & = & C \end{array}$$

where ν is the normalization morphism and $\widetilde{q}$ is the quotient morphism.

Proof. (1) Note that the function field K is a regualr extension of k, hence $L := Q(B)$ is an algebraic function field of dimension one over

k. Let $\widetilde{C}$ be the normalization of C in L and let $\widetilde{P}$ be a point of $\widetilde{C}$ lying over P. Then the residue fields of P and $\widetilde{P}$ coincide with k. Let $\overline{S} = S \times_C \widetilde{C}$. Then $\overline{S} = \text{Spec}\,(\overline{A})$, where $\overline{A} = A \otimes_B \widetilde{B}$ and $\widetilde{B}$ is the integral closure of B in L. Since $\widetilde{C}$ is birational to C, taking the tensor product $\otimes_B \widetilde{B}$ is equivalent to adjoining finitely many fractions of B. Hence $\overline{A} = A[\widetilde{B}]$. Let $\mathfrak{m}$ (resp. $\widetilde{\mathfrak{m}}$) be the maximal ideal of B (resp. $\widetilde{B}$) corresponding to the point P (resp. $\widetilde{P}$). Then we have

$$\begin{aligned}
\overline{A}/\widetilde{\mathfrak{m}}\overline{A} &= (A \otimes_B \widetilde{B}) \otimes_{\widetilde{B}} (\widetilde{B}/\widetilde{\mathfrak{m}}) \\
&= (A \otimes_B (B/\mathfrak{m})) \otimes_{B/\mathfrak{m}} (\widetilde{B}/\widetilde{\mathfrak{m}}) \cong A \otimes_B (B/\mathfrak{m}) = A/\mathfrak{m}.
\end{aligned}$$

Namely, we have $f^{-1}(P) \cong \overline{f}^{-1}(\widetilde{P})$, where $\overline{f} : \overline{S} \to \widetilde{C}$ is the base change of S. Hence we may replace C by $\widetilde{C}$ and assume that C is normal.

(2) It is known by Seidenberg [42] that δ extends to a derivation $\widetilde{\delta}$ of $\widetilde{A}$. Namely, $\widetilde{\delta}(\widetilde{A}) \subseteq \widetilde{A}$. Assume that C is normal by (1). Since S contains a cylinderlike open set $U \times \mathbb{A}^1$ with an open set $U = D(b) = \{b \neq 0\}$ of C for $b \in B$, we have $\widetilde{\delta}|_{U\times\mathbb{A}^1} = \delta|_{U\times\mathbb{A}^1}$. Since $\widetilde{\delta}$ defines a G_a-action on this cylindelike open set, $\widetilde{\delta}$ is an lnd on $\widetilde{A}$. $\square$

Proof of Theorem 1.1.2. (1) Let $\widetilde{F} = \widetilde{q}^{-1}(P)$. Then $\widetilde{F} = \coprod_j \widetilde{Z}_j$, where $\widetilde{Z}_j \cong \mathbb{A}^1$ by Lemma 1.1.4. Note that $\nu|_{\widetilde{F}} : \widetilde{F} \to F$ is a finite morphism. Suppose that $Q \in Z_i \cap Z_{i'}$. Let $\nu^{-1}(Z_i) = \widetilde{Z}_{ij_1} \coprod \cdots \coprod \widetilde{Z}_{ij_r}$ and $\nu^{-1}(Z_{i'}) = \widetilde{Z}_{i'j_1} \coprod \cdots \coprod \widetilde{Z}_{i'j_s}$. Then a point $\widetilde{Q} \in \nu^{-1}(Q)$ lies in the intersection of $\widetilde{Z}_{ij}$ and $\widetilde{Z}_{i'j'}$ for some j and j'. Hence the components $\widetilde{Z}_{ij}$ and $\widetilde{Z}_{i'j'}$ meet each other. But this contradicts Lemma 1.1.4. So, F is a disjoint union $\coprod_{i=1}^r Z_i$. Note that, for each $1 \leq i \leq r$, a multiple of Z_i is locally defined by $t = 0$. By Nagata [36, p. 65, footnote], $S - (F \setminus Z_i)$ is affine, and $S - (F \setminus Z_i)$ has a nontrivial G_a-action since $F \setminus Z_i$ is G_a-stable. Thus we may assume that F consists of a single irreducible component Z_i. Then Z_i is a surjective image of the affine line $\widetilde{Z}_j$ for some j. Hence Z_i is an affine rational curve with one place at infinity.

(2) Let t be a generator of the maximal ideal of $\mathcal{O}_{C,P}$. By replacing C by an open neighborhood U of P such that t is regular on U and P is the only zero of t on U, we may assume that $\mathfrak{m}$ is a principal ideal tB of B. Furthermore, we may assume that $S \setminus F \cong \widetilde{S} \setminus \widetilde{F}$, i.e., $A[t^{-1}] = \widetilde{A}[t^{-1}]$. Then we know that $\widetilde{A}$ is obtained from A by adjoining element of $A[t^{-1}]$ of the form a/t^m with $a \in A$ and $m > 0$. Take such an element a/t^m of $\widetilde{A}$. Assume that $a \notin tA$. Then it satisfies a monic

relation

$$(a/t^m)^n + a_1(a/t^m)^{n-1} + \cdots + a_{n-1}(a/t^m) + a_n = 0, \quad a_i \in A\ .$$

Then we have

$$a^n + t^m a_1 a^{n-1} + \cdots + t^{m(n-1)} a_{n-1} a + t^{mn} a_n = 0.$$

Hence $a^n \in t^m A$. By the assertion (1), we now assume that F is irreducible. Let $R = A/tA$ and let $\mathfrak{p} = \sqrt{0}$ which represents the "thickning" of the component Z_i in F. If Z_i is a reduced component, $\mathfrak{p} = (0)$. Meanwhile, $\mathfrak{p} = \sqrt{tA}/tA$ and $a \in \sqrt{tA}$. Since $\mathfrak{p} = (0)$, $a \in tA$. This is a contradiction to the above assumption. Hence $A = \widetilde{A}$, and $Z_i \cong \mathbb{A}^1$. □

We need the following result in subection 1.3.

Theorem 1.1.6. *Let S be an affine algebraic surface and let $f : S \to C$ be a dominant morphism to an affine curve C. Assume that, for every closed point $P \in C$, the fiber $f^{-1}(P)$ is a disjoint union of the affine lines. Then the following assertions hold.*

(1) *There exists a cylinderlike affine open set $Z = U \times \mathbb{A}^1$ of S such that every fiber of the projection $p_1 : Z \to U$ is a fiber component of f.*
(2) *There exists a nontrivial G_a-action on S such that the morphism f is factored by the quotient morphism $q : S \to S/G_a$ as*

$$f : S \xrightarrow{q} S/G_a \xrightarrow{g} C,$$

where g is a quasi-finite morphism.

Proof. (1) Restricting C to the smooth part of C, we may assume that C is smooth. Let $\nu : \widetilde{S} \to S$ be the normalization morphism and let $\widetilde{f} = f \cdot \nu$. Removing the fibers of $\widetilde{f}$ which pass through the singular points of $\widetilde{S}$, we may assume that $\widetilde{S}$ is smooth. On the other hand, if the singular locus $\mathrm{Sing}(S)$ contains a fiber component of f, then we throw away the complete fiber containing the component. So, we may assume that if $\mathrm{Sing}(S)$ has dimension one, all the components of dimension one are horizontal to the fibration f. Let $\widetilde{F}$ be a fiber of $\widetilde{f}$. Since ν is birational, every irreducible component, say $\widetilde{F}_i$ of $\widetilde{F}$ is a rational curve, which surjects birationally onto the irreducible component F_i of $F := \nu(\widetilde{F})$. Since $F_i \cong \mathbb{A}^1$ by the assumption, it follows that $\widetilde{F}_i \cong F_i \cong \mathbb{A}^1$. Thus the fiber $\widetilde{F}$ is isomorphic to F. This implies that $\mathrm{Sing}(S) = \emptyset$, and S is smooth. Now take the normalization

$\widetilde{C}$ of C in the function field of S and consider the factorization

$$f : S \xrightarrow{\widehat{f}} \widetilde{C} \xrightarrow{g} C \quad \text{(Stein factorization)}.$$

Then the morphism $\widehat{f}$ is an $\mathbb{A}^1$-fibration. We may assume that this fibration is trivial, i.e., $S \cong \widetilde{C} \times \mathbb{A}^1$.

(2) Let $S = \operatorname{Spec} A$ and $C = \operatorname{Spec} B$. We show that there exists an element $b \in B$ such that $A[b^{-1}] = \widetilde{B}[b^{-1}][t]$, where $\widetilde{B}$ is the integral closure of B in $Q(A)$. In fact, by the construction in (1), there exists an element $\widetilde{b} \in \widetilde{B}$ such that $Z = \widehat{f}^{-1}(U)$, where $U = D(\widetilde{b})$ in $\widetilde{C}$. Let

$$\widetilde{b}^n + b_{n-1}\widetilde{b}^{n-1} + \cdots + b_1\widetilde{b} + b_0 = 0, \quad b_i \in B,\ b_0 \neq 0$$

be the monic equation for $\widetilde{b}$ over B. Since $g^{-1}(D(b_0)) \subseteq D(\widetilde{b})$, we may take $U = g^{-1}(D(b_0))$ in the assertion (1). Let $b = b_0$. Then $A[b^{-1}] = \widetilde{B}[b^{-1}][t]$. Let $\delta = b^N(\partial/\partial t)$, where we choose N a positive integer such that if $A = k[a_1, \ldots, a_r]$ then $\delta(a_i) \in A$ for all i. Then δ is an lnd of A such that $\widetilde{B} \subseteq \operatorname{Ker}\delta$. Consider the associated G_a-action on S and the quotient morphism $q : S \to \operatorname{Spec}\operatorname{Ker}\delta$. Then $g : S/G_a \to C$ is the morphism associated with the inclusion $B \hookrightarrow \widetilde{B} \hookrightarrow \operatorname{Ker}\delta$. □

1.2. **Hidden G_a-actions on multiple fiber components.** With the notations in subsection one, we consider the case where all points of the fiber F is fixed by the G_a-action. Since the G_a-action on $S = \operatorname{Spec} A$ is non-trivial, the following lemma shows that the G_a-action on the fiber F is hidden in the "thickning" part of F. To simplify the situation, we assume below that *the fiber F is irreducible* and that *the G_a-action on F_{red} is trivial.* We denote F_{red} by Z.

Lemma 1.2.1. *If F is reduced, then the G_a-action induces a nontrivial action on the mth infinitesimal neighborhood J_m of F for some $m > 0$, where $J_m := t^mA/t^{m+1}A$ which is isomorphic to A/tA as an A-module.*

Proof. Suppose that F is reduced. By Theorem 1.1.2, Z is isomorphic to $\mathbb{A}^1 = \operatorname{Spec} k[\overline{v}]$. Let Q be a closed point of Z. Choose an element v of A such that $\overline{v} = v \pmod{tA}$ and $v(Q) = 0$. Then S is smooth at Q and $\{t, v\}$ is a system of local parameters at Q. We show that $\delta(v) = bt^m$ for $m > 0$ and $b \in A$ such that $b \notin tA$ and $b \pmod{tA} \in k^*$. Since any element a is written as $a - f(v) \in tA$ for some $f(v) \in k[v]$, A is a subalgebra of the tA-adic completion $\widehat{A} = \varprojlim_n A/t^nA \cong k[v][[t]]$. Hence we can write

$$\delta(v) = t^m \left(\sum_{i \geq 0} f_i(v)t^i \right), \quad f_i(v) \in k[v],\ f_0(v) \neq 0.$$

Write

$$f_0(v) = c_0 v^n + c_1 v^{n-1} + \cdots + c_{n-1} v + c_n, \quad c_i \in k,\ c_0 \neq 0.$$

We may assume that $c_0 = 1$. Suppose $n > 0$. Then, by a straightforward computation, we have

$$\delta^\ell(v) = g_0(v) t^{\ell m} + g_1(v) t^{\ell m+1} + \cdots ,$$

where the highest degree v-term of $g_0(v)$ is

$$\left(\prod_{i=1}^{\ell-1} (in - (i-1)) \right) \cdot v^{\ell n - (\ell-1)},$$

which is nonzero for every $\ell > 0$. Since $\delta^\ell(v) = 0$ for some ℓ, we have $n = 0$. Thus the representation of $\delta(v)$ on $J_m = t^m A/t^{m+1}A \cong k[v]$ is given by $\delta(v) = \Delta(v) t^m \pmod{t^{m+1}A}$, where $\Delta(v) = 1$. By the construction, one can verify that Δ is an lnd on A/tA. So, Δ gives a nontrivial G_a-action on J_m. If $m = 0$, this Δ coincides with the lnd induced on A/tA by δ. Since $\Delta(v) = 1$, the action of G_a on A/tA is nontrivial. It contradicts the assumption that $F = Z$ is contained in the fixed point locus. So, $m > 0$. $\square$

Next, we consider the case F is non-reduced. Let $R = A/tA$ and let $\mathfrak{p} = \sqrt{0}$ in R. By the assumption, $\mathfrak{p} \neq (0)$, $\mathfrak{p}^{N-1} \neq 0$ and $\mathfrak{p}^N = (0)$. Since $\mathfrak{p}$ is a δ-ideal, i.e., $\delta(\mathfrak{p}) \subseteq \mathfrak{p}$, where we denote the induced lnd on R by the same letter δ. We assume that $\delta(\mathfrak{p}) \neq (0)$. Then there exists an integer d such that $\delta(\mathfrak{p}) \subseteq \mathfrak{p}^d$ and $\delta(\mathfrak{p}) \not\subseteq \mathfrak{p}^{d+1}$, where $d < N$. We call the integer d the *depth* of δ at F and denote $d = \mathrm{depth}_F(\delta)$.

We have the following result.

Lemma 1.2.2. *Assume that A is normal and F is irreducible and non-reduced. Let $R_0 = R/\mathfrak{p}$. Let $M = \mathfrak{p}^{d-1}/\mathfrak{p}^d$ and let M^{**} be the double dual of the R_0-module M. Then $M^{**} \cong R_0 e$ with a free generator e, M^{**} is a δ-module, i.e., $\delta(M^{**}) \subseteq M^{**}$ and the representation of δ on M^{**} is given by $\delta(a) = \Delta(a)e$ and Δ is a nonzero lnd on R_0.*

Proof. Let Q be a general smooth point of $Z \cong \mathbb{A}^1$. Then Q is a smooth point of S. Hence we can choose local parameters $\{u, v\}$ of S such that Z is defined by $u = 0$ and v is a fiber coordinate of Z. Then $t = u^N \xi$, where $\xi \in \mathcal{O}_{S,Q}^*$. Let $\mathfrak{p}_Q = \mathfrak{p} \otimes \mathcal{O}_{S,Q}$ and $R_Q = R \otimes \mathcal{O}_{S,Q}$. Then $\mathfrak{p}_Q = uR_Q$ and $\mathfrak{p}_Q^d = u^d R_Q$. We take the element u from A.

Here we remark that $\delta(R) \subseteq \mathfrak{p}^{d-1}$. In fact, for any element $a \in A$, we have $au \pmod{tA} \in \mathfrak{p}$. Since $\mathfrak{p}_Q = uR_Q$, we have $\delta(au) \pmod{tA} \in \mathfrak{p}^d$. Since $\mathfrak{p}_Q^d = u^d R_Q$, $s\delta(au) = u^d z$ in A for $s, z \in A$ and $s(Q) \neq 0$. We can write the last equation as $su\delta(a) = u^d z - sa\delta(u)$. Since $\delta(\mathfrak{p}_Q) \subseteq \mathfrak{p}_Q^d = u^d R_Q$, we have

$$ss'u\delta(a) = u^d(s'z - saw) \text{ in } A$$

for $s', w \in A$ with $s'(Q) \neq 0$. Since A is an integral domain, we have $ss'\delta(a) = u^{d-1}(s'z - saw)$ in A. This implies that $\delta(a) \in \mathfrak{p}_Q^{d-1} \cap R = \mathfrak{p}^{d-1}$. By the definition of d, we have an induced k-module homomorphism

$$\delta : R_0 := R/\mathfrak{p} \longrightarrow M := \mathfrak{p}^{d-1}/\mathfrak{p}^d, \quad a \mapsto \delta(a) \pmod{\mathfrak{p}^d}. \qquad (*)$$

Define

$$\widetilde{\delta} := \rho \cdot \delta : R_0 \xrightarrow{\delta} M \xrightarrow{\rho} M^{**},$$

where $\rho : M \to M^{**}$ is the canonical R_0-module homomorphism. Since M^{**} is a torsion-free module over $R_0 \cong k[z]$, M^{**} is a free R_0-module. Write $M^{**} = R_0 e$. For $a_1, a_2 \in R$, we have

$$\begin{aligned} \widetilde{\delta}(a_1 a_2) &= \rho \cdot \delta(a_1 a_2) = \rho(a_1\delta(a_2) + a_2\delta(a_1)) \\ &= a_1(\rho\cdot\delta)(a_2) + a_2(\rho\cdot\delta)(a_1) = a_1\widetilde{\delta}(a_2) + a_2\widetilde{\delta}(a_1). \end{aligned}$$

If we write $\overline{a}_i = a_i \pmod{\mathfrak{p}}$ $(i = 1, 2)$ and $\widetilde{\delta}(\overline{a}_i) = \Delta(\overline{a}_i)e$, we have

$$\Delta(\overline{a}_1\overline{a}_2) = \overline{a}_1\Delta(\overline{a}_2) + \overline{a}_2\Delta(\overline{a}_1).$$

It is easy to show that $\Delta : R_0 \to R_0$ is an lnd. Hence Δ gives a required representation on M^{**}. $\square$

If $\delta(R) = 0$ then $\delta(A) \subseteq tA$, and we replace δ by $t^{-1}\delta$. Hence we may assume that $\delta(R) \neq 0$. If $\delta(\mathfrak{p}) = 0$ then Lemma 1.2.2 implies that $\delta(R) \subseteq \mathfrak{p}^{N-1}$, and we can take M^{**} as the double dual of the R_0-module $\mathfrak{p}^{N-1}$ to obtain the hidden lnd on $R_0 = R/\mathfrak{p}$. If A is not normal, the argument of Lemma 1.2.2 does not work. The following example reflects well the above situation.

