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Criteria for algebraicity of analytic functions

Jacek Bochnak, Janusz Gwoździewicz and Wojciech Kucharz

Abstract

We consider functions defined on an open subset of a nonsingular, either real or complex, algebraic set. We give criteria for an analytic function to be a Nash (resp. regular, resp. polynomial) function. Our criteria depend only on the behavior of such a function along irreducible nonsingular algebraic curves passing trough a given point. In the proofs we use results on algebraicity of formal power series, which are also established in this paper.

1 Introduction

Throughout this paper we let $\mathbb{F}$ stand for either the field of real numbers $\mathbb{R}$ or the field of complex numbers $\mathbb{C}$. Unless explicitly stated otherwise, all subsets of $\mathbb{F}^n$ will be endowed with the Euclidean topology, induced by the standard norm. We give criteria for algebraicity of $\mathbb{F}$-analytic ($\mathbb{R}$-analytic = real analytic, $\mathbb{C}$-analytic = holomorphic) functions. If no confusion is possible we write *analytic* instead of $\mathbb{F}$-*analytic*. The term *algebraicity* corresponds to three classes of functions: Nash, regular, and polynomial.

Let X be a nonsingular algebraic subset of $\mathbb{F}^m$ and let $f : U \to \mathbb{F}$ be an analytic function defined on an open subset U of X. Recall that f is called a *Nash function* if every point $p \in U$ has a connected open neighborhood U_p in U such that

$$\sum_{i=0}^{k} \varphi_i(x) f(x)^{k-i} = 0 \quad \text{for all } x \in U_p,$$

where the $\varphi_i : \mathbb{F}^m \to \mathbb{F}$ are polynomial functions, not all identically equal to 0 on U_p. Equivalently, one can require that the Zariski closure of the graph of $f|_{U_p}$ in $X \times \mathbb{F}$ be of dimension $\dim U_p$.

We say that a function $f : A \to \mathbb{F}$, defined on some subset A of $\mathbb{F}^m$, is *regular at a point* $p \in A$ if there exist two polynomial functions φ, $\psi : \mathbb{F}^m \to \mathbb{F}$ such that

$$\psi(p) \neq 0 \text{ and } \psi(x) f(x) = \varphi(x) \text{ for all } x \in A.$$

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We say that f is a *regular function* if it is regular at every point of A. We call f a *polynomial function* if it is the restriction of a polynomial function from $\mathbb{F}^m$ to $\mathbb{F}$.

First we consider the case of Nash functions.

Theorem 1.1 *Let $X \subset \mathbb{F}^m$ be a nonsingular algebraic subset, $f : U \to \mathbb{F}$ an analytic function defined on a connected open subset U of X, and p a point in U. Assume that for every irreducible nonsingular algebraic curve C in X, with $p \in C$, the restriction $f|_{U\cap C}$ is a Nash function. Then f is a Nash function.*

For regular functions we have the following.

Theorem 1.2 *Let $X \subset \mathbb{F}^m$ be an irreducible nonsingular algebraic subset, $f : U \to \mathbb{F}$ an analytic function defined on an open subset U of X, and p a point in U. Assume that for every irreducible nonsingular algebraic curve C in X, with $p \in C$, the restriction $f|_{U\cap C}$ is a regular function. Then f is a regular function.*

In particular, Theorem 1.2 holds if $U = X$. Recall that an irreducible nonsingular algebraic subset of $\mathbb{R}^m$ need not to be connected.

Theorems 1.1 and 1.2 fit into the direction of research presented in [2], [3], [6], [8]–[10]. Their proofs, given in Section 3, depend on rather subtle arguments involving power series. In Section 2, we deal with problems of a local nature. Of independent interest is Theorem 2.1, which gives a criterion for algebraicity of formal power series. It enables us to obtain a criterion, stated as Theorem 2.4, for an analytic function defined on a connected open neighborhood of the origin in $\mathbb{F}^n$ to be a Nash (resp. regular, resp. polynomial) function. Various aspects of the local case were investigated by several authors, going back go Hurwitz, Kronecker and Weierstrass (see [4, pp. 199–203], [15], [16] and the references therein). The theory of Nash functions is elaborated in [1]. In older texts, for example [4], Nash functions appear under the name *algebraic functions*.

2 Algebraicity of power series

To begin with we review some terminology and notation. By a ring we always mean a commutative ring with identity. Let R be a subring of a ring S. An element $u \in S$ is said to be algebraic over R if it is a root of a nonzero polynomial P in $R[Z]$; if $\deg P = d$ and u is not a root of any nonzero polynomial in $R[Z]$ of degree strictly less than d, then u is said to be algebraic of degree d over R. Given elements $u_1, \dots, u_n$ in S, we denote by $R[u_1, \dots, u_n]$ the subring of S generated by R and $u_1, \dots, u_n$. A polynomial F in $R[X] = R[X_1, \dots, X_n]$ will be frequently denoted by $F(X)$ or $F(X_1, \dots, X_n)$, where $X = (X_1, \dots, X_n)$. We adopt the same convention for formal power series in $R[[X]] = R[[X_1, \dots, X_n]]$. We will also write $X^i = X_1^{i_1} \cdots X_n^{i_n}$ for $i = (i_1, \dots, i_n) \in \mathbb{N}^n$, where $\mathbb{N}$ denotes the set of nonnegative integers.

The following is the key result of this section.

Theorem 2.1 *Let $f(X_1, \ldots, X_n)$ be a formal power series in variables $X_1, \ldots,$ X_n with coefficients in a field $\mathbb{K}$. Let R be the subring of $\mathbb{K}$ generated by the coefficients of $f(X_1, \ldots, X_n)$. Assume that there exist elements $t_1, \ldots, t_{n-1}$ in $\mathbb{K}$ which are algebraically independent over R. If the power series*

$$f(t_1 Y, \ldots, t_{n-1} Y, Y)$$

in one variable Y is algebraic of degree d over $\mathbb{K}[Y]$, then $f(X_1, \ldots, X_n)$ is algebraic of degree d over $\mathbb{K}[X_1, \ldots, X_n]$.

The proof of this theorem requires some preparation. For the sake of clarity we record the following fact.

Lemma 2.2 *Let R be a subring of a field $\mathbb{K}$ and let $A = (a_{ij})$, where $i \in \mathbb{N}$ and $j \in \{1, \ldots, n\}$, be an infinite matrix with entries in R. Then the solution space in $\mathbb{K}^n$ of the system of linear equations*

$$\sum_{j=1}^{n} a_{ij} T_j = 0, \quad i \in \mathbb{N} \tag{2.1}$$

is spanned by solutions lying in R^n. In particular, if (2.1) has a solution $(u_1, \ldots, u_n) \in \mathbb{K}^n$ with $u_1 \neq 0$, then it also has a solution $(v_1, \ldots, v_n) \in R^n$ with $v_1 \neq 0$.

Proof. Let r be the number of linearly independent rows of the matrix A. Then the solution space of (2.1) in $\mathbb{K}^n$ is spanned by $n - r$ vectors in K^n, where K is the field of fractions of R. Multiplying each of these vectors by the common denominator of its entries, we may assume that they belong to R^n. The assertion readily follows. ■

Lemma 2.3 *Let $h(Y) = h(Y_1, \ldots, Y_r)$ be a formal power series in variables $Y = (Y_1, \ldots, Y_r)$ with coefficients in a field $\mathbb{K}$. Let R be a subring of $\mathbb{K}$ that contains the coefficients of $h(Y)$. Assume that $h(Y)$ is algebraic of degree d over $\mathbb{K}[Y]$. Then there exists a polynomial $H(Y, Z)$ in $R[Y, Z]$, where Z is a single variable, with the following properties:*

(i) *$H(Y, Z)$ is of degree d with respect to Z;*

(ii) *$H(Y, h(Y)) = 0$.*

Proof. Since $h(Y)$ is algebraic of degree d over $\mathbb{K}[Y]$, there exists an irreducible polynomial $G(Y, Z)$ in $\mathbb{K}[Y, Z]$ such that (i) and (ii) hold with $H(Y, Z)$ replaced by $G(Y, Z)$. Clearly, $G(Y, Z)$ is irreducible when regarded as a polynomial in Z with coefficients in the field of fractions of $\mathbb{K}[Y]$.

Claim. The polynomial $G(Y, Z)$ is uniquely determined up to multiplication by a nonzero constant in $\mathbb{K}$.

Indeed, let $F(Y,Z)$ be another irreducible polynomial in $\mathbb{K}[Y,Z]$ such that (i) and (ii) hold with $H(Y,Z)$ replaced by $F(Y,Z)$. By a standard fact from the theory of fields,

$$A(Y)F(Y,Z) = B(Y)G(Y,Z)$$

for some nonzero polynomials $A(Y)$, $B(Y)$ in $\mathbb{K}[Y]$. Hence $F(Y,Z)$ is a constant multiple of $G(Y,Z)$, these two polynomials being irreducible. This proves the Claim.

It remains to show that there is a nonzero constant $c \in \mathbb{K}$ such that the polynomial $H(Y,Z) = cG(Y,Z)$ has coefficients in R. To this end let

$$G(Y,Z) = \sum_{j,k} b_{jk} Y^j Z^k,$$

where $j \in \mathbb{N}^r$, $k \in \mathbb{N}$, and $b_{jk} \in \mathbb{K}$. We get

$$G(Y,h(Y)) = \sum_i c_i Y^i,$$

where $i \in \mathbb{N}^r$, and the coefficients $c_i \in \mathbb{K}$ are of the form

$$c_i = \sum_{j,k} a_{ijk} b_{jk} = 0,$$

where $a_{ijk} \in R$ (recall that the coefficients of $h(Y)$ belong to R). By replacing each nonzero element b_{jk} with variable X_{jk}, we obtain the linear forms

$$C_i = \sum_{j,k} a_{ijk} X_{jk}$$

in n variables, where n is the number of nonzero elements b_{jk}. Now the issue is to show that the system of linear equations

$$C_i = 0, \quad i \in \mathbb{N}^r \tag{2.2}$$

has a nonzero solution in R^n. In view of the Claim, the solution space of (2.2) in $\mathbb{K}^n$ is a 1-dimensional vector subspace. Since $a_{ijk} \in R$, the proof is complete in view of Lemma 2.2. ■