Example 1.2.3. Let S be an affine hypersurface $t^2 z - y^n = 0$ in $\mathbb{A}^3$, where $n \geq 2$. Define an lnd δ by $\delta(t) = 0, \delta(y) = t^2$ and $\delta(z) = ny^{n-1}$. Then S is non-normal with $\mathrm{Sing}(S) = \{t = 0\}$, $A = k[t, y, z]$ and $R = k[z][\varepsilon]$, where $\varepsilon = y \pmod{tA}$ and $\varepsilon^n = 0$. Then $\mathrm{depth}_F(\delta) = n$ and $M = k[z]\varepsilon^{n-1}$. Here we can argue as in the proof of Lemma 1.2.2 though S is not normal. Then the lnd Δ on $R_0 = k[z]$ is given by $\Delta(z) = n$.

1.3. G_a**-actions on affine threefolds.** Let X be a smooth affine threefold with a nontrivial G_a-action and let $q : X \to Y$ be the quotient morphism. Assume that the quotient surface $Y := X/G_a$ is smooth. Let $P \in Y$ and let $F = q^{-1}(P)$. We proved in [14] that every irreducible component of F of dimension one is isomorphic to $\mathbb{A}^1$ and disjoint from other irreducible components. Here we consider a two-dimensional component S and show that S contains a cylinderlike open set. We consider a linear pencil Λ of hyperplane sections of Y passing through the point P. Namely, if Y is embedded into an affine space $\mathbb{A}^N$ with a system of coodinates $\{x_1, \ldots, x_N\}$ so that P corresponds to $(0, \ldots, 0)$, then let $L_\lambda = Y \cap H_\lambda$, where H_λ is a hyperplane $a_1x_1 + \cdots + a_Nx_N = 0$ and the set $\{\lambda = (a_1, a_2, \ldots, a_N)\}$ corresponds to a line Λ in the dual projective space $\mathbb{P}^{N-1}$, which we take to be sufficiently general. Let T_λ be the closure in X of $q^{-1}(L_\lambda \setminus \{P\})$. *We assume the hypothesis* (H) *that* $T_\lambda \cap S \neq \emptyset$ *for general* $\lambda \in \Lambda$.

Theorem 1.3.1. *With the same notations and assumptions as above, the component* S *contains a cylinderrlike open set* $U \times \mathbb{A}^1$.

Proof. (1) Let $q_\lambda : T_\lambda \to L_\lambda$ be the restriction of q onto the affine surface T_λ. If λ is general in Λ, then T_λ is smooth by Bertini's second theorem and q_λ is an $\mathbb{A}^1$-fibration. Let $F_\lambda = S \cap T_\lambda$. Then F_λ is a disjoint union of the affine lines by Lemma 1.1.4.

(2) Let $\sigma : \widehat{Y} \to Y$ be the blowing-up of the point P, hence $E := \sigma^{-1}(P) \cong \mathbb{P}^1$. Consider the fiber product $X \times_Y \widehat{Y}$ and the base change $q_{\widehat{Y}} = q \times_Y \widehat{Y}$. Then $X \times_Y \widehat{Y}$ contains three-dimensional components whose underlying sets are isomorphic to $S_i \times E$, where the S_i are two-dimensional components of $F = q^{-1}(P)$. The open set $X \setminus q^{-1}(P)$ is contained in $X \times_Y \widehat{Y}$ as an open set. Let $\widehat{X}$ be the closure of this open set in $X \times_Y \widehat{Y}$ [3] and let $\widehat{q} : \widehat{X} \to \widehat{Y}$ be the restriction of $q_{\widehat{Y}}$ to $\widehat{X}$. By the definition, $\widehat{q}$ coincides with q on the open set $\widehat{X} \setminus \widehat{q}^{-1}(E)$. The closed set $\widehat{q}^{-1}(E)$ contains the *proper* transform $\widehat{S}$ which is biregular to S.

(3) Let $\widehat{L}_\lambda$ be the proper transform of L_λ on $\widehat{Y}$. Then $\widehat{L}_\lambda \cap \widehat{L}_\mu = \emptyset$ if $\lambda \neq \mu$. Let $\widehat{T}_\lambda$ be the proper transform of T_λ in $\widehat{X}$. Then $\widehat{T}_\lambda$ meets $\widehat{S}$ along a curve $\widehat{F}_\lambda$ because $T_\lambda \cap S \neq \emptyset$ by the hypothesis (H). Since $\widehat{S}$ is isomorphic to S, $\widehat{F}_\lambda$ is isomorphic F_λ, which is a disjoint union of the affine lines.

[3]Let M be the maximal ideal of $B := \Gamma(Y, \mathcal{O}_Y)$ corresponding to the point P. Then $\widehat{X}$ is the blow-up of X with repsect to MA, i.e., $\widehat{X} \cong \mathrm{Proj}_X(\oplus_{n \geq 0}(MA)^n)$. Let $\tau : \widehat{X} \to X$ be the canonical morphism. Then $\tau^{-1}(C_j) \cong C_j \times E$ for every one-dimensional component C_j of F. But $\tau^{-1}(S_i) \cong S_i$ for every two-dimensional component S_i of F. We note here that $S_i \cap C_j = \emptyset$ for all i and j.

(4) By Theorem 1.1.6 and its proof in the step (1) applied to the morphism $\widehat{q}|_{\widehat{S}} : \widehat{S} \to E$, we conclude that $\widehat{S}$, hence S, contains a cylinderlike open set. Note that in Theorem 1.1.6 the base curve C is assumed to be affine, whereas E is a complete curve. $\square$

The following is a simple example which illustrates our present situation, which is Example 3.14 in [14].

Example 1.3.2. Let X be a smooth hypersurface in $\mathbb{A}^4 = \mathrm{Spec}k[x, y, u, z]$ defined by $xu - y^2z = y$. Then X has a G_a-action defined by an lnd

$$\delta = x\frac{\partial}{\partial z} + y^2\frac{\partial}{\partial u}.$$

Then $\mathrm{Ker}\,\delta = k[x, y]$ and the quotient morphism $q : X \to Y \cong \mathbb{A}^2$ is the projection $(x, y, z, u) \mapsto (x, y)$. Hence $F := q^{-1}(0, 0) = \mathbb{A}^2 = \mathrm{Spec}\,k[x, y]$ and $q^{-1}(\alpha, \beta) \cong \mathbb{A}^1$ if $(\alpha, \beta) \neq (0, 0)$. With the above notations, let $L_\lambda = \{y = \lambda x\}$. Then $T_\lambda = q^{-1}(L_\lambda)$ meets F along the line $u = \lambda$. $\square$

Remark 1.3.3. The hypothesis (H) holds if the fiber $F := q^{-1}(P)$ consists of only one component S of dimension two. In fact, $T_\lambda \cap S \neq \emptyset$ clearly, and then $\dim(T_\lambda \cap S) = 1$.

1.4. **Singular fibers of $\mathbb{P}^1$-fibrations on smooth projective threefolds.** Let $f : X \to Y$ be a $\mathbb{P}^1$-fibration with smooth algebraic varieties X and Y of dimension n and $n - 1$, respectively. Namely, f is a projective morphism whose general fibers as well as the generic fiber $X_\eta := X \times_Y \mathrm{Spec}\,k(Y)$ are isomorphic to $\mathbb{P}^1$. In general, the general fibers being isomorphic to $\mathbb{P}^1$ implies that the generic fiber X_η is a form of $\mathbb{P}^1$, i.e., $X_\eta \otimes_{k(Y)} K$ is K-isomorphic to $\mathbb{P}^1$ for an algebraic extension K of $k(Y)$. The form X_η is trivial if $n = 2$ by Tsen's theorem. Meanwhile, if $n > 2$, this is not the case because the Brauer group of $k(Y)$ is not necessarily trivial.

Lemma 1.4.1. *Let $f : X \to Y$ be a $\mathbb{P}^1$-fibration with smooth algebraic varieties X and Y. Then the following assertions hold.*

(1) *Let S be the closure of the set of points $Q \in Y$ such that either the scheme-theoretic fiber $F_Q := X \times_Y \mathrm{Spec}\,k(Q)$ has an irreducible component of dimension > 1 or every irreducible component F_i of F_Q has multiplicity > 1, i.e., $\mathrm{length}\mathcal{O}_{F_Q,F_i} > 1$. Then $\mathrm{codim}\,_Y S > 1$.*
(2) *Let $n = 3$. Then every fiber F_Q is simply-connected.*

Proof. (1) Suppose that $\mathrm{codim}\,_Y S = 1$. Then there exists an irreducible subvariety Z of codimension 1 of Y such that, for a gneral point

Q of Z, either the fiber F_Q has an irreducible component of dimension > 1 or every irreducible component has multiplicity > 1. The first case is impossible. In fact, it then occurs that $\dim f^{-1}(Z) \geq \dim Z + 2 = n - 2 + 2 = n$ and $f^{-1}(Z) = X$. This is a contradiction. Consider the second case. Suppose that the fiber F_Q has only irreducible components of dimension one whose multiplicity is greater than one. We may assume that X and Y are projective. Let H be a hyperplane section of Y through the point Q. If H is general, the inverse image $X_H := f^{-1}(H)$ is smooth with the induced $\mathbb{P}^1$-fibration. In fact, by Bertini's second theorem, X_H has singularity along the base locus of the linear system $L := \{H \mid Q \in H\}$. Consider the blowing-up of X along the center $f^{-1}(Q)$. Since f is equi-dimensional locally over the point Q and since X and Y are smooth, f is a flat morphism locally over Q. Then the blowing-up of X along $f^{-1}(Q)$ is isomorphic to the fiber product $X \times_Y \widehat{Y}$ by the reasoning as in the footnote to the proof of Theorem 1.3.1, where $\widehat{Y} \to Y$ is the blowing-up of the point Q. Then the proper transforms of $f^{-1}(H)$ get separated from each other and have the same fiber $f^{-1}(Q)$ as the intersection with the exceptional divisor. Furthermore, we note that the blowing-up $\mathrm{B}\ell_{f^{-1}(Q)_{\mathrm{red}}}X$ of X with respect to the center $f^{-1}(Q)_{\mathrm{red}}$ which is defined by the Ideal $\sqrt{\mathcal{I}_{f^{-1}(Q)}}$ is smooth since the fiber $f_{\widehat{Y}}^{-1}(\widehat{Q})$ of $f_{\widehat{Y}} : X \times_Y \widehat{Y} \to \widehat{Y}$ is locally isomorphic to the fiber $f^{-1}(Q)$, where $\widehat{Q}$ is a point of the exceptional locus of $\widehat{Y} \to Y$. Since $\mathcal{I}_{f^{-1}(Q)_{\mathrm{red}}} = \sqrt{\mathcal{I}_{f^{-1}(Q)}}$, we have $\mathrm{B}\ell_{f^{-1}(Q)_{\mathrm{red}}}X = \mathrm{B}\ell_{f^{-1}(Q)}X$. Hence $\mathrm{B}\ell_{f^{-1}(Q)}X$ is smooth. By reducing dimension of Y by iterated hyperplane sections, we may assume that $\dim Y = 1$. Then $f^{-1}(H)$ is a smooth projective surface with a $\mathbb{P}^1$-fibration over a smooth projective curve H. Then it is well-known that the fiber $f^{-1}(Q)$ has a reduced component, which is a contradiction to the beginning hypothesis.

(2) By the assertion (1), the closed set S is a finite set. Let Q be a point of Y. We choose a small open disk D of Q in Y so that $D \cap S = \emptyset$ if $Q \notin S$ or $D \cap (S \setminus \{Q\}) = \emptyset$ if $Q \in S$. Then we may assume that the fiber F_Q is a strong deformation retract of $U := f^{-1}(D)$. Hence $\pi_1(U) = \pi_1(F_Q)$. On the other hand, we can apply Nori's result [14, Lemma 1.2.1] to $U \setminus F_Q \to D \setminus \{Q\}$ to obtain an exact sequence

$$\pi_1(\mathbb{P}^1) \to \pi_1(U \setminus F_Q) \to \pi_1(D \setminus \{Q\}) \to (1),$$

where $\pi_1(\mathbb{P}^1) = \pi_1(D \setminus \{Q\}) = (1)$. Hence $\pi_1(U \setminus F_Q) = (1)$. Since $U \setminus F_Q$ is an open set of U, we have a surjection $\pi_1(U \setminus F_Q) \to \pi_1(U)$. Hence we have $\pi_1(F_Q) = \pi_1(U) = (1)$. Hence F_Q is simply-connected.

□

If $n = 3$, write the fiber F_Q as F_0 for the sake of simplicity, which may contain irreducible components of dimension 2. Write $F_0 = S \cup C$, where S (resp. C) is the sum of irreducible components of dimension 2 (resp. 1).

Lemma 1.4.2. *With the above notations and assumptions, if $n = 3$, we have the assertions.*

(1) $H_1(S; \mathbb{Z}) = H_1(C; \mathbb{Z}) = 0$.
(2) *Each component of C is a rational curve, and each component of S is a rational surface or a rationally ruled surface.*

Proof. (1) As the Mayer-Vietoris exact sequence applied to $F_0 = S \cup C$, we have an exact sequence of integral homology groups

$$\cdots \longrightarrow H_2(F_0) \longrightarrow H_1(S \cap C) \longrightarrow H_1(S) \oplus H_1(C) \longrightarrow H_1(F_0) \longrightarrow \cdots,$$

where $H_1(F_0) = 0$ by the assertion (2) of Lemma 1.4.1 and $H_1(S \cap C) = 0$ because $S \cap C$ is a non-empty finite set. Hence $H_1(S) \oplus H_1(C) = 0$, whence $H_1(S) = H_1(C) = 0$.

(2) For the irreducible components of C, they have trivial first homology groups by the Mayer-Vietoris sequence. This implies that each irreducible component of C is at worst a cuspidal rational curve, i.e., a rational curve with only unibranch singularities. In order to show that each irreducible component of S is a rational surface or a rationally ruled surface, we apply Hironaka's flattening theorem [17]. We can thereby find a proper birational morphism $g : Y' \to Y$ with a smooth Y' such that the fiber product $X \times_Y Y'$ has an irreducible component X' with a flat $\mathbb{P}^1$-fibration $f' : X' \to Y'$. Let Q' be a point of Y', possibly in the exceptional locus of g as g is a composite of blowing-ups with centers at points. Let H' be a general hyperplane section of Y' passing through Q' and let V be the inverse image $f'^{-1}(H')$. Then $f'|_V : V \to H'$ is a $\mathbb{P}^1$-fibration. Let $\widetilde{V}$ be the normalization of V. Then $\widetilde{f} = (f'|_V) \cdot \nu : \widetilde{V} \to H'$ is also a $\mathbb{P}^1$-fibration, where $\nu : \widetilde{V} \to V$ is the normalization morphism. Then every fiber is a sum of smooth rational curves. In fact, the smoothness of each rational curve follows from [27, (2.8.6.3), p. 107]. This implies that every fiber of $f'|_V$ is a sum of rational curves. Let S_i be an irreducible component of S. Then there exists a closed subvariety S_i' of X' such that S_i is dominated by S_i'. Note that S_i' is a rationally ruled surface. If the image of the ruling on S_i has a base point, S_i is a rational surface. Otherwise, S_i is a rationally ruled surface. □

We occasionally have to consider an algebraic surface S with a $\mathbb{P}^1$-fibration like a subvariety of a smooth algebraic threefold $X$with a

$\mathbb{P}^1$-fibration which has the induced $\mathbb{P}^1$-fibration. In general, S has singularity worse than normal singularity. We consider first the case of a normal surface.

Lemma 1.4.3. *Let S be a normal projective surface with a $\mathbb{P}^1$-fibration $f : S \to C$, where C is a smooth projective curve. Then we have:*

1. *S has only rational singularities, whose resolution graph is a tree of smooth rational curves and is a part of a degenerate fiber of $\mathbb{P}^1$.*
2. *Every fiber F of f is a union of smooth rational curves, and its intersection dual graph is a tree in the sense that the dual graph of the inverse image of F in a minimal resolution of singularity of S is a tree.*
3. *$H_1(F; \mathbb{Z}) = 0$.*

Proof. (1) Let $\sigma : \widehat{S} \to S$ be a minimal resolution of singularities. Then the composite $f \cdot \sigma : \widehat{S} \to C$ is a $\mathbb{P}^1$-fibration. Since $f^{-1}(U) \cong \widehat{f}^{-1}(U)$ for an open set U of C, each connected component of the exceptional locus of σ is a part of a degenerate fiber of $\widehat{f}$. Then it is well-known that this part contracts to a rational singular point. In fact, by using the five-term exact sequence associated to a spectral sequence $E_2^{pq} = R^q f_* R^p \sigma_* \mathcal{O}_{\widehat{S}} \Rightarrow R^n \widehat{f}_* \mathcal{O}_{\widehat{S}}$, we obtain an exact sequence

$$0 \to f_* R^1 \sigma_* \mathcal{O}_{\widehat{S}} \to R^1 \widehat{f}_* \mathcal{O}_{\widehat{S}} \to R^1 f_* \sigma_* \mathcal{O}_{\widehat{S}},$$

where $R^1 \widehat{f}_* \mathcal{O}_{\widehat{S}} = 0$ by [27, (2.8.6.2),p.107]. Hence $f_* R^1 \sigma_* \mathcal{O}_{\widehat{S}} = 0$, whence $R^1 \sigma_* \mathcal{O}_{\widehat{S}} = 0$ because it is supported by a finite set of S.

(2) Let $\widehat{F} = \sigma^{-1}(F)$. Then F is the surjective image of $\widehat{F}$. Since $\widehat{F}$ is a union of rational curves, so is the image F. More precisely, $H^1(F_{\text{red}}, \mathcal{O}_{F_{\text{red}}}) = 0$ by [27, (2.8.6.3), p.107]. If $F_{\text{red}} = \cup_{i=1}^r C_i$ is the irreducible decomposition, it follows that $H^1(C_i, \mathcal{O}_{C_i}) = 0$ for every i. This in turn implies that $C_i \cong \mathbb{P}^1$ for every i.

(3) The fiber F is obtained topologically from $\widehat{F}$ by removing the sum of the exceptional loci of singular poins $Q_1, \ldots, Q_n$, which are quotient singularities and adding back the points $T = \{Q_1, \ldots, Q_n\}$. As long exact sequences of integral homology (cohomology) groups for the pairs $(\widehat{F}, \Gamma)$ and (F, T), we have

$$\begin{array}{ccccccccc} H_1(\Gamma) & \to & H_1(\widehat{F}) & \to & H^1(\widehat{F} \setminus \Gamma) & \to & H_0(\Gamma) & \to & H_0(\widehat{F}) \\ H_1(T) & \to & H_1(F) & \to & H^1(F \setminus T) & \to & H_0(T) & \to & H_0(F), \end{array}$$

where $H_1(\Gamma) = H_1(T) = 0$, $\widehat{F} \setminus \Gamma$ is homeomorphic to $F \setminus T$ and $H_0(\Gamma) \cong H_0(T) \cong \mathbb{Z}^{\oplus \#(T)}$. Hence $H_1(F; \mathbb{Z}) \cong H_1(\widehat{F}; \mathbb{Z}) = 0$. $\square$

Remark 1.4.4. Let $f : S \to C$ be a $\mathbb{P}^1$-fibration as in Lemma 1.4.3. If $\mathrm{Sing}(S) \neq \emptyset$, we may have either a multiple fiber or a fiber with more than two irreducible fiber components meeting in one point. We exhibit these phenomena by the following examples.