Proof of Theorem 2.1. Assume that the formal power series

$$h(Y) = f(t_1 Y, \ldots, t_{n-1} Y, Y) \tag{2.3}$$

is algebraic of degree d over $\mathbb{K}[Y]$. Since the coefficients of $h(Y)$ belong to the subring $S = R[t_1, \ldots, t_{n-1}]$ of $\mathbb{K}$, applying Lemma 2.3 we obtain a polynomial $H(Y,Z)$ in $S[Y,Z]$, of degree d with respect to Z, such that

$$H(Y,h(Y)) = 0. \tag{2.4}$$

The polynomial $H(Y, Z)$ can be expressed as a finite sum

$$H(Y, Z) = \sum_{i,j} A_{ij}(t_1, \ldots, t_{n-1}) Y^i Z^j,$$

where $i, j \in \mathbb{N}$, and the A_{ij} are polynomials in $R[T_1, \ldots, T_{n-1}]$. Defining the polynomial F in $R[T_1, \ldots, T_{n-1}, Y, Z]$ by

$$F(T_1, \ldots, T_{n-1}, Y, Z) = \sum_{i,j} A_{ij}(T_1, \ldots, T_{n-1}) Y^i Z^j,$$

we get

$$H(Y, Z) = F(t_1, \ldots, t_{n-1}, Y, Z) \tag{2.5}$$

Note that

$$F(T_1, \ldots, T_{n-1}, Y, f(T_1Y, \ldots, T_{n-1}Y, Y)) = \sum_k B_k(T_1, \ldots, T_{n-1}) Y^k,$$

where $k \in \mathbb{N}$, and the B_k are polynomials in $R[T_1, \ldots, T_{n-1}]$. In view of (2.3), (2.4), and (2.5), we get

$$B_k(t_1, \ldots, t_{n-1}) = 0 \quad \text{for all } k \in \mathbb{N},$$

and hence

$$F(T_1, \ldots, T_{n-1}, Y, f(T_1Y, \ldots, T_{n-1}Y, Y)) = 0, \tag{2.6}$$

the elements t_t, ..., t_{n-1} being algebraically independent over R. Now, if $l \in \mathbb{N}$ is sufficiently large, then

$$Y^l F(T_1, \ldots, T_{n-1}, Y, Z) = \tilde{F}(T_1Y, \ldots, T_{n-1}Y, Y, Z)$$

for some polynomial $\tilde{F}$ in $\mathbb{K}[X_1, \ldots, X_n, Z]$ of degree d with respect to Z. According to (2.6), we have

$$\tilde{F}(T_1Y, \ldots, T_{n-1}Y, Y, f(T_1Y, \ldots, T_{n-1}Y, Y)) = 0,$$

which in turn implies that

$$\tilde{F}(X_1, \ldots, X_n, f(X_1, \ldots, X_n)) = 0.$$

Therefore the power series $f(X_1, \ldots, X_n)$ is algebraic of degree e over the ring $\mathbb{K}[X_1, \ldots, X_n]$, for some $e \leq d$.

We now prove that $e = d$. Applying Lemma 2.3 to $f(X_1, \ldots, X_n)$ we obtain a polynomial G in $R[X_1, \ldots, X_n, Z]$, of degree e with respect to Z, such that

$$G(X_1, \ldots, X_n, f(X_1, \ldots, X_n)) = 0.$$

Obviously

$$G(t_1Y, \ldots, t_{n-1}Y, Y, h(Y)) = 0.$$

The polynomial $G(t_1Y, \ldots, t_{n-1}Y, Y, Z)$ has degree e with respect to Z, the elements t_1, ..., t_{n-1} being algebraically independent over R. Since $h(Y)$ is algebraic of degree d over $\mathbb{K}[Y]$, we obtain $e = d$, as required. ■

Theorem 2.1 enables us to prove the following result on analytic functions.