(1) Let L be a smooth fiber of a $\mathbb{P}^1$-fibration on a smooth surface which is isomorphic to $\mathbb{P}^1$. Blow up a point P_0 on L to obtain the exceptional curve E_1. Then blow up the intersection point P_1 of E_1 and the proper transform of L. Let E_2 be the exceptional curve. Then the inverse image of L is $L' + 2E_2 + E_1'$ with $(L')^2 = (E_1')^2 = -2$, where L' and E_1' are the proper transforms of L and E_1. Now contract L' and E_1' to rational double points. Let $\overline{E}_2$ be the image of E_2. Then $2\overline{E}_2$ is a fiber of the induced $\mathbb{P}^1$-fibration.

(2) In the above example, let P_2 be a point on E_2 which is different from the intersection points $L' \cap E_2$ and $E_1' \cap E_2$. Blow up P_2 and let E_3 be the exceptional curve. Then the obtained fiber is $L' + E_1' + 2E_2 + 2E_3$, where we denote the proper transforms of L', E_1', E_2 by the same letters. We have $(L')^2 = (E_1')^2 = (E_2)^2 = -2$. We contract the (-2)-curve E_2 to a rational double point. Then the images $\overline{L}', \overline{E}_1', \overline{E}_3$ of L', E_1', E_3 meet in the contracted singular point. According to [38], the intersection numbers are given as follows: $(\overline{E}_1')^2 = (\overline{L}')^2 = -\frac{3}{2}, (\overline{E}_3)^2 = -\frac{1}{2}, (\overline{E}_1' \cdot \overline{L}') = (\overline{E}_3 \cdot \overline{L}') = (\overline{E}_1' \cdot \overline{E}_3) = \frac{1}{2}$.

Lemma 1.4.5. *Let S be an algebraic surface, i.e., an integral k-scheme of finite type of dimension 2, and let $f : S \to C$ be a $\mathbb{P}^1$-fibration, i.e., f is a proper morphism whose general fibers are isomorphic to $\mathbb{P}^1$. Let F be a scheme-theoretic fiber of f over a closed point P. Then the following assertions hold.*

(1) *Let $\widetilde{C}$ be the normalization of C and $\widetilde{f} : S \times_C \widetilde{C} \to \widetilde{C}$ be the base change of f. Let $\widetilde{P}$ be a point of $\widetilde{C}$ lying over P and let $\widetilde{F}$ be the scheme-theoretic fiber of $\widetilde{f}$ over $\widetilde{P}$. Then $\widetilde{F}$ is isomorphic to F as k-schemes. Hereafter we assume that C is normal.*

(2) *The singular locus of S is contained in the union of finitely many fibers of f.*

(3) *F is a connected union of rational irreducible components.*

(4) *$\pi_1(F)$ is a cyclic group. If S is normal, F is simply-connected.*

Proof. (1) The proof for the assertion (1) of Lemma 1.1.5 applies.

(2) There exists an open set U of C such that $f^{-1}(U) \cong U \times \mathbb{P}^1$. Let $\mathrm{Sing}(S)$ be the singular locus of S which is a closed set. Suppose that $\mathrm{Sing}(S)$ contains an irreducible component T of dimension one. Then T cannot lie horizontally to the fibration f because $\mathrm{Sing}(S) \cap f^{-1}(U) = \emptyset$. Hence T lies vertically to f. Namely it is contained in a fiber.

(3) Let $\nu : \widehat{S} \to S$ be the normalization morphism. Then $\widehat{f} = f \cdot \nu : \widehat{S} \to C$ is a $\mathbb{P}^1$-fibration. Let $\widehat{F} := \widehat{f}^*(P)$ be the fiber of $\widehat{f}$ over P. Then $\widehat{F}$ is a union of irreducible components isomorphic to $\mathbb{P}^1$ by Lemma 1.4.3. Hence the fiber F is a connected union of rational curves.

(4) Let Δ be a small open disk (analytic neighborhood) of P and let $\Delta^* = \Delta \setminus \{P\}$. By the assertion (3), we can take Δ so small that $f^{-1}(\Delta^*) \cong \Delta^* \times \mathbb{P}^1$. Since F is a strong deformation retract of $f^{-1}(\Delta)$, we have a surjection $\pi_1(f^{-1}(\Delta^*)) \to \pi_1(f^{-1}(\Delta)) \to (1)$. Since $\pi_1(f^{-1}(\Delta^*)) \cong \pi_1(\Delta^*) \cong \mathbb{Z}$ and $\pi_1(f^{-1}(\Delta)) \cong \pi_1(F)$, it follows that $\pi_1(F)$ is a cyclic group. If S is normal, then $H_1(F) = 0$ by Lemma 1.4.3, (2). Hence F is simply connected. $\square$

Lemma 1.4.6. *Let $f : X \to C$ be a projective morphism which is a $\mathbb{P}^1$-fibration over a smooth algebraic curve C and let F_0 be a singular fiber of f. Then F_0 is simply connected. In particular, ecery irreducible component is homeomorphic to $\mathbb{P}^1$.*

Proof. We can replace C by its smooth complete model and assume that C is a smooth projective curve and X is a projective surface. Further, we may assume that the fiber F_0 is a unique singular fiber of f. By Lemma 1.4.5, F_0 is a connected union of rational irreducible components. Further, f has a cross-section T. For the existence of a cross-section, note that the generic fiber X_η of f is isomorphic to a smooth conic in $\mathbb{P}^2$ defined over the function field $k(C)$, where $k(C)$ is a C_1-field by Tsen's theorem. Hence X_η has a $k(C)$-rational point whose closure in X gives a cross-section of f. Since $k(C)$ is an infinite field, f has infinitely many cross-sections.

Suppose that F_0 is not simply-connected. We can assume, by blowing up points on F_0, that there is a loop in the dual graph of F_0. Let E be an irreducible component contained in a loop. We can assume that the section T meets some irreducible component of F_0 other than E. Consider the union D of T and all the components of F_0 except E. We may assume that D is connected. In fact, if F_0 contains an irreducible component E_1 which is not connected to the section T by a chain of irreducible components of F_0 unless it contains E, then we replace E by the exceptional curve arising by the blowing-up of the intersection point of E and an adjacent component of the loop. Now $E - D$ has at least two places at infinity.

Let $\widetilde{X}$ be the normalization of X and let $\widetilde{D}$, $\widetilde{E}$ be the inverse images of D, E in $\widetilde{X}$ respectively. Then $\widetilde{D}$ is connected. In fact, the section T is smooth because it dominates birationally the smooth curve C.

If F_1 is an irreducible component of D meeting T, every irreducible component of its inverse image $\widetilde{F_1}$ in $\widetilde{X}$ passes through the unique point of the inverse image $\widetilde{T}$ of T lying over $T \cap F_0$, whence $\widetilde{F_1}$ is connected. Since every irreducible component of the inverse image of $F_0 \cap D$ is connected to an irreducible component of $\widetilde{F_1}$, we know that $\widetilde{D}$ is connected. Note that $\widetilde{E}$ might be reducible.

Now consider the relative cohomology exact sequence of a pair $(\widetilde{X}, \widetilde{D})$ with integral coefficients,

$$\longrightarrow H^0(\widetilde{X}) \longrightarrow H^0(\widetilde{D}) \longrightarrow H^1(\widetilde{X}, \widetilde{D}) \longrightarrow H^1(\widetilde{X}) \longrightarrow H^1(\widetilde{D}) \longrightarrow .$$

Since $\widetilde{X}$ is normal, $\widetilde{X} \setminus \widetilde{F_0}$ is a $\mathbb{P}^1$-bundle and $\widetilde{F_0}$ is simply-connected by Lemma 1.4.5, we know that $H^1(\widetilde{X}) \cong H^1(C)$. Since $\widetilde{D}$ is contracted to C, we know that $H^1(\widetilde{D}) \cong H^1(C)$. Hence $H^1(\widetilde{X}) \to H^1(\widetilde{D})$ is an isomorphism. Also, the map $H^0(\widetilde{X}) \to H^0(\widetilde{D})$ is an isomorphism since both $\widetilde{X}$ and $\widetilde{D}$ are connected. The above exact sequence shows that $H^1(\widetilde{X}, \widetilde{D}) = 0$. Hence we have $H_3(\widetilde{X} - \widetilde{D}; \mathbb{Q}) = 0$ by Lefschetz duality.

We note that $\widetilde{X}$ has at worst rational singular points. Hence Poincare (and Lefschetz) duality with rational coefficients is valid as here and used below.

Consider the relative $\mathbb{Q}$-coefficient cohomology exact sequence with compact support for $(\widetilde{X} - \widetilde{D}, \widetilde{E} - \widetilde{D})$.

$$\begin{aligned} &\longrightarrow H_c^0(\widetilde{X} - \widetilde{D}) \longrightarrow H_c^0(\widetilde{E} - \widetilde{D}) \longrightarrow H_c^1(\widetilde{X} - \widetilde{D}, \widetilde{E} - \widetilde{D}) \\ &\longrightarrow H_c^1(\widetilde{X} - \widetilde{D}) \longrightarrow H_c^1(\widetilde{E} - \widetilde{D}) \longrightarrow H_c^2(\widetilde{X} - \widetilde{D}, \widetilde{E} - \widetilde{D}) \longrightarrow . \end{aligned}$$

By duality, $H_c^1(\widetilde{X} - \widetilde{D}, \widetilde{E} - \widetilde{D})$ is isomorphic to $H_3(\widetilde{X} - (\widetilde{D} \cup \widetilde{E})) = 0$, since $\widetilde{X} - (\widetilde{D} \cup \widetilde{E})$ is an $\mathbb{A}^1$-bundle over C_0 and hence $H_3(\widetilde{X} - (\widetilde{D} \cup \widetilde{E})) \cong H_3(C_0) = 0$. Similarly, $H_c^2(\widetilde{X} - \widetilde{D}, \widetilde{E} - \widetilde{D}) \cong H_2(C_0) = 0$ because C_0 is an affine curve. Further, by the duality, $H_c^1(\widetilde{X} - \widetilde{D}; \mathbb{Q}) \cong H_3(\widetilde{X} - \widetilde{D}; \mathbb{Q}) = 0$ as seen above. Hence $H_c^1(\widetilde{E} - \widetilde{D}; \mathbb{Q}) = 0$. This is a contradiction since $E - D$ has at least two places at infinity and $H_c^1(\widetilde{E} - \widetilde{D}; \mathbb{Q}) \cong H^1(\widetilde{E} - \widetilde{D}; \mathbb{Q})$. In fact, write $\widetilde{E} - \widetilde{D} = Z_1 \coprod \cdots \coprod Z_r$, where each Z_i is completed to a tree of smooth rational curves $\widetilde{Z}_i$. If none of Z_i has an irreducible component with two or more places missing, an invertible function on Z_i is a constant. Since $\Gamma(E - D, \mathcal{O}^*) \subseteq \Gamma(\widetilde{E} - \widetilde{D}, \mathcal{O}^*)$ and $\Gamma(E - D, \mathcal{O}^*)$ is not a finite sum of k, we have a contradiction. Hence some connected component Z_i has an irreducible component with at least two places missing. This irreducible component gives a non-zero element in $H^1(\widetilde{E} - \widetilde{D}; \mathbb{Q}) \cong H_1(\widetilde{E} - \widetilde{D}; \mathbb{Q})$. □

Theorem 1.4.7. *Let $f : X \to Y$ be an equi-dimensional $\mathbb{P}^1$-fibration over a smooth projective variety Y and let $F_0 := f^{-1}(P)$ be a closed fiber of f. Then F_0 is simply-connected.*

Proof. Suppose that $\dim Y \geq 2$. Let H be a hyperplane section of Y passing through the point P and let $X_H := f^{-1}(H)$. Then H is a smooth projective variety and $f_H := f|_{X_H} : X_H \to H$ is an equi-dimensional $\mathbb{P}^1$-fibration with the fiber F_0 over the point P. By taking hyperplane sections through P repeatedly, we are reduced to the case where $\dim Y = 1$. Then Lemma 1.4.6 gives the result. □

1.5. **Freeness Conjecture of G. Freudenburg.** In his talk at the international conference "Affine Geometry, hyperbolicity, complex spaces", October 4-28, 2016 in Grenoble, G. Freudenburg made the following:

Conjecture 1.5.1. *Let $A = k[x, y, z]$ be a polynomial ring in three variables with a nontrivial locally nilpotent derivation (lnd, for short) D and let $B = \mathrm{Ker}\, D$. Then A is a free B-module.*

We denote by $k^{[n]}$ a polynomial ring in n variables with n generating variables not specified. Let $A = k^{[n]}$ with a nontrivial lnd D and let $B = \mathrm{Ker}\, D$. We can ask a similar question for $k^{[n]}$ as in Conjecture 1.5.1. If $n \geq 4$, then A is not a free B-module. Even less, A is not B-flat. A counterexample is $A = k[x_1, x_2, x_3, x_4]$ and

$$D = x_1 \frac{\partial}{\partial x_2} + x_2 \frac{\partial}{\partial x_3} + (x_2^2 - 2x_1x_3 - 1)\frac{\partial}{\partial x_4}.$$

Then B is a hypersurface $\xi_1\xi_4 = \xi_3^2 - \xi_2(\xi_2 - 1)^2$. If one checks the singular fibers of the quotient morphism $q : X \to Y$ with $X = \mathrm{Spec}\, A$ and $Y = \mathrm{Spec}\, B$, there is a unique singular fiber $\mathbb{A}^2 + \mathbb{A}^2$ (see [14, Example 4.17]). Hence q is not equi-dimensional. This implies that A is not B-flat.

If $n = 2$, we can choose variables x, y of A so that $A = k[x, y]$, $B = k[x]$ and $D(y) = y + f(x)$ with $f(x) \in A$. Hence A is a free B-module with a free basis $\{y^i \mid i \geq 0\}$. Hence the case $n = 3$ is the only case remained open. Even in this case, it is known that $Y \cong \mathbb{A}^2$ and $q : X \to Y$ is equi-dimensional. Furthermore, every singular fiber is a disjoint union of the curves isomorphic to $\mathbb{A}^1$ by [14]. Hence we know that A is B-flat and that A is the inductive limit of a directed set of finitely generated free B-submodules by [30]. We can replace $A = k[x, y, z]$ in Conjecture 1.5.1 by an affine domain of dimension $n = 2$ or 3. We first consider the case $n = 2$.

Theorem 1.5.2. *Let X be an irreducible affine surface defined over k with an $\mathbb{A}^1$-fibration $f : X \to C$ onto a smooth affine curve C. Let A and B be the coordinate rings of X and C, respectively. Then A is a free B-module.*

Proof. We reduce to the case when X is normal as follows. Let $\widetilde{A}$ be the normalization of A in its quotient field. Then $\widetilde{A}$ is the coordinate ring of the normalization $\widetilde{X}$ of X in its function field. Suppose that $\widetilde{A}$ is a free B-module. Now A is an B-submodule of $\widetilde{A}$. For any localization $B_{\mathfrak{m}}$ of B with respect to a maximal ideal $\mathfrak{m}$ of B, the localization $A_{\mathfrak{m}} := A \otimes_B B_{\mathfrak{m}}$ is an $B_{\mathfrak{m}}$-submodule of $\widetilde{A}_{\mathfrak{m}}$. Since $B_{\mathfrak{m}}$ is a PID, $A_{\mathfrak{m}}$ is a free $B_{\mathfrak{m}}$-module because a submodule of a free module over a PID is free. This proves that A is a projective B-module of infinite rank. By a result of Bass [1, Theorem 4.3], A is B-free. Now we can assume that X, and hence A, is normal.

If f is an $\mathbb{A}^1$-bundle then A can be seen to be a projective B-module because A is locally a polynomial ring in one variable over B, hence also a free B-module by Bass' result above. We will reduce to the $\mathbb{A}^1$-bundle case by the usual ramified covering trick. There exists a finite covering $C' \to C$ of C with smooth C' such that

(1) C' is ramified with a suitable ramification index over every point P of C with $f^{-1}(P)$ containing no reduced irreducible components,
(2) the normalization of the fiber product $X \times_C C'$, say X', has a surjective $\mathbb{A}^1$-fibration $f' : X' \to C'$ such that every fiber of f' has a reduced irreducible component.

By removing all the irreducible components from all the singular fibers except one reduced irreducible component, we obtain an open subset X'' of X' such that the induced $\mathbb{A}^1$-fibration $f'' : X'' \to C'$ has no singular fibers, i.e., no fibers which are not isomorphic to $\mathbb{A}^1$. Then X'' is smooth and is an $\mathbb{A}^1$-bundle. The coordinate ring of X'' is a free B'-module, where B' is the coordinate ring of C'. Note that B' is a projective B-module because B' is a finitely generated, torsion-free module over a Dedekind domain B. Since the coordinate ring A' of X' is a B'-submodule of the coordinate ring of X'', A' is a projective B'-module, hence a projective B-module. Then the result follows from Bass theorem above. $\square$

2. Equivariant Abhyankar-Sathaye Conjecture in dimension three

The Abhyankar-Sathaye conjecture in dimension three is affirmatively proved if the given polynomial f is invariant under a nontrivial action of the additive group G_a on Spec $k[x, y, z]$ and some technical conditions on f are satisfied.

2.1. **Arguments on singular and plinth loci.** Let $A = k[x, y, z]$ be a polynomial ring in three variables. Let f be a non-constant element of A. Then the Abhyankar-Sathaye conjecture in dimension three asserts that *if the affine hypersurface* $X_0 = \{f = 0\}$ *in* $X = \operatorname{Spec} A \cong \mathbb{A}^3$ *is isomorphic to the affine plane* $\mathbb{A}^2$ *then so is the hypersurface* $X_c = \{f = c\}$ *for every* $c \in k$. The conjecture implies that f is a coordinate, that is to say, $A = k[f, g, h]$ with certain elements $g, h \in A$, and *vice versa.*

There are several partial results proving the conjecture in the general case with n variables, e.g., by Sathaye[40] and Popov [37]. We also add one partial affirmative result by assuming that the additive group G_a acts non-trivially on X and X_0 is G_a-stable. Let δ be a locally nilpotent derivation (*lnd*, in short) on A which corresponds to the G_a-action. Let $B = \operatorname{Ker} \delta$. Then B is a polynomial ring in two variables by [35], and B is factoraily closed in A. For the terminology, see [10]. Let $Y = \operatorname{Spec} B$. Then the inclusion $B \hookrightarrow A$ defines the quotient morphism $q : X \to Y$, which is equi-dimensional and surjective by [3]. Assume that $f \in B$.