Theorem 2.4 *Let $f : U \to \mathbb{F}$ be an analytic function defined on a connected open neighborhood U of the origin in $\mathbb{F}^n$. Let W be a nonempty open subset of $\mathbb{F}^n$ and assume that for every vector line $L \subset \mathbb{F}^n$, with $W \cap L \neq \emptyset$, the restriction $f|_{U\cap L}$ is a Nash, (resp. regular, resp. polynomial) function. Then f is a Nash, (resp. regular, resp. polynomial) function.*

Proof. In a neighborhood of $0 \in \mathbb{F}^n$, the function f is given as a convergent power series

$$f(x_1, \ldots, x_n) = \sum c_{i_1,\ldots,i_n} x_1^{i_1} \cdots x_n^{i_n},$$

where $i_1, \ldots, i_n$ are nonnegative integers. Denoting by R the subring of $\mathbb{F}$ generated by the coefficients $c_{i_1,\ldots,i_n}$, we can find a vector line $L = \mathbb{F}\tau \subset \mathbb{F}^n$, where $\tau = (t_1, \ldots, t_{n-1}, 1) \in \mathbb{F}^n$ with $t_1, \ldots, t_{n-1}$ algebraically independent over R, so that $L \cap W \neq \emptyset$.

We now consider three cases.

Case 1. If $f|_{U\cap L}$ is a Nash function, then the power series

$$f(t_1 x_n, \ldots, t_{n-1} x_n, x_n)$$

is algebraic over $\mathbb{F}[x_n]$. By Theorem 2.1, the power series $f(x_1, \ldots, x_n)$ is algebraic over $\mathbb{F}[x_1, \ldots, x_n]$. Thus f is a Nash function on U, the open set U being connected.

Case 2. If $f|_{U\cap L}$ is a regular function that is not a polynomial function, then the power series $f(t_1 x_n, \ldots, t_{n-1} x_n, x_n)$ is algebraic of degree 1 over $\mathbb{F}[x_n]$. Hence, by Theorem 2.1, the power series $f(x_1, \ldots, x_n)$ is algebraic of degree 1 over $\mathbb{F}[x_1, \ldots, x_n]$. It follows that

$$\psi f = \varphi \quad \text{on } U$$

for some nonzero relatively prime polynomial functions $\varphi, \psi : \mathbb{F}^n \to \mathbb{F}$.

Suppose that $\mathbb{F} = \mathbb{C}$. Since φ, ψ are relatively prime, any point $p \in U \cap \psi^{-1}(0)$ would be an accumulation point of the set $\psi^{-1}(0) \setminus \varphi^{-1}(0)$. This, however, cannot happen, the function f being holomorphic. Consequently, $\psi(p) \neq 0$ for all $p \in U$, which means that the function f is regular on U.

If $\mathbb{F} = \mathbb{R}$, we can extend f to a holomorphic function defined on a connected open subset of $\mathbb{C}^n$ that contains U. Therefore f is a regular function on U also for $\mathbb{F} = \mathbb{R}$.

Case 3. If $f|_{U\cap L}$ is a polynomial function, then the power series

$$f(t_1 x_n, \ldots, t_{n-1} x_n, x_n) = \sum_{k=0}^{\infty} \Big(\sum_{i_1+\cdots+i_n=k} c_{i_1,\ldots,i_n} t_1^{i_1} \cdots t_{n-1}^{i_{n-1}} \Big) x_n^k$$

reduces to a polynomial in x_n. Hence, for all sufficiently large k, the coefficient of x_n^k vanishes. Since $t_1, \ldots, t_{n-1}$ are algebraically independent over R, the coefficients $c_{i_1,\ldots,i_n}$ with $i_1 + \cdots + i_n = k$ large enough are all zero. Consequently, f is a polynomial function on U. ■

3 Global results

The projective space of vector lines in $\mathbb{F}^n$ will be denoted by $\mathbb{P}^{n-1}(\mathbb{F})$. We will make use of the following auxiliary result.

Lemma 3.1 *Let $X \subset \mathbb{F}^m$ be a nonsingular algebraic subset of pure dimension $n \geq 1$. Assume that X contains the origin 0 in $\mathbb{F}^m$. Then there exists a polynomial map $\pi : X \to \mathbb{F}^n$ such that the following hold:*

(i) *π is the restriction of a surjective linear map $\mathbb{F}^m \to \mathbb{F}^n$.*

(ii) *$0 \in \mathbb{F}^n$ is a regular value of π.*

Furthermore, for such a map π, there exists a nonempty Zariski open subset Ω of $\mathbb{P}^{n-1}(\mathbb{F})$ such that π is transverse to every vector line in Ω.

Proof. We may assume that after a linear change of coordinates in $\mathbb{F}^m$ the tangent space to X at 0 coincides with $\mathbb{F}^n \times \{0\} \subset \mathbb{F}^m$. Let $\varphi : X \to \mathbb{F}^n$ be the restriction of the canonical projection $\mathbb{F}^m = \mathbb{F}^n \times \mathbb{F}^{m-n} \to \mathbb{F}^n$. Clearly φ is a submersion at $0 \in X$.