Since $q : X \to Y$ is an $\mathbb{A}^1$-fibration, we define the *singular locus* of q by $\operatorname{Sing}(q) = \{Q \in Y \mid q^{-1}(Q) \not\cong \mathbb{A}^1\}$, where $q^{-1}(Q)$ is the scheme-theoretic fiber over a point $Q \in Y$. Then, by [11, Lemma 3.1], $\operatorname{Sing}(q)$ is a closed set of Y. By Dutta [6], $\operatorname{Sing}(q)$ has pure dimension one.

We define the *plinth ideal* of δ by $\operatorname{pl}(\delta) = B \cap \delta(A)$ which is an ideal of B, and the *plinth locus* of q by $\operatorname{pl}(q) = V(\operatorname{pl}(\delta))$. Then we have the following result which we state in a slightly more general situation.

Lemma 2.1.1. *Let A be a regular factorial affine domain of dimension 3 and let δ be a nonzero lnd on A. With the same notations as above,* $\operatorname{Sing}(q)$ *is the codimension one part of* $\operatorname{pl}(q)$.

Proof. (1) Let S be an irreducible component of $\operatorname{Sing}(q)$. Then it is defined by a prime element p of B. Suppose that $S \not\subset \operatorname{pl}(q)$. Then there exists a maximal ideal $\mathfrak{m}$ of B such that $p \in \mathfrak{m}$ but $\operatorname{pl}(\delta) \not\subset \mathfrak{m}$. Then there exist elements $b \in B \setminus \{0\}$ and $u \in A$ such that $b = \delta(u) \notin \mathfrak{m}$ and hence $A[b^{-1}] = B[b^{-1}][u]$. Since $b^{-1} \in B_{\mathfrak{m}}$, it follows that $A_{\mathfrak{m}} := A \otimes_B B_{\mathfrak{m}} = B_{\mathfrak{m}}[u]$. Namely, the point Q defined by $\mathfrak{m}$ is not a point of

$\operatorname{Sing}(q)$. This is a contradiction because $Q \in V(pB) \subset \operatorname{Sing}(q)$. Hence $\operatorname{Sing}(q) \subseteq \operatorname{pl}(q)$.

(2) Take an irreducible component $V(pB)$ of $\operatorname{pl}(q)$, where p is a prime element of B. Suppose that $V(pB) \not\subset \operatorname{Sing}(q)$. Take a closed point Q of $V(pB)$ such that $Q \not\in \operatorname{Sing}(q)$. Let $\mathfrak{m}$ be the maximal ideal of B corresponding to Q. Then $A_{\mathfrak{m}} = B_{\mathfrak{m}}[u]$ since the quotient morphism $q : X \to Y$ is an $\mathbb{A}^1$-bundle in a small open neighborhood of Q. We can take u to be an element of A such that $\delta(u) = b \in B \setminus \mathfrak{m}$. Hence $b \in \operatorname{pl}(\delta)$, and $\operatorname{pl}(\delta) \not\subset \mathfrak{m}$. This is a contradiction to the choice of $\mathfrak{m}$. Hence every irreducible component of $\operatorname{pl}(q)$ is contained in $\operatorname{Sing}(q)$. $\square$

Let $f \in B$ be an element such that the hypersurface Y_0 in $X = \mathbb{A}^3$ defined by the ideal fA is isomorphic to $\mathbb{A}^2$. Let $\varphi : X \to \mathbb{A}^1 = \operatorname{Spec} k[f]$ be the morphism defined by $P \mapsto f(P)$. The hypersurface X_c is the scheme-theoretic fiber $\operatorname{Spec} A/(f-c)A$ of the morphism φ over the point of $\mathbb{A}^1$ defined by $f = c$. The morphism φ is decomposed as

$$\varphi : X \xrightarrow{q} Y \xrightarrow{p} \mathbb{A}^1,$$

where p is induced by the inclusion $k[f] \hookrightarrow B$. The G_a-equivariant Abhyankar-Sathaye conjecture will be proved if the following two assertions hold true.

(1) The curve $Y_c = \operatorname{Spec} B/(f-c)B$ in Y is the affine line in the affine plane Y.
(2) The restriction of the quotient morphism $q|_{X_c} : X_c \to Y_c$ is an $\mathbb{A}^1$-bundle.

In fact, if these two assertions are proved, then X_c is an $\mathbb{A}^1$-bundle over $\mathbb{A}^1$, which is trivial. Thus, $X_c \cong \mathbb{A}^2$ for every $c \in k$. Then X is an $\mathbb{A}^2$-bundle over $\mathbb{A}^1$ by [39], where the local triviality in the sense of Zariski topology follows from [13, Theorem 3.10]. Then this $\mathbb{A}^2$-bundle is trivial by [2], i.e., $X \cong \mathbb{A}^2 \times \mathbb{A}^1$. This implies that f is a coordinate of A. We call (1) and (2) the rquirements.

We consider the fibration $p : Y \to \mathbb{A}^1$. We may assume that the derivation is irreducible (cf. [10]). Then the induced G_a-action on X_c is non-trivial. Hence $q|_{X_c}$ is decomposed as

$$q|_{X_c} : X_c \xrightarrow{q_c} X_c/\!/G_a \xrightarrow{p_c} Y_c,$$

where q_c is the quotient morphism and $X_c/\!/G_a$ is the algebraic quotient.

Lemma 2.1.2. *The following assertions hold.*

(1) *For every $c \in k$, the element $f - c$ is irreducible in B.*
(2) *The curve Y_c is isomorphic to $\mathbb{A}^1$ for every $c \in k$.*

Proof. (1) Consider $q|_{X_0} : X_0 \to Y_0$. Since $X_0 \cong \mathbb{A}^2$ by the assumption, it follows that $X_0//G_a \cong \mathbb{A}^1$. Then the morphism $p_0 : X_0//G_a \to Y_0$ is a finite morphism (Stein factorization of $q|_{X_0}$). Hence Y_0 is an irreducible curve in $Y \cong \mathbb{A}^2$ with only one place at infinity. Then by the irreducibility theorem [32, p.89], the curve $f - c = 0$ is an irreducible curve with only one place at infinity. Hence $f - c$ is an irreducible element of B.

(2) We show that Y_0 is a smooth curve. By the initial assumption that $X_0 \cong \mathbb{A}^2$, X_0 is a rational affine curve with one place at infinity. Hence if it is smooth, then $Y_0 \cong \mathbb{A}^1$. By Abhyankar-Moh-Suzuki theorem, it follows that $Y_c \cong \mathbb{A}^1$ for all $c \in k$. Now note that $B = k[v, w]$ a polynomial ring over k. Then we have

$$\begin{aligned}
\frac{\partial f}{\partial x} &= \frac{\partial f}{\partial v}\frac{\partial v}{\partial x} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial x}\\
\frac{\partial f}{\partial y} &= \frac{\partial f}{\partial v}\frac{\partial v}{\partial y} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial y}\\
\frac{\partial f}{\partial z} &= \frac{\partial f}{\partial v}\frac{\partial v}{\partial z} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial z}
\end{aligned}$$

If Q is a singular point of Y_0 then $\frac{\partial f}{\partial v}(Q) = \frac{\partial f}{\partial w}(Q) = f(v(Q), w(Q)) = f(P) = 0$, where P is a point of X such that $Q = q(P)$. Hence P is a singular point of X_0 which contradicts the smoothness of X_0. Hence Y_0 is smooth. □

Lemma 2.1.3. *The morphism $p_c : X_c//G_a \to Y_c$ for $c \in k$ is an isomorphism if the ideal $(f - c)B$ does not contain* $\mathrm{pl}(\delta)$*.*

By Lemma 2.1.1, the assumption that $(f - c)B \not\supset \mathrm{pl}(\delta)$ is equivalent to the condition that Y_c is not an irreducible component of $\mathrm{pl}(q)$, hence of $\mathrm{Sing}(q)$. This implies that $\mathrm{Sing}(q)$ lies horizontally along the fibration $p : Y \to \mathbb{A}^1$ if $\mathrm{Sing}(q) \neq \emptyset$.

Proof. To simplify the notations, we consider the case $c = 0$. The proof is the same in the general case.

The morphism $q|_{X_0} : X_0 \to Y_0$ is induced by the natural inclusion $B/fB \hookrightarrow A/fA$. Let δ_0 be the lnd on A/fA induced by δ. Then we have an inclusion $B/fB \hookrightarrow \mathrm{Ker}\,\delta_0$, which gives the morphism p_0 in the above decomposition of $q|_{X_0}$.

In order to show that p_0 is an isomorphism, it suffices to show that if $\delta(a) \in fA$ then there exists an element $b \in B$ such that $a - b \in fA$.

Suppose first that δ has a slice u, i.e., $\delta(u) = 1$. Then $A = B[u]$, and we can write

$$a = b_0 + b_1 u + \cdots + b_n u^n, \quad b_0, \ldots, b_n \in B.$$

Since $\delta(a) = b_1 + 2b_2u + \cdots + nb_nu^{n-1}$ is divisible by f in A, the coefficients $b_1, \ldots, b_n$ are divisible by f in B. Write

$$\delta(a) = f\left(b_1' + 2b_2'u + \cdots + nb_n'u^{n-1}\right).$$

Set

$$a' = b_1'u + b_2'u^2 + b_3'u^3 + \cdots + b_n'u^n.$$

Then $\delta(a) = f\delta(a')$, and hence $a - fa' \in B$. Set $b = a - fa'$. Then $a - b = fa' \in fA$.

Suppose next that u is a local slice. Namely $\alpha = \delta(u)$ is a nonzero element of B. Then δ extends to an lnd δ' of $A[\alpha^{-1}]$ and $B[\alpha^{-1}] = \operatorname{Ker}\delta'$. Since $\delta'(u/\alpha) = 1$, by the previous result, we have $a - (b/\alpha^r) \in fA[\alpha^{-1}]$. Hence, changing b by a multiple of b by an element of the form α^s, we have $\alpha^r a - b \in fA$.

Let $\mathfrak{p}$ be a prime ideal of B such that $f \in \mathfrak{p}$ and $\mathfrak{p} \notin V(\mathrm{pl}(\delta))$, i.e., $\mathfrak{p} \in Y_0 \setminus V(\mathrm{pl}(\delta))$. Such a prime ideal $\mathfrak{p}$ exists by the assumption. Let $\delta_{\mathfrak{p}}$ be the extension of δ to $A_{\mathfrak{p}} = A \otimes_B B_{\mathfrak{p}}$. Then we can find an element $\alpha \in \mathrm{pl}(\delta)$ such that $\alpha \notin \mathfrak{p}$. Let $(\delta_{\mathfrak{p}})_0$ be the restriction of $\delta_{\mathfrak{p}}$ onto $A_{\mathfrak{p}}/fA_{\mathfrak{p}}$. Then we have Ker $(\delta_{\mathfrak{p}})_0 = B_{\mathfrak{p}}/fB_{\mathfrak{p}}$ because u/α is a slice of $\delta_{\mathfrak{p}}$ if $\alpha = \delta(u)$. This implies that p_c is birational.

Since $Y_c \cong \mathbb{A}^1$ by Lemma 2.1.2, the birational morphism p_c is an isomorphism. $\square$

We call (H1) the condition that $(f - c)B \not\supset \mathrm{pl}(\delta)$ for every $c \in k$.

Lemma 2.1.4. *Assume the condition* (H1) *and that* $Y_0 \cap \mathrm{Sing}(q) = \emptyset$. *Then f is a variable of A, i.e., $A = k[f, g, h]$ for some $g, h \in A$.*

Proof. Since $Y_0 \cap \mathrm{Sing}(q) = \emptyset$ by the assumption, $\mathrm{Sing}(q)$ is either the empty set or a finite disjoint union $\coprod_i Y_{c_i}$ with $c_i \in k$. Meanwhile, the condition (H1) implies that the last case does not occur. Hence $\mathrm{Sing}(q) = \emptyset$. Hence the requirements (1) and (2) are fulfilled. $\square$

Lemma 2.1.5. *Assume that $Y_c \cap \mathrm{Sing}(q) = \emptyset$ for some $c \in k$. Then f is a varaible of A.*

Proof. Since $\mathrm{Sing}(q) \cap Y_c = \emptyset$, $q|_{X_c} : X_c \to Y_c$ is an $\mathbb{A}^1$-fibration which has no singular fibers. It implies, in particular, that $p_c : X_c//G_a \to Y_c$ is an isomorphism and $q_c : X_c \to Y_c$ is an $\mathbb{A}^1$-bundle. Furthermore, since $Y_c \cong \mathbb{A}^1$ as a parallel line by Lemma 2.1.2, it follows that $X_c \cong \mathbb{A}^2$, and $\mathrm{Sing}(q)$ is a disjoint sum of Y_{c_i} for $c_1, \ldots, c_r$ unless $\mathrm{Sing}(q) = \emptyset$. This implies that almost all fibers of $\varphi : X \to \mathbb{A}^1 = \operatorname{Spec} k[f]$ are isomorphic to $\mathbb{A}^2$. Then f is a variable by Kaliman [18]. $\square$

2.2. **Statement of Theorem.**

Theorem 2.2.1. *With the notations of section 2.1, the following assertions hold.*

(1) *Assume that the condition* (H1) *holds. Then the singular fibers of the quotient morphism* $q : X \to Y$ *are disjoint union of the affine lines, and* $\mathrm{Sing}(q)$ *lies horizontally along the* $\mathbb{A}^1$*-bundle* $p : Y \to \mathbb{A}^1 = \mathrm{Spec}\, k[f]$.
(2) f *is a variable of* A *if* $\mathrm{Sing}(q) \cap Y_c = \emptyset$ *for some* $c \in k$.

Proof. The first assertion follows from Lemma 2.1.3, and the second assertion from Lemma 2.1.5. □

3. Forms of $\mathbb{A}^3$ with unipotent group actions

We show that a form of $\mathbb{A}^3$ is trivial if it has either a fixed-point free G_a-action or an effective action of a unipotent algebraic group of dimension two.

3.1. **Preliminary results.** By the abuse of notations in the present section, we let k be a field of characteristic zero and let $\overline{k}$ be an algebraic closure of k. A geometrically integral algebraic scheme over k is simply called a *k-variety.* A k-variety X is smooth if $Y \otimes_k \overline{k}$ is smooth. An algebraic k-variety X is called a *k-from* of $\mathbb{A}^n$ (or simply, a form of $\mathbb{A}^n$) if $X \otimes_k \overline{k}$ is isomorphic to $\mathbb{A}^n$ over $\overline{k}$. A k-form X of $\mathbb{A}^n$ is *trivial* if Y itself is isomorphic to $\mathbb{A}^n$ over k. It is well-known that the Galois cohomology $H^1_{et}(\overline{k}/k, \mathrm{Aut}_{\mathbb{A}^n/k})$ vanishes for $n = 1, 2$ by the structure theorem of the automorphism group $\mathrm{Aut}_{\mathbb{A}^n/k}$ and the vanishing implies that every k-form of $\mathbb{A}^n$ is trivial if $n = 1, 2$ (see [24] for the case $n = 2$).

Koras and Russell [28] proved that a k-form of $\mathbb{A}^3$ with a nontrivial G_m-action is trivial. We are motivated by this result to ask if a k-form of $\mathbb{A}^3$ (or $\mathbb{A}^n$ with $n \geq 3$) is trivial provided it has a nontrivial unipotent group action. We shall show that this is the case if a k-form of $\mathbb{A}^3$ has either a fixed-point free G_a-action or an effective action of a unipotent group of dimension 2. These are essentially due to Kaliman [19] and Daigle-Kaliman [5]. The third author was informed by Neena Gupta of her recent work [7] which asserts that a (separable) k-form $X = \mathrm{Spec}\, A$ of $\mathbb{A}^3$ is trivial provided A is endowed with a locally nilpotent k-derivation D such that $\mathrm{rank}(D \otimes 1_{\overline{k}}) \leq 2$.

Lemma 3.1.1. *Let X be a smooth k-variety such that $\overline{X} := X \otimes_k \overline{k}$ is affine and factorial and that $\Gamma(\overline{X}, \mathcal{O}^*_{\overline{X}}) = \overline{k}^*$. Then X is an affine factorial k-variety with $\Gamma(X, \mathcal{O}^*_X) = k^*$.*

Proof. Let $\mathcal{F}$ be a quasi-coherent $\mathcal{O}_X$-sheaf. Then $\mathcal{F} \otimes_k \overline{k}$ is a quasi-coherent $\mathcal{O}_{\overline{X}}$-sheaf. Since $\overline{X}$ is affine, $H^i(\overline{X}, \mathcal{F} \otimes \overline{k}) \cong H^i(X, \mathcal{F}) \otimes_k \overline{k} = 0$ for all $i > 0$. Hence $H^i(X, \mathcal{F}) = 0$ for all $i > 0$, and X is affine by Serre's criterion of affineness. Write $A = \Gamma(X, \mathcal{O}_X)$. Then $\overline{A} := A \otimes_k \overline{k} = \Gamma(\overline{X}, \mathcal{O}_{\overline{X}})$. It is clear that $k^* \subseteq \Gamma(X, \mathcal{O}_X^*) \subseteq \Gamma(\overline{X}, \mathcal{O}_{\overline{X}}^*) = \overline{k}^*$. Since $\Gamma(X, \mathcal{O}_X)^*$ is invariant under the Galois group of $\overline{k}/k$, we obtain $\Gamma(X, \mathcal{O}_X^*) = k^*$.

Let D be a k-irreducible subvariety of codimension one in X and let $\overline{D} = D \otimes_k \overline{k}$. Let $\overline{D} = D_1 + \cdots + D_r$ be the irreducible decomposition of $\overline{D}$. If one fixes one irreducible component, say D_1, the other irreducible components are translates of D_1 under the Galois group of $\overline{k}/k$. Since $\overline{X}$ is factorial, D_1 is defined by an element f_1 of $\overline{A}$. Then the translates D_i of D_1 are defined by the translates f_i of f_1. Let $F = \prod_{i=1}^r f_i$. Then $g(F) = a(g)F$ for an element g of the Galois group $\mathfrak{G}$ [4]. Then $\{a(g)\}$ determines a cocycle of $\mathfrak{G}$ with values in $\overline{k}^*$. By Hilbert Theorem 90 (the multiplicative case), we find an element $b \in \overline{k}^*$ such that $a(g) = g(b)b^{-1}$. Then Fb^{-1} is invariant under $\mathfrak{G}$ and D is defined by $Fb^{-1} \in \Gamma(X, \mathcal{O}_X)$. □

An algebraic group G defined over k is called a*unipotent* group if so is $\overline{G} := G \otimes_k \overline{k}$.