Let M be the space of all n-by-m matrices with entries in $\mathbb{F}$. For any constant $\epsilon > 0$, we set

$$M_\epsilon = \{\, t = (t_{ij}) \in M : |t_{ij}| < \epsilon \text{ for } 1 \leq i \leq n, 1 \leq j \leq m \,\}$$

and consider the map $\Phi : X \times M_\epsilon \to \mathbb{F}^n$ defined by

$$\Phi(x,t) = \Big(x_1 + \sum_{j=1}^{m} t_{1j} x_j, \ldots, x_n + \sum_{j=1}^{m} t_{nj} x_j\Big),$$

where $x = (x_1, \ldots, x_m) \in X$ and $t = (t_{ij}) \in M_\epsilon$. If ϵ is sufficiently small, then the map Φ is a submersion, since for each point $x \neq 0$ the restriction of Φ to $\{x\} \times M_\epsilon$ is a submersion, and φ is a submersion at 0. Hence, according to the standard consequence of Sard's theorem [5, p. 79, Theorem 2.7], the point $0 \in \mathbb{F}^n$ is a regular value of the map $\Phi_t : X \to \mathbb{F}^n$, $\Phi_t(x) = \Phi(x,t)$ for some $t \in M_\epsilon$. For the map $\pi = \Phi_t$ both conditions (i) and (ii) hold.

The last assertion in Lemma 3.1 is proved in [2, Lemma 4.2] for $\mathbb{F} = \mathbb{R}$, and the same argument works also for $\mathbb{F} = \mathbb{C}$. ∎

Proof of Theorem 1.1. We may assume without loss of generality that X is irreducible of dimension $n \geq 1$ and $p = 0$ is the origin in $\mathbb{F}^m$. Let $\pi : X \to \mathbb{F}^n$ and $\Omega \subset \mathbb{P}^{n-1}(\mathbb{F})$ be as in Lemma 3.1. Choose an open neighborhood N of p in U so that the map

$$\varphi = \pi|_N : N \to \pi(N) = V$$

is a Nash diffeomorphism, where V is a connected open neighborhood of the origin $0 \in \mathbb{F}^n$. Define $g : V \to \mathbb{F}$ to be the composite function $g = f \circ \varphi^{-1}$. Since π is transverse to every vector line $L \in \Omega$, it follows that the inverse image $\pi^{-1}(L)$ is a nonsingular algebraic curve in X, passing through p. Let $C(L)$

denote the irreducible component of $\pi^{-1}(L)$ that contains p. By assumption, the restriction $f|_{U\cap C(L)}$ is a Nash function, which implies that the restriction $g|_{V\cap L}$ is a Nash function. Applying Theorem 2.4 we get that g itself is a Nash function. Consequently, $f|_N$ is a Nash function. If follows that f is a Nash function as well, the set U being connected. ■

It is worthwhile to record the following consequence of Theorem 1.1.

Corollary 3.2 *Let $f : X \to \mathbb{C}$ be a holomorphic function defined on an irreducible nonsingular algebraic subset $X \subset \mathbb{C}^m$, and let p be a point in X. Assume that for every irreducible nonsingular algebraic curve C in X, with $p \in C$, the restriction $f|_C$ is a Nash function. Then f is a polynomial function.*

Proof. Since we consider the complex case, the set X is connected. Hence, in view of Theorem 1.1 (with $\mathbb{F} = \mathbb{C}$), f is a Nash function. It readily follows that the graph of f is an algebraic subset of $X \times \mathbb{C}$. Consequently, f is a regular function by the theorem of Serre [14, Proposition 8] or [12]. It is well known that any regular function on an algebraic subset of $\mathbb{C}^m$ is polynomial. ■

The proof of Theorem 1.2 is more involved and requires additional preparation.

Let $X \subset \mathbb{F}^m$ be an irreducible nonsingular algebraic subset and let $f : U \to \mathbb{F}$ be a function defined on an open subset U of X. We say that f *admits a rational representation* if there exist two polynomial functions $\varphi, \psi : \mathbb{F}^m \to \mathbb{R}$ such that

$$\psi(x)f(x) = \varphi(x) \quad \text{for all } x \in U$$

and ψ is not identically 0 on X.

We will make use of the following fact (see also [7, Proposition 2.1]).

Lemma 3.3 *Let $X \subset \mathbb{F}^m$ be an irreducible nonsingular algebraic subset and let U be an open subset of X. If an analytic $\mathbb{F}$-valued function on U admits a rational representation, then it is a regular function.*

Proof. The conclusion holds since for any point $p \in U$, the ring of germs of analytic functions at p is faithfully flat over the ring of germs of regular functions at p (the algebraic assertion here follows from [13, Theorem 8.14]). ■

Now we are in a position to reduce Theorem 1.2 to a local assertion.