Lemma 3.1.2. *Let G be a unipotent algebraic group defined over k. Then the following assertions hold.*

(1) *If* $\dim G = 1$ *then G is k-isomorphic to the additive group scheme G_a. Hence a G-action on an affine k-scheme $X =$* Spec A *is given by a locally nilpotent k-derivation (k-lnd, for short) δ on A.*
(2) *If* $\dim G = 2$ *then G is k-isomorphic to a direct product $G_a \times G_a$. Hence a G-action on an affine k-scheme X as above is given by two k-lnds δ_1, δ_2 on A such that $\delta_1\delta_2 = \delta_2\delta_1$.*

Proof. (1) Since $\dim G = 1$, it follows that $\overline{G} \cong G_a$ over $\overline{k}$. Hence the underlying scheme of G is a k-form of $\mathbb{A}^1$, which is trivial. Thus we can write $G = \operatorname{Spec} k[x]$. Let Δ be the comultiplication of G. Then $\Delta(x) = x \otimes 1 + 1 \otimes x$ over $\overline{k}$ since this gives a unique group structure on $\overline{G} \cong G_{a,\overline{k}}$. Hence we have $\Delta(x) = x \otimes 1 + 1 \otimes x$ over k and $G \cong G_{a,k}$.

[4]More precisely, we take a finite Galois extension k'/k such that $D \otimes_k k'$ splits into a sum of geometrically irreducible components and consider the Galois group $\mathrm{Gal}(k'/k)$.

(2) Since $\dim G = 2$, it is known that $\overline{G}$ is $\overline{k}$-isomorphic to a direct product $G_a \times G_a$.[5] Hence G is commutative and the underlying scheme of G is k-isomorphic to $\mathbb{A}^2$. Let $L(G)$ be the Lie algebra of G. Then $L(G)$ is a k-vector space of rank 2 with bracket product. Hence there exist k-derivations Δ_1, Δ_2 of $R := \Gamma(G, \mathcal{O}_G)$ such that $L(G) = k\Delta_1 + k\Delta_2$. Since $L(\overline{G}) = L(G) \otimes_k \overline{k} = \overline{k}\partial_1 + \overline{k}\partial_2$ with $\overline{k}$-lnds ∂_1, ∂_2 of $\overline{R} = R \otimes_k \overline{k}$, then Δ_1, Δ_2 as elements of $L(\overline{G})$ are expressed as $\overline{k}$-linear combinations of ∂_1, ∂_2. Since $\partial_1\partial_2 = \partial_2\partial_1$, it is straightforward to verify that Δ_1, Δ_2 are k-lnds such that $\Delta_1\Delta_2 = \Delta_2\Delta_1$. This implies that G is k-isomorphic to $G_a \times G_a$. A G-action on X corresponds to an algebraic homomorphism $\rho : G \to \mathrm{Aut}_{X/k}$ whose associated homomorphism of Lie algebras is $L(\rho) : L(G) \to \mathrm{Der}_k(A)$. Then the images of Δ_1, Δ_2 by $L(\rho)$ are the lnds δ_1, δ_2 of A such that $\delta_1\delta_2 = \delta_2\delta_1$. □

Let G be a unipotent group of dimension 2 defined over k. A direct product decomposition $G = G_a \times G_a$ corresponds to the writing $L(G) = k\Delta_1 + k\Delta_2$ with the notations in the proof of Lemma 3.1.2, where Δ_1, Δ_2 are the k-lnds of $R = \Gamma(G, \mathcal{O}_G)$ with $\Delta_1\Delta_2 = \Delta_2\Delta_1$. In other words, $\Delta_1 = \partial/\partial x_1$ and $\Delta_2 = \partial/\partial x_2$, where x_1, x_2 are the coordinates of the underlying schemes of the direct factors G_a with $\Delta(x_i) = x_i \otimes 1 + 1 \otimes x_i$ $(i = 1, 2)$, where Δ is the comultiplication. Another splitting of G as a direct product $G = G_a \times G_a$ corresponds to the lnds Δ'_1, Δ'_2 with similar properties. Since $L(G) = k\Delta'_1 + k\Delta'_2$, we have $\Delta'_i = \alpha_{i1}\Delta_1 + \alpha_{i2}\Delta_2$ for $i = 1, 2$, we have

$$\begin{pmatrix} \alpha_{11} & \alpha_{12} \\ \alpha_{21} & \alpha_{22} \end{pmatrix} \in \mathrm{GL}(2, k).$$

Fix a direct product decomposition $G = G_a \times G_2$ corresponding to Δ_1, Δ_2. We denote the factor G_a corresponding to Δ_i by G_i for $i = 1, 2$.

Suppose, in general, that the additive group G_a acts on a factorial k-variety $X = \mathrm{Spec}\, A$ via a k-lnd δ. The fixed-point locus X^{G_a} is defined by the ideal $I := \sum_{a \in A} \delta(a)A$. Suppose that X^{G_a} contains an irreducible component F of codimension one. Then F is defined by $f \in A$ such that $\delta(a)$ is divisible by f for all $a \in A$. If we write $\delta = f\delta'$, then δ' is a k-lnd of A. By repeating this operation which changes the G_a-action only on the components of X^{G_a}, we may assume that X^{G_a}

[5] By V. Popov, this is not the case in positive characteristic p since $G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & a^p \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b \in k \right\}$ is a two-dimensional non-commutative unipotent group.

has no irreducible components of codimension one. Then we have the following result.

Lemma 3.1.3. *With the same notations and assumptions, we assume that X is a smooth, $\overline{k}$-factorial*[6]*, k-variety of dimension 3. Let $q : X \to Y$ be the quotient morphism, where $Y = \operatorname{Spec} B$ with $B = \operatorname{Ker} \delta$. Let C be an irreducible curve contained in X^{G_a}. Then C is a fiber component of q which is isomorphic to $\mathbb{A}^1$. Furthermore, X^{G_a} is a disjoint union of the affine lines.*

Proof. We may assume that $k = \overline{k}$. Since X is factorial, there are no fiber components of dimension 2 in the morphism q. Suppose that C is transversal to q, i.e., $q(C)$ is a curve. Let Z be an irreducible component of $q^{-1}(q(C))$ which contains the curve C. Since X^{G_a} contains no components of codimension one, Z is an algebraic surface which is G_a-stable but not contained in the fixed-point locus. The restriction of q to Z, $q_Z : Z \to q(C)$, is factored by an $\mathbb{A}^1$-fibration $\rho : Z \to C'$, which is the quotient morphism of the induced G_a-action on Z. Let F be a general fiber of ρ. Then G_a acts non-trivially on F, and F meeets the curve C. Then F is pointwise fixed by G_a. This is a contradiction. Hence C is contained in a fiber of q. By [14], C is isomorphic to $\mathbb{A}^1$ and X^{G_a} is a disjoint union of the affine lines. $\square$

Back to the original situation, we assume that a unipotent k-group G of dimension 2 acts on a smooth, $\overline{k}$-factorial k-variety $X = \operatorname{Spec} A$ of dimension 3. Let $G = G_1 \times G_2$ be a direct product decomposition with $G_1 \cong G_2 \cong G_a$. Let σ be the G-action on X and let σ_i be the restriction of G onto the factors G_i, where G_1 (resp. G_2) is identified with $G_1 \times \{1\}$ (resp. $\{1\} \times G_2$). Let $q_i : X \to Y_i$ be the quotient morphism of the G_i-action σ_i for $i = 1, 2$. We say that the G-action is *non-confluent* if there exists a decomposition $G = G_1 \times G_2$ as above such that the quotient morphisms q_1 and q_2 have no common irreducible components. Otherwise, the action is called *confluent*. In the rest of this section, we consider the tangential confluence in a similar setting.

Let $X = \operatorname{Spec} A$ be a k-variety of dimension n such that $\overline{X}$ is smooth and factorial. Suppose that G_a acts on X via a k-lnd δ and X^{G_a} has no components of codimension one. Concerning the lift of the G_a-action on $\overline{X}$, we define the tangential direction of the orbit at a closed point P of $\overline{X}$. Let $(\mathcal{O}_P, \mathfrak{m}_P)$ be the local ring at P and let $\{\xi_1, \ldots, \xi_n\}$ be a system of local coordinates. Then the tangent space $T_{\overline{X},P}$ of $\overline{X}$ at

[6]This means by definition that $\overline{X}$ is factorial.

P is the $\overline{k}$-vector space $\sum_{i=1}^n \overline{k}(\partial/\partial\xi_i)$. The tangential direction ℓ_P of the G_a-orbit is a line generated by $\sum_{i=1}^n \overline{\delta}(\xi_i)(P)(\partial/\partial\xi_i)$, where $\overline{\delta}$ is the lift of δ to $\overline{A}$ and $\overline{\delta}(\xi_i)(P)$ is the class of $\overline{\delta}(\xi_i)$ in the residue field $\mathcal{O}_P/\mathfrak{m}_P = \overline{k}$. The line ℓ_P is defined as a point of the projective space $\mathbb{P}(T_{\overline{X},P})$ unless $\overline{\delta}(\xi_i)(P) = 0$ for all i. The condition that $\overline{\delta}(\xi_i)(P) = 0$ for all i is equivalent to the condition that the point P is a fixed point. In fact, suppose that $\overline{\delta}(\xi_i)(P) = 0$ for all i. For an element $a \in \overline{A}$, there exists $\alpha \in \overline{k}$ such that $a - \alpha = \sum_{i=1}^n f_i\xi_i$ with $f_i \in \mathcal{O}_P$. Then we have

$$\overline{\delta}(a) = \sum_{i=1}^n \big((\overline{\delta}(f_i)\xi_i + f_i\overline{\delta}(\xi_i)\big) \in \mathfrak{m}_P.$$

This implies that $\sum_{a\in\overline{A}} \overline{\delta}(a)\overline{A} \subseteq \mathfrak{m}_P$. Hence P is the G_a-fixed point. Conversely, if P is a fixed point, then $\overline{\delta}(\overline{A}) \subseteq \mathfrak{m}_P$. Hence $\overline{\delta}(\xi_i) \in \mathfrak{m}_P$. Thus the tangential direction is not defined.

Given a unipotent group G of dimension 2 and its action on a smooth affine variety X, we consider a decomposition $G = G_1 \times G_2$ with G_1, G_2 isomorphic to G_a. With the previous notations, we consider the tangential directions of the G_i-action on $\overline{X}$.

Before going further, we give a remark on a nontrivial, non-effective $\overline{G}$-action on $\overline{X}$.

Lemma 3.1.4. *Assume that $k = \overline{k}$. Let $\overline{G} = G_1 \times G_2$ be a direct product decomposition with $G_1 \cong G_2 \cong G_a$ and let δ_1, δ_2 be the corresponding k-lnds. Then the following conditions on a nontrivial G-action on $X =$ Spec A are equivalent.*

(i) *The G-action is not effective, i.e., there exists an element $g \neq e$ of G which acts on X trivially.*
(ii) *There exists another direct product decomposition $G = G_1' \times G_2'$ such that G_2' acts trivially on X.*
(iii) *There exists an element $c \in k^*$ such that $\delta_1 = c\delta_2$.*

Proof. (i) $\Rightarrow$ (ii). Since the G-action is nontrivial but non-effective, the closure of the subgroup G_2' generated by an element g in the statement is a connected algebraic subgroup isomorphic to G_a, and the quotient G/G_2' is also isomorphic to G_a. In fact, by the argument using the Lie algebra of G in the proof of Lemma 3.1.2, we have a direct product decomposition $G = G_1' \times G_2'$ such that $G_1' \cong G/G_2'$.

(ii) $\Rightarrow$ (i). This is clear.

(ii) $\Rightarrow$ (iii). With the notations in the proof of Lemma 3.1.2, δ_i is the image of Δ_i by $L(\rho) : L(G) \to \mathrm{Der}_k(A)$ for $i = 1, 2$. Let δ_i' be the image of Δ_i' for $i = 1, 2$, where Δ_i' corresponds to the subgroup G_i'. We may assume that $\Delta_2' = \Delta_1 - c\Delta_2$ for $c \in k$. Since $\Delta_2' = 0$ as G_2' acts trivially on X, we have $\delta_1 = c\delta_2$.

(iii) $\Rightarrow$ (ii). Let $\Delta_2' := \Delta_1 - c\Delta_2 \in L(G)$. Then Δ_2' defines a subgroup G_2' of G which acts trivially on X. Further, $G = G_1' \times G_2'$ for some subgroup G_1' of G. $\square$

For a closed point P of $\overline{X} \setminus (\overline{X}^{G_1} \cup \overline{X}^{G_2})$, we can consider the tangential directions $\ell_P^{(1)}$ and $\ell_P^{(2)}$ for the G_1 and G_2 actions.

Lemma 3.1.5. *With the above notations, suppose that $\ell_P^{(1)} = \ell_P^{(2)}$ [7]. Then there exists another decomposition $\overline{G} = G_1 \times G_2'$ such that the G_a-action σ_2' on $\overline{X}$ induced by the G_2'-factor has the point P as a fixed point.*

Proof. Suppose that the G_a-actions σ_1, σ_2 correspond to k-lnd δ_1, δ_2. Since $\ell_P^{(1)}$ and $\ell_P^{(2)}$ are defined and $\ell_P^{(1)} = \ell_P^{(2)}$, there exists $c \in \overline{k}^*$ such that

$$\overline{\delta}_2(\xi_i)(P) = c\overline{\delta}_1(\xi_i)(P),\ (1 \le i \le n).$$

Hence $(\overline{\delta}_2 - c\overline{\delta}_1)(\mathcal{O}_P) \subseteq \mathfrak{m}_P$. Set $\delta_2' = \delta_2 - c\delta_1$. If $\delta_2' = 0$, Lemma 3.1.4 implies that the subgroup G_2' acts on $\overline{X}$ trivially. Suppose that $\delta_2' \ne 0$. Since δ_1, δ_2 commute, δ_2' ia a $\overline{k}$-lnd and $\delta_1\delta_2' = \delta_2'\delta_1$. Hence there is a direct product decomposition $\overline{G} = G_1 \times G_2'$ corresponding to $L(\overline{G}) = \overline{k}\Delta_1 + \overline{k}\Delta_2'$, where $\Delta_2' := \Delta_2 - c\Delta_1$. Now $\delta_2' = L(\overline{\rho})(\Delta_2')$ for the k-group homomorphism $\overline{\rho} : \overline{G} \to \mathrm{Aut}_{\overline{X}/\overline{k}}$ associated to the given $\overline{G}$-action. It is clear that the point P is a fixed point with respect to the σ_2'-action. Note that if $c \in k$ the above decomposition is over k as $G = G_1 \times G_2'$. $\square$

Together with the assumptions in Lemma 3.1.5, we further assume that X is $\overline{k}$-factorial and of dimension 3. Concerning a subgroup of G which is isomorphic to G_a, every fiber component C of the quotient morphism $\overline{q} : \overline{X} \to \overline{Y} := \overline{X}/\!/G_a$ is isomorphic to $\mathbb{A}^1$ if considered with a reduced structure (see [14]). If this curve C passes through a point P of $\overline{X}$, the tangential direction ℓ_P of C at P is well-defined since $\overline{X}$ is smooth. We say that ℓ_P is the tangential direction at P even if P is a fixed point. We say that the G-action is *tangentially non-confluent* if there exists a decomposition $G = G_1 \times G_2$ as above such that the tangential directions $\ell_P^{(1)}$ is different from $\ell_P^{(2)}$ for every closed point P of $\overline{Y}$. It is obvious that the G-action is non-confluent if it is tangentially non-confluent.

[7] This implies that the G_1-orbit G_1P and G_2-orbit G_2P are tangent at P.

At the end of the section we give an example of a unipotent algebraic group of dimension 2 being decomposed into a product $G_a \times G_a$.

Example 3.1.6. We show that a unipotent group G of dimension two defined by

$$G = \left\{ \begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a, b \in k \right\} \subset \mathrm{GL}(3, k)$$

is decomposed as a direct product $G = G_a \times G_a$. In fact, we change an expression of element of G by an automorphism

$$\begin{pmatrix} 1 & a & b \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \longmapsto \begin{pmatrix} 1 & a & b + \frac{1}{2}a^2 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix}.$$

Then we have

$$\begin{pmatrix} 1 & a & b + \frac{1}{2}a^2 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & a & \frac{1}{2}a^2 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where

$$\left\{ \begin{pmatrix} 1 & a & \frac{1}{2}a^2 \\ 0 & 1 & a \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; a \in k \right\} \quad \text{and} \quad \left\{ \begin{pmatrix} 1 & 0 & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \;\middle|\; b \in k \right\}$$

are isomorphic to G_a and commute each other.

3.2. **Case of a fixed-point free G_a-action.** Let $X = \operatorname{Spec} A$ be an affine k-variety and let $\sigma : G_a \times X \to X$ be an action defined over k. By Lemma 3.1.1, σ is induced by a k-lnd δ on the k-algebra A. Let $B = \operatorname{Ker} \delta$ and $Y = \operatorname{Spec} B$. Hereafter we assume that X is a k-form of $\mathbb{A}^3$. Then Y is the (algebraic) quotient variety of X with respect to σ. We denote by $q : X \to Y$ the quotient morphism. We say that the G_a-action σ is *free* if $X^{G_a} = \emptyset$, i.e., $\delta(A)A = A$ [8].

Lemma 3.2.1. *With the above notations, X is k-isomorphic to $\mathbb{A}^2$. Namely, A is a polynomial ring $k[x_1, x_2]$.*

[8]For a G_a-action $\sigma : G_a \times X \to X$, let $\Psi_X := (\sigma, p_2) : G_a \times X \to X \times X$ be the *graph morphism* of σ. We say that the action σ is *free* (resp. *proper*) if the graph morphism Ψ_X is a closed immersion (resp. a proper morphism). Then σ is free (resp. proper) if and only if $\overline{\sigma} := \sigma \otimes_k \overline{k}$ is free (resp. proper). By an argument as in the section five, we have the implications: σ is free $\Rightarrow$ σ is proper $\Rightarrow$ σ is fixed-point free. If X is a k-form of $\mathbb{A}^3$, a theorem of Kaliman [19] implies that these three conditions are equivalent. Hence we can use this terminology without confusion.

Proof. Let $\overline{A} = A \otimes_k \overline{k}$, which is isomorphic to a polynomial ring in dimension three over $\overline{k}$. Then the k-lnd δ lifts to a $\overline{k}$-lnd $\overline{\delta}$ on $\overline{A}$. On the other hand, the following sequence of k-vector spaces is exact

$$0 \longrightarrow B \longrightarrow A \stackrel{\varphi - \mathrm{id}}{\longrightarrow} A[t],$$

where $\varphi : A \to A[t]$ is defined by $\varphi(a) = \displaystyle\sum_{i \geq 0} \frac{1}{i!}\delta^i(a)t^i$. Then the tensor product $\otimes_k \overline{k}$ gives rise to an exact sequence of $\overline{k}$-vector spaces

$$0 \longrightarrow B \otimes_k \overline{k} \longrightarrow \overline{A} \stackrel{\overline{\varphi} - \mathrm{id}}{\longrightarrow} \overline{A}[t],$$

where $\overline{\varphi}$ is the $\overline{k}$-algebra homomorphism $\overline{A} \to \overline{A}[t]$ defined by $\overline{\delta}$ as φ for δ. Hence $B \otimes_k \overline{k} = \mathrm{Ker}\,\overline{\delta}$. By [34], $B \otimes_k \overline{k}$ is a polynomial ring of dimension two over $\overline{k}$. Since there is no nontrivial k-form of $\mathbb{A}^2$ (cf. [24]), it follows that B is a polynomial ring of dimension two over k. $\square$

Let $\overline{Y} = Y \otimes_k \overline{k}$ and let $\overline{q} : \overline{X} \to \overline{Y}$ be the quotient morphism of the G_a-action $\overline{\sigma} = \sigma \otimes_k \overline{k}$. We note that $\overline{\delta}(\overline{A}) = \delta(A) \otimes_k \overline{k}$ and hence $\overline{\delta}(\overline{A})\overline{A} = \delta(A)\overline{A} = \delta(A)A \otimes_k \overline{k}$. This implies that $\overline{X}^{G_a} = X^{G_a} \otimes_k \overline{k}$ and hence that $\overline{\sigma}$ is free if and only if σ is free. This remark yields the following result in view of Kaliman's theorem [19] which asserts the following:

Lemma 3.2.2. *Let $\overline{\sigma} : G_a \times \mathbb{A}^3 \to \mathbb{A}^3$ be a fixed-point free G_a-action. Then the quotient morphism $\overline{q} : \mathbb{A}^3 \to \overline{Y}$ defines an $\mathbb{A}^1$-bundle over $\overline{Y} \cong \mathbb{A}^2$ and the G_a-action is a translation along the fibers. In particular, the quotient morphism $\overline{q}$ is surjective and the fiber $\overline{q}^{-1}(\overline{Q})$ is isomorphic to $\mathbb{A}^1$ with multiplicity 1 for every (closed and non-closed) point $\overline{Q}$ of $\overline{Y}$.*

Lemma 3.2.3. *Assume that the action σ is free. Then the following assertions hold.*

(1) *Any closed fiber of the quotient morphism $q : X \to Y$ is isomorphic to $\mathbb{A}^1$.*
(2) *Let $\mathfrak{p}$ be a height one prime ideal of B and let $\kappa(\mathfrak{p})$ be the quotient field of $B/\mathfrak{p}$. Then the fiber of q over $\mathfrak{p}$ is isomorphic to $\mathbb{A}^1$ over $\kappa(\mathfrak{p})$. Namely, $(A/\mathfrak{p}A) \otimes_{B/\mathfrak{p}} \kappa(\mathfrak{p})$ is a polynomial ring of dimension one over $\kappa(\mathfrak{p})$.*
(3) *Let K be the quotient field of B. Then the generic fiber of q, $X_K = \mathrm{Spec}\,(A \otimes_A K)$, is isomorphic to $\mathbb{A}^1$ over K.*

Proof. (1) Let $\mathfrak{m}$ be the maximal ideal of B corresponding to the closed point of Y. Then $(A/\mathfrak{m}A) \otimes_k \overline{k} = \bigoplus_{i=1}^r \overline{A}/\mathfrak{m}_i\overline{A}$, where $\mathfrak{m}\overline{B} = \bigcap_{i=1}^r \mathfrak{m}_i$ and r is the extension degree of $B/\mathfrak{m}$ over k. Let $\lambda_i : B/\mathfrak{m} \to \overline{k}$ be an embedding of $B/\mathfrak{m}$ to a subfield of $\overline{k}$. Then we have $(A/\mathfrak{m}A) \otimes_{B/\mathfrak{m}}$

$(\overline{k}, \lambda_i)$ is a polynomial ring of dimension one over $\overline{k}$ by Kaliman's theorem. Since there is no nontrivial form of $\mathbb{A}^1$, it follows that $A/\mathfrak{m}A$ is a polynomial ring of dimension one over $B/\mathfrak{m}$.

(2) Let $\mathfrak{p}$ be a height one prime ideal of B. Since B is a polynomial ring, $\mathfrak{p}$ is principal. Write $\mathfrak{p} = (f)$ with a k-irreducible element f of B. With f considered in $\overline{B}$, it is a product $f = \prod_{i=1}^{s} f_i$, where the f_i are conjugate of each other under the Galois group of the splitting field of f over k. We have $\overline{A}/f_i\overline{A} = (A/fA)\otimes_{B/fB}(\overline{B}/f_i\overline{B})$, which is geometrically integral. Now the restriction of $\overline{q}$ onto the hypersurface $V(f_i\overline{A}) = \mathrm{Spec}\, \overline{A}/f_i\overline{A}$ is a fibration over the curve $V(f_i\overline{B}) = \mathrm{Spec}\, \overline{B}/f_i\overline{B}$ whose closed fibers are isomorphic to $\mathbb{A}^1$ and whose generic fiber is geometrically integral. By [25, Theorem 2], it is an $\mathbb{A}^1$-bundle. Hence the generic fiber of $\mathrm{Spec}\, \overline{A}/f_i\overline{A}$ is isomorphic to $\mathbb{A}^1$ over $Q(\overline{B}/f_i\overline{B})$ [9]. Since $Q(\overline{B}/f_i\overline{B})$ is an algebraic extension of $Q(B/fB)$, it follows that $(A/fA) \otimes_{B/fB} Q(B/fB)$ is a polynomial ring of dimension one over $Q(B/fB)$.

(3) By the slice construction, the generic fiber $\overline{X}_{\overline{\eta}} = \mathrm{Spec}\,(\overline{A} \otimes_{\overline{B}} Q(\overline{B}))$ is isomorphic to $\mathbb{A}^1$ over $Q(\overline{B})$. Since $\overline{A} \otimes_{\overline{B}} Q(\overline{B}) = (A \otimes_B Q(B)) \otimes_{Q(B)} Q(\overline{B})$ and since there is no nontrivial form of $\mathbb{A}^1$, it follows that $A \otimes_B Q(B)$ is a polynomial ring of dimension one over $Q(B)$. $\square$

Now the following theorem is an easy consequence of a result of Kambayashi-Wright [26].

Theorem 3.2.4. *Let X be a k-form of $\mathbb{A}^3$ having a free G_a-action σ. Then X is k-isomorphic to $\mathbb{A}^3$.*

Proof. By Lemma 3.2.3, every closed (and non-closed) fiber of the quotient morphism $q : X \to Y$ is isomorphic to $\mathbb{A}^1$ over the respective residue field. Hence q is an $\mathbb{A}^1$-bundle in the Zariski topology by [26]. Since A is factorial, it follows that this $\mathbb{A}^1$-bundle is trivial. Hence X is isomorphic to $\mathbb{A}^3$. $\square$

3.3. **Case of an effective $G_a \times G_a$-action.** We consider the following result.

Lemma 3.3.1. *Let A be an affine k-domain of dimension 2. Suppose that A is geometrically integral over k, i.e., $Q(A)$ is a regular extension of k, and that A has two nonzero k-lnds δ_1, δ_2. Suppose further that $\delta_1\delta_2 = \delta_2\delta_1$ and* $\mathrm{Ker}\,\delta_1 \neq \mathrm{Ker}\,\delta_2$. *Then A is k-isomorphic to $k[t_1, t_2]$.*

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[9]This follows easily from the fact that $\overline{q} : \overline{X} \to \overline{Y}$ is an $\mathbb{A}^1$-bundle. But we would like to make a detour to give an argument which can be applied in case we drop the hypothesis that the action σ is free.

Proof. Let $A_1 = \mathrm{Ker}\delta_2, A_2 = \mathrm{Ker}\delta_1$ and $B = A_1 \cap A_2$. Then $\dim A_i = 1$ for $i = 1, 2$ and $\dim B = 0$. In particular, B is a field extension of k. Since $B \subset Q(A)$ and k is algebraically closed in $Q(A)$ by the hypothesis, it follows that $B = k$. Since $\delta_1\delta_2 = \delta_2\delta_1$, the lnd δ_1 induces a k-lnd on A_1 which we denote by the symbol Δ_1. If the induced Δ_1 is zero on A_1, then $A_2 \supseteq A_1$. Since $\dim A_2 = \dim A_1 = 1$, it follows that $Q(A_2)$ is an algebraic extension of $Q(A_1)$. Then the induced derivation δ_2 is trivial on $Q(A_2)$ since δ_2 is trivial on $Q(A_1)$. Then $A_1 = \mathrm{Ker}\,\delta_2 \supseteq A_2$. Hence $A_1 = A_2$ which contradicts the hypothesis. Thus the induced Δ_1 on A_1 is nontrivial. Similarly, δ_2 induces a nontrivial k-lnd Δ_2 on A_2. Then $\ker \Delta_1 = \mathrm{Ker}\,\Delta_2 = k$. This implies that $A_1 = k[t_1]$ and $A_2 = k[t_2]$. We can choose t_1, t_2 so that $\Delta_1(t_1) = 1$ and $\Delta_2(t_2) = 1$. Then t_1 and t_2 are slices of the lnds δ_1 and δ_2 in A. Then $A = A_2[t_1] = A_1[t_2] = k[t_1, t_2]$. □

Let $X = \mathrm{Spec}\,A$ be an affine k-variety of dimension 3. By Lemma 3.1.2, a $G_a \times G_a$-action σ on X is given by two k-lnds δ_1, δ_2 on A such that $\delta_1\delta_2 = \delta_2\delta_1$. We prove the following result of Daigle-Kaliman [5].

Theorem 3.3.2. *Let $X = \mathrm{Spec}\,A$ be a k-form of $\mathbb{A}^3$. Suppose that X has an effective action of a unipotent group G of dimension 2. Then Y is k-isomorphic to $\mathbb{A}^3$.*

Proof. Our proof consists of three steps.

(I) There exists a direct product decomposition $G = G_1 \times G_2$ such that the two G_a-actions induced by G_1 and G_2 are non-confluent. By Lemma 3.1.1, A is an factorial k-domain of dimension 3 with $A^* = k^*$. Let $A_1 = \mathrm{Ker}\,\delta_2, A_2 = \mathrm{Ker}\,\delta_1$, and $B = A_1 \cap A_2$. Then B, A_1, A_2 are factorial affine k-domains of respective dimensions $1, 2, 2$. The units of these domains are all constants. By what we have discussed, $\overline{A}_i = A_i \otimes_k \overline{k}$ is a polynomial ring of dimension 2 over $\overline{k}$, and $\overline{B} = B \otimes_k \overline{k}$ is a polynomial ring of dimension 1 over $\overline{k}$. Since all k-forms of $\mathbb{A}^1$ and $\mathbb{A}^2$ are trivial, it follows by the construction of A_i, B that $B = k[u], A_1 = k[u, t_1]$, and $A_2 = k[u, t_2]$ with $\delta_1(t_1) = f_1(u)$ and $\delta_2(t_2) = f_2(u)$ for nonzero elements $f_1(u), f_2(u)$ of $k[u]$.

(II) Let $A_0 = A_1 \otimes_B A_2 = k[u, t_1, t_2]$, which is a $k[u]$-subalgebra of A. Let $X_0 = \mathrm{Spec}\,A_0$ and $Z = \mathrm{Spec}\,B$. The inclusion $A_0 \hookrightarrow A$ defines a Z-morphism $\varphi : X \to X_0$. Let $K = Q(B)$. Then $A_K := A \otimes_B K =$

$K[t_1, t_2]$ by Lemma 3.3.1. Hence φ is birational because

$$\begin{aligned} A[f_1(u)^{-1}, f_2(u)^{-1}] &= (A[f_1(u)^{-1}])[f_2(u)^{-1}] \\ &= (A_2[f_1(u)^{-1}][t_1])[f_2(u)^{-1}] \\ &= (A_2[f_2(u)^{-1}])[f_1(u)^{-1}, t_1] \\ &= (B[f_2(u)^{-1}, t_2][f_1(u)^{-1}, t_1] \\ &= (B[f_1(u)^{-1}, f_2(u)^{-1}][t_1, t_2]. \end{aligned}$$

(III) The morphism $\varphi : X \to X_0$ is G-equivariant by the construction. Since $\overline{X} := X \otimes_k \overline{k} \cong \mathbb{A}^3$ and $X_0 \cong \mathbb{A}^3 = \operatorname{Spec} k[u, t_1, t_2]$, the birational morphism $\overline{\varphi} : \overline{X} \to \overline{X}_0$ is a polynomial morphism

$$(x, y, z) \mapsto (f_1(x, y, z), f_2(x, y, z), f_3(x, y, z)).$$

Let $J = \det(\partial(f_1, f_2, f_3)/\partial(x, y, z))$ be the Jacobian determinant. Then $J = 0$ defines the locus E of exceptional varieties of $\overline{\varphi}$, which is G-stable. Hence J is G-invariant, i.e., $J \in B$. This implies that the subvarieties of $\overline{X}$ which are contracted to curves or points by $\overline{\varphi}$ are the fibers of $\overline{p} : \overline{X} \to \overline{C}$, where $\overline{p}$ is the base change of $p : X \to C = \operatorname{Spec} k[u]$ defined by the inclusion $B = k[u] \hookrightarrow A$. Furthermore, $\overline{\varphi} : \overline{X} \to \overline{X}_0$ is an isomorphism on $\overline{X} \setminus E$, where $E = \overline{p}^{-1}V(J)$. In particular, the generic fiber of $p : X \to C$ is isomorphic to $\mathbb{A}^2$ because a form of $\mathbb{A}^2$ is trivial, and all closed fibers $p^{-1}(P)$ are isomorphic to $\mathbb{A}^2$ for all $P \in C \setminus V(J)$. It follows then from Kaliman [18] that u is a coordinate of $\overline{X} \cong \mathbb{A}^3$ and hence the fibers $\overline{p}^{-1}(\overline{P}) \cong \mathbb{A}^2$ for $\overline{P} \in V(J)$. Then $X \cong \mathbb{A}^3$ by Sathaye [39]. $\square$

The morphism $\varphi : X \to X_0$ in the above proof is not an isomorphism in general as shown by the following example.

Example 3.3.3. Consider the following lnds on a polynomial ring $A = k[x, y, z]$,

$$\delta_1 = x\frac{\partial}{\partial y} + \frac{\partial}{\partial z}, \quad \delta_2 = x^2\frac{\partial}{\partial y} + \frac{\partial}{\partial z}\ .$$

Then $\delta_1\delta_2 = \delta_2\delta_1$. We can compute $A_1 = \operatorname{Ker} \delta_1$, $A_2 = \operatorname{Ker} \delta_2$ and $B = A_1 \cap A_2$ as follows.

$$A_1 = k[x, y - xz],\ A_2 = k[x, y - x^2 z],\ B = k[x]\ .$$

Hence $A_0 = k[x, y - xz, y - x^2 z]$ and $\varphi : X \to X_0$ is given by

$$(x, y, z) \mapsto (x, y - xz, y - x^2 z).$$

The Jacobian determinant J is $x(1 - x)$. On the fibers $p^{-1}(x = 0)$ and $p^{-1}(x = 1)$, the morphism φ is given by

$$(0, y, z) \mapsto (0, y, y),\ (1, y, z) \mapsto (1, y - z, y - z)\ .$$

In this example the planes $\{x = 0\}$ and $\{x = 1\}$ are mapped to lines. $\square$

Remark 3.3.4. (1) With the notations of the proof of Lemma 3.3.1, the case where a fiber $\overline{p}^{-1}(P)$ is contracted to a point by $\overline{\varphi}$ is impossible. We may assume that $x = u$ and the fiber $\overline{p}^{-1}(P_0)$ is sent to the point $(x, t_1, t_2) = (0, 0, 0)$. The morphism $\overline{\varphi}$ is given by a polynomial morphism

$$(x, y, z) \mapsto (x, t_1, t_2) = (x, f_2(x, y, z), f_3(x, y, z)) \ ,$$

where $f_1 \in \operatorname{Ker} \delta_2$ and $f_2 \in \operatorname{Ker} \delta_1$. By the assumption, $f_2(y, z) = 0$ and $f_3(y, z) = 0$. Then $f_2(x, y, z)$ and $f_3(x, y, z)$ are divisible by x, and hence $f_2(x, y, z)/x \in k[x, f_2(x, y, z)]$. This is a contradiction.

(2) If the action of $G = G_a \times G_a$ is non-confluent, then the birational morphism φ is an isomorphism. In fact, if D is an irreducible curve in $\overline{X}$ such that $\overline{\varphi}(D)$ is a point of $\overline{X}_0$. Then D is a fiber component of both quotinet morphism $\overline{q}_i : \overline{X} \to \overline{X}_i$ for $i = 1, 2$. Such a curve does not exist if the G-action is non-confluent. Hence $\overline{\varphi}$ is a quasi-finite morphism. Then$\overline{\varphi}$ is an isomorphism by Zariski main theorem. Hence φ is an isomorphism. $\square$

3.4. **A k-form of $\mathbb{A}^4$ with a proper action of a unipotent group of dimension 2.** Let $X = \operatorname{Spec} A$ be now an affine algebraic k-variety of dimension 4 such that $X \otimes_k \overline{k}$ is $\overline{k}$-isomorphic to $\mathbb{A}^4$. Let G be a two-dimensional unipotent group. By Lemma 3.1.2, G is k-isomorphic to $G_a \times G_a$. Assume that X has a proper G-action. Let B be the ring of G-invariant elements of A. By a lemma of Zariski, $B = A \cap K$ is an affine k-domain, where K is the field of G-invariant elements of the function field $k(X) = Q(A)$. Let $Y = \operatorname{Spec} B$ and let $q : X \to Y$ be the quotient morphism. We denote $\overline{q} : \overline{X} \to \overline{Y}$ be the base change of q by the field extension $\overline{k}/k$.