Lemma 3.4 *With notation and hypothesis as in Theorem 1.2 assume, in addition, that for some open neighborhood $U_p \subset U$ of the point p the restriction $f|_{U_p}$ admits a rational representation. Then f is a regular function.*

Proof. By Lemma 3.3, it suffices to prove that f admits a rational representation (this is not entirely obvious because the set U need not to be connected). Since $f|_{U_p}$ admits a rational representation, we can choose two polynomial functions $\varphi, \psi : \mathbb{F}^m \to \mathbb{F}$ such that ψ is not identically 0 on X and

$$\psi(x)f(x) = \varphi(x) \quad \text{for all } x \in U_p.$$

It remains to show that this equality actually holds for all $x \in U$. Suppose this is not the case. Then the set

$$W = \{ x \in U : \psi(x) f(x) \neq \varphi(x) \}$$

is nonempty and open in U. By a suitable variant of Bertini's theorem (see Lemma 3.5 below), there exists an irreducible nonsingular algebraic curve C in X with

$$p \in C \text{ and } W \cap C \neq \emptyset.$$

Regularity of the restriction $f|_{U \cap C}$ allows us to choose two polynomial functions $\alpha, \beta : \mathbb{F}^m \to \mathbb{F}$ with

$$\beta(p) \neq 0 \text{ and } \beta(x) f(x) = \alpha(x) \text{ for all } x \in U \cap C.$$

Thus

$$\psi(x) \frac{\alpha(x)}{\beta(x)} = \psi(x) f(x) = \varphi(x) \quad \text{for all } x \in C \text{ near } p.$$

Consequently

$$\psi(x)\alpha(x) = \varphi(x)\beta(x) \quad \text{for all } x \in C,$$

the curve C being irreducible. Now, choose a point $q \in W \cap C$ with $\beta(q) \neq 0$. Then we get

$$\psi(q) f(q) = \varphi(q),$$

a contradiction. This completes the proof. ■

We have used the following consequence of Bertini's theorem, which is included here for the sake of completeness.

Lemma 3.5 *Let $X \subset \mathbb{F}^m$ be an irreducible nonsingular algebraic subset of positive dimension, p a point in X, and W a nonempty open subset of X. Then there exists an irreducible nonsingular algebraic curve C in X such that $p \in C$ and $W \cap C \neq \emptyset$.*

Proof. We assume that $\dim X = n \geq 2$ and $p = 0$ is the origin in $\mathbb{F}^m$. Let $\pi : X \to \mathbb{F}^n$ and $\Omega \subset \mathbb{P}^{n-1}(\mathbb{F})$ be as in Lemma 3.1. Clearly, the subset $\pi(W)$ of $\mathbb{F}^n$ has nonempty interior.

Case 1. Suppose that $\mathbb{F} = \mathbb{C}$.

Since $0 \in \mathbb{C}^n$ is a regular value of the map $\pi : X \to \mathbb{C}^n$, it follows that for a general vector line $L \subset \mathbb{C}^n$ the inverse image $\pi^{-1}(L)$ is an irreducible algebraic curve in X (to see the validity of this assertion, one can view $\mathbb{C}^n$ as a subset of $\mathbb{P}^n(\mathbb{C})$, identify vector lines in $\mathbb{C}^n$ with projective lines in $\mathbb{P}^n(\mathbb{C})$ passing through the point $0 \in \mathbb{C}^n \subset \mathbb{P}^n(\mathbb{C})$, and consult the proof of Bertini's theorem in [11, Theorem 3.3.1]). Choosing such a line L so that $L \in \Omega$ and $L \cap \pi(W) \neq \emptyset$, we obtain an irreducible nonsingular algebraic curve $C := \pi^{-1}(L)$ in X with $p \in C$ and $C \cap W \neq \emptyset$.

Case 2. Suppose that $\mathbb{F} = \mathbb{R}$.

Choose an irreducible nonsingular Zariski locally closed subset $\mathbb{X} \subset \mathbb{C}^m$, defined over $\mathbb{R}$, so that its set of real points $\mathbb{X}(\mathbb{R})$ coincides with X. Denote again by $\pi : \mathbb{X} \to \mathbb{C}^n$ the restriction of the canonical projection $\mathbb{C}^m = \mathbb{C}^n \times \mathbb{C}^{m-n} \to \mathbb{C}^n$. Shrinking $\mathbb{X}$ if necessary, we may assume that $0 \in \mathbb{C}^n$ is a regular value of π. We conclude by proceeding as in Case 1, ■

In our last lemma we return to the notion of rational representation in the context of formal power series.

Lemma 3.6 *Let $\mathbb{K}$ be a field. Let $F \in \mathbb{K}[[V]]$ be a formal power series in variables $V = (V_1, \dots, V_m)$, and let $\varphi = (\varphi_1, \dots, \varphi_m)$ be an m-tuple of formal power series $\varphi_i \in \mathbb{K}[[T]]$ in one variable T, with $\varphi(0) = 0$. Assume that there exist two polynomials $g, h \in \mathbb{K}[V]$ for which*

$$(Fh - g) \circ \varphi = 0 \text{ and } h \circ \varphi \neq 0.$$

If the coefficients of the power series F, φ_1, $\dots$, φ_m are all in a subring R of $\mathbb{K}$, then there exist two polynomials $G, H \in R[V]$ for which

$$(FH - G) \circ \varphi = 0 \text{ and } H \circ \varphi \neq 0.$$

Proof. Suppose that the subring R contains the coefficients of the power series F, φ_1, $\dots$, φ_m. Let $\{g_\alpha\}$ and $\{h_\beta\}$, for some α and β in $\mathbb{N}^m$, be the collections of all nonzero coefficients of the polynomials g and h, respectively. By equating to 0 the coefficients of the power series $(Fh - g)(\varphi(T))$ in T, we obtain relations

$$\sum_\alpha a_{i\alpha} g_\alpha + \sum_\beta b_{i\beta} h_\beta = 0 \quad \text{for } i \in \mathbb{N},$$

where the $a_{i\alpha}$ and $b_{i\beta}$ belong to R.

Since the power series $h(\varphi(T)) = \sum_j d_j T^j$ is nonzero, for some $l \in \mathbb{N}$, we get a relation

$$d_l = \sum_\beta c_\beta h_\beta \neq 0,$$

where the c_β belong to R.

The system of linear equations

$$\begin{cases} \sum_\alpha a_{i\alpha} Y_\alpha + \sum_\beta b_{i\beta} Z_\beta = 0, & i \in \mathbb{N} \\ U - \sum_\beta c_\beta Z_\beta = 0 \end{cases}$$

in variables Y_α, Z_β, U has a solution

$$(\{Y_\alpha\}, \{Z_\beta\}, U) = (\{g_\alpha\}, \{h_\beta\}, d_l)$$

in $\mathbb{K}$, with $d_l \neq 0$. By Lemma 2.2, this system also has a solution

$$(\{G_\alpha\}, \{H_\beta\}, D)$$

in R, with $D \neq 0$. Hence the polynomials

$$G = \sum_{\alpha} G_{\alpha} V^{\alpha}, \quad H = \sum_{\beta} H_{\beta} V^{\beta}$$

belong to $R[V]$ and have the required properties. ■

Proof of Theorem 1.2. We may assume that $\dim X = n \geq 2$. In view of Lemma 3.1, we may also assume that after a translation and a linear change of coordinates in $\mathbb{F}^m$ the following hold:

(i) $p = 0 \in U$.

(ii) $0 \in \mathbb{F}^n$ is a regular value of the restriction $\pi : X \to \mathbb{F}^n$ of the canonical projection $\mathbb{F}^m = \mathbb{F}^n \times \mathbb{F}^{m-n} \to \mathbb{F}^n$.

(iii) There exists a nonempty Zariski open subset Ω of $\mathbb{P}^{n-1}(\mathbb{F})$ such that π is transverse to every vector line in Ω.

(iv) The vector line $L = \{ (y_1, \ldots, y_n) \in \mathbb{F}^n : y_1 = 0, \ldots, y_{n-1} = 0 \}$ is in Ω.

According to (iii) and (iv), if $\epsilon > 0$ is sufficiently small, then for any elements $t_1, \ldots, t_{n-1}$ in $\mathbb{F}$, with $|t_1| < \epsilon, \ldots, |t_{n-1}| < \epsilon$, the line

$$L(t_1, \ldots, t_{n-1}) = \{ (y_1, \ldots, y_n) \in \mathbb{F}^n : y_1 = t_1 y_n, \ldots, y_{n-1} = t_{n-1} y_n \}$$

is in Ω.

Let us set $k = m - n$, $Y = (Y_1, \ldots, Y_n)$, $Z = (Z_1, \ldots, Z_k)$. Denote by $I(X)$ the ideal of the polynomial ring $\mathbb{F}[Y, Z] = \mathbb{F}[Y_1, \ldots, Y_n, Z_1 \ldots, Z_k]$ that consists of all polynomials vanishing on X. Since (ii) holds, we can find polynomials F_1, $\ldots$, F_k in $I(X)$ such that

$$\det \left(\frac{\partial F_i}{\partial Z_j}(0) \right) \neq 0. \tag{3.1}$$

If $A \in I(X)$, then the germ of A at $0 \in \mathbb{F}^m$ is a linear combination of the germs of the F_i with coefficients that are analytic function-germs $(\mathbb{F}^m, 0) \to \mathbb{F}$. In particular, the germ of the algebraic set X at $0 \in \mathbb{F}^m$ coincides with that of the zero locus of the polynomials F_i.

It follows from (3.1) and the implicit function theorem for power series that there exists a unique k-tuple $\Phi = (\Phi_1, \ldots, \Phi_k)$, where each Φ_j is a formal power series in $\mathbb{F}[[Y]]$, such that

$$\Phi(0) = 0 \text{ and } F_i(Y, \Phi(Y)) = 0 \text{ for } i = 1, \ldots, k.$$

Actually, the Φ_j are convergent power series. Interpreting Φ as an analytic map-germ

$$\Phi : (\mathbb{F}^n, 0) \to (\mathbb{F}^k, 0),$$

we get $F_i(b, \Phi(b)) = 0$ for all b close to $0 \in \mathbb{F}^n$, $i = 1, \ldots, k$. It follows that

$$\operatorname{graph} \Phi = \text{the germ of } X \text{ at } 0 \in \mathbb{F}^m. \tag{3.2}$$

Now, let K be the subfield of $\mathbb{F}$ generated by the coefficients of the polynomials $F_1, \ldots, F_k$. By the implicit function theorem again, $\Phi_j \in K[[Y]]$ for $j = 1, \ldots, k$.