Lemma 3.4.1. *Y is k-isomorphic to $\mathbb{A}^2$.*

Proof. It suffices to show that $\overline{Y}$ is isomorphic to $\mathbb{A}^2$. Since $\overline{Y} \cong \mathbb{A}^4/\overline{G}$ with $\overline{G} = G \otimes_k \overline{k}$, $\overline{B} = B \otimes_k \overline{k}$ is factorial and $\overline{B}^* = \overline{k}^*$. By the argument in Lemma 1.1 in [15], $\overline{Y}$ is isomorphic to $\mathbb{A}^2$. In fact, $\operatorname{Sing}(\overline{Y})$ is a finite set. Let $\overline{Y}' = \overline{Y} \setminus \operatorname{Sing}(\overline{Y})$. Since $\overline{X} \cong \mathbb{A}^4$, there is a linear plane $\overline{L}$ in $\overline{X}$ such that $\overline{q}|_{\overline{L}} : \overline{L} \to \overline{Y}$ is a dominant morphism and $\dim \overline{q}^{-1}(\overline{P}) \cap \overline{L} \le 0$ for every point $\overline{P}$ of $\operatorname{Sing}(\overline{Y})$. Then $\overline{\kappa}(\overline{Y}') = -\infty$. Hence $\overline{Y}'$ is either affine 1-ruled, i.e., $\overline{Y}'$ has an $\mathbb{A}^1$-fibration, or contains a Platonic $\mathbb{C}^*$-fiber space (we may assume that $\overline{k} = \mathbb{C}$). In the first case, $\overline{Y} \cong \mathbb{A}^2$. In the second case, argue exactly in the same fashion as in the proof of Lemma 1.1 in [15]. $\square$

Lemma 3.4.2. *The following assertions hold.*

(1) *The graph morphism $\Psi_{\overline{X}/\overline{Y}} := (\overline{\sigma}, p_2) : G \times \overline{X} \to \overline{X} \times_{\overline{Y}} \overline{X}$ is a surjective morphism. Further, the generic fiber $X_\eta := X \times_Y \operatorname{Spec} k(Y)$ is $k(Y)$-isomorphic to $\mathbb{A}^2$.*

(2) *The $\overline{G}$-action on $\overline{X}$ is fixpoint-free, and each fiber of $\overline{q}$ is a $\overline{G}$-orbit, hence isomorphic to $\mathbb{A}^2$ if considered with reduced structure.*

(3) *For each closed point P of Y, the fiber $q^{-1}(P)_{\rm red}$ is isomorphic to $\mathbb{A}^2$.*

Proof. (1) The graph morphism $\Psi_{\overline{X}} : \overline{G} \times \overline{X} \to \overline{X} \times \overline{X}$ splits as

$$\Psi_{\overline{X}} : \overline{G} \times \overline{X} \xrightarrow{\Psi_{\overline{X}/\overline{Y}}} \overline{X} \times_{\overline{Y}} \overline{X} \xrightarrow{\overline{\iota}} \overline{X} \times \overline{X},$$

where $\overline{\iota}$ is a closed immersion. Hence $\overline{\iota}$ is a separated morphism and the morphism $\Psi_{\overline{X}/\overline{Y}}$ is a proper morphism since so is the composite $\overline{\iota} \cdot \Psi_{\overline{X}/\overline{Y}}$. On the other hand, let ζ (resp. η) be the generic point of X (resp. Y). Let $X_\eta := X \times_Y \operatorname{Spec} k(\eta)$ be the generic fiber of q. By taking the fiber product with $\operatorname{Spec} k(\zeta)$ over X, the graph morphism $\Psi_{X/Y} : G \times X \to X \times_Y X$ induces a $k(\zeta)$-morphism $\Psi_\zeta : G \otimes_k k(\zeta) \to X_\eta \otimes_{k(\eta)} k(\zeta)$ which is proper. Hence it is a finite morphism because Ψ_ζ is an affine morphism as well. Since G contains no finite subgroups, Ψ_ζ is birational. Since X_η is normal, Ψ_ζ is an isomorphism by Zariski's Main Theorem. This implies that X_η is $k(\eta)$-isomorphic to $\mathbb{A}^2$ because G is k-isomorphic to $\mathbb{A}^2$ and a form of $\mathbb{A}^2$ is trivial, and that $\Psi_{X/Y} : G \times X \to X \times_Y X$ is a dominant morphism. Hence $\Psi_{\overline{X}/\overline{Y}}$ is also dominant and proper. This implies that $\Psi_{\overline{X}}$ is surjective.

(2) Hence each fiber of $\overline{q}$ is a $\overline{G}$-orbit and two-dimensional by the upper-semicontinuity of dimension of fibers, i.e., each closed fiber of $\overline{q}$ has dimension not less than 2, and the $\overline{G}$-action is fixed-point free because proper subgroups of G have dimension one.

(3) As in the proof of Lemma 3.2.3, let $\lambda_i : k(P) \to \overline{k}$ be an embedding of fields. Let $F := q^{-1}(P)_{\rm red}$. Then $F \otimes_{k(P)} (\overline{k}, \lambda_i) \cong \mathbb{A}^2_{\overline{k}}$. Namely, F is a $k(P)$-form of $\mathbb{A}^2$. Hence F is isomorphic to $\mathbb{A}^2$ over $k(P)$. $\square$

Lemma 3.4.3. *$\overline{q}$ has no singular fibers over codimension one points of $\overline{Y}$. Hence all singular fibers are either empty fibers or irreducible multiple fibers whose reduced form is isomorphic to $\mathbb{A}^2$. Let $\overline{S}$ be the set of points $\overline{P}$ of $\overline{Y}$ such that $\overline{q}^{-1}(\overline{P})$ is either the empty set or a singular fiber of $\overline{q}$. Then $\overline{S}$ is a finite set.*

Proof. Note that $\overline{B}$ is factorial. If $\overline{p}$ is a prime element of $\overline{B}$, $\overline{p}$ is a prime element in $\overline{A}$. In fact, $\overline{p}$ is a prime element of $\overline{A}^{\overline{G}_1}$ which is factorial because $\overline{B} = \left(\overline{A}^{\overline{G}_1}\right)^{\overline{G}_2}$, hence a prime element of $\overline{A}$, where $\overline{G} = \overline{G}_1 \times \overline{G}_2$ with $\overline{G}_1 \cong \overline{G}_2 \cong G_a$. Let $\overline{C}$ be an irreducible curve on $\overline{Y}$ defined by $\overline{p} = 0$. Then $\overline{S} := \overline{q}^{-1}(\overline{C})$ is an irreducible threefold defined by $\overline{p} = 0$ in $\overline{X}$ which is G-stable, and the restriction of $\overline{q}$ onto $\overline{S}$ is factored by the quotient morphism, $\overline{q}|_{\overline{S}} : \overline{S} \to \overline{S}/G \to \overline{C}$. If $\overline{q}|_{\overline{S}} : \overline{S} \to \overline{C}$ is not geometrically integral, then general fibers are disjoint union of irreducible components which are isomorphic to $\mathbb{A}^2$. This contradicts the properness hypothesis of the $\overline{G}$-action by Lemma 3.4.2. Hence if $\overline{q}$ has singular fibers, they are irreducible multiple fibers over isolated points of $\overline{Y}$. $\square$

Lemma 3.4.4. *The quotient morphism $\overline{q} : \overline{X} \to \overline{Y}$ has no multiple fibers. Hence the set $\overline{S}$ consists of the points of $\overline{Y}$ over which the fiber is the empty set.*

Proof. Let $\overline{Q}$ be a point of $\overline{Y}$ such that $\overline{q}^{-1}$ is a multiple fiber, i.e., a non-reduced fiber. Let $\overline{P} \in \overline{q}^{-1}(\overline{Q})$. By Seidenberg [41], a general hyperplane section H of $\overline{X}$ which passes through the point $\overline{P}$, where we take $\overline{P}$ to be sufficiently general on $\overline{q}^{-1}(\overline{Q})$. By considering an embedding of $\overline{k}$ into $\mathbb{C}$, we may work over $\mathbb{C}$. We take a small Euclidean neighborhood T of $\overline{P}$ in H and call T a slice transverse to $\overline{q}^{-1}(\overline{Q})$. Then $\overline{q}|_T$ is smooth at a point $\overline{P}'$ near $\overline{P}$ if and only if $\overline{q}$ is smooth at $\overline{P}'$. By Lemma 3.4.3, $\overline{q}|_T : T \to \overline{Y}$ is unramified outside a closed subset of T of codimension ≥ 1. By purity of branch locus, $\overline{q}|_T : T \to \overline{Y}$ is also smooth at $\overline{P}$. This is a contradiction because $\overline{q}^{-1}(\overline{Q})$ is a multiple fiber. Hence there are no multiple fibers.

By Seshadri [43, Theorem 6.1], there exists a geometric quotient U of $\overline{X}$ by G. Let $\rho : \overline{X} \to U$ be the quotient morphism. Then ρ is a locally trivial G-torsor and $\overline{q}$ is decomposed by π as

$$\overline{q} : \overline{X} \xrightarrow{\rho} U \xrightarrow{\pi} \overline{Y}.$$

Then π is one-to-one by the correspondence of G-orbits and hence birational. Since U and $\overline{Y}$ are normal, π is an open immersion by Zariski's main theorem. We follow the argument in [14, Proof of Theorem 3.9]. Let $\overline{Q}$ be a point of a multiple fiber $\overline{F}_{\overline{P}}$, where $\square$

Lemma 3.4.5. *Let $\overline{Y}^0 := \overline{Y} \setminus \overline{S}$ and let $\overline{X}^0 := \overline{q}^{-1}(\overline{Y}^0)$. Then $\overline{X}^0 = \overline{X}$ and $\overline{X}^0$ is a G-torsor over $\overline{Y}^0$.*

Proof. The first assertion follows from Lemma 3.4.4. It suffices to show that $\overline{X}^0 \times_{\overline{Y}^0} \overline{X}^0$ is smooth. Then $\overline{\Psi}_{\overline{X}/\overline{Y}}$ restricted onto $G \times \overline{X}^0$ induces an isomorphism $G \times \overline{X}^0 \cong \overline{X}^0 \times_{\overline{Y}^0} \overline{X}^0$. Let $\overline{P}, \overline{P}'$ be closed points of $\overline{X}^0$ such that $\overline{q}(\overline{P}) = \overline{q}(\overline{P}') = \overline{Q}$. Then there exists $\overline{g} \in \overline{G}(\overline{k})$ such that $\overline{P}' = {}^{\overline{g}}\overline{P}$. Then $\overline{g} \times \mathrm{id}_{\overline{X}^0}$ induces an isomorphism between local rings $\mathcal{O}_{\overline{Z},(\overline{P},\overline{P})}$ and $\mathcal{O}_{\overline{Z},(\overline{P}',\overline{P})}$, where $\overline{Z} = \overline{X}^0 \times_{\overline{Y}^0} \overline{X}^0$. Hence it suffices to show that $\mathcal{O}_{\overline{Z},(\overline{P},\overline{P})}$ is a regular local ring. Let $\{x_1, x_2\}$ be a regular system of parameters of the maximal ideal of $\mathcal{O}_{\overline{Y}^0,\overline{Q}}$ and let $\{t_1, t_2\}$ be elements of $\mathcal{O}_{\overline{X}^0,\overline{P}}$ whose images in $\mathcal{O}_{\overline{q}^{-1}(\overline{Q}),\overline{P}}$ form a regular system of parameters. Then it is clear that $\{x_1, x_2, t_1, t_2, t_1', t_2'\}$ is a regular system of parameters of the local ring $\mathcal{O}_{\overline{Z},(\overline{P},\overline{P})}$, where $\{t_1', t_2'\}$ is a copy of $\{t_1, t_2\}$. Hence $\overline{Z}$ is smooth at the point $(\overline{P}, \overline{P})$. $\square$

We can now answer the triviality question of a k-form of $\mathbb{A}^4$ as follows.

Theorem 3.4.6. *Let $X = \operatorname{Spec} A$ be a k-form of $\mathbb{A}^4$ equipped with a proper G-action, where G is a unipotent group of dimension 2. Then X is k-isomorphic to $\mathbb{A}^4$.*

Proof. If $\overline{S} = \emptyset$ then $\overline{X}$ is a $\overline{G}$-torsor over $\overline{Y}$. Since the G-action is defined over the field k, X is a G-torsor over Y. Write $G = G_1 \times G_2$ with $G_1 \cong G_2 \cong G_a$. Let $X_1 = X/G_1$[10] . Then X_1 is a G-torsor over Y. Since the set of isomorphism classes of G_a-torsors over Y is bijective to $H^1_{\mathrm{fl}}(Y, G_a) \cong H^1(Y, \mathcal{O}_Y) = 0$, we know that $X_1 \cong Y \times G_a$. Similarly, we have $X \cong X_1 \times G_a$. Hence $X \cong \mathbb{A}^2 \times G \cong \mathbb{A}^4$.

We shall show that $\overline{S} = \emptyset$. Let $\overline{q} : \overline{X} \to \overline{Y}$ be the same as in Lemma 3.4.4. Then $\overline{X}$ is a $\overline{G}$-torsor over $\overline{Y}^0$. By Lefschetz principle, we may assume that $\overline{k} = \mathbb{C}$. By a long exact sequence of homotopy groups for a fiber bundle, we have $\pi_i(\overline{Y}^0) \cong \pi_i(\overline{X}) \cong \pi_i(\mathbb{A}^4) = 0$ for every $i > 0$. Since $\pi_1(\overline{Y}^0) = (1)$, we have $H_i(\overline{Y}^0; \mathbb{Z}) \cong H_i(\overline{X}; \mathbb{Z}) = 0$ for every $i > 0$ by Hurwicz's isomorphism theorem. But, if $\overline{S} \neq \emptyset$,

[10]Since $\dim A = 4$ and $A_1 = \operatorname{Ker} \Delta$ with an lnd Δ on A associated to the G_1-action, we need to show that A_1 is finitely generated over k. By Seshadri [43], X is a locally trivial G-principal fiber bundle. Hence there exists an open covering $\mathcal{U} = \{U_i\}_{i \in I}$ with a finite index set I such that $q^{-1}(U_i) \cong U_i \times G$ which is a G-equivalent isomorphism over U_i. We can take the open sets U_i of the form $D(b_i)$ with $b_i \in B$. Then $A_1[b_i^{-1}] \cong A[b_i^{-1}]^{G_1} \cong B[b_i^{-1}][t_i]$ since G_1 acts on G and $G/G_1 \cong G_a$. Hence $A_1[b_i^{-1}]$ is generated by a single element f_i over $B[b_i^{-1}]$. Since the positive powers of b_i generate the unitary ideal in B, it is then easy to show that A_1 is generated by $\{f_i\}_{i \in I}$ over B.

then $H_3(\overline{Y}^0; \mathbb{Z}) \cong \mathbb{Z}^{\oplus\#(\overline{S})} \neq 0$ by a long exact sequence of integral cohomologies for a pair $(\overline{Y}, \overline{S})$. This is a contradiction. □

Remark 3.4.7. Let n be an integer greater than 2. Let G a commutative unipotent group of dimension $n-2$. Since G has a subnormal series whose subquotient is isomorphic to G_a, it follows that the underlying space of G is isomorphic to $\mathbb{A}^{n-2}$. Let $X = \operatorname{Spec} A$ be a k-form of $\mathbb{A}^n$ with a proper G-action. Let $B = A^G$ and let $Y = \operatorname{Spec} B$. Then B is an affine domain of dimension 2, and the inclusion $B \hookrightarrow A$ defines the quotient morphism. To avoid the difficulty arising from the triviality of forms of $\mathbb{A}^{n-2}$, we assume that the quotient morphism q has a cross-section if $n \geq 5$. Putting this assumption, every fiber of q will be a non-multiple fiber. For a closed point Q of Y, let $F = q^{-1}(Q)_{\mathrm{red}}$. Then F is defined over the residue field $k(Q)$ and given a proper G-action. Since $\dim F = n-2$ and the G-action on F has no fixed points, it follows that $F \cong \mathbb{A}^{n-2}$. Under this situation, Lemmas 3.4.1 $\sim$ 3.4.5 holds with only $\mathbb{A}^2$ in the assertions replaced by $\mathbb{A}^{n-2}$. So, by pursuing each step of the proof of Theorem 3.4.6, we can show that X is k-isomorphic to $\mathbb{A}^n$.

Furthermore, we note that proper G_a-actions on smooth affine varieties of dimension 3 or 4 are studied by Kaliman [20, Theorem 0.1, Corollary 0.3]. □

3.5. **Forms of $\mathbb{A}^n \times \mathbb{A}^1_*$.** Let $X = \operatorname{Spec} A$ be a k-form of $\mathbb{A}^n \times \mathbb{A}^1_*$, where $\mathbb{A}^1_*$ is the affine line with one point punctured. Then there exists a finite Galois extension k' of k with Galois group $\mathfrak{G}$ such that $X \otimes_k k'$ is k'-isomorphic to $\mathbb{A}^n \times \mathbb{A}^1_*$. Namely $A_{k'} := A \otimes_k k' \cong k'[x_1, \ldots, x_n, y, y^{-1}]$. Since $A^*_{k'} = k'^* \times \{y^m \mid m \in \mathbb{Z}\}$, we have

$$^g y = \gamma(g) y^{m(g)},\ \gamma(g) \in k'^*, m(g) \in \mathbb{Z},\ g \in \mathfrak{G}$$

where

$$\gamma(g'g) = {}^{g'}\gamma(g)\gamma(g')^{m(g)},\ m(g'g) = m(g')m(g), \quad g', g \in \mathfrak{G}.$$

Since $\gamma(e) = 1$ and $m(e) = 1$ for the identity element e of $\mathfrak{G}$, we know that $m : \mathfrak{G} \to \mathbb{Z}^*$ is a multiplicative character. Hence $m(g) = \pm 1$ for $g \in \mathfrak{G}$. Let $N = \{g \in \mathfrak{G} \mid m(g) = 1\}$. Then N is a normal subgroup of $\mathfrak{G}$ such that $[\mathfrak{G} : N] \leq 2$.