Choose an analytic function-germ $F : (\mathbb{F}^m, 0) \to \mathbb{F}$ so that its restriction to $(X, 0)$ coincides with the germ of f at 0. We identify F with its power series expansion at 0. Thus F is a convergent power series in $Y_1, \ldots, Y_n, Z_1, \ldots, Z_k$. Denote by S the subfield of $\mathbb{F}$ generated by K and the coefficients of F.

Choose elements $t_1, \ldots, t_{n-1}$ in $\mathbb{F}$, with $|t_1| < \epsilon, \ldots, |t_{n-1}| < \epsilon$, that are algebraically independent over S. Since the vector line $L(t_1, \ldots, t_{n-1})$ is in Ω, the inverse image $\pi^{-1}(L(t_1, \ldots, t_{n-1}))$ is a nonsingular algebraic curve in X; denote by C its irreducible component that contains the point $p = 0$. Consider the m-tuple $\varphi = (\varphi_1, \ldots, \varphi_m)$ of convergent power series in one variable T, where

$$\varphi_1(T) = t_1 T, \ \ldots, \ \varphi_{n-1}(T) = t_{n-1} T, \ \varphi_n(T) = T,$$

$$\varphi_{n+j}(T) = \Phi_j(t_1 T, \ldots, t_{n-1} T, T) \quad \text{for } j = 1, \ldots, k.$$

Note that φ, regarded as an analytic map-germ $\varphi : (\mathbb{F}, 0) \to (\mathbb{F}^m, 0)$, is a local parametrization of the curve C near 0. The coefficients of the power series $\varphi_1, \ldots, \varphi_m$ belong to $K[t_1, \ldots, t_{n-1}]$, hence also to $S[t_1, \ldots, t_{n-1}]$.

Since the restriction $f|_{U \cap C}$ is a regular function, there exist two polynomials G, H in $\mathbb{F}[Y, Z]$ such that

$$(FH - G) \circ \varphi = 0 \text{ and } H \circ \varphi \neq 0.$$

By Lemma 3.6 (with $R = S[t_1, \ldots, t_{n-1}]$), we can choose such polynomials G, H in $S[t_1, \ldots, t_{n-1}][Y, Z]$. Define two polynomials $\tilde{G}$, $\tilde{H}$ in $S[t_1, \ldots, t_{n-1}][Y, Z]$ by

$$\tilde{G}(Y, Z) = G(t_1 Y_n, \ldots, t_{n-1} Y_n, Y_n, Z),$$

$$\tilde{H}(Y, Z) = H(t_1 Y_n, \ldots, t_{n-1} Y_n, Y_n, Z).$$

Then

$$(F\tilde{H} - \tilde{G}) \circ \varphi = 0 \text{ and } \tilde{H} \circ \varphi \neq 0.$$

If $l \in \mathbb{N}$ is sufficiently large, then there exist two polynomials P, Q in $S[Y, Z]$ such that

$$Y_n^l \tilde{G}(Y, Z) = P(t_1 Y_n, \ldots, t_{n-1} Y_n, Y_n, Z),$$

$$Y_n^l \tilde{H}(Y, Z) = Q(t_1 Y_n, \ldots, t_{n-1} Y_n, Y_n, Z).$$

Consequently

$$(FQ - P) \circ \varphi = 0 \text{ and } Q \circ \varphi \neq 0.$$

Since P, Q are in $S[Y, Z]$, we get

$$(FQ - P)(\varphi(T)) = \sum_r c_r(t_1, \ldots, t_{n-1}) T^r,$$

where $r \in \mathbb{N}$, and the c_r are polynomials in $S[T_1, \dots, T_{n-1}]$. The equality $(FQ - P) \circ \varphi = 0$ implies that $c_r(T_1, \dots, T_{n-1}) = 0$, the elements $t_1, \dots, t_{n-1}$ being algebraically independent over S. Thus

$$(FQ - P)(T_1 T, \dots, T_{n-1} T, T, \Phi(T_1 T, \dots, T_{n-1} T, T)) = 0$$

as formal power series in $T_1, \dots, T_{n-1}, T$, which implies that

$$(FQ - P)(Y, \Phi(Y)) = 0 \tag{3.3}$$

as formal power series in $Y_1, \dots, Y_n$. The property $Q \circ \varphi \neq 0$ implies that the polynomial Q does not vanish identically on C.

Since

$$F|_{(X,0)} = \text{the germ of } f \text{ at } 0 = p \in X,$$

combining (3.2) and (3.3), we see that the restriction $f|_{U_p}$ of f to some neighborhood $U_p \subset U$ of p admits a rational representation. The proof is complete in view of Lemma 3.4 ■

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