Lemma 3.5.1. *Assume that $N = \mathfrak{G}$. Then there exists a k-form Y of $\mathbb{A}^n$ such that X is k-isomorphic to $Y \times \mathbb{A}^1_*$.*

Proof. Since $m(g) = 1$ for every $g \in \mathfrak{G}$, the equality $\gamma(g'g) = {}^{g'}\gamma(g)\gamma(g')$ holds for $g', g \in \mathfrak{G}$. By Hilbert's theorem 90, there exists an element $c \in k'^*$ such that $\gamma(g) = {}^g c \cdot c^{-1}$. Replacing y by y/c, we may assume that ${}^g y = y$ for every $g \in \mathfrak{G}$. So, $y \in A = (A \otimes_k k')^{\mathfrak{G}}$. Now each element $a \in A$ is expressed in $A \otimes_k k' = k'[x_1, \ldots, x_n, y, y^{-1}]$ as a Laurent polynomial

$$a = \sum_{i \in \mathbb{Z}} a'_i y^i, \quad a'_i \in k'[x_1, \ldots, x_n].$$

Since ${}^g a = a$, we have ${}^g(a'_i) = a'_i$ for every $i \in \mathbb{Z}$. Hence $a'_i \in A$. Let B be the set of elements $a \in A$ such that ${}^g a = a$ for $g \in \mathfrak{G}$ and a is constant as a Laurent polynomial in y. Then B is a k-algebra such that $(B[y, y^{-1}]) \otimes_k k' = A \otimes_k k'$. Then it follows that $A = B[y, y^{-1}]$. Since

$$(B \otimes_k k')[y, y^{-1}] = k'[x_1, \ldots, x_n][y, y^{-1}],$$

comparison of the constant rings of Laurent polynomial rings on both sides implies that $B \otimes_k k' = k'[x_1, \ldots, x_n]$. Thus $Y := \operatorname{Spec} B$ is a k-form of $\mathbb{A}^n$ trivialized by the field extension k'/k. □

Corollary 3.5.2. *Assume that $N = \mathfrak{G}$ and $n = 1$ or 2. Then X is k-isomorphic to $\mathbb{A}^n \times \mathbb{A}^1_*$.*

Proof. Since there are no non-trivial k-forms of $\mathbb{A}^n$ if $n = 1, 2$, the assertion follows from Lemma 3.5.1. □

Assume that $N \neq \mathfrak{G}$ and $k'' = (k')^N$. Then k'' is a quadratic extension with Galois group $\mathfrak{G}/N \cong \mathbb{Z}/2\mathbb{Z}$. Let σ be a generator of the group $\mathfrak{G}/N$. Write $k'' = k(\alpha)$ with $a = \alpha^2 \in k$. By Lemma 3.5.1, there exists a k''-algebra A'' such that $A \otimes_k k'' = A''[y, y^{-1}]$ and $A'' \otimes_{k''} k' \cong k'[x_1, \ldots, x_n]$. Since ${}^\sigma y = cy^{-1}$ with $c \in (k'')^*$ and $\sigma^2 = e$, it follows that ${}^\sigma c = c$, whence $c \in k$. Let $C = \operatorname{Spec} R$ be an affine plane curve defined by

$$R = k[Y, Y']/(Y^2 - aY'^2 = 4c).$$

Note that C is k-isomorphic to $\mathbb{A}^1_*$ if and only if $\alpha \in k$. Then we have the following result.

Lemma 3.5.3. *With the above notations and assumptions, the following assertions hold.*

1. *The k-algebra R is identified with a subalgebra of A. Hence there exists a morphism $f : X \to C$.*
2. *Each fiber of f is a k-form of $\mathbb{A}^n$.*
3. *If $n = 1, 2$, the morphism f defines an $\mathbb{A}^n$-bundle over C.*

Proof. (1) Set

$$Y = y + \frac{c}{y}, \quad Y' = \frac{1}{\alpha}\left(y - \frac{c}{y}\right).$$

Then ${}^{\sigma}Y = Y$ and ${}^{\sigma}Y' = Y'$. Hence $Y, Y' \in A$ and there is a relation $Y^2 - aY'^2 = 4c$. Further, since we have

$$y = \frac{1}{2}(Y + \alpha Y'), \quad y^{-1} = \frac{1}{2c}(Y - \alpha Y'),$$

it follows that $R := k[Y, Y']/(Y^2 - aY'^2 = 4c)$ is a k-subalgebra of A which is a k-form of $k[y, y^{-1}]$.

(2) Let P be a closed (or generic) point of C and let $k'(P) = k' \otimes_k k(P)$, where $k(P)$ is the reidue field of C at P. Then we have

$$\begin{aligned} A \otimes_R k'(P) &= (A \otimes_k k') \otimes_{R \otimes_k k'} k'(P) \\ &= k'[x_1, \ldots, x_n, y, y^{-1}] \otimes_{k'[y, y^{-1}]} k'(P) \\ &= k'[x_1, \ldots, x_n] \otimes_{k'} k'(P). \end{aligned}$$

This implies that the fiber $f^{-1}(P)$, which is defined over $k(P)$, is a $k(P)$-form of $\mathbb{A}^n$.

(3) If $n = 1, 2$, the generic fiber as well as general fibers of f is a form of $\mathbb{A}^n$, which is trivial. Hence f is an $\mathbb{A}^n$-bundle. □

4. Cancellation problem in dimension three

We can apply the arguments in the previous subsection 3.4 to the cancellation problem in dimension 3.

4.1. **Statement of a theorem.**

Theorem 4.1.1. *Let X be an affine variety of dimension 3 such that $X \times \mathbb{A}^{n-3} \cong \mathbb{A}^n$ with $n \geq 4$. Suppose that X has a proper G_a-action. Then X is isomorphic to $\mathbb{A}^3$.*

The proof will be given in this subsection. Since the projection $\mathbb{A}^n \to X$ is faithfully flat, X is smooth, contractible and factorial. Let A be the coordinate ring of X. We have further $A^* = k^*$. We assume that X has a proper G_a-action $\sigma : G_a \times X \to X$. Since the graph morphism $\Psi := (\sigma, \mathrm{id}_X) : G_a \times X \to X \times X$ is proper, the action is nontrivial. Let $q : X \to Y$ be the quotient morphism, where Y is the algebraic quotient Spec B, where $B = A^{G_a}$. Note that B is a factorial affine domain of dimension two such that $B^* = k^*$ and q is an $\mathbb{A}^1$-fibration. As in the proof of Lemma 3.4.2, the action σ is fixed-point free, and every non-empty fiber of q is a G_a-orbit. We set $S = Y \setminus q(X)$.

Lemma 4.1.2. *The following assertions hold.*

(1) *Let $g = q \cdot p : \mathbb{A}^n \to Y$ with the canonical projection $p : \mathbb{A}^n \to X$. Then g has no fibers which contains an irreducible component of dimension $n - 1$.*

(2) *Let P_i $(1 \le i \le r)$ be a finite set of closed points of Y. Then there exists a linear plane L of $\mathbb{A}^n$ such that $g|_L : L \to Y$ is a dominant morphism and $\dim g^{-1}(P_i) \cap L \le 0$ for every $1 \le i \le r$.*
(3) *Let $Y' = Y - \mathrm{Sing}(Y)$. Then $\overline{\kappa}(Y') = -\infty$.*
(4) *Y is isomorphic to $\mathbb{A}^2$.*

Proof. (1) There is a $G_a^{n-2} = G_a^{n-3} \times G_a$-action on $\mathbb{A}^n$ such that one factor G_a^{n-3} acts along the fibers of the trivial $\mathbb{A}^{n-3}$-bundle $\mathbb{A}^n \to X$ and another G_a is the given G_a-action on X. Suppose that g has an irreducible component F of dimension $n-1$. Then F is defined by an equation $b = 0$ for $b \in \Gamma(\mathbb{A}^n, \mathcal{O}_{\mathbb{A}^n})$. Since $g(F)$ is a point, F is stable by the G_a^{n-2}-action on $\mathbb{A}^n$. Hence $b \in B$. This gives a contradiction because the subvariety of Y defined by $b = 0$ has dimension one and $g(F)$ is a point.

(2) This follows from [35, Lemma 4].

(3) Note that Y is factorial because Y is the algebraic quotient X/G_a. Hence $\mathrm{Sing}(Y)$ is either empty or a finite set. Let $Y' = Y - \mathrm{Sing}(Y)$. Then $(g|_L)^{-1}(\mathrm{Sing}Y)$ is an empty set or a finite set by the assertion (2). Hence $L' := L - (g|_L)^{-1}(\mathrm{Sing}Y)$ has log Kodaira dimension $-\infty$ as the complement of a finite set of closed points in L. Then $\overline{\kappa}(Y') = -\infty$.

(4) By the condition $\overline{\kappa}(Y') = -\infty$, either Y' contains an open set U which is either a cylinderlike open set or a Platonic $\mathbb{C}^*$-fiber spce [34, Chapter 2, Theorem 5.1.2]. If Y' contains a cylinderlike open set, so does Y. Hence $Y \cong \mathbb{A}^2$ by the algebraic characterization of $\mathbb{A}^2$ [34, Chapter 3, Theorem 2.2.1]. If Y' contains a Platonic fiber space U, $Y \setminus U$ is a disjoint union of $\mathbb{A}^1$s. Then the class group $\mathrm{C}\ell(Y)$ has rank greater than or equal to the number of the affine lines attached to $Y \setminus U$. Since Y is factorial, it follows that Y is isomorphic to $\mathbb{A}^2/G$ with a finite subgroup G of $\mathrm{GL}(2, \mathbb{C})$. Since Y is factorial, Y is, in fact, isomorphic to the hypersurface $x^2 + y^3 + z^5 = 0$. Let Y^0 be the smooth part of Y. Then $\pi_1(Y^0) \cong G$, which is a binary icosahedral group. Let $Z^0 = g^{-1}(Y^0)$. Then $\mathbb{A}^n \setminus Z_0$ is the inverse image by g of the unique singular point of Y, which has dimension not greater than $n-2$. Hence Z^0 is simply-connected. Let $\widetilde{Y}^0$ be the universal covering of Y^0. Then the morphism $g|_{Z^0} : Z^0 \to Y^0$ decomposes as

$$g|_{Z^0} : Z^0 \xrightarrow{g_1} \widetilde{Y}^0 \xrightarrow{g_2} Y^0 \ .$$

Then general fibers of g are not connected. Since a general fiber of g is isomorphic to $\mathbb{A}^{n-2}$. This is a contradiction. Hence $Y \cong \mathbb{A}^2$. $\square$

We need the following result of A. Dutta [6].

Lemma 4.1.3. *Let $q : X \to Y$ be a faithfully flat affine morphism of finite type of locally noetherian schemes. Assume that Y is normal and that the following conditions are satisfied.*

1. *The fiber of q over the generic point of Y is $\mathbb{A}^1$.*
2. *The fiber of q over the generic point of each irreducible reduced closed subscheme of Y of codimension one is geometrically integral.*

Then X is an $\mathbb{A}^1$-bundle over Y. In particular, if Y is an affine scheme then X is a line bundle over Y.

In order to use the result of Dutta, we have to show the following result.

Lemma 4.1.4. *In our setting of $q : X \to Y$, the condition (1) and (2) in Lemma 4.1.3 are satisfied. Hence X is an $\mathbb{A}^1$-bundle over the open set $q(X)$ of Y, where $S := X \setminus q(X)$ is a finite set.*

Proof. We apply Lemma 4.1.3 to the morphism $q : X \to q(Y)$. Since X and Y are smooth and since q is equi-dimensional, it follows that q is a flat morphism. The local slice theorem (see [10]) implies the condition (1). Since every non-empty fiber of q ia a G_a-orbit, the fiber of q over the generic point of an irreducible curve C of Y is geometrically irreducible (see the proof of Lemma 3.4.3). If it is not geometrically reduced, a general fiber X_y with $y \in C$ is a multiple fiber of q. By the argument of Lemma 3.4.4 implies that q has no multiple fibers. Hence the fiber of q over the generic point of C is geometrically integral, and the condition (2) is satisfied.

Since $q : X \to Y$ is a flat morphism, the set S is a closed set. If it contains an irreducible component C of dimension one, C is defined by a prime element p. Since the empty set $q^{-1}(C)$ is defined by $p = 0$ in X, it follows that $p \in A^*$. Since $A^* = k^*$, this is impossible. So, S is a finite set. $\square$

Remark 4.1.5. Lemma 4.1.4 is not true without the properness assumption of the G_a-action. In fact, let δ be an LND on $A = \mathbb{C}[x, y, z]$ defined by $\delta(x) = 0, \delta(y) = 2z$ and $\delta(z) = x$. Then $h = xy - z^2 \in \mathrm{Ker}\,\delta$, where $\mathrm{Ker}\,\delta$ is eventually a polynomial ring $B = \mathbb{C}[x, h]$. In B, x is a prime element and $T := f^{-1}(L) = \mathbb{A}^2 = \mathrm{Spec}\,\mathbb{C}[y, z]$, where $L = \{x = 0\}$ on $\mathbb{A}^2 = \mathrm{Spec}\,B$. But the restriction $f|_T : T \to L$ is not a fibration. For a point $h = c \in \mathbb{C}$ on the line L, the inverse image of thies point in T is two lines $z = \pm\sqrt{c}$ parametrized by y if $c \neq 0$ and the double line if $c = 0$.

We need to show the following result.

Lemma 4.1.6. *The morphism $q : X \to Y$ is surjective.*

Proof. This is proved by the same argument as in Theorem 3.4.6 if one notes that X is contractible and $q : X \to q(X)$ is a locally trivial $\mathbb{A}^1$-bundle. $\square$

By combining the above lemmas together, we know that X is an $\mathbb{A}^1$-bundle over $Y \cong \mathbb{A}^1$. Hence $X \cong \mathbb{A}^3$.

4.2. **When does X have a geometric quotient?** Let X be a smooth affine threefold with a non-trivial G_a-action and let $q : X \to Y$ be the algebraic quotient with $Y = \mathrm{Spec} B$, where $B = \mathrm{Ker}\delta$ in $A := \Gamma(X, \mathcal{O}_X)$ and the lnd δ associated to the G_a-action.

Definiton 4.2.1. $q : X \to Y$ is the *geometric quotient* if the following three conditions are satisfied.

1. For every closed point $Q \in Y$, the fiber $q^{-1}(Y)$ is a G_a-orbit.
2. The topology of Y is the quotient topology of X. Namely, for a subset V of Y, V is open in Y if and only if $q^{-1}(V)$ is an open set in X.
3. For any affine open set $V = \mathrm{Spec}\, R$ of Y, the canonical homomorphism $R \to R \otimes_B A$ induces an isomorphism $R \cong \mathrm{Ker}(R \otimes_B \delta)$.

Lemma 4.2.2. *With the above notations and assumptions, we assume that the G_a-action is free and $q : X \to Y$ is a geometric quotient. Then X is an $\mathbb{A}^1$-bundle over Y. In particular, Y is smooth.*

Proof. By the condition (1), every fiber with reduced structure is irreducible and hence isomorphic to $\mathbb{A}^1$ by the assumption that G_a-action is free. Suppose that the fiber $f^*(Q)$ is a multiple fiber for a closed point $Q \in Y$. Let $B_n = \mathcal{O}_{Y,Q}/\mathfrak{m}_{Y,Q}^{n+1}$ and let $X_n := X \times_Y \mathrm{Spec}\, B_n$. Then the induced G_a-action on X_n is free. Hence, by [33], $Y_n \cong \mathbb{A}^1 \times \mathrm{Spec}\, R_n$. Here Y_0 is non-reduced because Y_0 is a multiple fiber. Considering the inverse limeits, we have

$$\widehat{X} = \varprojlim_{n\to\infty} Y_n \cong \mathbb{A}^1 \times \varprojlim_{n\to\infty} \mathrm{Spec}\, B_n \cong \mathbb{A}^1 \times \widehat{Y}.$$

Hence $Y_0 \cong \mathbb{A}^1$, which is a contradiction. We can argue in a different way. By [14, Theorem 3.11], $f^*(Q)_{\mathrm{red}}$ is contained in the fixed-point locus which is an empty set. We have therefore shown that $q : X \to Y$ has only irreducible reduced fibers isomorphic to $\mathbb{A}^1$. This implies that X is an $\mathbb{A}^1$-bundle over Y. $\square$

Free G_a-action does not necessarily imply that $q : X \to Y$ is a geometric quotient.

Example 4.2.3. Let $X = Z \times \mathbb{A}^1$, where Z is a Danielewski surface defined by $xy = z^2 - 1$. Consider a G_a-action on Z defined by an lnd δ such that $\delta(x) = 0, \delta(y) = 2z$ and $\delta(z) = x$. Then G_a acts as

$$^t x = x, \quad ^t y = y + 2zt + xt^2, \quad ^t z = z + xt.$$

Hence the action is free. However, the quotient morphism $q_0 : Z \to \mathbb{A}^1 = \operatorname{Spec} k[x]$ has a reducible fiber $q_0^{-1}(0)$. Now, take $X = Z \times \mathbb{A}^1$ and the G_a-action which is the given one on Z and trivial on the direct product factor $\mathbb{A}^1$. Then $q = q_0 \times \mathrm{id}_{\mathbb{A}^1}$ and $Y = \mathbb{A}^1 \times \mathbb{A}^1$. So the G_a-action on X is free but the quotient is not a geometric quotient.

Definiton 4.2.4. Let $X = \operatorname{Spec} A$ be an affine variety with a nontrivial G_a-action. We assume that the ring of invariants $B := A^{G_a}$ is finitely generated over k. Hence we can define the quotient morphism $q : X \to Y$, where $Y = \operatorname{Spec} B$ is the algebraic quotient. The G_a-action σ on X is called *free* if the morphism $\Psi := (\sigma, \mathrm{id}_X) : G_a \times X \to X \times X$ is a closed immersion. Since Ψ is factored by $\Psi_Y := (\sigma, X)_Y : G_a \times X \to X \times_Y X$, σ is free if Ψ_Y is a closed immersion.

Lemma 4.2.5. *With the above notations, we have the following implications on the G_a-action σ.*

$$\textit{free} \implies \textit{proper} \implies \textit{fixed-point free.}$$

If σ is free, then Y is the geometric quotient.

Proof. Note that $X \times_Y X = \operatorname{Spec}(A \otimes_B A)$ and $A[b^{-1}] = B[b^{-1}][u]$ for some $b \in B \setminus \{0\}$. Hence $(A \otimes_B A) \otimes_B B[b^{-1}] \cong A[b^{-1}][t]$. Namely, Ψ_Y is an isomorphism over an open set $D(b)$ of Y. Hence $\Psi^* : A \otimes_B A \to A \otimes_k k[t] = A[t]$ is injective. Hence Ψ is a dominant morphism. Suppose that σ is free. Since Ψ_Y is a closed immersion, $\Psi^* : A \otimes_B A \to A[t]$ is a surjective homomorphism. Hence Ψ^* is an isomorphism. This implies that $\Psi_Y : G_a \times X \to X \times_Y X$ is an isomorphism. In particular, X is a G_a-torsor over Y. Hence Y is a geometric quotient. It is clear that if σ is free, then σ is proper because the closed immersion is a proper morphism.

If σ is proper, every fiber of $q : X \to Y$ is a one-dimensional G_a-orbit. In particular, σ is fixed-point free. $\square$

In [19], it is proved that if σ is a fixed-point free action on $X = \mathbb{A}^3$, it is a free action.

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